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On commutator fully transitive Abelian groups

Andrey R. Chekhlov and Peter V. Danchev

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Abstract. There are two rather natural questions which arise in connection with the endomorphism ring of an Abelian group: when is the ring generated by its commutators, and when is the ring additively generated by its commutators? The current work explores these two problems for arbitrary Abelian groups. This leads in a standard way to consideration of two improved versions of Kaplansky's notion of full transitivity, which we call *commutator full transitivity* and *strongly commutator full transitivity*. We establish, *inter alia*, that these notions are strictly stronger than the classical concept of full transitivity, but there are nonetheless many parallels between these things.

1 Introduction

Throughout the present paper, let all groups be additive Abelian groups and let all unexplained notions and notation follow those from [19] and [23].

To simplify the notation, and to avoid any risk of confusion, we shall write $\mathrm{E}(G)$ for the endomorphism *ring* of a group G, and $\mathrm{End}(G) = \mathrm{E}(G)^+$ for the endomorphism *group* of a group G. Likewise, the endomorphism ψ is called *commutator* if it can be represented as $\psi = [\alpha, \beta] = \alpha\beta - \beta\alpha$ for some endomorphisms α, β of G. Commutators of endomorphisms rings of groups and certain other questions connected with them were studied in [5–11].

Moreover, we shall denote by $\mathrm{Comm}(G)$ the *subring* of $\mathrm{E}(G)$ containing the same identity and generated by the commutator endomorphisms. In view of the equality $[\alpha, \beta] = -[\beta, \alpha]$, an element $\phi \in \mathrm{Comm}(G)$ will have the form

$$\phi = \sum_{\text{finite}} c_{i_1} c_{i_2} \dots c_{i_k},$$

where every c_{i_j} is a commutator in $\mathrm{E}(G)$ for $i_j \in \mathbb{N}$ and $1 \le j \le k \in \mathbb{N}$.

Analogously, we let $\mathrm{comm}(G)$ denote the *subgroup* of $\mathrm{End}(G)$ generated by the commutator endomorphisms; so $\varphi \in \mathrm{comm}(G)$ has the form

$$\varphi = \sum_{i=1}^{n} c_i$$

for some finite n, where each c_i is a commutator in $\mathrm{End}(G)$. Since 1 can be represented as a finite sum of finite products of commutators, it is immediately seen that the same holds for $c_i = 1 \cdot c_i = c_i \cdot 1$ and thus $\mathrm{comm}(G) \subseteq \mathrm{Comm}(G)$.

As usual, mimicking [24, Section 27], $H_G(g)$ denotes the *height matrix* of the element g of a group G. In the case where the group G is a p-group, one may consider the *Ulm indicator* $U_G(g)$ of the element g instead of $H_G(g)$, while if the group G is torsion-free, one can consider the *characteristic* $\chi_G(g)$. Also, $o(g)$ will denote the *order* of the element g, i.e., the least $n \in \mathbb{N}$ with $ng = 0$ or ∞ if such an n does not exist. We also define the relation $\preceq$ as follows: for $m, n \in \mathbb{N} \cup \{\infty\}$ we suppose that $m \preceq n \Leftrightarrow$ either $n \mid m$ or $m = \infty$.

Let R be an associative unital ring, let G be a group, and let $f\colon R \to \mathrm{E}(G)$ be a ring homomorphism. We shall define the action of R on G by the equality $r(g) = f(r)(g)$. Similarly to above, we denote by $\mathrm{Comm}(R)$ and $\mathrm{comm}(R)$ the subring of R and the subgroup of R^+, respectively, generated by all commutators of R. So, we come to the following notion:

Main Definition. A group G is said to be *R-commutator fully transitive* if, given $0 \neq x, y \in G$ with $H_G(x) \leqslant H_G(y)$ and $o(x) \preceq o(y)$, there exists an element $f \in \mathrm{Comm}(R)$ with $f(x) = y$. If f is chosen from $\mathrm{comm}(R)$, then the group is called *R-strongly commutator fully transitive*.

In what follows we will consider several times the examined group as a module on its endomorphism ring. In particular, when $R = \mathrm{E}(G)$ and $R^+ = \mathrm{End}(G)$, one can obtain the following two concepts:

Definition 1. A group G is said to be *commutator fully transitive* (briefly, a cft-group) if, given $0 \neq x, y \in G$ with $H_G(x) \leqslant H_G(y)$ and $o(x) \preceq o(y)$, there exists an endomorphism $\phi \in \mathrm{Comm}(G)$ with $\phi(x) = y$.

Definition 2. A group G is said to be *strongly commutator fully transitive* (briefly, an scft-group) if, given $0 \neq x, y \in G$ with $H_G(x) \leqslant H_G(y)$ and $o(x) \preceq o(y)$, there exists an endomorphism $\varphi \in \mathrm{comm}(G)$ with $\varphi(x) = y$.

Note that if the group is reduced, then the condition $o(x) \preceq o(y)$ in both Definitions 1 and 2 can be eliminated in conjunction with [21, Proposition 2.23]. However, the later usage of that condition is basically motivated by the existence of divisible direct factors. It is also clear that any scft-group is a cft-group.

Notice that in [16] the authors studied the so-called *projectively fully transitive* p-groups, i.e., the p-groups G having the property that, for any $x, y \in G$ with $U_G(x) \leqslant U_G(y)$, there exists some $\varphi \in \mathrm{Proj}(G)$ such that $\varphi(x) = y$, where $\mathrm{Proj}(G)$ is the subring of $\mathrm{E}(G)$ generated by the idempotents of $\mathrm{E}(G)$. They also explored *strongly projectively fully transitive p-groups* defined in a similar way, replacing Proj(G) by $\Pi(G)$, which is the subgroup of End(G) additively generated by all the idempotents. In what follows we shall often cite and use results from [16].

Once again, throughout the text, the word group will denote an *additively* written *Abelian* group. In this context, our terminology, if not explicitly explained herein, is standard and follows Fuchs [19] and Kaplansky [23], where all mappings are written on the left. Another good sources on this subject are [3, 14, 15]. Likewise, if A, B are groups and $H \subseteq A$, let $\mathrm{Hom}(A, B)H = \sum_{f \in \mathrm{Hom}(A,B)} f(H)$. As usual, $\mathbb{Z}_n$ denotes the cyclic group of order n, whereas the ring of integers modulo n is denoted by $\mathbb{Z}_{(n)}$.

Our work is motivated mainly by [16] and [17]. Here we wish to consider the situation where the projection endomorphisms are replaced by commutator endomorphisms and thus to find the similarities and the discrepancies between them. We emphasize that there is no absolute analogy in both cases.

2 Elementary results

It is clear that if $\mathrm{Comm}(G) = \mathrm{E}(G)$ (resp., $\mathrm{comm}(G) = \mathrm{End}(G)$), then the fully transitive group G is a cft-group (resp., an scft-group), so we will first consider this situation. We shall say that a group G is a *commutator-generated* group (or a CG-group for short) if $\mathrm{Comm}(G) = \mathrm{E}(G)$; similarly, we say that G is a *commutator-sum* group (or a CS-group for short) if $\mathrm{comm}(G) = \mathrm{End}(G)$. It is self-evident that a CS-group is a CG-group because $\mathrm{End}(G) \subseteq \mathrm{E}(G)$. Likewise, it is apparent that a group with commutative endomorphism ring is neither a CG-group nor a CS-group; for more concrete information concerning groups with commutative endomorphism ring, we refer the interested reader to both [26] and [2].

However, the next construction demonstrates that there exist CG-groups which are not CS-groups.

Example 2.1. There is a CG-group that is not a CS-group.

Proof. Suppose $R = \{\frac{m}{2^n} \mid m, n \in \mathbb{Z}\}$ is the ring of the rational fractions whose denominators are powers of 2, and let $S = R \oplus Ri \oplus Rj \oplus Rk$ with $i^2 = j^2 = k^2 = -1$ be the ring of quaternions for the ring R. For any element $r \in R$ we have $[\frac{r}{2}i, j] = rk$, so $Rk \subseteq \mathrm{comm}(S)$. Similarly, $Ri \subseteq \mathrm{comm}(S)$ and $Rj \subseteq \mathrm{comm}(S)$. Thereby $\mathrm{comm}(S) = Ri \oplus Rj \oplus Rk$. Since $k^2 = [\frac{1}{2}i, j]^2 = -1$, one sees that $R \subseteq \mathrm{Comm}(S)$, i.e., $S = \mathrm{Comm}(S)$. Furthermore, according to Corner's realization theorem [24, Theorem 29.2] there exists a countable torsion-free group G with $\mathrm{E}(G) \cong S$. Hence $\mathrm{Comm}(G) = \mathrm{E}(G)$ and, consequently, G is a CG-group.

However, it is routinely checked that $1 \notin \mathrm{comm}(S)$, whence $\mathrm{comm}(G) \neq \mathrm{End}(G)$, and therefore G is not a CS-group, as asserted. □

The following fact is rather elementary but is crucial for our purposes.

Remark. Notice also that if $G = A \oplus B$, then any endomorphism $\delta \in \mathrm{E}(G)$ such that $\delta \restriction A = f \in \mathrm{Hom}(A, B)$ and $\delta \restriction B = 0_B$ can be represented like this:

$$\delta = \begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix}.$$

This observation suggests the following obvious technical result which will be used in the sequel.

Lemma 2.2. *If $G = A_1 \oplus \cdots \oplus A_n$, where all A_i are CG-groups (resp., CS-groups), then G is a CG-group (resp., a CS-group).*

The next statement is also of interest.

Lemma 2.3 ([25]). *If $G = A^{(\kappa)}$, where the cardinal κ is infinite, then*

$$\mathrm{comm}(G) = \mathrm{End}(G).$$

We sketch here an idea of the proof only for the sake of completeness and the reader's convenience: In fact, one may apply [25, Theorem 13], where it was proved that if N is a right R-module over a ring R, I is an infinite set and $M = N^{(I)}$, then the equality $\mathrm{End}_R(M) = [\mathrm{End}_R(M), \mathrm{End}_R(M)]$ holds, where $[S, S]$ is the additive subgroup in a ring S, generated by all commutators of the elements of the ring S.

As a useful consequence, we derive:

Corollary 2.4. *The following statements hold.*

(1) *If $G = A \oplus B$, where the component A is a fully invariant subgroup of G, then G is a CG-group (resp., a CS-group) if and only if both A and B are CG-groups (resp., CS-groups). In particular, if $G = D \oplus R$, where D is divisible and R is reduced, then G is a CG-group (resp., a CS-group) if and only if both D and R are CG-groups (resp., CS-groups).*

(2) *If $G = \bigoplus_i A_i$, where each A_i is a fully invariant subgroup of G, then G is a CG-group (resp., a CS-group) if and only if every component A_i is a CG-group (resp., a CS-group).*

Proof. (1) Since A is a fully invariant subgroup in G, any $\varphi \in \mathrm{E}(G)$ can be represented as $\varphi = \left(\begin{smallmatrix} \alpha & \gamma \\ 0 & \beta \end{smallmatrix}\right)$, where $\alpha \in \mathrm{E}(A)$, $\beta \in \mathrm{E}(B)$ and $\gamma \in \mathrm{Hom}(B, A)$. But we have $\alpha \in \mathrm{Comm}(A)$ and $\beta \in \mathrm{Comm}(B)$, so the Remark before Lemma 2.2 works to conclude that $\left(\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{Comm}(G)$. Thus $\varphi \in \mathrm{Comm}(G)$ and hence we obtain both claims, as desired.

(2) This is elementary. □

It is worthwhile noting that point (2) of Corollary 2.4 reduces the study of torsion CG-groups and CS-groups to the primary case.

Proposition 2.5. *The following statements hold.*

(1) *Let $G = C \oplus B$, where $C \neq 0$ is a free group and B is a CG-group having a direct summand isomorphic to C. Then G is a CG-group.*

(2) *If A is a free group, then $A^{(\kappa)}$ is a CG-group for any cardinal $\kappa \geqslant 2$.*

Proof. (1) Since B has a direct summand isomorphic to C, it follows that for each $\alpha \in \mathrm{E}(C)$ there exist $\zeta \in \mathrm{Hom}(C, B)$ and $\xi \in \mathrm{Hom}(B, C)$ such that $\alpha = \xi\zeta$. But

$$\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \zeta & 0 \end{pmatrix}.$$

Thus it is easily seen that $\mathrm{E}(G) = \mathrm{Comm}(G)$, as required.

Point (2) follows in the same manner. □

It is well known that every divisible group D has the following representation:

$$D = D_0 \oplus \left(\bigoplus_{p \in \Pi} D_p \right),$$

where D_0 is its torsion-free part such that $D_0 = 0$ or $D_0 \cong \mathbb{Q}^{(m)}$ for some cardinal $m \geqslant 1$, while Π is a subset of the set of all prime numbers such that if $\Pi \neq \varnothing$ and $p \in \Pi$, then $D_p \cong \mathbb{Z}_{p^\infty}^{(k_p)}$ for some cardinal $k_p \geqslant 1$, where $m = \mathrm{rank}(D_0)$ and $k_p = \mathrm{rank}(D_p)$.

Combining Corollary 2.4 and Proposition 2.5, we immediately deduce:

Corollary 2.6. *A divisible group D is a CG-group if and only if either $D_0 = 0$ or* $\mathrm{rank}(D_0) \geqslant 2$*, and if $\Pi \neq \varnothing$, then* $\mathrm{rank}(D_p) \geqslant 2$ *for any $p \in \Pi$.*

Note that the papers [6, 8, 10] investigated the E-commutant G' of a group G, that is, $G' = \langle [\alpha, \beta]G \mid \alpha, \beta \in \mathrm{E}(G) \rangle$. According to [9, Lemma 8] if $G = A \oplus B$, then $G' = \langle \mathrm{Hom}(A, B)A, \mathrm{Hom}(B, A)B, A', B' \rangle$. It is clear that if G is a CG-group or a CS-group, then we have that $G = G'$ whereas the converse fails. Indeed, if $G = \mathbb{Q} \oplus (\mathbb{Z} \oplus \mathbb{Z})$, then $G = G'$, but by Corollary 2.4 the group G is not a CG-group (and hence it is not a CS-group as well).

Notice also that if $G = A \oplus A$, where A is a group with commutative endomorphism ring, then G is not a CS-group. In fact, if $\varphi, \psi \in \mathrm{E}(G)$ with

$$\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix},$$

then

$$[\varphi, \psi] = \begin{pmatrix} \beta\gamma_1 - \beta_1\gamma & * \\ * & \gamma\beta_1 - \gamma_1\beta \end{pmatrix}.$$

Since the ring $\mathrm{E}(A)$ is commutative, we have $\beta\gamma_1 - \beta_1\gamma = -(\gamma\beta_1 - \gamma_1\beta)$. It is now plainly seen that the matrices of this type do not additively generate all of the ring $M(2, \mathrm{E}(A))$.

On the other hand, any bounded p-group A represents as

$$A = A_1 \oplus \cdots \oplus A_n, \tag{2.1}$$

where each subgroup A_i is isomorphic to a direct sum of some number of the group $\mathbb{Z}_{p^{n_i}}$ $(i = 1, \ldots, k)$ and $1 \leqslant n_1 < \cdots < n_k$.

So, we come to

Proposition 2.7. *The bounded p-group from* (2.1) *is a CG-group if and only if every its component A_i is a decomposable group, that is, none of its components A_i is a cyclic group.*

Proof. Necessity. Assume that some subgroup A_i is indecomposable, i.e., it is a cyclic group of order p^{n_i} (note that its endomorphism ring is commutative). Therefore $A = A_i \oplus B$, where $B = B_1 \oplus B_2$, $B_1 = \bigoplus_{j=1}^{i-1} A_j$, $B_2 = \bigoplus_{j=i+1}^{k} A_j$ ($B_1 = 0$ or $B_2 = 0$ if, resp., $i = 1$ or $i = k$). If $\varphi, \psi \in \mathrm{E}(A)$ with

$$\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} \varepsilon & \zeta \\ \eta & \theta \end{pmatrix},$$

then in view of commutativity of the ring $\mathrm{E}(A_i)$ we obtain that

$$[\varphi, \psi] = \begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix}, \tag{2.2}$$

where κ is a composition of homomorphisms from $\mathrm{Hom}(A_i, B)$ and $\mathrm{Hom}(B, A_i)$, respectively, $\lambda \in \mathrm{Hom}(B, A_i)$, $\mu \in \mathrm{Hom}(A_i, B)$, $\nu \in \mathrm{E}(B)$. It is easy to check that any finite product of commutators is of the form of (2.2). However,

$$\mathrm{Hom}(A_i, B_2)A_i \subseteq p^{n_{i+1}-n_i} B_2, \quad \mathrm{Hom}(B_1, A_i)B_1 \subseteq p^{n_i - n_{i-1}} A_i,$$

where $n_{i+1} - n_i \geqslant 1$ and $n_i - n_{i-1} \geqslant 1$. Hence $\mathrm{Im}\,\kappa \subseteq pA_i$, which ensures that A is not a CG-group.

Sufficiency. This follows directly from Lemma 2.2 and Proposition 2.5. □

Now we will exhibit separable p-groups which are not CG-groups (and, consequently, are not CS-groups).

Example 2.8. Suppose R is a commutative ring whose additive group is the completion of a free p-adic module of at most countable rank. Then there exists an unbounded separable p-group G_R which is not a CG-group.

Proof. With Corner's realization theorem from [13] at hand (see also [12] and [24, Theorem 28.11]), we conclude that there is an unbounded separable p-group G_R with $\mathrm{E}(G_R) = R \oplus E_S(G_R)$, where $E_S(G_R)$ is the ideal of small endomorphisms of G_R. Since $\mathrm{E}(G_R)/E_S(G_R) \cong R$ is a commutative ring, we deduce that G_R is not a CS-group, because if we assume that the ring $\mathrm{E}(G_R)$ is generated as a ring (resp., additively) by its commutators, then the same is true for $\mathrm{E}(G_R)/E_S(G_R)$, which is obviously false.

Such a ring R, for instance, can be taken to be $\widehat{\mathbb{Z}}_p \times \cdots \times \widehat{\mathbb{Z}}_p = \widehat{\mathbb{Z}}_p^{(n)}$, for a finite n, where $\widehat{\mathbb{Z}}_p$ is the ring of all p-adic integers. Notice that if $R = \widehat{\mathbb{Z}}_p$, then with the aid of [24, Proposition 28.12] the group G_R has to be an essentially indecomposable p-group. □

Proposition 2.9. *If A is a reduced separable p-group with a basic subgroup of $2 \leqslant \mathrm{rank} \leqslant 2^{\aleph_0}$, then for any infinite ordinal $\alpha < \omega^2$ there is a p-group G with $p^\alpha G = A$ such that G is not a CG-group.*

Proof. Again using Corner's realization theorem from [13], we construct a group G with $p^\alpha G = A$ and $\mathrm{E}(G)_A = \{\varphi \restriction A \mid \varphi \in \mathrm{E}(G)\} = \Phi$, where Φ is any complete separable p-adic subalgebra of $\mathrm{E}(A)$. If A is unbounded, then the choice $\Phi = \widehat{\mathbb{Z}}_p$ is possible, too. Since $\widehat{\mathbb{Z}}_p$ is commutative, the ring $\mathrm{E}(G)$ cannot be generated by its commutators since $\mathrm{E}(G)_A$ is a ring homomorphic image of $\mathrm{E}(G)$.

Suppose, instead, that the group A is bounded, and write $A = B \oplus C$, where $B = \mathbb{Z}_{p^{n_1}} \oplus \mathbb{Z}_{p^{n_2}}$ and $n_1 \leqslant n_2$. Let Φ be the algebra of matrices of the form $\left(\begin{smallmatrix} r & 0 \\ 0 & s \end{smallmatrix}\right)$, where $r \in \mathbb{Z}_{(p^{n_1})}$ and $s \in \mathbb{Z}_{(p^{n_2})}$; because of finiteness Φ is complete and separable. Once again in view of the commutativity of Φ, the ring $\mathrm{E}(G)$ cannot be generated by its commutators. Consequently, in either case, G is not a CG-group, as expected. □

Proposition 2.10. *The following statements hold.*

(i) *If a p-group G is a CG (resp., a CS)-group, then so is $p^n G$ for any finite n.*

(ii) *If a p-group G is a CG (resp., a CS)-group and the quotient $G/p^\alpha G$ is totally projective for some ordinal α, then $p^\alpha G$ is a CG (resp., a CS)-group.*

Proof. (i) This follows from the fact that the mapping $\phi\colon \mathrm{E}(G) \to \mathrm{E}(p^n G)$, defined by $\phi(f) = f \restriction p^n G$, is a ring epimorphism in accordance with [19, Proposition 113.3].

(ii) This can be verified similarly by referring to [22]. □

To give an example of a splitting group which is not a CG (resp., a CS)-group, say $G = A \oplus T$, where A is a torsion-free group and T is a torsion group, it is enough to choose either A or T to be not a CG (resp., a CS)-group.

Standardly, the letters $G_t = t(G)$ stand for the *torsion part* of any group G.

Proposition 2.11. *The following statements hold.*

(1) *If R is a countable commutative ring, the additive group R^+ of which is reduced torsion-free, then there exists a countable reduced mixed group G_R such that the factor-group $G_R/t(G_R)$ is divisible and G_R is not a CG-group. Moreover, if R^+ has rank n, then the torsion-free rank of G_R is equal to $2n$.*

(2) *For any infinite cardinal m there are 2^m reduced mixed non-isomorphic groups G such that $G/t(G)$ is divisible and G is not a CG-group.*

Proof. (1) According to [24, Corollary 30.5], there exists a group G such that $\mathrm{E}(G) = R \oplus \mathrm{E}_t(G)$, where $\mathrm{E}_t(G)$ is an ideal of $\mathrm{E}(G)$. If $\mathrm{Comm}(G) = \mathrm{E}(G)$, then the factor-ring $\mathrm{E}(G)/\mathrm{E}_t(G)$ also possesses this property that contradicts its commutativity.

(2) Referring to [24, Corollary 30.6], we can take $R = \mathbb{Z}$. □

It is well known that any completely decomposable torsion-free group G can uniquely be decomposed up to isomorphism as $G = \bigoplus_{s \in \Omega} G_s$, where G_s are homogeneous completely decomposable groups called *homogeneous components* of G, and Ω is some set of types.

Proposition 2.12. *The following statements hold.*

(1) *A completely decomposable torsion-free group G is a CG-group if and only if each of its homogeneous component has rank $\geqslant 2$.*

(2) *A vector torsion-free group $G = \prod_{s \in \Omega} G_s$, where G_s is a direct product of groups of rank 1 and type s (Ω is some set of types), is a CG-group if and only if* $\mathrm{rank}(G_s) \geqslant 2$ *for each $s \in \Omega$.*

Proof. (1) *Necessity*. The subgroup $G(s) = \bigoplus_{\tau \geqslant s} G_\tau = G_s \oplus (\bigoplus_{\tau > s} G_\tau)$ is a fully invariant direct summand of G. So, by Corollary 2.4, both $G(s)$ and G_s are CG-groups. Consequently, $\mathrm{rank}(G_s) \geqslant 2$.

Sufficiency. Let $G = G_1 \oplus G_2$, where G_1 is the direct sum of those G_s that have either infinite rank or finite rank that is even, and G_2 is the direct sum of those G_s that have odd rank. According to Lemma 2.2, it is enough to show that G_1 and G_2 are CG-groups. The group G_1 is a direct sum of two mutually isomorphic direct summands and thus Proposition 2.5 allows us to conclude that G_1 is a CG-group. The group G_2 can be presented in the form $G_2 = A_1 \oplus A_2 \oplus A_3 \oplus B$, where $A_1 \cong A_2 \cong A_3$ and each homogeneous component of B (if $B \neq 0$) has even rank, so G_2 is also a CG-group.

(2) This can be verified similarly. □

3 Basic results

This section is devoted to the exploration of the two new classes of groups named (strongly) commutator fully transitive groups. So, the main results will be divided into two corresponding subsections as follows.

3.1 Commutator fully transitive groups

We begin with a trivial but useful assertion.

Lemma 3.1. *The following statements hold.*

(1) *Let $G = A \oplus B$ be a cft-group and let A be a direct summand such that either* $\mathrm{Hom}(A, B) = 0$ *or* $\mathrm{Hom}(B, A) = 0$. *Then A is also a cft-group.*

(2) *If $G = \bigoplus_{i \in I} A_i$ is a reduced torsion-free group and either* $\mathrm{Hom}(A_i, A_j) = 0$ *or* $\mathrm{Hom}(A_j, A_i) = 0$ *for any $i, j \in I$ with $i \neq j$, then G is a cft-group if and only if $pA_i \neq A_i$ implies $pA_j = A_j$ for each prime p and all $i, j \in I$ with $i \neq j$.*

Proof. Point (1) is obvious. Since each cft-group is fully transitive, the necessity of (2) follows from the corresponding result for fully transitive groups (see, for example, [21, Theorem 3.20]). As for the sufficiency, we make the elementary observation that these groups are of necessity fully transitive and hence we refer to Lemma 3.5 below. □

We are now able to prove the following:

Proposition 3.2. *A divisible group D is a cft-group if and only if $D_0 = 0$ or* $\mathrm{rank}(D_0) \geqslant 2$, *and if $\Pi \neq \varnothing$, then* $\mathrm{rank}(D_p) \geqslant 2$ *for any $p \in \Pi$.*

Proof. Necessity follows from Lemma 3.1, whereas to treat the sufficiency we employ the fact that any divisible group is a fully transitive group and since by

Corollary 2.6 such a group is a CG-group, it follows immediately that it must be a cft-group. □

It is worthwhile noticing that neither $\mathbb{Q}$ nor $\mathbb{Z}_{p^\infty}$ are cft-groups. In fact, [26] and [2] show that these two divisible groups have commutative endomorphism rings, as the first one is the only divisible torsion-free group having this property (for any set Π of prime numbers, the group $\bigoplus_{p\in\Pi}\mathbb{Z}_{p^\infty}$ also has commutative endomorphism ring). Since these two groups are fully transitive, we thus obtain two examples of fully transitive groups that are not cft-groups.

It is well known that any separable p-group is a fully transitive group. But each fully transitive group with commutative endomorphism ring is obviously not a cft-group; for instance, owing to [26], $\mathbb{Z}_{p^n}$ is a separable (and even a p^n-bounded) p-group that is not cft.

So it is interesting to find a concrete example of a reduced inseparable fully transitive group which is not a cft-group. This is subsumed by the following two constructions:

Example 3.3. There exist two types of non-separable fully transitive p-groups which are not cft-groups.

Proof. (i) Using Corner's realization theorem from [13], we construct a p-group G such that $p^\omega G = \mathbb{Z}_{p^n}$ and $\mathrm{E}(G)\restriction p^\omega G = \mathbb{Z}_{(p^n)}$. Since $\mathrm{E}(G)$ acts fully transitively on $p^\omega G$, the group G is fully transitive. However, $\mathrm{E}(G)\restriction p^\omega G$ is commutative and $p^\omega G$ is fully invariant in G. Therefore, $\mathrm{Comm}(G)\restriction p^\omega G = 0$, i.e., G is not a cft-group, as expected.

(ii) Let $H = \mathbb{Z}_p \oplus \mathbb{Z}_p = \langle a\rangle \oplus \langle b\rangle$ and $\phi \in \mathrm{E}(H)$ such that $\phi(a) = b$ and $\phi(b) = a + b$; Φ is a subring in $\mathrm{E}(H)$ generated by I, ϕ, where I is the identity on H and p is a prime of the form $p = 5n+2$. If G is a group such that $p^\omega G = H$ and $\mathrm{E}(G)\restriction H = \Phi$, it was shown in [16, Proposition 3.5 (ii)] that G is a fully transitive group. Arguing as in (i), we detect that G is not a cft-group, as promised. □

The next statement illustrates that cft-groups are not closed under the formation of direct summands.

Corollary 3.4. *A direct summand of a cft-group need not necessarily be a cft-group.*

Proof. By virtue of Proposition 3.2, the two sums $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ and $\mathbb{Q} \oplus \mathbb{Q}$ are cft-groups, but as we commented above neither $\mathbb{Z}_{p^\infty}$ nor $\mathbb{Q}$ are cft-groups.

On the other hand, concerning the reduced inseparable case, let G be one of the non-cft-groups exhibited in Example 3.3. Then $G \oplus G$ is a cft-group by Theorem 3.8 below, as needed. □

It is clear that the direct sum $\bigoplus_{i \in I} A_i$ of cft-groups A_i $(i \in I)$ with infinite I is a cft-group if and only if for each finite subset $J \subseteq I$ there is a finite $S \subseteq I$ such that $J \subseteq S$ and $\bigoplus_{i \in S} A_i$ is a cft-group.

Lemma 3.5. *Let $A = \bigoplus_{i \in I} A_i$ be a fully transitive group, where every component A_i is a cft-group for $i \in I$. Then A is a cft-group.*

Proof. Assume that $H_A(a) \leqslant H_A(b)$ for some $0 \neq a, b \in A$. It is necessary to show that there exists an $\alpha \in \mathrm{Comm}(A)$ with the property that $\alpha(a) = b$. Since a and b can be written as a finite sum of elements of some A_i, it is possible to assume that I is finite and, in particular, that $|I| = 2$; whence we write $A = A_1 \oplus A_2$. But by assumption $\alpha(a) = b$ for some $\alpha \in \mathrm{E}(A)$. Given $\pi_i \colon A \to A_i$ are projections for $i = 1, 2$, we have that $a = a_1 + a_2$, $b = b_1 + b_2$, where $a_i, b_i \in A_i$ such that

$$(\pi_1 + \pi_2)\alpha(a) = \pi_1\alpha(a_1) + \pi_2\alpha(a_1) + \pi_1\alpha(a_2) + \pi_2\alpha(a_2) = b_1 + b_2,$$

$$\pi_1\alpha(a_1) + \pi_1\alpha(a_2) = b_1 \quad \text{and} \quad \pi_2\alpha(a_1) + \pi_2\alpha(a_2) = b_2.$$

However,

$$\pi_i\alpha\pi_i \in \mathrm{E}(A_i) = \mathrm{Comm}(A_i) \subseteq \mathrm{Comm}(A),$$

and

$$\pi_1\alpha\pi_2, \pi_2\alpha\pi_1 \in \mathrm{Comm}(A)$$

by the Remark stated before Lemma 2.2, as required. □

As two helpful consequences, we yield:

Proposition 3.6. *If $G = D \oplus R$ is a group, where D is a divisible subgroup and R is a reduced subgroup, then G is a cft-group if and only if D and R are cft-groups.*

Proof. Necessity follows from Lemma 3.1. As for the sufficiency, any divisible group is fully transitive and by hypothesis R is also fully transitive. So G is fully transitive, and it remains only to apply Lemma 3.5. □

Corollary 3.7. *Let G be either a p-group or a homogeneous torsion-free group. If G is a cft-group, then so is $G^{(\kappa)}$ for any cardinal κ.*

Proof. Every commutator fully transitive group is obviously fully transitive and hence we apply either [18] or [24] to get that $G^{(\kappa)}$ is fully transitive. Henceforth, Lemma 3.5 applies to infer that $G^{(\kappa)}$ is, in fact, a cft-group, as required. □

The last assertion can be somewhat refined like this:

Theorem 3.8. *Let $\kappa > 1$ and let G be either a p-group or a torsion-free homogeneous group. Then the following condition are equivalent:*

(a) *G is fully transitive,*

(b) *$G^{(\kappa)}$ is fully transitive,*

(c) *$G^{(\kappa)}$ is cft.*

Proof. Firstly, assume that G is a p-group. The equivalence between (a) and (b) was proved in [18]. The implication (c) $\Rightarrow$ (b) is obvious. Now, to show (b) $\Rightarrow$ (c), we employ Proposition 2.5 to get that $G^{(\kappa)}$ is a CG-group, so that $G^{(\kappa)}$ as a fully transitive CG-group must be a cft-group.

Next, assume that G is a torsion-free group. The same method as in the primary case also works, as the equivalence of (a) and (b) was noted in [24, Section 25, Exercise 12]. □

With Proposition 2.5 (2) at hand we can deduce the following statement.

Proposition 3.9. *Let $\kappa > 1$. Then the group $G^{(\kappa)}$ is cft if and only if $G^{(\kappa)}$ is fully transitive.*

Now, we need the following preliminary technical claim.

Lemma 3.10. *Let G be a separable p-group and let $B = \bigoplus_{i=1}^{\infty} B_i$ be its basic subgroup, where $B_i \cong \bigoplus_{m_i} \mathbb{Z}_{p^{n_i}}$ and $n_1 < n_2 < \cdots$. Then G is a cft-group if and only if $m_i > 1$ for every i such that $B_i \neq 0$.*

Proof. Necessity. This can be proved in the same manner as Proposition 2.7.

Sufficiency. Assuming that $U_G(a) \leqslant U_G(b)$, we can embed a and b in a finite direct summand A of G, say $G = A \oplus B$, because G is separable (see [19]). By adding to A, if necessary, a cyclic direct summand from B, we can derive that A is a CG-group by Proposition 2.5. Since A is also fully transitive, it follows that $\alpha(a) = b$ for some $\alpha \in \mathrm{Comm}(A) \subseteq \mathrm{Comm}(G)$, as desired. □

Remember that the n-th *invariant* $f_n(A)$ *of Ulm–Kaplansky* of a p-group A is the cardinal number $f_n(A) = \mathrm{rank}((p^n A)[p]/(p^{n+1}A)[p])$. From this point of view, Lemma 3.10 confirms that a separable p-group G is a cft-group if and only if $f_n(B) \neq 0$ implies that $f_n(B) > 1$ for each n, where B is its basic subgroup (see [19, Section 37, Exercise 9]). Since by [19, Section 34, Exercise 2] we know that $f_n(G) = f_n(B)$, we get the following useful consequences:

Corollary 3.11. *The following statements hold.*

(1) *Suppose G is a separable p-group. Then G is cft if and only if for each natural n, $f_n(G) \neq 0$ implies that $f_n(G) > 1$.*

(2) *A separable p-group is cft if and only if its basic subgroup is cft.*

We shall say that $\mathrm{E}(G)$ acts *commutator fully transitively* on the first Ulm subgroup $p^\omega G$ of a p-group G (resp., a torsion-free group) if, given $x, y \in p^\omega G$ with $U_G(x) \leqslant U_G(y)$ (resp., $\chi_G(x) \leqslant \chi_G(y)$), there exists some $\phi \in \mathrm{Comm}(G)$ with $\phi(x) = y$.

The following is a key technical instrument for our further applications.

Lemma 3.12. *A p-group G is a cft-group if and only if $G/p^\omega G$ is a cft-group and $\mathrm{E}(G)$ acts commutator fully transitively on $p^\omega G$.*

Proof. Since G is cft, it readily follows that the ring $\mathrm{E}(G)$ should act commutator fully transitively on $p^\omega G$ and that the basic subgroup B is cft. In fact, the latter follows directly from the proof of Proposition 2.7 because the bounded direct summands B_i of B are also direct summands of G, and thus they must be decomposable, and hence Lemma 3.10 applies to get the pursued claim. Next, according to Corollary 3.11, a separable p-group is a cft-group if and only if its basic subgroup is a cft-group. Since any basic subgroup of $G/p^\omega G$ is isomorphic to a basic subgroup of G (see, e.g., [19]), the *necessity* is proved.

[Correction added after online publication 21 May 2015: In the second line of the proof of Lemma 3.12 the text passage has been added from "and that the basic subgroup B is" to "the pursued claim".]

In order to prove *sufficiency*, we use the idea of the proofs of [13, Lemma 2.1] and [16, Lemma 3.11]; in fact, it is necessary only to make some small changes in the argument. To that end, consider $x, y \in G$ with $U_G(x) \leqslant U_G(y)$. Let r, s be the least non-negative integers such that $p^r x, p^s y \in p^\omega G$; if $r = 0$, then $x, y \in p^\omega G$ and we are done, so let $r > 0$. We may choose an integer

$$m > \max\{\mathrm{ht}_G(p^{r-1}x), \mathrm{ht}_G(p^{s-1}y)\};$$

if $s - 1 < 0$, we omit the final term $\mathrm{ht}_G(p^{s-1}y)$.

Furthermore, if $p^r x = p^{r+m}x_0$, then $x = x_1 + p^m x_0$, where $p^r x_1 = 0$. Note that $o(x_1) = p^r$ since $p^t x = p^{t+m}x_0$ for $t < r$ is a contradiction to the choice of m and $\mathrm{ht}_G(p^{r-1}x_1) = \mathrm{ht}_G(p^{r-1}x)$. Thus $\langle x_1 \rangle \cap p^m G = 0$. Now let A be a $p^m G$-high subgroup with $x_1 \in A$, and hence A, being a bounded pure subgroup of G, is its direct summand, say $G = A \oplus H$ for some complement $H \supseteq p^m G$. Let $\pi\colon G \to H$ be the projection corresponding to this decomposition. Since A is isomorphic to a direct summand of $G/p^\omega G$, by what we have shown in the proof of Lemma 3.10, A is a cft-group. Note that $s \leqslant r$, $p^r x, p^r y \in p^\omega G \subseteq H$ and $U_G(p^r x) \leqslant U_G(p^r y)$, so $\phi_0(p^r x) = p^r y$ for some $\phi_0 \in \mathrm{Comm}(G)$; moreover we consider that $\phi_0 \restriction A = 0$, i.e., $\phi_0 = \phi_0 \pi$. If $y_0 = \phi_0(x_0)$, then

$$p^r y = \phi_0(p^r x) = p^{r+m} y_0$$

and $y = y_1 + p^m y_0$ for certain y_1 with the property $p^r y_1 = 0$.

Let $y_1 = a_1 + h_1$, where $a_1 \in A$, $h_1 \in H$. Then

$$U_G(y_1) = U_G(a_1) \cap U_G(h_1) \geqslant U_G(x_1)$$

and so $U_G(x_1) \leqslant U_G(a_1), U_G(h_1)$. Thus $\theta(x_1) = a_1$ for some $\theta \in \mathrm{Comm}(A)$, so that $\theta \in \mathrm{Comm}(G)$ with $\theta \restriction H = 0$.

Since A is a bounded summand of G, we can certainly find an endomorphism ϕ' of G with $\phi'(x_1) = h_1$. Set $\psi = \pi\phi'(1-\pi)$ and observe that $\psi(x_1) = h_1$. Since $\psi(H) = 0$ and $\psi(A) \subseteq H$, it follows from the Remark before Lemma 2.2 that $\psi \in \mathrm{Comm}(G)$. Set $\phi_1 = \theta + \psi$ and note that

$$\phi_1(x_1) = a_1 + h_1 = y_1, \quad \phi_1 = \phi_1(1-\pi).$$

Finally, we set $\phi = \phi_0 + \phi_1$, so that $\phi \in \mathrm{Comm}(G)$. Now

$$\phi(x) = \phi_0(x) + \phi_1(x)$$

and, because $x = x_1 + p^m x_0$, we obtain

$$\phi_0(x) = \phi_0\pi(x) = \phi_0(p^m x_0) = p^m \phi_0(x_0) = p^m y_0$$

and

$$\phi_1(x) = \phi_1(1-\pi)(x) = \phi_1(x_1) = y_1.$$

Thus $\phi(x) = y_1 + p^m y_0 = y$. □

As two immediate consequences, we derive the following:

Corollary 3.13. *The following statements hold.*

(1) *Let the p-groups A and B be cft-groups. If B is separable, then $A \oplus B$ is a cft-group.*

(2) *Let A be a cft p-group and B be its basic subgroup. Then $A \oplus B$ is a cft-group.*

(3) *If A is a separable p-group, then $A^{(\kappa)}$ is a cft-group for any cardinal $\kappa > 1$.*

Proof. (1) It is enough to check that the separable p-group $(A/p^\omega A) \oplus B$ satisfies condition (1) of Corollary 3.11, but this follows immediately from Lemma 3.12 because $A/p^\omega A$ and B are cft-groups. Points (2) and (3) follow from (1) and Lemma 3.12. □

Corollary 3.14. *Let G be a p-group such that $p^{\omega}G \cong \mathbb{Z}_{p^{\kappa}}$, where $1 \leqslant \kappa \leqslant \infty$. Then G is not a cft-group.*

Using Corollaries 3.11 and 3.14 it is not difficult to construct a totally projective p-group which is not a cft-group satisfying a specific condition of its Ulm–Kaplansky invariants. In fact, construct a group G as in Example 3.3 (ii) such that the factor-group $G/p^{\omega}G$ is a direct sum of cyclic groups which is cft. Thus we get a fully transitive p-group G, which is necessarily totally projective and which is not cft, with the property that if the σ-th invariant of Ulm–Kaplansky $f_{\sigma}(G) \neq 0$, then $f_{\sigma}(G) > 1$, where $1 \leqslant \sigma \leqslant \omega$.

Proposition 3.15. *If G is a cft p-group, then $p^{\beta}G$ is a cft-group for all ordinals β.*

Proof. This follows directly from the facts that the inequality

$$U_{p^{\beta}G}(x) \leqslant U_{p^{\beta}G}(y)$$

holds precisely when $U_G(x) \leqslant U_G(y)$ holds for any $x, y \in p^{\beta}G$, and that $p^{\beta}G$ is a fully invariant subgroup of G. □

Proposition 3.16. *Suppose that G is a p-group, B is its basic subgroup and n is a natural number. If both p^nG and B are cft-groups, then G is a cft-group, and vice versa.*

[Correction added after online publication 21 May 2015: In the second line of the statement of Proposition 3.16 the text “and vice versa” has been added after “then G is a cft-group”.]

Proof. Set $H = p^nG$. In view of Lemma 3.12, it suffices to show that $\mathrm{E}(G)$ acts commutator fully transitively on the subgroup $p^{\omega}G = p^{\omega}H$. If $x, y \in p^{\omega}G$, then we have $U_G(x) \leqslant U_G(y)$ uniquely when $U_H(x) \leqslant U_H(y)$, and so $\alpha(x) = y$ for some $\alpha \in \mathrm{Comm}(H)$. According to [19, Proposition 113.3], every endomorphism of H is induced by some endomorphism of G, which ensures that each element of $\mathrm{Comm}(H)$ is induced by some element of $\mathrm{Comm}(G)$.

The converse follows immediately from Lemma 3.12 and Proposition 3.15. □

[Correction added after online publication 21 May 2015: The last sentence of the proof of Proposition 3.16 has been added.]

Recall that the p-groups G_1 and G_2 form a *fully transitive pair* if for every non-zero $x \in G_i$, $y \in G_j$ $(i, j \in \{1, 2\})$, with $U_{G_i}(x) \leqslant U_{G_j}(y)$, there exists an $\alpha \in \mathrm{Hom}(G_i, G_j)$ such that $\alpha(x) = y$. In [18], it was proved that if $\{G_i\}_{i \in I}$ is a family of p-groups such that for each $i, j \in I$ the pair $\{G_i, G_j\}_{i,j \in I}$ is fully transitive, then $\bigoplus_{i\in I} G_i$ is a fully transitive group. Notice that in [21], in order to describe direct sums of fully transitive groups, the notion of *systems of groups with condition of monotonicity* for height matrix was introduced. Likewise, in [4] some sufficient conditions were specified under which any system of torsion-free groups satisfied the condition of monotonicity.

Note also that it can be proved as in [20, Lemma 2.2] that the p-group G is a cft-group if and only if for all $0 \neq x, y \in G$ with $py = 0$ and $U_G(x) \leqslant U_G(y)$ there is an $\alpha \in \mathrm{E}(G)$ such that $\alpha(x) = y$ (the statement holds by induction on the order of y thus: supposing $o(y) = p^{n+1}$ and $U_G(x) \leqslant U_G(y)$, if $\varphi(px) = py$ and $\psi(x) = y - \varphi(x)$, then $(\varphi + \psi)x = y$; it is similarly seen that if G is a p-group, then for all $a \in A$, $b \in G$ with $H_A(a) \leqslant H_G(b)$ there exists an $f \in \mathrm{Hom}(A, G)$ with the property that $f(a) = b$ if and only if such a homomorphism f exists for all $a \in A$, $b \in G[p]$ with $H_A(a) \leqslant H_G(b)$).

The following somewhat strengthens [20, Theorem 1.1].

Proposition 3.17. *For every $i \in I$, let G_i be a cft p-group. Then the torsion group $H = t(\prod_{i\in I} G_i)$ is a cft p-group if and only if for each $i, j \in I$ the pair (G_i, G_j) is fully transitive.*

Proof. Necessity is obvious.

Sufficiency. Suppose that

$$U_H(x) \leqslant U_H(y)$$

for $x = (\ldots, x_i, \ldots)$, $y = (\ldots, y_i, \ldots) \in H$ and $py = 0$ (see the remark in the previous paragraph). Since $\mathrm{ht}_H(x) = \inf\{\mathrm{ht}_{G_i}(x_i) \mid i \in I\}$, there exists an $i \in I$ such that $\mathrm{ht}_H(x) = \mathrm{ht}_{G_i}(x_i)$, so we consider that $i = 1$. Since $py = 0$, it follows that $U_{G_1}(x_1) \leqslant U_H(y) \leqslant U_{G_i}(y_i)$ for all i and so there are $\alpha_i \colon G_1 \to G_i$ such that $\alpha_i(x_1) = y_i$, $i \in I$.

For the matrix

$$\varphi = \begin{pmatrix} 0 & 0 & \ldots \\ \alpha_2 & 0 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \ldots \\ \alpha_2 & 0 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix} \begin{pmatrix} 1 & 0 & \ldots \\ 0 & 0 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix} - \begin{pmatrix} 1 & 0 & \ldots \\ 0 & 0 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix} \begin{pmatrix} 0 & 0 & \ldots \\ \alpha_2 & 0 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

we have $\varphi \in \mathrm{comm}(\prod_{i\in I} G_i)$. Henceforth $\alpha_1 \in \mathrm{Comm}(G_1) \subseteq \mathrm{Comm}(\prod_{i\in I} G_i)$, $(\alpha_1 + \varphi)x = y$ and the restriction $\alpha_1 + \varphi$ to H is an endomorphism of H. □

Since any two separable or totally projective p-groups form a fully transitive pair, we have the following:

Corollary 3.18. *For every $i \in I$ let G_i be a cft separable or totally projective p-group. Then the torsion group $H = t(\prod_{i \in I} G_i)$ is cft.*

Proposition 3.19. *Let $G = A \oplus T$, where A is a torsion-free reduced group and T is a torsion reduced group. Then G is a cft-group if and only if both A and T are cft-groups.*

Proof. Necessity. Follows from Lemma 3.1.

Sufficiency. Assume that $H(x) \leqslant H(y)$ for some $x = a+b$, $y = c+d$, where $a, c \in A$ and $b, d \in T$. Then $H(x) \leqslant H(c)$, $H(x) \leqslant H(d)$. It is enough to show that there exist such $\alpha, \beta \in \mathrm{Comm}(G)$ that $\alpha(x) = c$, $\beta(x) = d$.

To that end, let $H(a+b) \leqslant H(c)$. Notice that we have $H(a) \leqslant H(c)$. In fact, if $h_p(b) = \infty$, then $H_p(a+b) = H_p(a)$; but if $h_p(b) < \infty$, then $h_p(p^k b) = \infty$ for some natural k, whence $h_p(p^k(a+b)) = h_p(p^k a) \leqslant h_p(p^k c)$ gives

$$h_p(a) \leqslant h_p(c) \quad \text{and} \quad H_p(a) \leqslant H_p(c).$$

So $\alpha(a) = c$ for some $\alpha \in \mathrm{Comm}(A) \subseteq \mathrm{Comm}(G)$.

Suppose now $H(a+b) \leqslant H(d)$. Then $d = d_1 + \cdots + d_k$, where $d_i \in t_{p_i}(T)$ and $H(d) \leqslant H(d_i)$ for each $i = 1, \ldots, k$, so we can consider that $d \in t_p(T)$ for some p. According to the comments before Proposition 3.17, we may assume that $d \in T[p]$.

If $H(b) \leqslant H(d)$, then the condition on T forces that such a β can be found.

Assume now that $H(a) \leqslant H(d)$. If $h_p(d) = \infty$, then $H(b) \leqslant H(d)$, so let $h_p(d) < \infty$. Set $h_p(a) = n$. Since T is reduced, in $t_p(T)$ there exists a cyclic direct summand $\langle z \rangle$ such that $o(z) > p^n$. Thus $H(a) \leqslant H(p^n z) \leqslant H(d)$. Consequently, there exists a homomorphism $f\colon \langle a \rangle_* \to \langle z \rangle \subseteq T$ defined by $f(a) = p^n z$, where $\langle a \rangle_*$ is the pure subgroup in A containing a. Since $\langle z \rangle$ as a bounded group is algebraically compact, the homomorphism f extends to a homomorphism $\varphi \in \mathrm{Hom}(A, T)$. But $\gamma(p^n z) = d$ for some $\gamma \in \mathrm{E}(T)$, so that $\gamma\varphi(a) = d$ and, according to the Remark before Lemma 2.2, we obtain that $\gamma\varphi \in \mathrm{comm}(G)$.

Finally, note that since $h_q(d) = \infty$ for any prime $q \neq p$ and

$$H_p(d) = (h_p(d), \infty, \ldots),$$

the inequalities $H(a) \leqslant H(d)$, $H(b) \leqslant H(d)$ are impossible. □

We conclude this subsection with the following observation: Imitating [5] or [9], a subgroup C of a group G is said to be *commutator invariant* if $f(C) \subseteq C$ for every $f \in \mathrm{E}(G)$ which is of the form $f = [\phi, \psi]$, where $\phi, \psi \in \mathrm{E}(G)$. Moreover, following [17], a p-group G is said to be *commutator socle-regular* if, for each commutator invariant subgroup C of G, there exists an ordinal α (depending on C) such that the equality $C[p] = (p^\alpha G)[p]$ holds.

What we now offer is the following property of cft-groups:

Proposition 3.20. *Every cft p-group is commutator socle-regular.*

Proof. Suppose that C is an arbitrary commutator invariant subgroup of G and $\alpha = \min\{\mathrm{ht}_G(z) \mid z \in C[p]\}$, whence $C[p] \leqslant (p^\alpha G)[p]$. Next, choose $x \in C[p]$ with $\mathrm{ht}_G(x) = \alpha$, so that $U_G(x) = (\alpha, \infty, \dots)$. Letting now $y \in (p^\alpha G)[p]$ be an arbitrary element, we deduce that $U_G(y) = (\beta, \infty, \dots)$, where $\beta \geqslant \alpha$. Since G is a cft-group, there is $\phi \in \mathrm{Comm}(G)$ such that $\phi(x) = y$. But, because ϕ is a linear combination of products of commutators and C is commutator invariant in G, we have that $y = \phi(x) \in C[p]$. Since y was arbitrary, we infer that $(p^\alpha G)[p] \leqslant C[p]$ and hence we obtain the desired equality. □

3.2 Strongly commutator fully transitive groups

Many of the results from Section 3.1 can be proved for scft-groups as well. In fact, this can be said for Lemma 3.1, Proposition 3.2, Lemma 3.5, Proposition 3.6, Corollary 3.7 and Proposition 3.15 – see the corresponding statements below, formulated for scft-groups.

The next lemma shows that there exists an scft-group which is not a CS-group (compare with remarks after Corollary 2.6 too).

Lemma 3.21. *The following statements hold.*

(1) *If $G = (\mathbb{Z}_{p^n})^{(\kappa)}$, where $\kappa > 1$, then G is an scft-group.*

(2) *If G is a homogeneous torsion-free separable group and* $\mathrm{rank}(G) > 1$, *then G is an scft-group.*

Proof. (1) Let $U_G(a) \leqslant U_G(b)$ for $0 \neq a, b \in G$ and write $a = a_1 + \dots + a_n$, $b = b_1 + \dots + b_m$, where $a_i \in A_{j_i}$, $b_s \in A_{j_s}$ and $A_{j_i}, A_{j_s} \cong \mathbb{Z}_{p^n}$. If

$$\mathrm{ht}(a_{i_0}) = \min\{\mathrm{ht}(a_1), \dots, \mathrm{ht}(a_n)\},$$

then $U_G(a_{i_0}) = U_G(a) \leqslant U_G(b_s)$ for each $s = 1, \dots, m$. If $i_0 \neq s$ for some s, then according to the Remark before Lemma 2.2 we have $\phi(a_0) = b_s$ for some $\phi \in \mathrm{comm}(G)$. But if $s_0 = i_0$ for some $1 \leqslant s_0 \leqslant m$, then since the additional direct summand B contains a direct summand isomorphic to A_{i_0}, as in Proposition 2.5 there exist $\zeta \in \mathrm{Hom}(A_{i_0}, B)$ and $\xi \in \mathrm{Hom}(B, A_{i_0})$ such that

$$b_{s_0} = \alpha(a_{s_0}) = \xi\zeta(a_{s_0}).$$

If now

$$\varphi = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \zeta & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \zeta & 0 \end{pmatrix} \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix},$$

then $\varphi(a_{i_0}) = b_{i_0}$, as required.

(2) If $\chi(a) \leqslant \chi(b)$, then a can be embedded in a direct summand of rank 1. So the proof is similar to that in (1). □

If G is a separable torsion-free group, then according to [1, Corollary 7.12] it is fully transitive if and only if for all its direct summands A and B of rank 1, if type $t(A) \neq t(B)$, then the condition $pA \neq A$ implies $pB = B$ for each prime number p. According to [24, Section 19, Exercise 7] any fully transitive separable group G can be presented as $G = \bigoplus_{i \in I} G_i$, where all G_i are homogeneous separable groups and the condition $pG_i \neq G_i$ implies $pG_j = G_j$ for each $i, j \in I$ with $i \neq j$.

So, we come now to the following result.

Corollary 3.22. *The following statements hold.*

(1) *Let G be a separable p-group and let $B = \bigoplus_{i=1}^{\infty} B_i$ be its basic subgroup, where $B_i \cong \bigoplus_{m_i} \mathbb{Z}_{p^{n_i}}$ and $n_1 < n_2 < \cdots$. Then G is an scft-group if and only if $m_i > 1$ for every i such that $B_i \neq 0$.*

(2) *Let A be an scft p-group and let B be its basic subgroup. Then $A \oplus B$ is an scft-group.*

(3) *If G is a separable p-group or a torsion-free fully transitive separable group (in particular, G is a homogeneous separable group), then the group $G^{(\kappa)}$ is scft for any cardinal $\kappa > 1$.*

(4) *A separable torsion-free group G is scft if and only if $G = \bigoplus_{i \in I} G_i$, where all G_i are decomposable homogeneous separable groups and the condition $pG_i \neq G_i$ implies $pG_j = G_j$ for each $i, j \in I$ with $i \neq j$.*

(5) *A vector non-zero torsion-free group G is scft if and only if $G = \prod_{i \in I} G_i$, where each component G_i is a direct product of groups of rank 1 and same type,* $\operatorname{rank}(G_i) > 1$ *and the condition $pG_i \neq G_i$ implies $pG_j = G_j$ for each $i, j \in I$ with $i \neq j$.*

Proof. Points (1) and (2) have similar proof to that of Lemma 3.10. Point (3) follows from (1) and Lemma 3.21. To show the validity of clause (4), it is necessary to use certain well-known facts about fully transitive torsion-free groups (see, for instance, [4, Theorem 11]). Finally, point (5) can be proved similarly to (4). □

Proposition 3.23. *Let A be a homogeneous torsion-free group and κ be an infinite cardinal. Then $A^{(\kappa)}$ is an scft-group if and only if A is fully transitive.*

Proof. The statement follows from Lemma 2.3 and from the fact that the group $A^{(\kappa)}$ is fully transitive if and only if the group A is fully transitive. □

Recall that if p is a prime number, then the p-rank $\operatorname{rank}_p(A)$ of the group A is identified as the rank of its factor-group A/pA. In conjunction with [19], any reduced algebraically compact torsion-free group $G \neq 0$ can be represented as $G = \prod_{p\in\Pi} G_p$, where $G_p \neq 0$ is a p-adic algebraically compact group and Π is a certain set of prime numbers.

Corollary 3.24. *The reduced algebraically compact torsion-free group*

$$G = \prod_{p\in\Pi} G_p$$

is scft if and only if $\operatorname{rank}_p(G_p) > 1$ *for each $p \in \Pi$.*

Proposition 3.25. *A divisible group D is an scft-group if and only if $D_0 = 0$ or* $\operatorname{rank}(D_0) \geqslant 2$*, and if $\Pi \neq \varnothing$, then* $\operatorname{rank}(D_p) \geqslant 2$ *for any $p \in \Pi$.*

Proof. Necessity follows from Lemma 3.1.

Sufficiency. As in Lemma 3.21, D_0 and D_p are scft-groups, so that $\bigoplus_p D_p$ is an scft-group. Since any divisible group is fully transitive, by Lemma 3.5 we obtain that $D = D_0 \oplus (\bigoplus_p D_p)$ is an scft-group. □

By a simple combination of the methods in the proofs of Corollary 3.22 and Lemma 3.12, one can prove the following.

Lemma 3.26. *A p-group G is an scft-group if and only if $G/p^\omega G$ is an scft-group and* $\mathrm{E}(G)$ *acts strongly commutator fully transitively on $p^\omega G$.*

Corollary 3.27. *If A is a bounded scft p-group, then there is an scft p-group G with $p^\omega G = A$.*

Proof. As in Proposition 2.9, we use Corollary 3.14 to construct a group G with the properties that $p^\omega G = A$ and $\{\varphi \restriction A \mid \varphi \in \mathrm{E}(G)\} = \mathrm{E}(A)$ such that $G/p^\omega G$ is an scft-group. □

It follows from Lemma 3.10 and Corollary 3.22 (1) that a separable p-group is cft if and only if it is scft. Under certain additional circumstances on the endomorphism ring of the group, this can be slightly extended to the following:

Proposition 3.28. *Let G be a p-group and let $\mathrm{E}(G) \restriction p^{\omega}G = \mathrm{E}(p^{\omega}G)$. Then the following two points hold:*

(1) *G is cft if and only if $G/p^{\omega}G$ and $p^{\omega}G$ are cft.*

(2) *G is scft if and only if $G/p^{\omega}G$ and $p^{\omega}G$ are scft.*

Proof. (1) Combining both Lemma 3.12 and Proposition 3.15, the necessity follows at once.

As for the sufficiency, one sees that $\mathrm{E}(p^{\omega}G)$ and thus $\mathrm{E}(G)$ both act commutator fully transitively on $p^{\omega}G$. So, again Lemma 3.12 applies to show that G is cft, as claimed.

(2) This equivalence follows by the same token with the aid of Lemma 3.26 accomplished with a similar statement for Ulm subgroups of scft-groups as that of Proposition 3.15. □

As a consequence, we get the following.

Corollary 3.29. *Suppose G is a p-group of length $\leq \omega \cdot 2$ such that*

$$\mathrm{E}(G) \restriction p^{\omega}G = \mathrm{E}(p^{\omega}G).$$

Then G is cft if and only if G is scft.

Proof. Since $p^{\omega}G$ is separable, we just apply Proposition 3.28 and the comments on separable groups stated before it. □

It seems, at present, to be extremely difficult if not impossible to construct a cft-group that is not scft. It is worthwhile noting that the same problem is currently unresolved for projectively fully transitive and strongly projectively fully transitive p-groups, respectively (cf. [16]).

Nevertheless, we can show the following:

Example 3.30. There exists a ring S such that there is an S-commutator fully transitive group which is not S-strongly commutator fully transitive.

Proof. Let p be a prime number and set $T = \{\frac{m}{n} \mid m, n \in \mathbb{Z},\ n \neq 0,\ (n, p) = 1\}$; it is obvious that T is a subring of the ring $\mathbb{Q}$ consisting of all rational numbers. Putting $S = T \oplus Ti \oplus Tj \oplus Tk$ with $i^2 = j^2 = k^2 = -1$ as the ring of quaternions of the ring T, it is not too hard to verify that the group S^+ is homogeneous completely decomposable of rank 4. Therefore, S^+ is strongly commutator fully transitive by Lemma 3.21. Further, as in Example 2.1, one may infer that $\mathrm{Comm}(S) = S$. It is well known that $S \cong \mathrm{E}_S(S_S)$. Since any endomorphism of the module S_S acts as left multiplicand on elements of the ring S and since for each non-zero element of S there is an integer multiple invertible, all non-zero endomorphisms of S_S are monomorphisms. In addition, the group S^+ is torsion-free.

Let now $0 \neq a, b \in S^+$ and $\chi(a) \leqslant \chi(b)$. Then $ua = p^n \cdot 1$ and $vb = p^m \cdot 1$ for some invertible elements $u, v \in S$, where $n \leqslant m$. Thus $b = (p^{m-n}v^{-1}u)a$, i.e., $b \in \mathrm{Comm}(\mathrm{E}_S(S^+))a$. But $b \notin \mathrm{comm}(\mathrm{E}_S(S^+))a$ for any $0 \neq a \in S^+$ since only $1 \in S$ sends a to a. So, we obtain the construction of a group which is S-commutator fully transitive but not S-strongly commutator fully transitive, where $S \cong \mathrm{E}_S(S_S)$ and id: $\mathrm{E}_S(S_S) \to \mathrm{E}(S^+)$ is the identical embedding. □

In contrast to fully transitive groups, for projectively fully transitive groups not every direct summand is projectively fully transitive (see [16, Corollary 3.9]). The same appears for scft-groups, so the direct summand of an scft-group is also not an scft-group; for a proof we use ideas from [16, Propositions 4.10 and 4.11].

Proposition 3.31. *If $p^\omega G$ is an elementary group for a p-group G, then G is fully transitive if and only if $G \oplus G$ is an scft-group.*

Proof. The sufficiency is immediate since direct summands of fully transitive groups are fully transitive.

Suppose now that G is fully transitive. Set $H = G \oplus G$ and consider the elements $(a, b), (c, d) \in p^\omega H$. Assume first that $a, b \neq 0$. Since all non-zero elements of $p^\omega G$ have the same Ulm sequence $(\omega, \infty, \dots)$, there are endomorphisms $\gamma, \delta \in \mathrm{E}(G)$ with the property $\gamma(b) = c$ and $\delta(a) = d$. The matrix

$$\Delta = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix}$$

maps (a, b) to (c, d). According to the Remark before Lemma 2.2, one sees that $\Delta \in \mathrm{comm}(H)$. Let now $a \neq 0, b = 0$ and $\alpha(a) = c, \delta(a) = d$, where $\alpha \in \mathrm{E}(G)$. Then the matrix

$$\Lambda = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix}$$

maps $(a, 0)$ to (c, d). Here as in Lemma 3.21 we observe that

$$\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{comm}(H) \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix} \in \mathrm{comm}(H).$$ □

Proposition 3.32. *There is a non-scft p-group G with elementary first Ulm subgroup such that $G \oplus G$ is an scft-group.*

Proof. It is enough to take any group from Example 3.3 (thus $n = 1$ if consider point (i)) and then we refer to Proposition 3.31. □

We close the work with some questions of interest.

4 Open results

Problem 1. Find conditions on a (torsion-free) fully transitive group A under which A is a cft (resp., an scft)-group.

Problem 2. Construct, if possible, a cft-group which is not an scft-group.

Problem 3. Find conditions on a totally projective p-group G such that it is a cft (resp., an scft)-group.

Problem 4. Find conditions on a CG-group (resp., a CS-group) such that it is a cft (resp., an scft)-group.

Problem 5. To what extent do there exist indecomposable torsion-free CG-groups which are not CS-groups?

Problem 6. Let A_i $(i \in I)$ be a system of reduced groups, and let $\mathbf{K}$ be an ideal of the Boolean algebra of all subsets of I. Find a suitable necessary and/or sufficient condition for the $\mathbf{K}$-direct sum $\bigoplus_{\mathbf{K}} A_i$ (in particular, the direct product $\prod_{i \in I} A_i$) to be a cft-group (resp., an scft-group).

Remark. In the proof of [16, Proposition 3.3] on lines 5–6 the word "idempotent" should be "product of idempotents".

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Author information

Andrey R. Chekhlov, Department of Mathematics, Tomsk State University,
Tomsk 634050, Russia.
E-mail: cheklov@math.tsu.ru

Peter V. Danchev, Department of Mathematics, Plovdiv State University,
Plovdiv 4000, Bulgaria.
E-mail: pvdanchev@yahoo.com