Asymptotic derivation of models for materials with small length scales

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Contents

Introduction

In Elasticity Theory an important role is played by structures presenting small length-scales of geometric or constitutive nature. In particular, understanding the behaviour of such materials and modeling them in an efficient way have recently become a very active research line in Materials Science, especially in view of the development of new technologies. Significant examples are thin objects, like membranes, shells or rods, and systems which are microscopically heterogeneous, like porous media or finely mixed composites. The major task in the study of these problems concerns the derivation of auxiliary *simpler* models which capture the overall properties of the initial ones.

In the case of *thin structures*, in which one or more dimensional extensions are small compared to the others, it is natural to reduce the original problem to a new one in a lower dimensional space, where the small dimensions disappear. Hence, lower dimensional theories are deduced by taking the limit as h goes to zero, where h is the parameter describing the smallness of the object, that is, the thickness in the case of membranes and the diameter of the cross-section in the case of rods.

In a similar way, in the case of *fine scale mixtures* one tries to replace the original heterogeneous material with a homogeneous *fictious* one (the homogenized material) as the size ε of the microstructure goes to zero. Indeed, in a composite the heterogeneities are small compared to its global dimension, and the limit process represents the transition from a microscopic to a macroscopic description of the material. We notice that, while in dimension reduction problems the small length scale is of geometric nature only, composites and highly heterogeneous media can exhibit several scales, of both geometric (size of the heterogeneities) and constitutive (toughness of the material) nature.

In mathematical terms, a common approach to these problems is the study of the asymptotic behaviour of integral functionals depending on a small-scale parameter, as this parameter goes to zero. We focus on the variational method, which is based on the analysis of the limit, in the sense of Γ-convergence, of the elastic energy associated to a deformation (in the nonlinear setting) or to a displacement (in the linear case) of the domain.

We now give an overview of the content of this thesis, which consists of two parts.

In the first part we present some results concerning the derivation of asymptotic models for thin curved rods (see Chapters 2 and 3).

The second part is devoted to the study of homogenization problems for composite materials (see Chapters 4 and 5) and for porous media (see Chapter 6).

Part I: Asymptotic models for thin curved rods

One of the main problems in nonlinear elasticity is to understand the relation between the three-dimensional theory and lower dimensional models for thin structures. In the classical approach these theories are usually deduced via formal asymptotic expansions or adding extra assumptions on the kinematics of the three-dimensional deformations (see, e.g., [16]). Recently the problem of the rigorous derivation of lower dimensional theories has been studied using a variational approach, which is based on the analysis of the limit of the 3D elastic energy in the sense of Γ-convergence. The first step in this direction is due to E. Acerbi, G. Buttazzo and D. Percivale (see [1]), who deduced a nonlinear model for elastic strings by means of a 3D-1D reduction. The analogue in 3D-2D reduction was studied by H. Le Dret and A. Raoult, who derived a nonlinear model for elastic membranes (see [37]). The more delicate case of plates was justified more recently by G. Friesecke, R.D. James and S. Müller in [30] (see also [32] for a complete survey on plate theories). The case of shells was considered in [38] and [31].

As for one-dimensional models, nonlinear theories for elastic rods have been deduced by M.G. Mora, S. Müller (see [40], [41]) and, independently, by O. Pantz (see [45]). In all these results, as in [1], the beam is assumed to be straight in the unstressed configuration.

In this part of the thesis we study the case of a heterogeneous *curved* beam made of a hyperelastic material.

In the following we shall denote by Ω the set $(0,L)\times D$, where $L>0$ and D is a bounded Lipschitz domain in $\mathbb{R}^2$ with $\mathcal{L}^2(D)=1$. Given $h>0$, we shall consider a beam, whose reference configuration is given by

$$\widetilde{\Omega}_h := \{\gamma(s) + h\,\xi\,\nu_2(s) + h\,\zeta\,\nu_3(s) : (s,\xi,\zeta)\in\Omega\},$$

where $\gamma:(0,L)\to\mathbb{R}^3$ is a smooth simple curve describing the mid-fiber of the beam, and $\nu_2,\nu_3:(0,L)\to\mathbb{R}^3$ are two smooth vectors such that $\{\gamma',\nu_2,\nu_3\}$ provides an orthonormal frame along the curve. In particular, the shape of the cross-section of the beam is constant along γ and is given by the set hD. Its orientation in the normal plane to γ, which may vary along the curve, is determined by the orientation of the two vectors $\nu_2(s),\nu_3(s)$.

A natural parametrization of $\widetilde{\Omega}_h$ is given by

$$\Psi^{(h)}:\Omega\to\widetilde{\Omega}_h,\quad (s,\xi,\zeta)\mapsto\gamma(s)+h\,\xi\,\nu_2(s)+h\,\zeta\,\nu_3(s),$$

which is one-to-one for h small enough.

The starting point of the variational approach is the elastic energy per unit cross-section

$$\tilde{I}^{(h)}(\tilde{y}) := \frac{1}{h^2}\int_{\widetilde{\Omega}_h} W\big(\big(\Psi^{(h)}\big)^{-1}(x),\nabla\tilde{y}(x)\big)dx$$

of a deformation $\tilde{y}\in W^{1,2}(\widetilde{\Omega}_h;\mathbb{R}^3)$. The stored energy density $W:\Omega\times\mathbb{M}^{3\times3}\to[0,+\infty]$ is required to satisfy some natural properties:

- W is frame indifferent: $W(z,RF)=W(z,F)$ for a.e. $z\in\Omega$, every $F\in\mathbb{M}^{3\times3}$, and every $R\in SO(3)$;
- $W(z,F)\geq C\,\mathrm{dist}^2(F,SO(3))$ for a.e. $z\in\Omega$ and every $F\in\mathbb{M}^{3\times3}$;
- $W(z,R)=0$ for a.e. $z\in\Omega$ and every $R\in SO(3)$.

For the complete list of assumptions on W we refer to Section 2.1 in Chapter 2.

We provide a description of the asymptotic behaviour of the different scalings of $\tilde{I}^{(h)}$, as $h\to0$, by means of Γ-convergence (see Section 1.1 in Chapter 1).

In Chapter 2 we study the case of energies $\tilde{I}^{(h)}$ of order h^β with $\beta\in[0,2]$. This is done by considering the Γ-limit of $h^{-\beta}\tilde{I}^{(h)}$ as $h\to0$.

As suggested by heuristic arguments, different scalings of the energy in terms of the thickness parameter h may correspond to different elastic behaviours. By means of Γ-convergence we shall provide an asymptotic description of all the meaningful scalings of $\tilde{I}^{(h)}$, as $h\to0$. This will lead to the identification of a complete hierarchy of one-dimensional models for curved beams.

We prove that, as for straight beams, the case $\beta = 0$ corresponds to stretching and shearing deformations, leading to a *string theory* as Γ-limit, while the case $\beta = 2$ corresponds to bending flexures and torsions keeping the mid-fiber unextended, leading to a *rod theory* as Γ-limit. This last result has been obtained also by P. Seppecher and C. Pideri in [51], independently. Finally, we also show that the case $\beta \in (0, 2)$ provides a degenerate model.

The main results of Chapter 2 are contained in Section 2.2, where we identify the Γ-limit of the sequence of functionals $\big(\tilde{I}^{(h)}/h^2\big)$. We first show a compactness result for sequences of deformations having equibounded energies (Theorem 2.3). More precisely, given a sequence $\big(\tilde{y}^{(h)}\big) \subset W^{1,2}(\widetilde{\Omega}_h; \mathbb{R}^3)$ with $\tilde{I}^{(h)}(\tilde{y}^{(h)})/h^2 \leq C$, we prove that there exist a subsequence (not relabelled) and some constants $c^{(h)} \in \mathbb{R}^3$ such that

$$
\begin{aligned}
&\tilde{y}^{(h)} \circ \Psi^{(h)} - c^{(h)} \to y \quad \text{strongly in } W^{1,2}(\Omega; \mathbb{R}^3),\\
&\frac{1}{h}\, \partial_\xi \big(\tilde{y}^{(h)} \circ \Psi^{(h)}\big) \to d_2 \quad \text{strongly in } L^2(\Omega; \mathbb{R}^3),\\
&\frac{1}{h}\, \partial_\zeta \big(\tilde{y}^{(h)} \circ \Psi^{(h)}\big) \to d_3 \quad \text{strongly in } L^2(\Omega; \mathbb{R}^3),
\end{aligned}
$$

where (y, d_2, d_3) belongs to the class

$$
\begin{aligned}
\mathcal{A} := \{(y, d_2, d_3) \in W^{2,2}((0,L); \mathbb{R}^3) \times W^{1,2}((0,L); \mathbb{R}^3) \times W^{1,2}((0,L); \mathbb{R}^3) :\\
(y'(s) \,|\, d_2(s) \,|\, d_3(s)) \in SO(3) \text{ for a.e. } s \text{ in } (0, L)\}.
\end{aligned}
$$

The key ingredient in the proof is the Geometric Rigidity Theorem proved by G. Friesecke, R.D. James and S. Müller in [30]. In Theorems 2.5 and 2.6 we show that the Γ-limit of the sequence $\big(\tilde{I}^{(h)}/h^2\big)$ is given by

$$
I(y, d_2, d_3) := \begin{cases} \dfrac{1}{2} \displaystyle\int_0^L Q\big(s, \big(R^T(s) R'(s) - R_0^T(s) R_0'(s)\big)\big) ds & \text{if } (y, d_2, d_3) \in \mathcal{A},\\ +\infty & \text{otherwise,} \end{cases} \tag{1}
$$

where $R := (y' \,|\, d_2 \,|\, d_3)$, $R_0 := (\gamma' \,|\, \nu_2 \,|\, \nu_3)$, and Q is a quadratic form arising from a minimization procedure involving the quadratic form of linearized elasticity (see (2.28)). We point out that in Theorems 2.5 and 2.6 we do not require any growth condition from above on the energy density W.

We notice that in the limit problem the behaviour of the rod is described by a triple (y, d_2, d_3). The function y represents the deformation of the mid-fiber, which satisfies $|y'| = 1$ a.e., because of the constraint $(y' \,|\, d_2 \,|\, d_3) \in SO(3)$ a.e.. Therefore, the admissible deformations are only those leaving the mid-fiber unextended. Moreover, the triple (y, d_2, d_3) provides an orthonormal frame along the deformed curve; in particular, d_2 and d_3 belong to the normal plane to the deformed curve and describe the rotation undergone by the cross section.

Since $R = (y' \,|\, d_2 \,|\, d_3)$ is a rotation a.e., the matrix $R^T R'$ is skew-symmetric a.e. and its entries are given by

$$
(R^T R')_{1k} = -(R^T R')_{k1} = y' \cdot d_k' \quad \text{for } k = 2, 3, \quad (R^T R')_{23} = -(R^T R')_{32} = d_2 \cdot d_3'.
$$

It is easy to see that the scalar products $y' \cdot d_k'$ are related to curvature and therefore, to bending effects, while $d_2 \cdot d_3'$ is related to torsion and twist. We remark also that the energy depends explicitly on the reference state of the beam through the quantity $R_0^T R_0'$, which encodes information about the bending and torsion of the beam in the initial configuration. Hence, due to the nontrivial geometry of the body, the limit energy depends on the position over the curve γ even for a homogeneous material.

We notice that, specifying $R_0 = Id$ in (1), we recover the result for straight rods obtained in [40] and [45].

The last section of Chapter 2 is devoted to the study of lower scalings of the energy, i.e., $\beta \in [0,2)$. Assuming that the energy density W satisfies a growth condition from above, we prove the Γ-convergence of the sequence $\big(\tilde{I}^{(h)}\big)$ to a functional corresponding to a string model. Finally we show that the intermediate scalings of the energy corresponding to $\beta \in (0,2)$ lead to a degenerate Γ-limit.

In Chapter 3 we consider the scalings h^β with $\beta > 2$. More precisely, we prove that in the case $\beta = 4$, the corresponding relevant deformations are close to a rigid motion, so that the Γ-limit describes a partially linearized model. This result generalizes to the case of curved rods what was proved in [41] for straight rods. Furthermore, we show that the scalings $\beta > 4$ lead to the linearized theory for rods, while the scalings $\beta \in (2,4)$ correspond to a constrained linearized theory.

We first present a compactness result for sequences of deformations having equibounded energies $h^{-\beta}\tilde{I}^{(h)}$ with $\beta > 2$ (Theorem 3.1). The key tool is again the Geometric Rigidity Theorem which ensures that, as in the case treated in Chapter 2, the limit of the rescaled gradients of the deformations is a rotation. Moreover, since we are dealing with higher scalings of the energy, we obtain the additional information that this limit rotation is constant. More precisely, we prove that if $\tilde{I}^{(h)}\big(\tilde{y}^{(h)}\big) \le c\,h^\beta$, $\beta > 2$, then there exist some constants $\bar{R}^{(h)} \in SO(3)$ such that $\bar{R}^{(h)} \to \bar{R}$ and, up to subsequences,

$$\nabla\big(\big(\bar{R}^{(h)}\big)^T\,\tilde{y}^{(h)}\big) \circ \Psi^{(h)} \to Id \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}).$$

In other words, up to a rigid motion, the deformations $\tilde{y}^{(h)}$ converge to the identity. This naturally leads to introduce a new sequence of scaled deformations $Y^{(h)}$, given by $\big(\bar{R}^{(h)}\big)^T\tilde{y}^{(h)} \circ \Psi^{(h)}$ (up to an additive constant) and to study the deviation of $Y^{(h)}$ from $\Psi^{(h)}$. To this aim, we define the scaled averaged displacement

$$v^{(h)}(s) := \frac{1}{h^{(\beta-2)/2}} \int_D \big(Y^{(h)}(s,\xi,\zeta) - \Psi^{(h)}(s,\xi,\zeta)\big)\,d\xi\,d\zeta$$

and the twist angle of the cross-section

$$w^{(h)}(s) := \frac{1}{h^{\beta/2}}\left(\frac{1}{\mu(D)}\int_D \big(Y^{(h)}(s,\xi,\zeta) - \Psi^{(h)}(s,\xi,\zeta)\big)\cdot(\xi\,\nu_3(s) - \zeta\,\nu_2(s))\;d\xi\,d\zeta\right),$$

where $\mu(D) := \int_D \big(\xi^2 + \zeta^2\big)\,d\xi\,d\zeta$. Finally, we introduce a function $u^{(h)}$, which measures the extension of the mid-fiber and is given by

$$u^{(h)}(s) := \begin{cases} \dfrac{1}{h^{\beta-2}} \displaystyle\int_{s_h}^{s} \Big(\int_D \partial_s\big(Y^{(h)}(s,\xi,\zeta) - \Psi^{(h)}(s,\xi,\zeta)\big)\cdot\tau(\sigma)\;d\xi\,d\zeta\Big)\,d\sigma & \text{if } 2 < \beta < 4, \\ \dfrac{1}{h^{\beta/2}} \displaystyle\int_{s_h}^{s} \Big(\int_D \partial_s\big(Y^{(h)}(s,\xi,\zeta) - \Psi^{(h)}(s,\xi,\zeta)\big)\cdot\tau(\sigma)\;d\xi\,d\zeta\Big)\,d\sigma & \text{if } \beta \ge 4, \end{cases}$$

where $s_h \in (0,L)$ is chosen in such a way that $u^{(h)}$ has zero average on $(0,L)$.

In Theorem 3.1 it is then shown that, up to subsequences, the following convergence properties are satisfied:

- $v^{(h)} \to v$ strongly in $W^{1,2}((0,L);\mathbb{R}^3)$, for some $v \in W^{2,2}((0,L);\mathbb{R}^3)$ with $v'\cdot\tau = 0$;
- $w^{(h)} \rightharpoonup w$ weakly in $W^{1,2}(0,L)$, for some $w \in W^{1,2}(0,L)$;
- $u^{(h)} \rightharpoonup u$ weakly in $W^{1,2}(0,L)$, for some $u \in W^{1,2}(0,L)$.

In Theorems 3.5 and 3.6 the Γ-limit of the functionals $\big(\tilde{I}^{(h)}/h^\beta\big)$, for $\beta \ge 4$, is identified.
In the case $\beta = 4$ we show that it is an integral functional depending on u, v and w, of the form

$$I_4(u,v,w) := \frac{1}{2}\int_0^L Q^0\Big(s, u' + \frac{1}{2}\big((v'\cdot\nu_2)^2 + (v'\cdot\nu_3)^2\big), B' + 2\,\mathrm{skw}\big(R_0^T R_0' B\big)\Big)\,ds,$$

where $B \in W^{1,2}((0,L);\mathbb{M}^{3\times3})$ denotes the matrix

$$B := \begin{pmatrix} 0 & -v' \cdot \nu_2 & -v' \cdot \nu_3 \\ v' \cdot \nu_2 & 0 & -w \\ v' \cdot \nu_3 & w & 0 \end{pmatrix} \tag{2}$$

and Q^0 is a quadratic form arising from a minimization problem involving the quadratic form of linearized elasticity (see (3.31)).

If $\beta > 4$ the limit functional is fully linearized and it is given by

$$I_\beta(u,v,w) := \frac{1}{2}\int_0^L Q^0\big(s, u', B' + 2\,\mathrm{skw}\big(R_0^T R_0' B\big)\big)\, ds,$$

where B and Q^0 are defined as before. We notice that I_β coincides with the functional obtained by dimension reduction starting from linearized elasticity (see Remark 3.3).

Finally, in the case $\beta \in (2,4)$, it turns out that v and u are linked by the following nonlinear constraint:

$$u' = -\frac{1}{2}\big((v' \cdot \nu_2)^2 + (v' \cdot \nu_3)^2\big). \tag{3}$$

Therefore, the function u is completely determined, once v is known, and hence the limit functional depends on v and w only. More precisely, it is given by

$$I_\beta(v,w) := \frac{1}{2}\int_0^L Q\big(s, B' + 2\,\mathrm{skw}\big(R_0^T R_0' B\big)\big)\, ds,$$

where B is defined as in (2) and Q is obtained by minimizing the quadratic form Q^0 with respect to its second argument (see (3.32)).

The last section of Chapter 3 is devoted to the extension of the previous results to the case of a thin ring. In other words, the mid-fiber of the beam is assumed to be a closed curve in $\mathbb{R}^3$. We prove that in this case the limiting functionals are finite only on the class of triples (u,v,w) such that v and w satisfy the periodic boundary conditions $v(0) = v(L)$ and $w(0) = w(L)$ (see Theorem 3.8). Moreover, on this class the Γ-limits coincide with the previous functionals I_β (see Theorem 3.9).

Part II: homogenization

Composite materials are widely used since they have very interesting properties. Indeed, they often combine the attributes of the constituents but sometimes the properties of the composite can be strikingly different from the properties of the constituent materials (see [39]).

In a good composite, the heterogeneities are very small compared with the global dimension of the sample. Heuristically, as the size of the microstructure becomes smaller and smaller, the microscopic structure of the material becomes finer and finer, while, on the other hand, from a macroscopic point of view the behaviour of the composite tends to be simpler. So we expect the limit behaviour of the material to be described in terms of a different *homogeneous* material, that captures the main features of the original constituents.

The results contained in Chapters 4 and 5 describe the homogenization of a material composed of two constituents which have a very different elastic behaviour. More precisely, we consider the case of an unbreakable elastic material presenting disjoint brittle inclusions arranged in a periodic way. In other words, we assume that cracks can appear and grow only in a prescribed disconnected region of the material, composed of a large number of small components with small toughness.

In what follows, let $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, be the region occupied by the material and let $\varepsilon > 0$ be a small parameter. Let $Q := (0,1)^n$ be the periodicity cell and let $Q_\delta \subset Q$ denote the concentric cube $(\delta, 1-\delta)^n$ for $0 < \delta < \frac{1}{2}$. We define the set $I^\varepsilon_\delta \subset \Omega$ representing the brittle inclusions in the material as

$$I^\varepsilon_\delta := \Omega \cap \bigcup_{h \in \mathbb{Z}^n} \varepsilon\,(Q_\delta + h). \tag{4}$$

In Chapter 4 we assume the material to be linearly elastic, and we restrict our analysis to the case of anti-plane shear. More precisely, we assume that the reference configuration is an infinite cylinder $\Omega \times \mathbb{R}$ and the displacement $v : \Omega \times \mathbb{R} \to \mathbb{R}^{n+1}$ has the special form $v(x,y) := (0, \dots, 0, u(x))$ for every $(x,y) \in \Omega \times \mathbb{R}$, where $u : \Omega \to \mathbb{R}$.

Since we are taking into account the possibility of creating cracks, displacements are allowed to have discontinuities. Therefore, the natural functional setting for the problem is the space of special functions with bounded variation $SBV(\Omega)$. More precisely, we consider displacements $u \in SBV^2(\Omega)$, that is, we assume in addition that the approximate gradient ∇u is in L^2 and that the $(n-1)$-dimensional Hausdorff measure of the jump set S_u is finite.

The elastic energy $\mathcal{F}^\varepsilon$ associated to a displacement $u \in SBV^2(\Omega)$ is defined as

$$\mathcal{F}^\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 dx + \alpha_\varepsilon \mathcal{H}^{n-1}(S_u) & \text{if } S_u \subset I^\varepsilon_\delta, \\ +\infty & \text{otherwise in } SBV^2(\Omega), \end{cases}$$

where α_ε is a positive parameter depending on ε.

The volume term in the expression of $\mathcal{F}^\varepsilon$ represents the linearly elastic energy of the body, while the surface integral describes the energy needed in order to open a crack in a material with toughness α_ε, according to Griffith's model of brittle fractures (see [33]).

We are interested in the asymptotic behaviour of the sequence $\mathcal{F}^\varepsilon$ as ε goes to zero, in the framework of Γ-convergence. We consider the case in which δ is fixed and independent of ε, while α_ε converges to zero as $\varepsilon \to 0$. We show that the limit model depends on the behaviour of the ratio $\frac{\alpha_\varepsilon}{\varepsilon}$ as ε goes to zero. However, it turns out that the different limiting models present a common feature: they describe an unbreakable material. This means that, even if at scale ε many microscopic cracks are present in the material, they are not equivalent in the limit model to a macroscopic crack, due to the fact that they are well separated from one another. Indeed, in the periodicity cell εQ the brittle inclusion εQ_δ is set at a distance $\varepsilon\delta$ from the boundary $\partial(\varepsilon Q)$, with $\delta > 0$ independent of ε. The size of the separation between different inclusions prevents the small cracks contained in the brittle region of the material from having the same asymptotic effect of a macroscopic fracture.

A different situation occurs when the parameter δ depends on ε and converges to zero as $\varepsilon \to 0$. This case has been partially solved in [10], assuming $\alpha_\varepsilon = 1$.

We show that three different limit models can arise, corresponding to the limit $\frac{\alpha_\varepsilon}{\varepsilon}$ being zero (subcritical case), finite (critical case) or $+\infty$ (supercritical case).

In the subcritical case $\alpha_\varepsilon << \varepsilon$, the limit functional turns out to be

$$\mathcal{F}^0(u) = \begin{cases} \displaystyle\int_\Omega f_0(\nabla u)\, dx & \text{in } H^1(\Omega), \\ +\infty & \text{otherwise in } L^2(\Omega), \end{cases}$$

where f_0 is a coercive quadratic form given by the cell formula

$$f_0(\xi) = \min \Big\{ \int_{Q \setminus Q_\delta} |\,\xi + \nabla w(y)|^2 dy : w \in H^1_\#(Q \setminus Q_\delta) \Big\}, \tag{5}$$

and $H^1_\#(Q \setminus Q_\delta)$ denotes the space of $H^1(Q \setminus Q_\delta)$ functions with periodic boundary values on ∂Q. Hence there exists a positive definite matrix $A_0 \in \mathbb{R}^{n \times n}$ with constant coefficients such that $f_0(\xi) = A_0\xi \cdot \xi$

for every $\xi \in \mathbb{R}^n$. Notice that $\mathcal{F}^0$ represents the energy of a linearly elastic homogeneous anisotropic material. Moreover, since $w \equiv 0$ is a competitor for the minimum in (5), the density f_0 satisfies

$$A_0\xi \cdot \xi = f_0(\xi) \leq \big(1 - \mathcal{L}^n(Q_\delta)\big)|\xi|^2 \leq |\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^n,$$

and the second inequality is strict for $\xi \neq 0$. This means that "$A_0 \lneqq Id$" in the usual sense of quadratic forms. This is due to the fact that in this regime, for the problem at fixed ε, displacements presenting discontinuities are energetically convenient. Hence, although the limit energy $\mathcal{F}^0$ describes an unbreakable material, the possibility to create a high number of microfractures in the approximating problems leads to a damaged limit material, that is, a material whose elastic properties are weaker than the original ones.

In the supercritical regime $\alpha_\varepsilon >> \varepsilon$ the limit model is described by the functional

$$\mathcal{F}^\infty(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\,dx & \text{in } H^1(\Omega), \\ +\infty & \text{otherwise in } L^2(\Omega). \end{cases}$$

Hence, the (possible) presence of small cracks in the problems at scale ε does not affect the elastic properties of the original material. Indeed, in this regime the formation of microfractures is penalized by the energy, that is, displacements presenting jumps are not energetically convenient. Therefore the macroscopic result describes an undamaged material.

The critical regime corresponds to the case where α_ε is of the same order as ε, so we can assume without loss of generality that $\alpha_\varepsilon = \varepsilon$. The limit functional is

$$\mathcal{F}^{hom}(u) = \begin{cases} \displaystyle\int_\Omega f_{hom}(\nabla u)\,dx & \text{in } H^1(\Omega), \\ +\infty & \text{otherwise in } L^2(\Omega), \end{cases}$$

where the density f_{hom} is given by the asymptotic cell formula

$$f_{hom}(\xi) := \lim_{t\to+\infty} \frac{1}{t^n} \inf\left\{ \int_{(0,t)^n} |\,\xi + \nabla w\,|^2 d\,x + \mathcal{H}^{n-1}(S_w) : w \in SBV_0^2\big((0,t)^n\big), S_w \subset I_\delta \right\}. \tag{6}$$

According to the notation used so far, the set I_δ is defined as

$$I_\delta := (0,t)^n \cap \bigcup_{h\in\mathbb{Z}^n} (Q_\delta + h). \tag{7}$$

Notice that, since in this case the coefficient α_ε and the size ε of the microstructure have the same order, there is a competition between the bulk energy and the surface term. Indeed they both contribute to the expression of the limit density.

Moreover, the limit functional describes an intermediate model with respect to the subcritical and the supercritical regimes. More precisely, the limit density satisfies

$$f_0(\xi) \lneqq f_{hom}(\xi) \leq \min\big\{|\xi|^2, f_0(\xi) + c(\delta)\big\}, \tag{8}$$

for every $\xi \in \mathbb{R}^n \setminus \{0\}$, where $c(\delta)$ is the $(n-1)$-dimensional measure of $\partial(Q_\delta)$ (see Lemma 4.17).

Notice that (8) entails that for $|\xi|$ large enough $f_{hom}(\xi) \lneqq |\xi|^2$. Therefore, the limit functional describes a damaged material. Using estimate (8) it is also possible to show that the limit density f_{hom} is *not* two-homogeneous, and hence it is not a quadratic form (see again Lemma 4.17).

The analysis developed so far can be applied also to the case in which the brittle region is an arbitrary Lipschitz domain well contained in Q, or more in general, when it consists of an n-dimensional Lipschitz set together with an $(n-1)$-dimensional component, as shown in Chapter 4.

Chapter 5 is devoted to the extension of the homogenization results presented in Chapter 4 to the vector-valued case in linearized (possibly anisotropic) elasticity. As before, we consider a linearly elastic material presenting brittle inclusions arranged in a periodic structure. Moreover, we impose a linearized non-interpenetration constraint between the lips of the fracture. We notice that in the case of anti-planar shear treated in Chapter 4 the non-interpenetration constraint is automatically satisfied.

Since also in this case the displacements are allowed to have discontinuities, the natural functional setting for the problem is the space $SBD(\Omega)$ of special functions with bounded deformation. Moreover, we consider as admissible the functions $u \in SBD^2(\Omega)$ satisfying the infinitesimal non-interpenetration condition $[u]\cdot\nu_u \geq 0$ $\mathcal{H}^{n-1}$-a.e. on the jump set J_u, where $[u]$ is the jump of u and ν_u is the normal to the jump set.

Let $\mathbb{C} = (\mathbb{C}_{ijkl})$ be the elasticity tensor and let $u \in SBD^2(\Omega)$ be a displacement. We denote by $\mathcal{E}u$ the absolutely continuous part of the symmetric gradient of u.

The energy associated to u is given by the functional $\mathcal{F}^\varepsilon$ defined as

$$\mathcal{F}^\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \mathbb{C}\mathcal{E}u : \mathcal{E}u\,dx + \alpha_\varepsilon \mathcal{H}^{n-1}(J_u) & \text{if } S_u \subset I^\varepsilon_\delta, [u]\cdot\nu_u \geq 0 \text{ a.e. on } J_u, \\ +\infty & \text{otherwise in } SBD^2(\Omega), \end{cases}$$

where the set I^ε_δ is defined as in (4), and α_ε is a positive parameter depending on ε. The volume term in the expression of $\mathcal{F}^\varepsilon$ represents the elastic energy, while the surface integral describes the energy needed to open a crack.

The overall properties of the composite material described by the functional $\mathcal{F}^\varepsilon$ can be expressed in terms of a *homogenized* simpler integral, which is given by the Γ-limit of $\mathcal{F}^\varepsilon$, as ε goes to zero. As in the case treated in Chapter 4, we assume that $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$, and we show that the limit model depends on the behaviour of the ratio $\frac{\alpha_\varepsilon}{\varepsilon}$ as ε goes to zero. Moreover, also in the present case the limit functionals describe an unbreakable material.

We will show that three different limit models can arise, corresponding to the limit $\frac{\alpha_\varepsilon}{\varepsilon}$ being zero (subcritical case), finite (critical case) or $+\infty$ (supercritical case).

In the subcritical case $\alpha_\varepsilon << \varepsilon$, the limit functional is given by

$$\mathcal{F}^0(u) = \begin{cases} \displaystyle\int_\Omega f_0(\mathcal{E}u)\,dx & \text{in } H^1(\Omega;\mathbb{R}^n), \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n). \end{cases} \tag{9}$$

The density f_0 is given by the cell formula

$$f_0(\xi) := \inf\left\{ \int_Q \mathbb{C}(\xi^s + \mathcal{E}w) : (\xi^s + \mathcal{E}w)\,dx : w \in SBD^2_\#(Q), J_w \subset Q_\delta,\ [w]\cdot\nu_w \geq 0 \text{ a.e. on } J_w \right\}, \tag{10}$$

where $SBD^2_\#(Q)$ denotes the space of $SBD^2(Q)$ functions with periodic boundary conditions on ∂Q and ξ^s denotes the symmetric part of ξ.

An interesting remark is that in general f_0 is not a quadratic form. Indeed, if we assume $\mathbb{C}$ to be isotropic, that is,

$$\mathbb{C} = 2\,\mu\,\mathbb{I} + \lambda\, Id \otimes Id,$$

where $\lambda, \mu > 0$, $(\mathbb{I})_{ijkl} = \delta_{ik}\delta_{jl}$, and $(Id \otimes Id)_{ijkl} = \delta_{ij}\delta_{kl}$, then it turns out that $f_0(Id) \neq f_0(-Id)$ (see Lemma 5.8).

This is in contrast with the situation in which the non-interpenetration constraint in not assumed. Indeed, in that case, proceeding as in Chapter 4, one can prove that the density function $\hat{f}_0$ is defined as

$$\hat{f}_0(\xi) := \inf\left\{ \int_Q \mathbb{C}(\xi^s + \mathcal{E}w) : (\xi^s + \mathcal{E}w)\,dx : w \in SBD^2_\#(Q), J_w \subset Q_\delta \right\}, \tag{11}$$

and is a quadratic form for every choice of the tensor $\mathbb{C}$.

A possible interpretation of this result is the following. For $\xi = Id$ the body is subject to a boundary deformation of pure extension in all directions. In this case, the solutions to (10) present discontinuities, since the non-interpenetration constraint is compatible with the boundary conditions and it is energetically convenient to have a nonempty jump set.

On the contrary, when $\xi = -Id$, i.e., in a regime of pure compression, it turns out that the optimal w in (10) is $w = 0$. This happens because the minimizers of the problem (11) corresponding to $\xi = -Id$ are not admissible for (10), since they do not satisfy the non-interpenetration constraint.

In the critical regime, corresponding to $\alpha_\varepsilon = \varepsilon$, the limit functional is

$$\mathcal{F}^{hom}(u) = \begin{cases} \displaystyle\int_\Omega f_{hom}(\mathcal{E}u)\,dx & \text{in } H^1(\Omega;\mathbb{R}^n), \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n), \end{cases} \tag{12}$$

where the density f_{hom} is given by the asymptotic cell problem

$$f_{hom}(\xi) := \lim_{t\to+\infty} \frac{1}{t^n} \inf\Big\{ \int_{(0,t)^n} \mathbb{C}(\xi^s + \mathcal{E}w) : (\xi^s + \mathcal{E}w)\,dx + \mathcal{H}^{n-1}(J_w) : w \in SBD^2_0\big((0,t)^n\big), \\ J_w \subset I_\delta,\ [w]\cdot\nu_w \geq 0\ \mathcal{H}^{n-1}\text{-a.e. on } J_w \Big\},$$

and the set I_δ is defined as in (7).

Since in this case the coefficient α_ε and the size ε of the microstructure are of the same order, there is a competition between the bulk energy and the surface term. Hence the limit functional describes an intermediate model with respect to the subcritical and the supercritical regimes.

In the supercritical regime $\alpha_\varepsilon >> \varepsilon$, the limit model is given by the functional

$$\mathcal{F}^{\infty}(u) = \begin{cases} \displaystyle\int_\Omega \mathbb{C}\mathcal{E}u : \mathcal{E}u\,dx & \text{in } H^1(\Omega;\mathbb{R}^n), \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n). \end{cases} \tag{13}$$

Therefore, the (possible) presence of cracks in the approximating problems has no effect on the limit. Indeed, as one may expect, in this case the energy penalizes the jumps of the deformations, so that the limit material has the same elastic properties as the original one and no damage occurs.

We want to underline that in this regime the Γ-limit is the same as if the non-interpenetration constraint were not imposed. The feature which makes this case mathematically different from the corresponding one in Chapter 4 is the lack of a lower semicontinuity result in SBD when no a priori bound for the L^∞ norm of the deformations is given. Hence, in order to prove the Γ-convergence result for this scaling, we need a modified version of the proof of lower semicontinuity in SBD given in [11], where the assumption of the equiboundedness of the L^∞ norm of the deformations is replaced by the fact that the measure of their jump sets goes to zero (see Lemma 5.12).

The methods of homogenization can also be applied to describe the asymptotic behaviour of degenerate structures, like periodically perforated domains. A wide literature deals with these models under the assumption that the material is unbreakable, that is, in the Sobolev setting. We recall that completely different situations are produced according to the type of boundary condition (Dirichlet or Neumann) imposed on the boundaries of the holes.

Indeed, in the case of Dirichlet boundary conditions, a typical phenomenon is that the limit energy contains an extra term of capacitary type, called *strange term* in [18, 19], that can be interpreted as a relaxation of the original constraint imposed on the displacements in the holes (see [22] and references therein).

On the other hand, the study of the case of Neumann boundary conditions requires the construction of suitable extension operators in order to fill the holes and to deal with displacements defined in the whole domain (see [2, 36]).

These results may be extended to more general functionals, as those describing the elastic energy of a brittle material, in the SBV setting. Recently, this problem has been addressed in [29], where the authors study the limiting behaviour of the Mumford-Shah functional in periodically perforated domains, under homogeneous Dirichlet conditions on the boundaries of the holes. In Chapter 6 we continue this analysis, considering the case of the same energy treated in [29], but imposing homogeneous Neumann conditions on the perforations.

The main result of Chapter 6 is the existence of an extension operator for special functions with bounded variation with a careful energy estimate. Our motivation comes from [2], where the same problem is addressed in the context of Sobolev spaces (see also [36]).

The main achievement in the quoted paper is the existence of a suitable extension operator in periodic domains, with extension constants invariant under homothety. This result turns out to be the fundamental tool for the analysis of the asymptotic behaviour of integral functionals on perforated domains.

It seems very natural to look for an extension of the homogenization results in [2] to non-coercive functionals consisting of a volume and a surface integral, as those occurring in computer vision and in the mathematical theory of elasticity for brittle materials. More precisely, we are interested in the study of the asymptotic behaviour of the Mumford-Shah functional on a periodically perforated domain, as the size of the holes and the periodicity parameter of the structure tend to zero.

Therefore, we are led to consider energies of the form

$$\mathcal{F}^\varepsilon(u,\Omega) := \begin{cases} MS(u,\Omega^\varepsilon) & \text{if } u \in L^\infty(\Omega^\varepsilon) \cap SBV^2(\Omega^\varepsilon), \\ +\infty & \text{otherwise in } L^2(\Omega), \end{cases} \tag{14}$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded, and Ω^ε is obtained by removing from Ω a periodic array of holes. More precisely, let $Q := (0,1)^n$ denote the periodicity cell, and let $E \subset\subset Q$ be a Lipschitz set. We define $\Omega^\varepsilon := \Omega \setminus E^\varepsilon$, where

$$E^\varepsilon := \Omega \cap \bigcup_{h \in \mathbb{Z}^n} \varepsilon\,(E+h). \tag{15}$$

In order to analyze the behaviour of the family $(\mathcal{F}^\varepsilon)$ as $\varepsilon \to 0$, we need the analogue in the SBV framework of the extension estimates obtained in [2].

This will be a direct consequence of the following theorem that is the main result of the chapter.

Theorem (Extension Theorem). *Let $D, A \subset \mathbb{R}^n$ be bounded open sets with Lipschitz boundary and assume that $D \subset A$ and $\partial D \cap A \subset\subset A$. Then there exists a constant $c = c(n, D, A) > 0$ and an extension operator $T : SBV^2(D) \cap L^\infty(D) \to SBV^2(A) \cap L^\infty(A)$ such that*

$$(i)\ Tu = u \quad a.e.\ in\ D,$$
$$(ii)\ ||Tu||_{L^\infty(A)} = ||u||_{L^\infty(D)},$$
$$(iii)\ MS(Tu, A) \leq c\, MS(u, D).$$

The constant c in invariant under homotheties.

In the case of Sobolev spaces, the classical argument to prove the estimate

$$\int_A |\nabla(Tu)|^2 dx \leq c \int_D |\nabla u|^2 \tag{16}$$

for the extension relies on Poincaré-Wirtinger inequality (see [36] and [52]). In the SBV case (16) cannot be obtained. Indeed, it is possible to construct non constant SBV functions whose absolutely continuous

gradient is zero almost everywhere. Moreover, the available extension of the Poincaré-Wirtinger inequality (see [26]) does not lead directly to (iii).
For this reason we decided to follow a different approach.

To prove the theorem, we first consider a local minimizer of MS, that is a solution $\hat{v}$ of the following problem:

$$\min\left\{MS(w, D\cup W) : w\in SBV^2(D\cup W), w=u \text{ in } D\right\},$$

where $W\subset\subset A$ is a sufficiently small neighbourhood of $\partial D\cap A$. Then, we carry out a delicate analysis of the behaviour of the function $\hat{v}$ in the set W. More precisely, we define the extension Tu in $A\setminus D$ modifying the function $\hat{v}$ in different ways, according to the measure of the set $S_{\hat{v}}\cap W$.

If this measure is *large* enough, then we consider the extension of u defined as $\hat{v}$ in $D\cup W$ and zero in the remaining part of A. In this way we have essentially increased the energy in the surface term only, of an amount that is comparable to the measure of $S_u\cap D$. This guarantees that properties (i)–(iii) are satisfied in this case.

On the other hand, if $\mathcal{H}^{n-1}(S_{\hat{v}}\cap W)$ is *small* then we may use the *elimination property* proved in [26, 23] to detect a subset of $W\setminus D$ where the function $\hat{v}$ has no jump (see also Theorem 1.13). This allows us to apply the extension property proved in the Sobolev setting.

As already mentioned, the previous result finds an immediate application in the study of the asymptotic behaviour of the functionals $\mathcal{F}^\varepsilon$ defined in (14).
Indeed, for every $u\in SBV^2(\Omega^\varepsilon)\cap L^\infty(\Omega^\varepsilon)$, the Extension Theorem provides an extension $\tilde{u}$ of u to the whole of Ω, such that $\tilde{u}\in SBV^2(\Omega)\cap L^\infty(\Omega)$ and

$$MS(\tilde{u},\Omega)\leq c\,\mathcal{F}^\varepsilon(u,\Omega), \tag{17}$$

where the constant $c=c(n,E,Q)$ depends on n, E and Q, but is *independent of* Ω, ε *and* u.
This means that we can *fill* the holes of Ω^ε by means of an extension of u, whose Mumford-Shah energy is kept bounded by $c\mathcal{F}^\varepsilon(u,\Omega)$. Inequality (17) is the key estimate to prove compactness of minimizing sequences for $(\mathcal{F}^\varepsilon)$ and thus, to identify a class of functions where the Γ-limit is finite. Within this class, we give a more explicit expression for the Γ-limit, characterizing the volume and the surface densities by means of two separate homogenization formulas (see Theorem 6.6).

The results of Chapter 2 are published in [49], while the results of Chapter 3 correspond to [46].
The content of Chapters 4 and 5 are contained in [47] and [48], respectively.
Finally, in Chapter 6 we present [14], a work in progress in collaboration with Filippo Cagnetti.

Chapter 1

Preliminaries

The purpose of this chapter is to present some known results that will be used in the thesis.

1.1 Γ-convergence

In this section we introduce the notion of Γ-convergence and state its main properties. For an exhaustive treatment of this topic we refer to [21].

Definition 1.1 *Let (X,d) be a metric space. We say that a sequence $F_h : X \to \overline{\mathbb{R}}$ Γ-converges to $F : X \to \overline{\mathbb{R}}$ with respect to the convergence induced by the metric d (or simply that F_h $\Gamma(d)$-converges to F) if for all $x \in X$ we have*

(i) ($\liminf$ inequality) for every sequence (x_h) converging to x

$$F(x) \leq \liminf_{h\to+\infty} F_h(x_h);$$

(ii) (existence of a recovery sequence) there exists a sequence (x_h) converging to x such that

$$F(x) = \lim_{h\to+\infty} F_h(x_h).$$

The function F is uniquely determined by conditions (i) and (ii) and it is called the Γ-limit of (F_h).

More in general, given a family of functionals (F_ε) labelled by a real parameter $\varepsilon > 0$, we say that F_ε Γ-converges to F if F is the Γ-limit of (F_{ε_h}) for every sequence $\varepsilon_h \to 0^+$.

Proposition 1.2 (Comparison with pointwise convergence) *If (F_h) is an increasing sequence of lower semicontinuous functionals which converges pointwise to a functional F, then F is lower semicontinuous and (F_h) Γ-converges to F. If (F_h) is a decreasing sequence of functionals which converges pointwise to a functional F and F is lower semicontinuous, then (F_h) Γ-converges to F.*

Proposition 1.3 (Convergence of minima and of minimizers) *Assume that (F_h) Γ-converges to a functional F. For every $h \in \mathbb{N}$ let x_h be a minimizer of F_h in X. If x is a cluster point of (x_h), then x is a minimizer of F in X, and*

$$F(x) = \limsup_{h\to+\infty} F_h(x_h).$$

If (x_h) *converges to* x *in* X*, then* x *is a minimizer of* F *in* X*, and*

$$F(x) = \lim_{h\to+\infty} F_h(x_h).$$

Definition 1.4 *Let* (X,d) *be a metric space, let* Ω *be an open subset of* $\mathbb{R}^n$*, and let* $\mathcal{E}$ *be an arbitrary class of subsets of* Ω *containing* $\mathcal{A}_0$*, where* $\mathcal{A}_0$ *is the class of all subsets* A *of* Ω *such that* $A \subset\subset \Omega$*.*

We say that a functional $F : X \times \mathcal{E} \to [0,+\infty]$ *is increasing (on* $\mathcal{E}$*) if for every* $x \in X$ *the set function* $F(x,\cdot)$ *is increasing on* $\mathcal{E}$*.*

Definition 1.5 *Given a functional* $F : X \times \mathcal{E} \to [0,+\infty]$*, we define its inner regularization as*

$$F_-(x,A) := \sup\big\{F(x,B) : B \in \mathcal{E}, B \subset\subset A\big\}.$$

Observe that if F *is increasing, then also* F_- *is increasing.*

Definition 1.6 *Let* (F_h) *be a sequence of increasing functionals defined on* $X \times \mathcal{E}$*, and let* $F', F'' : X \times \mathcal{E} \to \overline{\mathbb{R}}$ *be the functionals defined by*

$$F'(\cdot,A) := \Gamma - \liminf_{h\to+\infty} F_h(\cdot,A) \quad \text{and} \quad F''(\cdot,A) := \Gamma - \limsup_{h\to+\infty} F_h(\cdot,A), \tag{1.1}$$

for every $A \in \mathcal{E}$*.*

Definition 1.7 *We say that a sequence* (F_h) *is* $\overline{\Gamma}$*-convergent to a functional* F *whenever*

$$F = (F')_- = (F'')_-.$$

We have the following compactness theorem.

Theorem 1.8 *Every sequence of increasing functionals has a* $\overline{\Gamma}$*-convergent subsequence.*

1.2 Functions with bounded variation

We need to recall some properties of rectifiable sets and of the space SBV of special functions with bounded variation. We refer the reader to [8] for a complete treatment of these subjects.

A set $\Gamma \subset \mathbb{R}^n$ is rectifiable if there exist $N_0 \subset \Gamma$ with $\mathcal{H}^{n-1}(N_0) = 0$, and a sequence $(M_i)_{i\in\mathbb{N}}$ of C^1-submanifolds of $\mathbb{R}^n$ such that

$$\Gamma \setminus N_0 \subset \bigcup_{i\in\mathbb{N}} M_i.$$

For every $x \in \Gamma \setminus N_0$ we define the normal to Γ at x as $\nu_{M_i}(x)$. It turns out that the normal is well defined (up to the sign) for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$.

Let $U \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary. We define $SBV(U)$ as the set of functions $u \in L^1(U)$ such that the distributional derivative Du is a Radon measure which, for every open set $A \subset U$, can be represented as

$$Du(A) = \int_A \nabla u \, dx + \int_{A\cap S_u} [u](x)\, \nu_u(x)\, d\mathcal{H}^{n-1}(x),$$

where ∇u is the approximate differential of u, S_u is the set of jump of u (which is a rectifiable set), $\nu_u(x)$ is the normal to S_u at x, and $[u](x)$ is the jump of u at x.

For every $p \in]1, +\infty[$ we set

$$SBV^p(U) = \big\{u \in SBV(U) : \nabla u \in L^p(U;\mathbb{R}^n), \mathcal{H}^{n-1}(S_u) < +\infty\big\}.$$

If $u \in SBV(U)$ and $\Gamma \subset U$ is rectifiable and oriented by a normal vector field ν, then we can define the traces u^+ and u^- of $u \in SBV(U)$ on Γ which are characterized by the relations

$$\lim_{r\to 0} \frac{1}{r^n} \int_{\Omega\cap B_r^{\pm}(x)} |u(y) - u^{\pm}(x)|\, dy = 0 \quad \text{for } \mathcal{H}^{n-1} - \text{a.e. } x \in \Gamma,$$

where $B_r^{\pm}(x) := \{y \in B_r(x) : (y - x)\cdot \nu \gtrless 0\}$.
A set $E \subset U$ has finite perimeter in U if the characteristic function χ_E belongs to $SBV(U)$. We denote by $\partial^* E$ the set of jumps of χ_E and by $P(E,U)$ the total variation of the measure $D\chi_E$, that is, the perimeter of E in U.

Finally, if $E \subset U$, we denote with $E(\sigma)$ the set of points of density $\sigma \in [0,1]$ for E, i.e.,

$$E(\sigma) := \big\{x \in U : \lim_{r\to 0} \mathcal{L}^n(E\cap B_r(x))/\mathcal{L}^n(B_r(x)) = \sigma\big\}.$$

Theorem 1.9 (Closure of SBV) *Let $\varphi : [0,+\infty) \to [0,+\infty]$, $\vartheta : (0,+\infty) \to (0,+\infty]$ be lower semi-continuous increasing functions and assume that*

$$\lim_{t\to+\infty} \frac{\varphi(t)}{t} = +\infty, \quad \lim_{t\to 0} \frac{\vartheta(t)}{t} = +\infty. \tag{1.2}$$

Assume moreover that φ is convex and that ϑ is concave. Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $(u_h) \subset SBV(\Omega)$ such that

$$\sup_h \left\{ \int_\Omega \varphi(|\nabla u_h|)\, dx + \int_{S_{u_h}} \vartheta(|[u_h]|)\, d\mathcal{H}^{n-1} \right\} < +\infty. \tag{1.3}$$

If (u_h) weakly* converges in $BV(\Omega)$ to u, then $u \in SBV(\Omega)$, the approximate gradients ∇u_h weakly converge to ∇u in $L^1(\Omega;\mathbb{R}^n)$, $D^j u_h$ weakly* converge to $D^j u$ in Ω and

$$\int_\Omega \varphi(|\nabla u|)\, dx \le \liminf_{h\to+\infty} \int_\Omega \varphi(|\nabla u_h|)\, dx \tag{1.4}$$

$$\int_{S_u} \vartheta(|[u]|)\, d\mathcal{H}^{n-1} \le \liminf_{h\to+\infty} \int_{S_{u_h}} \vartheta(|[u_h]|)\, d\mathcal{H}^{n-1}. \tag{1.5}$$

Theorem 1.10 (Compactness in SBV) *Let $\varphi, \vartheta, \Omega$ be as in Theorem 1.9. Let $(u_h) \subset SBV(\Omega)$ be a sequence satisfying (1.3) and assume, in addition, that $||u_h||_{L^\infty}$ is uniformly bounded in h. Then, there exists a subsequence $(u_{h(k)})$ weakly* converging in $BV(\Omega)$ to $u \in SBV(\Omega)$.*

1.2.1 The Mumford-Shah functional

Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $w \in SBV^2(\Omega)$. For every $A \subset \Omega$ open and bounded, the Mumford-Shah functional at (w,A) is defined as

$$MS(w,A) := \int_A |\nabla w|^2\, dx + \mathcal{H}^{n-1}(A\cap S_w). \tag{1.6}$$

We give now the definition of local minimizer for the Mumford-Shah functional.

Definition 1.11 *We say that $w \in SBV^2(\Omega)$ is a local minimizer of MS in Ω if $MS(w,A) \le MS(v,A)$ for every open set $A \subset\subset \Omega$, whenever $v \in SBV^2(\Omega)$ and $\{v \neq w\} \subset\subset A \subset\subset \Omega$.*

Next theorem provides an estimate of the measure of the jump set for a local minimizer of the Mumford-Shah functional (see [26]).

Theorem 1.12 (Density lower bound) *There exists a strictly positive dimensional constant $\vartheta_0 = \vartheta_0(n)$ with the property that if $u \in SBV^2(\Omega)$ is a local minimizer of the functional $MS(w,\Omega)$ defined in (1.6) with $\Omega \subset \mathbb{R}^n$ open set, $n \ge 2$, then*

$$\mathcal{H}^{n-1}(S_u \cap B_\varrho(x)) > \vartheta_0 \varrho^{n-1} \tag{1.7}$$

for all balls $B_\varrho(x) \subset \Omega$ with centre $x \in S_u$ and radius $\varrho > 0$.

An equivalent but more appealing formulation of the previous theorem is the following elimination property (see [23]).

Theorem 1.13 (Elimination property) *There exists a constant $\beta > 0$ independent of Ω such that, if u is a local minimizer in $SBV^2(\Omega)$ of the functional $MS(w,\Omega)$ defined in (1.6) and $B_\varrho(x_0) \subset \Omega$ is any ball with centre $x_0 \in \Omega$ with*

$$\mathcal{H}^{n-1}(S_u \cap B_\varrho(x_0)) < \beta \varrho^{n-1}, \tag{1.8}$$

then $S_u \cap B_{\varrho/2}(x_0) = \emptyset$.

We state a theorem which provides an approximation result for SBV functions with the property that the Mumford-Shah functional along the approximating sequence converges to the Mumford-Shah functional on the limit function. For the proof we refer to [20].

Theorem 1.14 *Assume that $\partial\Omega$ is locally Lipschitz and let $u \in SBV^2(\Omega)$. Then there exists a sequence $(u_h) \subset SBV^2(\Omega)$ such that for every $h \in \mathbb{N}$*

(i) S_{u_h} is essentially closed;
(ii) $\bar{S}_{u_h}$ is a polyhedral set;
(iii) $u_h \in W^{k,\infty}(\Omega \setminus \bar{S}_{u_h})$ for every $k \in \mathbb{N}$;

and such that (u_h) approximates u in the following sense:

(iv) $u_h \to u$ strongly in $L^2(\Omega)$,
(v) $\nabla u_h \to \nabla u$ strongly in $L^2(\Omega)$,
(vi) $\mathcal{H}^{n-1}(S_{u_h}) \to \mathcal{H}^{n-1}(S_u)$.

1.3 Functions with bounded deformation

Let $U \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary. We define $BD(U)$ as the set of functions $u \in L^1(U;\mathbb{R}^n)$ such that the symmetric part of the distributional derivative Du is a Radon measure with bounded total variation.

We denote with Eu the symmetric part of Du, that is,

$$Eu := \{(Eu)_{ij}\}, \quad (Eu)_{ij} := \frac{1}{2}(D_i u_j + D_j u_i).$$

We can split the symmetric gradient into its the absolutely continuous, jump and Cantor parts, as

$$Eu = E^a u + E^j u + E^c u = \mathcal{E}u\,dx + E^j u + E^c u.$$

Now we summarize some results on functions of bounded deformation which will be useful in the sequel.

Sections of $BD(U)$ functions. Let $u \in BD(U)$, let $\xi \in S^{n-1}$ and let $y \in \mathbb{R}^n$. We denote by π_ξ the hyperplane orthogonal to ξ and by U^ξ the orthogonal projection of U on π_ξ. Moreover the section of U corresponding to y is denoted by U^ξ_y, that is, $U^\xi_y := \{t \in \mathbb{R} : y + t\,\xi \in \Omega\}$.

We can define the section $u^\xi_y : U^\xi_y \to \mathbb{R}$ as $u^\xi_y(t) := u(y + t\,\xi)\cdot\xi$, for every $t \in U^\xi_y$. Then

(i) for $\mathcal{H}^{n-1}$-a.e. $y \in U^\xi$ the function u^ξ_y belongs to $BV(U^\xi_y)$;

(ii) $(\mathcal{E}u(y + t\,\xi)\,\xi, \xi) = \nabla u^\xi_y(t)$;

(iii) $(\mathcal{E}u\xi, \xi) = \int_{U^\xi} \nabla u^\xi_y d\mathcal{H}^{n-1}(y)$, $|(\mathcal{E}u\xi, \xi)| = \int_{U^\xi} |\nabla u^\xi_y|\, d\mathcal{H}^{n-1}(y)$;

(iv) $(E^j u\xi, \xi) = \int_{U^\xi} D^j u^\xi_y d\mathcal{H}^{n-1}(y)$, $|(E^j u\xi, \xi)| = \int_{U^\xi} |D^j u^\xi_y|\, d\mathcal{H}^{n-1}(y)$;

(v) $(E^c u\xi, \xi) = \int_{U^\xi} D^c u^\xi_y d\mathcal{H}^{n-1}(y)$, $|(E^c u\xi, \xi)| = \int_{U^\xi} |D^c u^\xi_y|\, d\mathcal{H}^{n-1}(y)$.

$SBD(U)$ functions. We define $SBD(U)$ as the set of functions $u \in L^1(U; \mathbb{R}^n)$ such that the symmetric part of their distributional derivative Du, that is Eu, is a Radon measure which, for every open set $A \subset U$, can be represented as

$$Eu(A) = E^a u(A) + E^j u(A) = \int_A \mathcal{E}u\,dx + \int_{A\cap J_u} [u](x) \odot \nu_u(x)\, d\mathcal{H}^{n-1}(x),$$

where J_u is the set of jump of u (which is a rectifiable set), $\nu_u(x)$ is the normal to J_u at x, and $[u](x)$ is the jump of u at x. For every $p \in]1, +\infty[$ we set

$$SBD^p(U) = \{u \in SBD(U) : \mathcal{E}u \in L^p(U; \mathbb{M}^{n\times n}_{sym})\}.$$

We have that if $u \in SBD(U)$, then its sections are in $SBV(U^\xi_y)$ for every $\xi \neq 0$ and for $\mathcal{H}^{n-1}$-a.e. $y \in U^\xi$.

Theorem 1.15 (Compactness in SBD) *Let $\varphi : [0, +\infty) \to [0, +\infty]$ be a non-decreasing function such that*

$$\lim_{t\to+\infty} \frac{\varphi(t)}{t} = +\infty. \tag{1.9}$$

Let (u_h) be a sequence in $SBD(\Omega)$ such that

$$\int_\Omega |u_h|\, dx + |E^j u_h|(\Omega) + \int_\Omega \varphi(|\mathcal{E}u_h|)\, dx + \mathcal{H}^{n-1}(J_{u_h}) \le K, \tag{1.10}$$

for some constant $K > 0$ independent of h. Then there exist a subsequence, still denoted by (u_h), and a function $u \in SBD(\Omega)$ satisfying

$$\begin{aligned}
&u_h \to u \quad \textit{strongly in } L^1_{loc}(\Omega, \mathbb{R}^n),\\
&\mathcal{E}u_h \rightharpoonup \mathcal{E}u \quad \textit{weakly in } L^1(\Omega, \mathbb{M}^{n\times n}_{sym}),\\
&E^j u_h \rightharpoonup \mathcal{E}u \quad \textit{weakly}^* \textit{ in } \mathcal{M}_b(\Omega, \mathbb{M}^{n\times n}_{sym}),\\
&\mathcal{H}^{n-1}(J_u) \le \liminf_{h\to+\infty} \mathcal{H}^{n-1}(J_{u_h}).
\end{aligned}$$

Theorem 1.16 (Lower semicontinuity in SBD**)** *Let* $f:\Omega\times\mathbb{M}^{n\times n}_{sym}\to[0,+\infty]$ *and* $g:\overline{\Omega}\times\mathbb{M}^{n\times n}_{sym}\to[0,+\infty]$ *satisfy the following properties for every* $x\in\Omega$ *(resp.* $x\in\overline{\Omega}$*) and for every* $A\in\mathbb{M}^{n\times n}_{sym}$*:*

- $f(x,\cdot)$ *is convex and lower semicontinuous on* $\mathbb{M}^{n\times n}_{sym}$*;*
- $f(\cdot,A)$ *is measurable on* Ω*;*
- g *is lower semicontinuous on* $\overline{\Omega}\times\mathbb{M}^{n\times n}_{sym}$*;*
- $g(x,\cdot)$ *is convex and positively 1-homogeneous on* $\mathbb{M}^{n\times n}_{sym}$*.*

Let (u_h) *be a sequence in* $SBD(\Omega)$ *satisfying (1.10) and converging to a function* $u\in SBD(\Omega)$ *in* L^1_{loc}*. Then*

$$\int_\Omega f(x,\mathcal{E}u)\,dx\le\liminf_{h\to+\infty}\int_\Omega f(x,\mathcal{E}u_h)\,dx, \tag{1.11}$$

$$\int_{J_u} g(x,[u]\odot\nu_u)\,d\mathcal{H}^{n-1}\le\liminf_{h\to+\infty}\int_{J_{u_h}} g(x,[u_h]\odot\nu_{u_h})\,d\mathcal{H}^{n-1}, \tag{1.12}$$

$$\mathcal{H}^{n-1}(J_u)\le\liminf_{h\to+\infty}\mathcal{H}^{n-1}(J_{u_h}). \tag{1.13}$$

1.4 Extension operators in the Sobolev setting

We state some extension results for H^1 functions defined on perforated domains. They will be an important tool in the study of the homogenization problems treated in Chapters 4-6.

Theorem 1.17 (Existence of an extension operator) *Let* E *be a periodic, connected, open subset of* $\mathbb{R}^n$*, with Lipschitz boundary, let* $\varepsilon>0$*, and set* $E^\varepsilon:=\varepsilon E$*. Given a bounded open set* $\Omega\subset\mathbb{R}^n$*, there exist a linear and continuous extension operator* $T^\varepsilon: H^1(\Omega\cap E^\varepsilon)\to H^1_{loc}(\Omega)$ *and three constants* $k_0,k_1,k_2>0$ *depending on* E *and* n*, but not on* ε *and* Ω*, such that*

$$T^\varepsilon u=u \ \text{a.e. in}\ \Omega\cap E^\varepsilon,$$

$$\int_{\Omega(\varepsilon k_0)}|T^\varepsilon u|^2dx\le k_1\int_{\Omega\cap E^\varepsilon}|u|^2dx,$$

$$\int_{\Omega(\varepsilon k_0)}|D(T^\varepsilon u)|^2dx\le k_2\int_{\Omega\cap E^\varepsilon}|Du|^2dx,$$

for every $u\in H^1(\Omega\cap E^\varepsilon)$*. Here we used the notation* $\Omega(\varepsilon k_0):=\{x\in\Omega:\mathit{dist}(x,\partial\Omega)>\varepsilon k_0\}$*.*

For the proof we refer to [2].

Remark 1.18 Theorem 1.17 applies to a very large class of domains E. In particular, it covers the case in which E is obtained by removing from the periodicity cell $Q:=(0,1)^n$ a set B with Lipschitz boundary such that $\mathrm{dist}(B,\partial Q)>0$, and repeating this structure by periodicity (see also [36]).

Definition 1.19 *Let* ω *be an unbounded domain of* $\mathbb{R}^n$ *with a* Q*-periodic structure, where* $Q:=(0,1)^n$*. Assume that the cell of periodicity* $\omega\cap Q$ *is a domain with a Lipschitz boundary. Given a bounded open set* $\Omega\subset\mathbb{R}^n$ *and a positive parameter* $\varepsilon>0$*, we set* $\Omega^\varepsilon:=\Omega\cap\varepsilon\,\omega$*. Moreover, we set* $\gamma^\varepsilon:=\partial\Omega\cap\varepsilon\,\omega$*. We define the space* $H^1(\Omega^\varepsilon,\gamma^\varepsilon)$ *as*

$$H^1(\Omega^\varepsilon,\gamma^\varepsilon):=\{v\in H^1(\Omega^\varepsilon): v=0 \ \text{a.e. on}\ \gamma^\varepsilon\}. \tag{1.14}$$

Theorem 1.20 *Let* Ω_0 *be a bounded domain such that* $\Omega^\varepsilon \subset \Omega_0$ *and* $dist(\partial\Omega_0, \Omega) > 1$. *Then for every sufficiently small* ε *there exists a linear extension operator* $T^\varepsilon : H^1(\Omega^\varepsilon, \gamma^\varepsilon) \to H^1_0(\Omega_0)$ *and three constants* $k_0, k_1, k_2 > 0$ *such that*

$$\begin{aligned} ||T^\varepsilon u||_{H^1(\Omega_0)} &\leq k_1 ||u||_{H^1(\Omega^\varepsilon)}, \\ ||D(T^\varepsilon u)||_{L^2(\Omega_0)} &\leq k_2 ||Du||_{L^2(\Omega^\varepsilon)}, \\ ||E(T^\varepsilon u)||_{L^2(\Omega_0)} &\leq k_3 ||Eu||_{L^2(\Omega^\varepsilon)}, \end{aligned}$$

for any $u \in H^1(\Omega^\varepsilon, \gamma^\varepsilon)$, *where the constants* k_0, k_1, k_2 *do not depend on* ε.

Moreover, $(T^\varepsilon u)_{|A} = 0$ *for any open set* A *such that* $\bar{A} \subset \Omega_0 \setminus \Omega$, *if* ε *is sufficiently small.*

For the proof we refer to [44].

1.5 Integral representation

In this section we present some classical results concerning the integral representation of Γ-limits, both in the Sobolev and in the SBV settings.

Let Ω be a bounded open set of $\mathbb{R}^n$, let $\mathcal{A}(\Omega)$ be the class of all open subsets of Ω, and let $\mathcal{A}_0(\Omega)$ denote the class of all open subsets of Ω which are well contained in Ω.

Theorem 1.21 *Let* $F : L^2(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ *be an increasing functional satisfying the following properties:*

(a) F *is local;*

(b) F *is a measure;*

(c) F *is lower semicontinuous;*

(d) $F(u + c, A) = F(u, A)$ *for every* $u \in L^2(\Omega)$, $A \in \mathcal{A}(\Omega)$, *and* $c \in \mathbb{R}$;

(e) there exist $b \in \mathbb{R}$ *and* $a \in L^1_{loc}(\Omega)$ *such that*

$$0 \leq F(u, A) \leq \int_A \left(a(x) + b|Du|^2\right) dx$$

for every $u \in H^1(\Omega)$ *and for every* $A \in \mathcal{A}(\Omega)$.

Then there exists a Borel function $f : \Omega \times \mathbb{R}^n \to [0, +\infty]$ *such that*

(i) for every $u \in L^2(\Omega)$ *and for every* $A \in \mathcal{A}(\Omega)$ *such that* $u_{|A} \in H^1_{loc}(\Omega)$ *we have*

$$F(u, A) = \int_A f(x, Du(x))\, dx;$$

(ii) for almost every $x \in \Omega$ *the function* $f(x, \cdot)$ *is convex in* $\mathbb{R}^n$;

(iii) for almost every $x \in \Omega$ *we have*

$$0 \leq f(x, \xi) \leq a(x) + b|\xi|^2$$

for every $\xi \in \mathbb{R}^n$.

Let T be a finite set and denote with $BV(\Omega,T)$ the class of Borel functions $u:\Omega\to T$ such that $\{u=t\}$ is a set of finite perimeter in Ω for every $t\in T$.

Theorem 1.22 *Let $F:BV(\Omega,T)\times\mathcal{A}(\Omega)\to[0,+\infty[$ be a functional satisfying the following conditions:*

(a) F is local;

(b) F is a measure for every $u\in BV(\Omega,T)$;

(c) there exist a constant $\lambda>0$ such that

$$0\le F(u,A)\le\lambda\mathcal{H}^{n-1}(A\cap S_u)$$

for every $u\in BV(\Omega,T)$ and for every $A\in\mathcal{A}(\Omega)$;

(d) if $u_h\to u$ almost everywhere in A, then $F(u,A)\le\liminf_{h\to+\infty}F(u_h)$ for every $A\in\mathcal{A}(\Omega)$;

(e) for every $A\in\mathcal{A}_0(\Omega)$ there exists a continuous function $\omega_A:[0,+\infty[\to[0,+\infty[$ such that $\omega_A(0)=0$ and

$$|F(u,B)-F(v,B+z)|\le\omega_A(|z|)\mathcal{H}^{n-1}(B\cap S_u)$$

whenever $B\in\mathcal{A}(A)$, $z\in\mathbb{R}^n$, $|z|<dist(A,\partial\Omega)/2$ and $v(x+z)=u(x)$ in B.

Then, there exists a unique continuous function $f:\Omega\times T\times T\times S^{n-1}\to[0,\lambda]$ such that for every $x\in\Omega$, $i,j\in T$ and $\nu\in S^{n-1}$ $f(x,i,j,\nu)=f(x,j,i,-\nu)$,

$$p\mapsto f\Big(x,i,j,\frac{p}{|p|}\Big)|p| \quad \textit{is convex in } \mathbb{R}^n,$$

and $F(u,A)$ is representable as

$$F(u,A)=\int_{A\cap S_u}f(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{n-1}$$

for every $u\in BV(\Omega,T)$ and for every A.

Part I

Asymptotic models for thin curved rods

Chapter 2

The nonlinear bending-torsion theory for curved rods as Γ-limit of three-dimensional elasticity

In the first part of the thesis (Chapter 2 and Chapter 3) we study the case of a heterogeneous curved beam made of a hyperelastic material. We will consider different scalings of the elastic energy associated to a deformation of the beam and we derive a complete hierarchy of one-dimensional models for curved rods.

2.1 Notations and formulation of the problem

Let us describe the geometry of the unstressed curved beam.

Let $\gamma : [0, L] \to \mathbb{R}^3$ be a simple regular curve of class C^3 parametrized by the arc-length and let $\tau = \dot{\gamma}$ be its unit tangent vector. We assume that there exists an orthonormal frame of class C^2 along the curve. More precisely, we assume that there exists $R_0 \in C^2([0, L]; \mathbb{M}^{3\times 3})$ such that $R_0(s) \in SO(3)$ for every $s \in [0, L]$ and $R_0(s)\, e_1 = \tau(s)$ for every $s \in [0, L]$, where e_i, for $i = 1, 2, 3$, denotes the i-th vector of the canonical basis of $\mathbb{R}^3$ and $SO(3) = \{R \in \mathbb{M}^{3\times 3} : R^T R = Id,\ \det R = 1\}$. We set

$$\nu_k(s) := R_0(s)\, e_k, \text{ for } k = 2, 3.$$

We can introduce three scalar functions ϱ, k_2 and k_3 in $C^1([0, L])$ such that

$$\begin{aligned} \tau'(s) &= k_2(s)\, \nu_2(s) + k_3(s)\, \nu_3(s), \\ \nu_2'(s) &= -\, k_2(s)\, \tau(s) + \varrho(s)\, \nu_3(s), \\ \nu_3'(s) &= -\, k_3(s)\, \tau(s) - \varrho(s)\, \nu_2(s). \end{aligned} \tag{2.1}$$

Note that the curvature of γ can be easily recognized as $\sqrt{k_2^2 + k_3^2}$ and the torsion of γ as $\varrho + \dfrac{k_2 k_3' - k_3 k_2'}{k_2^2 + k_3^2}$.

Let $D \subset \mathbb{R}^2$ be a bounded open connected set with Lipschitz boundary such that

$$\int_D \xi\, \zeta\, d\xi\, d\zeta = 0 \tag{2.2}$$

and

$$\int_D \xi\, d\xi\, d\zeta = \int_D \zeta\, d\xi\, d\zeta = 0, \tag{2.3}$$

where (ξ, ζ) stands for the coordinates of a generic point of D. Without loss of generality, we can also assume $\mathcal{L}^2(D) = 1$. We set $\Omega := (0, L) \times D$.
The reference configuration of the thin beam is given by

$$\widetilde{\Omega}_h := \{\gamma(s) + h\,\xi\,\nu_2(s) + h\,\zeta\,\nu_3(s) : (s, \xi, \zeta) \in \Omega\},$$

where h is a small positive parameter. Clearly the curve γ and the set D represent the mid-fiber and the cross-section of the beam, respectively. The set $\widetilde{\Omega}_h$ is parametrized by the C^2 map

$$\Psi^{(h)} : \Omega \to \widetilde{\Omega}_h : \quad (s, \xi, \zeta) \mapsto \gamma(s) + h\,\xi\,\nu_2(s) + h\,\zeta\,\nu_3(s),$$

which is one-to-one for h small enough.

We assume that the thin beam is made of a hyperelastic material whose stored energy density $W : \Omega \times \mathbb{M}^{3\times 3} \to [0, +\infty]$ is a Carathéodory function satisfying the following hypotheses:

(i) there exists $\delta > 0$ such that the function $F \mapsto W(z, F)$ is of class C^2 on the set $\{F \in \mathbb{M}^{3\times 3} : \operatorname{dist}(F, SO(3)) < \delta\}$ for a.e. $z \in \Omega$;

(ii) the second derivative $\partial^2 W / \partial F^2$ is a Carathéodory function on the set

$$\Omega \times \{F \in \mathbb{M}^{3\times 3} : \operatorname{dist}(F, SO(3)) < \delta\} \tag{2.4}$$

and there exists a constant $C_1 > 0$ such that

$$\left| \frac{\partial^2 W}{\partial F^2}(z, F)[G, G] \right| \leq C_1 |\,G\,|^2 \text{ for a.e. } z \in \Omega, \text{ every } F \text{ with } \operatorname{dist}(F, SO(3)) < \delta$$

and every $G \in \mathbb{M}^{3\times 3}_{sym}$;

(iii) W is frame indifferent, i.e., $W(z, RF) = W(z, F)$ for a.e. $z \in \Omega$, every $F \in \mathbb{M}^{3\times 3}$ and every $R \in SO(3)$;

(iv) $W(z, R) = 0$ for every $R \in SO(3)$;

(v) $\exists\ C_2 > 0$ independent of z such that $W(z, F) \geq C_2 \operatorname{dist}^2(F, SO(3))$ for a.e. $z \in \Omega$ and every $F \in \mathbb{M}^{3\times 3}$.

Notice that, since we do not require any growth condition from above, W is allowed to assume the value $+\infty$ outside a neighborhood of the set (2.4). Therefore our treatment covers the physically relevant case in which $W = +\infty$ for $\det F < 0$, $W \to +\infty$ as $\det F \to 0^+$.
We conclude this section by analyzing some properties of the map $\Psi^{(h)}$, which will be useful in the sequel. We will use the following notation: for any function $z \in W^{1,2}(\Omega; \mathbb{R}^3)$ we set

$$\nabla_h z := \left(\partial_s z \,\Big|\, \frac{1}{h}\,\partial_\xi z \,\Big|\, \frac{1}{h}\,\partial_\zeta z \right).$$

We observe that $\nabla_h \Psi^{(h)}$ can be written as the sum of the rotation R_0 and a perturbation of order h, that is,

$$\nabla_h \Psi^{(h)}(s, \xi, \zeta) = R_0(s) + h\ (\xi\,\nu_2'(s) + \zeta\,\nu_3'(s)) \otimes e_1. \tag{2.5}$$

From this fact it follows that, as $h \to 0$,

$$\nabla_h \Psi^{(h)}(s, \xi, \zeta) \to R_0(s) \quad \text{and} \quad \det\left(\nabla_h \Psi^{(h)}\right) \to 1 = \det R_0 \text{ uniformly.} \tag{2.6}$$

This implies that for h small enough $\nabla_h\Psi^{(h)}$ is invertible at each point of Ω. Since the inverse of $\nabla_h\Psi^{(h)}$ can be written as

$$\big(\nabla_h\Psi^{(h)}\big)^{-1}(s,\xi,\zeta) = R_0^T(s) - h\,R_0^T(s)\,\big[\,(\xi\,\nu_2'(s)+\zeta\,\nu_3'(s))\otimes\tau(s)\big] + O(h^2) \tag{2.7}$$

with $O(h^2)/h^2$ uniformly bounded, $\big(\nabla_h\Psi^{(h)}\big)^{-1}$ converges to R_0^T uniformly.

Let $\tilde{y}\in W^{1,2}(\widetilde{\Omega}_h;\mathbb{R}^3)$ be a deformation of $\widetilde{\Omega}_h$. The elastic energy per unit volume associated to $\tilde{y}$ is defined by

$$\tilde{I}^{(h)}(\tilde{y}) := \frac{1}{h^2}\int_{\widetilde{\Omega}_h} W\big(\big(\Psi^{(h)}\big)^{-1}(x),\nabla\tilde{y}(x)\big)dx. \tag{2.8}$$

The main part of this chapter is devoted to the study of the asymptotic behaviour as $h\to 0$ of the sequence of functionals $\tilde{I}^{(h)}/h^2$. In the final part we will also discuss the scaling $\tilde{I}^{(h)}/h^\alpha$ for $0\le\alpha<2$. Furthermore, we show that the scalings $\beta>4$ lead to the linearized theory for rods, while the scalings $\beta\in(2,4)$ correspond to a constrained linearized theory.

2.2 Derivation of the bending-torsion theory for curved rods

The aim of this section is the study of the asymptotic behaviour of the sequence of functionals

$$\frac{1}{h^2}\,\tilde{I}^{(h)}(\tilde{y}) = \frac{1}{h^4}\int_{\widetilde{\Omega}_h} W\big(\big(\Psi^{(h)}\big)^{-1}(x),\nabla\tilde{y}(x)\big)dx$$

under the assumptions (i)-(v) of Section 2.1.

2.2.1 Compactness

We will show a compactness result for sequences of deformations having equibounded energy $\tilde{I}^{(h)}/h^2$. A key ingredient in the proof is the following rigidity result, proved by G. Friesecke, R.D. James and S. Müller in [30].

Theorem 2.1 *Let U be a bounded Lipschitz domain in $\mathbb{R}^n$, $n\ge 2$. Then there exists a constant $C(U)$ with the following property: for every $u\in W^{1,2}(U;\mathbb{R}^n)$ there is an associated rotation $R\in SO(n)$ such that*

$$\|\nabla u - R\|_{L^2(U)} \le C(U)\,\|\mathrm{dist}(\nabla u, SO(n))\|_{L^2(U)}\,.$$

Remark 2.2 The constant $C(U)$ can be chosen independent of U for a family of sets that are bilipschitz images of a cube (with uniform Lipschitz constants), as remarked in [31].

We introduce the class of limiting admissible deformations

$$\begin{aligned}\mathcal{A} := \{(y,d_2,d_3)\in W^{2,2}((0,L);\mathbb{R}^3)\times W^{1,2}((0,L);\mathbb{R}^3)\times W^{1,2}((0,L);\mathbb{R}^3):\\ (y'(s)\,|\,d_2(s)\,|\,d_3(s))\in SO(3)\text{ for a.e. s in }(0,L)\}.\end{aligned} \tag{2.9}$$

Now we are ready to state and prove the main result of this subsection.

Theorem 2.3 *Let $\big(\tilde{y}^{(h)}\big)$ be a sequence in $W^{1,2}\big(\widetilde{\Omega}_h;\mathbb{R}^3\big)$ such that*

$$\frac{1}{h^2}\,\tilde{I}^{(h)}(\tilde{y}^{(h)}) \le c < +\infty. \tag{2.10}$$

Then there exist a triple $(y, d_2, d_3) \in \mathcal{A}$*, a map* $\bar{R} \in W^{1,2}((0,L);\mathbb{M}^{3\times 3})$ *with* $\bar{R}(s) \in SO(3)$ *for a.e.* $s \in [0,L]$*, and some constants* $c^{(h)} \in \mathbb{R}^3$ *such that, up to subsequences,*

$$\tilde{y}^{(h)} \circ \Psi^{(h)} - c^{(h)} \to y \quad \textit{strongly in } W^{1,2}(\Omega;\mathbb{R}^3), \tag{2.11}$$

$$\frac{1}{h}\,\partial_\xi\big(\tilde{y}^{(h)} \circ \Psi^{(h)}\big) \to d_2 \quad \textit{strongly in } L^2(\Omega;\mathbb{R}^3), \tag{2.12}$$

$$\frac{1}{h}\,\partial_\zeta\big(\tilde{y}^{(h)} \circ \Psi^{(h)}\big) \to d_3 \quad \textit{strongly in } L^2(\Omega;\mathbb{R}^3), \tag{2.13}$$

$$\nabla\tilde{y}^{(h)} \circ \Psi^{(h)} \to \bar{R} \quad \textit{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}). \tag{2.14}$$

Moreover, for a.e. $s \in [0,L]$*, we have* $(y'(s)\,|\,d_2(s)\,|\,d_3(s)) = \bar{R}(s)\,R_0(s)$*, where* $R_0 = (\tau\,|\,\nu_2\,|\,\nu_3)$*.*

Proof. – Let $\big(\tilde{y}^{(h)}\big)$ be a sequence in $W^{1,2}(\widetilde{\Omega}_h;\mathbb{R}^3)$ satisfying (2.10). The assumption (v) on W implies that

$$\int_{\widetilde{\Omega}_h} \operatorname{dist}^2\big(\nabla\tilde{y}^{(h)}(x), SO(3)\big)dx < C\,h^4$$

for a suitable constant C. Using the change of variables $\Psi^{(h)}$, we have

$$\int_\Omega \operatorname{dist}^2\big(\nabla\tilde{y}^{(h)} \circ \Psi^{(h)}, SO(3)\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta \le c\,h^2. \tag{2.15}$$

From (2.6) and the estimate

$$\operatorname{dist}^2(F, SO(3)) \ge \frac{1}{2}\,|\,F\,|^2 - 3,$$

we get the bound

$$\int_\Omega \big|\,\nabla\tilde{y}^{(h)} \circ \Psi^{(h)}\big|^2 ds\,d\xi\,d\zeta \le c. \tag{2.16}$$

Define the sequence $F^{(h)} := \nabla\tilde{y}^{(h)} \circ \Psi^{(h)}$; from (2.16) it follows that there exists a function $F \in L^2(\Omega;\mathbb{M}^{3\times 3})$ such that, up to subsequences,

$$F^{(h)} \rightharpoonup F \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}). \tag{2.17}$$

Using Theorem 2.1, we will show that this convergence is in fact strong in L^2 and that the limit function F is a rotation a.e. depending only on the variable along the mid-fiber and belonging to $W^{1,2}((0,L);\mathbb{M}^{3\times 3})$. The idea is to divide the domain $\widetilde{\Omega}_h$ in small curved cylinders, which are images of homotetic straight cylinders through the same bilipschitz function. Then, we can apply the rigidity theorem to each small curved cylinder with the same constant. In this way we construct a piecewise constant rotation, which is close to the deformation gradient $\nabla\tilde{y}^{(h)}$ in the L^2 norm.

For every small enough $h > 0$, let $K_h \in \mathbb{N}$ satisfy

$$h \le \frac{L}{K_h} < 2\,h.$$

For every $a \in [0,L) \cap \dfrac{L}{K_h}\,\mathbb{N}$, define the segments

$$S_{a,K_h} := \begin{cases} (a, a+2\,h) & \text{if } a < L - \dfrac{L}{K_h}, \\ (L-2\,h, L) & \text{otherwise.} \end{cases}$$

Now consider the cylinders $C_{a,h} := S_{a,K_h} \times D$ and the subsets of $\widetilde{\Omega}_h$ defined by $\widetilde{C}_{a,h} := \Psi^{(h)}(C_{a,h})$. Remark that $\widetilde{C}_{a,h}$ is a bilipschitz image of a cube of size h, that is $(a,0,0) + h\,\big((0,2)\times D\big)$, through the map Ψ defined as

$$\Psi : [0,L]\times\mathbb{R}^2 \to \mathbb{R}^3, \quad (s, v_2, v_3) \mapsto \gamma(s) + v_2\,\nu_2(s) + v_3\,\nu_3(s).$$

By Theorem 2.1 we obtain that there exists a constant rotation $\tilde{R}^{(h)}_a$ such that

$$\int_{\widetilde{C}_{a,h}} \big|\,\nabla\tilde{y}^{(h)} - \tilde{R}^{(h)}_a\big|^2 dx \le c\int_{\widetilde{C}_{a,h}} \operatorname{dist}^2(\nabla\tilde{y}^{(h)}, SO(3))dx. \tag{2.18}$$

The subscript a in $\tilde{R}^{(h)}_a$ is used to remember that the rotation depends on the cylinder $\widetilde{C}_{a,h}$. In particular, since $\Psi^{(h)}\big(\big(a, a+\frac{L}{K_h}\big)\times D\big) \subset \widetilde{C}_{a,h}$, we get

$$\int_{\Psi^{(h)}\left(\left(a,a+\frac{L}{K_h}\right)\times D\right)} \big|\,\nabla\tilde{y}^{(h)} - \tilde{R}^{(h)}_a\big|^2 dx \le c\int_{\widetilde{C}_{a,h}} \operatorname{dist}^2(\nabla\tilde{y}^{(h)}, SO(3))dx. \tag{2.19}$$

Changing variables in the integral on the left-hand side, inequality (2.19) becomes

$$\begin{aligned}\int_{\left(a,a+\frac{L}{K_h}\right)\times D} \big|\,\nabla\tilde{y}^{(h)}\circ\Psi^{(h)} - \tilde{R}^{(h)}_a\big|^2 \det\big(\nabla\Psi^{(h)}\big) ds\,d\xi\,d\zeta \\ \le c\int_{\widetilde{C}_{a,h}} \operatorname{dist}^2\big(\nabla\tilde{y}^{(h)}, SO(3)\big)dx \\ \le c\int_{\widetilde{C}_{a,h}} W\big(\big(\Psi^{(h)}\big)^{-1}(x), \nabla\tilde{y}^{(h)}(x)\big)dx.\end{aligned}$$

Notice that $\det\big(\nabla\Psi^{(h)}\big) = h^2\det\big(\nabla_h\Psi^{(h)}\big)$ and, since $\det\big(\nabla_h\Psi^{(h)}\big) \to 1$ uniformly,

$$\int_{\left(a,a+\frac{L}{K_h}\right)\times D} \big|\,\nabla\tilde{y}^{(h)}\circ\Psi^{(h)} - \tilde{R}^{(h)}_a\big|^2 ds\,d\xi\,d\zeta \le \frac{c}{h^2}\int_{\widetilde{C}_{a,h}} W\big(\big(\Psi^{(h)}\big)^{-1}(x), \nabla\tilde{y}^{(h)}(x)\big)dx. \tag{2.20}$$

Now define the map $R^{(h)} : [0,L) \to SO(3)$ given by

$$R^{(h)}(s) := \tilde{R}^{(h)}_a \quad \text{for } s \in \Big[a, a+\frac{L}{K_h}\Big),\ a\in[0,L)\cap\frac{L}{K_h}\,\mathbb{N}.$$

Summing (2.20) over $a \in [0,L)\cap\frac{L}{K_h}\,\mathbb{N}$ leads to

$$\int_\Omega \big|\,\nabla\tilde{y}^{(h)}\circ\Psi^{(h)} - R^{(h)}\big|^2 ds\,d\xi\,d\zeta \le \frac{c}{h^2}\int_{\widetilde{\Omega}_h} W\big(\big(\Psi^{(h)}\big)^{-1}(x), \nabla\tilde{y}^{(h)}(x)\big)dx \tag{2.21}$$

for a suitable constant independent of h. By (2.10) we obtain

$$\int_\Omega \big|\,\nabla\tilde{y}^{(h)}\circ\Psi^{(h)} - R^{(h)}\big|^2 ds\,d\xi\,d\zeta \le c\,h^2. \tag{2.22}$$

Now, applying iteratively estimate (2.20) in neighbouring cubes, one can prove the following difference quotient estimate for $R^{(h)}$: for every $I' \subset\subset [0,L]$ and every $\delta\in\mathbb{R}$ with $|\,\delta| \le \operatorname{dist}(I', \{0,L\})$

$$\int_{I'} \big|\,R^{(h)}(s+\delta) - R^{(h)}(s)\,\big|^2 ds \le c\,(|\delta\,| + h)^2, \tag{2.23}$$

with c independent of I' and δ (see [40], proof of Theorem 2.1). Using the Fréchet-Kolmogorov criterion, we deduce that, for every sequence $(h_j) \to 0$, there exists a subsequence of $R^{(h_j)}$ which converges strongly in $L^2(I';\mathbb{M}^{3\times 3})$ to some $\bar{R} \in L^2(I';\mathbb{M}^{3\times 3})$, with $\bar{R}(s) \in SO(3)$ for a.e. $s \in I'$. From (2.17) and (2.22) it follows that $F = \bar{R}$ a.e.. Moreover (2.6) and (2.15) imply the convergence of the L^2 norm of $F^{(h)}$ to the L^2 norm of $\bar{R}$, hence

$$F^{(h)} \to \bar{R} \quad \text{strongly in}\, L^2(\Omega;\mathbb{M}^{3\times 3}).$$

This proves (2.14), once the regularity of the function $\bar{R}$ is shown. To this aim, divide both sides of the inequality (2.23) by $(|\delta| + h)^2$ and let $h \to 0$; then

$$\int_{I'} \frac{\left|\bar{R}(s+\delta) - \bar{R}(s)\right|^2}{|\delta|^2}\, ds \le c \tag{2.24}$$

and so $\bar{R} \in W^{1,2}(I';\mathbb{M}^{3\times 3})$. But this holds for every $I' \subset\subset [0,L]$ with a constant independent of the subset I', hence $\bar{R} \in W^{1,2}((0,L);\mathbb{M}^{3\times 3})$.
Now notice that

$$\nabla_h\big(\tilde{y}^{(h)} \circ \Psi^{(h)}\big) = \big(\nabla\tilde{y}^{(h)} \circ \Psi^{(h)}\big)\nabla_h\Psi^{(h)} = F^{(h)}\nabla_h\Psi^{(h)}; \tag{2.25}$$

by (2.6) and (2.14) we deduce that

$$\nabla_h\big(\tilde{y}^{(h)} \circ \Psi^{(h)}\big) \longrightarrow \bar{R}\,R_0 \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}). \tag{2.26}$$

In particular, we have

$$\nabla\big(\tilde{y}^{(h)} \circ \Psi^{(h)}\big) \longrightarrow \big(\bar{R}\,R_0 e_1\big) \otimes e_1 \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}). \tag{2.27}$$

By Poincaré inequality there exist some constants $c^{(h)} \in \mathbb{R}^3$ and a function y in $W^{1,2}(\Omega;\mathbb{R}^3)$ such that (2.11) is satisfied. Moreover (2.27) entails that the function y depends only on the variable s in $[0,L]$ and satisfies $y' = \bar{R}\,R_0 e_1$. Setting $d_k := \bar{R}\,R_0 e_k$ for $k = 2,3$, we have that $(y, d_2, d_3) \in \mathcal{A}$ and (2.12), (2.13) are satisfied by (2.26). □

2.2.2 Bound from below

Let $Q_3 : \Omega \times \mathbb{M}^{3\times 3} \longrightarrow [0,+\infty)$ be twice the quadratic form of linearized elasticity; i.e.,

$$Q_3(z,G) := \frac{\partial^2 W}{\partial F^2}(z, Id)[G,G]$$

for a.e. $z \in \Omega$ and every $G \in \mathbb{M}^{3\times 3}$. We introduce the quadratic form $Q : [0,L] \times \mathbb{M}^{3\times 3}_{\text{skew}} \to [0,+\infty)$ defined by

$$Q(s,P) := \inf_{\substack{\hat{\alpha}\in W^{1,2}(D;\mathbb{R}^3)\\ \hat{g}\in\mathbb{R}^3}} \left\{ \int_D Q_3\left(s,\xi,\zeta,R_0(s)\left(P\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix} + \hat{g}\,\Big|\,\partial_\xi\hat{\alpha}\,\Big|\,\partial_\zeta\hat{\alpha}\right)R_0^T(s)\right)d\xi\,d\zeta\right\}. \tag{2.28}$$

Remark 2.4 It is easy to check that the minimum in (2.28) is attained; moreover the minimizers depend linearly on P, hence Q is a quadratic form of P. Notice also that if $P \in L^2((0,L);\mathbb{M}^{3\times 3})$, then $\hat{\alpha} \in L^2(\Omega;\mathbb{R}^3)$ with $\partial_\xi\hat{\alpha}, \partial_\zeta\hat{\alpha} \in L^2(\Omega;\mathbb{R}^3)$, and $\hat{g} \in L^2((0,L);\mathbb{R}^3)$ (see [41, Remarks 4.1 - 4.3]).

In the following theorem we prove a lower bound for the energies $\tilde{I}^{(h)}/h^2$ in terms of the functional

$$I(y, d_2, d_3) := \begin{cases} \dfrac{1}{2}\displaystyle\int_0^L Q\big(s, \big(R^T(s)R'(s) - R_0^T(s)R_0'(s)\big)\big)ds & \text{if } (y, d_2, d_3) \in \mathcal{A}, \\ +\infty & \text{otherwise}, \end{cases} \tag{2.29}$$

where $R \in W^{1,2}((0, L); \mathbb{M}^{3\times3})$ denotes the matrix $R := (y' \,|\, d_2 \,|\, d_3)$ and $\mathcal{A}$ is the class defined in (2.9).

Theorem 2.5 *Let $y \in W^{1,2}(\Omega; \mathbb{R}^3)$ and let $d_2, d_3 \in L^2(\Omega; \mathbb{R}^3)$. Then, for every positive sequence (h_j) converging to zero and every sequence $\big(\tilde{y}^{(h_j)}\big) \subset W^{1,2}(\widetilde{\Omega}_{h_j}; \mathbb{R}^3)$ such that*

$$\tilde{y}^{(h_j)} \circ \Psi^{(h_j)} \to y \quad \textit{strongly in } W^{1,2}(\Omega; \mathbb{R}^3), \tag{2.30}$$

$$\frac{1}{h_j}\,\partial_\xi\big(\tilde{y}^{(h_j)} \circ \Psi^{(h_j)}\big) \to d_2 \quad \textit{strongly in } L^2(\Omega; \mathbb{R}^3), \tag{2.31}$$

$$\frac{1}{h_j}\,\partial_\zeta\big(\tilde{y}^{(h_j)} \circ \Psi^{(h_j)}\big) \to d_3 \quad \textit{strongly in } L^2(\Omega; \mathbb{R}^3), \tag{2.32}$$

it turns out that

$$I(y, d_2, d_3) \le \liminf_{j\to\infty} \frac{1}{h_j^4} \int_{\widetilde{\Omega}_{h_j}} W\big(\big(\Psi^{(h_j)}\big)^{-1}(x), \nabla\tilde{y}^{(h_j)}(x)\big)dx. \tag{2.33}$$

Proof. – In the following, for notational brevity, we will write simply h instead of h_j. Let $\big(\tilde{y}^{(h)}\big)$ be a sequence satisfying (2.30), (2.31) and (2.32). We can assume that

$$\liminf_{j\to\infty} \frac{1}{h_j^4} \int_{\widetilde{\Omega}_{h_j}} W\big(\big(\Psi^{(h_j)}\big)^{-1}(x), \nabla\tilde{y}^{(h_j)}(x)\big)dx \le C < +\infty,$$

otherwise (2.33) is trivial. Therefore, up to subsequences, (2.10) is satisfied. By Theorem 2.3 we deduce that $(y, d_2, d_3) \in \mathcal{A}$,

$$F^{(h)} := \nabla\tilde{y}^{(h)} \circ \Psi^{(h)} \longrightarrow \bar{R} \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times3}) \tag{2.34}$$

with $\bar{R} \in W^{1,2}((0, L); \mathbb{M}^{3\times3})$, $\bar{R} \in SO(3)$ a.e., and

$$R := (y' \,|\, d_2 \,|\, d_3) = \bar{R}\,R_0. \tag{2.35}$$

Moreover, as in the proof of Theorem 2.3, we can construct a piecewise constant approximation $R^{(h)} : [0, L] \to SO(3)$ such that

$$\int_\Omega \big|F^{(h)} - R^{(h)}\big|^2 ds\, d\xi\, d\zeta \le c\, h^2 \tag{2.36}$$

and $R^{(h)} \to \bar{R}$ strongly in $L^2(I'; \mathbb{M}^3)$ for every $I' \subset\subset [0, L]$. Define the functions $G^{(h)} : \Omega \to \mathbb{M}^{3\times3}$ as

$$G^{(h)} := \frac{1}{h}\Big((R^{(h)})^T F^{(h)} - Id\Big) = \frac{1}{h}\Big((R^{(h)})^T \nabla_h y^{(h)} \big(\nabla_h \Psi^{(h)}\big)^{-1} - Id\Big). \tag{2.37}$$

By (2.36) they are bounded in $L^2(\Omega; \mathbb{M}^{3\times3})$, so there exists $G \in L^2(\Omega; \mathbb{M}^{3\times3})$ such that $G^{(h)} \rightharpoonup G$ weakly in $L^2(\Omega; \mathbb{M}^{3\times3})$. We claim that

$$\liminf_{h\to0} \frac{1}{h^4} \int_{\widetilde{\Omega}_h} W\Big(\big(\Psi^{(h)}\big)^{-1}(x), \nabla\tilde{y}^{(h)}(x)\Big)dx \ge \frac{1}{2}\int_\Omega Q_3(s, \xi, \zeta, G)ds\, d\xi\, d\zeta. \tag{2.38}$$

Performing the change of variables $\Psi^{(h)}$, we have

$$\frac{1}{h^4}\int_{\widetilde{\Omega}_h} W\Big(\big(\Psi^{(h)}\big)^{-1}(x), \nabla\tilde{y}^{(h)}(x)\Big)dx = \frac{1}{h^2}\int_\Omega W\big(s,\xi,\zeta,F^{(h)}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta$$
$$= \frac{1}{h^2}\int_\Omega W\big(s,\xi,\zeta,\big(R^{(h)}\big)^T F^{(h)}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta \tag{2.39}$$

where the last equality follows from the frame indifference of W. Define the family of functions

$$\chi^{(h)}(s,\xi,\zeta) := \begin{cases} 1 & \text{in } \ \Omega\cap\big\{(s,\xi,\zeta) : \big|G^{(h)}(s,\xi,\zeta)\big| \le h^{-\frac{1}{2}}\big\}, \\ 0 & \text{otherwise.}\end{cases}$$

From the boundedness of $G^{(h)}$ in $L^2(\Omega;\mathbb{M}^{3\times3})$ we get that $\chi^{(h)}\to 1$ boundedly in measure, so that

$$\chi^{(h)}G^{(h)} \rightharpoonup G \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}). \tag{2.40}$$

By expanding W around the identity, we obtain that for every $(s,\xi,\zeta)\in\Omega$ and $A\in\mathbb{M}^{3\times3}$

$$W\big(s,\xi,\zeta,Id+A\big) = \frac{1}{2}\frac{\partial^2 W}{\partial F^2}\,(s,\xi,\zeta,Id+t\,A)[A,A]$$

where $0<t<1$ depends on the point (s,ξ,ζ) and on A. By (2.39) and by the definition of $G^{(h)}$

$$\frac{1}{h^2}\,\tilde{I}^{(h)}\big(\tilde{y}^{(h)}\big) = \frac{1}{h^2}\int_\Omega W\big(s,\xi,\zeta,Id+h\,G^{(h)}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta$$
$$\ge \frac{1}{h^2}\int_\Omega \chi^{(h)} W\big(s,\xi,\zeta,Id+h\,G^{(h)}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta$$
$$= \frac{1}{2}\int_\Omega \chi^{(h)}\left(\frac{\partial^2 W}{\partial F^2}\,\big(s,\xi,\zeta,Id+h\,t(h)\,G^{(h)}\big)\big[G^{(h)},G^{(h)}\big]\right)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta \tag{2.41}$$

where $0<t(h)<1$ depends on (s,ξ,ζ) and on $G^{(h)}$. The last integral in the previous formula can be written as

$$\frac{1}{2}\int_\Omega \chi^{(h)}\left(\frac{\partial^2 W}{\partial F^2}\,\big(s,\xi,\zeta,Id+h\,t(h)\,G^{(h)}\big)\big[G^{(h)},G^{(h)}\big]\right)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta$$
$$= \frac{1}{2}\int_\Omega \left(\chi^{(h)}\left(\frac{\partial^2 W}{\partial F^2}\,\big(s,\xi,\zeta,Id+h\,t(h)\,G^{(h)}\big)\big[G^{(h)},G^{(h)}\big] - Q_3\big(s,\xi,\zeta,G^{(h)}\big)\right)\right)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta$$
$$+\frac{1}{2}\int_\Omega Q_3\big(s,\xi,\zeta,\chi^{(h)}\,G^{(h)}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta. \tag{2.42}$$

By Scorza-Dragoni theorem there exists a compact set $K\subset\Omega$ such that the function $\partial^2 W/\partial F^2$ restricted to $K\times\overline{B_\delta(Id)}$ is continuous, hence uniformly continuous. Since $h\,t(h)\,\chi^{(h)}\,G^{(h)}$ is uniformly small for h small enough, for every $\varepsilon>0$ we have

$$\frac{1}{2}\int_\Omega \chi^{(h)}\left(\frac{\partial^2 W}{\partial F^2}\,\big(s,\xi,\zeta,Id+h\,t(h)\,G^{(h)}\big)\big[G^{(h)},G^{(h)}\big] - Q_3\big(s,\xi,\zeta,G^{(h)}\big)\right)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta$$
$$\ge -\frac{\varepsilon}{2}\int_K \chi^{(h)}\big|\,G^{(h)}\big|^2\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta \ge -\,C\,\varepsilon$$

for h small enough. As for the second integral in (2.42), by (2.6) and (2.39) we get

$$\liminf_{h\to0}\frac{1}{2}\int_\Omega Q_3\big(s,\xi,\zeta,\chi^{(h)}\,G^{(h)}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta \ge \frac{1}{2}\int_\Omega Q_3\big(s,\xi,\zeta,G\big)ds\,d\xi\,d\zeta \tag{2.43}$$

since Q_3 is a nonnegative quadratic form. Combining (2.41), (2.42) and (2.43) we have

$$\liminf_{h\to 0} \frac{1}{h^2}\,\tilde I^{(h)}(\tilde y^{(h)}) \geq \frac{1}{2}\int_\Omega Q_3(s,\xi,\zeta,G)ds\,d\xi\,d\zeta \,-\, C\,\varepsilon$$

and, since ε is arbitrary, (2.38) is proved. It remains to identify G.

Fix $(\xi_0,\zeta_0)\in D$; let $\delta_0=\delta_0(\xi_0,\zeta_0)>0$ be such that $B_{2\,\delta_0}(\xi_0,\zeta_0)\subset D$ and let $U_0:=(0,L)\times B_{\delta_0}(\xi_0,\zeta_0)$. Fix $t\in\mathbb{R}-\{0\}$, $|\,t\,|<\delta_0$. For every $(s,\xi,\zeta)\in U_0$ we can define the difference quotients of the functions $G^{(h)}$ with respect to the variables ξ and ζ along the direction τ, given by

$$\begin{cases} H_t^{(h)}(s,\xi,\zeta) := & \dfrac{1}{t}\Big(G^{(h)}(s,\xi+t,\zeta)-G^{(h)}(s,\xi,\zeta)\Big)\,\tau(s),\\[2ex] K_t^{(h)}(s,\xi,\zeta) := & \dfrac{1}{t}\Big(G^{(h)}(s,\xi,\zeta+t)-G^{(h)}(s,\xi,\zeta)\Big)\,\tau(s),\end{cases}$$

and the corresponding difference quotients of the limit function G

$$\begin{cases} H_t(s,\xi,\zeta) := & \dfrac{1}{t}\Big(G(s,\xi+t,\zeta)-G(s,\xi,\zeta)\Big)\,\tau(s),\\[2ex] K_t(s,\xi,\zeta) := & \dfrac{1}{t}\Big(G(s,\xi,\zeta+t)-G(s,\xi,\zeta)\Big)\,\tau(s).\end{cases}$$

Since $G^{(h)}\rightharpoonup G$ in $L^2(\Omega;\mathbb{M}^{3\times 3})$ and $R^{(h)}\longrightarrow \bar R$ boundedly in measure, we have

$$\begin{aligned} H_t^{(h)} &\rightharpoonup H_t \quad \text{weakly in}\, L^2(U_0;\mathbb{R}^3) \text{ and}\\ R^{(h)}\,H_t^{(h)} &\rightharpoonup \bar R\,H_t \quad \text{weakly in}\, L^2(U_0;\mathbb{R}^3). \end{aligned} \tag{2.44}$$

In terms of $F^{(h)}$ the left-hand side of (2.44) reads as

$$R^{(h)}(s)H_t^{(h)}(s,\xi,\zeta) = \frac{1}{h\,t}\Big(F^{(h)}(s,\xi+t,\zeta)-F^{(h)}(s,\xi,\zeta)\Big)\,\tau(s). \tag{2.45}$$

Now recall that, if we set $y^{(h)}:=\tilde y^{(h)}\circ\Psi^{(h)}$, we have

$$\nabla y^{(h)} = F^{(h)}\,\nabla\Psi^{(h)}; \tag{2.46}$$

in particular, taking the first column of the two matrices, we obtain

$$F^{(h)}(s,\xi,\zeta)\,\tau(s) = \partial_s y^{(h)}(s,\xi,\zeta) - h\,F^{(h)}(s,\xi,\zeta)\,(\xi\,\nu_2'(s)+\zeta\,\nu_3'(s)).$$

By the last equality and (2.45) we get

$$\begin{aligned} R^{(h)}(s)H_t^{(h)}(s,\xi,\zeta) &= \frac{1}{h\,t}\Big(\partial_s y^{(h)}(s,\xi+t,\zeta)-\partial_s y^{(h)}(s,\xi,\zeta)\Big)\\ &\quad -\frac{1}{t}\Big((\xi+t)\,F^{(h)}(s,\xi+t,\zeta)-\xi\,F^{(h)}(s,\xi,\zeta)\Big)\,\nu_2'(s)\\ &\quad -\frac{1}{t}\Big(\zeta\,F^{(h)}(s,\xi+t,\zeta)-\zeta\,F^{(h)}(s,\xi,\zeta)\Big)\,\nu_3'(s). \end{aligned} \tag{2.47}$$

For the first term we have

$$\begin{aligned} \frac{1}{h\,t}\,\partial_s\Big(y^{(h)}(s,\xi+t,\zeta)-y^{(h)}(s,\xi,\zeta)\Big) &= \frac{1}{h\,t}\,\partial_s\bigg(\int_\xi^{\xi+t}\partial_\xi y^{(h)}(s,\vartheta,\zeta)\,d\vartheta\bigg)\\ &= \partial_s\bigg(\frac{1}{t}\int_0^t\frac{1}{h}\,\partial_\xi y^{(h)}(s,\xi+\vartheta,\zeta)\,d\vartheta\bigg), \end{aligned}$$

so by (2.32) and (2.35)

$$\frac{1}{h\,t}\,\partial_s\Big(y^{(h)}(s,\xi+t,\zeta)-y^{(h)}(s,\xi,\zeta)\Big)\rightharpoonup d_2'(s)=\partial_s(\bar R(s)\,\nu_2(s))\quad\text{weakly in}\,W^{-1,2}(U_0;\mathbb{R}^3). \tag{2.48}$$

By (2.34) the second term in (2.47) converges to

$$\frac{1}{t}\,\Big((\xi+t)\,\bar R(s)-\xi\,\bar R(s)\Big)\,\nu_2'(s)=\bar R(s)\,\nu_2'(s)\quad\text{strongly in}\,L^2(U_0;\mathbb{R}^3) \tag{2.49}$$

and the last term to

$$\frac{1}{t}\,\Big(\zeta\,\bar R(s)-\zeta\,\bar R(s)\Big)\,\nu_3'(s)=0\quad\text{strongly in}\,L^2(U_0;\mathbb{R}^3). \tag{2.50}$$

Putting together (2.48), (2.49), (2.50) and (2.44)

$$\bar R(s)\,H_t(s,\xi,\zeta)=\partial_s(\bar R(s)\,\nu_2(s))-\bar R(s)\,\nu_2'(s)\ \text{a.e. in}\ U_0$$

and so

$$H_t(s,\xi,\zeta)=(\bar R(s))^T\,\bar R'(s)\,\nu_2(s)\ \text{a.e. in}\ U_0. \tag{2.51}$$

Repeating the same argument for $K_t^{(h)}$ we get

$$K_t(s,\xi,\zeta)=(\bar R(s))^T\,\bar R'(s)\,\nu_3(s)\ \text{a.e. in}\ U_0. \tag{2.52}$$

From the last two equalities we deduce that the functions H_t and K_t depend only on the variable s. Moreover, letting t go to 0 both in (2.51) and in (2.52), we get that the gradient of $G\,\tau$ w.r.to the variables (ξ,ζ) depends only on s, i.e.,

$$\nabla_{(\xi,\zeta)}\big(G(s,\xi,\zeta)\,\tau(s)\big)=(\bar R(s))^T\,\bar R'(s)\,(\nu_2(s)\,|\,\nu_3(s))\ \text{a.e. in}\ U_0. \tag{2.53}$$

Being this equality valid in $U_0=(0,L)\times B_{\delta_0}(\xi_0,\zeta_0)$, for an arbitrary $(\xi_0,\zeta_0)\in D$, we can conclude that it holds a.e. in the whole Ω. Since D is connected, we obtain that for a.e. $(s,\xi,\zeta)\in\Omega$

$$G(s,\xi,\zeta)\,\tau(s)=(\bar R(s))^T\,\bar R'(s)\,(\xi\,\nu_2(s)+\zeta\,\nu_3(s))+g(s)$$

with $g:[0,L]\to\mathbb{R}^3$. Remark that from the previous formula $g\in L^2((0,L);\mathbb{R}^3)$.
It remains to identify the components $G(s,\xi,\zeta)\,\nu_2(s)$ and $G(s,\xi,\zeta)\,\nu_3(s)$. By (2.46) we have

$$\begin{aligned}G^{(h)}(s,\xi,\zeta)\,\nu_2(s)&=\frac{1}{h}\,\Big((R^{(h)}(s))^TF^{(h)}(s,\xi,\zeta)\,\nu_2(s)-\nu_2(s)\Big)\\&=\frac{1}{h}\,\Big(h^{-1}(R^{(h)}(s))^T\partial_\xi y^{(h)}(s,\xi,\zeta)-\nu_2(s)\Big)\end{aligned}$$

and

$$\begin{aligned}G^{(h)}(s,\xi,\zeta)\,\nu_3(s)&=\frac{1}{h}\,\Big((R^{(h)}(s))^TF^{(h)}(s,\xi,\zeta)\,\nu_3(s)-\nu_3(s)\Big)\\&=\frac{1}{h}\,\Big(h^{-1}(R^{(h)}(s))^T\partial_\zeta y^{(h)}(s,\xi,\zeta)-\nu_3(s)\Big),\end{aligned}$$

so, if we define

$$\alpha^{(h)}(s,\xi,\zeta):=\frac{1}{h}\,\Big(h^{-1}(R^{(h)})^Ty^{(h)}(s,\xi,\zeta)-\xi\,\nu_2(s)-\zeta\,\nu_3(s)\Big)$$

it turns out that

$$\partial_\xi\alpha^{(h)}(s,\xi,\zeta)=G^{(h)}(s,\xi,\zeta)\,\nu_2(s)\quad\text{and}\quad\partial_\zeta\alpha^{(h)}(s,\xi,\zeta)=G^{(h)}(s,\xi,\zeta)\,\nu_3(s). \tag{2.54}$$

Applying the Poincaré inequality to the functions $\alpha^{(h)}$ for fixed s we obtain that for a.e. $s \in [0, L]$

$$\int_D \left| \alpha^{(h)}(s,\xi,\zeta) - \alpha_0^{(h)}(s) \right|^2 d\xi\, d\zeta \leq c \int_D \left(\left| \partial_\xi \alpha^{(h)}(s,\xi,\zeta) \right|^2 + \left| \partial_\zeta \alpha^{(h)}(s,\xi,\zeta) \right|^2 \right) d\xi\, d\zeta,$$

where $\alpha_0^{(h)}(s) := \int_D \alpha^{(h)}(s,\xi,\zeta)\, d\xi\, d\zeta$. Integrating over $[0, L]$, we have

$$||\alpha^{(h)} - \alpha_0^{(h)}||^2_{L^2(\Omega)} \leq c \left(||\partial_\xi \alpha^{(h)}||^2_{L^2(\Omega)} + ||\partial_\zeta \alpha^{(h)}||^2_{L^2(\Omega)} \right).$$

Since the right-hand side is bounded by (2.54), there exists a function $\alpha \in L^2(\Omega;\mathbb{R}^3)$ such that, up to subsequences,

$$\alpha^{(h)} - \alpha_0^{(h)} \rightharpoonup \alpha \quad \text{weakly in}\, L^2(\Omega;\mathbb{R}^3).$$

Moreover, from (2.54) we conclude that

$$\partial_\xi \alpha(s,\xi,\zeta) = G(s,\xi,\zeta)\, \nu_2(s) \quad \text{and} \quad \partial_\zeta \alpha(s,\xi,\zeta) = G(s,\xi,\zeta)\, \nu_3(s), \tag{2.55}$$

therefore $\partial_\xi \alpha, \partial_\zeta \alpha \in L^2(\Omega;\mathbb{R}^3)$. Now, define the functions $\hat{\alpha}(s,\xi,\zeta) := R_0^T(s)\, \alpha(s,\xi,\zeta)$ and $\hat{g}(s) := R_0^T(s)\, g(s)$. Thanks to these definitions and to (2.35), G can be written as

$$\begin{aligned} G &= \left(\left(R\, R_0^T\right)^T \left(R\, R_0^T\right)' R_0 \begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} + g \,\Big|\, \partial_\xi \alpha \,\Big|\, \partial_\zeta \alpha \right) R_0^T \\ &= R_0 \left(\left(R^T R' + (R_0^T)' R_0 \right) \begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} + \hat{g} \,\Big|\, \partial_\xi \hat{\alpha} \,\Big|\, \partial_\zeta \hat{\alpha} \right) R_0^T \\ &= R_0 \left(\left(R^T R' - R_0^T R_0' \right) \begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} + \hat{g} \,\Big|\, \partial_\xi \hat{\alpha} \,\Big|\, \partial_\zeta \hat{\alpha} \right) R_0^T, \end{aligned} \tag{2.56}$$

where the last equality follows from the identity $\left(R_0^T\right)' R_0 + R_0^T R_0' = 0$. Combining (2.38) and (2.56), we obtain

$$\liminf_{h \to 0} \frac{1}{h^2}\, \tilde{I}^{(h)}(\tilde{y}^{(h)}) \geq \frac{1}{2} \int_\Omega Q_3 \left(s,\xi,\zeta, R_0(s) \left(P(s) \begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} + \hat{g} \,\Big|\, \partial_\xi \hat{\alpha} \,\Big|\, \partial_\zeta \hat{\alpha} \right) R_0^T(s) \right) ds\, d\xi\, d\zeta,$$

with $P(s) := R^T(s) R'(s) - R_0^T(s) R_0'(s)$. By the definition of the quadratic form Q in (2.28) we clearly have $\int_D Q_3(s,\xi,\zeta,G) d\xi\, d\zeta \geq Q(s, P(s))$, and so

$$\liminf_{h \to 0} \frac{1}{h^4} \int_{\tilde{\Omega}_h} W\left(\left(\Psi^{(h)}\right)^{-1}(x), \nabla \tilde{y}^{(h)}(x) \right) dx \geq \frac{1}{2} \int_0^L Q\left(s, \left(R^T(s) R'(s) - R_0^T(s) R_0'(s) \right)\right) ds.$$

□

2.2.3 Bound from above

In this subsection we show that the lower bound proved in Theorem 2.5 is optimal.

Theorem 2.6 *For every sequence of positive* (h_j) *converging to* 0 *and for every* $(y, d_2, d_3) \in \mathcal{A}$ *there exists a sequence* $\big(\tilde y^{(h_j)}\big) \subset W^{1,2}\big(\widetilde\Omega_{h_j};\mathbb{R}^3\big)$ *such that*

$$\tilde y^{(h_j)} \circ \Psi^{(h_j)} \to y \quad \textit{strongly in } W^{1,2}(\Omega;\mathbb{R}^3), \tag{2.57}$$

$$\frac{1}{h_j}\,\partial_\xi\big(\tilde y^{(h_j)} \circ \Psi^{(h_j)}\big) \to d_2 \quad \textit{strongly in } L^2(\Omega;\mathbb{R}^3), \tag{2.58}$$

$$\frac{1}{h_j}\,\partial_\zeta\big(\tilde y^{(h_j)} \circ \Psi^{(h_j)}\big) \to d_3 \quad \textit{strongly in } L^2(\Omega;\mathbb{R}^3), \tag{2.59}$$

and

$$I(y, d_2, d_3) = \lim_{j\to\infty} \frac{1}{h_j^4} \int_{\widetilde\Omega_{h_j}} W\big(\big(\Psi^{(h_j)}\big)^{-1}(x), \nabla\tilde y^{(h_j)}(x)\big)dx, \tag{2.60}$$

where the class $\mathcal{A}$ *and the functional* I *are defined in (2.9) and (2.29), respectively.*

Proof. – Let $(y, d_2, d_3) \in \mathcal{A}$. Assume in addition that $y \in C^2([0,L];\mathbb{R}^3)$ and $d_2, d_3 \in C^1([0,L];\mathbb{R}^3)$. Consider the functions $y^{(h)} : \Omega \to \mathbb{R}^3$ defined by

$$y^{(h)}(s,\xi,\zeta) := y(s) + h\,\xi\, d_2(s) + h\,\zeta\, d_3(s) + h\, q(s) + h^2\,\beta(s,\xi,\zeta),$$

with $q \in C^1([0,L];\mathbb{R}^3)$ and $\beta \in C^1(\overline\Omega;\mathbb{R}^3)$. We define $\tilde y^{(h)} := y^{(h)} \circ \big(\Psi^{(h)}\big)^{-1}$; these functions clearly satisfy (2.57). Moreover, since

$$\nabla_h\big(\tilde y^{(h)} \circ \Psi^{(h)}\big) = \nabla_h y^{(h)} = (y' \,|\, d_2 \,|\, d_3) + h\,\big(\xi\, d_2' + \zeta\, d_3' + q' \,|\, \partial_\xi\beta \,|\, \partial_\zeta\beta\big) + h^2\partial_s\beta \otimes e_1, \tag{2.61}$$

also (2.58) and (2.59) follow easily. In order to prove (2.60), we first observe that, performing the change of variables $(s,\xi,\zeta) = \big(\Psi^{(h)}\big)^{-1}(x)$, we obtain

$$\begin{aligned}\frac{1}{h^2}\,\tilde I^{(h)}\big(\tilde y^{(h)}\big) =& \frac{1}{h^2}\int_\Omega W\big(s,\xi,\zeta,\nabla\tilde y^{(h)} \circ \Psi^{(h)}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\, d\xi\, d\zeta \\ =& \frac{1}{h^2}\int_\Omega W\big(s,\xi,\zeta,\nabla_h\big(\tilde y^{(h)} \circ \Psi^{(h)}\big)\,\big(\nabla_h\Psi^{(h)}\big)^{-1}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\, d\xi\, d\zeta,\end{aligned} \tag{2.62}$$

where the last equality is justified observing that

$$\nabla_h\big(\tilde y^{(h)} \circ \Psi^{(h)}\big) = \big(\nabla\tilde y^{(h)} \circ \Psi^{(h)}\big)\,\big(\nabla_h\Psi^{(h)}\big).$$

Then, by the definition of $\tilde y^{(h)}$,

$$\frac{1}{h^2}\,\tilde I^{(h)}\big(\tilde y^{(h)}\big) = \frac{1}{h^2}\int_\Omega W\big(s,\xi,\zeta,\big(\nabla_h y^{(h)}\big)\,\big(\nabla_h\Psi^{(h)}\big)^{-1}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\, d\xi\, d\zeta. \tag{2.63}$$

Using (2.7) and (2.61) we get

$$\begin{aligned}\nabla_h y^{(h)}\,\big(\nabla_h\Psi^{(h)}\big)^{-1} =& R\,R_0^T + h\,(\xi\, d'_2 + \zeta\, d'_3 + q' \,|\, \partial_\xi\beta \,|\, \partial_\zeta\beta)\,R_0^T \\ &- h\,R\,R_0^T\big[(\xi\,\nu_2' + \zeta\,\nu_3') \otimes e_1\big]\,R_0^T + O(h^2),\end{aligned}$$

where $R = (y'|d_2|d_3)$ and $O(h^2)/h^2$ is uniformly bounded. Now consider the rotation $\bar R(s) = R(s)\,R_0^T(s)$. Then

$$\bar R^T\nabla_h y^{(h)}\,\big(\nabla_h\Psi^{(h)}\big)^{-1} = Id + h\,\bar R^T(\xi\, d'_2 + \zeta\, d'_3 + q' \,|\, \partial_\xi\beta \,|\, \partial_\zeta\beta)\,R_0^T - h\,\big[(\xi\,\nu_2' + \zeta\,\nu_3') \otimes e_1\big]\,R_0^T + O(h^2).$$

If we define the functions

$$B^{(h)}(s,\xi,\zeta) := \frac{1}{h}\Big(\bar{R}^T\,\nabla_h y^{(h)}\,\big(\nabla_h\Psi^{(h)}\big)^{-1} - Id\Big),$$

it turns out that

$$\begin{aligned}
B^{(h)} &= (R_0\,R^T)(\xi\,d_2' + \zeta\,d_3' + q'\,|\,\partial_\xi\beta\,|\,\partial_\zeta\beta)R_0^T - \big[(\xi\,\nu_2' + \zeta\,\nu_3')\otimes e_1\big]\,R_0^T + O(h)\\
&= R_0\,R^T\left(R'\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix} + q'\,\Big|\,\partial_\xi\beta\,\Big|\,\partial_\zeta\beta\right)R_0^T - \left[\left(R_0'\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix}\right)\otimes e_1\right]R_0^T + O(h)\\
&= R_0\left(\Big(R^TR' - R_0^TR_0'\Big)\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix} + R^T q\,\Big|\,R^T\partial_\xi\beta\,\Big|\,R^T\partial_\zeta\beta\right)R_0^T + O(h)\\
&=: G_{q,\beta} + O(h)
\end{aligned} \tag{2.64}$$

where $O(h)/h$ is uniformly bounded. By frame indifference and the definition of $B^{(h)}$, we have

$$\begin{aligned}
\frac{1}{h^2}\,W\big(s,\xi,\zeta,\nabla_h y^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1}\big) &= \frac{1}{h^2}\,W\big(s,\xi,\zeta,\bar{R}^T\nabla_h y^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1}\big)\\
&= \frac{1}{h^2}\,W\big(s,\xi,\zeta,Id + h\,B^{(h)}\big).
\end{aligned}$$

Using (2.64) and the expansion of W around the identity, we obtain

$$\frac{1}{h^2}\,W\big(s,\xi,\zeta,\nabla_h y^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1}\big) \to \frac{1}{2}\,Q_3(s,\xi,\zeta,G_{q,\beta}) \quad \text{a.e..}$$

Moreover, the assumption (ii) gives the uniform bound

$$\frac{1}{h^2}\,W\big(s,\xi,\zeta,\nabla_h y^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1}\big) \le \frac{1}{2}\,C_1\,|\,G_{q,\beta}\,|^2 + C \in L^1(\Omega),$$

so, by the dominated convergence theorem and by (2.63) we conclude that

$$\lim_{h\to 0}\frac{1}{h^4}\int_{\widetilde{\Omega}_h} W\Big(\big(\Psi^{(h)}\big)^{-1}(x),\nabla\tilde{y}^{(h)}(x)\Big)dx = \frac{1}{2}\int_\Omega Q_3(s,\xi,\zeta,G_{q,\beta})\,ds\,d\xi\,d\zeta. \tag{2.65}$$

This holds for every $q \in C^1([0,L];\mathbb{R}^3)$ and for every $\beta \in C^1(\overline{\Omega};\mathbb{R}^3)$.

Consider now the general case. Let $(y,d_2,d_3) \in \mathcal{A}$, and let $\hat{\alpha}(s,\cdot) \in W^{1,2}(D;\mathbb{R}^3)$, $\hat{g}(s)$ be a solution to the minimum problem (2.28) for $P = R^TR' - R_0^TR_0'$. By Remark 2.4, $\hat{\alpha} \in L^2(\Omega;\mathbb{R}^3)$ with $\partial_\xi\hat{\alpha}, \partial_\zeta\hat{\alpha} \in L^2(\Omega;\mathbb{R}^3)$ and $\hat{g} \in L^2((0,L);\mathbb{R}^3)$. In order to conclude the proof it is enough to construct a sequence of smooth deformations converging to (y,d_2,d_3), on which the energy $\tilde{I}^{(h)}/h^2$ converges to the right-hand side of (2.65) with q and β replaced by $R^T\hat{g}$ and $R^T\hat{\alpha}$, respectively. This can be done by repeating the same construction as in [40]. □

Remark 2.7 (Homogeneous rods) If the rod is made of a homogeneous material, i.e., $W(z,F) = W(F)$, for a.e. z in Ω and every $F \in \mathbb{M}^{3\times 3}$, then the limiting energy density Q is given by the simpler formula

$$Q(s,P) = \inf_{\hat{\alpha}\in W^{1,2}(D;\mathbb{R}^3)}\left\{\int_D Q_3\left(R_0(s)\left(P\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix}\,\Big|\,\partial_\xi\hat{\alpha}\,\Big|\,\partial_\zeta\hat{\alpha}\right)R_0^T(s)\right)d\xi\,d\zeta\right\}. \tag{2.66}$$

In other words the optimal choice for $\hat{g}$ in (2.28) is $\hat{g} = 0$.
In order to show this, let $\hat{\alpha} \in W^{1,2}(D;\mathbb{R}^3)$ and let $\hat{g} \in \mathbb{R}^3$. We introduce the function

$$\tilde{\alpha}(s,\xi,\zeta) := \hat{\alpha}(s,\xi,\zeta) - \xi \int_D \partial_\xi \hat{\alpha}\, d\xi\, d\zeta - \zeta \int_D \partial_\zeta \hat{\alpha}\, d\xi\, d\zeta. \tag{2.67}$$

Then,

$$\begin{aligned} R_0 \left(P \begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} + \hat{g} \,\Big|\, \partial_\xi \hat{\alpha} \,\Big|\, \partial_\zeta \hat{\alpha} \right) R_0^T &= R_0 \left(P \begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} \Big|\, \partial_\xi \tilde{\alpha} \,\Big|\, \partial_\zeta \tilde{\alpha} \right) R_0^T \\ &\quad + R_0 \left(\hat{g} \,\Big|\, \int_D \partial_\xi \hat{\alpha}\, d\xi\, d\zeta \,\Big|\, \int_D \partial_\zeta \hat{\alpha}\, d\xi\, d\zeta \right) R_0^T \\ &=: \tilde{G} + Z. \end{aligned}$$

By expanding the quadratic form Q_3, we have

$$\int_D Q_3(G) d\xi\, d\zeta = \int_D Q_3(\tilde{G}) d\xi\, d\zeta + \int_D Q_3(Z) d\xi\, d\zeta \geq \int_D Q_3(\tilde{G}) d\xi\, d\zeta, \tag{2.68}$$

where we used (2.3), the fact that $\partial_\xi \tilde{\alpha}$ and $\partial_\zeta \tilde{\alpha}$ have zero average on D and the non negativity of Q_3. From this inequality the thesis follows immediately.
Notice that, due to the nontrivial geometry of the body, the limit energy depends on the position over the curve γ even for a homogeneous material.

Remark 2.8 (Homogeneous and isotropic rods) Assume the density W is homogeneous and isotropic, that is,

$$W(F) = W(FR) \quad \text{for every } R \in SO(3).$$

Then the quadratic form Q_3 is given by

$$Q_3(G) = 2\,\mu \left| \frac{G + G^T}{2} \right|^2 + \lambda\, (\operatorname{tr} G)^2$$

for some constants $\lambda, \mu \in \mathbb{R}$. It is easy to show that for all $G \in \mathbb{M}^{3\times 3}$ and $R \in SO(3)$

$$Q_3(R\, G\, R^T) = Q_3(G),$$

and so, formula (2.66) reduces to

$$\begin{aligned} Q(P) &= \inf_{\hat{\alpha} \in W^{1,2}(D;\mathbb{R}^3)} \left\{ \int_D Q_3 \left(P \begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} \Big|\, \partial_\xi \hat{\alpha} \,\Big|\, \partial_\zeta \hat{\alpha} \right) d\xi\, d\zeta \right\} \\ &= \frac{1}{2\,\pi} \frac{\mu(3\,\lambda + 2\,\mu)}{\lambda + \mu} (p_{12}^2 + p_{13}^2) + \frac{\mu}{2\,\pi} p_{23}^2, \end{aligned}$$

where the last equality follows from [40, Remark 3.5]. This means that in the case of a homogeneous and isotropic material the quadratic form Q is exactly the same as in the case of a straight rod treated in [40].

Remark 2.9 (Homogeneous and isotropic rods with a circular cross section) Assume that the cross section D is a circle of radius $\frac{1}{\sqrt{\pi}}$ centred at the origin. In this case, the quadratic form Q can be computed by a pointwise minimization. More precisely, for every s and for every P,

$$Q(s,P) = \frac{1}{4\pi} \min_{u,v,w} \left\{ Q_3\left(R_0(s) \left(\begin{array}{c|c|c} p_{12} & & \\ 0 & u & v \\ -p_{23} & & \end{array} \right) R_0^T(s) \right) + Q_3\left(R_0(s) \left(\begin{array}{c|c|c} p_{13} & & \\ p_{23} & v & w \\ 0 & & \end{array} \right) R_0^T(s) \right) \right\}.$$

The proof is completely analogous to [40, Remark 3.6].

2.3 Lower scalings of the energy

The content of this section is the study of the asymptotic behaviour of the functionals $\tilde{I}^{(h)}/h^\alpha$ for $0 \leq \alpha < 2$, as $h \to 0$. In addition to conditions (i)-(v) of Section 2 we assume also that $W(z,F) = W(z_1,F)$ for every $z = (z_1,z_2,z_3) \in \mathbb{R}^3$ and every $F \in \mathbb{M}^{3\times 3}$, and that

(vi) $\exists\, C_3 > 0$ independent of z_1 such that $W(z_1,F) \leq C_3 \operatorname{dist}^2(F, SO(3))$ for a.e. z_1 and every $F \in \mathbb{M}^{3\times 3}$.

It is convenient to write the functionals $\tilde{I}^{(h)}$ as integrals over the fixed domain $\Omega = \big(\Psi^{(h)}\big)^{-1}\big(\tilde{\Omega}_h\big)$. Changing variables as in (2.62) and setting $y := \tilde{y} \circ \Psi^{(h)}$, we have

$$\tilde{I}^{(h)}(\tilde{y}) = \int_\Omega W\big(s, \big(\nabla_h y\big)\big(\nabla_h \Psi^{(h)}\big)^{-1}\big) \det\big(\nabla_h \Psi^{(h)}\big) ds\, d\xi\, d\zeta =: \tilde{J}^{(h)}(y).$$

We extend the functional to the space $L^2(\Omega;\mathbb{R}^3)$, setting

$$J^{(h)}(y) = \begin{cases} \tilde{J}^{(h)}(y) & \text{if } y \in W^{1,2}(\Omega;\mathbb{R}^3), \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^3). \end{cases}$$

The aim of this section is to determine the Γ-limit of $J^{(h)}/h^\alpha$, for $0 \leq \alpha < 2$, as $h \to 0$, with respect to the strong topology of L^2.

2.3.1 Derivation of the nonlinear theory for curved strings

For this first part we specify $\alpha = 0$, so we are interested in the asymptotic behaviour of the functionals representing the energy per unit volume associated to a deformation of the reference configuration.

Theorem 2.10 (Compactness) *For every sequence* $\big(y^{(h)}\big)$ *in* $L^2(\Omega;\mathbb{R}^3)$ *such that*

$$J^{(h)}\big(y^{(h)}\big) \leq c < +\infty \tag{2.69}$$

there exist a function $y \in W^{1,2}((0,L);\mathbb{R}^3)$ *and some constants* $c^{(h)} \in \mathbb{R}^3$ *such that, up to subsequences,*

$$y^{(h)} - c^{(h)} \rightharpoonup y \quad \textit{weakly in } W^{1,2}(\Omega;\mathbb{R}^3).$$

Proof. – Let $\big(y^{(h)}\big)$ be a sequence in $L^2(\Omega;\mathbb{R}^3)$ satisfying (2.69). From the definition of the functional we have immediately that $y^{(h)} \in W^{1,2}(\Omega;\mathbb{R}^3)$. The assumptions on W and the uniform boundedness of $\left(\nabla_h \Psi^{(h)}\right)^{-1}$ and of $\det\left(\nabla_h \Psi^{(h)}\right)$ give the boundedness in $L^2(\Omega;\mathbb{M}^{3\times3})$ of $\left(\nabla_h y^{(h)}\right)$ and hence of $\left(\nabla y^{(h)}\right)$. Therefore, using the Poincaré inequality

$$||y^{(h)} - c^{(h)}||_{L^2(\Omega;\mathbb{R}^3)} \le ||\nabla y^{(h)}||_{L^2(\Omega;\mathbb{M}^{3\times3})},$$

where $c^{(h)} \in \mathbb{R}^3$ is the mean value of $y^{(h)}$ over Ω, it turns out that the sequence $y^{(h)} - c^{(h)}$ is bounded in $W^{1,2}(\Omega;\mathbb{R}^3)$; hence there exists a function $y \in W^{1,2}(\Omega;\mathbb{R}^3)$ such that, up to subsequences,

$$y^{(h)} - c^{(h)} \rightharpoonup y \quad \text{weakly in}\, W^{1,2}(\Omega;\mathbb{R}^3).$$

Moreover since $\left(\nabla_h y^{(h)}\right)$ is bounded in $L^2(\Omega;\mathbb{M}^{3\times3})$, we have

$$\partial_\xi y^{(h)} \to 0 \quad \text{and} \quad \partial_\zeta y^{(h)} \to 0 \quad \text{strongly in}\, L^2(\Omega;\mathbb{R}^3).$$

Therefore the limit function y depends only on the first variable. □

Theorem 2.11 (Γ-convergence) *Let I be the functional defined as*

$$I(y) = \begin{cases} \displaystyle\int_0^L W_0^{**}(s, y'(s))\, ds & \textit{if } y \in W^{1,2}((0,L);\mathbb{R}^3), \\ +\infty & \textit{otherwise in } L^2(\Omega;\mathbb{R}^3), \end{cases} \tag{2.70}$$

where W_0^{**} is given by the convex envelope of the function $W_0 : [0,L]\times\mathbb{R}^3 \to \mathbb{R}$ defined as

$$W_0(s,z) := \inf\left\{W\left(s, (z\,|\,v_2\,|\,v_3)R_0^T(s)\right) : v_2, v_3 \in \mathbb{R}^3\right\}.$$

Then

$$\Gamma - \lim_{h\to 0} J^{(h)} = I,$$

i.e., the following conditions are satisfied:
(i)(liminf inequality) for every $y \in L^2(\Omega;\mathbb{R}^3)$ and every sequence $\left(y^{(h)}\right) \subset L^2(\Omega;\mathbb{R}^3)$ such that $y^{(h)} \to y$ strongly in $L^2(\Omega;\mathbb{R}^3)$, it turns out that

$$I(y) \le \liminf_{h\to 0} J^{(h)}\left(y^{(h)}\right); \tag{2.71}$$

(ii)(limsup inequality) for every $y \in L^2(\Omega;\mathbb{R}^3)$ there exists a sequence $\left(y^{(h)}\right) \subset L^2(\Omega;\mathbb{R}^3)$ converging strongly to y in $L^2(\Omega;\mathbb{R}^3)$ such that

$$\limsup_{h\to 0} J^{(h)}\left(y^{(h)}\right) \le I(y). \tag{2.72}$$

Remark 2.12 Notice that, if $A := (z\,|\,v_2\,|\,v_3)\,R_0^T$, then $A\,\tau = z$ and $A\,\nu_k = v_k$ for $k = 2,3$. In other words, in the definition of W_0, the minimization is done with respect to the normal components of the matrix in the argument of W, keeping equal to z the tangential component.

Remark 2.13 Observe that conditions (iv) and (v) imply that for a.e. $s \in [0,L]$,

$$W_0^{**}(s,z) = 0 \quad \text{if and only if} \quad |z| \le 1, \tag{2.73}$$

(see [1]).

PROOF. – (of Theorem 2.11) (i) Let y and $y^{(h)}$ be as in the statement. We can assume that

$$\liminf_{h\to 0} J^{(h)}\big(y^{(h)}\big) < +\infty,$$

otherwise (2.71) is trivial. Therefore, up to subsequences, (2.69) is satisfied. From Theorem 2.10 we deduce that $y \in W^{1,2}((0,L);\mathbb{R}^3)$ and that the convergence is indeed weak in $W^{1,2}(\Omega;\mathbb{R}^3)$.
Now define the function $W_0 : [0,L]\times\mathbb{R}^3 \to \mathbb{R}$ as

$$W_0(s,z) := \inf\big\{W\big(s,(z\,|\,v_2\,|\,v_3)R_0^T(s)\big) : v_2,v_3 \in \mathbb{R}^3\big\}.$$

Due to the coerciveness assumptions this function is finite.

Notice that, since $R_0\,R_0^T = Id$, we can write

$$W\big(s,\nabla_h y^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1}\big) = W\Big(s,\nabla_h y^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1}R_0R_0^T\Big)$$

and using the explicit expression of $\big(\nabla_h\Psi^{(h)}\big)^{-1}$ given in (2.7), i.e.,

$$\big(\nabla_h\Psi^{(h)}\big)^{-1}(s,\xi,\zeta) = R_0^T(s) - h\,R_0^T(s)\,\big[(\xi\,\nu_2'(s)+\zeta\,\nu_3'(s))\otimes e_1\big]R_0^T(s) + O(h^2),$$

we have

$$\nabla_h y^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1}R_0e_1 \rightharpoonup y' \quad \text{weakly in } L^2(\Omega;\mathbb{R}^3). \tag{2.74}$$

So, from the definition of W_0

$$\begin{aligned} J^{(h)}\big(y^{(h)}\big) &\geq \int_\Omega W_0\big(s,\nabla_h y^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1}R_0e_1\big)\,\det\big(\nabla_h\Psi^{(h)}\big)\,ds\,d\xi\,d\zeta \\ &\geq \int_\Omega W_0^{**}\big(s,\nabla_h y^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1}R_0e_1\big)\,\det\big(\nabla_h\Psi^{(h)}\big)\,ds\,d\xi\,d\zeta. \end{aligned}$$

Now we pass to the $\liminf$ in both sides of the previous inequality, using the uniform convergence of the determinant remarked in (2.6), and we get

$$\begin{aligned} \liminf_{h\to 0}\,J^{(h)}\big(y^{(h)}\big) &\geq \liminf_{h\to 0}\int_\Omega W_0^{**}\big(s,\big(\nabla_h y^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1}R_0\big)e_1\big)\,\det\big(\nabla_h\Psi^{(h)}\big)\,ds\,d\xi\,d\zeta \\ &= \liminf_{h\to 0}\int_\Omega W_0^{**}\big(s,\big(\nabla_h y^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1}R_0\big)e_1\big)\,ds\,d\xi\,d\zeta. \end{aligned}$$

Since the functional

$$G(u) := \int_\Omega W_0^{**}(s,u)\,ds\,d\xi\,d\zeta$$

is convex, it is sequentially weakly lower semicontinuous in $L^2(\Omega;\mathbb{R}^3)$; so, by (2.74) we can conclude that

$$\liminf_{h\to 0}\,J^{(h)}\big(y^{(h)}\big) \geq \int_0^L W_0^{**}(s,y'(s))\,ds. \tag{2.75}$$

(ii) Let y be a function in $W^{1,2}((0,L);\mathbb{R}^3)$, otherwise the bound in (2.72) is trivial. Let $w_2,w_3 \in W^{1,2}((0,L);\mathbb{R}^3)$ be arbitrary functions and consider $y^{(h)} : \Omega \to \mathbb{R}^3$ defined by

$$y^{(h)}(s,\xi,\zeta) := y(s) + h\,\xi\,w_2(s) + h\,\zeta\,w_3(s).$$

Clearly, as $\nabla y^{(h)} = y'\otimes e_1 + h\,\big(\xi\,w_2'+\zeta\,w_3'\,|\,w_2\,|\,w_3\big)$, we have that

$$y^{(h)} \to y \quad \text{strongly in}\, W^{1,2}(\Omega;\mathbb{R}^3). \tag{2.76}$$

Now we want to study the behaviour of the sequence

$$J^{(h)}\big(y^{(h)}\big) = \int_\Omega W\big(s, (\nabla_h y^{(h)})\big(\nabla_h \Psi^{(h)}\big)^{-1}\big) \det\big(\nabla_h \Psi^{(h)}\big) ds\, d\xi\, d\zeta$$

when $h \to 0$. Notice that the scaled gradient of $y^{(h)}$ satisfies

$$\nabla_h y^{(h)} = (y' \,|\, w_2 \,|\, w_3) + h\,(\xi\, w_2' + \zeta\, w_3') \otimes e_1 \to (y' \,|\, w_2 \,|\, w_3) \text{ a.e..} \tag{2.77}$$

So, by (2.6) and (vi), using the dominated convergence theorem we get

$$\begin{aligned}\lim_{h\to 0} J^{(h)}\big(y^{(h)}\big) &= \lim_{h\to 0} \int_\Omega W\big(s, (\partial_s y^{(h)} \,|\, w_2 \,|\, w_3)\big(\nabla_h \Psi^{(h)}\big)^{-1}\big) \det\big(\nabla_h \Psi^{(h)}\big)\, ds\, d\xi\, d\zeta \\ &= \int_0^L W\big(s, (y' \,|\, w_2 \,|\, w_3)\, R_0^T\big)\, ds.\end{aligned}$$

Up to now we have shown that for every choice of $w_2, w_3 \in W^{1,2}((0,L);\mathbb{R}^3)$, there exists a sequence $\big(y^{(h)}\big)$ such that (2.76) is satisfied and

$$\lim_{h\to 0} J^{(h)}\big(y^{(h)}\big) = \int_0^L W\big(s, (y' \,|\, w_2 \,|\, w_3)\, R_0^T\big) ds.$$

Therefore,

$$\begin{aligned}\Gamma - \limsup_{h\to 0} J^{(h)}(v) &:= \inf \Big\{ \limsup_{h\to 0} J^{(h)}\big(u^{(h)}\big) : u^{(h)} \to y \text{ strongly in } L^2(\Omega;\mathbb{R}^3) \Big\} \\ &\le \inf \Big\{ \int_0^L W\big(s, (y' \,|\, w_2 \,|\, w_3)\, R_0^T\big)\, ds : w_2, w_3 \in W^{1,2}((0,L);\mathbb{R}^3) \Big\} \\ &= \inf \Big\{ \int_0^L W\big(s, (y' \,|\, w_2 \,|\, w_3)\, R_0^T\big)\, ds : w_2, w_3 \in L^2((0,L);\mathbb{R}^3) \Big\},\end{aligned} \tag{2.78}$$

where the last equality is a consequence of the dominated convergence theorem and of the density of $W^{1,2}((0,L);\mathbb{R}^3)$ in $L^2((0,L);\mathbb{R}^3)$.

By the measurable selection lemma (see for example [27]) applied to the Carathéodory function

$$g : [0,L] \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}, \quad (s, v_2, v_3) \mapsto g(s, v_2, v_3) := W\big(s, (y'(s) \,|\, v_2 \,|\, v_3) R_0^T(s)\big)$$

we obtain the existence of two measurable functions $w_2^0, w_3^0 : [0,L] \to \mathbb{R}^3$ satisfying

$$W\big(s, (y'(s) \,|\, w_2^0(s) \,|\, w_3^0(s)) R_0^T(s)\big) = \inf_{v_2, v_3 \in \mathbb{R}^3} W\big(s, (y'(s) \,|\, v_2 \,|\, v_3) R_0^T(s)\big) = W_0(s, y'(s)).$$

Moreover, from the coerciveness of W it follows that w_2^0, w_3^0 belong indeed to $L^2((0,L);\mathbb{R}^3)$ and so they are in competition for the infimum in (2.78). Hence, for every $y \in W^{1,2}((0,L);\mathbb{R}^3)$ we have

$$\Gamma - \limsup_{h\to 0} J^{(h)}(y) \le \int_0^L W_0(s, y'(s))\, ds =: \tilde{J}(y).$$

Now define the functional

$$J(y) = \begin{cases} \tilde{J}(y) & \text{if } y \in W^{1,2}((0,L);\mathbb{R}^3), \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^3); \end{cases} \tag{2.79}$$

clearly it turns out that

$$\Gamma - \limsup_{h\to 0} J^{(h)}(y) \le J(y) \quad \text{for every } y \in L^2(\Omega;\mathbb{R}^3). \tag{2.80}$$

As the lower semicontinuous envelope of J with respect to the strong topology of $L^2(\Omega;\mathbb{R}^3)$ is given by the functional I (see [21] and [37, Lemma 5]), the thesis follows immediately from (2.80). □

2.3.2 Intermediate scaling

In this subsection we show that scalings of the energy of order h^α, with $\alpha \in (0,2)$, lead to a trivial Γ-limit.

Theorem 2.14 (Compactness and Γ- convergence) *Let $\mathcal{W}_1$ be the class of functions defined as*

$$\mathcal{W}_1 := \{y \in W^{1,2}((0,L);\mathbb{R}^3) : |y'(s)| \le 1 \,\text{a.e.}\}. \tag{2.81}$$

For every sequence $\big(y^{(h)}\big)$ in $L^2(\Omega;\mathbb{R}^3)$ such that

$$\frac{1}{h^\alpha}\, J^{(h)}\big(y^{(h)}\big) \le c < +\infty \tag{2.82}$$

there exist a function $y \in \mathcal{W}_1$ and some constants $c^{(h)} \in \mathbb{R}$ such that, up to subsequences,

$$y^{(h)} - c^{(h)} \rightharpoonup y \quad \textit{weakly in } W^{1,2}(\Omega;\mathbb{R}^3).$$

Moreover,

$$\Gamma - \lim_{h\to 0} \frac{1}{h^\alpha}\, J^{(h)} = \begin{cases} 0 & \text{in } \mathcal{W}_1, \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^3). \end{cases} \tag{2.83}$$

Proof. – Let $\big(y^{(h)}\big)$ be such that (2.82) is satisfied. Then

$$J^{(h)}\big(y^{(h)}\big) < c\, h^\alpha. \tag{2.84}$$

By Theorem 2.10 this implies that there exist $y \in W^{1,2}((0,L);\mathbb{R}^3)$ and some constants $c^{(h)} \in \mathbb{R}$ such that the sequence $y^{(h)} - c^{(h)}$ converges to y weakly in $W^{1,2}(\Omega;\mathbb{R}^3)$. Moreover by Theorem 2.11 and by (2.84)

$$0 = \liminf_{h\to 0} J^{(h)}\big(y^{(h)}\big) \ge \int_0^L W_0^{**}(s, y'(s))ds,$$

and this gives the additional condition that $|y'(s)| \le 1$ for almost every $s \in [0,L]$, thanks to Remark 2.13. Therefore $v \in \mathcal{W}_1$.
Let us prove (2.83). The liminf inequality follows directly from the fact that the energy density W is nonnegative and from the compactness. As for the limsup inequality we first notice that we can restrict our analysis to functions $y \in \mathcal{W}_1$, being the other case trivial. Since $|y'(s)| \le 1$ for a.e. $s \in [0,L]$, there exist two measurable functions $d_2, d_3 : [0,L] \to \mathbb{R}^3$ such that

$$(y'(s)\,|\,d_2(s)\,|\,d_3(s)) \in Co(SO(3)) \quad \text{for a.e. } s \in [0,L],$$

where $Co(SO(3))$ denotes the convex hull of $SO(3)$. As first step, we assume in addition that $(y'\,|\,d_2\,|\,d_3)$ is a piecewise constant rotation; for simplicity we can limit ourselves to the case

$$(y'(s)\,|\,d_2(s)\,|\,d_3(s)) = \begin{cases} R_1 & \text{if } s \in [0, s_0[, \\ R_2 & \text{if } s \in [s_0, L] \end{cases}$$

with $R_1, R_2 \in SO(3)$. Now, let $\omega(h)$ be a sequence converging to zero, as $h \to 0$, and let P be a smooth function $P : [0,1] \longrightarrow SO(3)$, such that $P(0) = R_1$ and $P(1) = R_2$. Now consider a reparametrization of P, denoted by $P^{(h)}$ and given by

$$P^{(h)}(s) := P\bigg(\frac{s - s_0}{\omega(h)}\bigg).$$

Define the sequence $y^{(h)} : \Omega \to \mathbb{R}^3$ as

$$y^{(h)}(s,\xi,\zeta) := \begin{cases} R_1 \begin{pmatrix} s \\ h\,\xi \\ h\,\zeta \end{pmatrix} & \text{on } s \in [0, s_0[\times D, \\ \displaystyle\int_{s_0}^{s} \big(P^{(h)}\big)(\sigma) e_1 \, d\sigma + P^{(h)}(s) \begin{pmatrix} 0 \\ h\,\xi \\ h\,\zeta \end{pmatrix} + b^{(h)} & \text{on } \big[s_0, s_0 + \omega(h)\big] \times D, \\ R_2 \begin{pmatrix} s \\ h\,\xi \\ h\,\zeta \end{pmatrix} + d^{(h)} & \text{on } \big]s_0 + \omega(h), L\big] \times D, \end{cases}$$

where the constants $b^{(h)}$ and $d^{(h)}$ are chosen in order to make $y^{(h)}$ continuous. It turns out that the scaled gradient has the following expression:

$$\nabla_h y^{(h)} = \begin{cases} R_1 & \text{on } [0, s_0[\times D, \\ P^{(h)}(s) + \left(\big(P^{(h)}\big)'(s) \begin{pmatrix} 0 \\ h\,\xi \\ h\,\zeta \end{pmatrix} \right) \otimes e_1 & \text{on } \big[s_0, s_0 + \omega(h)\big] \times D, \\ R_2 & \text{on } \big]s_0 + \omega(h), L\big] \times D; \end{cases} \tag{2.85}$$

moreover $\nabla_h y^{(h)} \to (y' \,|\, d_2 \,|\, d_3)$ strongly in $L^2(\Omega; \mathbb{R}^3)$. In order to evaluate the functional on this sequence we use the fact that, by (v) and (2.6),

$$\frac{1}{h^\alpha} \, J^{(h)}\big(y^{(h)}\big) \le \frac{c}{h^\alpha} \int_\Omega \operatorname{dist}^2\Big(\nabla_h y^{(h)} \, \big(\nabla_h \Psi^{(h)}\big)^{-1}, SO(3)\Big) \, ds \, d\xi \, d\zeta. \tag{2.86}$$

From (2.85) the integral on the right-hand side of the previous expression can be written as

$$\int_0^{s_0} \int_D \operatorname{dist}^2\Big(R_1 \, \big(\nabla_h \Psi^{(h)}\big)^{-1}, SO(3)\Big) \, ds \, d\xi \, d\zeta + \int_{s_0+\omega(h)}^{L} \int_D \operatorname{dist}^2\Big(R_2 \, \big(\nabla_h \Psi^{(h)}\big)^{-1}, SO(3)\Big) \, ds \, d\xi \, d\zeta$$
$$+ \int_{s_0}^{s_0+\omega(h)} \int_D \operatorname{dist}^2\Big(\nabla_h y^{(h)} \, \big(\nabla_h \Psi^{(h)}\big)^{-1}, SO(3)\Big) \, ds \, d\xi \, d\zeta. \tag{2.87}$$

The first two terms in (2.87) give a contribution of order h^2 since, by (2.7), for $i = 1, 2$,

$$\begin{aligned} \operatorname{dist}^2\Big(R_i \, \big(\nabla_h \Psi^{(h)}\big)^{-1}, SO(3)\Big) &\le h^2 \operatorname{dist}^2\Big(R_i \, R_0^T \, \big[(\xi\,\nu_2' + \zeta\,\nu_3') \otimes e_1\big] R_0^T, SO(3)\Big) \\ &\le C\,h^2 \operatorname{dist}^2\Big(\big[(\xi\,\nu_2' + \zeta\,\nu_3') \otimes e_1\big], SO(3)\Big), \end{aligned}$$

so they can be neglected in the computation of the limit of (2.86). The only term we have to analyze is the last integral in (2.87). Set

$$A^{(h)}(s,\xi,\zeta) := \left(\big(P^{(h)}\big)' \begin{pmatrix} 0 \\ h\,\xi \\ h\,\zeta \end{pmatrix} \right) \otimes e_1.$$

Using again (2.7) we have that

$$\operatorname{dist}^2\Big(\nabla_h y^{(h)} \, \big(\nabla_h \Psi^{(h)}\big)^{-1}, SO(3)\Big) \le \operatorname{dist}^2\Big(A^{(h)} \, \big(\nabla_h \Psi^{(h)}\big)^{-1}, SO(3)\Big) \le C\,h^2 \, (\xi^2 + \zeta^2) \, \big| \big(P^{(h)}\big)' \big|^2,$$

so we get the following estimate:

$$\int_{s_0}^{s_0+\omega(h)} \int_D \operatorname{dist}^2\big(\nabla_h y^{(h)} \big(\nabla_h \Psi^{(h)}\big)^{-1}, SO(3)\big)\, ds\, d\xi\, d\zeta \le C\, h^2 \int_{s_0}^{s_0+\omega(h)} \big|\, \big(P^{(h)}\big)' \,\big|^2\, ds$$
$$= C\, \frac{h^2}{\omega(h)} \int_0^1 \big|\, P' \big|^2 ds.$$

Notice that, if we choose $\omega(h) \sim h^\beta$, with $0 < \beta < 2 - \alpha$, also this term can be neglected in (2.86), hence

$$\lim_{h\to 0} \frac{1}{h^\alpha}\, J^{(h)}\big(y^{(h)}\big) = 0$$

and this concludes the proof in the case $(y' \,|\, d_2 \,|\, d_3)$ is a piecewise constant rotation.

Consider now the general case. Since $(y' \,|\, d_2 \,|\, d_3) \in Co(SO(3))$ a.e., there exists a sequence of piecewise constant rotations $R_j : [0, L] \longrightarrow SO(3)$ such that $R_j \to (y' \,|\, d_2 \,|\, d_3)$ strongly in $L^2((0, L); \mathbb{M}^{3\times 3})$. For each element R_j of the sequence we can repeat the same construction done in the previous case and find a sequence $y_j^{(h)}$ whose scaled gradients $\nabla_h y_j^{(h)}$ converge to R_j as $h \to 0$ and such that for every j

$$\lim_{h\to 0} \frac{1}{h^\alpha} \int_\Omega W\big(s, \nabla_h y_j^{(h)} \big(\nabla_h \Psi^{(h)}\big)^{-1}\big) \det\big(\nabla_h \Psi^{(h)}\big) ds\, d\xi\, d\zeta = 0. \tag{2.88}$$

Now we can choose, for every j, an element of the sequence $y_j^{(h)}$, say $y_j^{(h_j)}$, in such a way that

$$\Big\| \nabla_{h_j} y_j^{(h_j)} - R_j \Big\|_{L^2(\Omega; \mathbb{M}^{3\times 3})} < \frac{1}{j} \tag{2.89}$$

and

$$\frac{1}{h_j^\alpha} \int_\Omega W\big(s, \nabla_{h_j} y_j^{(h_j)} \big(\nabla_{h_j} \Psi^{(h_j)}\big)^{-1}\big) \det\big(\nabla_{h_j} \Psi^{(h_j)}\big) ds\, d\xi\, d\zeta < \frac{1}{j}. \tag{2.90}$$

These estimates show that the sequence $y_j^{(h_j)}$ converges to $(y' \,|\, d_2 \,|\, d_3)$ strongly in $L^2((0, L); \mathbb{M}^{3\times 3})$ and that

$$\lim_{j\to\infty} \frac{1}{h_j^\alpha} \int_\Omega W\big(s, \nabla_{h_j} y^{(h_j)} \big(\nabla_{h_j} \Psi^{(h_j)}\big)^{-1}\big) \det\big(\nabla_{h_j} \Psi^{(h_j)}\big) ds\, d\xi\, d\zeta = 0. \tag{2.91}$$

This concludes the proof. □

Chapter 3

Asymptotic models for curved rods derived from nonlinear elasticity by Γ-convergence

In this chapter we continue the analysis of the asymptotic behaviour of the elastic energy associated to a displacement of a curved thin rod started in Chapter 2, considering the higher scalings of the energy. For the notation we refer to Section 2.1 at the beginning of Chapter 2.

Let $\tilde{y} \in W^{1,2}(\widetilde{\Omega}_h;\mathbb{R}^3)$ be a deformation of $\widetilde{\Omega}_h$. The elastic energy per unit cross-section associated to $\tilde{y}$ is defined by

$$\tilde{I}^{(h)}(\tilde{y}) := \frac{1}{h^2}\int_{\widetilde{\Omega}_h} W\big((\Psi^{(h)})^{-1}(x), \nabla\tilde{y}(x)\big)dx.$$

We study the sequence of functionals $\tilde{I}^{(h)}/h^\beta$, with $\beta > 2$.

3.1 Compactness results

In this section we analyze the compactness properties of sequences of deformations having energy $\tilde{I}^{(h)}$ of order h^β with $\beta > 2$. For notational convenience we prefer to write $\beta > 2$ as $2\alpha - 2$ with $\alpha > 2$. The main ingredient in the proof is the rigidity result, proved by G. Friesecke, R.D. James and S. Müller in [30]. For the statement of the theorem we refer to Thorem 2.1 in the previous chapter.

Before stating the compactness theorem, let us introduce some sequences which will be widely used in the sequel. Given a sequence of deformations $Y^{(h)} : \Omega \to \mathbb{R}^3$, we consider the functions $v^{(h)} : (0,L) \to \mathbb{R}^3$, $w^{(h)}, u^{(h)} : (0,L) \to \mathbb{R}$, defined as

$$v^{(h)}(s) := \frac{1}{h^{\alpha-2}}\int_D \big(Y^{(h)}(s,\xi,\zeta) - \Psi^{(h)}(s,\xi,\zeta)\big)\,d\xi\,d\zeta, \tag{3.1}$$

$$w^{(h)}(s) := \frac{1}{h^{\alpha-1}}\left(\frac{1}{\mu(D)}\int_D \big(Y^{(h)}(s,\xi,\zeta) - \Psi^{(h)}(s,\xi,\zeta)\big)\cdot(\xi\,\nu_3(s) - \zeta\,\nu_2(s))\,d\xi\,d\zeta\right), \tag{3.2}$$

$$u^{(h)}(s) := \begin{cases} \dfrac{1}{h^{2(\alpha-2)}}\displaystyle\int_{s_h}^{s}\Big(\int_D \partial_s\big(Y^{(h)}(s,\xi,\zeta) - \Psi^{(h)}(s,\xi,\zeta)\big)\cdot\tau(\sigma)\,d\xi\,d\zeta\Big)d\sigma & \text{if } 2<\alpha<3,\\[2ex] \dfrac{1}{h^{\alpha-1}}\displaystyle\int_{s_h}^{s}\Big(\int_D \partial_s\big(Y^{(h)}(s,\xi,\zeta) - \Psi^{(h)}(s,\xi,\zeta)\big)\cdot\tau(\sigma)\,d\xi\,d\zeta\Big)d\sigma & \text{if } \alpha\geq 3,\end{cases} \tag{3.3}$$

where $s_h \in (0,L)$ is chosen in such a way that $u^{(h)}$ has zero average on $(0,L)$ and $\mu(D) := \int_D \big(\xi^2 + \zeta^2\big)\, d\xi\, d\zeta$. Notice that $v^{(h)}$ is the averaged displacement associated with the deformation $Y^{(h)}$. The function $w^{(h)}$ describes the twist of the cross-section. Finally, $u^{(h)}$ is related to the tangential component of the displacement. More precisely, up to a suitable scaling, its derivative $\big(u^{(h)}\big)'$ coincides with the average on the cross-section D of the tangential divergence of $Y^{(h)} - \Psi^{(h)}$.

We are now in a position to prove the compactness result.

Theorem 3.1 *Let* $\big(\tilde{y}^{(h)}\big) \subset W^{1,2}\big(\widetilde{\Omega}_h;\mathbb{R}^3\big)$ *be a sequence verifying*

$$\frac{1}{h^{2\,\alpha-2}}\, \tilde{I}^{(h)}(\tilde{y}^{(h)}) \le c < +\infty \tag{3.4}$$

for every $h > 0$. *Then there exist an associated sequence* $R^{(h)} \subset C^\infty((0,L);\mathbb{M}^{3\times 3})$ *and constants* $\bar{R}^{(h)} \in SO(3)$, $c^{(h)} \in \mathbb{R}^3$ *such that, if we define* $Y^{(h)} := \big(\bar{R}^{(h)}\big)^T \tilde{y}^{(h)} \circ \Psi^{(h)} - c^{(h)}$, *we have*

$$R^{(h)}(s) \in SO(3) \quad \textit{for every } s \in (0,L), \tag{3.5}$$

$$\big|\big|R^{(h)} - Id\big|\big|_{L^\infty(0,L)} \le C\, h^{\alpha-2}, \quad \big|\big|\big(R^{(h)}\big)'\big|\big|_{L^2(0,L)} < C\, h^{\alpha-2}, \tag{3.6}$$

$$\big|\big|\nabla_h Y^{(h)} \big(\nabla_h \Psi^{(h)}\big)^{-1} - R^{(h)}\big|\big|_{L^2(\Omega)} \le C\, h^{\alpha-1}. \tag{3.7}$$

Moreover, defining $v^{(h)}$, $w^{(h)}$ *and* $u^{(h)}$ *as in (3.1), (3.2) and (3.3), we have that, up to subsequences, the following properties are satisfied:*

(a) $v^{(h)} \to v$ *strongly in* $W^{1,2}((0,L);\mathbb{R}^3)$, *with* $v \in W^{2,2}((0,L);\mathbb{R}^3)$ *and* $v' \cdot \tau = 0$;

(b) $w^{(h)} \rightharpoonup w$ *weakly in* $W^{1,2}(0,L)$;

(c) $\begin{cases} u^{(h)} \to u \textit{ strongly in } W^{1,2}(0,L) & \textit{if } 2 < \alpha < 3, \\ u^{(h)} \rightharpoonup u \textit{ weakly in } W^{1,2}(0,L) & \textit{if } \alpha \ge 3. \end{cases}$

In addition, for $2 < \alpha < 3$ *the function* u *satisfies the following constraint:*

$$u' = -\frac{1}{2}\Big((v' \cdot \nu_2)^2 + (v' \cdot \nu_3)^2\Big); \tag{3.8}$$

(d) $\big(\nabla_h Y^{(h)} \big(\nabla_h \Psi^{(h)}\big)^{-1} - Id\big)/h^{\alpha-2} \to A$ *strongly in* $L^2(\Omega;\mathbb{M}^{3\times 3})$, *where the matrix* $A \in W^{1,2}((0,L))$ *is given by*

$$A = R_0 \begin{pmatrix} 0 & -\,v' \cdot \nu_2 & -\,v' \cdot \nu_3 \\ v' \cdot \nu_2 & 0 & -\,w \\ v' \cdot \nu_3 & w & 0 \end{pmatrix} R_0^T; \tag{3.9}$$

(e) $\big(R^{(h)} - Id\big)/h^{\alpha-2} \rightharpoonup A$ *weakly in* $W^{1,2}((0,L);\mathbb{M}^{3\times 3})$;

(f) $sym\big(R^{(h)} - Id\big)/h^{2(\alpha-2)} \to A^2/2$ *uniformly on* $(0,L)$.

PROOF. – Let $(\tilde{y}^{(h)})$ be a sequence in $W^{1,2}(\widetilde{\Omega}_h;\mathbb{R}^3)$ satisfying (3.4); using the change of variables $\Psi^{(h)}$ and the fact that $\nabla\Psi^{(h)} = h^2\nabla_h\Psi^{(h)}$, this estimate becomes

$$\frac{1}{h^{2\alpha-2}}\int_\Omega W\big(s,\xi,\zeta,\nabla\tilde{y}^{(h)}\circ\Psi^{(h)}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta \le c.$$

The coercivity assumption (v) and (2.6) imply that

$$\frac{1}{h^{2\alpha-2}}\int_\Omega \mathrm{dist}^2\big(\nabla\tilde{y}^{(h)}\circ\Psi^{(h)},SO(3)\big)ds\,d\xi\,d\zeta \le c.$$

Step 1: Construction of the approximating sequence of rotations.
Proceeding exactly as in Theorem 2.3, we can construct a sequence of piecewise constant rotations $Q^{(h)}:[0,L]\to SO(3)$ (denoted by $R^{(h)}$ in the cited Theorem) satisfying the estimate (2.21), that is,

$$\int_\Omega \big|\,\nabla\tilde{y}^{(h)}\circ\Psi^{(h)} - Q^{(h)}\big|^2 ds\,d\xi\,d\zeta \le \frac{c}{h^2}\int_{\widetilde{\Omega}_h} W\big(\big(\Psi^{(h)}\big)^{-1}(x),\nabla\tilde{y}^{(h)}(x)\big)dx$$

for a suitable constant independent of h. By (3.4) we obtain

$$\int_\Omega \big|\nabla\tilde{y}^{(h)}\circ\Psi^{(h)} - Q^{(h)}\big|^2 ds\,d\xi\,d\zeta < c\,h^{2\alpha-2}. \tag{3.10}$$

Moreover, as in Theorem 2.3, for every $s\in(h,L-h)$ and every $|\delta|<h$ it turns out that

$$\big|Q^{(h)}(s+\delta)-Q^{(h)}(s)\big|^2 \le c\,h^{2\alpha-3}, \tag{3.11}$$

and for every $I'\subset\subset(0,L)$ and every $\delta\in\mathbb{R}$ with $|\delta|<\mathrm{dist}(I',\{0,L\})$ we have

$$\int_{I'}\big|Q^{(h)}(s+\delta)-Q^{(h)}(s)\big|^2 ds \le c\,h^{2(\alpha-2)}(|\delta|+h)^2, \tag{3.12}$$

with c independent of I' and δ. Now, let $\eta\in C_0^\infty(0,1)$ be such that $\eta\ge 0$, and $\int_0^1\eta(t)\,dt = 1$. We set $\eta^{(h)}(t) := \frac{1}{h}\eta(\frac{t}{h})$ and we define, as in the proof of [41, Theorem 2.2],

$$\tilde{Q}^{(h)}(s) := \int_0^h \eta^{(h)}(t)Q^{(h)}(s-t)\,dt, \quad s\in[0,L],$$

where we have extended $Q^{(h)}$ out of $(0,L)$ putting $Q^{(h)}(s) := Q^{(h)}(0)$ for $s\le 0$ and $Q^{(h)}(s) := Q^{(h)}(L)$ for $s\ge L$.

By (3.11) and (3.12) it easily follows that, for every $h>0$,

$$||\tilde{Q}^{(h)}-Q^{(h)}||_{L^2(0,L)} \le C\,h^{\alpha-1},\ ||\big(\tilde{Q}^{(h)}\big)'||_{L^2(0,L)} \le c\,h^{\alpha-2}, \tag{3.13}$$

$$||\tilde{Q}^{(h)}-Q^{(h)}||^2_{L^\infty(0,L)} \le C\,h^{2\alpha-3}. \tag{3.14}$$

In particular, estimates (3.10) and (3.13) yield

$$||\nabla\tilde{y}^{(h)}\circ\Psi^{(h)}-\tilde{Q}^{(h)}||_{L^2(\Omega)} \le c\,h^{\alpha-1}. \tag{3.15}$$

Let $\pi: U\to SO(3)$ be a smooth projection from a neighborhood U of $SO(3)$ onto $SO(3)$. From (3.14) it is clear that the functions $\tilde{Q}^{(h)}$ take values in U for h small enough; therefore, we can define

$\tilde{R}^{(h)} := \pi(\tilde{Q}^{(h)})$. Since $||(\tilde{R}^{(h)})'||_{L^2(0,L)} \leq C\,h^{\alpha-2}$ by (3.13), using Sobolev-Poincaré inequality we deduce

$$||\tilde{R}^{(h)} - P^{(h)}||_{L^\infty(0,L)} \leq ||(\tilde{R}^{(h)})'||_{L^2(0,L)} \leq c\,h^{\alpha-2}, \tag{3.16}$$

where $P^{(h)}$ is the mean value of $R^{(h)}$ over $(0,L)$. This implies that

$$\mathrm{dist}\big(P^{(h)}, SO(3)\big) \leq c\,h^{\alpha-2},$$

so there exists a sequence of constant rotations $(\bar{R}^{(h)})$ such that $|\,P^{(h)} - \bar{R}^{(h)}| \leq c\,h^{\alpha-2}$. By this and (3.16) we get

$$||\tilde{R}^{(h)} - \bar{R}^{(h)}||_{L^\infty(0,L)} \leq ||\tilde{R}^{(h)} - P^{(h)}||_{L^\infty(0,L)} + |\,P^{(h)} - \bar{R}^{(h)}| \leq c\,h^{\alpha-2}.$$

Finally, define $R^{(h)} := (\bar{R}^{(h)})^T\tilde{R}^{(h)}$; this sequence is of class C^∞ and satisfies (3.5) and (3.6). Moreover, from (3.15) we obtain

$$||\nabla\big((\bar{R}^{(h)})^T\tilde{y}^{(h)}\big)\circ\Psi^{(h)} - R^{(h)}||_{L^2(\Omega)} < C\,h^{\alpha-1}. \tag{3.17}$$

Let $c^{(h)} \in \mathbb{R}^3$ be the average of the function $(\bar{R}^{(h)})^T\,\tilde{y}^{(h)}\circ\Psi^{(h)} - \Psi^{(h)}$ on Ω and let us define the sequence $Y^{(h)} := (\bar{R}^{(h)})^T\,\tilde{y}^{(h)}\circ\Psi^{(h)} - c^{(h)}$. Then we can write (3.17) in terms of $\nabla_h Y^{(h)}$ and we get

$$||\nabla_h Y^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1} - R^{(h)}||_{L^2(\Omega)} \leq C\,h^{\alpha-1}, \tag{3.18}$$

which is exactly (3.7).

Step 2: Definition of the matrix A.
As in the case of a straight rod treated in [41], we consider the sequence $A^{(h)}$ defined as

$$A^{(h)}(s) := \frac{1}{h^{\alpha-2}}\big(R^{(h)}(s) - Id\big),$$

which converges uniformly and weakly in $W^{1,2}$ to a matrix $A \in W^{1,2}((0,L);\mathbb{M}^{3\times3})$. This is exactly property (e). Since $R^{(h)} \in SO(3)$, we have

$$A^{(h)} + \big(A^{(h)}\big)^T = -\,h^{\alpha-2}\,\big(A^{(h)}\big)^T A^{(h)}. \tag{3.19}$$

Passing to the limit as $h \to 0$, we deduce that A is skew-symmetric. Moreover, after division by $2\,h^{\alpha-2}$ in (3.19), we get

$$\frac{1}{h^{2(\alpha-2)}}\,\mathrm{sym}\,\big(R^{(h)} - Id\big) \;\to\; \frac{A^2}{2} \quad \text{uniformly},$$

so property (f) follows. The convergence of the sequence $A^{(h)}$, together with the estimate (3.7), imply that

$$\frac{1}{h^{\alpha-2}}\Big(\nabla_h Y^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1} - Id\Big) \to A \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}). \tag{3.20}$$

Step 3: Identification of A via limiting deformations v and w.
Now we characterize the elements of A in terms of some limiting deformations. By (2.6) and (3.20) we get

$$\frac{1}{h^{\alpha-2}}\,\nabla_h\big(Y^{(h)} - \Psi^{(h)}\big) \to A\,R_0 \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}), \tag{3.21}$$

so, in particular,

$$\frac{1}{h^{\alpha-2}}\,\partial_s\big(Y^{(h)} - \Psi^{(h)}\big) \to A\,\tau \quad \text{strongly in } L^2(\Omega;\mathbb{R}^3). \tag{3.22}$$

Let $v^{(h)}$ be the sequence introduced in (3.1). By the choice of $c^{(h)}$, it has zero average on $(0,L)$ and by (3.22) its derivative is bounded in $L^2((0,L);\mathbb{R}^3)$. Therefore, by Poincaré inequality, there exists a function $v \in W^{1,2}((0,L);\mathbb{R}^3)$ such that

$$v^{(h)} \to v \quad \text{strongly in } W^{1,2}((0,L);\mathbb{R}^3).$$

Moreover, by (3.22) we obtain that $v' = A\,\tau$. As A belongs to $W^{1,2}((0,L);\mathbb{M}^{3\times 3})$ and is skew-symmetric, we deduce that $v \in W^{2,2}((0,L);\mathbb{R}^3)$ and $v'\cdot\tau = 0$. Then (a) is proved.

Considering the second and the third columns in (3.21) we have

$$\frac{1}{h^{\alpha-1}}\,\partial_\xi\big(Y^{(h)} - \Psi^{(h)}\big) \to A\,\nu_2 \quad \text{and} \quad \frac{1}{h^{\alpha-1}}\,\partial_\zeta\big(Y^{(h)} - \Psi^{(h)}\big) \to A\,\nu_3 \quad \text{strongly in } L^2(\Omega;\mathbb{R}^3). \tag{3.23}$$

If we apply Poincaré inequality to the function $Y^{(h)} - \Psi^{(h)}$ on D, we get

$$\big|\big|\, Y^{(h)} - \Psi^{(h)} - \big(Y^{(h)} - \Psi^{(h)}\big)_D \,\big|\big|^2_{L^2(D)} \le c\Big(\big|\big|\,\partial_\xi\big(Y^{(h)} - \Psi^{(h)}\big)\,\big|\big|^2_{L^2(D)} + \big|\big|\,\partial_\zeta\big(Y^{(h)} - \Psi^{(h)}\big)\,\big|\big|^2_{L^2(D)}\Big) \tag{3.24}$$

for a.e. $s \in (0,L)$, where $\big(Y^{(h)} - \Psi^{(h)}\big)_D(s) := \int_D \big(Y^{(h)} - \Psi^{(h)}\big)\,d\xi\,d\zeta$. Integrating both sides of (3.24) with respect to s, we obtain that the sequence $\big(Y^{(h)} - \Psi^{(h)} - \big(Y^{(h)} - \Psi^{(h)}\big)_D\big)/h^{\alpha-1}$ is bounded in $L^2(\Omega;\mathbb{R}^3)$; moreover, (3.23) yields that there exists a function $q \in L^2((0,L);\mathbb{R}^3)$ such that

$$\frac{1}{h^{\alpha-1}}\big(Y^{(h)} - \Psi^{(h)} - \big(Y^{(h)} - \Psi^{(h)}\big)_D\big) \to \xi\,A\,\nu_2 + \zeta\,A\,\nu_3 \quad \text{strongly in } L^2(\Omega;\mathbb{R}^3). \tag{3.25}$$

Let $w^{(h)}$ be the sequence defined in (3.2). Thanks to (2.3), it can be rewritten as

$$w^{(h)} = \frac{1}{h^{\alpha-1}}\,\frac{1}{\mu(D)}\int_D \Big(Y^{(h)} - \Psi^{(h)} - (Y^{(h)} - \Psi^{(h)})_D\Big)\cdot(\xi\,\nu_3 - \zeta\,\nu_2)\,d\xi\,d\zeta. \tag{3.26}$$

From this expression it is clear that, using (3.25),

$$w^{(h)} \to w = \frac{1}{\mu(D)}\int_D \big(\xi\,A\,\nu_2 + \zeta\,A\,\nu_3\big)\cdot(\xi\,\nu_3 - \zeta\,\nu_2)\,d\xi\,d\zeta = (A\,\nu_2)\cdot\nu_3 \tag{3.27}$$

strongly in $L^2(0,L)$, where the last equality follows from (2.2) and from the fact that A is skew-symmetric. It remains to show that the convergence in (3.27) is actually weak in $W^{1,2}(0,L)$. To this aim it is enough to verify the boundedness of the derivative of $w^{(h)}$ in the L^2- norm. We get

$$\begin{aligned}\big(w^{(h)}\big)' = {}& \frac{1}{h^{\alpha-1}}\Big(\frac{1}{\mu(D)}\int_D \partial_s(Y^{(h)} - \Psi^{(h)})\cdot(\xi\,\nu_3 - \zeta\,\nu_2)\,d\xi\,d\zeta\Big) + \\ & + \frac{1}{h^{\alpha-1}}\Big(\frac{1}{\mu(D)}\int_D \big(Y^{(h)} - \Psi^{(h)}\big)\cdot(\xi\,\nu_3' - \zeta\,\nu_2')\,d\xi\,d\zeta\Big).\end{aligned} \tag{3.28}$$

For the last integral on the right-hand side of (3.28) the required bound can be proved using the convergence in (3.25), arguing in a similar way to (3.26)-(3.27). For the first integral notice that

$$\begin{aligned}\frac{1}{h^{\alpha-1}}\int_D \partial_s(Y^{(h)} - \Psi^{(h)})\cdot(\xi\,\nu_3 - \zeta\,\nu_2)\,d\xi\,d\zeta = {}& \frac{1}{h^{\alpha-1}}\int_D \big(\partial_s Y^{(h)} - R^{(h)}\partial_s\Psi^{(h)}\big)\cdot(\xi\,\nu_3 - \zeta\,\nu_2)\,d\xi\,d\zeta \\ & + \frac{1}{h^{\alpha-1}}\int_D \big(R^{(h)}\partial_s\Psi^{(h)} - \partial_s\Psi^{(h)}\big)\cdot(\xi\,\nu_3 - \zeta\,\nu_2)\,d\xi\,d\zeta.\end{aligned}$$

In virtue of (3.18) and (2.6), the first term on right-hand side is bounded in L^2, hence it remains to control the L^2-norm of the second integral. Now, using (2.5), we have

$$\int_D \big(R^{(h)}\partial_s\Psi^{(h)} - \partial_s\Psi^{(h)}\big)\cdot(\xi\,\nu_3 - \zeta\,\nu_2)\,d\xi\,d\zeta = h\int_D \Big[\big(R^{(h)} - Id\big)(\xi\,\nu_2' + \zeta\,\nu_3')\Big]\cdot(\xi\,\nu_3 - \zeta\,\nu_2)\,d\xi\,d\zeta.$$

The required bound follows from (3.6), hence (b) is shown.

As A is skew-symmetric and $A\tau = v'$, $(A\nu_2)\cdot\nu_3 = w$, we conclude that

$$R_0^T A R_0 = \begin{pmatrix} 0 & -v'\cdot\nu_2 & -v'\cdot\nu_3 \\ v'\cdot\nu_2 & 0 & -w \\ v'\cdot\nu_3 & w & 0 \end{pmatrix},$$

which gives (3.9).

Step 4: Convergence of the sequence $(u^{(h)})$.

Let $(u^{(h)})$ be the sequence defined in (3.3).

Consider first the case $2 < \alpha < 3$. It is easy to verify that its derivative is bounded in $L^2(0,L)$. Indeed,

$$\begin{aligned}(u^{(h)})' = \frac{1}{h^{2(\alpha-2)}}\int_D \partial_s\big(Y^{(h)} - \Psi^{(h)}\big)\cdot\tau\, d\xi\, d\zeta &= \frac{1}{h^{2(\alpha-2)}}\int_D \big(\partial_s Y^{(h)} - R^{(h)}\partial_s\Psi^{(h)}\big)\cdot\tau\, d\xi\, d\zeta \\ &+ \frac{1}{h^{2(\alpha-2)}}\int_D \big(R^{(h)}\partial_s\Psi^{(h)} - \partial_s\Psi^{(h)}\big)\cdot\tau\, d\xi\, d\zeta.\end{aligned}$$

Since $\alpha < 3$, the first term converges to zero strongly in L^2 by (2.6) and (3.18). As for the second term, using (2.5), the fact that $R^{(h)}$ is independent of ξ and ζ and (2.3), we have

$$\begin{aligned}\frac{1}{h^{2(\alpha-2)}}\int_D \big(R^{(h)}\partial_s\Psi^{(h)} - \partial_s\Psi^{(h)}\big)\cdot\tau\, d\xi\, d\zeta &= \frac{1}{h^{2(\alpha-2)}}\big(R^{(h)}\tau - \tau\big)\cdot\tau \\ &= \frac{1}{h^{2(\alpha-2)}}\operatorname{sym}\big(R^{(h)} - Id\big)\tau\cdot\tau.\end{aligned}$$

By property (f) this converges to $(A^2\tau)\cdot\tau/2$ uniformly on $(0,L)$. As $u^{(h)}$ has zero average, by Poincaré inequality we deduce that $u^{(h)}$ converges to u strongly in $W^{1,2}$, where u satisfies

$$u' = \left(\frac{A^2}{2}\tau\right)\cdot\tau = -\frac{1}{2}\Big((v'\cdot\nu_2)^2 + (v'\cdot\nu_3)^2\Big). \tag{3.29}$$

In the case $\alpha \geq 3$ the derivative of $(u^{(h)})$ can be written as

$$(u^{(h)})' = \frac{1}{h^{\alpha-1}}\int_D \big(\partial_s Y^{(h)} - R^{(h)}\partial_s\Psi^{(h)}\big)\cdot\tau\, d\xi\, d\zeta + \frac{1}{h^{\alpha-1}}\operatorname{sym}\big(R^{(h)} - Id\big)\tau\cdot\tau.$$

The first term is bounded in $L^2(0,L)$ by (2.6) and (3.18), while the second term converges to zero uniformly by (f).

This concludes the proof of (c) and of the theorem. □

3.2 Liminf inequalities

In this section we will show a lower bound for the energy $(\tilde{I}^{(h)})/h^{(2\alpha-2)}$, for all the scalings $\alpha > 2$, and we will describe the limiting functionals.

Let $Q_3 : \Omega \times \mathbb{M}^{3\times3} \longrightarrow [0,+\infty)$ be twice the quadratic form of linearized elasticity, i.e.,

$$Q_3(z,G) := \frac{\partial^2 W}{\partial F^2}(z, Id)[G,G] \tag{3.30}$$

for every $z \in \Omega$ and every $G \in \mathbb{M}^{3\times 3}$. Let $Q^0 : (0,L)\times\mathbb{R}\times\mathbb{M}^{3\times 3}_{skew} \to [0,+\infty)$ and $Q : (0,L)\times M^{3\times 3} \to [0,+\infty)$ be defined as

$$Q^0(s,t,F) := \min_{\varphi\in W^{1,2}(D;\mathbb{R}^3)} \int_D Q_3\Big(s,\xi,\zeta,R_0\Big(F\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix} + t\,e_1\,\Big|\,\partial_\xi\varphi\,\Big|\,\partial_\zeta\varphi\Big)\,R_0^T\Big)\,d\xi\,d\zeta \tag{3.31}$$

and

$$Q(s,F) := \min_{t\in\mathbb{R}} Q^0(s,t,F), \tag{3.32}$$

respectively. For $u,w \in W^{1,2}(0,L)$ and $v \in W^{2,2}((0,L);\mathbb{R}^3)$ we introduce the functionals

$$I_\alpha(u,v,w) := \begin{cases} \dfrac12\displaystyle\int_0^L Q^0\Big(s, u' + \frac12\big((v'\cdot\nu_2)^2 + (v'\cdot\nu_3)^2\big), B' + 2\,\mathrm{skw}\big(R_0^T R_0' B\big)\Big)\,ds & \text{if } \alpha = 3,\\[2ex] \dfrac12\displaystyle\int_0^L Q^0\big(s, u', B' + 2\,\mathrm{skw}\big(R_0^T R_0' B\big)\big)\,ds & \text{if } \alpha > 3,\end{cases} \tag{3.33}$$

and, for $2<\alpha<3$,

$$I_\alpha(v,w) := \frac12\int_0^L Q\big(s, B' + 2\,\mathrm{skw}\big(R_0^T R_0' B\big)\big)\,ds, \tag{3.34}$$

where $B \in W^{1,2}((0,L);\mathbb{M}^{3\times 3})$ denotes the matrix

$$B := \begin{pmatrix} 0 & -v'\cdot\nu_2 & -v'\cdot\nu_3\\ v'\cdot\nu_2 & 0 & -w\\ v'\cdot\nu_3 & w & 0\end{pmatrix}. \tag{3.35}$$

Remark 3.2 It is easy to see that the minimum in (3.31) is attained, it is unique and it can be computed on the subspace

$$\mathcal{V} := \Big\{\varphi \in W^{1,2}(D;\mathbb{R}^3) : \int_D \varphi\,d\xi\,d\zeta = 0,\ \int_D \varphi\cdot(\zeta\,\nu_2 - \xi\,\nu_3)\,d\xi\,d\zeta = 0\Big\}, \tag{3.36}$$

(see [41, Remark 4.1]). Moreover the minimizer φ depends linearly on the data t and F. More precisely, if $t \in L^2(0,L)$ and $F \in L^2((0,L);\mathbb{M}^{3\times 3}_{skew})$, then denoting with $\varphi(s,\cdot) \in \mathcal{V}$ the solution of the problem (3.31) with data $t(s)$ and $F(s)$, we have that $\varphi \in L^2(\Omega;\mathbb{R}^3)$ and also $\partial_\xi\varphi, \partial_\zeta\varphi \in L^2(\Omega;\mathbb{R}^3)$. Analogously, if t solves (3.32), then t depends linearly on F. So, if $F \in L^2((0,L);\mathbb{M}^{3\times 3}_{skew})$ and t is the solution to (3.32) corresponding to $F(s)$, then $t \in L^2(0,L)$.

Remark 3.3 The limit functionals corresponding to the scalings $2<\alpha<3$ and $\alpha>3$ turn out to be linear. Notice that, in the case $2<\alpha<3$, the deformation u is completely determined by v in virtue of the constraint (3.8) in Theorem 3.1. This explains the reason why the Γ-limit obtained for this scaling does not depend on u. On the other hand, for $\alpha>3$, the function u is independent of v and w and the functional I_α describing the one-dimensional problem coincides with the one obtained by dimension reduction, starting from 3D linearized elasticity (see [34], [35] and [50]).

More precisely, if we assume in addition that the density W is homogeneous and isotropic, that is,

$$W(F) = W(FR) \quad \text{for every } R \in SO(3),$$

then the quadratic form Q_3 is given by

$$Q_3(G) = 2\,\mu\left|\frac{G+G^T}{2}\right|^2 + \lambda\,(\mathrm{tr}\,G)^2$$

for some constants $\lambda, \mu \in \mathbb{R}$. Since for all $G \in \mathbb{M}^{3\times 3}$ and $R \in SO(3)$ we have

$$Q_3(R\,G\,R^T) = Q_3(G),$$

by [40, Remark 3.5] formula (3.31) reduces to

$$Q^0(s,t,F) = \frac{\mu(3\,\lambda + 2\,\mu)}{\lambda + \mu}\,(t^2 + I_3 F_{12}^2 + I_2 F_{13}^2) + \mu\,T\,F_{23}^2, \tag{3.37}$$

where $I_3 = \int_D \xi^2 d\xi\, d\zeta$, $I_2 = \int_D \zeta^2 d\xi\, d\zeta$ and T is the so-called torsional rigidity, which depends on the section. Therefore by (3.37), (2.2) and (2.3) the limit functional reads as follows

$$I_\alpha(u,v,w) = \frac{1}{2}\,\frac{\mu(3\,\lambda + 2\,\mu)}{\lambda + \mu}\int_0^L \big((u')^2 + I_2 q_2^2 + I_3 q_3^2\big)\,ds + \frac{1}{2}\,\mu\,T\int_0^L q_1^2 ds,$$

where

$$\begin{aligned}
q_1 &:= w' + k_2(v\cdot\nu_3)' - k_3(v\cdot\nu_2)' + \varrho\,\big(k_2(v\cdot\nu_2) + k_3(v\cdot\nu_3)\big),\\
q_2 &:= k_2 w - (v\cdot\nu_3)'' - 2\,\varrho\,(v\cdot\nu_2)' - (v\cdot\tau)\,\big(\varrho\,k_2 + k_3'\big) + (v\cdot\nu_2)\,\big(k_2 k_3 + \varrho'\big) + (v\cdot\nu_3)\,\big(\varrho^2 - k_3^2\big),\\
q_3 &:= k_3\,w + (v\cdot\nu_2)'' - 2\,\varrho\,(v\cdot\nu_3)' - (v\cdot\tau)\,\big(\varrho\,k_3 - k_2'\big) - (v\cdot\nu_2)\,\big(\varrho^2 - k_2^2\big) + (v\cdot\nu_3)\,\big(k_2\,k_3 - \varrho'\big).
\end{aligned}$$

This is the functional derived in [34], [35] and [50], starting from linearized elasticity.

Now we are ready to show a lower bound for the functionals $h^{-\alpha}\tilde{I}^h$ with $2 < \alpha < 3$.

Theorem 3.4 (Case $2 < \alpha < 3$) *Let $w \in W^{1,2}(0,L)$ and let $v \in W^{2,2}((0,L);\mathbb{R}^3)$ be such that $v'\cdot\tau = 0$. Then, for every positive sequence (h_j) converging to zero and every sequence $\big(\tilde{y}^{(h_j)}\big) \subset W^{1,2}(\widetilde{\Omega}_{h_j};\mathbb{R}^3)$ such that the sequence $Y^{(h_j)} := \tilde{y}^{(h_j)} \circ \Psi^{(h_j)}$ satisfies the properties (a), (b) and (d) of Theorem 3.1, it turns out that*

$$\liminf_{j\to\infty} \frac{1}{h_j^{2\,\alpha}} \int_{\widetilde{\Omega}_{h_j}} W\Big(\big(\Psi^{(h_j)}\big)^{-1}(x), \nabla\tilde{y}^{(h_j)}(x)\Big)dx \geq I_\alpha(v,w), \tag{3.38}$$

where I_α is introduced in (3.34).

Proof. – In the following, we will write simply h instead of h_j. Let $\big(\tilde{y}^{(h)}\big)$ be a sequence such that $Y^{(h)} := \tilde{y}^{(h)} \circ \Psi^{(h)}$ satisfies the required assumptions.

First step: lower bound for the energy. We can suppose that

$$\liminf_{h\to 0} \frac{1}{h^{2\,\alpha}} \int_{\widetilde{\Omega}_h} W\Big(\big(\Psi^{(h)}\big)^{-1}(x), \nabla\tilde{y}^{(h)}(x)\Big)dx \leq c < +\infty,$$

otherwise (3.38) is trivial. Therefore, up to subsequences, (3.4) is satisfied. By Theorem 3.1 we get the existence of a sequence $R^{(h)} : [0,L] \to SO(3)$ such that

$$||\nabla_h Y^{(h)}\big(\nabla_h \Psi^{(h)}\big)^{-1} - R^{(h)}||_{L^2(\Omega)} \leq c\,h^{\alpha-1} \tag{3.39}$$

and $R^{(h)} \to Id$ uniformly. Define the functions $G^{(h)} : \Omega \to \mathbb{M}^{3\times 3}$ as

$$G^{(h)} := \frac{1}{h^{\alpha-1}}\Big((R^{(h)})^T \nabla_h Y^{(h)}\big(\nabla_h \Psi^{(h)}\big)^{-1} - Id\Big). \tag{3.40}$$

By (3.39) they are bounded in $L^2(\Omega;\mathbb{M}^{3\times 3})$, so there exists $G \in L^2(\Omega;\mathbb{M}^{3\times 3})$ such that $G^{(h)} \rightharpoonup G$ weakly in $L^2(\Omega;\mathbb{M}^{3\times 3})$. We claim that

$$\liminf_{h\to 0} \frac{1}{h^{2\alpha}} \int_{\widetilde{\Omega}_h} W\Big(\big(\Psi^{(h)}\big)^{-1}(x), \nabla\tilde{y}^{(h)}(x)\Big)dx \geq \frac{1}{2}\int_\Omega Q_3(s,\xi,\zeta,G)ds\,d\xi\,d\zeta. \tag{3.41}$$

Performing the change of variables $\Psi^{(h)}$ and using the frame indifference of W, we have

$$\begin{aligned}\frac{1}{h^{2\alpha}} \int_{\widetilde{\Omega}_h} W\Big(\big(\Psi^{(h)}\big)^{-1}(x), \nabla\tilde{y}^{(h)}\Big)dx &= \frac{1}{h^{2\alpha-2}}\int_\Omega W\big(s,\xi,\zeta,\nabla\tilde{y}^{(h)}\circ\Psi^{(h)}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta\\ &= \frac{1}{h^{2\alpha-2}}\int_\Omega W\Big(s,\xi,\zeta,\big(\nabla_h Y^{(h)}\big)\big(\nabla_h\Psi^{(h)}\big)^{-1}\Big)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta. \end{aligned}\tag{3.42}$$

We introduce the functions

$$\chi^{(h)}(s,\xi,\zeta) := \begin{cases} 1 & \text{if } \ \big|G^{(h)}(s,\xi,\zeta)\big| \leq h^{2-\alpha},\\ 0 & \text{otherwise.}\end{cases}$$

From the boundedness of $G^{(h)}$ in $L^2(\Omega;\mathbb{M}^{3\times 3})$ we get that $\chi^{(h)} \to 1$ boundedly in measure, so that

$$\chi^{(h)}G^{(h)} \rightharpoonup G \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}). \tag{3.43}$$

By expanding W around the identity, we obtain that for every $(s,\xi,\zeta)\in\Omega$ and $A\in\mathbb{M}^{3\times 3}$

$$W\big(s,\xi,\zeta,Id+A\big) = \frac{1}{2}\frac{\partial^2 W}{\partial F^2}(s,\xi,\zeta,Id+t\,A)[A,A],$$

where $0<t<1$ depends on the point (s,ξ,ζ) and on A. By (3.42) and by the definition of $G^{(h)}$ we have

$$\begin{aligned}\frac{1}{h^{2\alpha}}\tilde{I}^{(h)}\big(\tilde{y}^{(h)}\big) &= \frac{1}{h^{2\alpha-2}}\int_\Omega W\big(s,\xi,\zeta,Id+h^{\alpha-1}G^{(h)}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta\\ &\geq \frac{1}{h^{2\alpha-2}}\int_\Omega \chi^{(h)}W\big(s,\xi,\zeta,Id+h^{\alpha-1}G^{(h)}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta\\ &= \frac{1}{2}\int_\Omega \chi^{(h)}\bigg(\frac{\partial^2 W}{\partial F^2}\big(s,\xi,\zeta,Id+h^{\alpha-1}t(h)\,G^{(h)}\big)\big[G^{(h)},G^{(h)}\big]\bigg)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta,\end{aligned}$$

where $0<t(h)<1$ depends on (s,ξ,ζ) and on $G^{(h)}$. For the last integral in the previous formula we have that

$$\begin{aligned}&\int_\Omega \chi^{(h)}\bigg(\frac{\partial^2 W}{\partial F^2}\big(s,\xi,\zeta,Id+h^{\alpha-1}t(h)\,G^{(h)}\big)\big[G^{(h)},G^{(h)}\big]\bigg)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta =\\ &\int_\Omega \chi^{(h)}\bigg(\frac{\partial^2 W}{\partial F^2}\big(s,\xi,\zeta,Id+h^{\alpha-1}t(h)\,G^{(h)}\big)\big[G^{(h)},G^{(h)}\big] - Q_3\big(s,\xi,\zeta,G^{(h)}\big)\bigg)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta\\ &\qquad + \int_\Omega Q_3\big(s,\xi,\zeta,\chi^{(h)}G^{(h)}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta.\end{aligned}\tag{3.44}$$

Notice that the second integral is lower semicontinuous with respect to the weak topology of L^2; so, the claim follows from (3.43), once we prove that the first term in (3.44) can be neglected for h small enough. To this aim, we apply Scorza-Dragoni theorem to the function $\partial^2 W/\partial F^2$ and we have that there exists a compact set $K\subset\Omega$ such that the function $\partial^2 W/\partial F^2$ restricted to $K\times\overline{B_\delta(Id)}$ is continuous, hence uniformly continuous. Since $h\,t(h)\,\chi^{(h)}\,G^{(h)}$ is uniformly small for h small enough, for every $\varepsilon > 0$ we have

$$\frac{1}{2}\int_{\Omega}\chi^{(h)}\left(\frac{\partial^2 W}{\partial F^2}\big(s,\xi,\zeta,Id + h\,t(h)\,G^{(h)}\big)\big[G^{(h)},G^{(h)}\big] - Q_3\big(s,\xi,\zeta,G^{(h)}\big)\right)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta$$
$$\geq -\frac{\varepsilon}{2}\int_K \chi^{(h)}\big|\,G^{(h)}\big|^2\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta \geq -C\,\varepsilon$$

for h small enough. Hence, being ε arbitrary, (3.41) is proved.

Since, by frame indifference, the quadratic form Q_3 depends only on the symmetric part of G, we obtain the bound

$$\liminf_{h\to 0}\frac{1}{h^{2\,\alpha}}\int_{\tilde{\Omega}_h} W\Big(\big(\Psi^{(h)}\big)^{-1}(x),\nabla\tilde{y}^{(h)}(x)\Big)dx \geq \frac{1}{2}\int_{\Omega} Q_3(s,\xi,\zeta,\tilde{G})ds\,d\xi\,d\zeta, \tag{3.45}$$

where $\tilde{G}$ denotes the symmetric part of G.

Second step: identification of $\tilde{G}$. In order to identify $\tilde{G}$, we first notice that, since $R^{(h)} \to Id$ uniformly,

$$R^{(h)}G^{(h)} = \frac{1}{h^{\alpha-1}}\Big(\nabla_h Y^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1} - R^{(h)}\Big) \rightharpoonup G$$

weakly in $L^2(\Omega;\mathbb{M}^{3\times 3})$; moreover, by (2.6),

$$R^{(h)}G^{(h)}\nabla_h\Psi^{(h)} = \frac{1}{h^{\alpha-1}}\Big(\nabla_h Y^{(h)} - R^{(h)}\nabla_h\Psi^{(h)}\Big) \rightharpoonup G\,R_0. \tag{3.46}$$

In particular, considering the second and the third columns in (3.46) we get

$$R^{(h)}G^{(h)}\nu_2 = \frac{1}{h^{\alpha}}\,\partial_\xi\big(Y^{(h)} - R^{(h)}\Psi^{(h)}\big) = \frac{1}{h^{\alpha}}\Big(\partial_\xi Y^{(h)} - h\,R^{(h)}\nu_2\Big) \rightharpoonup G\,\nu_2$$

and

$$R^{(h)}G^{(h)}\nu_3 = \frac{1}{h^{\alpha}}\,\partial_\zeta\big(Y^{(h)} - R^{(h)}\Psi^{(h)}\big) = \frac{1}{h^{\alpha}}\Big(\partial_\zeta Y^{(h)} - h\,R^{(h)}\nu_3\Big) \rightharpoonup G\,\nu_3.$$

Let us define the functions $\tilde{\beta}^{(h)} : \Omega \to \mathbb{R}^3$ as

$$\tilde{\beta}^{(h)}(s,\xi,\zeta) := \frac{1}{h^{\alpha}}\Big(Y^{(h)} - h\,\xi\,R^{(h)}\nu_2 - h\,\zeta\,R^{(h)}\nu_3\Big).$$

Easy computations show that

$$\partial_\xi\tilde{\beta}^{(h)} = R^{(h)}G^{(h)}\nu_2 \quad\text{and}\quad \partial_\zeta\tilde{\beta}^{(h)} = R^{(h)}G^{(h)}\nu_3, \tag{3.47}$$

hence $\partial_\xi\tilde{\beta}^{(h)}$ and $\partial_\zeta\tilde{\beta}^{(h)}$ are bounded in $L^2(\Omega)$. By Poincaré inequality, this implies that

$$\big|\big|\tilde{\beta}^{(h)} - \tilde{\beta}^{(h)}_D\big|\big|^2_{L^2(\Omega)} \leq C\left(\big|\big|\partial_\xi\tilde{\beta}^{(h)}\big|\big|^2_{L^2(\Omega)} + \big|\big|\partial_\zeta\tilde{\beta}^{(h)}\big|\big|^2_{L^2(\Omega)}\right) \leq c,$$

where $\tilde{\beta}^{(h)}_D(s) := \int_D \tilde{\beta}^{(h)}(s,\xi,\zeta)\,d\xi\,d\zeta$. Therefore, there exists a function $\beta \in L^2(\Omega;\mathbb{R}^3)$ such that

$$\beta^{(h)} := \tilde{\beta}^{(h)} - \tilde{\beta}^{(h)}_D \rightharpoonup \beta \text{ weakly in } L^2(\Omega;\mathbb{R}^3). \tag{3.48}$$

From (3.47), as $h \to 0$, we get

$$G\,\nu_2 = \partial_\xi\beta \quad\text{and}\quad G\,\nu_3 = \partial_\zeta\beta. \tag{3.49}$$

Considering the first columns in (3.46) we have

$$R^{(h)}G^{(h)}\partial_s\Psi^{(h)} = \frac{1}{h^{\alpha-1}}\Big(\partial_s Y^{(h)} - R^{(h)}\partial_s\Psi^{(h)}\Big) \rightharpoonup G\,\tau. \tag{3.50}$$

Using (2.5) and the definitions of $\tilde\beta_D^{(h)}$ and $\beta^{(h)}$, we can write

$$\begin{aligned} R^{(h)}G^{(h)}\partial_s\Psi^{(h)} &= \frac{1}{h^{\alpha-1}}\Big(\partial_s Y^{(h)} - h\,\xi R^{(h)}\nu_2' - h\,\zeta R^{(h)}\nu_3'\Big) - \frac{1}{h^{\alpha-1}}\,R^{(h)}\tau \\ &= h\,\partial_s\beta^{(h)} + \frac{1}{h^{\alpha-2}}\,\big(R^{(h)}\big)'(\xi\,\nu_2+\zeta\,\nu_3) + \frac{1}{h^{\alpha-1}}\int_D \big(\partial_s Y^{(h)} - R^{(h)}\tau\big)\,d\xi\,d\zeta. \end{aligned} \tag{3.51}$$

By (3.48) it follows that

$$h\,\partial_s\beta^{(h)} \rightharpoonup 0 \quad \text{weakly in } W^{-1,2}(\Omega;\mathbb{R}^3). \tag{3.52}$$

Moreover, from (3.39), it turns out that there exists $g\in L^2((0,L);\mathbb{R}^3)$ such that

$$\frac{1}{h^{\alpha-1}}\int_D \big(\partial_s Y^{(h)} - R^{(h)}\tau\big)\,d\xi\,d\zeta = \frac{1}{h^{\alpha-1}}\int_D \big(\partial_s Y^{(h)} - R^{(h)}\partial_s\Psi^{(h)}\big)\,d\xi\,d\zeta \rightharpoonup g \tag{3.53}$$

weakly in $L^2((0,L);\mathbb{R}^3)$. Passing to the limit in (3.51) and using (3.50), (3.53), (3.52), and property (e) of Theorem 3.1, we obtain

$$G\,\tau = A'(\xi\,\nu_2+\zeta\,\nu_3) + g. \tag{3.54}$$

Finally, by (3.49) and (3.54) we have that

$$G\,R_0 = \left(A'\,R_0\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix} + g\,\middle|\,\partial_\xi\beta\,\middle|\,\partial_\zeta\beta\right).$$

As $\operatorname{sym}\big(R_0^T G\,R_0\big) = R_0^T\tilde G\,R_0$, we deduce that

$$R_0^T\tilde G\,R_0 = \operatorname{sym}\left(R_0^T A'\,R_0\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix} + \hat g\,\middle|\,\partial_\xi\hat\beta\,\middle|\,\partial_\zeta\hat\beta\right),$$

where $\hat\beta := R_0^T\beta$ and $\hat g := R_0^T g$. If we define $\varphi := \hat\beta + \xi\,(\hat g\cdot e_2)\,e_1 + \zeta\,(\hat g\cdot e_3)\,e_1$, we obtain the expression

$$R_0^T\tilde G\,R_0 = \operatorname{sym}\left(R_0^T A'\,R_0\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix} + (\hat g\cdot e_1)\,e_1\,\middle|\,\partial_\xi\varphi\,\middle|\,\partial_\zeta\varphi\right). \tag{3.55}$$

Now, let us rewrite the previous expression in terms of the matrix B defined in (3.35), noticing that $A = R_0 B R_0^T$. It turns out that

$$A' = R_0' B R_0^T + R_0 B' R_0^T + R_0 B\big(R_0^T\big)',$$

hence

$$R_0^T A'\,R_0 = R_0^T R_0' B + B' + B\big(R_0^T\big)' R_0.$$

Since B is skew-symmetric, we deduce

$$R_0^T A'\,R_0 = B' + 2\operatorname{skw}\big(R_0^T R_0' B\big). \tag{3.56}$$

Using this identity in (3.55) we have

$$R_0^T \tilde{G}\, R_0 = \mathrm{sym}\left(\left(B' + 2\,\mathrm{skw}\big(R_0^T R_0' B\big)\right)\begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} + (\hat{g}\cdot e_1)\, e_1 \Big| \partial_\xi \varphi \Big| \partial_\zeta \varphi \right).$$

Finally, as

$$\hat{g}\cdot e_1 = \big(R_0^T g\big)\cdot e_1 = g\cdot (R_0 e_1) = g\cdot\tau,$$

we conclude that

$$R_0^T \tilde{G}\, R_0 = \mathrm{sym}\left(\left(B' + 2\,\mathrm{skw}\big(R_0^T R_0' B\big)\right)\begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} + (g\cdot\tau)\, e_1 \Big| \partial_\xi \varphi \Big| \partial_\zeta \varphi \right). \tag{3.57}$$

Third step: description of the limit functional. Since $\varphi(s,\cdot) \in W^{1,2}(D;\mathbb{R}^3)$ for a.e. $s \in (0,L)$, using (3.45), (3.57) and the definition of Q, we obtain exactly (3.38). □

It remains to show the lower bound for the functionals $h^{-\alpha}\tilde{I}^h$ with $\alpha \geq 3$.

Theorem 3.5 (Case $\alpha \geq 3$) *Let $u, w \in W^{1,2}(0,L)$ and let $v \in W^{2,2}((0,L);\mathbb{R}^3)$ be such that $v'\cdot\tau = 0$. Then, for every positive sequence (h_j) converging to zero and every sequence $\big(\tilde{y}^{(h_j)}\big) \subset W^{1,2}(\widetilde{\Omega}_{h_j};\mathbb{R}^3)$ such that the sequence $Y^{(h_j)} := \tilde{y}^{(h_j)} \circ \Psi^{(h_j)}$ satisfies the properties (a)-(d) of Theorem 3.1, it turns out that*

$$\liminf_{j\to\infty} \frac{1}{h_j^{2\alpha}} \int_{\widetilde{\Omega}_{h_j}} W\Big(\big(\Psi^{(h_j)}\big)^{-1}(x), \nabla\tilde{y}^{(h_j)}(x)\Big)dx \geq I_\alpha(u,v,w), \tag{3.58}$$

where I_α is defined as in (3.33).

Proof. – We can repeat exactly what we did in the first two steps of the proof of Theorem 3.4. At this point, let us distinguish the cases $\alpha = 3$ and $\alpha > 3$.
Case $\alpha = 3$.
Starting from (3.53), we can identify the tangential component of g. Indeed, observe that, if we write

$$\int_D \big(\partial_s Y^{(h)} - R^{(h)}\tau\big)\cdot\tau\, d\xi\, d\zeta = \int_D \partial_s\big(Y^{(h)} - \Psi^{(h)}\big)\cdot\tau\, d\xi\, d\zeta - \int_D \big(R^{(h)}\tau - \tau\big)\cdot\tau\, d\xi\, d\zeta,$$

by the definition of $\big(u^{(h)}\big)$ we get

$$\frac{1}{h^{\alpha-1}}\int_D \big(\partial_s Y^{(h)} - R^{(h)}\tau\big)\cdot\tau\, d\xi\, d\zeta = \big(u^{(h)}\big)' - \frac{1}{h^{\alpha-1}}\int_D \big(R^{(h)}\tau - \tau\big)\cdot\tau\, d\xi\, d\zeta. \tag{3.59}$$

If we let $h \to 0$ in (3.59) we obtain, from (3.53) and in virtue of property (f) in Theorem 3.1,

$$g\cdot\tau = u' - \frac{1}{2}\big(A^2\,\tau\big)\cdot\tau. \tag{3.60}$$

Notice that, using the explicit expression of A given in (3.9), we have

$$\frac{1}{2}\big(A^2\,\tau\big)\cdot\tau = -\frac{1}{2}\big((v'\cdot\nu_2)^2 + (v'\cdot\nu_3)^2\big). \tag{3.61}$$

Now, by (3.55),(3.60) and (3.61), we can write the expression of $\tilde{G}$ in this case, which turns to be

$$\tilde{G} = R_0\,\mathrm{sym}\left(\left(B' + 2\,\mathrm{skw}\big(R_0^T R_0' B\big)\right)\begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} + \left(u' + \frac{1}{2}\big((v'\cdot\nu_2)^2 + (v'\cdot\nu_3)^2\big)\right) e_1 \Big| \partial_\xi \varphi \Big| \partial_\zeta \varphi \right) R_0^T. \tag{3.62}$$

Since $\varphi(s,\cdot) \in W^{1,2}(D;\mathbb{R}^3)$ for a.e. $s \in (0,L)$, using the definition of Q^0 the bound (3.45) becomes, as we claimed,

$$\liminf_{h\to 0} \frac{1}{h^6} \int_{\tilde{\Omega}_h} W\Big(\big(\Psi^{(h)}\big)^{-1}(x), \nabla \tilde{y}^{(h)}(x)\Big) dx \geq I_3^0(u,v,w),$$

with I_3^0 defined in (3.33).

Case $\alpha > 3$.

If we let $h \to 0$ in (3.59) we obtain from (3.53) and in virtue of property (f) in Theorem 3.1,

$$g \cdot \tau = u'.$$

In fact, being $\alpha > 3$, it turns out that $\alpha - 1 < 2(\alpha - 2)$, so

$$\mathrm{sym}\big(R^{(h)} - Id\big)/h^{\alpha-1} \to 0 \quad \text{uniformly on } (0,L).$$

Now we can write down the expression of $\tilde{G}$ for $\alpha > 3$, that is

$$\tilde{G} = R_0\,\mathrm{sym}\left(\Big(B' + 2\,\mathrm{skw}\big(R_0^T R_0' B\big)\Big)\begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} + u'\,e_1 \Big| \partial_\xi \varphi \Big| \partial_\zeta \varphi \right) R_0^T. \tag{3.63}$$

Since $\varphi(s,\cdot) \in W^{1,2}(D;\mathbb{R}^3)$ for a.e. $s \in (0,L)$, using (3.45), (3.63) and the definition of Q^0, we obtain exactly (3.58), as we claimed. □

3.3 Construction of the recovery sequences

In this section we show that the lower bounds obtained in Theorems 3.4 and 3.5 are optimal. Also in this case the scalings $2 < \alpha < 3$ and $\alpha \geq 3$ will be treated separately. However, we will first consider the higher scalings h^α with $\alpha \geq 3$, since as in in [32], the case $2 < \alpha < 3$ turns out to be very delicate and requires a more detailed analysis.

3.3.1 Higher scaling.

Let us consider now the higher scalings of the energy, that is the case $\alpha \geq 3$.

Theorem 3.6 (Case $\alpha \geq 3$) *For every $u, w \in W^{1,2}(0,L)$ and $v \in W^{2,2}((0,L);\mathbb{R}^3)$ such that $v' \cdot \tau = 0$ there exists a sequence $\big(\tilde{y}^{(h)}\big) \subset W^{1,2}(\tilde{\Omega}_h;\mathbb{M}^{3\times 3})$ such that, setting $Y^{(h)} := \tilde{y}^{(h)} \circ \Psi^{(h)}$, we have*

(i) $\big(\nabla_h Y^{(h)} \big(\nabla_h \Psi^{(h)}\big)^{-1} - Id\big)/h^{\alpha-2} \to A$ *strongly in* $L^2(\Omega;\mathbb{M}^{3\times 3})$;

(ii) $v^{(h)} \to v$ *strongly in* $W^{1,2}((0,L);\mathbb{R}^3)$;

(iii) $w^{(h)} \rightharpoonup w$ *weakly in* $W^{1,2}(0,L)$;

(iv) $u^{(h)} \rightharpoonup u$ *weakly in* $W^{1,2}(0,L)$,

where A, $v^{(h)}$, $u^{(h)}$, and $w^{(h)}$ are defined as in (3.9), (3.1), (3.3) and (3.2). Moreover,

$$\limsup_{h\to 0} \frac{1}{h^{2\alpha}} \int_{\tilde{\Omega}_h} W\Big(\big(\Psi^{(h)}\big)^{-1}(x), \nabla \tilde{y}^{(h)}(x)\Big) dx \leq I_\alpha(u,v,w), \tag{3.64}$$

where I_α is defined in (3.33).

Proof. – As first step we assume to deal with more regular functions; more precisely, we require that $u, w \in C^1[0,L]$ and $v \in C^2([0,L];\mathbb{R}^3)$.

As in [41], let us define the functions $\gamma_2, \gamma_3, \kappa^{(h)} : [0,L] \to \mathbb{R}^3$ in the following way:

$$\gamma_2(s) := 2\,w\,(v'\cdot\nu_3)\,e_1 + \big(w^2 + (v'\cdot\nu_2)^2\big)\,e_2 + (v'\cdot\nu_2)\,(v'\cdot\nu_3)\,e_3, \tag{3.65}$$

$$\gamma_3(s) := -\,2\,w\,(v'\cdot\nu_2)\,e_1 + (v'\cdot\nu_2)\,(v'\cdot\nu_3)\,e_2 + \big(w^2 + (v'\cdot\nu_3)^2\big)\,e_3, \tag{3.66}$$

$$\kappa^{(h)}(s,\xi,\zeta) := (1 - h\,\xi\,k_2 - h\,\zeta\,k_3)\,\tau, \tag{3.67}$$

where k_2 and k_3 are the scalar functions defined in (2.1). Finally, let $\varphi \in C^1(\bar\Omega;\mathbb{R}^3)$ and let $\beta : \Omega \to \mathbb{R}^3$ be

$$\beta(s,\xi,\zeta) := \begin{cases} R_0(s)\varphi(s,\xi,\zeta) - \dfrac{1}{2}\,\xi\,R_0(s)\gamma_2(s) - \dfrac{1}{2}\,\zeta\,R_0(s)\gamma_3(s) & \text{if } \alpha = 3, \\ R_0(s)\varphi(s,\xi,\zeta) & \text{if } \alpha > 3. \end{cases} \tag{3.68}$$

For every $h > 0$ consider the function $Y^{(h)} : \Omega \to \mathbb{R}^3$ defined as

$$Y^{(h)} = \Psi^{(h)} + h^{\alpha-2}\,v + h^{\alpha-1}\,u\,\kappa^{(h)} + h^{\alpha-1}\xi\,A\,\nu_2 + h^{\alpha-1}\zeta\,A\,\nu_3 + h^{\alpha}\beta, \tag{3.69}$$

where the matrix A is defined as in (3.9).

Let us compute the scaled gradient of the deformation $Y^{(h)}$. First of all notice that $\nabla_h\kappa^{(h)} = \big(\tau' \,\big|\, -\big(\tau'\cdot\nu_2\big)\,\tau \,\big|\, -\big(\tau'\cdot\nu_3\big)\,\tau\big) + O(h)$, and that

$$\Big(\tau' \,\Big|\, -\big(\tau'\cdot\nu_2\big)\,\tau \,\Big|\, -\big(\tau'\cdot\nu_3\big)\,\tau\Big) = \big(\tau'\otimes\tau - \tau\otimes\tau'\big)\,R_0. \tag{3.70}$$

Hence, the scaled gradient turns out to be

$$\begin{aligned} \nabla_h Y^{(h)} = {} & \nabla_h\Psi^{(h)} + h^{\alpha-2}\,A\,R_0 + h^{\alpha-1}\left(\big(A\,R_0\big)'\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix} + u'\,\tau\,\Big|\,\partial_\xi\beta\,\Big|\,\partial_\zeta\beta\right) + \\ & + h^{\alpha-1}\,u\,\big(\tau'\otimes\tau - \tau\otimes\tau'\big)\,R_0 + O(h^{\alpha}). \end{aligned} \tag{3.71}$$

So we have that, by (2.7),

$$\frac{1}{h^{\alpha-2}}\Big(\nabla_h Y^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1} - Id\Big) = A + O(h)$$

and this proves (i). Now remark that, if we define $v^{(h)}$ as in (3.1), we have, using (2.3),

$$v^{(h)} = v + h\,u\,\tau + h^2\int_D \beta\,d\xi\,d\zeta,$$

so also (ii) follows. For the sequence $w^{(h)}$ defined as in (3.2) we get, by (2.2) and (2.3),

$$\begin{aligned} w^{(h)} &= \frac{1}{\mu(D)}\int_D (\xi\,A\,\nu_2 + \zeta\,A\,\nu_3 + h\,\beta)\cdot(\xi\,\nu_3 - \zeta\,\nu_2)\,d\xi\,d\zeta \\ &= (A\,\nu_2)\cdot\nu_3 + O(h), \end{aligned}$$

which is exactly w, up to a perturbation of order h. This proves (iii).

Moreover, if we define $u^{(h)}$ as in (3.3) we have

$$\big(u^{(h)}\big)' = \frac{1}{h}\int_D \big(v'\cdot\tau + h\,u' + h^2\partial_s\beta\cdot\tau\big)\,d\xi\,d\zeta = u' + h\int_D \partial_s\beta\cdot\tau\,d\xi\,d\zeta \tag{3.72}$$

hence the convergence property in (iv) is also proved.

Once all these properties are satisfied, we can show (3.64). Using (2.7) and (3.71) we have

$$Z^{(h)} := \nabla_h Y^{(h)}\big(\nabla_h \Psi^{(h)}\big)^{-1} = Id + h^{\alpha-2}\left(A + h\,u'\,\tau\otimes\tau + h\left(A'\,R_0\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix}\Bigg|\,\partial_\xi\beta\,\Bigg|\,\partial_\zeta\beta\right)R_0^T\right) + h^{\alpha-1}\,u\,\big(\tau'\otimes\tau - \tau\otimes\tau'\big) + O(h^\alpha). \tag{3.73}$$

Using the identity $(Id+B^T)(Id+B) = Id + 2\,\mathrm{sym}\,B + B^T B$, we obtain for the nonlinear strain

$$\big(Z^{(h)}\big)^T Z^{(h)} = Id + 2\,h^{\alpha-1}\,\mathrm{sym}\left(\left(A'\,R_0\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix}\Bigg|\,\partial_\xi\beta\,\Bigg|\,\partial_\zeta\beta\right)R_0^T\right) + 2\,h^{\alpha-1}\,u'\,\tau\otimes\tau + h^{2(\alpha-2)}A^TA + \sigma\big(h^{2(\alpha-2)}\big), \tag{3.74}$$

where $\sigma(h^\gamma)/h^\gamma \to 0$ uniformly as $h\to 0$.

Now, let us distinguish the cases $\alpha=3$ and $\alpha>3$.

Case $\alpha = 3$.

Notice that if we specify $\alpha=3$ in (3.74), all the terms are of the same order with respect to h, that is of order 2. Taking the square root we have that

$$\Big[\big(Z^{(h)}\big)^T Z^{(h)}\Big]^{1/2} = Id + h^2\tilde G + O(h^3), \tag{3.75}$$

where

$$\tilde G := u'\tau\otimes\tau + \mathrm{sym}\left(\left(A'\,R_0\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix}\Bigg|\,\partial_\xi\beta\,\Bigg|\,\partial_\zeta\beta\right)R_0^T\right) - \frac{A^2}{2}.$$

In order to write $\tilde G$ in a more useful way, notice that, by (3.9),

$$\tilde G = R_0\left[\mathrm{sym}\left(R_0^TA'\,R_0\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix} + \left(u' + \frac12\big((v'\cdot\nu_2)^2 + (v'\cdot\nu_3)^2\big)\right)e_1\,\Bigg|\,\partial_\xi(R_0^T\beta)\,\Bigg|\,\partial_\zeta(R_0^T\beta)\right)\right]R_0^T + \frac12\,R_0\begin{pmatrix}0 & w\,(v'\cdot\nu_3) & -w\,(v'\cdot\nu_2)\\ w\,(v'\cdot\nu_3) & w^2+(v'\cdot\nu_2)^2 & (v'\cdot\nu_2)\,(v'\cdot\nu_3)\\ -w\,(v'\cdot\nu_2) & (v'\cdot\nu_2)\,(v'\cdot\nu_3) & w^2+(v'\cdot\nu_3)^2\end{pmatrix}R_0^T. \tag{3.76}$$

We can rewrite (3.76) in terms of φ and B, using (3.56) and (3.68), as

$$\tilde G = \mathrm{sym}\left[R_0\left(\Big(B' + 2\,\mathrm{skw}\big(R_0^TR_0'B\big)\Big)\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix} + \left(u' + \frac12\big((v'\cdot\nu_2)^2 + (v'\cdot\nu_3)^2\big)\right)e_1\,\Bigg|\,\partial_\xi\varphi\,\Bigg|\,\partial_\zeta\varphi\right)R_0^T\right].$$

From the frame-indifference of the energy density W, since $\det\big(\nabla_hY^{(h)}\big)\big(\nabla_h\Psi^{(h)}\big)^{-1} > 0$ for sufficiently small h, we have

$$W\big(s,\xi,\zeta,Z^{(h)}\big) = W\Big(s,\xi,\zeta,\big[\big(Z^{(h)}\big)^TZ^{(h)}\big]^{1/2}\Big).$$

Thus, by (3.75) and Taylor expansion, we obtain

$$\frac{1}{h^4}\,W\big(s,\xi,\zeta,Z^{(h)}\big) \to \frac12\,Q_3(s,\xi,\zeta,\tilde G)\ \text{a.e.},$$

and

$$\frac{1}{h^4}\,W\big(s,\xi,\zeta,Z^{(h)}\big) \le \frac{1}{2}\,\gamma\,|\,\tilde G\,|^2 + C\,h \le C\,\big(|\,B\,|^4 + |\,B'|^2 + |\,\partial_\xi\varphi\,|^2 + |\,\partial_\zeta\varphi\,|^2 + |\,u'|^2 + 1\big) \in L^1(\Omega).$$

Set $\check y^{(h)} := Y^{(h)} \circ \big(\Psi^{(h)}\big)^{-1}$; by the dominated convergence theorem we get the following equality:

$$\limsup_{h\to 0} \frac{1}{h^6}\int_{\tilde\Omega_h} W\Big(\big(\Psi^{(h)}\big)^{-1}(x), \nabla \check y^{(h)}(x)\Big)dx = \frac{1}{2}\int_\Omega Q_3(s,\xi,\zeta,\tilde G)\,ds\,d\xi\,d\zeta. \tag{3.77}$$

Consider the general case. Let $u,w \in W^{1,2}(0,L)$ and $v \in W^{2,2}((0,L);\mathbb{R}^3)$. Let $\varphi(s,\cdot) \in \mathcal{V}$ be the solution of the minimum problem (3.31) defining Q^0, with $t := u' + \frac{1}{2}\big((v'\cdot\nu_2)^2 + (v'\cdot\nu_3)^2\big)$ and $F := B' + 2\,\mathrm{skw}\big(R_0^T R_0' B\big)$, where B is introduced in (3.35). As we have already noticed in Remark 3.2, φ and its derivatives with respect to ξ and ζ belong to $L^2(\Omega;\mathbb{R}^3)$.

Now, we can smoothly approximate u,w in the strong topology of $W^{1,2}$, v in the strong topology of $W^{2,2}$, and φ, $\partial_\xi\varphi$ and $\partial_\zeta\varphi$ in the strong topology of L^2. Since the approximating sequences satisfy (3.77), and the right-hand side of (3.77) is continuous with respect to the mentioned topologies, we conclude that (3.77) holds also in the general case. Hence, using the minimality of φ, we obtain (3.64).

Case $\alpha > 3$.
In this case, in the expression (3.74), the term of order $2(\alpha-2)$ in h can be neglected, since $2(\alpha-2) > \alpha-1$ when $\alpha > 3$. Hence we can write

$$\big(Z^{(h)}\big)^T Z^{(h)} = Id + 2\,h^{\alpha-1}\,\mathrm{sym}\left(\left(A'\,R_0\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix}\Bigg|\,\partial_\xi\beta\,\Bigg|\,\partial_\zeta\beta\right)R_0^T\right) + 2\,h^{\alpha-1}\,u'\,\tau\otimes\tau + \sigma(h^{\alpha-1}),$$

where $\sigma(h^\gamma)/h^\gamma \to 0$ uniformly as $h\to 0$. Taking the square root we have that

$$\Big[\big(Z^{(h)}\big)^T Z^{(h)}\Big]^{1/2} = Id + h^{\alpha-1}\tilde G + \sigma\big(h^{\alpha-1}\big), \tag{3.78}$$

where

$$\tilde G := u'\tau\otimes\tau + \mathrm{sym}\left(\left(A'\,R_0\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix}\Bigg|\,\partial_\xi\beta\,\Bigg|\,\partial_\zeta\beta\right)R_0^T\right).$$

We can rewrite $\tilde G$ in terms of φ and B as

$$\tilde G = \mathrm{sym}\left[R_0\left(\Big(B' + 2\,\mathrm{skw}\big(R_0^T R_0' B\big)\Big)\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix} + u'\,e_1\,\Bigg|\,\partial_\xi\varphi\,\Bigg|\,\partial_\zeta\,\varphi\right)R_0^T\right].$$

From the frame-indifference of the energy density W, since $\det\big(\nabla_h Y^{(h)}\big)\big(\nabla_h\Psi^{(h)}\big)^{-1} > 0$ for sufficiently small h, we have

$$W\big(s,\xi,\zeta,Z^{(h)}\big) = W\Big(s,\xi,\zeta,\big[\big(Z^{(h)}\big)^T Z^{(h)}\big]^{1/2}\Big);$$

thus, by (3.78) and Taylor expansion, we obtain

$$\frac{1}{h^{2\,\alpha-2}}\,W\big(s,\xi,\zeta,Z^{(h)}\big) \to \frac{1}{2}\,Q_3(s,\xi,\zeta,\tilde G)\ \text{a.e.},$$

and

$$\frac{1}{h^{2\,\alpha-2}}\,W\big(s,\xi,\zeta,Z^{(h)}\big) \le \frac{1}{2}\,\gamma\,|\,\tilde G\,|^2 + C\,h \le C\,\big(|\,B'|^2 + |\,B\,|^2 + |\,\partial_\xi\varphi\,|^2 + |\,\partial_\zeta\varphi\,|^2 + |\,u'|^2 + 1\big) \in L^1(\Omega).$$

Set $\check{y}^{(h)} := Y^{(h)} \circ \big(\Psi^{(h)}\big)^{-1}$; by the dominated convergence theorem we get the following equality:

$$\limsup_{h\to 0} \frac{1}{h^{2\,\alpha}} \int_{\widetilde{\Omega}_h} W\Big(\big(\Psi^{(h)}\big)^{-1}(x), \nabla \check{y}^{(h)}(x)\Big) dx = \frac{1}{2}\int_\Omega Q_3(s,\xi,\zeta,\tilde{G})\, ds\, d\xi\, d\zeta. \tag{3.79}$$

Consider the general case. Let $u, w \in W^{1,2}(0,L)$ and $v \in W^{2,2}((0,L);\mathbb{R}^3)$. Let $\varphi(s,\cdot) \in \mathcal{V}$ be the solution of the minimum problem (3.31) defining Q^0, with $t := u'$ and $F := B' + 2\,\mathrm{skw}\big(R_0^T R_0' B\big)$, where B is defined as in (3.35). It is easy to show that (3.79) remains true, following the same approximation arguments used in the proof of Theorem 3.7. Hence, using the minimality of φ, we obtain (3.64). □

3.3.2 Intermediate scaling

We now consider the scalings h^α with $2 < \alpha < 3$. As in [32], this case turns out to be very delicate and requires a detailed analysis.

Theorem 3.7 (Case $2<\alpha<3$**)** *For every* $w \in W^{1,2}(0,L)$ *and* $v \in W^{2,2}((0,L);\mathbb{R}^3)$ *such that* $v'\cdot\tau = 0$ *there exists a sequence* $\big(\check{y}^{(h)}\big) \subset W^{1,2}(\tilde{\Omega}_h;\mathbb{M}^{3\times 3})$ *such that, setting* $Y^{(h)} := \check{y}^{(h)} \circ \Psi^{(h)}$*, we have*

(i) $\Big(\big(\nabla_h Y^{(h)}\big)\big(\nabla_h \Psi^{(h)}\big)^{-1} - Id\Big)/h^{\alpha-2} \to A$ *strongly in* $L^2(\Omega;\mathbb{M}^{3\times3})$*;*

(ii) $v^{(h)} \to v$ *strongly in* $W^{1,2}((0,L);\mathbb{R}^3)$*;*

(iii) $w^{(h)} \rightharpoonup w$ *weakly in* $W^{1,2}(0,L)$*,*

with $A, v^{(h)}$ *and* $w^{(h)}$ *defined as in (3.9), (3.1) and (3.2). Moreover*

$$\limsup_{h\to 0} \frac{1}{h^{2\,\alpha}} \int_{\widetilde{\Omega}_h} W\Big(\big(\Psi^{(h)}\big)^{-1}(x), \nabla \check{y}^{(h)}(x)\Big) dx \le I_\alpha(v,w), \tag{3.80}$$

where I_α *is introduced in (3.34).*

Proof. – As in Theorem 2.6, we preliminarily assume that $w \in C^1[0,L]$ and $v \in C^2([0,L];\mathbb{R}^3)$. Let $g \in C^0[0,L]$ and $\varphi \in C^1(\bar{\Omega};\mathbb{R}^3)$. Denote by β the function $\beta(s,\xi,\zeta) := R_0(s)\varphi(s,\xi,\zeta)$ and by $\tilde{g}$ a primitive of the function g.

Define the functions $\gamma_2, \gamma_3, \kappa^{(h)}$ as in the proof of Theorem 2.6. Finally define the function $u \in C^1[0,L]$ as a primitive of

$$-\frac{1}{2}\Big((v'\cdot\nu_2)^2 + (v'\cdot\nu_3)^2\Big).$$

In analogy with the cases $\alpha \ge 3$, one could make the ansatz

$$\begin{aligned} Y^{(h)} = {} & \Psi^{(h)} + h^{\alpha-2} v + h^{\alpha-1}\xi\, A\,\nu_2 + h^{\alpha-1}\zeta\, A\,\nu_3 + \big(h^{2(\alpha-2)}u + h^{\alpha-1}\tilde{g}\big)\,\kappa^{(h)} + \\ & -\frac{1}{2}\, h^{(2\,\alpha-3)} R_0\big(\xi\,\gamma_2 + \zeta\,\gamma_3\big) + h^\alpha \beta. \end{aligned} \tag{3.81}$$

Hence, by (3.70) the scaled gradient of the deformation $Y^{(h)}$ is

$$\begin{aligned} \nabla_h Y^{(h)} = {} & \nabla_h \Psi^{(h)} + h^{\alpha-2}\, A\, R_0 + h^{\alpha-1}\left(\big(A\,R_0\big)' \begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} \Bigg| \partial_\xi \beta \Bigg| \partial_\zeta \beta \right) + h^{\alpha-1} g\,\tau\otimes e_1 + \\ & + \big(h^{2(\alpha-2)}u + h^{\alpha-1}\tilde{g}\big)\,\big(\tau'\otimes\tau - \tau\otimes\tau'\big)\, R_0 + \frac{1}{2}\, h^{2(\alpha-2)}\, R_0\,\big(2\,u'\,e_1 \,|\, -\gamma_2 \,|\, -\gamma_3\big) + \sigma(h^{\alpha-1}). \end{aligned} \tag{3.82}$$

Now, using (2.7) and (3.82) we have

$$Z^{(h)} := \nabla_h Y^{(h)} \big(\nabla_h \Psi^{(h)}\big)^{-1} = Id + h^{\alpha-2} A + h^{\alpha-1} \left(A' R_0 \begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} + g\,\tau \Big| \partial_\xi \beta \Big| \partial_\zeta \beta \right) R_0^T +$$
$$+ \big(h^{2(\alpha-2)} u + h^{\alpha-1} \tilde{g}\big) \big(\tau' \otimes \tau - \tau \otimes \tau'\big) + \frac{1}{2} h^{2(\alpha-2)} R_0 \big(2\,u'\,e_1 \,|\, -\gamma_2 \,|\, -\gamma_3\big) R_0^T + \sigma(h^{\alpha-1}). \tag{3.83}$$

This procedure leads to the desired conclusion for $\alpha > 5/2$, but our ansatz cannot work for α close to 2. Indeed, for $\alpha > 5/2$, using the identity $(Id + P^T)(Id + P) = Id + 2\,\mathrm{sym}\,P + P^T P$, and noticing that some of the matrices on the right-hand side of (3.83) are skew-symmetric, we obtain for the nonlinear strain

$$\big(Z^{(h)}\big)^T Z^{(h)} = Id + 2\,h^{\alpha-1}\,\mathrm{sym}\left(\left(A' R_0 \begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} \Big| \partial_\xi \beta \Big| \partial_\zeta \beta \right) R_0^T \right) + 2\,h^{\alpha-1}\, g\,\tau \otimes \tau$$
$$+ h^{2(\alpha-2)} R_0 \Big(\mathrm{sym}\,(2\,u'\,e_1 \,|\, -\gamma_2 \,|\, -\gamma_3) \Big) R_0^T + h^{2(\alpha-2)} A^T A + \sigma(h^{\alpha-1}). \tag{3.84}$$

Moreover, using (3.9) and our definition of u, γ_2 and γ_3, we have that

$$\Big[\big(Z^{(h)}\big)^T Z^{(h)}\Big]^{1/2} = Id + h^{\alpha-1} \tilde{G} + \sigma\big(h^{\alpha-1}\big), \tag{3.85}$$

where

$$\tilde{G} := g\,\tau \otimes \tau + \mathrm{sym}\left(\left(A' R_0 \begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} \Big| \partial_\xi \beta \Big| \partial_\zeta \beta \right) R_0^T \right).$$

As in Theorem 2.6, the frame-indifference of the energy density W and the dominated convergence theorem give the following equality:

$$\limsup_{h \to 0} \frac{1}{h^{2\alpha}} \int_{\tilde{\Omega}_h} W\Big(\big(\Psi^{(h)}\big)^{-1}(x), \nabla \check{y}^{(h)}(x) \Big) dx = \frac{1}{2} \int_\Omega Q_3(s, \xi, \zeta, \tilde{G})\, ds\, d\xi\, d\zeta, \tag{3.86}$$

and the general case can be proved by approximation. Then, using the minimality assumptions on g and φ, we obtain (3.80) and so the claim.

Unfortunately, this procedure fails for α close to 2, since in that case terms of order $h^{4(\alpha-2)}$ appear in the expression of the nonlinear strain $\big(Z^{(h)}\big)^T Z^{(h)}$, and they cannot be absorbed in $o(h^{\alpha-1})$.

Therefore, in the spirit of the proof of [32, Theorem 6.2], we modify the ansatz (3.81) in order to get an exact isometry. Let us define for every $h > 0$, the sequence

$$Y^{(h)} := \int_0^s (R_\varepsilon \tau)\, d\sigma + h\xi R_\varepsilon \nu_2 + h\zeta R_\varepsilon \nu_3 + h^\alpha \beta, \tag{3.87}$$

where $R_\varepsilon := e^{\varepsilon A}$, with A defined as in (3.9), and $\varepsilon := h^{\alpha-2}$. Notice that, due to the fact that A is skew-symmetric, the matrix R_ε turns out to be a rotation.

The scaled gradient of the deformation $Y^{(h)}$ is given by

$$\nabla_h Y^{(h)} = R_\varepsilon R_0 + h\, R_\varepsilon (\xi \nu_2' + \zeta \nu_3') \otimes e_1 + h \left(R_\varepsilon' R_0 \begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} \Big| h^{\alpha-2} \partial_\xi \beta \Big| h^{\alpha-2} \partial_\zeta \beta \right) + O(h^\alpha). \tag{3.88}$$

Now, using (2.5), the expression (3.88) becomes

$$\nabla_h Y^{(h)} = R_\varepsilon \nabla_h \Psi^{(h)} + h\left(R'_\varepsilon\, R_0 \begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} \Bigg| h^{\alpha-2}\partial_\xi \beta \Bigg| h^{\alpha-2}\partial_\zeta \beta \right) + O(h^\alpha),$$

and hence, by (2.7) we have

$$\begin{aligned} Z^{(h)} :=\ & \nabla_h Y^{(h)} (\nabla_h \Psi^{(h)})^{-1} = R_\varepsilon + h\left(R'_\varepsilon\, R_0 \begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} \Bigg| h^{\alpha-2}\partial_\xi \beta \Bigg| h^{\alpha-2}\partial_\zeta \beta \right) R_0^T + o(h^{\alpha-1}), \\ =\ & R_\varepsilon \left(Id + h\left(R_\varepsilon^T R'_\varepsilon\, R_0 \begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} \Bigg| h^{\alpha-2}\partial_\xi (R_\varepsilon^T \beta) \Bigg| h^{\alpha-2}\partial_\zeta (R_\varepsilon^T \beta) \right) R_0^T \right) + \sigma(h^{\alpha-1}). \end{aligned} \tag{3.89}$$

Now notice that, by definition, the rotation R_ε verifies the identities:

$$R_\varepsilon(s) = Id + \varepsilon\, A(s) + \sigma(\varepsilon), \quad R'_\varepsilon(s) = \varepsilon \int_0^1 e^{(1-\sigma)\varepsilon\, A(s)} A'(s)\, e^{\sigma\, \varepsilon\, A(s)} d\sigma.$$

Therefore we have in particular

$$R_\varepsilon^T R'_\varepsilon = \varepsilon\, A' + \sigma(\varepsilon). \tag{3.90}$$

Hence, using (3.90) and the fact that $\varepsilon = h^{\alpha-2}$, (3.89) simplifies as follows

$$Z^{(h)} =\ R_\varepsilon \left(Id + h^{\alpha-1}\left(A'\, R_0 \begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} \Bigg| \partial_\xi \beta \Bigg| \partial_\zeta \beta \right) R_0^T \right) + \sigma(h^{\alpha-1}).$$

Thus, using frame indifference, we obtain

$$\begin{aligned} \frac{1}{h^{2\,\alpha-2}}\, W\big(s,\xi,\zeta,Z^{(h)}\big) &= \frac{1}{h^{2\,\alpha-2}}\, W\Big(s,\xi,\zeta,\big(R_\varepsilon\big)^T Z^{(h)}\Big) \\ &\to \frac{1}{2}\, Q_3(s,\xi,\zeta,\tilde{G}) \text{ a.e.,} \end{aligned}$$

and proceeding as before we get the desired claim.

□

3.4 The case of a closed thin beam

It appears natural to ask whether the same analysis that we have developed so far can be extended to the case of a thin rod whose mid-fiber is a closed curve. In this section we will show that this additional requirement imposes a restriction on the class of admissible limit deformations, while the expression of the limiting functional is not affected by this constraint.

Throughout this section we will assume $\alpha = 3$ for simplicity, but the results can be easily extended to the other cases.

The setting of the problem is exactly the same as before. The additional assumptions are

$$\gamma(0) = \gamma(L), \gamma'(0) = \gamma'(L) \quad \text{and } \nu_k(0) = \nu_k(L), \text{for } k = 2,3. \tag{3.91}$$

Notice that, from (3.91) it easily follows that $\Psi^{(h)}(0,\xi,\zeta) = \Psi^{(h)}(L,\xi,\zeta)$ for every $(\xi,\zeta) \in D$.

Now we will state and prove a compactness result which allows to identify the domain of the Γ-limit.

Theorem 3.8 *Let* $\big(\tilde y^{(h)}\big) \subset W^{1,2}\big(\widetilde\Omega_h;\mathbb{R}^3\big)$ *be a sequence verifying*

$$\frac{1}{h^4}\,\tilde I^{(h)}(\tilde y^{(h)}) \le c < +\infty \tag{3.92}$$

for every $h > 0$. *Then there exist an associated sequence* $R^{(h)} \subset C^\infty((0,L);\mathbb{M}^{3\times 3})$ *and constants* $\bar R^{(h)} \in SO(3)$, $c^{(h)} \in \mathbb{R}^3$ *such that, if we define* $Y^{(h)} := \big(\bar R^{(h)}\big)^T\,\tilde y^{(h)}\circ\Psi^{(h)} - c^{(h)}$, *we have*

$$R^{(h)}(s) \in SO(3) \quad \text{for every } s \in (0,L), \tag{3.93}$$

$$\big|\big|R^{(h)} - Id\big|\big|_{L^\infty(0,L)} \le C\,h, \quad \big|\big|\big(R^{(h)}\big)'\big|\big|_{L^2(0,L)} < C\,h, \tag{3.94}$$

$$\big|\big|\nabla_h Y^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1} - R^{(h)}\big|\big|_{L^2(\Omega)} \le C\,h^2, \tag{3.95}$$

$$\big|R^{(h)}(0) - R^{(h)}(L)\big| \le c\,h^{3/2}. \tag{3.96}$$

Moreover, defining $v^{(h)}$, $w^{(h)}$ *and* $u^{(h)}$ *as in (3.1), (3.2) and (3.3), we have that, up to subsequences, the following properties are satisfied:*

(a) $v^{(h)} \to v$ *strongly in* $W^{1,2}((0,L);\mathbb{R}^3)$; *moreover,* $v \in W^{2,2}((0,L);\mathbb{R}^3)$, $v'\cdot\tau = 0$, $v(0) = v(L)$, *and* $v'(0) = v'(L)$;

(b) $w^{(h)} \rightharpoonup w$ *weakly in* $W^{1,2}(0,L)$, *with* $w(0) = w(L)$;

(c) $u^{(h)} \rightharpoonup u$ *weakly in* $W^{1,2}(0,L)$;

(d) $\big(\nabla_h Y^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1} - Id\big)/h \to A$ *strongly in* $L^2(\Omega;\mathbb{M}^{3\times 3})$, *where the matrix* $A \in W^{1,2}((0,L))$ *is defined in (3.9);*

(e) $\big(R^{(h)} - Id\big)/h \rightharpoonup A$ *weakly in* $W^{1,2}((0,L);\mathbb{M}^{3\times 3})$;

(f) $sym\big(R^{(h)} - Id\big)/h^2 \to A^2/2$ *uniformly on* $(0,L)$.

Proof. – The argument follows the proof of Proposition 4.1 in [42], but we will include the details for the convenience of the reader. As in the proof of Theorem 3.1, the rigidity theorem provides the existence of a sequence of piecewise constant rotations $Q^{(h)} : (0,L) \to SO(3)$ such that, for every small cylinder $\widetilde C_{a,h}$ we have

$$\int_{\widetilde C_{a,h}} \big|\,\nabla\tilde y^{(h)} - Q^{(h)}\big|^2 dx \le c\int_{\widetilde C_{a,h}} \operatorname{dist}^2(\nabla\tilde y^{(h)}, SO(3))dx.$$

Changing variables, the previous inequality becomes

$$\int_{\left(a,a+\frac{L}{K_h}\right)\times D} \big|\,\nabla\tilde y^{(h)}\circ\Psi^{(h)} - Q^{(h)}\big|^2 ds\,d\xi\,d\zeta \le c\int_{\left(a,a+\frac{L}{K_h}\right)\times D} \operatorname{dist}^2\big(\nabla\tilde y^{(h)}\circ\Psi^{(h)}, SO(3)\big)ds\,d\xi\,d\zeta. \tag{3.97}$$

Let us define $\bar Q := Q^{(h)}(0)$. If we specify the relation for $a = 0$ we have

$$\int_{\left(0,\frac{L}{K_h}\right)\times D} \big|\,\nabla\tilde y^{(h)}\circ\Psi^{(h)} - \bar Q\big|^2 ds\,d\xi\,d\zeta \le c\int_{\left(0,\frac{L}{K_h}\right)\times D} \operatorname{dist}^2(\nabla\tilde y^{(h)}\circ\Psi^{(h)}, SO(3))ds\,d\xi\,d\zeta. \tag{3.98}$$

(3.99)

In order to establish (3.96), we start from the trace inequality

$$\int_D |\, v(0,\xi,\zeta) - \bar{v}\,|^2 \, d\xi \, d\zeta \le c \int_{(0,l)\times D} |\, \nabla v\,|^2 \, ds \, d\xi \, d\zeta,$$

which holds uniformly for $1 \le l \le 2$, with $\bar{v} = \int_D v(0,\xi,\zeta)\, d\xi\, d\zeta$. If we write this estimate for

$$v(s,\xi,\zeta) := \frac{1}{h}\left(\tilde{y}^{(h)} \circ \Psi^{(h)}\right)(hs,\xi,\zeta) - \frac{1}{h}\,\bar{Q}\Psi^{(h)}(hs,\xi,\zeta),$$

we obtain the following relation:

$$\begin{aligned} &\int_D \Big|\left(\tilde{y}^{(h)} \circ \Psi^{(h)} - \bar{Q}\Psi^{(h)}\right)(0,\xi,\zeta) - \int_D \left(\tilde{y}^{(h)} \circ \Psi^{(h)} - \bar{Q}\Psi^{(h)}\right)(0,\xi,\zeta)\, d\xi\, d\zeta \Big|^2 d\xi\, d\zeta \\ &\le c\, h \int_{(0,lh)\times D} \big|\, \nabla_h\left(\tilde{y}^{(h)} \circ \Psi^{(h)}\right) - \bar{Q}\nabla_h \Psi^{(h)}\,\big|^2 ds\, d\xi\, d\zeta. \end{aligned} \tag{3.100}$$

Putting together (3.100) and (3.98) we have, after easy computations,

$$\int_D \Big|\left(\tilde{y}^{(h)} \circ \Psi^{(h)}\right)(0,\xi,\zeta) - \int_D \left(\tilde{y}^{(h)} \circ \Psi^{(h)}\right)(0,\xi,\zeta)\, d\xi\, d\zeta - h\,\bar{Q}(\xi\,\nu_2(0) + \zeta\,\nu_3(0))\Big|^2 d\xi\, d\zeta \le c\, h^5. \tag{3.101}$$

In a similar way, if we define $\bar{\bar{Q}} := Q^{(h)}(L)$, we deduce

$$\int_D \Big|\left(\tilde{y}^{(h)} \circ \Psi^{(h)}\right)(L,\xi,\zeta) - \int_D \left(\tilde{y}^{(h)} \circ \Psi^{(h)}\right)(L,\xi,\zeta)\, d\xi\, d\zeta - h\,\bar{\bar{Q}}(\xi\,\nu_2(L) + \zeta\,\nu_3(L))\Big|^2 d\xi\, d\zeta \le c\, h^5. \tag{3.102}$$

Now, subtracting (3.102) from (3.101) and taking into account (3.91), we obtain

$$\int_D \big|\left[\bar{Q} - \bar{\bar{Q}}\right](\xi\,\nu_2(0) + \zeta\,\nu_3(0))\big|^2 d\xi\, d\zeta \le c\, h^3,$$

which leads to

$$\big|Q^{(h)}(0) - Q^{(h)}(L)\big| \le c\, h^{3/2}. \tag{3.103}$$

If we define the sequences $\tilde{Q}^{(h)}$ and $R^{(h)}$ as in Theorem 3.1, it is easy to check that they also satisfy (3.103), hence (3.96) is proved. For the estimates (3.93), (3.94), and (3.95) we proceed exactly as in Theorem 3.1.

Let us define the sequences $v^{(h)}$, $w^{(h)}$ and $u^{(h)}$ as in (3.1), (3.2) and (3.3). The convergence properties follow from Theorem 3.1. It remains only to verify the boundary conditions for the limiting functions v and w. Since $\Psi^{(h)}(0,\xi,\zeta) = \Psi^{(h)}(L,\xi,\zeta)$ and $Y^{(h)}(0,\xi,\zeta) = Y^{(h)}(L,\xi,\zeta)$ for every $(\xi,\zeta) \in D$, we have by definition that $v^{(h)}(0) = v^{(h)}(L)$ and $w^{(h)}(0) = w^{(h)}(L)$. Hence we directly obtain that v and w satisfy

$$v(0) = v(L) \quad \text{and} \quad w(0) = w(L). \tag{3.104}$$

Now notice that, by definition,

$$\nabla_h Y^{(h)} \left(\nabla_h \Psi^{(h)}\right)^{-1} = \left(\bar{R}^{(h)}\right)^T \left(\nabla_h \tilde{y}^{(h)}\right) \circ \Psi^{(h)}. \tag{3.105}$$

Therefore, using (3.105) and the fact that $\Psi^{(h)}(0,\xi,\zeta) = \Psi^{(h)}(L,\xi,\zeta)$ for every $(\xi,\zeta) \in D$, we have in particular that

$$\frac{\left(\nabla_h Y^{(h)} \left(\nabla_h \Psi^{(h)}\right)^{-1} - Id\right)}{h}(0,\xi,\zeta) = \frac{\left(\nabla_h Y^{(h)} \left(\nabla_h \Psi^{(h)}\right)^{-1} - Id\right)}{h}(L,\xi,\zeta)$$

for every $(\xi,\zeta)\in D$. The last relation, together with property (d), implies that $A(0)=A(L)$. Hence $v'(0)=v'(L)$ and so the proof is concluded. $\square$

Now we are in a position to prove the Γ-convergence of the sequence $\big(\tilde I^{(h)}\big)/h^4$. As we have already noticed, the limit functional has the same expression as in (3.33), but the class of deformations on which it is finite includes the boundary conditions. More precisely we have the following convergence result.

Theorem 3.9 *(1) Let $u,w\in W^{1,2}(0,L)$ and let $v\in W^{2,2}((0,L);\mathbb{R}^3)$ be such that $v'\cdot\tau=0$. Assume also that v and w satisfy the boundary conditions (3.104). Then, for every positive sequence (h_j) converging to zero and every sequence $\big(\tilde y^{(h_j)}\big)\subset W^{1,2}(\widetilde\Omega_{h_j};\mathbb{R}^3)$ such that the sequence $Y^{(h_j)}:=\tilde y^{(h_j)}\circ\Psi^{(h_j)}$ satisfies the properties (a)-(d) of Theorem 3.8, it turns out that*

$$\liminf_{j\to\infty}\frac{1}{h_j^6}\int_{\widetilde\Omega_{h_j}}W\Big(\big(\Psi^{(h_j)}\big)^{-1}(x),\nabla\tilde y^{(h_j)}(x)\Big)dx\geq I_3^0(u,v,w), \tag{3.106}$$

where I_3^0 is defined in (3.33).

(2) For every sequence of positive (h_j) converging to 0 and for every $u,w\in W^{1,2}(0,L)$ and $v\in W^{2,2}((0,L);\mathbb{R}^3)$ satisfying the boundary conditions and such that $v'\cdot\tau=0$, there exists a sequence $\big(\check y^{(h_j)}\big)\subset W^{1,2}(\tilde\Omega_{h_j};\mathbb{M}^{3\times3})$ such that, setting $Y^{(h_j)}:=\check y^{(h_j)}\circ\Psi^{(h_j)}$, we have

(i) $\big(\nabla_{h_j}Y^{(h_j)}\big(\nabla_{h_j}\Psi^{(h_j)}\big)^{-1}-Id\big)/h\to A$ *strongly in* $L^2(\Omega;\mathbb{M}^{3\times3})$*;*

(ii) $v^{(h_j)}\to v$ *strongly in* $W^{1,2}((0,L);\mathbb{R}^3)$*;*

(iii) $w^{(h_j)}\rightharpoonup w$ *weakly in* $W^{1,2}(0,L)$*;*

(iv) $u^{(h_j)}\rightharpoonup u$ *weakly in* $W^{1,2}(0,L)$*,*

where A, $v^{(h_j)}$, $w^{(h_j)}$, and $u^{(h_j)}$ are defined as in (3.9), (3.1), (3.2) and (3.3). Moreover,

$$\limsup_{j\to\infty}\frac{1}{h_j^6}\int_{\widetilde\Omega_{h_j}}W\Big(\big(\Psi^{(h_j)}\big)^{-1}(x),\nabla\check y^{(h_j)}(x)\Big)dx\leq I_3^0(u,v,w), \tag{3.107}$$

where I_3^0 is defined in (3.33).

Proof. – (1) The proof of this part can be done repeating exactly the proof of Theorem 3.5.
(2) As in Theorem 3.7, we preliminarly assume that $u,w\in C^1[0,L]$ and $v\in C^2([0,L];\mathbb{R}^3)$. Let $\varphi\in C^1(\bar\Omega;\mathbb{R}^3)$ and define $\beta:\Omega\to\mathbb{R}^3$ as $\beta(s,\xi,\zeta):=R_0(s)\varphi(s,\xi,\zeta)$.
Let γ_2, γ_3 and $\kappa^{(h)}$ be as in the proof of Theorem 3.7. For every $h>0$ let us consider a function $\vartheta^{(h)}\in C^1[0,L]$ supported in $[L-\sqrt{h},L]$, such that $\vartheta^{(h)}(L)=1$ and $\big|\big(\vartheta^{(h)}\big)'\big|\leq\frac{c}{\sqrt{h}}$. Then let us define the function $Y^{(h)}:\Omega\to\mathbb{R}^3$ as

$$\begin{aligned}Y^{(h)}=\;&\Psi^{(h)}+h\,v+h^2\,u\,\kappa^{(h)}+h^2\,\frac{u(L)-u(0)}{L}\left(\int_0^s(L-\sigma)\,\tau'(\sigma)\,d\sigma-h\,(L-s)\,(\xi\,k_2+\zeta\,k_3)\,\tau\right)\\&+h^2\xi\,A\,\nu_2+h^2\zeta\,A\,\nu_3+h^3\beta^{(h)},\end{aligned}$$

where $\beta^{(h)}(s,\xi,\zeta):=\beta(s,\xi,\zeta)+\vartheta^{(h)}(s)(\beta(0,\xi,\zeta)-\beta(L,\xi,\zeta))$. It turns out that the function $Y^{(h)}$ satisfies periodic boundary conditions in $(0,L)$. Indeed,

$$\begin{aligned}Y^{(h)}(0,\xi,\zeta)=\Psi^{(h)}(0,\xi,\zeta)+h\,v(0)+h^2\,u(0)\,\tau(0)-h^3u(L)\,\tau(0)(\xi\,k_2(0)+\zeta\,k_3(0))+\\+h^2\xi\,A(0)\,\nu_2(0)+h^2\zeta\,A(0)\,\nu_3(0)+h^3\beta(0,\xi,\zeta),\end{aligned} \tag{3.108}$$

and, using the assumptions (3.91) and (3.104), we have

$$\begin{aligned}Y^{(h)}(L,\xi,\zeta) =\ & \Psi^{(h)}(0,\xi,\zeta) + h\,v(0) + h^2\,u(L)\,\tau(0)\,(1 - h\,\xi\,k_2(0) - h\,\zeta\,k_3(0)) + \\ & + h^2\,\frac{u(L)-u(0)}{L}\int_0^L (L-\sigma)\,\tau'(\sigma)\,d\sigma + h^2\xi\,A(0)\,\nu_2(0) + h^2\zeta\,A(0)\,\nu_3(0) + h^3\beta(0,\xi,\zeta).\end{aligned} \tag{3.109}$$

Now notice that, using $\tau = \gamma'$ and $\gamma(0) = \gamma(L)$, we have

$$\int_0^L (L-\sigma)\,\tau'(\sigma)\,d\sigma = -L\,\tau(0) + \int_0^L \tau(\sigma)\,d\sigma = -L\,\tau(0).$$

Plugging this equality into (3.109) we obtain

$$\begin{aligned}Y^{(h)}(L,\xi,\zeta) =\ & \Psi^{(h)}(0,\xi,\zeta) + h\,v(0) + h^2\,u(L)\,\tau(0)\,(1 - h\,\xi\,k_2(0) - h\,\zeta\,k_3(0)) + \\ & - h^2\,(u(L)-u(0))\,\tau(0) + h^2\xi\,A(0)\,\nu_2(0) + h^2\zeta\,A(0)\,\nu_3(0) + h^3\beta(0,\xi,\zeta)\end{aligned} \tag{3.110}$$

which is the same expression in (3.108).

Moreover, the convergence properties (i)-(iv) can be deduced as in Theorem 3.6. For the scaled gradient of $Y^{(h)}$ we have, using (3.70)

$$\begin{aligned}\nabla_h Y^{(h)} =\ & \nabla_h \Psi^{(h)} + h\,A\,R_0 + h^2\left(\left(A\,R_0\right)'\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix} + u'\,\tau\,\Big|\,\partial_\xi \beta^{(h)}\,\Big|\,\partial_\zeta \beta^{(h)}\right) + \\ & + h^2\left(u + \frac{u(L)-u(0)}{L}\,(L-s)\right)\left(\tau'\otimes\tau - \tau\otimes\tau'\right)R_0 + h^3\partial_s\beta^{(h)}\otimes e_1 + O(h^3),\end{aligned}$$

where $\left|\partial_s\beta^{(h)}\right| \le \frac{c}{\sqrt{h}}$. Now, since $\beta^{(h)} \to \beta$, $\partial_\xi\beta^{(h)} \to \partial_\xi\beta$ and $\partial_\zeta\beta^{(h)} \to \partial_\zeta\beta$ strongly in $L^2(\Omega;\mathbb{R}^3)$, one can prove a convergence result like (3.77) and obtain the general case by approximation. □

Part II

Homogenization results for composite materials and porous media

Chapter 4

Damage as Γ-limit of microfractures in anti-planar linear elasticity

In the second part of the thesis we state some homogenization results for functionals describing the elastic energy of a body where a fracture can occur. Hence the natural setting of the problems we treat is the space of special functions with bounded variation (in the scalar case) or the space of special functions with bounded deformation (in the vectorial case).

In this chapter we consider the case of an unbreakable elastic material presenting disjoint brittle inclusions arranged in a periodic way. In other words, we assume that cracks can appear and grow only in a prescribed disconnected region of the material, composed of a large number of small components with small toughness.

4.1 Formulation of the problem

Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. In the following we will denote by Q the unit cube $(0,1)^n$ and by Q_ϱ the inner cube $(\varrho, 1-\varrho)^n$, for some $\varrho \in (0,1)$.
Let $\delta > 0$ and $E, F \subset Q_\delta$ be defined in the following way:

- E is a finite union of disjoint sets given by the closure of domains with Lipschitz boundary;
- F is a finite union of disjoint closed $(n-1)$-dimensional smooth manifolds.

Assume also that E and F are disjoint.

For every $\varepsilon > 0$ let us consider the periodic structure in $\mathbb{R}^n$ generated by an ε-homothetic of the basic cell Q. For notational brevity we will use the superscript ε to denote the ε-homothetic of any domain. In particular, $Q^\varepsilon := \varepsilon\, Q$.

Let us write the domain Ω as union of cubes of side ε:

$$\Omega = \left(\bigcup_{h \in \mathbb{Z}_\varepsilon} (Q+h)^\varepsilon \right) \cup R(\varepsilon),$$

where $\mathbb{Z}_\varepsilon := \{h \in \mathbb{Z}^n : (Q+h)^\varepsilon \subset \Omega\}$, and $R(\varepsilon)$ is the remaining part of Ω. Notice that $\mathcal{L}^n(R(\varepsilon))$ is of order ε. Let $N(\varepsilon)$ be the cardinality of the set $\mathbb{Z}_\varepsilon$; notice that $N(\varepsilon)$ is of order $1/\varepsilon^n$.

We denote by $\{Q_k^\varepsilon\}_{k=1,\dots,N(\varepsilon)}$ an enumeration of the family of cubes $(Q+h)^\varepsilon$ covering Ω, so that we can rewrite Ω as

$$\Omega = \left(\bigcup_{k=1}^{N(\varepsilon)} Q_k^\varepsilon \right) \cup R(\varepsilon). \tag{4.1}$$

In the same way we can define the sets $E_k^\varepsilon, F_k^\varepsilon \subset Q_k^\varepsilon$ and then $\tilde{E}^\varepsilon, \tilde{F}^\varepsilon \subset \Omega$ as

$$\tilde{E}^\varepsilon := \bigcup_{k=1}^{N(\varepsilon)} E_k^\varepsilon, \quad \tilde{F}^\varepsilon := \bigcup_{k=1}^{N(\varepsilon)} F_k^\varepsilon. \tag{4.2}$$

The starting point of the problem is the energy associated to a function $u \in SBV^2(\Omega)$, that is

$$\mathcal{F}^\varepsilon(u) = \int_\Omega |\nabla u|^2 dx + \int_{S_u} f_\alpha\Big(\frac{x}{\varepsilon}\Big)\, d\mathcal{H}^{n-1}(x),$$

where $f_\alpha : \mathbb{R}^n \to [0,+\infty]$ is a Q-periodic function defined as

$$f_\alpha(y) = \begin{cases} \alpha & \text{in } E \cup F, \\ +\infty & \text{otherwise in } Q, \end{cases}$$

and α is a positive parameter. Clearly, being f_α Q-periodic, the function

$$x \mapsto f_\alpha\Big(\frac{x}{\varepsilon}\Big)$$

turns out to be Q^ε-periodic.

We are interested in the case in which δ is fixed and independent of ε, while $\alpha = \alpha_\varepsilon$ depends on ε and goes to zero as $\varepsilon \to 0$.

We will study three different cases, i.e.,

1. Subcritical regime $\quad \dfrac{\alpha_\varepsilon}{\varepsilon} \to 0 \quad$ as $\varepsilon \to 0$,
2. Supercritical regime $\quad \dfrac{\alpha_\varepsilon}{\varepsilon} \to +\infty \quad$ as $\varepsilon \to 0$,
3. Critical regime $\quad \dfrac{\alpha_\varepsilon}{\varepsilon} \to c \in (0,+\infty) \quad$ as $\varepsilon \to 0$.

4.2 Subcritical regime: very brittle inclusions

In this section we assume $\alpha_\varepsilon << \varepsilon$ in the expression of the energy $\mathcal{F}^\varepsilon$.

We define the functional $\mathcal{F}^0 : L^2(\Omega) \to [0,+\infty]$ as

$$\mathcal{F}^0(u) = \begin{cases} \displaystyle\int_\Omega f_0(\nabla u)\, dx & \text{if } u \in H^1(\Omega), \\ +\infty & \text{otherwise in } L^2(\Omega), \end{cases} \tag{4.3}$$

where f_0 solves the cell problem

$$f_0(\xi) = \min\left\{ \int_{Q\setminus E} |\xi + \nabla w(y)|^2 dy : w \in H^1_\#(Q \setminus (E \cup F)) \right\}. \tag{4.4}$$

The functional $\mathcal{F}^0$ will turn out to be the Γ-limit of the sequence $(\mathcal{F}^\varepsilon)$ in this case, that is for $\alpha_\varepsilon << \varepsilon$. It is convenient to introduce the auxiliary functionals $\mathcal{G}^\varepsilon : L^2(\Omega) \to [0, +\infty]$ defined by

$$\mathcal{G}^\varepsilon(v) = \begin{cases} \displaystyle\int_\Omega a\Big(\frac{x}{\varepsilon}\Big)|\nabla v|^2 dx & \text{if } v \in H^1(\Omega \setminus \tilde{F}^\varepsilon), \\ +\infty & \text{otherwise in } L^2(\Omega), \end{cases} \tag{4.5}$$

where a is a Q-periodic function given by

$$a(y) = \begin{cases} 0 & \text{in } E, \\ 1 & \text{in } Q \setminus E. \end{cases}$$

As a preliminary result, we show that $\mathcal{G}^\varepsilon$ Γ-converges to $\mathcal{F}^0$ with respect to the strong topology of L^2.

Theorem 4.1 *The sequence of functionals* $(\mathcal{G}^\varepsilon)$ *Γ-converges to* $\mathcal{F}^0$ *with respect to the strong topology of* L^2.

Proof. – Let $\eta > 0$ and let F_η be a neighbourhood of F with Lipschitz boundary such that $\operatorname{dist}(F_\eta, F) \le \eta$ and $\operatorname{dist}(F_\eta, E) > 0$. Now we define the functionals $\mathcal{G}^\varepsilon_\eta : L^2(\Omega) \to [0, +\infty]$ as

$$\mathcal{G}^\varepsilon_\eta(v) = \begin{cases} \displaystyle\int_\Omega a_\eta\Big(\frac{x}{\varepsilon}\Big)|\nabla v|^2 dx & \text{if } v \in H^1(\Omega) \\ +\infty & \text{otherwise in } L^2(\Omega), \end{cases} \tag{4.6}$$

where a_η is a Q-periodic function given by

$$a_\eta(y) = \begin{cases} 0 & \text{if } y \in E \cup F_\eta, \\ 1 & \text{otherwise in } Q. \end{cases}$$

From the standard theory for non-coercive convex homogenization (see e.g. [9] and [12]), we know that

$$\Gamma(L^2) - \lim_{\varepsilon \to 0} \mathcal{G}^\varepsilon_\eta = \mathcal{G}_\eta, \tag{4.7}$$

where the functional $\mathcal{G}_\eta : L^2(\Omega) \to [0, +\infty]$ is defined as

$$\mathcal{G}_\eta(v) = \begin{cases} \displaystyle\int_\Omega f_\eta(\nabla v)\, dx & \text{if } v \in H^1(\Omega), \\ +\infty & \text{otherwise in } L^2(\Omega), \end{cases}$$

and f_η solves for every $\xi \in \mathbb{R}^n$ the cell problem

$$\begin{aligned} f_\eta(\xi) &= \min\Big\{ \int_{Q \setminus (E \cup F_\eta)} |\xi + \nabla w(y)|^2 dy : w \in H^1_\#(Q \setminus (E \cup F_\eta)) \Big\} \\ &= \min\Big\{ \int_{Q \setminus (E \cup F_\eta)} |\xi + \nabla w(y)|^2 dy : w \in H^1_\#(Q) \Big\}. \end{aligned}$$

Notice that the last equality is due to classical extension theorems (see, for instance, [3]).

Comparison between $\mathcal{G}^\varepsilon$ *and* $\mathcal{G}^\varepsilon_\eta$. Let v^ε be a sequence having equibounded energies $\mathcal{G}^\varepsilon$ and such that v^ε converges strongly to some v in L^2. Then we claim that $v \in H^1(\Omega)$ and that

$$\liminf_{\varepsilon \to 0} \mathcal{G}^\varepsilon(v^\varepsilon) \ge \mathcal{G}_\eta(v). \tag{4.8}$$

By the fact that $\mathcal{G}^\varepsilon(v^\varepsilon)$ are bounded we deduce in particular that the $H^1\big(\Omega \setminus \big(\tilde{E}^\varepsilon \cup \tilde{F}^\varepsilon_\eta\big)\big)$ norm of v^ε is equibounded.

Therefore, Theorem 1.17 ensures that for every $\varepsilon > 0$ there exists an extension of v^ε, that is a function $\tilde{v}^\varepsilon_\eta$ such that

$$\tilde{v}^\varepsilon_\eta = v^\varepsilon \quad \text{in } \Omega \setminus \big(\tilde{E}^\varepsilon \cup \tilde{F}^\varepsilon_\eta\big), \tag{4.9}$$

with the property that the $H^1(\Omega)$ norm of the sequence $(\tilde{v}^\varepsilon_\eta)$ is equibounded. Hence there exists a function $v^* \in H^1(\Omega)$ such that

$$\tilde{v}^\varepsilon_\eta \rightharpoonup v^* \quad \text{weakly in } H^1(\Omega) \quad \text{as } \varepsilon \to 0,$$

hence strongly in $L^2(\Omega)$. Let χ_η be the characteristic function of the set $Q \setminus (E \cup F_\eta)$, extended by periodicity with period Q; then we have that

$$\chi^\varepsilon_\eta := \chi_\eta\Big(\frac{\cdot}{\varepsilon}\Big) \rightharpoonup \frac{\mathcal{L}^n(Q \setminus (E \cup F_\eta))}{\mathcal{L}^n(Q)} =: \vartheta > 0 \quad \text{weakly}^* \text{ in } L^\infty(\Omega).$$

Using the relation (4.9) we get, when $\varepsilon \to 0$,

$$0 = \int_\Omega \big(\tilde{v}^\varepsilon_\eta - v^\varepsilon\big)\chi^\varepsilon_\eta \, dx \to \int_\Omega \vartheta\,(v^* - v)\, dx,$$

which entails $v = v^*$, being $\vartheta > 0$. Hence $v \in H^1(\Omega)$.
Moreover, the extension we have built allows us to write the estimate

$$\mathcal{G}^\varepsilon(v^\varepsilon) \geq \mathcal{G}^\varepsilon_\eta(\tilde{v}^\varepsilon_\eta), \tag{4.10}$$

and in virtue of the result (4.7) we get (4.8).

It remains to show that on $H^1(\Omega)$ the Γ-limit of the sequence $\big(\mathcal{G}^\varepsilon\big)$ is given by $\mathcal{F}^0$, where $\mathcal{F}^0$ is defined by (4.3) and (4.4).

Liminf inequality.
Let $v \in H^1(\Omega)$ and let (v^ε) be a sequence having equibounded energy $\mathcal{G}^\varepsilon$, such that v^ε converges to v strongly in L^2. Then (4.8) holds for every $\eta > 0$.

Since f_η converges increasingly to f_0, then $f_0 = \sup_\eta f_\eta = \lim_{\eta\to 0} f_\eta$. Hence

$$\sup_\eta \mathcal{G}_\eta = \mathcal{F}^0,$$

and then from (4.8) we get the bound

$$\liminf_{\varepsilon\to 0} \mathcal{G}^\varepsilon(v^\varepsilon) \geq \mathcal{F}^0(v).$$

Limsup inequality. Let $\xi \in \mathbb{R}^n$ and let us define $v_\xi(x) := \xi \cdot x$. Let w be the solution of the minimum problem defining $f_0(\xi)$, that is, $w \in H^1_\#(Q \setminus (E \cup F))$, and

$$f_0(\xi) = \int_{Q\setminus E} |\xi + \nabla w|^2 dx.$$

Let $\tilde{w}$ be the periodic extension of w to $\mathbb{R}^n$ and let us define the sequence $v^\varepsilon := v_\xi + \varepsilon\, \tilde{w}\Big(\dfrac{x}{\varepsilon}\Big)$; clearly it converges to v_ξ strongly in L^2. Moreover

$$\begin{aligned}\mathcal{G}^\varepsilon(v^\varepsilon) &= \int_\Omega a\Big(\frac{x}{\varepsilon}\Big)|\nabla v^\varepsilon|^2 dx = \varepsilon^n \int_{\Omega/\varepsilon} a(x)\,|\xi + \nabla\tilde{w}|^2 dx = \mathcal{L}^n(\Omega)\int_Q a(x)\,|\xi + \nabla w|^2 dx + o(\varepsilon)\\ &= \mathcal{L}^n(\Omega)\int_{Q\setminus E} |\xi + \nabla w|^2 dx + o(\varepsilon) = \mathcal{L}^n(\Omega)\, f_0(\xi) + o(\varepsilon) = \mathcal{F}^0(v_\xi) + o(\varepsilon),\end{aligned}$$

where $o(\varepsilon)$ is a small error that disappears when $\varepsilon \to 0$ and which is due to the fact that in general Ω/ε is not given by an exact number of unit cubes.

We have therefore proved the existence of a recovery sequence for affine functions. We can extend the result to piecewise affine continuous functions, thanks to the local character of $\mathcal{G}^\varepsilon$. Then, using the density in $H^1(\Omega)$ of the piecewise affine continuous functions and the continuity of $\mathcal{F}^0$ on $H^1(\Omega)$, we get the claim in the general case. □

Remark 4.2 From the previous result we deduce immediately that f_0 is a quadratic form, being $\mathcal{F}^0$ the Γ-limit of the quadratic forms $\mathcal{G}^\varepsilon$. Hence there exists a matrix $A_0 \in \mathbb{R}^{n\times n}$ with constant coefficients such that

$$f_0(\xi) = A_0\xi \cdot \xi \quad \text{for every } \xi \in \mathbb{R}^n. \tag{4.11}$$

Now we can prove the Γ-convergence result for the sequence $\mathcal{F}^\varepsilon$.

Theorem 4.3 (Bound from below) *Let $u \in L^2(\Omega)$ and let (u^ε) be a sequence with equibounded energy $\mathcal{F}^\varepsilon$ such that $u^\varepsilon \to u$ strongly in L^2. Then $u \in H^1(\Omega)$ and*

$$\liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) \geq \mathcal{F}^0(u). \tag{4.12}$$

Proof. – Let $u \in L^2(\Omega)$ and let (u^ε) be a sequence converging to u strongly in $L^2(\Omega)$ and such that $\mathcal{F}^\varepsilon(u^\varepsilon) \leq c < +\infty$. From the definition of the functional this implies in particular that the $H^1\big(\Omega \setminus \big(\tilde{E}^\varepsilon \cup \tilde{F}^\varepsilon\big)\big)$ norm of (u^ε) is equibounded.

By Theorem 1.17 it is possible to extend every u^ε to a new function $\tilde{u}^\varepsilon$ in such a way that the resulting sequence $(\tilde{u}^\varepsilon)$ has $H^1(\Omega \setminus \tilde{F}^\varepsilon)$ norm equibounded. We claim that $\tilde{u}^\varepsilon \to u$ strongly in $L^2(\Omega)$.

As first step, fix $\eta > 0$ and define for every $\varepsilon > 0$ an extension $\tilde{u}^\varepsilon_\eta$ of $\tilde{u}^\varepsilon$ to the whole Ω, which coincides with $\tilde{u}^\varepsilon$ out of an η-neighborhood of $\tilde{F}^\varepsilon$. As in Theorem 4.1 it turns out that the $H^1(\Omega)$-norm of the sequence $(\tilde{u}^\varepsilon_\eta)$ is equibounded, and that $\tilde{u}^\varepsilon_\eta \rightharpoonup u$ weakly in H^1. This proves in particular that $u \in H^1(\Omega)$. Moreover,

$$\begin{aligned}\int_\Omega |\,\tilde{u}^\varepsilon - u\,|^2 dx &= \int_{\Omega\setminus\tilde{E}^\varepsilon} |\,\tilde{u}^\varepsilon - u\,|^2 dx + \int_{\tilde{E}^\varepsilon} |\,\tilde{u}^\varepsilon - u\,|^2 dx \\ &= \int_{\Omega\setminus\tilde{E}^\varepsilon} |\,u^\varepsilon - u\,|^2 dx + \int_{\tilde{E}^\varepsilon} |\,\tilde{u}^\varepsilon_\eta - u\,|^2 dx \\ &\leq \int_\Omega |\,u^\varepsilon - u\,|^2 dx + \int_\Omega |\,\tilde{u}^\varepsilon_\eta - u\,|^2 dx,\end{aligned} \tag{4.13}$$

and since the right-hand side in (4.13) converges to zero as $\varepsilon \to 0$, we can conclude that

$$\tilde{u}^\varepsilon \to u \quad \text{strongly in } L^2(\Omega).$$

Using the sequence $\tilde{u}^\varepsilon$ we can write

$$\mathcal{F}^\varepsilon(u^\varepsilon) \geq \mathcal{G}^\varepsilon(\tilde{u}^\varepsilon), \tag{4.14}$$

where the functional $\mathcal{G}^\varepsilon$ is defined as in (4.5). Hence by Theorem 4.1 we obtain (4.12). □

Remark 4.4 We underline that the bound (4.12) holds true independently of the rate of convergence of α_ε and implies in particular that the Γ-limit of $\mathcal{F}^\varepsilon$ is finite only in $H^1(\Omega)$.

Theorem 4.5 (Bound from above) *For every* $u \in H^1(\Omega)$ *there exists a sequence* $(u^\varepsilon) \subset SBV^2(\Omega)$, *with* $S_u \subset \tilde{E}^\varepsilon \cup \tilde{F}^\varepsilon$, *such that*

$$(i) \quad u^\varepsilon \to u \quad \textit{strongly in } L^2(\Omega), \tag{4.15}$$

$$(ii) \quad \lim_{\varepsilon \to 0} \mathcal{F}^\varepsilon(u^\varepsilon) = \mathcal{F}^0(u). \tag{4.16}$$

Proof. – Let $u \in H^1(\Omega)$. The Γ-convergence result in Theorem 4.1 guarantees the existence of a sequence $(v^\varepsilon) \subset L^2(\Omega)$ such that

$$\begin{cases} v^\varepsilon \to u \quad \text{strongly in } L^2(\Omega), \\ \mathcal{G}^\varepsilon(v^\varepsilon) \to \mathcal{F}^0(u). \end{cases}$$

A recovery sequence for $\mathcal{F}^\varepsilon$ will be constructed by modifying properly (v^ε).
Notice that, by the definition of $\mathcal{G}^\varepsilon$, it turns out that the $H^1\big(\Omega \setminus \big(\tilde{E}^\varepsilon \cup \tilde{F}^\varepsilon\big)\big)$ norm of v^ε is equibounded. We split the proof into three steps.

First step. There exists a sequence $\big(\tilde{v}^\varepsilon\big) \subset H^1(\Omega \setminus \tilde{F}^\varepsilon)$ such that

$$(1) \quad \tilde{v}^\varepsilon = v^\varepsilon \quad \text{in } \Omega \setminus \big(\tilde{E}^\varepsilon \cup \tilde{F}^\varepsilon\big), \tag{4.17}$$

$$(2) \quad ||\,\tilde{v}^\varepsilon||_{H^1(\Omega\setminus\tilde{F}^\varepsilon)} \le c\,||\,v^\varepsilon||_{H^1(\Omega\setminus(\tilde{E}^\varepsilon\cup\tilde{F}^\varepsilon))}, \tag{4.18}$$

where the constant c is independent of ε. This can be done exactly as in Theorem 4.3.

Second step. The sequence $\big(\tilde{v}^\varepsilon\big) \subset H^1(\Omega \setminus \tilde{F}^\varepsilon)$ of the previous step is still a recovery sequence for $\mathcal{G}^\varepsilon$, i.e.,

$$(3) \quad \tilde{v}^\varepsilon \to u \quad \text{strongly in } L^2(\Omega), \tag{4.19}$$

$$(4) \quad \mathcal{G}^\varepsilon(\tilde{v}^\varepsilon) \to \mathcal{F}^0(u). \tag{4.20}$$

Property (3) can be proved as in Theorem 4.3 while condition (4) follows immediately, since $\mathcal{G}^\varepsilon$ depends only on the behaviour of its argument in $\Omega \setminus \tilde{E}^\varepsilon$ and v^ε and $\tilde{v}^\varepsilon$ agree on that set.

Third step. There exists a sequence $(u^\varepsilon) \subset SBV^2(\Omega)$ with $S_{u^\varepsilon} \subset \tilde{E}^\varepsilon \cup \tilde{F}^\varepsilon$ such that

$$(\bar{i}) \quad ||u^\varepsilon - \tilde{v}^\varepsilon||_{L^2(\Omega)} = o(\varepsilon), \tag{4.21}$$

$$(\overline{ii}) \quad \mathcal{F}^\varepsilon(u^\varepsilon) = \mathcal{G}^\varepsilon(\tilde{v}^\varepsilon) + o(\varepsilon) \tag{4.22}$$

as $\varepsilon \to 0$. Define

$$u^\varepsilon(x) := \begin{cases} \tilde{v}^\varepsilon(x) & \text{if } x \in \Omega \setminus \tilde{E}^\varepsilon, \\ \tilde{v}^\varepsilon_k & \text{if } x \in E^\varepsilon_k, \end{cases}$$

where $\tilde{v}^\varepsilon_k$ is the mean value of $\tilde{v}^\varepsilon$ over E^ε_k, for $k = 1, \dots, N(\varepsilon)$. Then

$$||u^\varepsilon - \tilde{v}^\varepsilon||^2_{L^2(\Omega)} = \sum_{k=1}^{N(\varepsilon)} \int_{E^\varepsilon_k} |\tilde{v}^\varepsilon(x) - \tilde{v}^\varepsilon_k|^2 dx.$$

By Poincaré inequality, for every k we have

$$\int_{E^\varepsilon_k} |\tilde{v}^\varepsilon(x) - \tilde{v}^\varepsilon_k|^2 dx \le c\left(\mathcal{L}^n(E^\varepsilon_k)\right)^{2/n} \int_{E^\varepsilon_k} |\nabla \tilde{v}^\varepsilon(x)|^2 dx,$$

and $\mathcal{L}^n(E_k^\varepsilon)$ is of order ε^n, hence

$$|| u^\varepsilon - \tilde{v}^\varepsilon||^2_{L^2(\Omega)} \le c\,\varepsilon^2 \sum_{k=1}^{N(\varepsilon)} \int_{E_k^\varepsilon} |\nabla \tilde{v}^\varepsilon(x)|^2 dx \le c\,\varepsilon^2 \int_\Omega |\nabla \tilde{v}^\varepsilon(x)|^2 dx \le c\,\varepsilon^2,$$

and this proves $(\bar{i})$. Now, we prove $(\overline{ii})$. Let us write explicitly the expression of $\mathcal{F}^\varepsilon(u^\varepsilon)$,

$$\begin{aligned}\mathcal{F}^\varepsilon(u^\varepsilon) &= \int_\Omega |\nabla u^\varepsilon|^2 dx + \int_{S_{u^\varepsilon}} f_{\alpha_\varepsilon}\Big(\frac{x}{\varepsilon}\Big)\, d\mathcal{H}^{n-1}(x) = \int_{\Omega\setminus\tilde{E}^\varepsilon} |\nabla u^\varepsilon|^2 dx + \alpha_\varepsilon \mathcal{H}^{n-1}\big(S_{u^\varepsilon}\big)\\ &= \int_{\Omega\setminus\tilde{E}^\varepsilon} |\nabla \tilde{v}^\varepsilon|^2 dx + \alpha_\varepsilon \mathcal{H}^{n-1}\big(S_{u^\varepsilon}\big) = \mathcal{G}^\varepsilon\big(\tilde{v}^\varepsilon\big) + \alpha_\varepsilon \mathcal{H}^{n-1}\big(S_{u^\varepsilon}\cap \tilde{E}^\varepsilon\big).\end{aligned}$$

Notice that if we show that $\alpha_\varepsilon \mathcal{H}^{n-1}\big(S_{u^\varepsilon}\cap \tilde{E}^\varepsilon\big) = o(\varepsilon)$ as $\varepsilon \to 0$, then $(\overline{ii})$ follows directly. Actually, we have

$$\alpha_\varepsilon \mathcal{H}^{n-1}\big(S_{u^\varepsilon}\cap \tilde{E}^\varepsilon\big) \le \alpha_\varepsilon N(\varepsilon)\, P(E^\varepsilon, Q^\varepsilon) = C\,\alpha_\varepsilon \frac{1}{\varepsilon^n}\,\varepsilon^{n-1} = C\,\frac{\alpha_\varepsilon}{\varepsilon},$$

and $\frac{\alpha_\varepsilon}{\varepsilon} = o(\varepsilon)$ as $\varepsilon \to 0$ by assumption.

Finally, notice that conditions $(\bar{i})$ and $(\overline{ii})$ are equivalent to (i) and (ii) respectively, hence the theorem is proved. □

4.3 Supercritical regime: stiffer inclusions

In this section we consider the case $\alpha_\varepsilon >> \varepsilon$. We have previously shown that for $\alpha_\varepsilon << \varepsilon$ configurations exhibiting a high number of discontinuities are favoured by the energy. We will prove that on the contrary in this regime the energy penalizes the presence of jumps in the displacements.

Before studying this case, we state and prove some technical lemmas which will be used in the following.

Lemma 4.6 *Let us consider a sequence of measurable functions $a_k : \Omega \to \mathbb{R}_+$ such that*

$$a_k \to a \quad \textit{in measure.}$$

Then, for every $v \in L^2(\Omega;\mathbb{R}^m)$ and for every sequence $(v_k) \subset L^2(\Omega;\mathbb{R}^m)$ such that

$$v_k \rightharpoonup v \quad \textit{weakly in } L^2(\Omega;\mathbb{R}^m),$$

it turns out that

$$\int_\Omega a|v|^2 dx \le \liminf_{k\to+\infty} \int_\Omega a_k |v_k|^2 dx.$$

Proof. – Let $v \in L^2(\Omega;\mathbb{R}^m)$ and $v_k \rightharpoonup v$ weakly in $L^2(\Omega;\mathbb{R}^m)$.

We can extract a subsequence (k_j) such that

$$\liminf_{k\to+\infty} \int_\Omega a_k |v_k|^2 dx = \lim_{j\to+\infty} \int_\Omega a_{k_j} |v_{k_j}|^2 dx. \tag{4.23}$$

From the convergence in measure of a_k to a we deduce that for every $\eta > 0$ there exists a measurable set $E_\eta \subset \Omega$ such that $\mathcal{L}^n(E_\eta) < \eta$ and

$$\big|\, a_{k_{j_i}} - a \,\big| \le \frac{1}{i} \quad \text{a.e. on } \Omega\setminus E_\eta$$

for a suitable subsequence $(a_{k_{j_i}})$ of (a_{k_j}). By (4.23) we get

$$\liminf_{k\to+\infty}\int_\Omega a_k|v_k|^2dx = \lim_{i\to+\infty}\int_\Omega a_{k_{j_i}}|v_{k_{j_i}}|^2dx \ge \lim_{i\to+\infty}\int_{\Omega\setminus E_\eta} a_{k_{j_i}}|v_{k_{j_i}}|^2dx \tag{4.24}$$

$$\ge \liminf_{i\to+\infty}\left\{\int_{\Omega\setminus E_\eta} a\,|v_{k_{j_i}}|^2dx - \frac{1}{i}\int_\Omega |v_{k_{j_i}}|^2dx\right\}. \tag{4.25}$$

Using the lower semicontinuity of the functional $L^2(\Omega,\mathbb{R}^m)\ni v\to\int_{\Omega\setminus E_\eta} a\,|v|^2dx$ with respect to the weak topology of L^2, we have

$$\liminf_{k\to+\infty}\int_\Omega a_k|v_k|^2dx \ge \int_{\Omega\setminus E_\eta} a\,|v|^2dx$$

for every $\eta>0$. Letting $\eta\to0$ the claim follows. □

In the next lemma we state and prove a Γ-convergence result for an auxiliary functional that will appear in the proof of the main theorem of this section.

Lemma 4.7 *Let us fix* $0<\bar\delta<\delta<\frac12$ *such that* $Q_\delta\subset\subset Q_{\bar\delta}$. *For every* $h\in\mathbb{N}$, *let* $\mathcal{I}^h : L^2(Q_{\bar\delta})\to[0,+\infty]$ *be the functional defined as*

$$\mathcal{I}^h(w) := \begin{cases} \displaystyle\int_{Q_{\bar\delta}} |\nabla w|^2dx + \mathcal{H}^{n-1}(S_w) & \textit{if } w\in SBV^2(Q_{\bar\delta}), S_w\subset Q_\delta, \mathcal{H}^{n-1}(S_w)\le\frac1h,\\ +\infty & \textit{otherwise in } L^2(Q_{\bar\delta}).\end{cases}$$

Then the sequence $\mathcal{I}^h$ Γ*-converges with respect to the strong topology of* L^2 *to the functional* $\mathcal{I}$: $L^2(Q_{\bar\delta})\to[0,+\infty]$ *given by*

$$\mathcal{I}(w) := \begin{cases} \displaystyle\int_{Q_{\bar\delta}} |\nabla w|^2dx & \textit{if } w\in H^1(Q_{\bar\delta}),\\ +\infty & \textit{otherwise in } L^2(Q_{\bar\delta}).\end{cases}$$

Proof. – Let $w\in L^2(Q_{\bar\delta})$ and let (w_h) be a sequence converging to w strongly in L^2 and having equibounded energy $\mathcal{I}^h$. We claim that $w\in H^1(Q_{\bar\delta})$ and that

$$\liminf_{h\to+\infty}\mathcal{I}^h(w_h)\ge\mathcal{I}(w). \tag{4.26}$$

Without loss of generality we can assume that $||w_h||_{L^\infty}\le c<+\infty$.

Indeed, if the claim (4.26) is proved in this case, then we can recover the general result in the following way. Let $w\in L^2(Q_{\bar\delta})$ and $(w_h)\subset L^2(Q_{\bar\delta})$ converging to w strongly in L^2 and having equibounded energy. For every $l\in\mathbb{N}$ let us define $T_l(w_h) := \big(w_h\wedge l\big)\vee(-l)$. Since $T_l(w_h)$ converges to T_lw strongly in L^2, as $h\to+\infty$ and $||T_l(w_h)||_{L^\infty}\le l$, we have by (4.26) that $T_lw\in H^1(Q_{\bar\delta})$ and

$$\liminf_{h\to+\infty}\mathcal{I}^h\big(T_l(w_h)\big)\ge\mathcal{I}(T_lw).$$

Now, by

$$\mathcal{I}^h\big(T_l(w_h)\big)\le\mathcal{I}^h(w_h),$$

we have that for every $l\in\mathbb{N}$

$$\liminf_{h\to+\infty}\mathcal{I}^h(w_h)\ge\mathcal{I}(T_lw). \tag{4.27}$$

Since (w_h) has equibounded energy, this inequality implies that $(T_l w)$ is equibounded in $H^1(Q_{\bar\delta})$. Hence, there exists a subsequence (l_k) and a function $v \in H^1(Q_{\bar\delta})$ such that $T_{l_k}w$ converges to v weakly in $H^1(Q_{\bar\delta})$, hence strongly in $L^2(Q_{\bar\delta})$, as $k \to +\infty$. From the uniqueness of the limit, since w is the pointwise limit of $T_l w$, it follows that $v = w$, which entails that $w \in H^1(Q_{\bar\delta})$.

In view of these remarks and of the lower semicontinuity of the Dirichlet functional, in (4.27) we obtain the chain of inequalities

$$\liminf_{h\to+\infty} \mathcal{I}^h(w_h) \geq \limsup_{l\to+\infty} \mathcal{I}(T_l w) \geq \limsup_{k\to+\infty} \mathcal{I}(T_{l_k} w) \geq \liminf_{k\to+\infty} \mathcal{I}(T_{l_k} w) \geq \mathcal{I}(w),$$

which is exactly (4.26).

So, from now on we will assume that $||w_h||_{L^\infty} \leq c < +\infty$. Under this further assumption we can apply directly Ambrosio's compactness and lower semicontinuity theorems (see for instance [5] and [4]) in order to deduce the compactness for the sequence w_h having equibounded energy and the liminf inequality. The fact that $\mathcal{H}^{n-1}(S_{w_h}) \leq \frac{1}{h}$ ensures in particular that the limit function belongs to the Sobolev space H^1.

Finally, the existence of a recovery sequence for a function $w \in H^1(Q_{\bar\delta})$ follows immediately by taking $w_h = w$ for every $h \in \mathbb{N}$. □

Next lemma contains a Γ-convergence result for the same functionals as in Lemma 4.7, but taking into account Dirichlet boundary conditions.

Lemma 4.8 *Let* $(\varphi_h), \varphi \in H^{1/2}(\partial Q_{\bar\delta})$ *be such that* $\varphi_h \to \varphi$ *strongly in* $H^{1/2}(\partial Q_{\bar\delta})$. *For every* $h \in \mathbb{N}$, *let* $\mathcal{I}^h_{\varphi_h} : L^2(Q_{\bar\delta}) \to [0, +\infty]$ *be the functional defined by*

$$\mathcal{I}^h_{\varphi_h}(w) := \begin{cases} \displaystyle\int_{Q_{\bar\delta}} |\nabla w|^2 dx + \mathcal{H}^{n-1}(S_w) & \text{if } w \in SBV^2(Q_{\bar\delta}), S_w \subset Q_\delta, \mathcal{H}^{n-1}(S_w) \leq \frac{1}{h}, \\ & \quad w = \varphi_h \text{ on } \partial Q_{\bar\delta}, \\ +\infty & \text{otherwise in } L^2(Q_{\bar\delta}). \end{cases} \tag{4.28}$$

Then the sequence $(\mathcal{I}^h_{\varphi_h})$ Γ*-converges with respect to the strong topology of* L^2 *to the functional* $\mathcal{I}_\varphi : L^2(Q_{\bar\delta}) \to [0, +\infty]$ *given by*

$$\mathcal{I}_\varphi(w) := \begin{cases} \displaystyle\int_{Q_{\bar\delta}} |\nabla w|^2 dx & \text{if } w \in H^1(Q_{\bar\delta}),\ w = \varphi \text{ on } \partial Q_{\bar\delta}, \\ +\infty & \text{otherwise in } L^2(Q_{\bar\delta}). \end{cases}$$

Proof. – *First step: proof of compactness and liminf.* Let $(w_h), w \in L^2(Q_{\bar\delta})$ be such that $w_h \to w$ strongly in L^2 and $\mathcal{I}^h_{\varphi_h}(w_h) \leq c < +\infty$. From the equality $\mathcal{I}^h_{\varphi_h}(w_h) = \mathcal{I}^h(w_h)$ and the previous lemma, we get that $w \in H^1(Q_{\bar\delta})$; moreover,

$$\liminf_{h\to\infty} \mathcal{I}^h_{\varphi_h}(w_h) = \liminf_{h\to\infty} \mathcal{I}^h(w_h) \geq \mathcal{I}(w).$$

It remains to show that $w = \varphi$ on $\partial Q_{\bar\delta}$. First of all we can notice that the bound $\mathcal{I}^h_{\varphi_h}(w_h) \leq c < +\infty$ implies that $w_h = \varphi_h$ on $\partial Q_{\bar\delta}$. Moreover we have $||w_h||_{H^1(Q_{\bar\delta}\setminus Q_\delta)} \leq c$, hence $w_h \rightharpoonup w$ weakly in $H^1(Q_{\bar\delta} \setminus Q_\delta)$. This convergence entails in particular the convergence of the traces on $\partial Q_{\bar\delta}$, that is,

$$\varphi_h = (w_h)_{|\partial Q_{\bar\delta}} \to w_{|\partial Q_{\bar\delta}} \quad \text{strongly in } L^2(\partial Q_{\bar\delta}). \tag{4.29}$$

Since $\varphi_h \to \varphi$ strongly in $H^{1/2}(\partial Q_{\bar\delta})$, from (4.29) we get the equality $w = \varphi$ on $\partial Q_{\bar\delta}$.

Second step: limsup. Let $w \in H^1(Q_{\bar\delta})$ be such that $w = \varphi$ on $\partial Q_{\bar\delta}$. The surjectivity of the trace operator onto $H^{1/2}$ ensures that for every $h \in \mathbb{N}$ there exists $v_h \in H^1(Q_{\bar\delta})$ verifying the equality $v_h = \varphi_h - \varphi$ on $\partial Q_{\bar\delta}$ and the bound

$$||v_h||_{H^1(Q_{\bar\delta})} \le c\,||\varphi_h - \varphi||_{H^{1/2}(\partial Q_{\bar\delta})}.$$

From the assumption we have $v_h \to 0$ strongly in H^1. Let us define the sequence $w_h = w + v_h$. It turns out that $w_h = \varphi_h$ on $\partial Q_{\bar\delta}$ and that $w_h \to w$ strongly in H^1. Therefore w_h is a recovery sequence for $\mathcal{I}^h_{\varphi_h}$.

□

Now we are ready to state and prove the main result of this section.

Define the functional $\mathcal{F}^\infty : L^2(\Omega) \to [0, +\infty]$ as

$$\mathcal{F}^\infty(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\,dx & \text{in } H^1(\Omega), \\ +\infty & \text{otherwise in } L^2(\Omega). \end{cases}$$

We will show that $\mathcal{F}^\infty$ is the Γ-limit of the sequence $(\mathcal{F}^\varepsilon)$ in this case, that is, when $\alpha_\varepsilon >> \varepsilon$.

Theorem 4.9 (Bound from below) *Let $u \in L^2(\Omega)$ and let (u^ε) be a sequence converging to u strongly in L^2 and having equibounded energy $\mathcal{F}^\varepsilon$. Then $u \in H^1(\Omega)$ and*

$$\liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) \ge \mathcal{F}^\infty(u). \tag{4.30}$$

Proof. – We remark that, as $\mathcal{F}^\varepsilon(u^\varepsilon)$ is bounded, the functions u^ε can have jumps only in the set $\tilde E^\varepsilon \cup \tilde F^\varepsilon$ defined in (4.2).

We now classify the cubes Q^ε_k according to the measure of the jump set that they contain. More precisely, let us introduce a positive parameter $\beta > 0$ that will be chosen later in a suitable way. We say that a cube Q^ε_k is *good* whenever $\mathcal{H}^{n-1}\big(S_{u^\varepsilon} \cap Q^\varepsilon_k\big) \le \beta\,\varepsilon^{n-1}$, and *bad* otherwise and we denote with $N_g(\varepsilon)$ and $N_b(\varepsilon)$ the number of *good* and *bad* cubes, respectively. First of all we can notice that, by the fact that the sequence (u^ε) has equibounded energy, we have in particular that there exists a constant $c > 0$ such that $\alpha_\varepsilon \mathcal{H}^{n-1}(S_{u^\varepsilon}) \le c$. From this we deduce an important bound for the number of bad cubes, that is $N_b(\varepsilon) \le \dfrac{c}{\alpha_\varepsilon \varepsilon^{n-1}}$. We can write, from (4.1),

$$\Omega = \left(\bigcup_{k=1}^{N_g(\varepsilon)} Q^\varepsilon_k\right) \cup \left(\bigcup_{k=1}^{N_b(\varepsilon)} Q^\varepsilon_k\right) \cup R(\varepsilon) =: (Q^\varepsilon)^g \cup (Q^\varepsilon)^b \cup R(\varepsilon). \tag{4.31}$$

First step: energy estimate on good cubes. Let Q^ε_k be a good cube and consider

$$\mathcal{F}^\varepsilon\big(u^\varepsilon, Q^\varepsilon_k\big) := \int_{Q^\varepsilon_k} |\nabla u^\varepsilon|^2 dx + \alpha_\varepsilon \mathcal{H}^{n-1}\big(S_{u^\varepsilon} \cap Q^\varepsilon_k\big). \tag{4.32}$$

Define the function v^ε in the unit cube Q_k as $u^\varepsilon(\varepsilon\, y) =: \sqrt{\alpha_\varepsilon \varepsilon}\, v^\varepsilon(y)$. In terms of v^ε, (4.32) becomes

$$\mathcal{F}^\varepsilon\big(u^\varepsilon, Q^\varepsilon_k\big) = \alpha_\varepsilon \varepsilon^{n-1}\bigg\{\int_{Q_k} |\nabla v^\varepsilon|^2 dx + \mathcal{H}^{n-1}(S_{v^\varepsilon} \cap Q_k)\bigg\}, \tag{4.33}$$

with $\mathcal{H}^{n-1}(S_{v^\varepsilon} \cap Q_k) \le \beta$. In other words, by means of a change of variables we have reduced the problem to the study of the Mumford-Shah functional over a fixed domain, with some constraints on the jump set. From now on we will omit the subscript k. Let $\bar\delta, \hat\delta$ be such that $Q_\delta \subset\subset Q_{\bar\delta} \subset\subset Q_{\hat\delta} \subset\subset Q$.

Let us consider the problem of finding local minimizers for the Mumford-Shah functional under the required conditions, that is

$$\text{(LMS)}\quad \operatorname{loc\,min}\Big\{\int_{Q_{\hat\delta}}|\nabla w|^2dx+\mathcal{H}^{n-1}(S_w): w\in SBV^2(Q_{\hat\delta}), S_w\subset E\cup F, \mathcal{H}^{n-1}(S_w)\le\beta\Big\}.$$

According to the definition given in [22], we recall that a local minimizer is a function which minimizes the given functional with respect to all perturbations with compact support. Let us denote by $\mathcal{M}_\beta$ the class of solutions of (LMS).

For a given $\hat v\in\mathcal{M}_\beta$, let us consider the function $\tilde v$ solving

$$\text{(Dir)}\quad \begin{cases}\Delta w=0 & \text{in } Q_{\bar\delta}\\ w=\hat v & \text{in } Q_{\hat\delta}\setminus Q_{\bar\delta}.\end{cases}$$

We want to prove that for every $\eta>0$ there exists $\beta>0$ such that for every $\hat v\in\mathcal{M}_\beta$ and for the corresponding $\tilde v$ we have

$$\int_{Q_{\hat\delta}}|\nabla\tilde v|^2dx\le(1+\eta)\int_{Q_{\hat\delta}}|\nabla\hat v|^2dx. \tag{4.34}$$

Hence we will take such a β in the definition of good and bad cubes.

Let us prove (4.34) by contradiction. Suppose (4.34) is false. Then there exists $\eta>0$ such that for every $\beta>0$ there exists $\hat v\in\mathcal{M}_\beta$ and a corresponding $\tilde v$ for which

$$\int_{Q_{\hat\delta}}|\nabla\tilde v|^2dx>(1+\eta)\int_{Q_{\hat\delta}}|\nabla\hat v|^2dx. \tag{4.35}$$

In particular (4.35) implies that for every $h>0$ there exists $\hat v_h\in\mathcal{M}_{\frac1h}$ and $\tilde v_h$ solution of (Dir) with $\hat v$ replaced by $\hat v_h$ for which

$$\int_{Q_{\hat\delta}}|\nabla\tilde v_h|^2dx>(1+\eta)\int_{Q_{\hat\delta}}|\nabla\hat v_h|^2dx. \tag{4.36}$$

Since $Q_{\hat\delta}=\big(Q_{\hat\delta}\setminus Q_{\bar\delta}\big)\cup Q_{\bar\delta}$, we can split the previous integrals and, using the fact that $\tilde v_h=\hat v_h$ in $Q_{\hat\delta}\setminus Q_{\bar\delta}$ we obtain from (4.36)

$$\int_{Q_{\bar\delta}}|\nabla\tilde v_h|^2dx>(1+\eta)\int_{Q_{\bar\delta}}|\nabla\hat v_h|^2dx+\eta\int_{Q_{\hat\delta}\setminus Q_{\bar\delta}}|\nabla\hat v_h|^2dx. \tag{4.37}$$

Since the problem defining $\tilde v_h$ is linear, we can normalize the left-hand side of (4.37), so that we can assume

$$1=\int_{Q_{\bar\delta}}|\nabla\tilde v_h|^2dx>(1+\eta)\int_{Q_{\bar\delta}}|\nabla\hat v_h|^2dx+\eta\int_{Q_{\hat\delta}\setminus Q_{\bar\delta}}|\nabla\hat v_h|^2dx. \tag{4.38}$$

This means that, in particular,

$$\int_{Q_{\hat\delta}}|\nabla\hat v_h|^2dx\le\frac{1}{\eta}<+\infty. \tag{4.39}$$

Without loss of generality we can assume that $\int_{Q_{\hat\delta}\setminus Q_\delta}\hat v_h dx=0$; therefore, since $S_{\hat v_h}\subset Q_\delta$, inequality (4.39) implies that $||\hat v_h||_{H^1(Q_{\hat\delta}\setminus Q_\delta)}\le c$. Using the fact that $\hat v_h$ is harmonic in $Q_{\hat\delta}\setminus Q_\delta$ we get the convergence of the traces of $\hat v_h$ on $\partial Q_{\bar\delta}$, that is

$$\varphi_h:=(\hat v_h)_{|\partial Q_{\bar\delta}}\to\varphi\quad\text{strongly in } H^{1/2}(\partial Q_{\bar\delta}). \tag{4.40}$$

At this point, let us consider the following problems:

$$(\mathrm{Dir})_{\varphi_h}\quad \begin{cases}\Delta w = 0 & \text{in } Q_{\bar\delta}\\ w=\varphi_h & \text{on } \partial Q_{\bar\delta},\end{cases}\qquad (\mathrm{Dir})_{\varphi}\quad \begin{cases}\Delta w = 0 & \text{in } Q_{\bar\delta}\\ w=\varphi & \text{on } \partial Q_{\bar\delta}.\end{cases}$$

Clearly, $\tilde v_h$ is the only solution to $(\mathrm{Dir})_{\varphi_h}$ for every h. Let us call $\tilde v$ the solution to $(\mathrm{Dir})_{\varphi}$. From (4.40) it turns out that $\tilde v_h \to \tilde v$ strongly in $H^1(Q_{\bar\delta})$, hence,

$$1=\int_{Q_{\bar\delta}}|\nabla \tilde v_h|^2\,dx \to \int_{Q_{\bar\delta}}|\nabla \tilde v|^2\,dx = 1. \tag{4.41}$$

Notice that the functions $\hat v_h$ defined by the minimum problem (LMS) are absolute minimizers of the same functional over the same class once we fix the boundary data φ_h. Therefore they are absolute minimizers for the functional $\mathcal{I}^h_{\varphi_h}$ defined in (4.28). The Γ-convergence result proved in Lemma 4.8 gives the L^2 convergence of the sequence $\hat v_h$ to the only minimizer of the functional $\mathcal{I}_\varphi$, that is exactly $\tilde v$, and the convergence of the energies.

Now, if we let $h\to+\infty$ in (4.38) we obtain that

$$1=\int_{Q_{\bar\delta}}|\nabla \tilde v|^2 dx \geq (1+\eta)\int_{Q_{\bar\delta}}|\nabla \tilde v|^2 dx,$$

which gives the contradiction, therefore (4.34) is proved.

Let $\eta>0$ be fixed; we choose $\beta>0$ such that the property (4.34) is satisfied and for every $\varepsilon>0$ we consider the problem

$$(\mathrm{MS})\ \min\Big\{\int_{Q_{\hat\delta,k}}|\nabla w|^2dx+\mathcal{H}^{n-1}(S_w):\quad w\in SBV^2(Q_{\hat\delta,k}), S_w\subset E\cup F,\\ \mathcal{H}^{n-1}(S_w)\leq\beta, w=v^\varepsilon \text{ on } \partial Q_{\hat\delta,k}\Big\}.$$

For a minimizer $\hat v^\varepsilon$ of (MS), let $\tilde v^\varepsilon$ be the corresponding function defined by (Dir), with $\hat v$ replaced by $\hat v^\varepsilon$. We have that, as before,

$$\int_{Q_{\hat\delta,k}}|\nabla\tilde v^\varepsilon|^2dx\leq(1+\eta)\int_{Q_{\hat\delta,k}}|\nabla \hat v^\varepsilon|^2dx. \tag{4.42}$$

Hence, in particular,

$$\begin{aligned}\int_{Q_{\hat\delta,k}}|\nabla v^\varepsilon|^2dx+\mathcal{H}^{n-1}(S_{v^\varepsilon}\cap Q_{\hat\delta,k}) &\geq \int_{Q_{\hat\delta,k}}|\nabla \hat v^\varepsilon|^2dx+\mathcal{H}^{n-1}(S_{\hat v^\varepsilon}\cap Q_{\hat\delta,k})\\ &\geq\Big(1-\frac{\eta}{1+\eta}\Big)\int_{Q_{\hat\delta,k}}|\nabla\tilde v^\varepsilon|^2dx,\end{aligned} \tag{4.43}$$

where v^ε is the function in (4.33). Now define $\tilde u^\varepsilon$ as $\tilde u^\varepsilon(\varepsilon y):=\sqrt{\alpha_\varepsilon\varepsilon}\,\tilde v^\varepsilon(y)$. By (4.33) and (4.43) we obtain

$$\int_{Q^\varepsilon_{\hat\delta,k}}|\nabla u^\varepsilon|^2dx+\alpha_\varepsilon\mathcal{H}^{n-1}\big(S_{u^\varepsilon}\cap Q^\varepsilon_{\hat\delta,k}\big)\geq\Big(1-\frac{\eta}{1+\eta}\Big)\int_{Q^\varepsilon_{\hat\delta,k}}|\nabla\,\tilde u^\varepsilon|^2dx. \tag{4.44}$$

Second step: energy estimate on bad cubes. Let Q^ε_k be a bad cube. This means that $\mathcal{H}^{n-1}\big(S_{u^\varepsilon}\cap Q^\varepsilon_k\big)>\beta\,\varepsilon^{n-1}$. First of all, recall that we have a control on the number of bad cubes, that is, $N_b(\varepsilon)\leq\dfrac{c}{\alpha_\varepsilon\varepsilon^{n-1}}$.

The idea is to use the obvious inequality

$$\int_{Q_k^\varepsilon} |\nabla u^\varepsilon|^2 dx + \alpha_\varepsilon \mathcal{H}^{n-1}\big(S_{u^\varepsilon} \cap Q_k^\varepsilon\big) \geq \int_{Q_k^\varepsilon} \chi_\delta^\varepsilon \, |\nabla \check{u}^\varepsilon|^2 dx,$$

where χ_δ^ε is the characteristic function of the set $Q_k^\varepsilon \setminus Q_{\delta,k}^\varepsilon$ and the function $\check{u}^\varepsilon$ coincides with u^ε in $Q_k^\varepsilon \setminus Q_{\delta,k}^\varepsilon$ and is extended to $Q_{\delta,k}^\varepsilon$ in a way that keeps its H^1 norm bounded.

Third step: final estimate. Let us define a new sequence $w^\varepsilon \in SBV^2(\Omega)$ as

$$w^\varepsilon := \begin{cases} \tilde{u}^\varepsilon & \text{in } \big(Q_{\hat{\delta}}^\varepsilon\big)^g, \\ u^\varepsilon & \text{in } \big((Q^\varepsilon)^g \setminus (Q_{\hat{\delta}}^\varepsilon)^g\big) \cup R(\varepsilon), \\ \check{u}^\varepsilon & \text{in } (Q^\varepsilon)^b, \end{cases}$$

where $(Q^\varepsilon)^g, (Q^\varepsilon)^g$ and $R(\varepsilon)$ are given in (4.31) and $\big(Q_{\hat{\delta}}^\varepsilon\big)^g$ denotes the set

$$\big(Q_{\hat{\delta}}^\varepsilon\big)^g := \bigcup_{k=1}^{N_g(\varepsilon)} Q_{\hat{\delta},k}^\varepsilon.$$

Define also the function $a^\varepsilon : \Omega \to \mathbb{R}$ as

$$a^\varepsilon(x) := \begin{cases} 0 & \text{in } (Q_\delta^\varepsilon)^b, \\ 1 & \text{otherwise in } \Omega, \end{cases}$$

where the set $(Q_\delta^\varepsilon)^b$ is defined as

$$(Q_\delta^\varepsilon)^b := \bigcup_{k=1}^{N_b(\varepsilon)} Q_{\delta,k}^\varepsilon.$$

From what we proved in the previous steps we can write

$$\mathcal{F}^\varepsilon(u^\varepsilon) \geq \left(1 - \frac{\eta}{1+\eta}\right) \int_\Omega a^\varepsilon(x) \, |\nabla w^\varepsilon|^2 dx. \tag{4.45}$$

It remains to apply Lemma 4.6 to (4.45). First of all we show the convergence of a^ε. We have

$$\int_\Omega |\, a^\varepsilon - 1| \, dx = \mathcal{L}^n\big((Q_\delta^\varepsilon)^b\big) = N_b(\varepsilon)\, \varepsilon^n \mathcal{L}^n(Q_\delta) \leq c\, \frac{\varepsilon}{\alpha_\varepsilon},$$

hence $a^\varepsilon \to 1$ strongly in $L^1(\Omega)$. Once we prove that $w^\varepsilon \rightharpoonup u$ weakly in $H^1(\Omega)$, it turns out that

$$\liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) \geq \left(1 - \frac{\eta}{1+\eta}\right) \int_\Omega |\nabla u|^2 dx,$$

and the thesis follows letting η converge to zero.

Fourth step: convergence of w^ε. First of all it is clear from (4.45) and the choice of $\check{u}^\varepsilon$ that $||\nabla w^\varepsilon||_{L^2(\Omega)} \leq c$. Then, as in the proof of Theorem 4.1, the fact that w^ε and u^ε coincide in a set with positive measure ensures the convergence.

□

Theorem 4.10 (Bound from above) *For every $u \in H^1(\Omega)$ there exists a sequence (u^ε) such that*

$$(i) \quad u^\varepsilon \to u \quad \text{strongly in } L^2(\Omega),$$
$$(ii) \quad \lim_{\varepsilon \to 0} \mathcal{F}^\varepsilon\big(u^\varepsilon\big) = \mathcal{F}^\infty(u).$$

Proof. – The thesis follows trivially by choosing $u^\varepsilon = u$ for every $\varepsilon > 0$. □

4.4 Critical regime: intermediate case

In this section we will analyze the case in which the fragility coefficient of the inclusions in the material and the size ε of the periodic structure are of the same order. We can assume, without loss of generality, that $\alpha_\varepsilon = \varepsilon$. So, the functional we are interested in is given by

$$\mathcal{F}^\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 dx + \varepsilon\,\mathcal{H}^{n-1}(S_u) & \text{if } u \in SBV^2(\Omega), S_u \subset \tilde{E}^\varepsilon \cup \tilde{F}^\varepsilon, \\ +\infty & \text{otherwise in } L^2(\Omega). \end{cases}$$

As first step, we localize the sequence $\big(\mathcal{F}^\varepsilon\big)$, introducing an explicit dependence on the set of integration. More explicitly, for every $u \in L^2(\Omega)$ and for every open set $A \in \mathcal{A}(\Omega)$ we define

$$\mathcal{F}^\varepsilon(u, A) := \begin{cases} \displaystyle\int_A |\nabla u|^2 dx + \varepsilon\,\mathcal{H}^{n-1}(S_u \cap A) & \text{if } u \in SBV^2(A),\ S_u \subset \big(\tilde{E}^\varepsilon \cup \tilde{F}^\varepsilon\big) \cap A, \\ +\infty & \text{otherwise in } L^2(\Omega). \end{cases}$$

For a fixed $u \in L^2(\Omega)$ we can extend the localized functional we have just defined to a measure $\big(\mathcal{F}^\varepsilon\big)^*(u,\cdot)$ on the class of Borel sets $\mathcal{B}(\Omega)$ in the usual way:

$$\big(\mathcal{F}^\varepsilon\big)^*(u, B) := \inf\big\{\mathcal{F}^\varepsilon(u, A) : A \in \mathcal{A}(\Omega), B \subseteq A\big\}.$$

4.4.1 Integral representation of the Γ-limit

In this subsection we are going to prove that the sequence $\big(\mathcal{F}^\varepsilon\big)$ Γ-converges to a functional $\mathcal{F}^{hom}$, and that this limit functional admits an integral representation. A preliminary result is given by next theorem, in which we prove the Γ-convergence of a suitable subsequence of $\big(\mathcal{F}^\varepsilon\big)$.

Theorem 4.11 *Let ε be a sequence converging to zero. Then there exist a subsequence $(\sigma(\varepsilon))$ and a functional $\mathcal{F}^{hom}_\sigma : L^2(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ such that, for every $A \in \mathcal{A}(\Omega)$,*

$$\mathcal{F}^{hom}_\sigma(\cdot, A) = \Gamma - \lim_{\varepsilon \to 0} \mathcal{F}^{\sigma(\varepsilon)}(\cdot, A)$$

in the strong L^2-topology. Moreover, for every $u \in L^2(\Omega)$, the set function $\mathcal{F}^{hom}_\sigma(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on Ω.

Next Theorem provides an extension of the fundamental estimate to SBV^2. The proof follows easily from [13, Proposition 3.1], but we will include the details for the convenience of the reader.

Theorem 4.12 (Fundamental estimate in SBV^2) *For every $\eta > 0$ and for every A', A'' and $B \in \mathcal{A}(\Omega)$, with $A' \subset\subset A''$, there exists a constant $M > 0$ with the following property: for every $\varepsilon > 0$*

and for every $u \in SBV^2(A'')$ *such that* $S_u \subset \big(\tilde{E}^\varepsilon \cup \tilde{F}^\varepsilon\big) \cap A''$, *and for every* $v \in SBV^2(B)$ *such that* $S_v \subset \big(\tilde{E}^\varepsilon \cup \tilde{F}^\varepsilon\big) \cap B$ *there exists a function* $\varphi \in C_0^\infty(\Omega)$ *with* $\varphi = 1$ *in a neighbourhood of* $\bar{A}'$, *spt* $\varphi \subset A''$ *and* $0 \le \varphi \le 1$ *such that*

$$\mathcal{F}^\varepsilon(\varphi\, u + (1-\varphi)\, v, A' \cup B) \le (1+\eta)\, \mathcal{F}^\varepsilon(u, A'') + (1+\eta)\, \mathcal{F}^\varepsilon(v, B) + M \int_T |u-v|^2 dx,$$

where $T := (A'' \setminus A') \cap B$.

Proof. – Let $\eta > 0$, A', A'' and B be as in the statement. Let $A_1, \dots A_{k+1}$ be open subsets of $\mathbb{R}^n$ such that $A' \subset\subset A_1 \subset\subset A_2 \subset\subset \cdots \subset\subset A_{k+1} \subset\subset A''$. For every $i = 1, \dots, k$ let φ_i be a function in $C_0^\infty(\Omega)$ with $\varphi_i = 1$ on a neighborhood of $\bar{A}_i$ and $\operatorname{spt} \varphi \subset A_{i+1}$.

Now, let u and v be as in the statement and define the function w_i on $A' \cup B$ as $w_i := \varphi_i u + (1-\varphi_i)\, v$ (where u and v are arbitrarily extended outside A'' and B, respectively). For $i = 1, \dots, k$ set $T_i := (A_{i+1} \setminus \bar{A}_i) \cap B$. We can write, for fixed $\varepsilon > 0$,

$$\begin{aligned}
\mathcal{F}^\varepsilon(w_i, A' \cup B) &= \int_{A' \cup B} |\nabla w_i|^2 dx + \varepsilon\, \mathcal{H}^{n-1}\big(S_{w_i} \cap (A' \cup B)\big) \\
&= \big(\mathcal{F}^\varepsilon\big)^*(u, (A' \cup B) \cap \bar{A}_i) + \big(\mathcal{F}^\varepsilon\big)^*(v, B \setminus A_{i+1}) + \mathcal{F}^\varepsilon(w_i, T_i) \\
&\le \mathcal{F}^\varepsilon(u, A'') + \mathcal{F}^\varepsilon(v, B) + \mathcal{F}^\varepsilon(w_i, T_i).
\end{aligned} \tag{4.46}$$

We can write more explicitly the last term in the previous expression as

$$\begin{aligned}
\mathcal{F}^\varepsilon(w_i, T_i) &= \int_{T_i} |\varphi_i \nabla u + (1-\varphi_i)\, \nabla v + \nabla \varphi_i (u-v)|^2 dx + \varepsilon\, \mathcal{H}^{n-1}\big(S_{w_i} \cap T_i\big) \\
&\le \int_{T_i} |\varphi_i \nabla u + (1-\varphi_i)\, \nabla v + \nabla \varphi_i (u-v)|^2 dx + \varepsilon\, \mathcal{H}^{n-1}\big(S_u \cap T_i\big) + \varepsilon\, \mathcal{H}^{n-1}\big(S_v \cap T_i\big) \\
&=: I_i^\varepsilon(T_i).
\end{aligned} \tag{4.47}$$

We would like to control $I_i^\varepsilon(T_i)$ by means of $\mathcal{L}^n(T_i)$. Let us define $M_k := \max_{1 \le i \le k} ||\nabla \varphi_i||^2_{L^\infty}$. Hence

$$\begin{aligned}
I_i^\varepsilon(T_i) &\le 2 \int_{T_i} |\varphi_i \nabla u + (1-\varphi_i)\, \nabla v|^2 dx + 2 \int_{T_i} |\nabla \varphi_i (u-v)|^2 dx + \\
&\quad + \varepsilon\, \mathcal{H}^{n-1}\big(S_u \cap T_i\big) + \varepsilon\, \mathcal{H}^{n-1}\big(S_v \cap T_i\big) \\
&\le 2 \int_{T_i} |\nabla u|^2 dx + 2 \int_{T_i} |\nabla v|^2 dx + 2 \int_{T_i} |\nabla \varphi_i|^2 |u-v|^2 dx + \\
&\quad + \varepsilon\, \mathcal{H}^{n-1}\big(S_u \cap T_i\big) + \varepsilon\, \mathcal{H}^{n-1}\big(S_v \cap T_i\big) \\
&\le 2\, \mathcal{F}^\varepsilon(u, T_i) + 2\, \mathcal{F}^\varepsilon(v, T_i) + 2\, M_k \int_{T_i} |u-v|^2 dx =: J^\varepsilon(T_i).
\end{aligned} \tag{4.48}$$

Now, let $i_0 \in \{1, \dots, k\}$ be such that T_{i_0} realizes $\min_{1 \le i \le k} J^\varepsilon(T_i)$. Then, being J^ε a measure, we have

$$J^\varepsilon(T_{i_0}) \le \frac{1}{k} \sum_{i=1}^{k} J^\varepsilon(T_i) \le \frac{1}{k}\, J^\varepsilon(T). \tag{4.49}$$

Notice that $i_0 = i_0(\varepsilon)$, it depends on ε.

Combining together (4.46)-(4.49), we get

$$\begin{aligned}\mathcal{F}^\varepsilon(w_{i_0}, A'\cup B) &\le \mathcal{F}^\varepsilon(u,A'') + \mathcal{F}^\varepsilon(v,B) + \frac{1}{k}\,J^\varepsilon(T)\\ &= \mathcal{F}^\varepsilon(u,A'') + \mathcal{F}^\varepsilon(v,B) + \frac{2}{k}\,\mathcal{F}^\varepsilon(u,T) + \frac{2}{k}\,\mathcal{F}^\varepsilon(v,T) + \frac{2}{k}\,M_k\int_T |u-v|^2dx\\ &\le \mathcal{F}^\varepsilon(u,A'') + \mathcal{F}^\varepsilon(v,B) + \frac{2}{k}\,\mathcal{F}^\varepsilon(u,A'') + \frac{2}{k}\,\mathcal{F}^\varepsilon(v,B) + \frac{2}{k}\,M_k\int_T |u-v|^2dx. \qquad (4.50)\end{aligned}$$

Now, since the choice of the number k of the stripes between A' and A'' is completely free, we can assume that k is such that $\frac{2}{k} < \eta$. Hence $k = k(\eta)$. Let us define $\overline{M}_\eta := \frac{2}{k}M_k$; then in (4.50) we have

$$\mathcal{F}^\varepsilon(w_{i_0}, A'\cup B) \le (1+\eta)\,\mathcal{F}^\varepsilon(u,A'') + (1+\eta)\,\mathcal{F}^\varepsilon(v,B) + \overline{M}_\eta\int_T |u-v|^2dx,$$

which is exactly the claim. □

Now we are ready to give the proof of Theorem 4.11.

Proof. – [Proof of Theorem 4.11] Since for every $\varepsilon > 0$ the functional $\mathcal{F}^\varepsilon$ is increasing, we deduce by Theorem 1.8 that there exist a subsequence $(\sigma(\varepsilon))$ and a functional $\mathcal{F}^{hom}_\sigma : L^2(\Omega)\times\mathcal{A}(\Omega) \to [0,+\infty]$ such that $\mathcal{F}^{hom}_\sigma = \overline{\Gamma}(L^2) - \lim_{\varepsilon\to 0}\mathcal{F}^{\sigma(\varepsilon)}$. We put a subscript σ in order to underline that the limit functional may depend on the subsequence. Now define the nonnegative increasing functional $J : L^2(\Omega)\times\mathcal{A}(\Omega) \to [0,+\infty]$ as

$$J(u,A) := \begin{cases} \displaystyle\int_A |\nabla u|^2 dx & \text{if } u_{|A} \in H^1(A),\\ +\infty & \text{otherwise.}\end{cases}$$

Clearly, J is a measure with respect to A. Moreover $0 \le \mathcal{F}^{\sigma(\varepsilon)} \le J$ for every $\varepsilon > 0$ and the fundamental estimate holds uniformly for the subsequence $\big(\mathcal{F}^{\sigma(\varepsilon)}\big)$ by Theorem 4.12. Then we can proceed as in [21, Proposition 18.6] and we obtain that

$$\mathcal{F}^{hom}_\sigma(u,A) = (\mathcal{F}^{hom}_\sigma)'(u,A) = (\mathcal{F}^{hom}_\sigma)''(u,A)$$

for every $u \in L^2(\Omega)$ and for every $A \in \mathcal{A}(\Omega)$ such that $J(u,A) < +\infty$.

Fix $A \in \mathcal{A}(\Omega)$. As we noticed in Theorem 4.3, we have the bound $\mathcal{F}^{\sigma(\varepsilon)}(\cdot,A) \ge \mathcal{G}^{\sigma(\varepsilon)}(\cdot,A)$, with $\mathcal{G}^{\sigma(\varepsilon)}$ defined in (4.5). Hence by Theorem 4.1 the Γ-limit of $\mathcal{F}^{\sigma(\varepsilon)}(\cdot,A)$ is finite only on $H^1(A)$, which is the same domain where $J(\cdot,A)$ is finite, and is given by $\mathcal{F}^{hom}_\sigma(\cdot,A)$. This proves the stated convergence of a subsequence $\big(\mathcal{F}^{\sigma(\varepsilon)}\big)$.

Finally, $\mathcal{F}^\varepsilon(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on Ω. Then, by Theorem 4.12 and [21, Theorem 18.5] we have that for every $u \in L^2(\Omega)$ the set function $\mathcal{F}^{hom}_\sigma(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on Ω. □

Now we show some general properties for the Γ-limit of $\mathcal{F}^\varepsilon$, even if, up to now, we have proved the convergence only for a subsequence. The fact that the whole sequence converges will follow from the characterization of the Γ-limit, which will depend only on the gradient of the displacement and not on the subsequence $\sigma(\varepsilon)$. From now on let us assume that we have already proved it and postpone the proof to the end of the section. Hence we can omit the subscript σ and call $\mathcal{F}^{hom}$ the Γ-limit of the whole sequence $\big(\mathcal{F}^\varepsilon\big)$.

Lemma 4.13 *The restriction of the functional $\mathcal{F}^{hom} : L^2(\Omega)\times\mathcal{A}(\Omega) \to [0,+\infty]$ to $H^1(\Omega)\times\mathcal{A}(\Omega)$ satisfies the following properties: for every $u,v \in H^1(\Omega)$ and for every $A \in \mathcal{A}(\Omega)$*

(a) $\mathcal{F}^{hom}$ *is local, i.e.,* $\mathcal{F}^{hom}(u,A) = \mathcal{F}^{hom}(v,A)$ *whenever* $u_{|A} = v_{|A}$ *;*

(b) the set function $\mathcal{F}^{hom}(u,\cdot)$ *is the restriction to* $\mathcal{A}(\Omega)$ *of a Borel measure on* Ω *;*

(c) $\mathcal{F}^{hom}(\cdot,A)$ *is sequentially weakly lower semicontinuous on* $H^1(\Omega)$ *;*

(d) for every $a \in \mathbb{R}$ *we have* $\mathcal{F}^{hom}(u,A) = \mathcal{F}^{hom}(u+a,A)$ *;*

(e) $\mathcal{F}^{hom}$ *satisfies the bound*

$$0 \le \mathcal{F}^{hom}(u,A) \le \int_A |\nabla u|^2 dx.$$

Proof. – Properties (a) and (c) follow from the fact that $\mathcal{F}^{hom}(\cdot,A)$ is the Γ-limit of the sequence $\mathcal{F}^\varepsilon(\cdot,A)$, while (b) comes from Theorem 4.11. For property (d) we can proceed as follows. Let $u \in H^1(\Omega)$, $A \in \mathcal{A}(\Omega)$ and consider a recovery sequence $(u^\varepsilon) \subset L^2(\Omega) \cap SBV^2(A)$ satisfying the usual constraints for the jump set, converging to u strongly in $L^2(\Omega)$ and such that $\big(\mathcal{F}^\varepsilon(u^\varepsilon,A)\big)$ converges to $\mathcal{F}^{hom}(u,A)$. Then $(u^\varepsilon + a)$ converges to $u+a$ in $L^2(\Omega)$ and

$$\mathcal{F}^{hom}(u+a,A) \le \liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon + a, A) = \liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon, A) = \mathcal{F}^{hom}(u,A).$$

On the other hand, $\mathcal{F}^{hom}(u,A) = \mathcal{F}^{hom}((u+a)+(-a),A) \le \mathcal{F}^{hom}(u+a,A)$, hence (d) is proved. For property (e), we just recall that the Γ-limit of the sequence $\big(\mathcal{F}^\varepsilon\big)$ is bounded from above by the Dirichlet functional, since that value is reached by a special sequence. □

Next theorem shows that the functional $\mathcal{F}^{hom}$ admits an integral representation.

Theorem 4.14 *There exists a unique convex function* $f : \mathbb{R}^n \to [0,+\infty[$ *with the following properties:*

(i) $0 \le f(\xi) \le |\xi|^2$ *for every* $\xi \in \mathbb{R}^n$ *;*

(ii) $\mathcal{F}^{hom}(u,A) = \int_A f(\nabla u)\,dx$ *for every* $A \in \mathcal{A}(\Omega)$ *and for every* $u \in H^1(A)$*.*

Proof. – Notice that the functional $\mathcal{F}^{hom}$ satisfies all the assumptions of [21, Theorem 20.1], so thanks to Lemma 4.13 the Carathéodory function $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(y,\xi) := \limsup_{\varrho\to 0} \frac{\mathcal{F}^{hom}(\xi\cdot x, B_\varrho(y))}{\mathcal{L}^n(B_\varrho(y))} \tag{4.51}$$

provides the integral representation

$$\mathcal{F}^{hom}(u,A) = \int_A f(x,\nabla u)\,dx$$

for every $A \in \mathcal{A}(\Omega)$ and for every $u \in L^2(\Omega)$ such that $u_{|A} \in H^1(A)$. Moreover the same theorem ensures that for a.e. $x \in \Omega$ the function $f(x,\cdot)$ is convex on $\mathbb{R}^n$ and that

$$0 \le f(x,\xi) \le |\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and for every } \xi \in \mathbb{R}^n.$$

It remains to show that f is independent of the first variable. Using the definition (4.51), it is sufficient to prove that for every $y,z \in \Omega$ and $\xi \in \mathbb{R}^n$ and for every $\varrho > 0$, we have

$$\mathcal{F}^{hom}(\xi\cdot x, B_\varrho(y)) = \mathcal{F}^{hom}(\xi\cdot x, B_\varrho(z)). \tag{4.52}$$

Hence, let us fix $y, z \in \Omega$ and $\xi \in \mathbb{R}^n$ and $\varrho > 0$; being $\mathcal{F}^{hom}(\cdot, B_\varrho(y))$ a Γ-limit, there exists a recovery sequence $(u^\varepsilon) \subset SBV^2(B_\varrho(y))$ satisfying the usual constraint on the jump set, such that $u^\varepsilon \to 0$ strongly in $L^2(\Omega)$ and

$$\lim_{\varepsilon \to 0} \mathcal{F}^\varepsilon(\xi \cdot x + u^\varepsilon, B_\varrho(y)) = \mathcal{F}^{hom}(\xi \cdot x, B_\varrho(y)).$$

Without loss of generality we can assume $(u^\varepsilon) \subset SBV^2_0(B_\varrho(y))$, where the subscript 0 denotes the functions vanishing on the boundary. Indeed we can always reduce to this case by means of a cut-off function. Now let us define the vector $\tau^\varepsilon \in \mathbb{R}^n$ as

$$\tau^\varepsilon := \varepsilon \left[\frac{z-y}{\varepsilon} \right],$$

where the symbol $[\cdot]$ denotes the integer part. Extend u^ε by zero out of $B_\varrho(y)$ and define the new sequence $v^\varepsilon(x) := u^\varepsilon(x - \tau^\varepsilon)$. It turns out that $S_{v^\varepsilon} \subset \tilde{E}^\varepsilon \cup \tilde{F}^\varepsilon$; moreover v^ε is identically zero out of $B_\varrho(y) + \tau^\varepsilon$ and it converges to zero strongly in $L^2(\Omega)$. Observe that for small enough ε and for every $r > 1$ we have that $B_\varrho(y) + \tau^\varepsilon \subset B_{r\varrho}(z)$. Hence the sequence $\xi \cdot x + v^\varepsilon$ gives a bound for $\mathcal{F}^{hom}(\xi \cdot x, B_\varrho(z))$, that is

$$\begin{aligned} \mathcal{F}^{hom}(\xi \cdot x, B_\varrho(z)) &\le \mathcal{F}^{hom}(\xi \cdot x, B_{r\varrho}(z)) \le \liminf_{\varepsilon \to 0} \mathcal{F}^\varepsilon(\xi \cdot x + v^\varepsilon, B_{r\varrho}(z)) \\ &= \liminf_{\varepsilon \to 0} \left\{ \int_{B_{r\varrho}(z)} |\,\xi + \nabla v^\varepsilon|^2 dx + \varepsilon\, \mathcal{H}^{n-1}(S_{v^\varepsilon} \cap B_{r\varrho}(z)) \right\}. \end{aligned} \tag{4.53}$$

We can rewrite the last line of (4.53) in terms of u^ε, and so we get

$$\begin{aligned} \mathcal{F}^{hom}(\xi \cdot x, B_\varrho(z)) &\le \liminf_{\varepsilon \to 0} \left\{ \int_{B_\varrho(y)} |\,\xi + \nabla u^\varepsilon|^2 dx + |\,\xi|^2 \mathcal{L}^n(B_{r\varrho} \setminus B_\varrho) + \varepsilon\, \mathcal{H}^{n-1}(S_{u^\varepsilon} \cap B_\varrho(y)) \right\} \\ &= \mathcal{F}^{hom}(\xi \cdot x, B_\varrho(y)) + |\,\xi|^2 \mathcal{L}^n(B_{r\varrho} \setminus B_\varrho). \end{aligned} \tag{4.54}$$

Now, if we let $r \to 1$ we have that $\mathcal{F}^{hom}(\xi \cdot x, B_\varrho(z)) \le \mathcal{F}^{hom}(\xi \cdot x, B_\varrho(y))$. The reverse inequality can be deduced in the same way, hence the claim follows. □

4.4.2 Homogenization formula

Once we have shown that the Γ-limit of the sequence $(\mathcal{F}^\varepsilon)$ admits an integral representation, it remains to characterize the limit density. We will prove that it solves an asymptotic cell problem.

We define the function $f_{hom} : \mathbb{R}^n \to [0, +\infty)$ as

$$f_{hom}(\xi) := \lim_{t \to +\infty} \frac{1}{t^n} \inf \left\{ \int_{(0,t)^n} |\,\xi + \nabla w|^2 d\,x + \mathcal{H}^{n-1}(S_w) : w \in SBV^2_0\big((0,t)^n\big), S_w \subset \tilde{E} \cup \tilde{F} \right\} \tag{4.55}$$

where, according to the notation used so far, we have

$$\tilde{E} := \Omega \cap \bigcup_{h \in \mathbb{Z}^n} (E + h), \quad \tilde{F} := \Omega \cap \bigcup_{h \in \mathbb{Z}^n} (F + h). \tag{4.56}$$

Theorem 4.15 *The function f_{hom} in (4.55) is well defined, that is the function*

$$g(t) := \frac{1}{t^n} \inf \left\{ \int_{(0,t)^n} |\,\xi + \nabla w|^2 dx + \mathcal{H}^{n-1}(S_w) : w \in SBV^2_0\big((0,t)^n\big), S_w \subset \tilde{E} \cup \tilde{F} \right\} \tag{4.57}$$

admits a limit as $t \to +\infty$.

PROOF. – Let $\xi \in \mathbb{R}^n$ and let $t > 0$; by definition of g, there exists a function $u_t \in SBV_0^2\big((0,t)^n\big)$ with $S_{u_t} \subset \tilde{E} \cup \tilde{F}$ such that

$$\frac{1}{t^n}\left\{\int_{(0,t)^n} |\,\xi + \nabla u_t|^2 d\,x + \mathcal{H}^{n-1}(S_{u_t})\right\} \le g(t) + \frac{1}{t}.$$

Fix $s > t$ and define a subset of $\mathbb{N}^n$ as

$$K := \big\{\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n : 0 < ([t]+1)\,k_j < s, \text{ for } j = 1, \ldots, n\big\}.$$

Then, we define the set $I := ([t]+1)K$. Now, consider the function $u_s : \mathbb{R}^n \to \mathbb{R}$ defined in the following way:

$$u_s(x) := \begin{cases} u_t(x - \mathbf{i}) & \text{if } x \in \mathbf{i} + (0,t)^n, \mathbf{i} \in I, \\ 0 & \text{otherwise.} \end{cases}$$

The fact that we performed a translation by integers and the Q-periodicity of the jumps for the function u_t entail $S_{u_s} \subset \tilde{E} \cup \tilde{F}$. Moreover, u_s vanishes on the boundary of $(0,s)^n$. Hence, u_s is a competitor for $g(s)$, and so

$$g(s) \le \frac{1}{s^n}\left\{\int_{(0,s)^n} |\,\xi + \nabla u_s|^2 d\,x + \mathcal{H}^{n-1}(S_{u_s})\right\}.$$

Define the set $R_t^s \subset (0,s)^n$ as

$$R_t^s := (0,s)^n \setminus \bigcup_{\mathbf{i} \in I} \big(\mathbf{i} + (0,t)^n\big).$$

Since for the cardinality of the set I we have

$$\frac{s^n}{([t]+1)^n} - 1 < |I| = \Big(\Big[\frac{s}{[t]+1}\Big]\Big)^n \le \frac{s^n}{([t]+1)^n}, \tag{4.58}$$

then it turns out that

$$\mathcal{L}^n(R_t^s) = s^n - \Big(\Big[\frac{s}{[t]+1}\Big]\Big)^n t^n \le s^n - \Big(\frac{s - ([t]+1)}{[t]+1}\Big)^n t^n. \tag{4.59}$$

Notice that $u_s = 0$ on R_t^s and that $S_{u_s} \cap R_t^s = \emptyset$; therefore

$$\begin{aligned} g(s) &\le \frac{1}{s^n}\Big\{\mathcal{L}^n(R_t^s)\,|\,\xi|^2 + \sum_{\mathbf{i}\in I}\int_{\mathbf{i}+(0,t)^n} |\,\xi + \nabla u_s|^2 d\,x + \sum_{\mathbf{i}\in I}\mathcal{H}^{n-1}\big(S_{u_s} \cap (\mathbf{i} + (0,t)^n)\big)\Big\} \\ &= \frac{1}{s^n}\Big\{\mathcal{L}^n(R_t^s)\,|\,\xi|^2 + \sum_{\mathbf{i}\in I}\int_{(0,t)^n} |\,\xi + \nabla u_t|^2 d\,x + \sum_{\mathbf{i}\in I}\mathcal{H}^{n-1}\big(S_{u_t} \cap (0,t)^n\big)\Big\}. \end{aligned}$$

Using (4.58) and (4.59) we obtain, finally,

$$g(s) \le \frac{t^n}{([t]+1)^n}\Big(g(t) + \frac{1}{t}\Big) + |\,\xi\,|^2\Big(1 - \Big(\frac{s-t-1}{s}\Big)^n\Big(\frac{t}{t+1}\Big)^n\Big).$$

Taking first the upper limit as $s \to +\infty$ and then the lower limit as $t \to +\infty$ we get

$$\limsup_{s\to+\infty} g(s) \le \liminf_{t\to+\infty} g(t),$$

and this concludes the proof. □

Next theorem shows that the Γ-limit of the sequence $\big(\mathcal{F}^\varepsilon\big)$ can be expressed in terms of the homogenization formula (4.55).

Theorem 4.16 *The function f appearing in the expression of the limit functional $\mathcal{F}^{hom}$ and the function f_{hom} defined by the asymptotic cell problem coincide, i.e., for every $\xi \in \mathbb{R}^n$ it turns out that*

$$f(\xi) = f_{hom}(\xi).$$

Proof. – *First step: $f \geq f_{hom}$.* Let $\xi \in \mathbb{R}^n$ and define $u_\xi(x) := \xi \cdot x$ for every $x \in \mathbb{R}^n$. By definition of Γ-convergence, there exists a recovery sequence $u^\varepsilon \subset SBV^2(Q)$ with $S_{u^\varepsilon} \subset \big(\tilde{E}^\varepsilon \cup \tilde{F}^\varepsilon\big) \cap Q$, such that $u^\varepsilon \to u_\xi$ strongly in $L^2(Q)$ and

$$\lim_{\varepsilon \to 0} \mathcal{F}^\varepsilon(u^\varepsilon, Q) = \mathcal{F}^{hom}(u_\xi, Q) = f(\xi).$$

Let us write $u^\varepsilon =: u_\xi + v^\varepsilon$, where $v^\varepsilon \subset SBV^2(Q)$ and $v^\varepsilon \to 0$ strongly in $L^2(Q)$. Without loss of generality we can assume $v^\varepsilon \in SBV_0^2(Q)$. Hence

$$f(\xi) = \lim_{\varepsilon \to 0} \mathcal{F}^\varepsilon(u_\xi + v^\varepsilon, Q) = \lim_{\varepsilon \to 0} \bigg\{ \int_Q |\xi + \nabla v^\varepsilon|^2 d\,x + \varepsilon\, \mathcal{H}^{n-1}(S_{v^\varepsilon}) \bigg\}. \tag{4.60}$$

Now, let us define the function $w^\varepsilon \in SBV_0^2(Q/\varepsilon)$ as

$$v^\varepsilon(x) =: \varepsilon\, w^\varepsilon\Big(\frac{x}{\varepsilon}\Big).$$

Remark that $S_{w^\varepsilon} \subset \tilde{E} \cup \tilde{F}$. Then, rewriting (4.60) in terms of w^ε we obtain

$$\begin{aligned} f(\xi) &= \lim_{\varepsilon \to 0} \varepsilon^n \bigg\{ \int_{Q/\varepsilon} |\xi + \nabla w^\varepsilon|^2 dx + \mathcal{H}^{n-1}(S_{w^\varepsilon}) \bigg\} \\ &\geq \lim_{\varepsilon \to 0} \varepsilon^n \inf \bigg\{ \int_{(0,\frac{1}{\varepsilon})^n} |\xi + \nabla w|^2 dx + \mathcal{H}^{n-1}(S_w) : w \in SBV_0^2\big((0,1/\varepsilon)^n\big),\, S_w \subset \tilde{E} \cup \tilde{F} \bigg\} \\ &= f_{hom}(\xi). \end{aligned}$$

Second step: $f \leq f_{hom}$. Let $\xi \in \mathbb{R}^n$ and $l \in \mathbb{N}$; consider a function $w \in SBV_0^2((0,l)^n)$, with $S_w \subset \tilde{E} \cup \tilde{F}$, such that

$$\begin{aligned} &\int_{(0,l)^n} |\xi + \nabla w|^2 dx + \mathcal{H}^{n-1}(S_w) \\ &\leq \inf \bigg\{ \int_{(0,l)^n} |\xi + \nabla v|^2 dx + \mathcal{H}^{n-1}(S_v) : v \in SBV_0^2((0,l)^n), S_v \subset \tilde{E} \cup \tilde{F} \bigg\} + 1. \end{aligned} \tag{4.61}$$

Let us define the sequence $u^\varepsilon : Q \to \mathbb{R}$ as

$$u^\varepsilon(x) := \xi \cdot x + \varepsilon\, \tilde{w}\Big(\frac{x}{\varepsilon}\Big),$$

where $\tilde{w}$ denotes the function defined in the whole $\mathbb{R}^n$, obtained through a periodic extension of w. We have that $\mathcal{F}^\varepsilon(u^\varepsilon, Q) < +\infty$, being $S_{u^\varepsilon} \subset \tilde{E}^\varepsilon \cup \tilde{F}^\varepsilon$, and that u^ε converges to $\xi \cdot x$ strongly in $L^2(Q)$. Moreover

$$\mathcal{F}^\varepsilon(u^\varepsilon, Q) = \int_Q |\nabla u^\varepsilon|^2 dx + \varepsilon\, \mathcal{H}^{n-1}(S_{u^\varepsilon}) = \varepsilon^n \bigg\{ \int_{Q/\varepsilon} |\xi + \nabla \tilde{w}|^2 dx + \mathcal{H}^{n-1}(S_{\tilde{w}}) \bigg\}.$$

Now, in order to use the periodicity of $\tilde{w}$, we can write the domain Q/ε as union of (suitably translated) periodicity cells $(0,l)^n$. Assume for simplicity that Q/ε is covered exactly by an integer number of these cells, that is by $1/(l\,\varepsilon)^n$ cells. Indeed, in the general case the integral over the remaining part of Q/ε is a negligible term.

Using (4.61), we get

$$\mathcal{F}^\varepsilon(u^\varepsilon, Q) = \frac{1}{l^n}\Big\{\int_{(0,l)^n} |\xi + \nabla w|^2 dx + \mathcal{H}^{n-1}(S_w)\Big\}$$
$$\leq \frac{1}{l^n}\inf\Big\{\int_{(0,l)^n} |\xi + \nabla v|^2 dx + \mathcal{H}^{n-1}(S_v) : v \in SBV^2_0((0,l)^n), S_w \subset \tilde{E} \cup \tilde{F}\Big\} + \frac{1}{l^n}.$$

Taking first the lim sup of both sides as $\varepsilon \to 0$ and then letting $l \to +\infty$ we obtain

$$\limsup_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon, Q) \leq f_{hom}(\xi),$$

hence the claim is proved. □

Notice that from this theorem we deduce that the whole sequence $\big(\mathcal{F}^\varepsilon\big)$ Γ-converges, since the formula for the limit energy density does not depend on the subsequence.
Up to now we have proved that the Γ-limit of the sequence $\mathcal{F}^\varepsilon$ can be expressed through an asymptotic cell problem. Nevertheless it is desirable to give a more explicit description of the density f_{hom} and this will be partially done in the next lemmas.

Lemma 4.17 *The functional $\mathcal{F}^{hom}$ is not a quadratic form.*

Proof. – *First step.* For every $\xi \in \mathbb{R}^n$ the following estimate holds:

$$A_0\xi\cdot\xi \leq f_{hom}(\xi) \leq A_0\xi\cdot\xi + P(E,Q), \tag{4.62}$$

where $P(E,Q)$ denotes the perimeter of E in Q, according to the notation introduced in Chapter 1.

Indeed, the lower bound follows from (4.12) and Remark 4.4. For the upper bound, by the definition of Γ-limit it is sufficient to find a sequence $u^\varepsilon \subset SBV^2(\Omega)$ with $S_{u^\varepsilon} \subset \tilde{E}^\varepsilon \cup \tilde{F}^\varepsilon$ and converging to $u_\xi := \xi\cdot x$ strongly in $L^2(\Omega)$, such that

$$\lim_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) = A_0\xi\cdot\xi + P(E,Q).$$

To this aim, we just take as u^ε the recovery sequence introduced in the proof of Theorem 4.5.

Second step. For every $\xi \in \mathbb{R}^n\setminus\{0\}$, we have

$$A_0\xi\cdot\xi \lneqq |\xi|^2. \tag{4.63}$$

Indeed, for $\xi \neq 0$, we have

$$A_0\xi\cdot\xi = \min\Big\{\int_{Q\setminus E} |\,\xi + \nabla w(y)|^2 dy : w \in SBV^2_\#(Q), S_w \subset E\cup F\Big\}$$
$$\leq \int_{Q\setminus E} |\xi|^2 dy = \mathcal{L}^n(Q\setminus E)\,|\xi|^2 < |\xi|^2,$$

since $0 < \mathcal{L}^n(Q\setminus E) < \mathcal{L}^n(Q) = 1$.

Third step. For every $\xi \in \mathbb{R}^n\setminus\{0\}$ we have

$$f_{hom}(\xi) \gneqq A_0\xi\cdot\xi. \tag{4.64}$$

To prove (4.64) it is enough to show that, for every $\xi \neq 0$ and for every admissible sequence u^ε converging to $u_\xi = \xi \cdot x$ strongly in $L^2(\Omega)$, we have

$$\limsup_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) > \mathcal{L}^n(\Omega)\, A_0\xi \cdot \xi. \tag{4.65}$$

We can restrict to the case $\mathcal{F}^\varepsilon(u^\varepsilon) < +\infty$, otherwise there is nothing to prove. For the sake of simplicity, let us assume that $\Omega = Q$. We will treat separately the case in which u^ε has no jumps and the general case.

Case $S_{u^\varepsilon} = \emptyset$ *for every* $\varepsilon > 0$. Being $\mathcal{F}^\varepsilon(u^\varepsilon) = \int_Q |\nabla u^\varepsilon|^2 dx < +\infty$, we have that the sequence (u^ε) is bounded in $H^1(Q)$. In particular this implies that $\nabla u^\varepsilon \rightharpoonup \xi$ weakly in $L^2(Q)$. By the weakly lower semicontinuity of the Dirichlet integral we deduce that

$$|\xi|^2 \le \liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon),$$

which together with (4.63), gives (4.65).

Case $S_{u^\varepsilon} \neq \emptyset$ *for some* $\varepsilon > 0$. Let us fix $\beta > 0$ independent of ε and classify the cubes Q_k^ε according to $\mathcal{H}^{n-1}(S_{u^\varepsilon} \cap Q_k^\varepsilon)$ being smaller or larger than $\beta\,\varepsilon^{n-1}$. From what we proved in Theorem 4.9, it is possible to choose the parameter β in such a way that the cubes where $\mathcal{H}^{n-1}(S_{u^\varepsilon} \cap Q_k^\varepsilon) \le \beta\,\varepsilon^{n-1}$ can be assumed to be undamaged.

Hence we can divide the cubes Q_k^ε in two classes: the undamaged cubes and the ones such that $\mathcal{H}^{n-1}(S_{u^\varepsilon} \cap Q_k^\varepsilon) > \beta\,\varepsilon^{n-1}$, where $\beta > 0$ is a small constant, independent of ε. Denote by $N_d(\varepsilon)$ the number of damaged cubes. From the expression of the functional no bound for $N_d(\varepsilon)$ can be derived, i.e., it may happen that $\mathcal{H}^{n-1}(S_{u^\varepsilon} \cap Q_k^\varepsilon) > \beta\,\varepsilon^{n-1}$ for every $k = 1.\ldots, N(\varepsilon)$. In any case it is clear that $\varepsilon^n N_d(\varepsilon)$ is a bounded quantity. According to the behaviour of $N_d(\varepsilon)$ as $\varepsilon \to 0$, three different cases may arise.

1) Assume that the number of damaged cube is small, that is

$$\limsup_{\varepsilon\to 0} \varepsilon^n N_d(\varepsilon) = 0. \tag{4.66}$$

Define the function $a^\varepsilon : Q \to \mathbb{R}$ as

$$a^\varepsilon(x) := \begin{cases} 0 & \text{in the damaged } Q_k^\varepsilon, \\ 1 & \text{otherwise in } Q. \end{cases}$$

From (4.66) we have that $a^\varepsilon \to 1$ strongly in $L^1(Q)$. Now,

$$\begin{aligned} \mathcal{F}^\varepsilon(u^\varepsilon) &= \int_Q |\nabla u^\varepsilon|^2 dx + \varepsilon\,\mathcal{H}^{n-1}(S_{u^\varepsilon}) \\ &\ge \int_Q a^\varepsilon(x)\,|\nabla u^\varepsilon|^2 dx + \beta\,\varepsilon^n N_d(\varepsilon). \end{aligned}$$

Then, taking the $\liminf$ as $\varepsilon \to 0$ we get

$$\liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) \ge |\xi|^2,$$

so also in this case (4.65) follows from (4.63).

2) Assume that the number of damaged cube is high, that is

$$\liminf_{\varepsilon\to 0} \varepsilon^n N_d(\varepsilon) = C > 0. \tag{4.67}$$

In this case we can say that, for ε small enough, we have $\varepsilon^n N_d(\varepsilon) > C/2$. Hence, recalling the definition (4.5) after a suitable extension of u^ε in $\tilde{E}^\varepsilon$, we have

$$\mathcal{F}^\varepsilon(u^\varepsilon) = \int_Q |\nabla u^\varepsilon|^2 dx + \varepsilon\,\mathcal{H}^{n-1}(S_{u^\varepsilon}) \geq \mathcal{G}^\varepsilon(u^\varepsilon) + \beta\,\varepsilon^n N_d(\varepsilon) \geq \mathcal{G}^\varepsilon(u^\varepsilon) + \beta\,\frac{C}{2}.$$

Then, taking the $\liminf$ as $\varepsilon \to 0$ we get by Theorem 4.1

$$\liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) \geq A_0\xi\cdot\xi + \beta\,\frac{C}{2},$$

so also in this case (4.65) holds.

3) Finally, let us analyze the intermediate case. Assume that

$$\liminf_{\varepsilon\to 0} \varepsilon^n N_d(\varepsilon) = 0.$$

and

$$\limsup_{\varepsilon\to 0} \varepsilon^n N_d(\varepsilon) = C > 0.$$

Consider a subsequence ε_k such that

$$\lim_{k\to\infty} \varepsilon_k^n N_d(\varepsilon_k) = \limsup_{\varepsilon\to 0} \varepsilon^n N_d(\varepsilon).$$

Then, we can apply the result of the previous case to this subsequence and we get

$$\limsup_{k\to\infty} \mathcal{F}^{\varepsilon_k}(u^{\varepsilon_k}) \geq A_0\xi\cdot\xi + \beta\,\frac{C}{2}.$$

Being the $\limsup$ of the whole sequence bigger or equal to the $\limsup$ of a subsequence, we have the thesis (4.65).

Fourth step. Assume by contradiction that f_{hom} is 2-homogeneous. Hence replacing ξ with $\lambda\,\xi$ in (4.62) we have that, for every $\lambda \in \mathbb{R}$,

$$\lambda^2 A_0\xi\cdot\xi \leq \lambda^2 f_{hom}(\xi) \leq \lambda^2 A_0\xi\cdot\xi + P(E,Q). \tag{4.68}$$

Dividing by λ^2 and letting $\lambda \to +\infty$ one gets

$$f_{hom}(\xi) = A_0\xi\cdot\xi,$$

which is in contrast with (4.64). This shows that f_{hom} is not 2-homogeneous and therefore $\mathcal{F}^{hom}$ is not a quadratic form. □

Remark 4.18 The estimates (4.62) and (4.64) proved in the previous lemma can be summarized by the formula

$$A_0\xi\cdot\xi \lneqq f_{hom}(\xi) \leq \min\big\{|\xi|^2, A_0\xi\cdot\xi + P(E,Q)\big\}, \tag{4.69}$$

that holds true for every $\xi \in \mathbb{R}^n \setminus \{0\}$.

It is clear that there exists a threshold $M > 0$ such that

$$A_0\xi\cdot\xi + P(E,Q) \lneqq |\xi|^2 \quad \text{for every } |\xi| > M. \tag{4.70}$$

Condition (4.70) together with (4.69) entail in particular that

$$f_{hom}(\xi) \lneqq |\xi|^2 \quad \text{for every } |\xi| > M,$$

that is, for $|\xi|$ sufficiently big, the limit density is strictly smaller than $|\xi|^2$.

It is not yet clear the behaviour of $f_{hom}(\xi)$ for $|\xi|$ very small, but we expect that

$$\lim_{|\xi|\to 0} \frac{f_{hom}(\xi)}{|\xi|^2} = 1.$$

Lemma 4.17 shows also that the functional $\mathcal{F}^{hom}$ is not a quadratic form and it is not even 2-homogeneous. Next lemma clarifies how 2-homogeneity is violated.

Lemma 4.19 *For every $\xi \in \mathbb{R}^n$ and every $\lambda \geq 1$ we have the inequality*

$$f_{hom}(\lambda\,\xi) \leq \lambda^2 f_{hom}(\xi), \tag{4.71}$$

while for every $\xi \in \mathbb{R}^n$ and every $0 < \lambda \leq 1$ we have the reverse inequality

$$f_{hom}(\lambda\,\xi) \geq \lambda^2 f_{hom}(\xi). \tag{4.72}$$

Proof. – Let $\xi \in \mathbb{R}^n$ be given and let $w \in SBV_0^2((0,t)^n)$ with $S_w \subset \tilde{E} \cup \tilde{F}$. Consider $\lambda \geq 1$ and set $w_\lambda := \lambda\, w$. Clearly it turns out that $w_\lambda \in SBV_0^2((0,t)^n)$ and $S_{w_\lambda} \subset \tilde{E} \cup \tilde{F}$. Moreover

$$\int_{(0,t)^n} |\xi + \nabla w|^2 dx + \mathcal{H}^{n-1}(S_w) \geq \frac{1}{\lambda^2} \left\{ \int_{(0,t)^n} |\,\lambda\,\xi + \nabla w_\lambda|^2 dx + \mathcal{H}^{n-1}\big(S_{w_\lambda}\big) \right\}. \tag{4.73}$$

Now, if we take the infimum of both sides of (4.73) over all $w \in SBV_0^2((0,t)^n)$ with $S_w \subset \tilde{E} \cup \tilde{F}$, we divide by t^n the resulting expression and let $t \to +\infty$, we obtain exactly (4.71), using the definition (4.55).

Proceeding in a similar way we get the reverse inequality (4.72) in the case $\lambda \leq 1$. □

4.5 Appendix

In this appendix we present an alternative proof of Theorem 4.9 in the case of a bidimensional domain Ω. This proof is based on the maximum principle, which allows us to estimate the local opening of the crack in a small ball surrounding the crack. It is therefore strictly bidimensional. A similar method can be found in [15] and in [23].

We use the same notation as in the previous sections. In particular we denote with $Q := (0,1)^2$ the unit cube and with $Q_\delta \subset\subset Q_{\hat{\delta}} \subset\subset Q$ the concentric cubes with distance δ and $\hat{\delta}$ from ∂Q, respectively. Let $E, F \subset Q_\delta$ be the sets where a crack may appear, satisfying the assumptions required in (4.1). Let us fix a boundary displacement on $\partial Q_{\hat{\delta}}$, given by the trace of a function $\varphi \in H^1(Q)$, and let $0 < \beta < (\delta - \hat{\delta})/2$ be a parameter.

Let $\tilde{v}$ be the elastic solution corresponding to the datum φ, that is the solution to the problem

$$\text{(Dir)} \quad \min\left\{ \int_{Q_{\hat{\delta}}} |\nabla w|^2 dx : w \in H^1(Q_{\hat{\delta}}), w = \varphi \,\text{on}\, \partial Q_{\hat{\delta}} \right\},$$

and let $\hat{v}$ be a solution to the problem

$$\text{(MS)} \;\min\left\{ \int_{Q_{\hat{\delta}}} |\nabla w|^2 dx + \mathcal{H}^1(S_w) : w \in SBV^2(Q_{\hat{\delta}}), S_w \subset E \cup F, \mathcal{H}^1(S_w) \leq \beta, w = \varphi \,\text{on}\, \partial Q_{\hat{\delta}} \right\}.$$

The main result of this section is the following.

Theorem 4.20 *For every β small enough, there exists a constant $\omega(\beta) > 0$ with $\omega(\beta) \to 0$ as $\beta \to 0$ such that the functions $\tilde{v}$ and $\hat{v}$ defined by the problems (Dir) and (MS), respectively, satisfy the following relation:*

$$\int_{Q_{\hat{\delta}}} |\nabla \hat{v}|^2 dx + \mathcal{H}^1(S_{\hat{v}}) \geq (1 - \omega(\beta)) \int_{Q_{\hat{\delta}}} |\nabla \tilde{v}|^2 dx. \tag{4.74}$$

Remark 4.21 Theorem 4.20 ensures that if a function has a “small” jump set, then it can be replaced with a function which has no discontinuities, up to a “small” error in terms of the energy, depending on the measure of the jump set.

This is exactly what we proved in (4.34) within Theorem 4.9. As we have already noticed, the proof of Theorem 4.20 works only in dimension 2, but it has the advantage of being more direct.

Proof. – [of Theorem 4.20] Let $\hat{v}$ be a minimizer for the problem (MS) and let us set

$$\Gamma := S_{\hat{v}}. \tag{4.75}$$

We notice that we can arbitraly change the (constant) values of the function $\hat{v}$ in the regions where the gradient is zero, and the resulting function is still a minimizer for the same problem. So our first step is to fix the constants in these regions.

Properties of Γ. As a first step, we shall split Γ in two parts, called Γ_* and $\Gamma \setminus \Gamma_*$, where Γ_* will be related to the sets on which $\hat{v}$ is constant.

Let $G \subset Q_{\hat{\delta}}$ be a set having finite perimeter in $Q(\hat{\delta})$, maximal with respect to inclusion, such that $\partial^* G \subset \Gamma$. Assume that $\mathcal{L}^2(G) > 0$. It is easy to show that the function $\hat{v}$ is constant in G. In fact otherwise we can define, for a constant $c \in \mathbb{R}$, the function

$$w := \begin{cases} \hat{v} & \text{in } Q(\hat{\delta}) \setminus G, \\ c & \text{in } G. \end{cases}$$

It turns out that w is still a competitor for (MS) and that its energy is strictly smaller than the energy of $\hat{v}$, which contradicts the minimality. Hence $\hat{v}$ is constant in G. In view of this, we may also assume that if $x \in \Gamma \setminus \partial^* G$, then x is not a point of density 1 for G. Otherwise we would get $[\hat{v}](x) = 0$, where $[\hat{v}](x)$ denotes the difference of the traces of $\hat{v}$ at x.

Let us divide G in the union of its *indecomposable components* according to [6, Theorem 1], i.e., let $(G_i)_{i\in\mathbb{N}}$ be a family of sets with finite perimeter such that $G = \cup_{i\in\mathbb{N}} G_i$, $\mathcal{H}^1(\partial G) = \sum_{i\in\mathbb{N}} \mathcal{H}^1(\partial G_i)$, $\mathcal{L}^2(G_h \cap G_k) = 0$, $\mathcal{H}^1(\partial^* G_h \cap \partial^* G_k) = 0$ for every $h \neq k$, and such that for every $k \in \mathbb{N}$ the set G_k cannot be written as $G_k = G_k^1 \cup G_k^2$ with $\mathcal{L}^2(G_k^1 \cap G_k^2) = 0$ and $\mathcal{H}^1(\partial^* G_k) = \mathcal{H}^1(\partial^* G_k^1) + \mathcal{H}^1(\partial^* G_k^2)$. Let us set

$$\Gamma_* := \partial^* G = \bigcup_{j=0}^{\infty} \partial^* G_j.$$

Choice of minimizers for (MS). Let us choose the minimizer $\hat{v}$ by requiring

$$\text{ess-}\inf_{\partial^* G_j} \hat{v}^+ \leq \hat{v}_{|G_j} \leq \text{ess-}\sup_{\partial^* G_j} \hat{v}^+, \tag{4.76}$$

where $\hat{v}^+$ denotes the trace of $\hat{v}$ external to G_j. In this way we have imposed a constraint on the constant values of $\hat{v}$ in the connected components of $Q(\hat{\delta})$ that do not touch $\partial Q(\hat{\delta})$.

Comparison between $\hat{v}$ and $\tilde{v}$. We now prove (4.74). First of all we have that

$$\begin{aligned} \int_{Q(\hat{\delta})} \left(|\nabla \tilde{v}|^2 - |\nabla \hat{v}|^2\right) dx &= \int_{Q(\hat{\delta})} (\nabla \tilde{v} - \nabla \hat{v}) (\nabla \tilde{v} + \nabla \hat{v}) \, dx \\ &= \int_{Q(\hat{\delta})} (\nabla \tilde{v} - \nabla \hat{v}) \nabla \tilde{v} \, dx. \end{aligned} \tag{4.77}$$

The last equality follows from

$$\int_{Q(\hat\delta)} (\nabla\tilde v - \nabla\hat v)\,\nabla\hat v\,dx = 0,$$

that is the Euler-Lagrange equation satisfied by $\hat v$, using as test function $\tilde v - \hat v$. Integrating by parts (4.77) we get

$$\int_{Q(\hat\delta)} \big(|\nabla\tilde v|^2 - |\nabla\hat v|^2\big)\,dx = -\int_{Q(\hat\delta)} (\tilde v - \hat v)\,\Delta\tilde v\,dx + \int_{\partial Q(\hat\delta)} (\tilde v - \hat v)\,\frac{\partial\tilde v}{\partial\nu}\,d\mathcal{H}^1 - \int_{S_{\hat v}} \frac{\partial\tilde v}{\partial\nu}\,[\hat v]\,d\mathcal{H}^1. \tag{4.78}$$

Notice that in the right-hand side of (4.78) the first two terms vanish because $\tilde v$ is harmonic and $\hat v = \tilde v$ on $\partial Q(\hat\delta)$. Therefore, (4.78) reduces to

$$\int_{Q(\hat\delta)} \big(|\nabla\tilde v|^2 - |\nabla\hat v|^2\big)\,dx = -\int_{S_{\hat v}} \frac{\partial\tilde v}{\partial\nu}\,[\hat v]\,d\mathcal{H}^1. \tag{4.79}$$

We want now to give an estimate of the last term in the previous expression. For the normal derivative of $\tilde v$, using the harmonicity of $\tilde v$ we get

$$\Big|\frac{\partial\tilde v}{\partial\nu}\Big| \le \sup_{Q_\delta} |\nabla\tilde v| \le C(\delta,\hat\delta)\,||\nabla\tilde v||_{L^2(Q(\hat\delta))}. \tag{4.80}$$

It remains to estimate $\int_{S_{\hat v}} |[\hat v]|\,d\mathcal{H}^1$.

Estimate for the jump of $\hat v$. Let us fix $x \in S_{\hat v}$ and let us define the set

$$C(x) := \big\{r \in [0, 2\,\beta] : \partial B_r(x) \cap S_{\hat v} = \emptyset\big\}.$$

As $\mathcal{H}^1(S_{\hat v}) < \beta$, we conclude that

$$\mathcal{H}^1(C(x)) \ge \beta$$

and this estimate holds true for every $x \in S_{\hat v}$.

Let us now take $r \in C(x)$, $\xi, \zeta \in \partial B_r(x)$. Let us consider the angles $\varphi, \psi \in [0, 2\pi)$ such that

$$\xi = x + (r\,\cos\varphi, r\,\sin\varphi), \quad \zeta = x + (r\,\cos\psi, r\,\sin\psi),$$

and assume for instance that $\psi < \varphi$. Then we can write

$$|\hat v(\xi) - \hat v(\zeta)| = \Big|\int_\psi^\varphi \partial_\vartheta \hat v(r,\vartheta)\,d\vartheta\Big| \le \sqrt{\varphi - \psi}\,\Big(\int_\psi^\varphi |\partial_\vartheta \hat v(r,\vartheta)|^2\,d\vartheta\Big)^{1/2}. \tag{4.81}$$

Using the fact that $\partial_\vartheta = -r\sin\vartheta\partial_1 + r\cos\vartheta\partial_2$ and the bound $(\varphi - \psi) < 2\pi$, we have

$$|\hat v(\xi) - \hat v(\zeta)| \le c\,\Big(\int_\psi^\varphi r^2|\nabla\hat v|^2 d\vartheta\Big)^{1/2} \le c\,\Big(\int_0^{2\,\pi} r^2|\nabla\hat v|^2 d\vartheta\Big)^{1/2}.$$

Hence, since the previous estimate holds true for every $\xi, \zeta \in \partial B_r(x)$, we have

$$\frac{1}{\sqrt{r}} \sup_{\xi,\zeta\in\partial B_r(x)} |\hat v(\xi) - \hat v(\zeta)| \le c\,\Big(\int_0^{2\,\pi} r|\nabla\hat v|^2 d\vartheta\Big)^{1/2}. \tag{4.82}$$

Maximum principle. For every $x \in S_{\hat{v}}$ and for a.e. $r \in C(x)$ we have

$$|[\hat{v}](x)| \leq \sup_{\xi,\zeta \in \partial B_r(x)} |\hat{v}(\xi) - \hat{v}(\zeta)|. \tag{4.83}$$

Indeed, we can define the new function

$$\hat{v}_r := \begin{cases} m_r \vee (M_r \wedge \hat{v}) & \text{in } B_r(x), \\ \hat{v} & \text{otherwise in } Q(\hat{\delta}), \end{cases}$$

where

$$m_r := \min_{\partial B_r(x)} \hat{v} \quad \text{and} \quad M_r := \max_{\partial B_r(x)} \hat{v}.$$

The function $\hat{v}_r$ is still a competitor for the minimum of (MS) and it coincides with $\hat{v}$ by (4.76). Hence either $\hat{v}_r = \hat{v}$, or the energy associated to $\hat{v}_r$ is greater or equal to the energy corresponding to $\hat{v}$. Since, by definition, the truncation reduces the energy, we conclude that $\hat{v}_r = \hat{v}$. This gives immediately that $\hat{v}$ satisfies the maximum principle in the ball $B_r(x)$, hence (4.83) is satisfied.

From (4.82) and (4.83) we obtain the inequality

$$\frac{1}{\sqrt{r}}\, |[\hat{v}](x)| \leq c \Big(\int_0^{2\pi} r |\nabla \hat{v}|^2 d\vartheta \Big)^{1/2}.$$

Squaring and integrating over $C(x)$ yields

$$|[\hat{v}](x)|^2 \int_{C(x)} \frac{1}{r}\, dr \leq c \int_{C(x)} \int_0^{2\pi} |\nabla \hat{v}|^2 r\, dr\, d\vartheta.$$

Since $C(x) \subset [0, 2\beta]$, we have

$$\int_{C(x)} \frac{1}{r}\, dr \geq \frac{1}{2\beta}\, \mathcal{H}^1(C(x)) \geq \frac{1}{2},$$

hence we deduce

$$|[\hat{v}](x)| \leq c \Big(\int_{B_{2\beta}(x)} |\nabla \hat{v}|^2 dz \Big)^{1/2}$$

for $\mathcal{H}^1$-a.e. $x \in S_{\hat{v}}$. Moreover, since $\beta < (\delta - \hat{\delta})/2$, we have that $B_{2\beta}(x) \subset Q(\hat{\delta})$ for every $x \in S_{\hat{v}}$, so that

$$|[\hat{v}](x)| \leq c \Big(\int_{Q(\hat{\delta})} |\nabla \hat{v}|^2 dz \Big)^{1/2}.$$

By integrating the previous expression over $S_{\hat{v}}$ we obtain

$$\int_{S_{\hat{v}}} |[\hat{v}]|\, d\mathcal{H}^1 \leq c\, \mathcal{H}^1(S_{\hat{v}}) ||\nabla \hat{v}||_{L^2(Q(\hat{\delta}))}. \tag{4.84}$$

Combining together (4.79), (4.80) and (4.84) we obtain

$$\int_{Q(\hat{\delta})} \big(|\nabla \tilde{v}|^2 - |\nabla \hat{v}|^2\big)\, dx \leq 2\, c\, C(\delta, \hat{\delta})\, \mathcal{H}^1(S_{\hat{v}})\, ||\nabla \tilde{v}||_{L^2(Q(\hat{\delta}))} ||\nabla \hat{v}||_{L^2(Q(\hat{\delta}))}. \tag{4.85}$$

Using in (4.85) the Young inequality $2\, ab \leq a^2 + b^2$, which holds true for every $a, b > 0$, we have

$$\int_{Q(\hat{\delta})} \big(|\nabla \tilde{v}|^2 - |\nabla \hat{v}|^2\big)\, dx \leq c\, C(\delta, \hat{\delta}) \mathcal{H}^1(S_{\hat{v}}) \big(||\nabla \tilde{v}||^2_{L^2(Q(\hat{\delta}))} + ||\nabla \hat{v}||^2_{L^2(Q(\hat{\delta}))}\big).$$

Being $\mathcal{H}^1(S_{\hat{v}}) < \beta$, we finally have

$$\int_{Q(\hat{\delta})} |\nabla \hat{v}|^2 \, dx \geq \left(\frac{1 - c\,\beta}{1 + c\,\beta} \right) \int_{Q(\hat{\delta})} |\nabla \tilde{v}|^2 \, dx, \tag{4.86}$$

where $c > 0$ is a constant depending only on the geometry of the problem. The estimate (4.86) gives (4.74) with $\omega(\beta) := 2c\beta/(1 + c\beta)$. □

Chapter 5

Damage as Γ-limit of microfractures in linearized elasticity under the non-interpenetration constraint

Chapter 5 is devoted to the extension of the homogenization results presented in Chapter 4 to the vector-valued case in linearized (possibly anisotropic) elasticity. As before, we consider a linearly elastic material presenting brittle inclusions arranged in a periodic structure. Moreover, we impose a linearized non-interpenetration constraint between the lips of the fracture.

5.1 Formulation of the problem

Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. In the following we will denote by Q the unit cube $(0,1)^n$ and with Q_ϱ the inner cube $(\varrho, 1-\varrho)^n$, for some $\varrho > 0$.
For every $\varepsilon > 0$, let us consider the periodic structure in $\mathbb{R}^n$ generated by an ε-homothetic of the basic cell Q. For notational brevity we will use the superscript ε to denote the ε-homothetic of any domain. In particular, $Q^\varepsilon := \varepsilon Q$.

Let us write the domain Ω as union of cubes of side ε:

$$\Omega = \left(\bigcup_{h \in \mathbb{Z}_\varepsilon} (Q+h)^\varepsilon \right) \cup R(\varepsilon),$$

where $\mathbb{Z}_\varepsilon := \{h \in \mathbb{Z}^n : (Q+h)^\varepsilon \subset \Omega\}$, and $R(\varepsilon)$ is the remaining part of Ω. Notice that $\mathcal{L}^n(R(\varepsilon))$ is of order ε. Let $N(\varepsilon)$ be the cardinality of the set $\mathbb{Z}_\varepsilon$; notice that $N(\varepsilon)$ is of order $1/\varepsilon^n$.

We denote by $\{Q_k^\varepsilon\}_{k=1,\dots,N(\varepsilon)}$ an enumeration of the family of cubes $(Q+h)^\varepsilon$ covering Ω, so that we can rewrite Ω as

$$\Omega = \left(\bigcup_{k=1}^{N(\varepsilon)} Q_k^\varepsilon \right) \cup R(\varepsilon). \tag{5.1}$$

In the same way we can define the sets $Q_{\delta,k}^\varepsilon$ and then I_δ^ε as

$$I_\delta^\varepsilon := \bigcup_{k=1}^{N(\varepsilon)} Q_{\delta,k}^\varepsilon. \tag{5.2}$$

Let $\mathbb{C} = (\mathbb{C}_{ijkl})$ be the elasticity tensor, considered as a symmetric positive definite linear operator from $\mathbb{M}^{n\times n}_{sym}$ into itself. It turns out that there exist two constants $0 < \lambda \leq \Lambda$ such that for any $\xi \in \mathbb{M}^{n\times n}_{sym}$, it holds

$$\lambda\,|\xi|^2 \leq \mathbb{C}\xi : \xi \leq \Lambda\,|\xi|^2, \tag{5.3}$$

where $\xi : \eta = \mathrm{trace}(\xi\eta^T) = \xi_{ij}\eta_{ij}$ and $|\xi|^2 = \xi : \xi$ is the standard Euclidean norm. Clearly, the tensor $\mathbb{C}$ is symmetric with respect to any interchange of indices, that is,

$$\mathbb{C}_{ijkl} = \mathbb{C}_{klij} = \mathbb{C}_{jikl}. \tag{5.4}$$

To every function $u \in SBD^2(\Omega)$ we associate the energy

$$\mathcal{F}^\varepsilon(u) = \int_\Omega \sigma(u) : \mathcal{E}u\,dx + \int_{J_u} f_\alpha\Big(\frac{x}{\varepsilon}, [u], \nu_u\Big)\,d\mathcal{H}^{n-1}(x),$$

where $\sigma(u) = \mathbb{C}\mathcal{E}u$, $f_\alpha : \mathbb{R}^n \times \mathbb{R}^n \times S^{n-1} \to [0, +\infty]$ is a Q-periodic function defined as

$$f_\alpha(y, z, \nu) = \begin{cases} \alpha & \text{if } y \in Q_\delta \text{ and } z \cdot \nu \geq 0, \\ +\infty & \text{otherwise,} \end{cases}$$

and α is a positive parameter. Clearly, being f_α Q-periodic, the function

$$x \mapsto f_\alpha\Big(\frac{x}{\varepsilon}, z, \nu\Big)$$

turns out to be Q^ε-periodic.
As in Chapter 4 we are interested in the case in which δ is fixed and independent of ε, while $\alpha = \alpha_\varepsilon$ depends on ε and goes to zero as $\varepsilon \to 0$. We will study three different cases, i.e.,

1. Subcritical regime $\quad \dfrac{\alpha_\varepsilon}{\varepsilon} \to 0 \quad$ as $\varepsilon \to 0$,
2. Supercritical regime $\quad \dfrac{\alpha_\varepsilon}{\varepsilon} \to +\infty \quad$ as $\varepsilon \to 0$,
3. Critical regime $\quad \dfrac{\alpha_\varepsilon}{\varepsilon} \to c \in (0, +\infty) \quad$ as $\varepsilon \to 0$.

5.2 Integral representation

The purpose of this section is to show that, independently of the rate at which α_ε converges to zero with respect to ε, the sequence $(\mathcal{F}^\varepsilon)$ admits a Γ-convergent subsequence. Moreover we will prove that the limit functionals can be written in an integral form. This will be done in an abstract setting. The characterization of the limit energy density for the different regimes will be done in the Sections 5.3-5.5.

In order to prove the Γ-convergence of a subsequence of $(\mathcal{F}^\varepsilon)$, a crucial step is to show that the functionals $\mathcal{F}^\varepsilon$ satisfy the so-called *fundamental estimate*, independently of the rate of convergence of α_ε.

As a first step, we localize the sequence $(\mathcal{F}^\varepsilon)$, introducing an explicit dependence on the set of integration. That is, for every $u \in L^2(\Omega;\mathbb{R}^n)$ and for every open set $A \in \mathcal{A}(\Omega)$ we define

$$\mathcal{F}^\varepsilon(u, A) := \begin{cases} \displaystyle\int_A \sigma(u) : \mathcal{E}u\,d\,x + \alpha_\varepsilon\,\mathcal{H}^{n-1}(J_u \cap A) & \text{if } u \in SBD^2(A), J_u \subset I^\varepsilon_\delta \cap A, \\ & [u]\cdot\nu_u \geq 0 \ \mathcal{H}^{n-1}\text{-a.e. on } J_u, \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n). \end{cases}$$

For a fixed $u \in L^2(\Omega;\mathbb{R}^n)$ we can extend the localized functional we have just defined to a measure $(\mathcal{F}^\varepsilon)^*(u,\cdot)$ on the class of Borel sets $\mathcal{B}(\Omega)$ in the usual way:

$$(\mathcal{F}^\varepsilon)^*(u,B) := \inf\big\{\mathcal{F}^\varepsilon(u,A) : A \in \mathcal{A}(\Omega), B \subseteq A\big\}. \tag{5.5}$$

Next theorem provides an extension of the fundamental estimate to SBD^2. The proof is obtained by modifying [13, Proposition 3.1], valid for SBV functions.

Theorem 5.1 (Fundamental estimate in SBD^2) *For every $\eta > 0$ and for every A', A'' and $B \in \mathcal{A}(\Omega)$, with $A' \subset\subset A''$, there exists a constant $M > 0$ with the following property: for every $\varepsilon > 0$ and for every $u \in SBD^2(A'')$ such that $J_u \subset I^\varepsilon_\delta \cap A''$ and $[u]\cdot\nu_u \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_u, and for every $v \in SBD^2(B)$ such that $J_v \subset I^\varepsilon_\delta \cap B$ and $[v]\cdot\nu_v \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_v, there exists a function $\varphi \in C^\infty_0(\Omega)$ with $\varphi = 1$ in a neighborhood of $\bar{A}'$, spt $\varphi \subset A''$ and $0 \leq \varphi \leq 1$ such that*

$$\mathcal{F}^\varepsilon(\varphi\, u + (1-\varphi)\, v, A' \cup B) \leq (1+\eta)\,\mathcal{F}^\varepsilon(u, A'') + (1+\eta)\,\mathcal{F}^\varepsilon(v,B) + M\int_T |u-v|^2 dx,$$

where $T := (A'' \setminus A') \cap B$.

Proof. – Let $\eta > 0$, A', A'' and B be as in the statement. Let $A_1,\ldots,A_{k+1}$ be open subsets of $\mathbb{R}^n$ such that $A' \subset\subset A_1 \subset\subset A_2 \subset\subset \cdots \subset\subset A_{k+1} \subset\subset A''$. For $i = 1,\ldots,k$, set $T_i := (A_{i+1} \setminus \bar{A}_i) \cap B$. For every $i = 1,\ldots,k$, let φ_i be a function in $C^\infty_0(\Omega)$ with $\varphi_i = 1$ on a neighborhood of $\bar{A}_i$ and spt $\varphi \subset A_{i+1}$.

Now, let u and v be as in the statement and define the function w_i on $A'\cup B$ as $w_i := \varphi_i u + (1-\varphi_i)\, v$ (where u and v are arbitrarily extended outside A'' and B, respectively). We need to verify that w_i belongs to the domain of $\mathcal{F}^\varepsilon(\cdot, A' \cup B)$. By definition we have that $w_i \in SBD^2(A' \cup B)$ and that $J_{w_i} \subset I^\varepsilon_\delta \cap (A' \cup B)$. Hence it remains to check that $[w_i]\cdot\nu_{w_i} \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_{w_i}. Clearly, for $x \in J_{w_i} \setminus T_i$ the condition is satisfied since it holds true for u and v. Hence we can restrict our attention to the case $x \in T_i \cap (J_u \cap J_w)$. If J_u and J_w intersect tangentially at x, then $\nu_{w_i} = \nu_u = \nu_v$ and the non-interpenetration condition is fulfilled, otherwise the normal ν_{w_i} is not defined at x.
Now we can write, for fixed $\varepsilon > 0$,

$$\begin{aligned}
\mathcal{F}^\varepsilon(w_i, A' \cup B) &= \int_{A'\cup B} \sigma(w_i) : \mathcal{E}w_i\, dx + \alpha_\varepsilon \mathcal{H}^{n-1}\big(J_{w_i} \cap (A' \cup B)\big)\\
&= \big(\mathcal{F}^\varepsilon\big)^*(u, (A'\cup B)\cap \bar{A}_i) + \big(\mathcal{F}^\varepsilon\big)^*(v, B \setminus A_{i+1}) + \mathcal{F}^\varepsilon(w_i, T_i)\\
&\leq \mathcal{F}^\varepsilon(u, A'') + \mathcal{F}^\varepsilon(v,B) + \mathcal{F}^\varepsilon(w_i, T_i).
\end{aligned} \tag{5.6}$$

Let us define $M_k := \max_{1\leq i\leq k} ||\nabla\varphi_i||^2_{L^\infty}$. Using (5.3), we can estimate the last term in (5.6) as

$$\begin{aligned}
\mathcal{F}^\varepsilon(w_i, T_i) &\leq \Lambda \int_{T_i} |\mathcal{E}(\varphi_i u + (1-\varphi_i)\, v)|^2 dx + \alpha_\varepsilon\, \mathcal{H}^{n-1}\big(J_{w_i} \cap T_i\big)\\
&\leq c\int_{T_i} |\mathcal{E}u|^2 dx + c\int_{T_i} |\mathcal{E}v|^2 dx + c\, M_k \int_{T_i} |u-v|^2 dx\\
&\quad + \alpha_\varepsilon\, \mathcal{H}^{n-1}\big(J_u \cap T_i\big) + \alpha_\varepsilon\, \mathcal{H}^{n-1}\big(J_v \cap T_i\big)\\
&\leq c\,\mathcal{F}^\varepsilon(u, T_i) + c\,\mathcal{F}^\varepsilon(v, T_i) + c\, M_k \int_{T_i} |u-v|^2 dx =: L^\varepsilon(T_i).
\end{aligned} \tag{5.7}$$

Now, let $i_0 \in \{1,\ldots,k\}$ be such that T_{i_0} realizes $\min_{1\leq i\leq k} L^\varepsilon(T_i)$. Then, being L^ε a measure, we have

$$L^\varepsilon(T_{i_0}) \leq \frac{1}{k}\sum_{i=1}^k L^\varepsilon(T_i) \leq \frac{1}{k}\, L^\varepsilon(T). \tag{5.8}$$

Notice that $i_0 = i_0(\varepsilon)$, it depends on ε. Combining together (5.6)-(5.8), we get

$$\begin{aligned}\mathcal{F}^\varepsilon(w_{i_0}, A' \cup B) \le\ & \mathcal{F}^\varepsilon(u, A'') + \mathcal{F}^\varepsilon(v, B) + \frac{1}{k}\, L^\varepsilon(T) \\ =\ & \mathcal{F}^\varepsilon(u, A'') + \mathcal{F}^\varepsilon(v, B) + \frac{c}{k}\,\mathcal{F}^\varepsilon(u, T) + \frac{c}{k}\,\mathcal{F}^\varepsilon(v, T) + \frac{c}{k}\, M_k \int_T |u - v|^2 dx \\ \le\ & \mathcal{F}^\varepsilon(u, A'') + \mathcal{F}^\varepsilon(v, B) + \frac{c}{k}\,\mathcal{F}^\varepsilon(u, A'') + \frac{c}{k}\,\mathcal{F}^\varepsilon(v, B) + \frac{c}{k}\, M_k \int_T |u - v|^2 dx. \qquad (5.9)\end{aligned}$$

Now, since the choice of the number k of the stripes between A' and A'' is completely free, we can assume that k is such that $\frac{c}{k} < \eta$. Hence $k = k(\eta)$. Let us define $\overline{M}_\eta := \frac{c}{k} M_k$; then in (5.9) we have

$$\mathcal{F}^\varepsilon(w_{i_0}, A' \cup B) \le (1+\eta)\,\mathcal{F}^\varepsilon(u, A'') + (1+\eta)\,\mathcal{F}^\varepsilon(v, B) + \overline{M}_\eta \int_T |u - v|^2 dx,$$

which is exactly the claim. □

Next theorem shows that the functional $\mathcal{F}' := \Gamma - \liminf_\varepsilon \mathcal{F}^\varepsilon$ is finite only on $H^1(\Omega; \mathbb{R}^n)$.

Theorem 5.2 *Let $\mathcal{G} : L^2(\Omega; \mathbb{R}^n) \to [0, +\infty]$ be the functional defined as*

$$\mathcal{G}(u) = \begin{cases} \displaystyle\int_\Omega A_0 \mathcal{E}u : \mathcal{E}u \, dx & \text{in } H^1(\Omega; \mathbb{R}^n), \\ +\infty & \text{otherwise in } L^2(\Omega; \mathbb{R}^n), \end{cases} \qquad (5.10)$$

where $A_0 = (A_{ijkh})$ is the fourth order tensor with constant coefficients given by the solution of the cell problem

$$A_0 \xi : \xi = \min \left\{ \int_{Q \setminus Q_\delta} \sigma(w) : \mathcal{E}w \, dy : w - \xi\, y \in H^1_\#(Q; \mathbb{R}^n) \right\},$$

for $\xi \in \mathbb{M}^{n\times n}_{sym}$. Then,

$$\mathcal{F}'(u) \ge \lambda\, \mathcal{G}(u) \quad \text{for every } u \in L^2(\Omega; \mathbb{R}^n), \qquad (5.11)$$

where $\mathcal{F}'$ is defined as in (1.1), with G^ε replaced by $\mathcal{F}^\varepsilon$ and λ is the constant in (5.3).

Proof. – Let $u \in L^2(\Omega; \mathbb{R}^n)$ and let (u^ε) be a sequence converging to u strongly in L^2 and such that $\mathcal{F}^\varepsilon(u^\varepsilon) \le c < +\infty$.

Let us define the auxiliary functional $\mathcal{G}^\varepsilon : L^2(\Omega; \mathbb{R}^n) \to [0, +\infty]$ as

$$\mathcal{G}^\varepsilon(v) = \begin{cases} \displaystyle\int_\Omega a\Big(\frac{x}{\varepsilon}\Big) |\mathcal{E}v|^2 dx & \text{if } v \in H^1(\Omega; \mathbb{R}^n), \\ +\infty & \text{otherwise in } L^2(\Omega; \mathbb{R}^n), \end{cases} \qquad (5.12)$$

where a is a Q-periodic function given by

$$a(y) = \begin{cases} 0 & \text{for } y \in Q_\delta, \\ 1 & \text{for } y \in Q \setminus Q_\delta. \end{cases}$$

It is well known that the sequence $(\mathcal{G}^\varepsilon)$ Γ-converges (with respect to the strong topology in L^2) to the functional $\mathcal{G}$ defined in (5.10). For further details we refer to [17].

We would like to compare $\mathcal{F}^\varepsilon(u^\varepsilon)$ with the value of $\mathcal{G}^\varepsilon$ on a suitable extension of u^ε. As $\mathcal{F}^\varepsilon(u^\varepsilon) \le +\infty$ we have in particular that the sequence $(\mathcal{E}u^\varepsilon)$ is equibounded in $L^2(\Omega^\varepsilon; \mathbb{R}^n)$, where $\Omega^\varepsilon := \Omega \setminus I^\varepsilon_\delta$. Hence, by Korn inequality we deduce that u^ε is equibounded in $H^1(\Omega^\varepsilon; \mathbb{R}^n)$.

Let us denote with $\tilde{u}^\varepsilon \subset H^1(\Omega;\mathbb{R}^n)$ the extension of u^ε, whose existence is guaranteed by Theorem 1.20. It turns out that $\tilde{u}^\varepsilon$ converges to u weakly in H^1, hence $u \in H^1(\Omega;\mathbb{R}^n)$. Moreover, from (5.3) we have

$$\mathcal{F}^\varepsilon(u^\varepsilon) \geq \lambda\, \mathcal{G}^\varepsilon(\tilde{u}^\varepsilon), \tag{5.13}$$

from which we deduce the bound (5.11). □

Notice that the estimate (5.11) holds true independently of the rate at which α_ε converges to zero and implies that the $\Gamma - \liminf$ of $\mathcal{F}^\varepsilon$ is finite only in $H^1(\Omega;\mathbb{R}^n)$.

We can finally state our Γ-convergence result for a subsequence of $(\mathcal{F}^\varepsilon)$.

Theorem 5.3 *Let ε be a sequence converging to zero. Then there exist a subsequence $(\sigma(\varepsilon))$ and a functional $\mathcal{F}_\sigma : L^2(\Omega;\mathbb{R}^n) \times \mathcal{A}(\Omega) \to [0, +\infty]$ such that, for every $A \in \mathcal{A}(\Omega)$,*

$$\mathcal{F}_\sigma(\cdot, A) = \Gamma - \lim_{\varepsilon \to 0} \mathcal{F}^{\sigma(\varepsilon)}(\cdot, A)$$

in the strong L^2-topology. Moreover, for every $u \in L^2(\Omega;\mathbb{R}^n)$, the set function $\mathcal{F}_\sigma(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on Ω.

Proof. – Since for every $\varepsilon > 0$ the functional $\mathcal{F}^\varepsilon$ is increasing, we deduce by Theorem 1.8 that there exist a subsequence $(\sigma(\varepsilon))$ and a functional $\mathcal{F}_\sigma : L^2(\Omega;\mathbb{R}^n) \times \mathcal{A}(\Omega) \to [0, +\infty]$ such that $\mathcal{F}_\sigma = \overline{\Gamma}(L^2) - \lim_{\varepsilon\to 0} \mathcal{F}^{\sigma(\varepsilon)}$. We put a subscript σ in order to underline that the limit functional may depend on the subsequence. Now define the nonnegative increasing functional $H : L^2(\Omega;\mathbb{R}^n) \times \mathcal{A}(\Omega) \to [0, +\infty]$ as

$$H(u, A) := \begin{cases} \displaystyle\int_A |\mathcal{E}u|^2 dx & \text{if } u_{|A} \in H^1(A;\mathbb{R}^n), \\ +\infty & \text{otherwise.} \end{cases}$$

Clearly, H is a measure with respect to A. Moreover, by (5.3) we have that $0 \leq \mathcal{F}^{\sigma(\varepsilon)} \leq \Lambda H$ for every $\varepsilon > 0$ and by Theorem 5.1 the fundamental estimate holds uniformly for the subsequence $\big(\mathcal{F}^{\sigma(\varepsilon)}\big)$. Therefore, we can proceed as in [21, Proposition 18.6] and we obtain that

$$\mathcal{F}_\sigma(u, A) = (\mathcal{F}_\sigma)'(u, A) = (\mathcal{F}_\sigma)''(u, A)$$

for every $u \in L^2(\Omega;\mathbb{R}^n)$ and for every $A \in \mathcal{A}(\Omega)$ such that $H(u, A) < +\infty$.

Fix $A \in \mathcal{A}(\Omega)$. We recall that in Theorem 5.2 we obtained the bound $\mathcal{F}'(\cdot, A) \geq \lambda\mathcal{G}(\cdot, A)$, where the functional $\mathcal{G}$ was defined in (5.10). Notice that, by definition,

$$\mathcal{F}_\sigma(\cdot, A) = (\mathcal{F}_\sigma)'(\cdot, A) \geq \mathcal{F}'(\cdot, A). \tag{5.14}$$

Hence we deduce that $\mathcal{F}_\sigma(\cdot, A) \geq \lambda\mathcal{G}(\cdot, A)$. This entails in particular that the Γ-limit of $\mathcal{F}^{\sigma(\varepsilon)}(\cdot, A)$ is finite only on $H^1(A;\mathbb{R}^n)$, which is the same domain where $J(\cdot, A)$ is finite, and is given by $\mathcal{F}_\sigma(\cdot, A)$. This proves the stated convergence of a subsequence $\big(\mathcal{F}^{\sigma(\varepsilon)}\big)$.

Finally, $\mathcal{F}^\varepsilon(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on Ω. Then, by Theorem 5.1 and [21, Theorem 18.5] we have that for every $u \in L^2(\Omega;\mathbb{R}^n)$ the set function $\mathcal{F}_\sigma(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on Ω. □

We now show general properties for the Γ-limit of $\mathcal{F}^\varepsilon$, even if, so far, we have only proved convergence of a subsequence. The fact that the whole sequence $(\mathcal{F}^\varepsilon)$ converges will follow from the characterization of the Γ-limit, which will depend only on the symmetric gradient of the deformation and not on the subsequence $\sigma(\varepsilon)$. This will be done separately for the different regimes in Theorems 5.6, 5.11, 5.15, respectively.

In the remaining part of this section we therefore assume that the whole sequence $(\mathcal{F}^\varepsilon)$ converges to a functional that we call $\mathcal{F}$, and we omit the subscript σ.

Lemma 5.4 *The restriction of the functional* $\mathcal{F} : L^2(\Omega;\mathbb{R}^n) \times \mathcal{A}(\Omega) \to [0,+\infty]$ *to* $H^1(\Omega;\mathbb{R}^n) \times \mathcal{A}(\Omega)$ *satisfies the following properties: for every* $u, v \in H^1(\Omega;\mathbb{R}^n)$ *and for every* $A \in \mathcal{A}(\Omega)$

(a) $\mathcal{F}$ *is local, i.e.,* $\mathcal{F}(u,A) = \mathcal{F}(v,A)$ *whenever* $u_{|A} = v_{|A}$*;*

(b) the set function $\mathcal{F}(u,\cdot)$ *is the restriction to* $\mathcal{A}(\Omega)$ *of a Borel measure on* Ω*;*

(c) $\mathcal{F}(\cdot,A)$ *is sequentially weakly lower semicontinuous on* $H^1(\Omega;\mathbb{R}^n)$*;*

(d) for every $a \in \mathbb{R}^n$ *we have* $\mathcal{F}(u,A) = \mathcal{F}(u+a,A)$*;*

(e) $\mathcal{F}$ *satisfies the bound*

$$0 \le \mathcal{F}(u,A) \le \Lambda \int_A |\mathcal{E}u|^2 dx.$$

Proof. – Properties (a) and (c) follow from the fact that $\mathcal{F}(\cdot,A)$ is the Γ-limit of the sequence $\mathcal{F}^\varepsilon(\cdot,A)$, while (b) comes from Theorem 5.3. For property (d) we can proceed as follows. Let $u \in H^1(\Omega;\mathbb{R}^n)$, $A \in \mathcal{A}(\Omega)$ and consider a recovery sequence $(u^\varepsilon) \subset L^2(\Omega;\mathbb{R}^n) \cap SBD^2(A)$ satisfying the usual constraints for the jump set, converging to u strongly in $L^2(\Omega;\mathbb{R}^n)$ and such that $\big(\mathcal{F}^\varepsilon(u^\varepsilon,A)\big)$ converges to $\mathcal{F}(u,A)$. Then $(u^\varepsilon + a)$ converges to $u+a$ in $L^2(\Omega;\mathbb{R}^n)$ and

$$\mathcal{F}(u+a,A) \le \liminf_{\varepsilon\to0} \mathcal{F}^\varepsilon(u^\varepsilon+a,A) = \liminf_{\varepsilon\to0} \mathcal{F}^\varepsilon(u^\varepsilon,A) = \mathcal{F}(u,A).$$

On the other hand, $\mathcal{F}(u,A) = \mathcal{F}((u+a)+(-a),A) \le \mathcal{F}(u+a,A)$, hence (d) is proved. For property (e), we just recall that the Γ-limit of the sequence $\big(\mathcal{F}^\varepsilon\big)$ is bounded from above by the functional $\Lambda \int_A |\mathcal{E}u|^2 dx$, by assumption (5.3). □

Next theorem shows that the functional $\mathcal{F}$ admits an integral representation.

Theorem 5.5 *There exists a unique convex function* $f : \mathbb{M}^{n\times n} \to [0,+\infty[$ *with the following properties:*

(i) $0 \le f(\xi) \le \Lambda|\xi|^2$ *for every* $\xi \in \mathbb{M}^{n\times n}$*;*

(ii) $\mathcal{F}(u,A) = \int_A f(\nabla u)\, dx$ *for every* $A \in \mathcal{A}(\Omega)$ *and for every* $u \in H^1(A;\mathbb{R}^n)$*.*

Proof. – Notice that the functional $\mathcal{F}$ satisfies all the assumptions of [21, Theorem 20.1], so thanks to Lemma 5.4 the Carathéodory function $f : \Omega \times \mathbb{M}^{n\times n} \to \mathbb{R}$ defined as

$$f(y,\xi) := \limsup_{\varrho\to0} \frac{\mathcal{F}(\xi\, x, B_\varrho(y))}{\mathcal{L}^n(B_\varrho(y))} \tag{5.15}$$

provides the integral representation

$$\mathcal{F}(u,A) = \int_A f(x,\nabla u)\, dx$$

for every $A \in \mathcal{A}(\Omega)$ and for every $u \in L^2(\Omega;\mathbb{R}^n)$ such that $u_{|A} \in H^1(A;\mathbb{R}^n)$. Moreover the same theorem ensures that for a.e. $x \in \Omega$ the function $f(x,\cdot)$ is convex on $\mathbb{M}^{n\times n}$ and that

$$0 \leq f(x,\xi) \leq \Lambda|\xi|^2 \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and for every } \xi \in \mathbb{M}^{n\times n}.$$

It remains to show that f is independent of the first variable and this can be done in the usual way (see for instance Theorem 4.14 in Chapter 4). □

In order to distinguish the different regimes, in the next sections we will use a different notation for the limit functional $\mathcal{F}$. It will be denoted by $\mathcal{F}^0$ in the subcritical case, by $\mathcal{F}^{hom}$ in the critical regime, and by $\mathcal{F}^\infty$ in the supercritical case.

5.3 Subcritical regime: very brittle inclusions

In this section we shall analyze the subcritical case, where the fragility coefficient of the inclusions in the material is much smaller than the size ε of the periodic structure. The energy of the material is thus given by

$$\mathcal{F}^\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \sigma(u) : \mathcal{E}u\,dx + \alpha_\varepsilon\,\mathcal{H}^{n-1}(J_u) & \text{if } u \in SBD^2(\Omega), J_u \subset I^\varepsilon_\delta, \\ & [u]\cdot\nu_u \geq 0 \ \mathcal{H}^{n-1}\text{-a.e. on } J_u, \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n), \end{cases}$$

with $\alpha_\varepsilon << \varepsilon$. It is convenient to localize the sequence $(\mathcal{F}^\varepsilon)$ by defining, for every $u \in L^2(\Omega;\mathbb{R}^n)$ and for every open set $A \in \mathcal{A}(\Omega)$

$$\mathcal{F}^\varepsilon(u,A) := \begin{cases} \displaystyle\int_A \sigma(u) : \mathcal{E}u\,dx + \alpha_\varepsilon\,\mathcal{H}^{n-1}(J_u\cap A) & \text{if } u \in SBD^2(A),\ J_u \subset I^\varepsilon_\delta \cap A, \\ & [u]\cdot\nu_u \geq 0 \ \mathcal{H}^{n-1}\text{-a.e. on } J_u\cap A, \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n). \end{cases}$$

5.3.1 Cell formula

We have already shown that the Γ-limit exists on a subsequence and it admits an integral representation. It remains to characterize the limit density. We shall prove that it is given by a cell problem.

Let $\xi \in \mathbb{M}^{n\times n}$; we will denote with ξ^s its symmetric part, that is,

$$\xi^s := \frac{\xi+\xi^T}{2} \in \mathbb{M}^{n\times n}_{sym}.$$

Define the function $f_0 : \mathbb{M}^{n\times n} \to [0,+\infty)$ as

$$f_0(\xi) := \inf\left\{ \int_Q \sigma(\xi\,x + w) : (\xi^s + \mathcal{E}w)\,dx : w \in SBD^2_\#(Q), J_w \subset Q_\delta,\ [w]\cdot\nu_w \geq 0 \text{ a.e. on } J_w \right\}. \quad (5.16)$$

Next theorem shows that the Γ-limit of the sequence $\big(\mathcal{F}^\varepsilon\big)$ can be expressed in terms of the cell formula (5.16).

Theorem 5.6 *The density f of the limit functional $\mathcal{F}$ (see Theorem 2.78) coincides with the function f_0 defined by the cell formula (5.16), i.e., for every $\xi \in \mathbb{M}^{n\times n}$*

$$f(\xi) = f_0(\xi).$$

PROOF. – *First step:* $f \geq f_0$. Let $\xi \in \mathbb{M}^{n\times n}$ and define $u_\xi(x) := \xi\, x$ for every $x \in \mathbb{R}^n$. By definition of Γ-convergence, there exists a recovery sequence $u^\varepsilon \subset SBD^2(Q)$ with $J_{u^\varepsilon} \subset I^\varepsilon_\delta$ and $[u^\varepsilon]\cdot\nu_{u^\varepsilon} \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_{u^ε}, such that $u^\varepsilon \to u_\xi$ strongly in $L^2(Q;\mathbb{R}^n)$ and

$$\lim_{\varepsilon\to 0}\mathcal{F}^\varepsilon(u^\varepsilon,Q) = \mathcal{F}^0(u_\xi,Q) = f(\xi).$$

Let us write $u^\varepsilon =: u_\xi + v^\varepsilon$, where $v^\varepsilon \in SBD^2(Q)$, $J_{v^\varepsilon} \subset I^\varepsilon_\delta$, $[v^\varepsilon]\cdot\nu_{v^\varepsilon} \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_{v^ε} and $v^\varepsilon \to 0$ strongly in $L^2(Q;\mathbb{R}^n)$. Without loss of generality we can assume $v^\varepsilon \in SBD^2_0(Q)$. Hence

$$f(\xi) = \lim_{\varepsilon\to 0}\mathcal{F}^\varepsilon(u_\xi + v^\varepsilon,Q) = \lim_{\varepsilon\to 0}\left\{\int_Q \sigma(\xi\,x + v^\varepsilon) : (\xi^s + \mathcal{E}v^\varepsilon)\,dx + \alpha_\varepsilon\mathcal{H}^{n-1}(J_{v^\varepsilon})\right\}. \tag{5.17}$$

Now, let us define the function $w^\varepsilon \in SBD^2_0(Q/\varepsilon)$ as

$$v^\varepsilon(x) =: \varepsilon\, w^\varepsilon\Big(\frac{x}{\varepsilon}\Big).$$

Remark that $J_{w^\varepsilon} \subset I_\delta$, where I_δ is defined as

$$I_\delta := \big(0,\frac{1}{\varepsilon}\big)^n \cap \bigcup_{h\in\mathbb{Z}^n}(Q_\delta + h). \tag{5.18}$$

Then, rewriting (5.17) in terms of w^ε we obtain

$$\begin{aligned} f(\xi) \;&=\; \lim_{\varepsilon\to 0}\varepsilon^n\left\{\int_{Q/\varepsilon}\sigma(\xi\,x + w^\varepsilon) : (\xi^s + \mathcal{E}w^\varepsilon)\,dx + \tfrac{\alpha_\varepsilon}{\varepsilon}\,\mathcal{H}^{n-1}(J_{w^\varepsilon})\right\} \\ &\geq \lim_{\varepsilon\to 0}\varepsilon^n\inf\left\{\int_{(0,\frac{1}{\varepsilon})^n}\sigma(\xi\,x + w) : (\xi^s + \mathcal{E}w)\,dx : w \in SBD^2_0\big((0,1/\varepsilon)^n\big),\ J_w \subset I_\delta \right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\left. [w]\cdot\nu_w \geq 0\ \mathcal{H}^{n-1}\text{-a.e. on } J_w\right\} \\ &=\; f_0(\xi), \end{aligned}$$

where the last equality follows by convexity (see [12, Theorem 14.7]). Indeed, the non-interpenetration condition is preserved under convex combinations.

Second step: $f \leq f_0$. Let $\xi \in \mathbb{M}^{n\times n}$ and $l \in \mathbb{N}$; consider a function $w \in SBD^2_0((0,l)^n)$, with $J_w \subset I_\delta$ and $[w]\cdot\nu_w \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_w, such that

$$\begin{aligned} &\int_{(0,l)^n}\sigma(\xi\,x + w) : (\xi^s + \mathcal{E}w)\,dx \\ &\leq \inf\left\{\int_{(0,l)^n}\sigma(\xi\,x + v) : (\xi^s + \mathcal{E}v)\,dx : v \in SBD^2_0((0,l)^n), J_v \subset I_\delta, [v]\cdot\nu_v \geq 0 \text{ a.e. on } J_v\right\} + 1. \end{aligned} \tag{5.19}$$

Let us define the sequence $u^\varepsilon : Q \to \mathbb{R}^n$ as

$$u^\varepsilon(x) := \xi\,x + \varepsilon\,\tilde{w}\Big(\frac{x}{\varepsilon}\Big),$$

where $\tilde{w}$ denotes the function defined in the whole $\mathbb{R}^n$, obtained through a periodic extension of w. We have that $\mathcal{F}^\varepsilon(u^\varepsilon,Q) < +\infty$, being $J_{u^\varepsilon} \subset I^\varepsilon_\delta$ and $[u^\varepsilon]\cdot\nu_{u^\varepsilon} \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_{u^ε}. Moreover u^ε converges to $\xi\,x$ strongly in $L^2(Q;\mathbb{R}^n)$. We can write

$$\mathcal{F}^\varepsilon(u^\varepsilon,Q) = \int_Q \sigma(u^\varepsilon) : \mathcal{E}u^\varepsilon\,dx + \alpha_\varepsilon\mathcal{H}^{n-1}(J_{u^\varepsilon}) \tag{5.20}$$

$$= \varepsilon^n\left\{\int_{Q/\varepsilon}\sigma(\xi\,x + \tilde{w}) : (\xi^s + \mathcal{E}\tilde{w})\,dx + \tfrac{\alpha_\varepsilon}{\varepsilon}\,\mathcal{H}^{n-1}(J_{\tilde{w}})\right\}. \tag{5.21}$$

Now, in order to use the periodicity of $\tilde{w}$, we can write the domain Q/ε as union of (suitably translated) periodicity cells $(0,l)^n$. Assume for simplicity that Q/ε is covered exactly by an integer number of these cells, that is by $1/(l\,\varepsilon)^n$ cells. Indeed, in the general case the integral over the remaining part of Q/ε is negligible. Then (5.20) reads as

$$\mathcal{F}^\varepsilon(u^\varepsilon, Q) = \frac{1}{l^n}\Big\{ \int_{(0,l)^n} \sigma(\xi\, x + w) : (\xi^s + \mathcal{E}w)\, dx + \tfrac{\alpha_\varepsilon}{\varepsilon}\, \mathcal{H}^{n-1}(J_w) \Big\}.$$

Passing to the lim sup as $\varepsilon \to 0$ and using the fact that we are in the subcritical regime, (5.3.1) gives

$$\limsup_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon, Q) = \frac{1}{l^n}\int_{(0,l)^n} \sigma(\xi\, x + w) : (\xi^s + \mathcal{E}w)\, dx. \tag{5.22}$$

Then, using (5.19) and (5.22) we get

$$\limsup_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon, Q) \le \frac{1}{l^n} \inf \Big\{ \int_{(0,l)^n} \sigma(\xi\, x + v) : (\xi^s + \mathcal{E}v)\, dx : v \in SBD^2_0((0,l)^n), \\ J_w \subset I_\delta, [v]\cdot \nu_v \ge 0 \ \mathcal{H}^{n-1}\text{-a.e. on } J_v \Big\} + \frac{1}{l^n}.$$

Letting $l \to +\infty$ in the previous expression and using again convexity, we obtain

$$\limsup_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon, Q) \le f_0(\xi),$$

hence the claim is proved. □

Remark 5.7 The previous theorem implies in particular that in the subcritical regime the whole sequence $(\mathcal{F}^\varepsilon)$ Γ-converges, since the formula for the limit energy density does not depend on the subsequence.

Moreover, from the cell formula we deduce that $f(\xi) = f(\xi^s)$, that is, the limit density function depends only on the symmetric part of its argument.

When the elasticity tensor $\mathbb{C}$ is isotropic, we can give a more explicit description of the density f_0, as shown in the following lemma.

Lemma 5.8 *Let $\mathbb{C}$ be of the special form $\mathbb{C} = 2\,\mu\,\mathbb{I} + \lambda\, Id \otimes Id$, $\mu, \lambda > 0$, and let f_0 be the corresponding limit density defined as in (5.16). Then it turns out that $f_0(Id) \ne f_0(-Id)$.*

Proof. – By the assumption on $\mathbb{C}$ we have that, for every $w \in SBD^2(Q)$

$$\sigma(w) = 2\,\mu\,\mathcal{E}w + \lambda(\mathcal{E}w : Id)\, Id = 2\,\mathcal{E}w + \lambda(\operatorname{tr}\mathcal{E}w)\, Id \in \mathbb{M}^{n\times n}_{sym}. \tag{5.23}$$

First step: $f_0(Id) \lneqq 2\,\mu\, n + \lambda\, n^2$.

First of all, we can notice that f_0 can be rewritten as

$$f_0(\xi) := \inf \Big\{ \int_Q \sigma(w) : \mathcal{E}w\, dx : w - \xi\, x \in SBD^2_{\#}(Q), J_w \subset Q_\delta,\, [w]\cdot \nu_w \ge 0 \ \mathcal{H}^{n-1}\text{-a.e. on } J_w \Big\}, \tag{5.24}$$

for every $\xi \in \mathbb{M}^{n\times n}_{sym}$.

For $i = 1, \dots, n$, let us denote with $\{\partial Q^i_{+\delta}, \partial Q^i_{-\delta}\}$ the opposite *hyperfaces* of ∂Q_δ which are orthogonal to the vector e_i. More precisely,

$$\partial Q^i_{\pm\delta} := \big\{ x \in \partial Q_\delta : x \cdot e_i \gtrless 0 \big\}.$$

Let $\xi \in \mathbb{M}^{n\times n}_{sym}$ and assume that there exists a constant $c_\xi = (c_1, \ldots, c_n) \in \mathbb{R}^n$ with the property

$$\max_{x\in\partial Q^i_{-\delta}} \big((\xi\, x)\cdot e_i\big) < c_i < \min_{x\in\partial Q^i_{-\delta}} \big((\xi\, x)\cdot e_i\big) \quad \text{for every } i = 1, \ldots, n. \tag{5.25}$$

Then, it turns out that the function w_ξ defined as

$$w_\xi(x) = \begin{cases} \xi\, x & \text{if } x \in Q \setminus Q_\delta, \\ c_\xi & \text{if } x \in Q_\delta, \end{cases}$$

is a competitor in (5.24). Indeed, $w_\xi - \xi\, x \in SBD^2_0(Q) \subset SBD^2_\#(Q)$ and $J_{w_\xi} \subset \partial Q_\delta$. It remains to check the non-interpenetration condition for every $x \in J_{w_\xi}$. Notice that if $\hat{x} \in \partial Q^i_{+\delta}$ for some i, then

$$[w_\xi(\hat{x})]\cdot \nu_{w_\xi}(\hat{x}) = (\xi\, \hat{x} - c_\xi)\cdot e_i \geq \min_{x\in\partial Q^i_{+\delta}} \big((\xi\, x)\cdot e_i\big) - c_i > 0,$$

by (5.25). On the other hand, if $\hat{x} \in \partial Q^i_{-\delta}$ for some i, then

$$[w_\xi(\hat{x})]\cdot \nu_{w_\xi}(\hat{x}) = (\xi\, \hat{x} - c_\xi)\cdot (-e_i) \geq c_i - \max_{x\in\partial Q^i_{-\delta}} \big((\xi\, x)\cdot e_i\big) > 0,$$

again by (5.25).

Since w_ξ is a competitor in (5.24), we obtain by comparison that

$$f_0(\xi) \leq \int_Q \sigma(w_\xi) : \mathcal{E}w_\xi\, dx = \mathcal{L}^n(Q\setminus Q_\delta)\big(2\,\mu\,|\xi|^2 + \lambda(\mathrm{tr}\xi)^2\big) \lneqq \big(2\,\mu\,|\xi|^2 + \lambda(\mathrm{tr}\xi)^2\big). \tag{5.26}$$

In particular, since for $\xi = Id$ the property (5.25) is clearly satisfied (it is enough to take $c_i = 0$ for every i), we have by (5.26) that

$$f_0(Id) \lneqq 2\,\mu\, n + \lambda\, n^2.$$

Second step: $f_0(-Id) = 2\,\mu\, n + \lambda\, n^2$.

In order to prove this relation it is more convenient to use the characterization of the density f_0 in the form (5.16).

Let us fix $\xi \in \mathbb{M}^{n\times n}_{sym}$. Since $\sigma(\xi\, x) = \mathbb{C}\xi \in \mathbb{M}^{n\times n}_{sym}$, we can assume without loss of generality that $\sigma(\xi\, x)$ is a diagonal matrix. Let us denote with $(\lambda_1, \ldots, \lambda_n)$ its eigenvalues.

We will derive a necessary and sufficient condition to have $w = 0$ as a minimizer of (5.16).

Let $v \in SBD^2_\#(Q)$ such that $J_v \subset Q_\delta$ and $[v]\cdot\nu_v \geq 0$ $\mathcal{H}^{n-1}$- a.e. on J_v, and let $\eta \geq 0$. We define

$$I(\eta) := \frac{1}{2}\int_Q \sigma(\xi\, x + \eta\, v) : (\xi + \eta\,\mathcal{E}v)\, dx$$

and we impose that

$$\left(\frac{d}{d\eta} I(\eta)\right)_{|\eta=0} = \frac{1}{2}\left(\frac{d}{d\eta}\int_Q \sigma(\xi\, x + \eta\, v) : (\xi + \eta\,\mathcal{E}v)\, dx\right)_{|\eta=0} \geq 0 \tag{5.27}$$

for every admissible v.

Since the functional in (5.16) is convex, we have indeed that (5.27) is a necessary and sufficient condition for minimality. We notice that condition (5.27) is equivalent to

$$\int_Q \sigma(\xi\, x) : \mathcal{E}v\, dx \geq 0 \tag{5.28}$$

for every admissible v. Integrating by parts and using the fact that $(\sigma(\xi\, x))_{ij} = \lambda_i \delta_{ij}$, the left hand side in the previous expression becomes

$$\int_Q \sigma(\xi\, x) : \mathcal{E}v\, dx = -\sum_{i,j=1}^{n} \int_{J_v} (\sigma(\xi\, x))_{ij}\, [v_j]\, \nu_{v_i} d\mathcal{H}^{n-1}(x) = -\sum_{i=1}^{n} \int_{J_v} \lambda_i\, [v_i]\, \nu_{v_i} d\mathcal{H}^{n-1}.$$

Therefore, (5.28) reduces to

$$-\sum_{i=1}^{n} \int_{J_v} \lambda_i\, [v_i]\, \nu_{v_i} d\mathcal{H}^{n-1} \geq 0 \tag{5.29}$$

for every admissible v. As v satisfies the non-interpenetration condition, that is,

$$\sum_{i=1}^{n} [v_i]\, \nu_{v_i} \geq 0. \tag{5.30}$$

and is arbitrary, we conclude that the eigenvalues λ_i of $\sigma(\xi\, x)$ are forced to be equal and negative, that is $\lambda_i = -\nu$ for every $i = 1, \ldots, n$ and $\nu > 0$. In practice this implies that

$$\frac{1}{2}\left(\frac{d}{d\eta} \int_Q \sigma(\xi\, x + \eta\, v) : (\xi + \eta\, \mathcal{E}v)\, dx\right)_{|\eta=0} \geq 0 \ \text{ for every admissible } v \quad \Longleftrightarrow \quad \sigma(\xi\, x) = -\nu\, Id, \tag{5.31}$$

that is, $w = 0$ is minimal if and only if $\sigma(\xi\, x) = -\nu\, Id$, with $\nu > 0$. By (5.23) this condition is fulfilled if and only if

$$2\,\mu\,\xi + \lambda(\mathrm{tr}\,\xi)\, Id = -\nu\, Id. \tag{5.32}$$

that is, ξ is a negative multiple of the identity. It is immediate to verify that $\xi = -\,Id$ satisfies (5.32), hence we have

$$f_0(-Id) = \int_Q (2\,\mu\, |Id|^2 + \lambda\, (\mathrm{tr} Id)^2)\, dx = 2\,\mu\, n + \lambda\, n^2.$$

□

Remark 5.9 As immediate corollary from the previous lemma, we can deduce that, in general, the limit density f_0 is not a quadratic form.

5.4 Critical regime: intermediate case

In this section we shall analyze the critical case where the fragility coefficient of the inclusions in the material is of the same order of the size ε of the periodic structure. We can assume, without loss of generality, that $\alpha_\varepsilon = \varepsilon$. The energy of the material is thus given by

$$\mathcal{F}^\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \sigma(u) : \mathcal{E}u\, dx + \varepsilon\, \mathcal{H}^{n-1}(J_u) & \text{if } u \in SBD^2(\Omega), J_u \subset I^\varepsilon_\delta, \\ & [u] \cdot \nu_u \geq 0 \ \mathcal{H}^{n-1}\text{-a.e. on } J_u, \\ +\infty & \text{otherwise in } L^2(\Omega; \mathbb{R}^n). \end{cases}$$

We localize the sequence $(\mathcal{F}^\varepsilon)$ by defining, for every $u \in L^2(\Omega; \mathbb{R}^n)$ and for every open set $A \in \mathcal{A}(\Omega)$

$$\mathcal{F}^\varepsilon(u, A) := \begin{cases} \displaystyle\int_A \sigma(u) : \mathcal{E}u\, dx + \varepsilon\, \mathcal{H}^{n-1}(J_u \cap A) & \text{if } u \in SBD^2(A),\ J_u \subset I^\varepsilon_\delta \cap A, \\ & [u] \cdot \nu_u \geq 0 \ \mathcal{H}^{n-1}\text{-a.e. on } J_u \cap A, \\ +\infty & \text{otherwise in } L^2(\Omega; \mathbb{R}^n). \end{cases}$$

5.4.1 Homogenization formula

We have already shown in Theorem 5.5 that the Γ-limit exists on a subsequence and it admits an integral representation. It remains to characterize the limit density. We shall prove that it is given by an asymptotic cell problem.

Define the function $f_{hom} : \mathbb{M}^{n\times n} \to [0,+\infty)$ as

$$f_{hom}(\xi) := \lim_{t\to+\infty} \frac{1}{t^n} \inf\Big\{ \int_{(0,t)^n} \sigma(\xi\, x + w) : (\xi^s + \mathcal{E}w)\, dx + \mathcal{H}^{n-1}(J_w) : w \in SBD^2_0\big((0,t)^n\big), \\ J_w \subset I_\delta, [w]\cdot\nu_w \geq 0 \ \mathcal{H}^{n-1}\text{-a.e. on } J_w \Big\} \tag{5.33}$$

where, according to the notation used so far, we have set

$$(0,t)^n = \Big(\bigcup_{h\in\mathbb{Z}_t} (Q+h)\Big) \cup R(t) \quad \text{and } I_\delta := \bigcup_{h\in\mathbb{Z}_t} (Q_\delta + h),$$

Theorem 5.10 *The function f_{hom} in (5.33) is well defined, that is the function*

$$g(t) := \frac{1}{t^n} \inf\Big\{ \int_{(0,t)^n} \sigma(\xi\, x + w) : (\xi^s + \mathcal{E}w)\, dx + \mathcal{H}^{n-1}(J_w) : w \in SBD^2_0\big((0,t)^n\big), \\ J_w \subset I_\delta, [w]\cdot\nu_w \geq 0 \ \mathcal{H}^{n-1}\text{-a.e. on } J_w \Big\} \tag{5.34}$$

admits a limit as $t\to+\infty$.

Proof. – For the proof we refer to Chapter 4. □

Next theorem shows that the Γ-limit of the sequence $(\mathcal{F}^\varepsilon)$ can be expressed in terms of the homogenization formula (5.33).

Theorem 5.11 *The density f of the limit functional $\mathcal{F}$ (see Theorem 5.5) coincides with the function f_{hom} defined by the cell formula (5.33), i.e., for every $\xi \in \mathbb{M}^{n\times n}$*

$$f(\xi) = f_{hom}(\xi).$$

Proof. – *First step: $f \geq f_{hom}$.* Let $\xi \in \mathbb{M}^{n\times n}$ and define $u_\xi(x) := \xi\, x$ for every $x\in\mathbb{R}^n$. By definition of Γ-convergence, there exists a recovery sequence $u^\varepsilon \subset SBD^2(Q)$ with $J_{u^\varepsilon} \subset I^\varepsilon_\delta$ and $[u^\varepsilon]\cdot\nu_{u^\varepsilon} \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_{u^ε}, such that $u^\varepsilon \to u_\xi$ strongly in $L^2(Q;\mathbb{R}^n)$ and

$$\lim_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon, Q) = \mathcal{F}^0(u_\xi, Q) = f(\xi).$$

Let us write $u^\varepsilon =: u_\xi + v^\varepsilon$, where $v^\varepsilon \in SBD^2(Q)$, $J_{v^\varepsilon} \subset I^\varepsilon_\delta$, $[v^\varepsilon]\cdot\nu_{v^\varepsilon} \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_{v^ε} and $v^\varepsilon \to 0$ strongly in $L^2(Q;\mathbb{R}^n)$. Without loss of generality we can assume $v^\varepsilon \in SBD^2_0(Q)$. Hence

$$f(\xi) = \lim_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u_\xi + v^\varepsilon, Q) = \lim_{\varepsilon\to 0} \Big\{ \int_Q \sigma(\xi\, x + v^\varepsilon) : (\xi^s + \mathcal{E}v^\varepsilon)\, dx + \varepsilon\, \mathcal{H}^{n-1}(J_{v^\varepsilon}) \Big\}. \tag{5.35}$$

Now, let us define the function $w^\varepsilon \in SBD^2_0(Q/\varepsilon)$ as

$$v^\varepsilon(x) =: \varepsilon\, w^\varepsilon\Big(\frac{x}{\varepsilon}\Big).$$

Remark that $J_{w^\varepsilon} \subset I_\delta$. Then, rewriting (5.35) in terms of w^ε we obtain

$$\begin{aligned} f(\xi) &= \lim_{\varepsilon\to 0} \varepsilon^n \Big\{ \int_{Q/\varepsilon} \sigma(\xi\, x + w^\varepsilon) : (\xi^s + \mathcal{E} w^\varepsilon)\, dx + \mathcal{H}^{n-1}(J_{w^\varepsilon}) \Big\} \\ &\geq \lim_{\varepsilon\to 0} \varepsilon^n \inf \Big\{ \int_{(0,\frac{1}{\varepsilon})^n} \sigma(\xi\, x + w) : (\xi^s + \mathcal{E} w)\, dx + \mathcal{H}^{n-1}(J_w) : w \in SBD^2_0\big((0,1/\varepsilon)^n\big), \\ &\qquad\qquad J_w \subset I_\delta, [w]\cdot\nu_w \geq 0 \ \mathcal{H}^{n-1}\text{-a.e. on } J_w \Big\} \\ &= f_{hom}(\xi). \end{aligned}$$

Second step: $f \leq f_{hom}$. Let $\xi \in \mathbb{M}^{n\times n}$ and $l \in \mathbb{N}$; then, consider a function $w \in SBD^2_0((0,l)^n)$, with $J_w \subset I_\delta$ and $[w]\cdot\nu_w \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_w, such that

$$\begin{aligned} \int_{(0,l)^n} \sigma(\xi\, x + w) : (\xi^s + \mathcal{E} w)\, dx + \mathcal{H}^{n-1}(J_w) \leq \inf \Big\{ \int_{(0,l)^n} \sigma(\xi\, x + v) : (\xi^s + \mathcal{E} v)\, dx + \mathcal{H}^{n-1}(J_v) \\ : v \in SBD^2_0((0,l)^n), J_v \subset I_\delta, [v]\cdot\nu_v \geq 0 \text{ a.e. on } J_v \Big\} + 1. \end{aligned} \tag{5.36}$$

Let us define the sequence $u^\varepsilon : Q \to \mathbb{R}^n$ as

$$u^\varepsilon(x) := \xi\, x + \varepsilon\, \tilde{w}\Big(\frac{x}{\varepsilon}\Big),$$

where $\tilde{w}$ denotes the function defined in the whole $\mathbb{R}^n$, obtained through a periodic extension of w. We have that $\mathcal{F}^\varepsilon(u^\varepsilon, Q) < +\infty$, being $J_{u^\varepsilon} \subset I^\varepsilon_\delta$ and $[u^\varepsilon]\cdot\nu_{u^\varepsilon} \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_{u^ε}. Moreover u^ε converges to $\xi\, x$ strongly in $L^2(Q;\mathbb{R}^n)$. We can write

$$\mathcal{F}^\varepsilon(u^\varepsilon, Q) = \int_Q \sigma(u^\varepsilon) : (\mathcal{E} u^\varepsilon)\, dx + \varepsilon\, \mathcal{H}^{n-1}(J_{u^\varepsilon}) = \varepsilon^n \Big\{ \int_{Q/\varepsilon} \sigma(\xi\, x + \tilde{w}) : (\xi^s + \mathcal{E}\tilde{w})\, dx + \mathcal{H}^{n-1}(J_{\tilde{w}}) \Big\}. \tag{5.37}$$

Now, in order to use the periodicity of $\tilde{w}$, we can write the domain Q/ε as union of (suitably translated) periodicity cells $(0,l)^n$. Assume for simplicity that Q/ε is covered exactly by an integer number of these cells, that is by $1/(l\,\varepsilon)^n$ cells. Indeed, in the general case the integral over the remaining part of Q/ε is a negligible term. Then, using (5.36), we get from (5.37)

$$\begin{aligned} \mathcal{F}^\varepsilon(u^\varepsilon, Q) &= \frac{1}{l^n} \Big\{ \int_{(0,l)^n} \sigma(\xi\, x + w) : (\xi^s + \mathcal{E} w)\, dx + \mathcal{H}^{n-1}(J_w) \Big\} \\ &\leq \frac{1}{l^n} \inf \Big\{ \int_{(0,l)^n} \sigma(\xi\, x + v) : (\xi^s + \mathcal{E} v)\, dx + \mathcal{H}^{n-1}(J_v) : v \in SBD^2_0((0,l)^n), \\ &\qquad\qquad J_w \subset I_\delta, [v]\cdot\nu_v \geq 0 \ \mathcal{H}^{n-1}\text{-a.e. on } J_v \Big\} + \frac{1}{l^n}. \end{aligned}$$

Passing to the lim sup as $\varepsilon \to 0$ and then letting $l \to +\infty$ we obtain

$$\limsup_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon, Q) \leq f_{hom}(\xi),$$

hence the claim is proved. □

Notice that from this theorem we deduce that also in the critical case the whole sequence $(\mathcal{F}^\varepsilon)$ Γ-converges, since the formula for the limit energy density does not depend on the subsequence.

Moreover, we deduce that $f(\xi) = f(\xi^s)$, that is the limit density function depends only on the symmetric part of its argument.

5.5 Supercritical regime: stiffer inclusions

In this section we shall analyze the supercritical case, where the fragility coefficient of the inclusions in the material is bigger than the size ε of the periodic structure.

In the sequel we present a proper modification of the argument used in [7] and in [11] to prove compactness and lower semicontinuity in SBD.

Lemma 5.12 *Let us fix* $0 < \bar{\delta} < \delta < \frac{1}{2}$ *such that* $Q_\delta \subset\subset Q_{\bar{\delta}}$. *Let* $w \in L^2(Q_{\bar{\delta}};\mathbb{R}^n)$ *and let* (w_h) *be a sequence converging strongly to* w *in* L^2. *Assume that* $||\mathcal{E}w_h||_{L^2(Q_{\bar{\delta}})} \leq c$ *and that* $\mathcal{H}^{n-1}(J_{w_h}) \to 0$ *as* $h \to 0$. *Then* $w \in H^1(Q_{\bar{\delta}};\mathbb{R}^n)$ *and*

$$\mathcal{E}w_h \rightharpoonup \mathcal{E}w \quad \textit{weakly in } L^2(Q_{\bar{\delta}};\mathbb{M}^{n\times n}).$$

Proof. – First of all, up to subsequences, we can assume that

$$\mathcal{H}^{n-1}(J_{w_h}) \leq \frac{1}{h^2}.$$

First step: $w \in H^1(Q_{\bar{\delta}};\mathbb{R}^n)$.
Let $\xi \in S^{n-1}$, $y \in \Pi^\xi$ and let us define for every $h \in \mathbb{N}$ the section $(w_h)^\xi_y(t) := w_h(y + t\,\xi)\cdot\xi$. It is well known that for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ the section $(w_h)^\xi_y \in SBV^2\big((Q_{\bar{\delta}})^\xi_y\big)$. Moreover, from the fact that $w_h \to w$ strongly in L^2, it follows that, up to subsequences,

$$(w_h)^\xi_y \to w^\xi_y \quad \text{strongly in } L^2\big((Q_{\bar{\delta}})^\xi_y\big) \quad \text{for } \mathcal{H}^{n-1} - \text{a.e. } y \in \Pi^\xi.$$

Let us denote with N_1 the set such that for every $y \in \Pi^\xi \setminus N_1$ we have $(w_h)^\xi_y \in SBV^2\big((Q_{\bar{\delta}})^\xi_y\big)$ and $(w_h)^\xi_y \to w^\xi_y$ strongly in L^2. As we have already noticed, $\mathcal{H}^{n-1}(N_1) = 0$.
Let us define the set E_h as

$$E_h := \bigcup_{j\geq h} J_{w_j}.$$

From the inequality $\mathcal{H}^{n-1}(J_{w_h}) \leq \frac{1}{h^2}$, it turns out that $\mathcal{H}^{n-1}(E_h) \to 0$ as $h \to +\infty$. Hence for every $\vartheta > 0$ there exists $h(\vartheta)$ such that $\mathcal{H}^{n-1}(E_{h(\vartheta)}) < \vartheta$. Clearly, $J_{w_h} \subset E_{h(\vartheta)}$ for every $h \geq h(\vartheta)$.

Let us denote with $(E_{h(\vartheta)})^\xi$ the projection of the set $E_{h(\vartheta)}$ on Π^ξ. By definition, it turns out that for every $y \in (\Pi^\xi \setminus (E_{h(\vartheta)})^\xi) \setminus N_1$ and for $h \geq h(\vartheta)$, the section $(w_h)^\xi_y \in H^1\big((Q_{\bar{\delta}})^\xi_y\big)$. Moreover, the H^1 norm of $(w_h)^\xi_y$ is equibounded.

Indeed, using Fubini we can write

$$\int_{Q_{\bar{\delta}}} |\mathcal{E}w_h\xi\cdot\xi|^2 dx = \int_{Q_{\bar{\delta}}} |\nabla w_h\xi\cdot\xi|^2 dx = \int_{\Pi^\xi}\left[\int_{(Q_{\bar{\delta}})^\xi_y} |\nabla(w_h)^\xi_y|^2\, dt\right] d\mathcal{H}^{n-1}(y). \tag{5.38}$$

From the fact that $\xi \in S^{n-1}$ we have

$$\int_{Q_{\bar{\delta}}} |\mathcal{E}w_h\xi\cdot\xi|^2 dx \leq \int_{Q_{\bar{\delta}}} |\mathcal{E}w_h|^2 dx, \tag{5.39}$$

and the right-hand side of (5.39) is equibounded by assumption. Hence from (5.38) we obtain

$$\int_{\Pi^\xi}\left[\int_{(Q_{\bar{\delta}})^\xi_y} |\nabla(w_h)^\xi_y|^2\, dt\right] d\mathcal{H}^{n-1}(y) \leq c. \tag{5.40}$$

Now, let $w_{k(y)}$ be a subsequence (depending on y) of w_h such that

$$\liminf_{h\to+\infty}\int_{(Q_{\bar\delta})^\xi_y}|\nabla(w_h)^\xi_y|^2\,dt=\lim_{k(y)\to+\infty}\int_{(Q_{\bar\delta})^\xi_y}|\nabla(w_{k(y)})^\xi_y|^2\,dt. \tag{5.41}$$

The bound (5.40) guarantees that there exists a function v such that, up to extracting a further subsequence $w_{j(y)}\subset w_{k(y)}$, we have

$$(w_{j(y)})^\xi_y\rightharpoonup v\quad\text{weakly in } H^1\big((Q_{\bar\delta})^\xi_y\big), \tag{5.42}$$

for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi\setminus(E_{h(\vartheta)})^\xi$. Since for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$ the whole sequence $(w_h)^\xi_y$ converges to w^ξ_y strongly in L^2, (5.42) implies that

$$(w_{j(y)})^\xi_y\rightharpoonup w^\xi_y\quad\text{weakly in } H^1\big((Q_{\bar\delta})^\xi_y\big). \tag{5.43}$$

By the lower semicontinuity in H^1 and (5.41) we obtain the inequality

$$\int_{(Q_{\bar\delta})^\xi_y}|\nabla(w^\xi_y)|^2\,dt\le\liminf_{j(y)\to+\infty}\int_{(Q_{\bar\delta})^\xi_y}|\nabla(w_{j(y)})^\xi_y|^2\,dt=\liminf_{h\to+\infty}\int_{(Q_{\bar\delta})^\xi_y}|\nabla(w_h)^\xi_y|^2\,dt, \tag{5.44}$$

which holds true for $\mathcal{H}^{n-1}$-a.e. $y\in(\Pi^\xi\setminus(E_{h(\vartheta)})^\xi)$. Integrating (5.44) with respect to y and using Fatou Lemma we get

$$\int_{\Pi^\xi\setminus(E_{h(\vartheta)})^\xi}\left[\int_{(Q_{\bar\delta})^\xi_y}|\nabla(w^\xi_y)|^2\,dt\right]d\mathcal{H}^{n-1}(y)\le\liminf_{h\to+\infty}\int_{\Pi^\xi\setminus(E_{h(\vartheta)})^\xi}\left[\int_{(Q_{\bar\delta})^\xi_y}|\nabla(w_h)^\xi_y|^2\,dt\right]d\mathcal{H}^{n-1}(y). \tag{5.45}$$

Hence, by (5.40) we obtain

$$\int_{\Pi^\xi\setminus(E_{h(\vartheta)})^\xi}\left[\int_{(Q_{\bar\delta})^\xi_y}|\nabla(w^\xi_y)|^2\,dt\right]d\mathcal{H}^{n-1}(y)\le c, \tag{5.46}$$

where the constant c is independent of ϑ.

The estimate (5.46), together with the fact that $w\in L^2(Q_{\bar\delta};\mathbb{R}^n)$ and that for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi\setminus(E_{h(\vartheta)})^\xi$ the section $w^\xi_y\in H^1\big((Q_{\bar\delta})^\xi_y\big)$, allow us to conclude that $w\in H^1(Q_{\bar\delta};\mathbb{R}^n)$.

Indeed, let us define the sets E_∞ and E_0 as

$$E_\infty:=\cap_h E_h\quad\text{and}\quad E_0:=\lim_h E_h,$$

where the convergence in the definition of E_0 is intended to be almost everywhere with respect to the Hausdorff measure.

From $\mathcal{H}^{n-1}(E_h)\le\frac{1}{h^2}$ and $E_{h+1}\subset E_h$, it turns out that

$$\mathcal{H}^{n-1}(E_\infty)=0=\mathcal{H}^{n-1}(E_0).$$

Now, since $\Pi^\xi\setminus(E_\infty)^\xi$ is contained in $\Pi^\xi\setminus(E_h)^\xi$ for h large enough, we have that for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi\setminus(E_\infty)^\xi$ the section $w^\xi_y\in H^1\big((Q_{\bar\delta})^\xi_y\big)$. Hence, being $\mathcal{H}^{n-1}(E_\infty)=0$, we conclude that $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$ the section $w^\xi_y\in H^1\big((Q_{\bar\delta})^\xi_y\big)$. On the other hand, using the monotone convergence in (5.46), we have

$$\lim_{h(\vartheta)\to\infty}\int_{\Pi^\xi\setminus(E_{h(\vartheta)})^\xi}\left[\int_{(Q_{\bar\delta})^\xi_y}|\nabla(w^\xi_y)|^2\,dt\right]d\mathcal{H}^{n-1}(y)=\int_{\Pi^\xi\setminus(E_0)^\xi}\left[\int_{(Q_{\bar\delta})^\xi_y}|\nabla(w^\xi_y)|^2\,dt\right]d\mathcal{H}^{n-1}(y)\le c. \tag{5.47}$$

Again, the fact that $\mathcal{H}^{n-1}(E_0)=0$, implies that

$$\int_{\Pi^\xi}\left[\int_{(Q_{\bar\delta})^\xi_y}|\nabla(w^\xi_y)|^2\,dt\right]d\mathcal{H}^{n-1}(y)\le c. \tag{5.48}$$

At this point we can apply [8, Proposition 3.105] to conclude that

$$\nabla(w^\xi_y)=D_t[w(y+t\xi)\cdot\xi]\in L^2(Q_{\bar\delta}),$$

that is, $Dw\xi\cdot\xi=Ew\xi\cdot\xi\in L^2(Q_{\bar\delta})$ and this is true for every ξ. Using the identity

$$Ew\xi\cdot\eta=\frac{1}{2}[Ew(\xi+\eta)\cdot(\xi+\eta)-Ew\xi\cdot\xi-Ew\eta\cdot\eta]\quad\forall\,\xi,\eta,$$

we conclude that $Ew\in L^2(Q_{\bar\delta};\mathbb{M}^{n\times n})$. Therefore, being $w\in L^2(Q_{\bar\delta};\mathbb{R}^n)$, Korn inequality ensures that $w\in H^1(Q_{\bar\delta};\mathbb{R}^n)$.

Second step: convergence of the symmetric gradient. Let us define, for a given scalar function $v\in L^2(Q_{\bar\delta})$, the functional

$$L^\xi_y(w_h,v):=\int_{(Q_{\bar\delta})^\xi_y}|\nabla(w_h)^\xi_y-v(t,y)|^2\,dt.$$

Using (5.39) and the fact that $v\in L^2(Q_{\bar\delta})$, we obtain the bound

$$\int_{\Pi^\xi}L^\xi_y(w_h,v)\,d\mathcal{H}^{n-1}(y)\le\int_{Q_{\bar\delta}}|\mathcal{E}w_h\xi\cdot\xi-v|^2dx\le c.$$

Now, let $w_{k(y)}$ be a subsequence (depending on y) of w_h such that

$$\liminf_{h\to+\infty}L^\xi_y(w_h,v)=\lim_{k(y)\to+\infty}L^\xi_y(w_{k(y)},v). \tag{5.49}$$

The bound (5.40) guarantees that, up to extracting a further subsequence $w_{j(y)}\subset w_{k(y)}$, we have

$$(w_{j(y)})^\xi_y\rightharpoonup w^\xi_y\quad\text{weakly in }H^1\big((Q_{\bar\delta})^\xi_y\big),$$

for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi\setminus(E_{h(\vartheta)})^\xi$, and in particular

$$\nabla(w_{j(y)})^\xi_y-v\rightharpoonup\nabla w^\xi_y-v\quad\text{weakly in }L^2\big((Q_{\bar\delta})^\xi_y\big).$$

Hence, by the lower semicontinuity of the functional L^ξ_y and by (5.49), we obtain

$$L^\xi_y(w,v)\le\liminf_{j(y)\to+\infty}L^\xi_y(w_{j(y)},v)=\liminf_{h\to+\infty}L^\xi_y(w_h,v).$$

Integrating the previous expression with respect to y leads to

$$\int_{\Pi^\xi\setminus(E_{h(\vartheta)})^\xi}L^\xi_y(w,v)\,d\mathcal{H}^{n-1}(y)\le\liminf_{h\to+\infty}\int_{\Pi^\xi\setminus(E_{h(\vartheta)})^\xi}L^\xi_y(w_h,v)\,d\mathcal{H}^{n-1}(y).$$

Being $w\in H^1(Q_{\bar\delta};\mathbb{R}^n)$ we can pass to the limit as $\vartheta\to0$ in the previous expression and we get

$$\int_{Q_{\bar\delta}}|\mathcal{E}w\xi\cdot\xi-v|^2dx\le\liminf_{h\to+\infty}\int_{Q_{\bar\delta}}|\mathcal{E}w_h\xi\cdot\xi-v|^2dx. \tag{5.50}$$

The fact that (5.50) holds true for every $v \in L^2(Q_{\bar\delta})$ implies that, for every $\xi \in S^{n-1}$

$$\mathcal{E}w_h\xi \cdot \xi \rightharpoonup \mathcal{E}w\xi \cdot \xi \quad \text{weakly in } L^2(Q_{\bar\delta}). \tag{5.51}$$

Now we consider a basis $\{\xi_1, \dots, \xi_n\}$ of $\mathbb{R}^n$ such that $\xi_i + \xi_j \in S^{n-1}$ for every $i \neq j$, and we specify $\xi = \xi_i + \xi_j$ in (5.51). Then we have

$$\mathcal{E}w_h \rightharpoonup \mathcal{E}w \quad \text{weakly in } L^2(Q_{\bar\delta}; \mathbb{M}^{n\times n}),$$

and this concludes the proof. □

In next lemma we give a Γ-convergence result for an auxiliary functional which will be used in the proof of the main result of this section.

Lemma 5.13 *Let us fix $0 < \bar\delta < \delta < \frac{1}{2}$ such that $Q_\delta \subset\subset Q_{\bar\delta}$. For every $h \in \mathbb{N}$, let $\mathcal{G}^h : L^2(Q_{\bar\delta}; \mathbb{R}^n) \to [0, +\infty]$ be the functional defined as*

$$\mathcal{G}^h(w) := \begin{cases} \displaystyle\int_{Q_{\bar\delta}} \sigma(w) : \mathcal{E}w\, dx + \mathcal{H}^{n-1}(J_w) & \text{if } w \in SBD^2(Q_{\bar\delta}),\ J_w \subset Q_\delta, \mathcal{H}^{n-1}(J_w) \leq \frac{1}{h^2}, \\ & \quad [w]\cdot \nu_w \geq 0\ \mathcal{H}^{n-1} \text{ a.e. on } J_w, \\ +\infty & \text{otherwise in } L^2(Q_{\bar\delta}; \mathbb{R}^n). \end{cases}$$

Then the sequence $(\mathcal{G}^h)$ Γ-converges with respect to the strong topology of L^2 to the functional $\mathcal{G} : L^2(Q_{\bar\delta}; \mathbb{R}^n) \to [0, +\infty]$ given by

$$\mathcal{G}(w) := \begin{cases} \displaystyle\int_{Q_{\bar\delta}} \sigma(w) : \mathcal{E}w\, dx & \text{if } w \in H^1(Q_{\bar\delta}; \mathbb{R}^n), \\ +\infty & \text{otherwise in } L^2(Q_{\bar\delta}; \mathbb{R}^n). \end{cases}$$

Proof. – Let $w \in L^2(Q_{\bar\delta}; \mathbb{R}^n)$ and let (w_h) be a sequence converging to w strongly in L^2 and having equibounded energy $\mathcal{G}_h$. Using the bounds (5.3) we can apply the previous lemma to obtain that $w \in H^1(Q_{\bar\delta}; \mathbb{R}^n)$ and that

$$\mathcal{E}w_h \rightharpoonup \mathcal{E}w \quad \text{weakly in } L^2(Q_{\bar\delta}; \mathbb{M}^{n\times n}). \tag{5.52}$$

Hence, by lower semicontinuity we obtain the inequality

$$\mathcal{G}(w) = \int_{Q_{\bar\delta}} \sigma(w) : \mathcal{E}w\, dx \leq \liminf_{h\to+\infty} \int_{Q_{\bar\delta}} \sigma(w_h) : \mathcal{E}w_h\, dx,$$

that implies in particular that

$$\mathcal{G}(w) \leq \liminf_{h\to+\infty} \mathcal{G}^h(w_h).$$

Finally, the existence of a recovery sequence for a function $w \in H^1(Q_{\bar\delta}; \mathbb{R}^n)$ follows immediately by taking $w_h = w$ for every $h \in \mathbb{N}$. □

Next lemma contains a Γ-convergence result for the same functionals as in Lemma 5.13, but taking into account Dirichlet boundary conditions.

Lemma 5.14 *Let $(\varphi_h), \varphi \in H^{1/2}(\partial Q_{\bar\delta}; \mathbb{R}^n)$ be such that $\varphi_h \to \varphi$ strongly in $H^{1/2}(\partial Q_{\bar\delta})$. For every $h \in \mathbb{N}$, let $\mathcal{G}^h_{\varphi_h} : L^2(Q_{\bar\delta}; \mathbb{R}^n) \to [0, +\infty]$ be the functional defined by*

$$\mathcal{G}^h_{\varphi_h}(w) := \begin{cases} \displaystyle\int_{Q_{\bar\delta}} \sigma(w) : \mathcal{E}w\, dx + \mathcal{H}^{n-1}(J_w) & \text{if } w \in SBD^2(Q_{\bar\delta}), J_w \subset Q_\delta, \mathcal{H}^{n-1}(J_w) \leq \frac{1}{h^2}, \\ & \quad [w]\cdot \nu_w \geq 0\ \mathcal{H}^{n-1}\text{-a.e. on } J_w,\ w = \varphi_h \text{ on } \partial Q_{\bar\delta}, \\ +\infty & \text{otherwise in } L^2(Q_{\bar\delta}; \mathbb{R}^n). \end{cases} \tag{5.53}$$

Then the sequence $(\mathcal{G}^h_{\varphi_h})$ Γ*-converges with respect to the strong topology of* L^2 *to the functional* $\mathcal{G}_\varphi : L^2(Q_{\bar\delta};\mathbb{R}^n) \to [0,+\infty]$ *given by*

$$\mathcal{G}_\varphi(w) := \begin{cases} \displaystyle\int_{Q_{\bar\delta}} \sigma(w) : \mathcal{E}w\,dx & \text{if } w \in H^1(Q_{\bar\delta};\mathbb{R}^n),\ w = \varphi \text{ on } \partial Q_{\bar\delta}, \\ +\infty & \text{otherwise in } L^2(Q_{\bar\delta};\mathbb{R}^n). \end{cases}$$

Proof. – *First step: proof of compactness and liminf.* Let $(w_h), w \in L^2(Q_{\bar\delta};\mathbb{R}^n)$ be such that $w_h \to w$ strongly in L^2 and $\mathcal{G}^h_{\varphi_h}(w_h) \le c < +\infty$. From the equality $\mathcal{G}^h_{\varphi_h}(w_h) = \mathcal{G}^h(w_h)$ and the previous lemma we get that $w \in H^1(Q_{\bar\delta};\mathbb{R}^n)$; moreover

$$\liminf_{h\to+\infty} \mathcal{G}^h_{\varphi_h}(w_h) = \liminf_{h\to+\infty} \mathcal{G}^h(w_h) \ge \mathcal{G}(w).$$

It remains to show that $w_{|\partial Q_{\bar\delta}} = \varphi$.

From $\mathcal{G}^h_{\varphi_h}(w_h) \le c$, we obtain the equiboundedness of w_h in $H^1(Q_{\bar\delta} \setminus Q_\delta;\mathbb{R}^n)$, and hence the convergence

$$w_h \rightharpoonup w \quad \text{weakly in } H^1(Q_{\bar\delta} \setminus Q_\delta;\mathbb{R}^n).$$

The compactness of the trace operator gives

$$\varphi_h = (w_h)_{|\partial Q_{\bar\delta}} \to w_{|\partial Q_{\bar\delta}} \quad \text{strongly in } L^2(\partial Q_{\bar\delta};\mathbb{R}^n).$$

On the other hand, by assumption, $\varphi_h \to \varphi$ strongly in $H^{1/2}(\partial Q_{\bar\delta};\mathbb{R}^n)$. Therefore, $w_{|\partial Q_{\bar\delta}} = \varphi$.

Second step: limsup. Let $w \in H^1(Q_{\bar\delta};\mathbb{R}^n)$ be such that $w_{|\partial Q_{\bar\delta}} = \varphi$. Let us consider the sequence $(v_h) \subset H^1(Q_{\bar\delta};\mathbb{R}^n)$ such that $(v_h)_{|\partial Q_{\bar\delta}} = \varphi_h - \varphi$; it turns out that $v_h \to 0$ strongly in H^1. We claim that $w_h := v_h + w$ is a recovery sequence. Indeed, $(w_h)_{|\partial Q_{\bar\delta}} = \varphi_h$ and $w_h \to w$ strongly in H^1, hence $\mathcal{E}w_h \to \mathcal{E}w$ strongly in L^2. Since the functional $\mathcal{G}^h_{\varphi_h}$ gives a norm equivalent to the standard L^2-norm, we have the desired convergence. □

Finally we are ready to state and prove the convergence result for the functional $\mathcal{F}^\varepsilon$, in the supercritical regime.

Define the functional $\mathcal{F}^\infty : L^2(\Omega;\mathbb{R}^n) \to [0,+\infty]$ as

$$\mathcal{F}^\infty(u) = \begin{cases} \displaystyle\int_\Omega \sigma(u) : \mathcal{E}u\,dx & \text{in } H^1(\Omega;\mathbb{R}^n), \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n). \end{cases}$$

Next theorem shows that $\mathcal{F}^\infty$ is the Γ-limit of the sequence $(\mathcal{F}^\varepsilon)$ in the case $\frac{\alpha_\varepsilon}{\varepsilon} \to +\infty$.

Theorem 5.15 (Γ-convergence) *(i) Let* $u \in L^2(\Omega;\mathbb{R}^n)$ *and let* (u^ε) *be a sequence converging to* u *strongly in* L^2 *and having equibounded energy* $\mathcal{F}^\varepsilon$*. Then* $u \in H^1(\Omega;\mathbb{R}^n)$ *and*

$$\liminf_{\varepsilon\to0} \mathcal{F}^\varepsilon(u^\varepsilon) \ge \mathcal{F}^\infty(u). \tag{5.54}$$

(ii) For every $u \in H^1(\Omega;\mathbb{R}^n)$ *there exists a sequence* (u^ε) *such that*

- $u^\varepsilon \to u$ *strongly in* $L^2(\Omega;\mathbb{R}^n)$, (5.55)
- $\lim_{\varepsilon\to0} \mathcal{F}^\varepsilon(u^\varepsilon) = \mathcal{F}^\infty(u)$. (5.56)

Proof. – (i) We remark that, as $\mathcal{F}^\varepsilon(u^\varepsilon)$ is bounded, the functions u^ε can have jumps only in the set I^ε_δ defined in (5.2).

We now classify the cubes Q^ε_k according to the measure of the jump set that they contain. More precisely, let us introduce a parameter $\beta > 0$ that will be chosen later in a suitable way. We say that a cube Q^ε_k is *good* whenever $\mathcal{H}^{n-1}\big(J_{u^\varepsilon} \cap Q^\varepsilon_k\big) \le \beta\,\varepsilon^{n-1}$, and *bad* otherwise, and we denote with $N_g(\varepsilon)$ and $N_b(\varepsilon)$ the number of *good* and *bad* cubes, respectively. We can notice that, since the sequence (u^ε) has equibounded energy, there exists a constant $c > 0$ such that $\alpha_\varepsilon \mathcal{H}^{n-1}(J_{u^\varepsilon}) \le c$. From this we deduce an important bound for the number of bad cubes, that is $N_b(\varepsilon) \le \dfrac{c}{\alpha_\varepsilon \varepsilon^{n-1}}$. We can write (5.1) in the form

$$\Omega = \left(\bigcup_{k=1}^{N_g(\varepsilon)} Q^\varepsilon_k\right) \cup \left(\bigcup_{k=1}^{N_b(\varepsilon)} Q^\varepsilon_k\right) \cup R(\varepsilon) =: Q^\varepsilon_g \cup Q^\varepsilon_b \cup R(\varepsilon). \tag{5.57}$$

First step: energy estimate on good cubes. Let Q^ε_k be a good cube and consider

$$\mathcal{F}^\varepsilon\big(u^\varepsilon, Q^\varepsilon_k\big) = \int_{Q^\varepsilon_k} \sigma(u^\varepsilon) : \mathcal{E}u^\varepsilon\,dx + \alpha_\varepsilon \mathcal{H}^{n-1}\big(J_{u^\varepsilon} \cap Q^\varepsilon_k\big). \tag{5.58}$$

Define the function v^ε in the unit cube Q_k as $u^\varepsilon(\varepsilon\,y) =: \sqrt{\alpha_\varepsilon \varepsilon}\, v^\varepsilon(y)$. In terms of v^ε, the energy (5.58) can be written as

$$\mathcal{F}^\varepsilon\big(u^\varepsilon, Q^\varepsilon_k\big) = \alpha_\varepsilon \varepsilon^{n-1}\left\{\int_{Q_k} \sigma(v^\varepsilon) : \mathcal{E}v^\varepsilon\,dx + \mathcal{H}^{n-1}(J_{v^\varepsilon} \cap Q_k)\right\}, \tag{5.59}$$

with $\mathcal{H}^{n-1}(J_{v^\varepsilon} \cap Q_k) \le \beta$. Therefore, by means of a change of variables we have reduced the problem to the study of a Mumford-Shah like functional over a fixed domain, with some constraints on the jump set. From now on we will omit the subscript k. Let $\bar\delta, \hat\delta$ be such that $Q_\delta \subset\subset Q_{\bar\delta} \subset\subset Q_{\hat\delta} \subset\subset Q$.

Let us consider the problem of finding local minimizers for the Mumford-Shah like functional under the required conditions, that is

$$\text{(LMin)}\quad \operatorname{loc\,min}\left\{\int_{Q_{\hat\delta}} \sigma(w) : \mathcal{E}w\,dx + \mathcal{H}^{n-1}(J_w) : \quad w \in SBD^2(Q_{\hat\delta}), J_w \subset Q_\delta, \mathcal{H}^{n-1}(J_w) \le \beta, \; [w]\cdot\nu_w \ge 0 \;\mathcal{H}^{n-1}\text{-a.e. on } J_w\right\}.$$

According to the definition given in [22], we recall that a local minimizer is a function which minimizes the given functional with respect to all perturbations with compact support. Let us denote by $\mathcal{M}_\beta$ the class of solutions of (LMin). For a given $\hat v \in \mathcal{M}_\beta$, let us consider the function $\tilde v$ solving

$$\text{(Eul)}\quad \begin{cases} \operatorname{div}\sigma(\tilde v) = 0 & \text{in } Q_{\bar\delta}, \\ \tilde v = \hat v & \text{in } Q_{\hat\delta} \setminus Q_{\bar\delta}. \end{cases}$$

We want to prove that for every $\eta > 0$ there exists $\beta > 0$ such that for every $\hat v \in \mathcal{M}_\beta$ and for the corresponding $\tilde v$ we have

$$\int_{Q_{\hat\delta}} \sigma(\tilde v) : \mathcal{E}\tilde v\,dx \le (1+\eta)\int_{Q_{\hat\delta}} \sigma(\hat v) : \mathcal{E}\hat v\,dx. \tag{5.60}$$

Hence we will take such a β in the definition of good and bad cubes.

Let us prove it by contradiction. Suppose (5.60) is false. Then there exists $\eta > 0$ such that for every $\beta > 0$ we can find $\hat v \in \mathcal{M}_\beta$ and a corresponding $\tilde v$ for which

$$\int_{Q_{\hat\delta}} \sigma(\tilde v) : \mathcal{E}\tilde v\,dx > (1+\eta)\int_{Q_{\hat\delta}} \sigma(\hat v) : \mathcal{E}\hat v\,dx. \tag{5.61}$$

In particular (5.61) implies that for every $h>0$ there exists $\hat{v}_h \in \mathcal{M}_{\frac{1}{h^2}}$ and $\tilde{v}_h$ solution of (Eul) for which

$$\int_{Q_{\hat{\delta}}} \sigma(\tilde{v}_h) : \mathcal{E}\tilde{v}_h dx > (1+\eta) \int_{Q_{\hat{\delta}}} \sigma(\hat{v}_h) : \mathcal{E}\hat{v}_h dx. \tag{5.62}$$

Since $Q_{\hat{\delta}} = \big(Q_{\hat{\delta}} \backslash Q_{\bar{\delta}}\big) \cup Q_{\bar{\delta}}$, we can split the previous integrals and, using the fact that $\tilde{v}_h = \hat{v}_h$ in $Q_{\hat{\delta}} \backslash Q_{\bar{\delta}}$ we obtain from (5.62),

$$\int_{Q_{\bar{\delta}}} \sigma(\tilde{v}_h) : \mathcal{E}\tilde{v}_h dx > (1+\eta) \int_{Q_{\bar{\delta}}} \sigma(\hat{v}_h) : \mathcal{E}\hat{v}_h dx + \eta \int_{Q_{\hat{\delta}} \backslash Q_{\bar{\delta}}} \sigma(\hat{v}_h) : \mathcal{E}\hat{v}_h dx. \tag{5.63}$$

Since the problem defining $\tilde{v}_h$ is linear, we can normalize the left-hand side of (5.63), so that we have

$$1 = \int_{Q_{\bar{\delta}}} \sigma(\tilde{v}_h) : \mathcal{E}\tilde{v}_h dx > (1+\eta) \int_{Q_{\bar{\delta}}} \sigma(\hat{v}_h) : \mathcal{E}\hat{v}_h dx + \eta \int_{Q_{\hat{\delta}} \backslash Q_{\bar{\delta}}} \sigma(\hat{v}_h) : \mathcal{E}\hat{v}_h dx. \tag{5.64}$$

This means that, in particular,

$$\int_{Q_{\hat{\delta}}} |\mathcal{E}\hat{v}_h|^2 dx \leq \frac{1}{\eta} < +\infty. \tag{5.65}$$

Without loss of generality we can assume that $\int_{Q_{\hat{\delta}} \backslash Q_\delta} \hat{v}_h dx = 0$; therefore, since $J_{\hat{v}_h} \subset Q_\delta$, (5.65) and Korn inequality imply that $||\hat{v}_h||_{(H^1(Q_{\hat{\delta}} \backslash Q_\delta)^n} \leq c$.

From this bound we deduce that there exists some $\hat{v} \in H^1(Q_{\hat{\delta}} \setminus Q_\delta; \mathbb{R}^n)$ such that $\hat{v}_h \rightharpoonup \hat{v}$ weakly in H^1 and, in particular, strongly in L^2. The local minimality of $\hat{v}_h$ implies that

$$\int_{Q_{\hat{\delta}} \backslash Q_\delta} \sigma(\hat{v}_h) : \mathcal{E}\phi \, dx = 0 \quad \text{for every } \phi \in H^1_0(Q_{\hat{\delta}} \setminus Q_\delta; \mathbb{R}^n). \tag{5.66}$$

Now, if we write (5.66) for a test function $\phi = \psi\,(\hat{v}_h - \hat{v})$, with $\psi \in C^1_0(Q_{\hat{\delta}} \setminus Q_\delta)$, we obtain

$$\int_{Q_{\hat{\delta}} \backslash Q_\delta} \psi\, \sigma(\hat{v}_h) : \mathcal{E}\hat{v}_h dx = \int_{Q_{\hat{\delta}} \backslash Q_\delta} \psi\, \sigma(\hat{v}_h) : \mathcal{E}\hat{v}\, dx - \int_{Q_{\hat{\delta}} \backslash Q_\delta} \sigma(\hat{v}_h) : \big((\hat{v}_h - \hat{v}) \nabla\psi\big)\, dx.$$

Since $\hat{v}_h \rightharpoonup \hat{v}$ weakly in $H^1(Q_{\hat{\delta}} \setminus Q_\delta; \mathbb{R}^n)$, if we let $h \to +\infty$ in the previous equation we get

$$\lim_{h\to+\infty} \int_{Q_{\hat{\delta}} \backslash Q_\delta} \psi\, \sigma(\hat{v}_h) : \mathcal{E}\hat{v}_h dx = \int_{Q_{\hat{\delta}} \backslash Q_\delta} \psi\, \sigma(\hat{v}) : \mathcal{E}\hat{v}\, dx. \tag{5.67}$$

This means in particular that for every $B \subset\subset Q_{\hat{\delta}} \setminus Q_\delta$

$$\mathcal{E}\hat{v}_h \to \mathcal{E}\hat{v} \quad \text{strongly in } L^2(B; \mathbb{M}^{n\times n}_{sym}). \tag{5.68}$$

Indeed, (5.67) together with the weak convergence of the sequence $\hat{v}_h$ in $H^1(Q_{\hat{\delta}} \setminus Q_\delta)$ imply that $\mathcal{E}\hat{v}_h$ converges strongly to $\mathcal{E}\hat{v}$ with respect to the norm induced on L^2 by the tensor $\mathbb{C}$ introduced in (5.3) and (5.4). The equivalence of this norm to the standard L^2 norm gives (5.68). Hence, by the strong convergence of $\hat{v}_h$ to $\hat{v}$ in L^2, (5.68) and Korn inequality, we deduce

$$\hat{v}_h \to \hat{v} \quad \text{strongly in } H^1(B; \mathbb{R}^n).$$

This entails the convergence of the traces of $\hat{v}_h$ on $\partial Q_{\bar{\delta}}$, that is,

$$\varphi_h := (\hat{v}_h)_{|\partial Q_{\bar{\delta}}} \to \varphi := (\hat{v})_{|\partial Q_{\bar{\delta}}} \quad \text{strongly in } H^{1/2}(\partial Q_{\bar{\delta}}; \mathbb{R}^n). \tag{5.69}$$

At this point, let us consider the following problems:

$$(\mathrm{Eul})_{\varphi_h} \quad \begin{cases} \operatorname{div} \sigma(w) = 0 & \text{in } Q_{\bar\delta} \\ w = \varphi_h & \text{on } \partial Q_{\bar\delta}, \end{cases} \qquad (\mathrm{Eul})_{\varphi} \quad \begin{cases} \operatorname{div} \sigma(w) = 0 & \text{in } Q_{\bar\delta} \\ w = \varphi & \text{on } \partial Q_{\bar\delta}. \end{cases}$$

Clearly, $\tilde v_h$ is the solution to $(\mathrm{Eul})_{\varphi_h}$ for every h. Let us call $\tilde v$ the solution to $(\mathrm{Eul})_{\varphi}$. From (5.69) it turns out that $\tilde v_h \to \tilde v$ strongly in $H^1(Q_{\bar\delta};\mathbb{R}^n)$, hence,

$$1 = \int_{Q_{\bar\delta}} \sigma(\tilde v_h) : \mathcal{E}\tilde v_h dx \to \int_{Q_{\bar\delta}} \sigma(\tilde v) : \mathcal{E}\tilde v \, dx = 1. \tag{5.70}$$

Notice that the functions $\hat v_h$ defined by the minimum problem (LMin) are absolute minimizers of the same functional once we fix the boundary data φ_h. Therefore they are absolute minimizers for the functional $\mathcal{G}^h_{\varphi_h}$ defined in (5.53). The Γ-convergence result proved in Lemma 5.14 ensures the L^2 convergence of the sequence $\hat v_h$ to the only minimizer of the functional $\mathcal{G}_\varphi$, that is exactly $\tilde v$, and the convergence of the energies.

Now, if we let $h \to +\infty$ in (5.64) we obtain

$$1 = \int_{Q_{\bar\delta}} \sigma(\tilde v) : \mathcal{E}\tilde v \, dx \geq (1+\eta) \int_{Q_{\bar\delta}} \sigma(\tilde v) : \mathcal{E}\tilde v \, dx,$$

which gives the contradiction, therefore (5.60) is proved.

Let $\eta > 0$ be fixed; we choose $\beta > 0$ such that the property (5.60) is satisfied and for every $\varepsilon > 0$ we consider the problem

$$(\mathrm{Min}) \ \min \Big\{ \int_{Q_{\hat\delta,k}} \sigma(w) : \mathcal{E}w \, dx + \mathcal{H}^{n-1}(J_w) \quad : w \in SBD^2(Q_{\hat\delta,k}), J_w \subset Q_{\delta,k}, \mathcal{H}^{n-1}(J_w) \leq \beta, \\ [w]\cdot\nu_w \geq 0 \ \mathcal{H}^{n-1}\text{-a.e. on } J_w, w = v^\varepsilon \text{ on } \partial Q_{\hat\delta,k} \Big\}.$$

For a minimizer $\hat v^\varepsilon$ in (Min), let us consider the corresponding $\tilde v^\varepsilon$ defined by (Eul), with $\hat v$ replaced by $\hat v^\varepsilon$. We have that, as before,

$$\int_{Q_{\hat\delta,k}} \sigma(\tilde v^\varepsilon) : \mathcal{E}\tilde v^\varepsilon dx \leq (1+\eta) \int_{Q_{\hat\delta,k}} \sigma(\hat v^\varepsilon) : \mathcal{E}\hat v^\varepsilon dx. \tag{5.71}$$

Hence, in particular,

$$\int_{Q_{\hat\delta,k}} \sigma(v^\varepsilon) : \mathcal{E}v^\varepsilon dx + \mathcal{H}^{n-1}(J_{v^\varepsilon} \cap Q_{\hat\delta,k}) \geq \int_{Q_{\hat\delta,k}} \sigma(\hat v^\varepsilon) : \mathcal{E}\hat v^\varepsilon dx + \mathcal{H}^{n-1}(J_{\hat v^\varepsilon} \cap Q_{\hat\delta,k}) \tag{5.72}$$

$$\geq \left(1 - \frac{\eta}{1+\eta}\right) \int_{Q_{\hat\delta,k}} \sigma(\tilde v^\varepsilon) : \mathcal{E}\tilde v^\varepsilon dx, \tag{5.73}$$

where v^ε is the function in (5.59).
Now we define $\tilde u^\varepsilon$ as $\tilde u^\varepsilon(\varepsilon y) := \sqrt{\alpha_\varepsilon \varepsilon}\ \tilde v^\varepsilon(y)$. By (5.59) and (5.72) we obtain

$$\int_{Q^\varepsilon_{\hat\delta,k}} \sigma(u^\varepsilon) : \mathcal{E}u^\varepsilon dx + \alpha_\varepsilon \mathcal{H}^{n-1}\big(J_{u^\varepsilon} \cap Q^\varepsilon_{\hat\delta,k}\big) \geq \left(1 - \frac{\eta}{1+\eta}\right) \int_{Q^\varepsilon_{\hat\delta,k}} \sigma(\tilde u^\varepsilon) : \mathcal{E}\,\tilde u^\varepsilon dx. \tag{5.74}$$

Second step: energy estimate on bad cubes. Let Q^ε_k be a bad cube. The idea is to use the trivial inequality

$$\int_{Q^\varepsilon_k} \sigma(u^\varepsilon) : \mathcal{E}u^\varepsilon dx + \alpha_\varepsilon \mathcal{H}^{n-1}\big(J_{u^\varepsilon} \cap Q^\varepsilon_k\big) \geq \int_{Q^\varepsilon_k} \chi^\varepsilon_\delta \, \sigma(\hat u^\varepsilon) : \mathcal{E}\hat u^\varepsilon dx,$$

where χ_δ^ε is the characteristic function of the set $Q_k^\varepsilon \setminus Q_{\delta,k}^\varepsilon$ and the function $\hat{u}^\varepsilon$ coincides with u^ε in $Q_k^\varepsilon \setminus Q_{\delta,k}^\varepsilon$ and is extended to $Q_{\delta,k}^\varepsilon$ in a way that keeps its H^1 norm bounded. We recall also that we have a control on the number of bad cubes, that is, $N_b(\varepsilon) \leq \dfrac{c}{\alpha_\varepsilon \varepsilon^{n-1}}$.

Third step: final estimate. Let us define the new sequence $w^\varepsilon \in SBD^2(\Omega)$ as

$$w^\varepsilon := \begin{cases} \tilde{u}^\varepsilon & \text{in } \big(Q_{\hat{\delta}}^\varepsilon\big)^g, \\ u^\varepsilon & \text{in } \big((Q^\varepsilon)^g \setminus \big(Q_{\hat{\delta}}^\varepsilon\big)^g\big) \cup R(\varepsilon), \\ \hat{u}^\varepsilon & \text{in } (Q^\varepsilon)^b, \end{cases}$$

where $(Q^\varepsilon)^g$, $(Q^\varepsilon)^b$ and $R(\varepsilon)$ are given in (5.57) and $\big(Q_{\hat{\delta}}^\varepsilon\big)^g$ denotes the set

$$\big(Q_{\hat{\delta}}^\varepsilon\big)^g := \bigcup_{k=1}^{N_g(\varepsilon)} Q_{\hat{\delta},k}^\varepsilon.$$

Define also the function $a^\varepsilon : \Omega \to \mathbb{R}$ as

$$a^\varepsilon(x) := \begin{cases} 0 & \text{in } (Q_\delta^\varepsilon)^b, \\ 1 & \text{otherwise in } \Omega. \end{cases}$$

From what we proved in the previous steps we can write

$$\mathcal{F}^\varepsilon(u^\varepsilon) \geq \left(1 - \frac{\eta}{1+\eta}\right) \int_\Omega a^\varepsilon(x)\, \sigma(w^\varepsilon) : \mathcal{E} w^\varepsilon \, dx. \tag{5.75}$$

It remains to apply Lemma 4.6 to (5.75). First of all we show the convergence of a^ε. We have

$$\int_\Omega |\, a^\varepsilon - 1|\, dx = \mathcal{L}^n\big((Q_\delta^\varepsilon)^b\big) = N_b(\varepsilon)\, \varepsilon^n \mathcal{L}^n(Q_\delta) \leq c \sqrt{\frac{\varepsilon}{\alpha_\varepsilon}},$$

hence $a^\varepsilon \to 1$ strongly in $L^1(\Omega)$. Once we prove that $w^\varepsilon \rightharpoonup u$ weakly in $H^1(\Omega;\mathbb{R}^n)$, it turns out that

$$\liminf_{\varepsilon \to 0} \mathcal{F}^\varepsilon(u^\varepsilon) \geq \left(1 - \frac{\eta}{1+\eta}\right) \int_\Omega \sigma(u) : \mathcal{E} u \, dx,$$

and the thesis follows letting η converge to zero.

Fourth step: convergence of w^ε. First of all it is clear that $||\mathcal{E} w^\varepsilon||_{(L^2(\Omega))^{n\times n}} \leq c$. Then, the fact that w^ε and u^ε coincide in a set with positive measure ensures the convergence.

(ii) The claim follows trivially by choosing $u^\varepsilon = u$ for every $\varepsilon > 0$. □

Chapter 6

An extension theorem in SBV and an application to the homogenization of the Mumford-Shah functional

In this last chapter we study the asymptotic behaviour of the Mumford-Shah functional in periodically perforated domains, under homogeneous Neumann conditions on the boundaries of the perforations.

6.1 Extension property

In this section we state and prove the main result of the chapter, that is, an extension property for SBV maps.

Theorem 6.1 *Let $D, A \subset \mathbb{R}^n$ be bounded open sets with Lipschitz boundary and assume that $D \subset A$ and $\partial D \cap A \subset\subset A$. Then there exists a constant $c = c(n, D, A) > 0$ such that for every $u \in SBV^2(D) \cap L^\infty(D)$ we can construct an extension $\tilde{u} \in SBV^2(A) \cap L^\infty(A)$ of u satisfying*

$$\begin{aligned} &(i)\ \tilde{u} = u \quad a.e.\ in\ D, \\ &(ii)\ \|\tilde{u}\|_{L^\infty(A)} = \|u\|_{L^\infty(D)}, \\ &(iii)\ MS(\tilde{u}, A) \le c\, MS(u, D). \end{aligned} \tag{6.1}$$

The constant c in invariant under homotheties.

In order to prove the extension result, we need to use a retraction property for Lipschitz domains.

Theorem 6.2 *Let $D \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary ∂D. Then there exist an open neighbourhood $W \subset \mathbb{R}^n$ of ∂D and a bilipschitz map $\phi : W \cap \overline{D} \to W \cap (\mathbb{R}^n \setminus D)$ such that $\phi_{|\partial D} = Id$.*

PROOF. – Since the set D has Lipschitz boundary, we can find a finite open cover $U_1, \ldots, U_m$ of ∂D such that we can associate to every U_j a vector $u_j^0 \in \mathbb{R}^n$ and a parameter $\eta_j \in (0, 1]$ with the following property. If $x \in \partial D \cap U_j$ for some j, then for every $t \in (0, 1]$ and for every $u_j \in \mathbb{R}^n$ such that $|u_j - u_j^0| < \eta_j$ it turns out that $x + t\, u_j \in D$ and $x - t\, u_j \in \mathbb{R}^n \setminus \overline{D}$.

Set $\eta := \min_j \eta_j$. Now, for every index j we fix an open set $V_j \subset\subset U_j$ such that $V_1, \dots, V_m$ is still a covering of ∂D. Let $(\psi_j)_{j=1,\dots,m}$ be a partition of unity for ∂D subordinate to $(V_j)_{j=1,\dots,m}$:

$$\psi_j \in C_0^\infty(\mathbb{R}^n),\ \text{supp}\ \psi_j \subset V_j,\ 0 \le \psi_j \le 1 \text{ on } \mathbb{R}^n,\ \sum_{j=1}^m \psi_j = 1 \,\text{on}\ \partial D.$$

Let us fix $\alpha_0 > 0$ so that for every collection of vectors $\{u_1, \dots, u_m\}$ satisfying $|u_i - u_i^0| < \eta$ for every i, we have

$$\alpha_0 \sum_{i=1}^m |u_i| < \text{dist}(V_j, \partial U_j) \quad \text{for } j = 1, \dots, m.$$

Let us define $B_\eta^m(u^0) := \{u = (u_1, \dots, u_m) \in (\mathbb{R}^n)^m : |u_i - u_i^0| < \eta \text{ for every } i\}$. For every $|\alpha| \le \alpha_0$ and for every $u \in B_\eta^m(u^0)$, we define the C^∞ function $r_u^\alpha : \mathbb{R}^n \to \mathbb{R}^n$

$$r_u^\alpha(x) := x + \alpha \sum_{j=1}^m \psi_j(x)\, u_j.$$

It turns out that by construction $r_u^\alpha - Id$ has compact support and $r_u^\alpha - Id \to 0$ in $C_0^\infty(\mathbb{R}^n; \mathbb{R}^n)$ as $\alpha \to 0$. Moreover, following the argument used in [22, Proposition 1.2], it is possible to show that $r_u^\alpha(\overline{D}) \subset D$ for $0 < \alpha \le \alpha_0$, while $r_u^\alpha(\mathbb{R}^n \setminus D) \subset \mathbb{R}^n \setminus \overline{D}$ for $-\alpha_0 \le \alpha < 0$.
Let us set $\Psi_u(x) := \sum_{j=1}^m \psi_j(x)\, u_j$ and $\Psi^0(x) := \sum_{j=1}^m \psi_j(x)\, u_j^0$.

We claim that there exists $\eta_0 \in (0, \eta]$ such that for every $x \in \partial D$ we have the following property

$$|v - \Psi^0(x)| < \eta_0 \ \Rightarrow \begin{cases} x + \alpha\, v \in D & \text{if} \quad 0 < \alpha \le \alpha_0, \\ x + \alpha\, v \in \mathbb{R}^n \setminus \overline{D} & \text{if} \ -\alpha_0 \le \alpha < 0. \end{cases} \tag{6.2}$$

We notice that in order to obtain (6.2) it is sufficient to prove that

$$\text{if } v \text{ satisfies } |v - \Psi^0(x)| < \eta_0, \text{ then } v = \Psi_u(x) \quad \text{for some } u \in B_\eta^m(u^0). \tag{6.3}$$

Indeed, we know that for every $u \in B_\eta^m(u^0)$ we have $x + \alpha\, \Psi_u(x) \in D$ if $0 < \alpha \le \alpha_0$ and $x + \alpha\, \Psi_u(x) \in \mathbb{R}^n \setminus \overline{D}$ if $-\alpha_0 \le \alpha < 0$.

Let us prove (6.3). Let us fix $x \in \partial D$; we define the linear map $L^x : \big(\mathbb{R}^n\big)^m \to \mathbb{R}^n$ as

$$u = (u_1, \dots, u_m) \mapsto L^x(u) := \Psi_u(x) = \sum_{j=1}^m \psi_j(x)\, u_j.$$

Since $x \in \partial D$, we have that $\sum_j \psi_j(x) = 1$. Hence, there exists $\bar\imath \in \{1, \dots, m\}$ such that $\psi_{\bar\imath}(x) \ge \frac{1}{m}$.

We claim that $L^x(B_\eta^m(u^0))$ contains a neighbourhood of $L^x(u^0)$. First of all, let us notice that

$$L^x(B_\eta^m(u^0)) = L^x\big(B_\eta(u_1^0) \times \cdots \times B_\eta(u_m^0)\big) \supseteq A, \tag{6.4}$$

where $A := L^x\big(\{u_1^0\} \times \cdots \times \{u_{\bar\imath-1}^0\} \times B_\eta(u_{\bar\imath}^0) \times \{u_{\bar\imath+1}^0\} \times \cdots \times \{u_m^0\}\big)$. Easy computations show that

$$\big\{y - L^x(u_0) : y \in A\big\} = B_{\eta\, \psi_{\bar\imath}(x)}(0).$$

Therefore we can rewrite A as

$$A = L^x(u^0) + B_{\eta\, \psi_{\bar\imath}(x)}(0) = B_{\eta\, \psi_{\bar\imath}(x)}(L^x(u^0)) \supseteq B_{\frac{\eta}{m}}(L^x(u^0)). \tag{6.5}$$

The same argument can be repeated for every $x \in \partial D$. Let us define $\eta_0 := \frac{\eta}{m}$. We want to verify (6.3).

Let $x \in \partial D$ and $v \in \mathbb{R}^n$ such that $|v - \Psi^0(x)| < \eta_0$, i.e., $v \in B_{\eta_0}(\Psi^0(x)) = B_{\eta_0}(L^x(u^0))$. From (6.4) and (6.5) we have that $v \in A \subset L^x(B^m_\eta(u^0))$, hence there exists $u \in B^m_\eta(u^0)$ such that $v = L^x(u) = \Psi_u(x)$. This proves (6.3).

For every $x_0 \in \partial D$ let us consider the following Cauchy problem:

$$\begin{cases} \dot{x}(t) = \Psi^0(x(t)) \\ x(0) = x_0 \end{cases} \tag{6.6}$$

We denote by $\Phi(x_0, t)$ the solution to (6.6). Using (6.2) and the compactness of ∂D, we have that there exists $t_0 > 0$ independent of $x_0 \in \partial D$ such that $\{\Phi(x_0,t) : t \in (0,t_0)\} \subset D$ and $\{\Phi(x_0,-t) : t \in (0,t_0)\} \subset \mathbb{R}^n \setminus \overline{D}$. Clearly, the restriction $\Phi_{|\partial D \times (-t_0,t_0)}$ is bijective. In particular we have that $\{\Phi(x_0,0) : x_0 \in \partial D\} = \partial D$.

By classical results, the set W defined as

$$W := \big\{\Phi(x_0,t) : (x_0,t) \in \partial D \times (-t_0,t_0)\big\} \tag{6.7}$$

is an open neighbourhood of ∂D. Now we define $W^+, W^- \subset W$ as

$$W^+ := W \cap D = \big\{\Phi(x_0,t) : (x_0,t) \in \partial D \times (0,t_0)\big\}, \tag{6.8}$$

$$W^- := W \cap (\mathbb{R}^n \setminus \overline{D}) = \big\{\Phi(x_0,t) : (x_0,t) \in \partial D \times (-t_0,0)\big\}. \tag{6.9}$$

Using classical properties of the flow it is possible to show that the map $\Phi_{|\partial D \times (-t_0,t_0)} : \partial D \times (-t_0,t_0) \to W$ is bilipschitz.

We define $\phi : W^+ \cup \partial D \to W^- \cup \partial D$ in the following way. Let $y \in W^+ \cup \partial D$. There exists a pair $(x_0,t) \in \partial D \times [0,t_0)$ such that $y = \Phi(x_0,t)$. We set $\phi(y) := \Phi(x_0,-t)$. This map is bijective and bilipschitz. Hence the theorem is proved. □

Proof. – [Proof of Theorem 6.1] Let $u \in SBV^2(D) \cap L^\infty(D)$.

By Theorem 6.2 we can find a neighbourhood W of $\partial D \cap A$, $W \subset\subset A$ and a bilipschitz map

$$\phi : W \cap (A \setminus D) \to W \cap \overline{D}, \quad \phi_{|\partial D \cap A} = Id.$$

Now we define $v : D \cup W \to \mathbb{R}$ as

$$v(x) = \begin{cases} u(x) & \text{if } x \in D, \\ u(\phi(x)) & \text{if } x \in W \cap (A \setminus D). \end{cases}$$

It turns out that $v \in SBV^2(D \cup W)$ and that the following estimates hold:

$$\int_{D\cup W} |\nabla v|^2 dx = \int_D |\nabla u|^2 dx + \int_{W\setminus D} |\nabla (u \circ \phi)|^2 dx \le C \int_D |\nabla u|^2 dx, \tag{6.10}$$

$$\mathcal{H}^{n-1}\big(S_v \cap (D \cup W)\big) = \mathcal{H}^{n-1}\big(S_u \cap D\big) + \mathcal{H}^{n-1}\big(S_v \cap (W \setminus D)\big) \le C\,\mathcal{H}^{n-1}\big(S_u \cap D\big). \tag{6.11}$$

For the rigorous proof of (6.10) and (6.11) we refer to Theorem 6.8 in the Appendix.

Now, let us consider a solution $\hat{v}$ of the following problem:

$$\text{(MMS)}\ \min\left\{ \int_{D\cup W} |\nabla w|^2 dx + \mathcal{H}^{n-1}(S_w) : w \in SBV^2(D \cup W), w = u \text{ in } D \right\}.$$

We have that $\hat{v} = v$ a.e. in D and that

$$MS(\hat{v}, D \cup W) \le MS(v, D \cup W). \tag{6.12}$$

Without loss of generality we can assume that $||\hat{v}||_{L^\infty(D\cup W)} = ||u||_{L^\infty(D)}$.

Let us analyze more carefully the structure of the neighbourhood W. By (6.7) we know that we can write it as

$$W = \big\{\Phi(x_0,t) : (x_0,t) \in \partial D \times (-t_0,t_0)\big\}.$$

Let W^+ and W^- be defined as in (6.8) and (6.9).

Now we set $\Gamma := \{\Phi(x_0,-t_0/2) : x_0 \in \partial D\}$. For every $z \in \Gamma$ let $\varrho(z)$ be defined as

$$\varrho(z) := \sup\big\{\varrho > 0 : B_\varrho(z) \subset W^-\big\},$$

and let

$$\gamma := \frac{1}{2}\inf_{z\in\Gamma} \varrho(z).$$

Clearly, $\gamma > 0$. Let $\omega > 0$ be defined as $\omega := \beta\gamma^{n-1}$, where $\beta > 0$ is the constant given by the Elimination Theorem 1.13.

In order to construct the required extension, we need to distinguish two cases, that will be treated in a different way.

First case: small jump set

We assume that $\mathcal{H}^{n-1}(S_{\hat{v}} \cap W^-) < \omega$. Let us fix $z \in \Gamma$ and let us consider the ball $B_\gamma(z) \subset W^-$. Clearly $\mathcal{H}^{n-1}(S_{\hat{v}} \cap B_\gamma(z)) \le \mathcal{H}^{n-1}(S_{\hat{v}} \cap W^-) < \omega$. By our definition of ω, this implies that

$$\mathcal{H}^{n-1}(S_{\hat{v}} \cap B_\gamma(z)) < \beta\gamma^{n-1}.$$

Hence, by Theorem (1.13) we have that $S_{\hat{v}} \cap B_{\gamma/2}(z) = \emptyset$.

The same argument can be repeated for every $z \in \Gamma$. Therefore we deduce that the set $\Delta \subset W^-$ defined as

$$\Delta := \bigcup_{z\in\Gamma} B_{\gamma/2}(z)$$

does not intersect the jump set of $\hat{v}$.

Notice that the set Δ disconnects W^-. We can write $W^- \setminus \Delta := \Delta_1 \cup \Delta_2$, where $\partial D \subset \partial\Delta_1$. Now, let us define the function $\tilde{u} : A \to \mathbb{R}$ as

$$\tilde{u}(x) := \begin{cases} u(x) & \text{if } x \in D,\\ \hat{v}(x) & \text{if } x \in \Delta_1 \cup \Delta,\\ \tilde{v}(x) & \text{otherwise in } A, \end{cases}$$

where $\tilde{v}$ denotes the H^1 extension of $\hat{v}$ from Δ to $A \setminus (D \cup \Delta_1)$.

It is well known that the function $\tilde{v}$ satisfies the estimate

$$\int_{A\setminus(D\cup\Delta_1)} |\nabla\tilde{v}|^2 dx \le \tilde{C}\int_\Delta |\nabla\hat{v}|^2 dx, \tag{6.13}$$

where the constant $\tilde{C}$ depends on Δ and on $A \setminus (D \cup \Delta_1)$, that is, $\tilde{C} = \tilde{C}(D,A)$. Furthermore, up to truncation, we can always assume that the L^∞ bound is preserved.

Then, it turns out that $\tilde{u} \in SBV^2(A)$ and that $||\tilde{v}||_{L^\infty(A)} = ||u||_{L^\infty(D)}$. Moreover, by (6.13), we have

$$\begin{aligned} MS(\tilde{u},A) &= MS(u,D) + MS(\hat{v},\Delta_1) + \int_{A\setminus(D\cup\Delta_1)} |\nabla\tilde{v}|^2 dx\\ &\le MS(u,D) + MS(\hat{v},\Delta_1) + \tilde{C}\, MS(\hat{v},\Delta). \end{aligned}$$

Finally, from the minimality of $\hat{v}$ and from the estimates (6.10) and (6.11) we obtain

$$MS(\tilde{u}, A) \leq MS(u, D) + (\tilde{C} + 1)\, MS(v, W^-) \leq (1 + C\,(\tilde{C} + 1))\, MS(u, D). \tag{6.14}$$

Second case: large jump set
We assume that $\mathcal{H}^{n-1}(S_{\hat{v}} \cap W^-) \geq \omega$. Let us define the function $\tilde{u} : A \to \mathbb{R}$ as

$$\tilde{u}(x) := \begin{cases} u(x) & \text{if } x \in D, \\ \hat{v}(x) & \text{if } x \in W^-, \\ 0 & \text{if } x \in A \setminus (D \cup W^-). \end{cases}$$

It turns out that $\tilde{u} \in SBV^2(A)$. Moreover,

$$\begin{aligned} MS(\tilde{u}, A) &= MS(u, D) + MS(\hat{v}, W^-) + \mathcal{H}^{n-1}(\partial W^- \setminus \partial D) \\ &\leq MS(u, D) + MS(v, W^-) + \bar{C}\,\mathcal{H}^{n-1}(S_{\hat{v}} \cap W^-), \end{aligned}$$

where we used (6.12) and the fact that, being $\omega > 0$, there exists a constant $\bar{C} > 0$ such that

$$\mathcal{H}^{n-1}(\partial W^- \setminus \partial D) < \bar{C}\,\omega.$$

Finally, using the estimates (6.10) and (6.11) we obtain

$$MS(\tilde{u}, A) \leq MS(u, D) + (\bar{C} + 1)\, MS(v, W^-) \leq (1 + C\,(\bar{C} + 1))\, MS(u, D). \tag{6.15}$$

Estimate in the general case.
Let us define $c(n, D, A) := \max\{(1 + C\,(\tilde{C} + 1)), (1 + C\,(\bar{C} + 1))\}$. By (6.14) and (6.15) we have that (6.1) holds in the general case. □

Remark 6.3 Estimate (6.1) guarantees that the constant $c(n, D, A)$ is invariant under dilations of the domain, as shown in Theorem 6.4.

6.2 Homogenization of Neumann problems

In this section we consider an application of the extension property to a non coercive homogenization problem.

Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let Q denote the unit cube $(0,1)^n$, and let $E \subset\subset Q$ be a Lipschitz set.

For every $\varepsilon > 0$ let us consider the periodic structure in $\mathbb{R}^n$ generated by an ε-homothetic of the basic cell Q. For notational brevity we will use the superscript ε to denote the ε-homothetic of any domain. In particular, $Q^\varepsilon := \varepsilon\, Q$. Let us write the domain Ω as union of cubes of side ε:

$$\Omega = \Omega \cap \left(\bigcup_{h \in \mathbb{Z}^n} (Q + h)^\varepsilon \right).$$

In the same way we can define the set $\tilde{E}^\varepsilon \subset \Omega$ as

$$\tilde{E}^\varepsilon := \Omega \cap \left(\bigcup_{h \in \mathbb{Z}^n} (E + h)^\varepsilon \right) \tag{6.16}$$

Finally, let $\Omega^\varepsilon := \Omega \setminus \tilde{E}^\varepsilon$.

The starting point of the problem is the energy associated to a function $u \in SBV^2(\Omega)$, that is

$$\mathcal{F}^\varepsilon(u) := \int_{\Omega^\varepsilon} |\nabla u|^2 dx + \mathcal{H}^{n-1}(\Omega^\varepsilon \cap S_u).$$

Notice that we can rewrite the functional $\mathcal{F}^\varepsilon$ as

$$\mathcal{F}^\varepsilon(u) = \int_\Omega a\Big(\frac{x}{\varepsilon}\Big)|\nabla u|^2 dx + \int_{S_u} a\Big(\frac{x}{\varepsilon}\Big)\, d\mathcal{H}^{n-1}(x),$$

where a is a Q-periodic function given by

$$a(y) = \begin{cases} 0 & \text{in } E, \\ 1 & \text{in } Q \setminus E. \end{cases}$$

6.2.1 Compactness

In this subsection we focus on the compactness for a sequence having equibounded energy $\mathcal{F}^\varepsilon$. As first result, we use Theorem 6.1 in order to obtain an extension result from the domain Ω^ε to the whole Ω.

Theorem 6.4 *Fix $\varepsilon > 0$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $E \subset\subset Q$ be an open set with Lipschitz boundary and consider the sets $\tilde{E}^\varepsilon$ defined in (6.16). Let $\Omega^\varepsilon = \Omega \setminus \tilde{E}^\varepsilon$.*

Then there exist an extension operator $T^\varepsilon : SBV^2(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon) \to SBV^2(\Omega) \cap L^\infty(\Omega)$ and a constant $k_0 > 0$ depending on E and n, but not on ε and Ω, such that

- $T^\varepsilon u = u$ *a.e. in* Ω^ε,
- $||T^\varepsilon u||_{L^\infty(\Omega)} = ||u||_{L^\infty(\Omega^\varepsilon)}$,
- $MS(T^\varepsilon u, \Omega) \leq k_0\big(MS(u, \Omega^\varepsilon) + \mathcal{H}^{n-1}(\partial\Omega)\big)$

for every $u \in SBV^2(\Omega^\varepsilon)$.

Proof. – Let $u \in SBV^2(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon)$ and let us consider a set $\Omega_0 \supset\supset \Omega$. We can write the set Ω_0 as union of cubes in the following way:

$$\Omega_0 = \Bigg(\bigcup_{h \in \mathbb{Z}_\varepsilon} \varepsilon(h + Q)\Bigg) \cup R(\varepsilon),$$

where $\mathbb{Z}_\varepsilon := \{h \in \mathbb{Z}^n : \varepsilon(h + Q) \subset \Omega_0\}$, and $R_0(\varepsilon)$ is the remaining part of Ω_0. We denote by $\{Q_k^\varepsilon\}_{k=1,\dots,N(\varepsilon)}$ an enumeration of the family of cubes $(Q + h)^\varepsilon$ covering Ω_0, so that we can rewrite Ω_0 as

$$\Omega_0 = \Bigg(\bigcup_{k=1}^{N(\varepsilon)} Q_k^\varepsilon\Bigg) \cup R(\varepsilon).$$

In the same way we can define the set $\tilde{E}_0^\varepsilon \subset \Omega_0$ as

$$\tilde{E}_0^\varepsilon := \bigcup_{k=1}^{N(\varepsilon)} E_k^\varepsilon.$$

Finally, let $\Omega_0^\varepsilon := \Omega_0 \setminus \tilde{E}_0^\varepsilon$. Clearly,

$$\Omega_0^\varepsilon = \Bigg(\bigcup_{k=1}^{N(\varepsilon)} (Q_k^\varepsilon \setminus E_k^\varepsilon)\Bigg) \cup R_0(\varepsilon). \tag{6.17}$$

Let $\tilde{u} : \Omega_0^\varepsilon \to \mathbb{R}$ be defined as

$$\tilde{u} := \begin{cases} u & \text{in } \Omega^\varepsilon \\ 0 & \text{otherwise in } \Omega_0^\varepsilon. \end{cases} \tag{6.18}$$

Clearly the function $\tilde{u}$ satisfies $\tilde{u} = u$ in Ω^ε, $||\tilde{u}||_{L^\infty(\Omega_0^\varepsilon)} = ||u||_{L^\infty(\Omega^\varepsilon)}$, and

$$MS(\tilde{u}, \Omega_0^\varepsilon) \leq MS(u, \Omega^\varepsilon) + \mathcal{H}^{n-1}(\partial\Omega). \tag{6.19}$$

Notice that, using (6.17) we can write

$$MS(\tilde{u}, \Omega_0^\varepsilon) = \sum_{k=1}^{N(\varepsilon)} MS(\tilde{u}, Q_k^\varepsilon \setminus E_k^\varepsilon) + MS(u, R_0(\varepsilon)).$$

Define the function v as $\tilde{u}(\varepsilon\, y) =: \sqrt{\varepsilon}\, v(y)$. Then for every $k \in \{1, \ldots, N(\varepsilon)\}$ we have

$$MS(\tilde{u}, Q_k^\varepsilon \setminus E_k^\varepsilon) = \varepsilon^{n-1} MS(v, Q_k \setminus E_k). \tag{6.20}$$

Now, we apply the result in Theorem 6.1 in every cube with the same constant and we obtain that there exists a function $\tilde{v}_k \in SBV^2(Q_k)$ such that

- $\tilde{v}_k = v$ a.e. in $Q_k \setminus E_k$,
- $MS(\tilde{v}_k, Q_k) \leq \tilde{k}_0\, MS(v, Q_k \setminus E_k)$,

where the constant $\tilde{k}_0$ depends only on Q and E.

Let us define $L^\varepsilon \tilde{u} : \Omega_0 \to \mathbb{R}$ as

$$(L^\varepsilon \tilde{u})(x) := \begin{cases} \sqrt{\varepsilon}\, \tilde{v}_k\Big(\dfrac{x}{\varepsilon}\Big) & \text{if } x \in Q_k^\varepsilon,\ k \in \{1, \ldots, N(\varepsilon)\}, \\ \tilde{u}(x) & \text{if } x \in R_0(\varepsilon). \end{cases}$$

It turns out that $L^\varepsilon \tilde{u} \in SBV^2(\Omega_0) \cap L^\infty(\Omega_0)$, $L^\varepsilon \tilde{u} = \tilde{u}$ in Ω_0^ε, and by (6.20)

$$\begin{aligned} MS(L^\varepsilon \tilde{u}, \Omega_0) &= \varepsilon^{n-1} \sum_{k=1}^{N(\varepsilon)} MS(\tilde{v}_k, Q_k) + MS(\tilde{u}, R_0(\varepsilon)) \\ &\leq \tilde{k}_0\, \varepsilon^{n-1} \sum_{k=1}^{N(\varepsilon)} MS(v, Q_k \setminus E_k) + MS(\tilde{u}, R_0(\varepsilon)) \\ &= \tilde{k}_0 \sum_{k=1}^{N(\varepsilon)} MS(\tilde{u}, Q_k^\varepsilon \setminus E_k^\varepsilon) + MS(\tilde{u}, R_0(\varepsilon)) \leq k_0 MS(\tilde{u}, \Omega_0^\varepsilon), \end{aligned}$$

where $k_0 := \tilde{k}_0 + 1$. Therefore, combining the previous expression with (6.19) we have

$$MS(L^\varepsilon \tilde{u}, \Omega_0) \leq k_0 \big(MS(u, \Omega^\varepsilon) + \mathcal{H}^{n-1}(\partial\Omega)\big), \tag{6.21}$$

therefore the claim follows defining $T^\varepsilon u := (L^\varepsilon \tilde{u})_{|\Omega}$. □

Now we prove the compactness result.

Theorem 6.5 *Let $(u^\varepsilon) \subset SBV^2(\Omega) \cap L^\infty(\Omega)$ be a sequence satisfying the following bounds:*

$$||u^\varepsilon||_{L^\infty(\Omega^\varepsilon)} \leq c \quad \textit{and} \quad \mathcal{F}^\varepsilon(u^\varepsilon) \leq c < +\infty,$$

where $c > 0$ is a constant independent of ε. Then there exist a sequence $(\tilde{u}^\varepsilon) \subset SBV^2(\Omega)$ and a function $u \in SBV^2(\Omega)$ such that $\tilde{u}^\varepsilon = u^\varepsilon$ a.e. in Ω^ε for every ε and $(\tilde{u}^\varepsilon)$ converges to u weakly* in $BV(\Omega)$.

Proof. – Let us define $\tilde{u}^\varepsilon := T^\varepsilon u^\varepsilon$, where T^ε is the extension operator defined in Theorem 6.4. Then, from the assumptions on the sequence (u^ε) and using the properties of T^ε we obtain

$$||\tilde{u}^\varepsilon||_{L^\infty(\Omega)} \le c \quad \text{and} \qquad MS(\tilde{u}^\varepsilon, \Omega) \le c < +\infty.$$

Hence, by Ambrosio's compactness Theorem 1.10 we have directly the claim. □

6.2.2 Integral representation

The present subsection is devoted to the identification of the Γ-limit of the sequence $(\mathcal{F}^\varepsilon)$ in $SBV^2(\Omega)$, with respect to the strong convergence in $L^2(\Omega)$.

Let us define for $u \in SBV^2(\Omega)$ the functional $\mathcal{F}^{hom}$ as

$$\mathcal{F}^{hom}(u) := \int_\Omega f^{hom}(\nabla u)\, dx + \int_{S_u} \varphi(\nu_u)\, d\mathcal{H}^{n-1}. \tag{6.22}$$

The limit densities $f^{hom} : \mathbb{R}^n \to [0, +\infty]$ and $\varphi : S^{n-1} \to [0, +\infty]$ are characterized by means of the following homogenization formulas:

$$f^{hom}(\xi) := \min\Big\{ \int_Q a(y) |\,\xi + \nabla w(y)|^2 dy : w \in H^1_\#(Q) \Big\},$$

where $H^1_\#(Q)$ denotes the space of $H^1(Q)$ functions with periodic boundary values on ∂Q, and

$$\varphi(\nu) := \lim_{T\to+\infty} \frac{1}{T^{n-1}} \inf\Big\{ \int_{S_w} a(y)\, d\mathcal{H}^{n-1} : w \in SBV(TQ_\nu), \nabla w = 0 \,\text{a.e.}, w = w_{1,\nu} \text{ on } \partial TQ_\nu \Big\}, \tag{6.23}$$

where Q_ν is any unit cube in $\mathbb{R}^n$ with centre at the origin and one face orthogonal to ν, and

$$w_{1,\nu}(x) = \begin{cases} 1 & \text{if } \langle x, \nu\rangle \ge 0, \\ 0 & \text{if } \langle x, \nu\rangle < 0. \end{cases}$$

For notational brevity we denote with $\mathcal{P}$ the class of admissible functions for the infimum in the definition of φ, that is,

$$\mathcal{P} := \big\{ w \in SBV(TQ_\nu) : \nabla w = 0 \,\text{a.e.}, w = w_{1,\nu} \text{ on } \partial TQ_\nu \big\}. \tag{6.24}$$

Theorem 6.6 *The family $(\mathcal{F}^\varepsilon)$ Γ-converges with respect to the strong topology of $L^2(\Omega)$ to the functional $\mathcal{F}^{hom}$ introduced in (6.22). More precisely for every $u \in SBV^2(\Omega)$ the following properties are satisfied:*

(i) for every $(u^\varepsilon) \subset SBV^2(\Omega)$ converging to u strongly in $L^2(\Omega)$

$$\mathcal{F}^{hom}(u) \le \liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon),$$

(ii) there exists a sequence $(u^\varepsilon) \subset SBV^2(\Omega)$ converging to u strongly in $L^2(\Omega)$ such that

$$\mathcal{F}^{hom}(u) \ge \limsup_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon).$$

For the proof of Theorem 6.6 we rely on [13, Theorem 2.3]. Due to the lack of coerciveness, we cannot apply the results in [13] directly to the functionals $\mathcal{F}^\varepsilon$. So we first modify the sequence to get the coerciveness we need, and then we obtain the stated Γ-convergence by approximation.

Let us define for $\eta > 0$ the approximating functionals $\mathcal{F}^{\varepsilon}_{\eta} : SBV^2(\Omega) \to [0,+\infty)$ as

$$\mathcal{F}^{\varepsilon}_{\eta}(u) = \int_{\Omega} a_{\eta}\Big(\frac{x}{\varepsilon}\Big)|\nabla u|^2 dx + \int_{S_u} a_{\eta}\Big(\frac{x}{\varepsilon}\Big)\, d\mathcal{H}^{n-1},$$

where a_{η} is a Q-periodic function given by

$$a_{\eta}(y) = \begin{cases} \eta & \text{if } y \in E, \\ 1 & \text{if } y \in Q \setminus E. \end{cases}$$

Theorem 6.7 *The family $(\mathcal{F}^{\varepsilon}_{\eta})$ Γ-converges with respect to the strong topology of $L^2(\Omega)$ to the functional $\mathcal{F}^{hom}_{\eta} : SBV^2(\Omega) \to [0,+\infty)$ defined as*

$$\mathcal{F}^{hom}_{\eta}(u) := \int_{\Omega} f^{hom}_{\eta}(\nabla u)\, dx + \int_{S_u} \varphi_{\eta}(\nu_u)\, d\mathcal{H}^{n-1}.$$

The limit densities $f^{hom}_{\eta} : \mathbb{R}^n \to [0,+\infty]$ and $\varphi_{\eta} : S^{n-1} \to [0,+\infty]$ are identified by means of the following homogenization formulas:

$$f^{hom}_{\eta}(\xi) := \min\Big\{ \int_{Q} a_{\eta}(y)|\,\xi + \nabla w(y)|^2 dy : w \in H^1_{\#}(Q)\Big\}, \tag{6.25}$$

$$\varphi_{\eta}(\nu) := \lim_{T\to+\infty} \frac{1}{T^{n-1}} \inf\Big\{ \int_{S_w} a_{\eta}(y)\, d\mathcal{H}^{n-1} : w \in \mathcal{P}\Big\}, \tag{6.26}$$

where $H^1_{\#}(Q)$ and $\mathcal{P}$ are defined as before.

Proof. – The functionals $\mathcal{F}^{\varepsilon}_{\eta}$ satisfy all the assumptions required in order to apply [13, Theorem 2.3] and hence the thesis follows directly. □

Now we are ready to give the proof of Theorem 6.6.

Proof. – [Proof of Theorem 6.6] We split the proof into three steps.

First step: approximation. It turns out that

$$\mathcal{F}^{hom} = \inf_{\eta} \mathcal{F}^{hom}_{\eta} = \lim_{\eta\to 0} \mathcal{F}^{hom}_{\eta}. \tag{6.27}$$

Indeed, since $a_{\eta} \downarrow a$ pointwise as $\eta \to 0$, using Proposition 1.2 one has

$$f^{hom}(\xi) = \inf_{\eta} f^{hom}_{\eta}(\xi) = \lim_{\eta\to 0} f^{hom}_{\eta}(\xi). \tag{6.28}$$

For the surface integral one can proceed as follows. Since (φ_{η}) is decreasing and $\varphi_{\eta} \geq \varphi$ for every $\eta > 0$, taking the limit as η goes to zero we have directly

$$\varphi(\nu) \leq \inf_{\eta} \varphi_{\eta}(\nu) = \lim_{\eta\to 0} \varphi_{\eta}(\nu).$$

On the other hand for every $w \in \mathcal{P}$, where the class $\mathcal{P}$ is defined in (6.24), the following estimate holds:

$$\frac{1}{T^{n-1}} \int_{S_w \cap TQ_{\nu}} a_{\eta}(y)\, d\mathcal{H}^{n-1} \leq \frac{1}{T^{n-1}} \int_{S_w \cap TQ_{\nu}} a(y)\, d\mathcal{H}^{n-1} + \frac{\eta}{T^{n-1}} \mathcal{H}^{n-1}(S_w \cap TQ_{\nu}). \tag{6.29}$$

In particular for a minimizing sequence $(w_h) \subset \mathcal{P}$ of the cell problem (6.23), by (6.29) we have

$$\frac{1}{T^{n-1}} \int_{S_{w_h} \cap TQ_{\nu}} a_{\eta}(y)\, d\mathcal{H}^{n-1} \leq \frac{1}{T^{n-1}} \int_{S_{w_h} \cap TQ_{\nu}} a(y)\, d\mathcal{H}^{n-1} + \frac{\eta}{T^{n-1}} \mathcal{H}^{n-1}(S_{w_h} \cap TQ_{\nu}). \tag{6.30}$$

Notice that, from the definition of the class $\mathcal{P}$, we have

$$\int_{S_{w_h}\cap TQ_\nu} a(y)\, d\mathcal{H}^{n-1} = MS(w_h, \tilde{E}_T), \tag{6.31}$$

where $\tilde{E}_T := TQ_\nu \cap (\mathbb{Z}^n + (Q\setminus E))$. At this point, we can apply Theorem 6.4 to obtain an extension $\tilde{w}_h$ of w_h to the whole domain TQ_ν satisfying

$$MS(\tilde{w}_h, TQ_\nu) \le k_0\big(MS(w_h, \tilde{E}_T) + \mathcal{H}^{n-1}(\partial TQ_\nu)\big).$$

This implies in particular that

$$\mathcal{H}^{n-1}(S_{\tilde{w}_h}\cap TQ_\nu) \le k_0 \int_{S_{w_h}\cap TQ_\nu} a(y)\, d\mathcal{H}^{n-1} + k_0 T^{n-1}, \tag{6.32}$$

where we used (6.31). The minimality of w_h ensures that there exists a constant $c>0$ such that

$$\frac{1}{T^{n-1}}\mathcal{H}^{n-1}(S_{\tilde{w}_h}\cap TQ_\nu) \le c. \tag{6.33}$$

Since $w_h = \tilde{w}_h$ a.e. in $\tilde{E}_T$, we can assume without loss of generality that (6.33) holds for the sequence w_h. Therefore in (6.30) we obtain

$$\frac{1}{T^{n-1}} \inf_{w\in\mathcal{P}} \int_{S_w\cap TQ_\nu} a_\eta(y)\, d\mathcal{H}^{n-1} \le \frac{1}{T^{n-1}} \inf_{w\in\mathcal{P}} \int_{S_w\cap TQ_\nu} a(y)\, d\mathcal{H}^{n-1} + c\,\eta.$$

If we let $T\to+\infty$ and then $\eta\to 0$ we get

$$\varphi(\nu) = \inf_\eta \varphi_\eta(\nu) = \lim_{\eta\to 0}\varphi_\eta(\nu). \tag{6.34}$$

Hence, from (6.28), (6.34) and monotone convergence we obtain (6.27).

Second step: liminf inequality (i). It is immediate to remark that for every $u\in SBV^2(\Omega)$

$$\mathcal{F}^\varepsilon_\eta(u) \le \mathcal{F}^\varepsilon(u) + \eta\, MS(u,\Omega). \tag{6.35}$$

Let $u\in SBV^2(\Omega)$ and let $(u^\varepsilon)\subset SBV^2(\Omega)$ be a sequence converging to u strongly in $L^2(\Omega)$ and having equibounded energy $\mathcal{F}^\varepsilon(u^\varepsilon)$. By (6.35) we have

$$\mathcal{F}^\varepsilon_\eta(u^\varepsilon) \le \mathcal{F}^\varepsilon(u^\varepsilon) + \eta\, MS(u^\varepsilon,\Omega). \tag{6.36}$$

Using Theorem 6.4 we can assume that the sequence (u^ε) has equibounded energy $\mathcal{F}^\varepsilon(u^\varepsilon)+\eta MS(u^\varepsilon,\Omega)$.

Hence, since in particular $MS(u^\varepsilon,\Omega)\le c$, we get from (6.36) and from Theorem 6.7

$$\mathcal{F}^{hom}_\eta(u) \le \liminf_{\varepsilon\to 0}\mathcal{F}^\varepsilon_\eta(u^\varepsilon) \le \liminf_{\varepsilon\to 0}\mathcal{F}^\varepsilon(u^\varepsilon) + \eta\, c,$$

that holds true for every $\eta>0$. If we now let $\eta\to 0$ in the previous expression we obtain the required bound

$$\mathcal{F}^{hom}(u) = \lim_{\eta\to 0}\mathcal{F}^{hom}_\eta(u) \le \liminf_{\varepsilon\to 0}\mathcal{F}^\varepsilon(u^\varepsilon).$$

Third step: limsup inequality (ii). In this case we simply use the trivial estimate

$$\mathcal{F}^\varepsilon_\eta \ge \mathcal{F}^\varepsilon. \tag{6.37}$$

Indeed, let $u \in SBV^2(\Omega)$ and let $(u^\varepsilon) \subset SBV^2(\Omega)$ be a recovery sequence for the functionals $\mathcal{F}^\varepsilon_\eta$. Then

$$\mathcal{F}^{hom}_\eta(u) = \limsup_{\varepsilon \to 0} \mathcal{F}^\varepsilon_\eta(u^\varepsilon) \geq \limsup_{\varepsilon \to 0} \mathcal{F}^\varepsilon(u^\varepsilon).$$

This implies in particular that

$$\mathcal{F}^{hom}(u) = \inf_\eta \mathcal{F}^{hom}_\eta(u) \geq \limsup_{\varepsilon \to 0} \mathcal{F}^\varepsilon(u^\varepsilon),$$

and therefore the proof is concluded. □

6.3 Appendix

In this last section we prove in a rigorous way an integral estimate for the composition of an SBV function with a bilipschitz map which provides a stability result for the Mumford-Shah functional under bilipschitz transformations of the domain. More precisely we have the following theorem.

Theorem 6.8 *Let W, W' be open subsets of $\mathbb{R}^n$ with compact Lipschitz boundary, let $\phi : W' \to W$ be a bilipschitz function and $u \in SBV^2(W)$. Then the function $v : W' \to \mathbb{R}$ defined as $v(x) := u(\phi(x))$ belongs to $SBV^2(W')$ and the following estimate holds:*

$$\int_{W'} |\nabla v|^2 dx + \mathcal{H}^{n-1}(S_v) \leq C \left(\int_W |\nabla u|^2 dx + \mathcal{H}^{n-1}(S_u) \right), \tag{6.38}$$

where the constant $C = C(\phi)$ depends only on the change of variables ϕ.

Proof. – It is well known that the function v belongs to $SBV(W')$ (see for example [8]). In order to prove the estimate (6.38), we split the proof into two steps.

First step: approximation of u.

As first step we approximate u with more regular functions and we prove the claim for the approximating functions. More precisely, let (u_h) be the sequence provided by Theorem 1.14, and set $v_h := u_h \circ \phi$. We claim that there exists a constant $C = C(\phi)$ such that

$$\int_{W'} |\nabla v_h|^2 dy + \mathcal{H}^{n-1}(S_{v_h}) \leq C \left(\int_W |\nabla u_h|^2 dx + \mathcal{H}^{n-1}(S_{u_h}) \right). \tag{6.39}$$

By property (iii) of Theorem 1.14 we can apply the standard chain rule and we get

$$\nabla v_h = (\nabla \phi)^T (\nabla u_h \circ \phi) \quad \mathcal{L}^n\text{-a.e. on } W' \setminus \phi^{-1}(\overline{S}_{u_h}),$$

that is, since ϕ^{-1} maps $\mathcal{L}^n$-negligible sets into $\mathcal{L}^n$-negligible sets,

$$\nabla v_h = (\nabla \phi)^T (\nabla u_h \circ \phi) \quad \mathcal{L}^n\text{-a.e. on } W'. \tag{6.40}$$

By (6.40) we have

$$\int_{W'} |\nabla v_h|^2 dy = \int_{W'} |(\nabla \phi)^T (\nabla u_h \circ \phi)|^2 dy \leq C_1(\phi) \int_W |\nabla u_h|^2 dx. \tag{6.41}$$

It remains to estimate the measure of the jump set of v_h. Notice that $S_{v_h} = \phi^{-1}(S_{u_h})$. Hence, passing to the measure we obtain

$$\mathcal{H}^{n-1}(S_{v_h}) = \int_{S_{v_h}} 1 \, d\mathcal{H}^{n-1} \leq C_2(\phi) \int_{S_{u_h}} 1 \, d\mathcal{H}^{n-1}. \tag{6.42}$$

Therefore (6.39) follows from (6.41) and (6.42).

Second step: limit estimate.
It remains to pass to the limit in (6.39) as $h \to +\infty$. For the right-hand side the convergence is given by property (v) of Theorem 1.14. So we reduced to prove the following result:

$$\int_{W'} |\nabla v|^2 dy + \mathcal{H}^{n-1}(S_v) \leq \liminf_{h\to+\infty} \left(\int_{W'} |\nabla v_h|^2 dy + \mathcal{H}^{n-1}(S_{v_h}) \right). \tag{6.43}$$

The lack of a uniform L^∞ bound for the sequence (v_h) forces us to use a truncation argument in order to apply Ambrosio's compactness theorem. Hence, let $M > 0$ and define $v_h^M := (v_h \wedge M) \vee (-M)$; clearly, $v_h^M \to v^M := (v \wedge M) \vee (-M)$ strongly in $L^2(W')$. By Ambrosio's compactness theorem we have that $v_h^M \rightharpoonup v^M$ weakly* in $BV(W')$. At this point, by Ambrosio's lower semicontinuity theorem we obtain the following inequality:

$$\int_{W'} |\nabla v^M|^2 dy + \mathcal{H}^{n-1}(S_{v^M}) \leq \liminf_{h\to+\infty} \left(\int_{W'} |\nabla v_h^M|^2 dy + \mathcal{H}^{n-1}(S_{v_h^M}) \right). \tag{6.44}$$

It is immediate to notice that

$$\int_{W'} |\nabla v_h^M|^2 dy + \mathcal{H}^{n-1}(S_{v_h^M}) \leq \int_{W'} |\nabla v_h|^2 dy + \mathcal{H}^{n-1}(S_{v_h}),$$

therefore we can write

$$\int_{W'} |\nabla v^M|^2 dy + \mathcal{H}^{n-1}(S_{v^M}) \leq \liminf_{h\to+\infty} \left(\int_{W'} |\nabla v_h|^2 dy + \mathcal{H}^{n-1}(S_{v_h}) \right). \tag{6.45}$$

Now we let M tend to $+\infty$ in order to pass from (6.45) to (6.43). We treat separately the volume term and the surface integral in the left-hand side of (6.45). For the jump set we simply notice that, being $M \mapsto S_{v^M}$ an increasing function and $S_v = \cup_M S_{v^M}$, we have the convergence

$$\mathcal{H}^{n-1}(S_v) = \lim_{M\to+\infty} \mathcal{H}^{n-1}(S_{v^M}).$$

For the volume integral we point out that, from the chain rule formula in BV, we can write the explicit expression of the absolutely continuous gradient of the truncated function v^M as

$$\nabla v^M = \begin{cases} \nabla v & \text{if } |v| < M, \\ 0 & \text{otherwise.} \end{cases}$$

At this point, by Lebesgue dominated convergence theorem we get

$$\int_{W'} |\nabla v|^2 dy = \lim_{M\to+\infty} \int_{W'} |\nabla v^M|^2 dy,$$

and the proof is concluded. □

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