PHYSICAL REVIEW D **91,** 085005 (2015)

Photon propagation in noncommutative QED with constant external field

R. Fresneda,[1,*] D. M. Gitman,[2,3,4,†] and A. E. Shabad[3,2,‡]

[1]*Federal University of ABC (UFABC), Avenida dos Estados, 5001, Bairro Bangu, CEP 09210-580, Santo André, São Paulo, Brazil*

[2]*P. Tomsk State University, Lenin Prospekt 36, Tomsk 634050, Russia*

[3]*P. N. Lebedev Physics Institute, Leninsky Prospekt 53, Moscow 117924, Russia*

[4]*Instituto de Física, Universidade de São Paulo, CEP 05508-090, São Paulo, São Paulo, Brazil*

(Received 20 January 2015; published 3 April 2015)

We find dispersion laws for the photon propagating in the presence of mutually orthogonal constant external electric and magnetic fields in the context of the θ-expanded noncommutative QED. We show that there is no birefringence to the first order in the noncommutativity parameter θ. By analyzing the group velocities of the photon eigenmodes we show that there occurs superluminal propagation for any direction. This phenomenon depends on the mutual orientation of the external electromagnetic fields and the noncommutativity vector. We argue that the propagation of signals with superluminal group velocity violates causality in spite of the fact that the noncommutative theory is not Lorentz invariant and speculate about possible workarounds.

DOI: 10.1103/PhysRevD.91.085005 PACS numbers: 11.30.Cp, 11.15.-q

I. INTRODUCTION

The general belief [1] that space-time at the Plank scale is quantized has materialized in the abundant development over the last decades in the field of noncommutative quantum field theory and quantum mechanics [2]. Despite the general failure of the hope to regularize quantum field theory [3–5] and the scarcity of renormalizable models [6,7], this field is still being actively researched [8–11]. With regard to the usual realization of noncommutativity in physics, where the star product in the deformed algebra of functions is given in terms of a constant antisymmetric matrix, numerous phenomenological consequences from noncommutativity have placed stringent conditions on the magnitude of the noncommutativity parameter [8,12,13].

Noncommutative theories constitute an example of theories with Lorentz symmetry violation, since they contain an "external" antisymmetric tensor $\theta_{\mu\nu}$ stemming from the commutation relation between the operator-valued coordinates $[X_\mu, X_\nu] = i\,\theta_{\mu\nu}$, and should be considered in this context [14]. To theories with broken relativistic invariance, superluminal propagation is generally peculiar, starting with the theoretical evidence in Ref. [15] of photons that propagate faster than light in the background, specially Schwarzschild, metrics. Another context for superluminal propagation is presented by quantum electrodynamics, when its Lorentz-invariance is violated by the presence of an exponentially high external (e.g., magnetic) field [16], owing to the lack of asymptotic freedom in that theory. There is vast literature where the superluminal signals in Lorentz-violated theories [17] are revealed and discussed.[1]

The dramatic role of the superluminal propagation in destroying the causality principle is often underestimated, following the view that once there is no relativistic invariance from the outset, one may not be upset by the appearance of a superluminal signal. Such an attitude is, certainly, too thoughtless. The simplest refutation may be found in the example of the electrodynamics in a medium, whose presence violates Lorentz invariance—formally by the involvement of an external vector of the 4-velocity of the medium. The group velocity of an electromagnetic wave in a moving medium is its group velocity in the medium at rest relativistically added to the speed of the medium. (This statement also can be extended to a Lorentz-non-invariant vacuum, characterized by the presence of any external vector or a tensor). If a superluminal signal should exist in the medium at rest, it would also be superluminal in the moving frame and might lead to the paradoxical reversal of the time coordinates of timelike separated events, following the same standard consideration of a Lorentz-invariant vacuum.

In the present paper we are dealing with the problem of propagation of electromagnetic waves when an external constant electromagnetic field is present, in the nonlinear electrodynamics to which the noncommutative theory with

*rodrigo.fresneda@ufabc.edu.br
†gitman@dfn.if.usp.br
‡shabad@lpi.ru

[1]Referring to the speed of the wave-front propagation following [18]. We consider the group velocity also as that of a signal following Refs. [19,20]. It must not exceed the speed of light. As for the phase velocity, its use by some authors as a criterion for superluminosity is not justified: it is permitted to outrun light without any contradiction with causality.

the Abelian gauge group is reduced by the Seiberg-Witten map [21]. In the lowest order of noncommutativity, the nonlinearity is only cubic in the electromagnetic field. We restrict ourselves to the so-called space-space noncommutativity, when the noncommutativity tensor has its time components vanishing in a certain reference frame, and we are working within the lowest nontrivial term in the expansion of the Taylor series in powers of the noncommutativity parameter. We impose mutually orthogonal electric and magnetic fields of arbitrary magnitudes, but constant in time and space (the property of orthogonality is retained in any Lorentz frame). Unlike the Lorentz-invariant vacuum, neither of these fields can be excluded by a Lorentz boost, since this boost would change the noncommutativity parameter (by supplying time components to it). Therefore, our case is more general, indeed, than the case of the magnetic field alone, considered previously in the literature [22], because it includes both fields, but it does not overlap with the external field of Ref. [23], where electric and magnetic fields are taken together, but restricted to the condition that—in the frame where the noncommutativity tensor has vanishing time components—they are mutually parallel and parallel to the noncommutativity (pseudo)vector.[2]

In the most general constant external field, we calculate the second- and third-rank polarization tensors (Sec. II), responsible, respectively, for the light propagation and for the electromagnetic wave splitting into two. We consider the general problem of light propagation by finding the eigenvalues of the second-rank polarization tensor in Secs. III and IV. The photon dispersion laws are found for the simplest special case, where a magnetic or electric field is parallel to the noncommutativity (pseudo)vector $\theta_i = \epsilon_{ijk}\theta_{jk}$ in Sec. IV. The dispersion laws for mutually perpendicular electric and magnetic fields are established in Sec. VI. Also the two intermediate special cases, where there is either only magnetic or only electric external fields are considered. In all cases the absence of birefringence[3] is noted and formulas for the group velocities are derived. In all cases the group velocities prove to exceed the speed of light c, taken as unity, when certain relations between the orientation of the external fields and the noncommutative vector take place. While in the case where only electric or magnetic fields are present, a direction of the wave vector (namely, the one along the coinciding directions of the fields and $\boldsymbol{\theta}$) exists, where the speed of propagation is simply unity, no such direction is found in the general context of Sec. VI, where both fields are present simultaneously.

In the concluding remarks of Sec. VII we discuss two possible scenarios intended to avoid the verdict—that suggests itself—on the inconsistency of the noncommutative theory due to the incompatibility with the causality principle.

II. INCLUSION OF AN EXTERNAL FIELD IN THE NONCOMMUTATIVE MAXWELL ACTION

Following [22,25] we consider the first-order Seiberg-Witten map [21] of the noncommutative Maxwell theory that results in the action for the electromagnetic field $F_{\mu\nu}$

$$S_{\mathrm{SW}} = -\frac{1}{4}\int d^4x F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\int d^4x \theta^{\alpha\beta}F^{\mu\nu}F_{\alpha\mu}F_{\beta\nu} + \frac{1}{8}\int d^4x\theta^{\alpha\beta}F_{\alpha\beta}F^{\mu\nu}F_{\mu\nu} + O(\theta^2). \quad (1)$$

It is understood that the coupling constant is included in θ. Throughout this paper we restrict ourselves to space-space noncommutativity. That is, in Lorentz invariant terms, we require the following relations[4]:

$$(\theta\tilde{\theta})_{\mu\nu} = 0, \qquad \theta^2 < 0, \quad (2)$$

to be obeyed by the noncommutativity tensor, which imply the existence of a Lorentz frame—the special frame—where the noncommutativity tensor only has space-space components: $\theta_{0i} = 0$.

In what follows, it is convenient to divide the Lagrangian density from (1) in two parts,

$$\mathcal{L}_{\mathrm{SW}} = \mathcal{L}_0 + \mathfrak{L}, \quad (3)$$

where

$$\mathcal{L}_0 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \quad \mathfrak{L} = -\frac{1}{2}\theta^{\alpha\beta}F^{\mu\nu}F_{\alpha\mu}F_{\beta\nu} + \frac{1}{8}\theta^{\alpha\beta}F_{\alpha\beta}F^{\mu\nu}F_{\mu\nu}.$$

Let us divide the gauge connection into two parts, a dynamic field a and an external field $\mathcal{A}$,

$$A_\mu = a_\mu + \mathcal{A}_\mu.$$

Then,

$$F_{\mu\nu} = f_{\mu\nu} + \mathcal{F}_{\mu\nu}, \quad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu, \quad \mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$$

[2]In that paper the author first points the superluminal propagation in the direction perpendicular to the fields.

[3]The phenomenon of birefringence can be generally stated as existence of distinct solutions to the dispersion equations of light for its different polarization modes (see e.g., [19]). An example of absence of birefringence in an anisotropic medium is given by [24] an external magnetic field in the vacuum treated via the Born-Infeld model, where the anisotropy of the medium, that the direction of the energy propagation does not coincide with that of the wave vector, is retained in that example.

[4]Here the contraction of the electromagnetic field strength tensor $F_{\mu\nu}$ with its dual $\tilde{F}_{\mu\nu}$ is to be understood as $F\tilde{F} = F_{\mu\nu}\tilde{F}^{\nu\mu}$. Likewise, $\theta^2 = \theta_{\mu\nu}\theta^{\nu\mu}$ and $(\theta\tilde{\theta})_{\mu\nu} \equiv \theta_{\mu\alpha}\tilde{\theta}^{\alpha}_{\nu}$. For a summary of our notational conventions, see the Appendix.

and the action $S_{\rm SW}$ (1) becomes

$$\begin{aligned}S_{\rm SW}=&-\frac{1}{4}\int d^4x(f^{\mu\nu}f_{\mu\nu}+\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}+2f^{\mu\nu}\mathcal{F}_{\mu\nu})\\&-\frac{1}{2}\int d^4x\theta^{\alpha\beta}(f^{\mu\nu}f_{\alpha\mu}f_{\beta\nu}+\mathcal{F}^{\mu\nu}\mathcal{F}_{\alpha\mu}\mathcal{F}_{\beta\nu}+2f^{\mu\nu}f_{\alpha\mu}\mathcal{F}_{\beta\nu}\\&+f^{\mu\nu}\mathcal{F}_{\alpha\mu}\mathcal{F}_{\beta\nu}+\mathcal{F}^{\mu\nu}f_{\alpha\mu}f_{\beta\nu}+2\mathcal{F}^{\mu\nu}f_{\alpha\mu}\mathcal{F}_{\beta\nu})\\&+\frac{1}{8}\int d^4x\theta^{\alpha\beta}f_{\alpha\beta}(f^{\mu\nu}f_{\mu\nu}+\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}+2f^{\mu\nu}\mathcal{F}_{\mu\nu})\\&+\frac{1}{8}\int d^4x\theta^{\alpha\beta}\mathcal{F}_{\alpha\beta}(f^{\mu\nu}f_{\mu\nu}+\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}+2f^{\mu\nu}\mathcal{F}_{\mu\nu}).\end{aligned}$$

Let us write the integrand here as

$$\mathcal{L}_{\rm SW}=\mathcal{L}(\mathcal{F})+\frac{1}{2}D^{-1}_{\mu\nu}a^\mu a^\nu+\frac{1}{6}\Pi_{\mu\nu\sigma}a^\mu a^\nu a^\sigma, \tag{4}$$

where

$$D^{-1}_{\mu\nu}=k^2\eta_{\mu\nu}-k_\mu k_\nu+\Pi_{\mu\nu} \tag{5}$$

is the photon propagator. The polarization second- and third-rank tensors are defined as

$$\Pi_{\mu\nu}=\frac{\partial^2\mathfrak{L}}{\partial A_\mu\partial A_\nu}\bigg|_{F=\mathcal{F}},\qquad \Pi_{\mu\nu\rho}=\frac{\partial^3\mathfrak{L}}{\partial A_\mu\partial A_\nu\partial A_\rho}\bigg|_{F=\mathcal{F}}. \tag{6}$$

These are transverse in every index: $\Pi_{\mu\nu}k^\nu=\Pi_{\mu\nu\rho}k^\rho=0$. These identities follow from the definition Eq. (6) and from the fact that due to the $U(1)$ gauge invariance of the action, Eq. (1), the latter depends on the vector potential only through the field intensity tensor. The mechanism may be traced as in Ref. [26].

The first term on the right-hand side of (4), $\mathcal{L}(\mathcal{F})$, is

$$\begin{aligned}\mathcal{L}(\mathcal{F})=&-\frac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}-\frac{1}{2}\theta^{\alpha\beta}\mathcal{F}^{\mu\nu}\mathcal{F}_{\alpha\mu}\mathcal{F}_{\beta\nu}\\&+\frac{1}{8}\theta^{\alpha\beta}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}.\end{aligned} \tag{7}$$

We did not include the term linear in a in (4), since it vanishes for constant external fields $\mathcal{F}_{\mu\nu}=\text{const}$, because $\frac{\partial\mathcal{L}_{\rm SW}}{\partial A_\mu}\Big|_{A=\mathcal{A}}=\frac{\partial}{\partial x_\nu}\frac{\partial\mathcal{L}_{\rm SW}}{\partial F_{\nu\mu}}\Big|_{F=\mathcal{F}}=0$. In other words, any constant field, in no way correlated with the tensor $\theta_{\alpha\beta}$, is an exact source-free solution to the equation of motion.

The part quadratic in the dynamic fields a^μ contains the terms

$$\begin{aligned}&-\frac{1}{4}f^{\mu\nu}f_{\mu\nu}-\theta^{\alpha\beta}f^{\mu\nu}f_{\alpha\mu}\mathcal{F}_{\beta\nu}-\frac{1}{2}\theta^{\alpha\beta}\mathcal{F}^{\mu\nu}f_{\alpha\mu}f_{\beta\nu}\\&+\frac{1}{4}\theta^{\alpha\beta}f_{\alpha\beta}f^{\mu\nu}\mathcal{F}_{\mu\nu}+\frac{1}{8}\theta^{\alpha\beta}\mathcal{F}_{\alpha\beta}f^{\mu\nu}f_{\mu\nu},\end{aligned}$$

while cubic contributions come from

$$-\frac{1}{2}\theta^{\alpha\beta}f^{\mu\nu}f_{\alpha\mu}f_{\beta\nu}+\frac{1}{8}\theta^{\alpha\beta}f_{\alpha\beta}f^{\mu\nu}f_{\mu\nu}.$$

Taking into account that the Fourier transform of $\partial_\mu a_\nu(x)$ is $ik_\mu a_\nu(k)$, the quadratic contribution gives rise to the expression for the photon propagator

$$\begin{aligned}D^{-1}_{\mu\nu}=&k^2\eta_{\mu\nu}-k_\mu k_\nu+\frac{1}{2}(\theta\mathcal{F})(k^2\eta_{\mu\nu}-k_\mu k_\nu)\\&-\{[(\theta\mathcal{F})_{\mu\nu}+(\theta\mathcal{F})_{\nu\mu}]k^2+2(k\theta\mathcal{F}k)\eta_{\mu\nu}\\&-(k[\theta\mathcal{F}+\mathcal{F}\theta])_\nu k_\mu-(k[\mathcal{F}\theta+\theta\mathcal{F}])_\mu k_\nu\}\end{aligned} \tag{8}$$

and to the polarization tensor quadratic in k. The third-rank polarization tensor is given by the expression

$$\begin{aligned}\Pi_{\mu\nu}=&\frac{1}{2}(\theta\mathcal{F})(k^2\eta_{\mu\nu}-k_\mu k_\nu)-[(\theta\mathcal{F})_{\mu\nu}+(\theta\mathcal{F})_{\nu\mu}]k^2\\&-2(k\theta\mathcal{F}k)\eta_{\mu\nu}+(k[\theta\mathcal{F}+\mathcal{F}\theta])_\nu k_\mu\\&+(k[\mathcal{F}\theta+\theta\mathcal{F}])_\mu k_\nu,\\ \Pi_{\mu\nu\sigma}=&3ik^\alpha\theta_{\alpha\sigma}(k^2\eta_{\mu\nu}-k_\mu k_\nu),\end{aligned} \tag{9}$$

cubic in k. This character of dependence on the 4-momentum vector is a direct consequence of the fact that the action (1) is local in the sense that it does not include space and time derivatives of the field intensities, the same as in the local or infrared, $k\to 0$, approximation of quantum electrodynamics with or without a constant external field, in which case the power of k coincides with the rank of the polarization tensor [26]. The equation of motion for the field a above the external field background, resulting from (4), is the classical quadratic equation of nonlinear optics

$$\frac{\partial\mathcal{L}}{\partial A_\mu}=\Pi_{\mu\nu}a_\mu+\frac{1}{2}\Pi_{\mu\nu\rho}a_\nu a_\rho=0.$$

It describes in a classical way a transformation of a photon into two (or vice versa), which may be also treated via classical scattering theory in probabilistic terms after we define one- or two-photon asymptotic states. The probabilities of photon splitting/merging are given in terms of the third-rank polarization tensor, whereas the photon normal modes, as well as integrals of motion associated with conserved Noether generators of canonical transformations within the Hamiltonian formalism are determined by the second-rank polarization tensor. Therefore the term quadratic in a of (4) is taken as the "free" Lagrangian. Then the cubic term should be treated as the "interaction" part.

Before proceeding, we find it interesting to mention the Lorentz-covariant treatment within the standard commutative electrodynamics of quite a different system where the Lorenz invariance is violated not only by an external field, but also by the "vacuum," whose role in that case was played by the electron-positron plasma in a magnetic field

[27,28]. Let us take the following antisymmetric tensor $k_\mu u_\nu - k_\nu u_\mu$ formed by the photon momentum and by the four-velocity of the medium u_μ and substitute it for the noncommutativity tensor $\theta_{\mu\nu}$ in the polarization operator (9). Then, apart from the trivial unit transverse tensor its tensor structure coincides with the matrix $\psi^{(8)}_{\mu\nu}$ from [27,28], which appears when the plasma with excess of electrons over positrons is taken. [Contrary to the latter case, however, there is no antisymmetric matrices in the expansion (9) that might be responsible for elliptical polarization of eigenmodes and hence for the Faraday rotation of the polarization plane.[5]] This demonstrates a way of how an analog of the noncommutative tensor may be modeled by such medium.

III. GENERAL COVARIANT DESCRIPTION OF PHOTON PROPAGATION

Since the external field and the noncommutativity parameter θ are constant, the second rank polarization tensor only depends on coordinate differences and hence on one momentum k_μ. By calculating the second derivative of $\mathcal{L}_{\rm SW}$ with respect to the potential, we arrive at the expression (9), which is a linear combination of symmetric transverse matrices. The symmetry is explicitly provided by the relation $(\mathcal{F}\theta)_{\mu\nu} = (\theta\mathcal{F})_{\nu\mu}$ that follows from the antisymmetry of the matrices θ and $\mathcal{F}$.

It is noteworthy that when the external field is absent, the polarization operator is zero in spite of the presence of the noncommutativity parameter θ. Thus, the dispersion laws are in that case all trivial $k^2 = \mathbf{k}^2 - k_0^2 = 0$.

The eigenvalues of the vacuum polarization tensor define the energy spectrum of free electromagnetic waves propagating in the external field. The most general gauge-invariant expression for the polarization tensor in a constant and homogeneous background (including an electromagnetic field and the noncommutativity tensor) is [27,28,30]

$$\Pi_{\mu\nu}(k) = \sum_{i=1}^{6} \Pi_i(I)\Psi^{(i)}_{\mu\nu}, \tag{10}$$

where $\Psi^{(i)}$ are linearly independent transverse symmetric matrices, and the coefficients $\Pi_i(I)$ depend on the invariants of the theory, such as the ones involving the vector k_μ and the electromagnetic field strength $\mathcal{F}_{\mu\nu}$, k^2, $k\mathcal{F}^2k$, $\mathfrak{F}$ and $\mathfrak{G}$, and also the ones depending on the external tensor $\theta_{\mu\nu}$, such as $k\mathcal{F}^2\theta\mathcal{F}k$. The number of independent matrices in (10) is 6, because the symmetry conditions $\Pi_{\mu\nu}(k) = \Pi_{\nu\mu}(k)$ following from the definition (6) leave 10 out of $16 = 4\times 4$ independent components of the polarization operator, whereas the four transversality conditions $\Pi_{\mu\nu}(k)k^\nu = 0$ reduce their number to 6.

[5]The Faraday rotation, as well as birefringence, was reported in [29] for a model with two noncommutativity tensors against the background of a plane wave and a magnetic field.

In general, in order to find the Green function D as well as dispersion laws for photon eigenmodes it is necessary to diagonalize the polarization tensor (9) following the method developed in [27,28,30]. The polarization operator has three nontrivial scalar eigenvalues $\varkappa_i$, $i = 1, 2, 3$ ($\varkappa_4 = 0$ due to its transversality)

$$\Pi^\nu_\mu d^{(i)}_\nu = \varkappa_i d^{(i)}_\mu, \qquad i = 1, 2, 3, 4 \tag{11}$$

(that may depend on the scalars available in the problem), which are linear in θ and $\mathcal{F}$, and quadratic in k. Then the dispersion equations are

$$\varkappa_i = k^2, \qquad i = 1, 2, 3, \tag{12}$$

while the three eigenvectors $d^{(i)}_\mu$ carry information about polarizations of the three (as a matter of fact, two) eigenmodes. To find the eigenvalues and eigenvectors one generally needs to solve a cumbersome cubic equation. More important is that the corresponding eigenvectors are not universal, but depend on dynamics, that is on θ, which includes the coupling constant.

We shall proceed with the general case in Sec V, where this cubic equation will be avoided by considering only the lowest order in θ. But preliminarily, in the next subsection, we consider the special case when the magnetic field is parallel to the pseudovector $\theta_i = \varepsilon_{ijk}\theta_{jk}$, in a frame where $\theta_{0i} = 0$. In this case, the diagonalization of the polarization tensor simplifies to a closed-form purely kinematic solution, the same as in the standard, commutative electrodynamics in external magnetic field [28,30].

IV. MAGNETIC FIELD PARALLEL TO THE NONCOMMUTATIVITY PSEUDOVECTOR $\boldsymbol{\theta}$

In the special case to be considered now the simplest relation between the external field and the noncommutativity tensor $\theta_{\mu\nu} = \frac{1}{2}\alpha\mathcal{F}_{\mu\nu}$ is adopted, where α is a dimensional parameter ($[\alpha] = \text{mass}^{-4}$). Consequently, in view of (2), also $\mathfrak{G} = 0$, $(k\theta\tilde{\mathcal{F}}k) = 0$, $\mathfrak{F} > 0$. As a result, the external field is purely magnetic in the special frame, $\mathcal{B}_i = \frac{1}{2}\epsilon_{ijk}\mathcal{F}_{jk}$, and parallel to the noncommutativity pseudovector $\boldsymbol{\theta}$, defined as $\theta_i = \epsilon_{ijk}\theta_{jk}$. We therefore align the axes of our special frame so that the third axis coincides with the direction of the magnetic field. Then, $\theta_1 = \theta_2 = \mathcal{B}_1 = \mathcal{B}_2 = 0$, in other words, $\theta_3 = \theta_{12}$ and $\mathcal{F}_{12}$ are the only nonvanishing components of the noncommutativity vector and of the external field strength tensor.

With the condition $\theta_{\mu\nu} = \frac{1}{2}\alpha\mathcal{F}_{\mu\nu}$ the field tensor remains the only external tensor in the problem, and the covariant expansion of the polarization operator is the same as in the problem of commutative electrodynamics in a constant field of the most general form. While $\mathfrak{G} \neq 0$, the expansion (10) contains four independent matrices $\Psi^{(i)}_{\mu\nu}$, given by [28]

$$\Psi^{(1)}_{\mu\nu} = k^2\eta_{\mu\nu} - k_\mu k_\nu, \qquad \Psi^{(2)}_{\mu\nu} = -(\mathcal{F}k)_\mu(\mathcal{F}k)_\nu,$$
$$\Psi^{(3)}_{\mu\nu} = -k^2\left(\delta^\sigma_\mu - \frac{k_\mu k^\sigma}{k^2}\right)F^2_{\sigma\kappa}\left(\delta^\kappa_\nu - \frac{k^\kappa k_\nu}{k^2}\right),$$
$$\Psi^{(4)}_{\mu\nu} = (\mathcal{F}k)_\mu\mathcal{F}^3_{\nu\sigma}k^\sigma + \mathcal{F}^3_{\mu\sigma}k^\sigma(\mathcal{F}k)_\nu. \tag{13}$$

In our present special case $\mathfrak{G} = 0$; therefore, one has $\mathcal{F}^3_{\mu\nu} = -2\mathfrak{F}\mathcal{F}_{\mu\nu}$. As a result, the set of matrices $\Psi^{(i)}$ is no longer linear independent ($\Psi^{(4)}$ is proportional to $\Psi^{(2)}$). Moreover, the eigenvectors $d^{(i)}_\mu$, $i = 1,2,3,4$, of the polarization tensor have the simple form $d^{(i)}_\mu = b^{(i)}_\mu$

$$b^{(1)}_\mu = (\mathcal{F}^2k)_\mu k^2 - k_\mu(k\mathcal{F}^2k), \qquad b^{(2)}_\mu = (\tilde{\mathcal{F}}k)_\mu,$$
$$b^{(3)}_\mu = (\mathcal{F}k)_\mu, \qquad b^{(4)}_\mu = k_\mu. \tag{14}$$

The nonvanishing eigenvalues are given in terms of the coefficients Π_i in (10) as

$$\varkappa_1 = \Pi_1k^2 + \Pi_3(k\mathcal{F}^2k + 2\mathfrak{F}k^2),$$
$$\varkappa_2 = \Pi_1k^2,$$
$$\varkappa_3 = \Pi_1k^2 + \Pi_2k\mathcal{F}^2k + \Pi_3 2\mathfrak{F}k^2. \tag{15}$$

The solutions of the dispersion equations (12) supply poles to the photon propagator. The equation with $i = 1$ has only $k^2 = 0$ as its solution, which is pure gauge due to the properties of the corresponding eigenvector $b^{(1)}_\mu$.

Using the condition $\theta_{\mu\nu} = \frac{1}{2}\alpha\mathcal{F}_{\mu\nu}$ we get for the polarization tensor (9)

$$\Pi_{\mu\nu}(k) = -\alpha\mathfrak{F}(k^2\eta_{\mu\nu} - k_\mu k_\nu) - \alpha\mathcal{F}^2_{\mu\nu}k^2 - \alpha\mathcal{F}^2_{\alpha\beta}k^\alpha k^\beta\eta_{\mu\nu}$$
$$+ \alpha k^\sigma\mathcal{F}^2_{\sigma\nu}k_\mu + \alpha k^\sigma\mathcal{F}^2_{\sigma\mu}k_\nu$$
$$= -\alpha\left(\mathfrak{F} + \mathcal{F}^2_{\sigma\kappa}\frac{k^\sigma k^\kappa}{k^2}\right)(k^2\eta_{\mu\nu} - k_\mu k_\nu)$$
$$- \alpha k^2\left(\delta^\sigma_\mu - \frac{k^\sigma k_\mu}{k^2}\right)\mathcal{F}^2_{\sigma\kappa}\left(\delta^\kappa_\nu - \frac{k^\kappa k_\nu}{k^2}\right).$$

Comparing this expression with the general expansion (10), where the $\Psi^{(i)}$ matrices are given by (13), one finds the coefficients Π_i to be

$$\Pi_1 = -\alpha\mathfrak{F}\left(\frac{k^2 + \frac{k\mathcal{F}^2k}{\mathfrak{F}}}{k^2}\right), \qquad \Pi_2 = 0, \qquad \Pi_3 = \alpha.$$

Therefore, the nonzero eigenvalues of the polarization tensor are readily obtained from (15) taking into account the above coefficients,

$$\varkappa_1 = \alpha\mathfrak{F}k^2, \qquad \varkappa_2 = -\alpha\mathfrak{F}\left(k^2 + \frac{k\mathcal{F}^2k}{\mathfrak{F}}\right),$$
$$\varkappa_3 = \alpha\mathfrak{F}\left(k^2 - \frac{k\mathcal{F}^2k}{\mathfrak{F}}\right).$$

The solutions to the equations $\varkappa_i = k^2$ can be found with the help of the relations

$$k^2 + \frac{k\mathcal{F}^2k}{2\mathfrak{F}} = \mathbf{k}^2_\parallel - k_0^2, \qquad \frac{k\mathcal{F}^2k}{2\mathfrak{F}} = -\mathbf{k}^2_\perp,$$

valid in the special frame. The two-dimensional vector $\mathbf{k}_\perp$ is the photon momentum projection onto the plane orthogonal to $\boldsymbol{\mathcal{B}}$, while $\mathbf{k}_\parallel$ is the photon momentum projection onto the direction of $\boldsymbol{\mathcal{B}}$.

The dispersion equations (12) with $i = 2, 3$ have the common solution

$$k_0^2 - \mathbf{k}^2_\parallel = (1 - \alpha\mathcal{B}_3^2)\mathbf{k}^2_\perp = (1 - 2\theta_3\mathcal{B}_3)\mathbf{k}^2_\perp$$

or

$$k_0 = |\mathbf{k}| - \theta_3\mathcal{B}_3\frac{\mathbf{k}^2_\perp}{|\mathbf{k}|} + \mathcal{O}(\theta^2).$$

In O(3)-invariant form this looks like

$$k_0 = |\mathbf{k}| - (\boldsymbol{\theta}\cdot\boldsymbol{\mathcal{B}})\frac{\boldsymbol{\mathcal{B}}^2\mathbf{k}^2 - (\boldsymbol{\mathcal{B}}\cdot\mathbf{k})^2}{\boldsymbol{\mathcal{B}}^2|\mathbf{k}|} + \mathcal{O}(\theta^2).$$

There is no birefringence (within the linear-in-θ accuracy adopted), since these solutions are the same. In this respect the situation is analogous to the Born-Infeld model of nonlinear electrodynamics, only there the absence of birefringence is an exact property of the model, distinguishing it from any other nonlinear electrodynamics.

In spite of the absence of birefringence, the anisotropic character of the propagation retains and manifests itself in the fact that the group velocity is not parallel to the wave vector $\mathbf{k}$, its modulus, the speed of the wave packet, being different for different orientations of the wave vector. Again there is complete analogy with the Born-Infeld model, see e.g., [16].

One can calculate the group velocity vector

$$\mathbf{v}_{\rm gr} = \frac{dk_0}{d\mathbf{k}} = \frac{\mathbf{k}}{|\mathbf{k}|}\left(1 - 2(\boldsymbol{\theta}\cdot\boldsymbol{\mathcal{B}}) + (\boldsymbol{\theta}\cdot\boldsymbol{\mathcal{B}})\frac{\boldsymbol{\mathcal{B}}^2\mathbf{k}^2 - (\boldsymbol{\mathcal{B}}\cdot\mathbf{k})^2}{\boldsymbol{\mathcal{B}}^2|\mathbf{k}|^2}\right)$$
$$+ 2\boldsymbol{\mathcal{B}}\frac{(\boldsymbol{\theta}\cdot\boldsymbol{\mathcal{B}})(\boldsymbol{\mathcal{B}}\cdot\mathbf{k})}{\boldsymbol{\mathcal{B}}^2|\mathbf{k}|^2} + \mathcal{O}(\theta^2)$$

and its modulus

$$v_{\rm gr} = \left|\frac{dk_0}{d\mathbf{k}}\right| = 1 - \theta_3\mathcal{B}_3\frac{\mathbf{k}^2_\perp}{\mathbf{k}^2} + \mathcal{O}(\theta^2). \tag{16}$$

The direction of propagation as identified with that of the group velocity does not, obviously, coincide with the direction of the wave vector $\mathbf{k}$, but makes the angle φ with it, such that $\cos\varphi = 1 - 2(\theta_3\mathcal{B}_3\cos\eta\sin\eta)^2$, $|\varphi| = |\theta_3\mathcal{B}_3\sin 2\eta|$, where $\cos\eta = \frac{\mathbf{k}_\parallel}{|\mathbf{k}|}$. The directions of $\mathbf{v}_{\rm gr}$ and $\mathbf{k}$ coincide for propagations parallel ($\mathbf{k}_\perp = 0$,

$\cos\eta = 1$) and perpendicular ($\mathbf{k}_\parallel = 0, \cos\eta = 0$) to the magnetic field.

It is seen from (16) that for any direction of propagation but parallel, the speed can be smaller or larger than one, depending on whether $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\theta}$ are parallel or antiparallel.

V. GENERAL COVARIANT DESCRIPTION OF PHOTON PROPAGATION CONTINUED

In this section we come back to the general case of Sec. III and proceed by diagonalization of the inverse propagator (5), instead of diagonalizing the polarization tensor, as in the previous section.

We need to solve the eigenvalue equation for the inverse propagator (8), which is equivalent to (11)

$$(D^{-1})_\mu{}^\sigma d_\sigma = \lambda d_\mu \tag{17}$$

where the vector d_σ is a linear combination of the vectors $b_\sigma^{(i)}$ (14), which, unlike the special case of Sec. IV, no longer are eigenvectors, but still form an orthogonal basis,

$$d_\sigma = \sum_{i=1}^{4} \alpha_i b_\sigma^{(i)}. \tag{18}$$

Contrary to the eigenvectors (14) of the problem in the previous section, the eigenvectors (18) will depend on the dynamics, i.e., on the noncommutativity parameter that contains the coupling constant.

One can check that the basis vectors $b^{(i)}$ are eigenvectors of the free part of the propagator,

$$(\delta_\mu^\sigma k^2 - k_\mu k^\sigma) b_\sigma^{(a)} = k^2 b_\mu^a, \quad \text{for } a = 1,2,3,$$
$$(\delta_\mu^\sigma k^2 - k_\mu k^\sigma) b_\sigma^{(4)} = 0.$$

Since the polarization vector is transverse, $\Pi_\mu{}^\sigma k_\sigma = 0$, as are the first three basis vectors, Eq. (17) reduces to

$$(D^{-1})_\mu{}^\sigma \sum_{a=1}^{3} \alpha_a b_\mu^{(a)} = (k^2 - \varkappa) \sum_{i=a}^{3} \alpha_a b_\mu^{(a)} \equiv \lambda \sum_{a=1}^{3} \alpha_a b_\mu^{(a)},$$

where $\varkappa$ is the eigenvalue of the polarization tensor whose eigenvector is $\sum_{a=1}^3 \alpha_a b_\mu^{(a)}$. Making use of the orthogonality of the eigenvectors $b^{(a)}$, $b^{(a)\mu} b_\mu^{(b)} = 0$ for $a \neq b$, one has

$$x_{ab}\alpha_b \equiv \frac{b^{(a)\mu}(D^{-1})_\mu{}^\sigma b_\sigma^{(b)}}{b^{(a)\nu} b_\nu^{(a)}} \alpha_b = \lambda \alpha_a.$$

Now in order to determine λ we need to solve the equation

$$\det|x_{ab} - \lambda\delta_{ab}| = 0,$$

which expresses λ in terms of the matrix elements x_{ab}. Taking into account that the off-diagonal components of x_{ab} are of order θ, they only contribute terms of order at least θ^2 in the characteristic polynomial. Therefore, the dominant contributions come from the diagonal part,

$$\det|x_{ab} - \lambda\delta_{ab}| = \prod_{a=1}^{3} (x_{aa} - \lambda) + O(\theta^2).$$

Therefore, to first order in θ, the dispersion equations (12) can be written in the form $\lambda_a = x_{aa} = 0$.

VI. MUTUALLY ORTHOGONAL ELECTRIC AND MAGNETIC FIELDS

We are going to solve the dispersion relations in the special case where the electric and magnetic fields are orthogonal, and a Lorentz frame exists where the noncommutativity tensor has vanishing time-space components. Therefore, we add the restriction $\mathfrak{G} = 0$ to conditions (2).

In calculating the diagonal components of the matrix x_{ab}, the only nontrivial contribution from the polarization operator (9) comes from the term $(\theta\mathcal{F})_{\mu\nu}$, due to the transversality of the basis vectors and the simplifying relations $(\mathcal{F}\tilde{\mathcal{F}})_{\mu\nu} = 0$ and $\mathcal{F}_{\mu\nu}^3 = -2\mathfrak{F} F_{\mu\nu}$, peculiar to the configuration $\mathfrak{G} = 0$ (see the Appendix).

$$x_{11} = k^2\left(1 - \frac{1}{2}(\theta\mathcal{F})\right) + 2\frac{k^2}{(k\mathcal{F}^2 k)}(k\mathcal{F}^2\theta\mathcal{F}k),$$
$$x_{22} = k^2\left(1 - \frac{1}{2}(\theta\mathcal{F})\right) + 2(k\theta\mathcal{F}k),$$
$$x_{33} = k^2\left(1 - \frac{1}{2}(\theta\mathcal{F})\right) + 2(k\theta\mathcal{F}k) + 2\frac{k^2}{(k\mathcal{F}^2 k)}(k\mathcal{F}^2\theta\mathcal{F}k). \tag{19}$$

As in Sec. IV, we go over to a special reference frame where the orientation of the spatial axes are such that the third axis is aligned to the magnetic field, i.e., $\mathcal{B}_{1,2} = 0$, $\mathcal{B}_3 \neq 0$. Then the electric field should lie in the (1,2) plane. Now we may rotate the spatial frame around the magnetic field (axis 3) to nullify θ_1. Hence the choices $\theta_2, \theta_3, \mathcal{B}_3 \neq 0$, and $\mathcal{E}_{1,2} \neq 0$ to be considered in the present section represent the most general case specified by the conditions (2) and $\mathfrak{G} = 0$.

With the help of the identities (A1) from the Appendix, we are able to write the eigenvalues (19) in the special reference frame as

$$x_{11} = k^2(1-\boldsymbol{\theta}\cdot\boldsymbol{\mathcal{B}}),$$
$$x_{22} = k^2(1+\boldsymbol{\theta}\cdot\boldsymbol{\mathcal{B}}) + 2[k^0\mathbf{k}\cdot(\boldsymbol{\theta}\times\boldsymbol{\mathcal{E}}) - \mathbf{k}^2(\boldsymbol{\theta}\cdot\boldsymbol{\mathcal{B}}) + (\mathbf{k}\cdot\boldsymbol{\mathcal{B}})(\mathbf{k}\cdot\boldsymbol{\theta})],$$
$$x_{33} = k^2(1-\boldsymbol{\theta}\cdot\boldsymbol{\mathcal{B}}) + 2[k^0\mathbf{k}\cdot(\boldsymbol{\theta}\times\boldsymbol{\mathcal{E}}) - \mathbf{k}^2(\boldsymbol{\theta}\cdot\boldsymbol{\mathcal{B}}) + (\mathbf{k}\cdot\boldsymbol{\mathcal{B}})(\mathbf{k}\cdot\boldsymbol{\theta})].$$

As a result, the equation $x_{11}=0$ implies $k^2=0$, while the two equations $x_{22}=x_{33}=0$ imply the common solution

$$k^2 = -2[k^0\mathbf{k}\cdot(\boldsymbol{\theta}\times\boldsymbol{\mathcal{E}}) - \mathbf{k}^2(\boldsymbol{\theta}\cdot\boldsymbol{\mathcal{B}}) + (\mathbf{k}\cdot\boldsymbol{\mathcal{B}})(\mathbf{k}\cdot\boldsymbol{\theta})].$$

This means again that birefringence is absent up to the adopted accuracy of $o(\theta^2)$. The positive branch of k^0 is

$$k^0 = -(\mathbf{k}\cdot(\boldsymbol{\theta}\times\boldsymbol{\mathcal{E}})) + \left[(1-\boldsymbol{\theta}\cdot\boldsymbol{\mathcal{B}})|\mathbf{k}| + \frac{1}{|\mathbf{k}(|}(\mathbf{k}\cdot\boldsymbol{\mathcal{B}})(\mathbf{k}\cdot\boldsymbol{\theta})\right]. \tag{20}$$

The group velocity is

$$\mathbf{v}_{\mathrm{gr}} \equiv \frac{dk_0}{d\mathbf{k}} = \frac{\mathbf{k}}{|\mathbf{k}|}\left(1-(\boldsymbol{\theta}\cdot\boldsymbol{\mathcal{B}}) - \frac{(\mathbf{k}\cdot\boldsymbol{\mathcal{B}})(\mathbf{k}\cdot\boldsymbol{\theta})}{\mathbf{k}^2}\right) + \boldsymbol{\theta}\frac{(\mathbf{k}\cdot\boldsymbol{\mathcal{B}})}{|\mathbf{k}|} + \boldsymbol{\mathcal{B}}\frac{(\mathbf{k}\cdot\boldsymbol{\theta})}{|\mathbf{k}|} - (\boldsymbol{\theta}\times\boldsymbol{\mathcal{E}}), \tag{21}$$

and its norm is

$$\begin{aligned} v_{\mathrm{gr}} &= 1-(\boldsymbol{\theta}\cdot\boldsymbol{\mathcal{B}}) + \frac{(\mathbf{k}\cdot\boldsymbol{\mathcal{B}})(\mathbf{k}\cdot\boldsymbol{\theta})}{\mathbf{k}^2} - \frac{(\mathbf{k}\cdot(\boldsymbol{\theta}\times\boldsymbol{\mathcal{E}}))}{|\mathbf{k}|} \\ &= 1 - \frac{\mathbf{k}_\perp^2}{\mathbf{k}^2}\theta_3\mathcal{B}_3 + \frac{k_2k_3}{\mathbf{k}^2}\theta_2\mathcal{B}_3 \\ &\quad + \frac{1}{|\mathbf{k}|}(k_3\theta_2\mathcal{E}_1 + k_1\theta_3\mathcal{E}_2 - k_2\theta_3\mathcal{E}_1). \end{aligned} \tag{22}$$

Since $\boldsymbol{\theta}$ and $\boldsymbol{\mathcal{B}}$ are pseudovectors, while $\boldsymbol{\mathcal{E}}$ is a vector, Eqs. (20), (21) are vectors and (22) is a scalar.

Special cases can be obtained from the above by removing field components. For instance, by setting $\boldsymbol{\mathcal{E}}=0$, one obtains the following modification:

$$v_{\mathrm{gr}} = 1 - \frac{\mathbf{k}_\perp^2}{\mathbf{k}^2}\theta_3\mathcal{B}_3 + \frac{k_2k_3}{\mathbf{k}^2}\theta_2\mathcal{B}_3 \tag{23}$$

of the group velocity (16), valid for the special case[6] $\theta_2, \theta_3, \mathcal{B}_3 \neq 0$, $\boldsymbol{\mathcal{E}}=0$. This case can be specified in an invariant manner by adding the invariant conditions $\mathcal{F}\tilde{\theta}=0$, $(F\theta\tilde{\theta})=0$ and $\mathfrak{F}>0$, which have the effect of removing the electric field from the special frame. Because of the factor of k_3 in the θ_2 term in (23), the propagation transverse to the magnetic field still gives $v_{\mathrm{gr}}^\perp = 1-\theta_3\mathcal{B}_3$, which may be greater than unity, like in (16), but it also remains equal to unity for propagation parallel to the magnetic field. This is no longer the case for the more general Eq. (22): for the parallel propagation, too, the group velocity (22)

$$v_{\mathrm{gr}} = 1+\theta_2\mathcal{E}_1$$

may exceed unity.

One can proceed in a similar fashion by specifying another particular case of interest, by setting $\boldsymbol{\mathcal{B}}=0$. Then, by rotating the coordinate system around the third axis we can annihilate the component $\mathcal{E}_2$ of the electric field, and annihilate the component θ_3 by the subsequent rotation around the first axis. Therefore, by setting $B_3=\mathcal{E}_2=\theta_3=0$ (22) we get the following expression:

$$v_{\mathrm{gr}} = 1+\theta_2\mathcal{E}_1\frac{k_3}{|\mathbf{k}|}$$

for the group velocity in an electric field directed along the first axis. The photon propagating in the plane spanned by the mutually orthogonal vectors of the electric field $\boldsymbol{\mathcal{E}}$ and the noncommutativity vector $\boldsymbol{\theta}$ does so with unit speed (that of light in the vacuum). For other directions it exceeds unity when the three vectors $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\theta}$, and $\mathbf{k}$ make a right triad, $\mathcal{E}_i = a\epsilon_{ijk}\theta_j k_k$ with $a>0$.

VII. CONCLUDING REMARKS

We have considered, in the lowest order of the noncommutativity parameter, photon propagation in an anisotropic medium, equivalent to the vacuum in the space-space noncommutative electrodynamics with external constant electric and magnetic fields. The most general case considered includes mutually perpendicular electric and magnetic fields, arbitrarily oriented with respect to the noncommutativity vector, taken together or—as special cases—separately. Our consideration is based on the Lorentz-covariant formalism of the Lorentz-noninvariant theory, that operates with the local action written as a Lorentz scalar including the noncommutativity tensor and the background constant electromagnetic fields, violating Lorentz invariance. We found the dispersion laws and we derived expressions for the group velocities. In neither case the birefringence occurs within the accuracy adopted, i.e., the solutions of dispersion equations for two propagating modes coincide. In neither case the direction of the wavevector, the photon 3-momentum, coincides with the direction of the energy-momentum propagation given by the group velocity (see [31] for the Lorentz-noninvariant case and nonsymmetric energy-momentum tensor). The direction of the wave vector should not be referred to as the direction of propagation. Also no conclusions on the speed of propagation can be arrived at based on the phase

[6]The dispersion law in this special case was obtained in [22].

velocity, contrary to what some authors are inclined to do. However, the group velocity should be considered as carrying information. Consequently it must not exceed the speed of light in vacuum, $c = 1$, without conflicting with the causality principle, even though we are dealing with a Lorentz-noninvariant theory. (The known exception to this rule made by the phenomenon of abnormal dispersion can be easily circumvented by a redefinition of the group velocity for complex energy instead of complex momentum [32].) Contrary to the causality requirement, we have found that the group velocity does exceed unity if special relations between the background electromagnetic field and the noncommutativity tensor are fulfilled. Moreover, in the general case considered, there is no special direction of propagation relative to the background fields that would exclude propagation with speed exceeding unity. We consider this situation as a serious indication of inconsistency of the theory.

What may the possible way out be?

In general physics courses, professors sometimes tell students that a perfectly rigid body cannot be built, because the speed of sound in it would be greater than the speed of light in vacuum. However, they do not explain what mechanisms can prevent one from constructing such a body. A positive example of such a mechanism may be found in quantum electrodynamics with external fields, where a super-Plank background field leads to the possibility of superluminal propagation. However, this field cannot be achieved, because the instability destroying that field occurs earlier [16]. Following that line we must admit that a certain mechanism should exist that would exclude or forbid causality violating relations between the background field and the noncommutativity tensor. Within the simplest case of Sec. IV this would mean a prohibition of the background magnetic field being opposite to the noncommutativity vector. Such a mechanism, however, is unknown; anyway it is not seen to be provided by the field equations.

On the other hand, we may speculate that the presence of a superluminal signal may not be considered a catastrophe, indeed, provided that the excess over the speed of light is extremely small. We should take into account that the realization of a time machine would require a Lorentz transformation with speed $V < 0$, enough to reverse the sign of the time coordinate. This means that the inequality $|V| > 1/v$ is at least necessary. (It is understood here that the signal is superluminal, $v > 1$, albeit the speed of the reference frame does not exceed the speed of light, $|V| < 1$.) Such must be also the speed of the device that registers the arrival of the superluminal signal and sends a superluminal signal back, as explained in [17]. Since the signal speed is expected to exceed unity only just a little, the speed of that device should closely approach the speed of light. Moreover, to get a sufficiently negative time interval, as it is desirable for achieving a sufficiently remote past, it is required that $|V|$ approach unity still closer. Since we never experimented with such devices, we cannot state that the whole manifold of the established physical facts contradicts the possibility of constructing a time machine once we have at our disposal a superluminal signal, whereas the logical paradox implied by this device is not alone sufficient for ruling it out.

Let us estimate the necessary speed of the detector/emitter in the case of noncommutative electrodynamics with external fields considered in the present paper. According to the present results, the excess of the group velocity over unity $\Delta v = v^{\mathrm{gr}} - 1$ is of the order of θB. It makes sense to take for the magnetic field its largest value known from pulsars and magnetars, which is of the order of magnitude of Schwinger's characteristic value $\frac{m^2}{e}$ or two orders higher. With m and e being the electron mass and charge this makes approximately $4.4 \times 10^{13} G$. According to the strongest estimate of the noncommutativity parameter found in [13], $\theta < (1000\ \mathrm{Tev})^{-2}$, then $\Delta v = \theta B < \sim 10^{-18}$. Hence the speed of the emitter/detector in the pulsar magnetosphere must be greater than 1–10^{-18} One cannot fathom this speed for a macroscopic body. This corresponds to the speed of a cosmic proton with energy of 10^{18} ev. The kinetic energy of the emitter/detector weighing one gram would be 10^{41} ev.

ACKNOWLEDGMENTS

Gitman thanks CNPq and FAPESP for permanent support. The work of A. S. and D. G. is also partially supported by the Tomsk State University Competitiveness Improvement Program, by a grant from "The Tomsk State University Academician D. I. Mendeleev Fund Program" and also by RFBR under the Project 15-02-00293a. A. S. also thanks USP for kind hospitality extended to him during his stay in São Paulo, Brazil, and acknowledges the support of FAPESP under Processo 2014/08970-1.

APPENDIX NOTATION AND USEFUL RELATIONS

We follow the conventions adopted in [28], in particular, the metric is given by $\eta = \mathrm{diag}(1,1,1,-1)$, $\mu,\nu,\rho,\ldots = 1,2,3,0$, and $\varepsilon_{1230} = 1$. We use the following conventions with regard to contractions:

$$F^n_{\alpha\beta} \equiv F_\alpha{}^{\alpha 1} F_{\alpha 1}{}^{\alpha 2} \ldots F_{\alpha_{n-1}\beta}, \qquad (\theta F^n)_{\alpha\beta} \equiv \theta_\alpha{}^{\alpha 1} F^n_{\alpha_1\beta},$$
$$(\theta K) = (\theta K)_\alpha{}^\alpha, \qquad n = 0,1,2,\ldots,$$
$$(Kk)_\alpha \equiv K_{\alpha\beta} k^\beta, \qquad (kK)_\alpha \equiv k^\beta K_{\beta\alpha},$$
$$(Kk)_\alpha = -(kK)_\alpha \quad \text{for } K^T = -K.$$

Thus, $(kF\theta k) \equiv k^\mu F_{\mu\nu} \theta^{\nu\sigma} k_\sigma$ and $(\theta F) = (F\theta) \equiv \theta_{\alpha\beta} F^{\beta\alpha}$. We also have

$$\mathfrak{F} = -\frac{1}{4}F^2 = \frac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2), \qquad \mathfrak{G} = -\frac{1}{4}(F\tilde{F}) = \mathbf{E}\cdot\mathbf{B},$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, $E^i = F^{i0}$ and $B^i = \frac{1}{2}\epsilon^{ijk}F^{jk}$. We also note the useful relations

$$(F\tilde{F})_{\mu\nu} = -\eta_{\mu\nu}\mathfrak{G},$$
$$F^3_{\mu\nu} = -2\mathfrak{F}F_{\mu\nu} - \mathfrak{G}\tilde{F}_{\mu\nu},$$
$$(\tilde{F}^2)_{\mu\nu} = \frac{1}{2}\eta_{\mu\nu}F^2 - (F^2)_{\mu\nu}.$$

We list some identities for the case where in the special frame one has $\theta_{0i} = 0$. The dual noncommutativity tensor is defined as $\tilde{\theta}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\theta^{\rho\sigma}$. The noncommutativity vector is $\theta_i = \epsilon_{ijk}\theta_{jk}$, and its dual is $\tilde{\theta}_i = \epsilon_{ijk}\tilde{\theta}_{jk}$. The following relations hold:

$$\begin{aligned}
(\theta F) &= -2\boldsymbol{\theta}\cdot\mathbf{B},\\
(kF^2k) &= -(k^0)^2\mathbf{E}^2 - 2k^0\mathbf{k}\cdot(\mathbf{E}\times\mathbf{B})\\
&\quad + (\mathbf{k}\cdot\mathbf{E})^2 - \mathbf{k}^2\mathbf{B}^2 + (\mathbf{k}\cdot\mathbf{B})^2,\\
(k\theta Fk) &= k^0\mathbf{k}\cdot(\boldsymbol{\theta}\times\mathbf{E}) - \mathbf{k}^2(\boldsymbol{\theta}\cdot\mathbf{B}) + (\mathbf{k}\cdot\mathbf{B})(\mathbf{k}\cdot\boldsymbol{\theta}),\\
(kF^2\theta Fk) &= [(k^0)^2\mathbf{E}^2 - (\mathbf{k}\cdot\mathbf{E})^2 + 2k^0\mathbf{k}\cdot(\mathbf{E}\times\mathbf{B})\\
&\quad + \mathbf{k}^2\mathbf{B}^2 - (\mathbf{k}\cdot\mathbf{B})^2](\boldsymbol{\theta}\cdot\mathbf{B}) - (k^0)^2(\mathbf{E}\cdot\mathbf{B})(\boldsymbol{\theta}\cdot\mathbf{E})\\
&\quad + (\mathbf{k}\cdot\mathbf{E})(\mathbf{E}\cdot\mathbf{B})(\mathbf{k}\cdot\boldsymbol{\theta}) - k^0(\mathbf{E}\cdot\mathbf{B})\mathbf{k}\cdot(\boldsymbol{\theta}\times\mathbf{B}).
\end{aligned} \tag{A1}$$

[1] S. Doplicher, K. Fredenhagen, and J. Roberts, Phys. Lett. B **331**, 39 (1994).
[2] M. Douglas and N. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001).
[3] S. Minwalla, M. V. Raamsdonk, and N. Seiberg, J. High Energy Phys. 02 (2000) 020.
[4] I. Y. Aref'eva, D. M. Belov, and A. S. Koshelev, Phys. Lett. B **476**, 431 (2000).
[5] I. Chepelev and R. Roiban, J. High Energy Phys. 03 (2001) 001.
[6] H. Grosse and R. Wulkenhaar, Lett. Math. Phys. **71**, 13 (2005).
[7] R. Fresneda, D. M. Gitman, and D. V. Vassilevich, Phys. Rev. D **78**, 025004 (2008).
[8] S. Aghababaei, M. Haghighat, and A. Kheirandish, Phys. Rev. D **87**, 047703 (2013); C. P. Martin, Classical Quantum Gravity, **30**, 155019 (2013).
[9] W. Wang, J.-H. Huang, and Z.-M. Sheng, Phys. Rev. D **86**, 025003 (2012);
[10] A. Ilderton, J. Lundin, and M. Marklund, SIGMA **6**, 041 (2010).
[11] J. Trampetic and J. You, arXiv:1501.00276; G. Deshpande and S. K. Garg, Phys. Lett. B **708**, 150 (2012); R. Horvat, A. Ilakovac, P. Schupp, J. Trampetic, and J. You, J. High Energy Phys. 04 (2012) 108; J. Trampetic and J. You, SIGMA **10**, 054 (2014); C. P. Martin, Phys. Rev. D **89**, 065018 (2014); **86**, 065010 (2012).
[12] R. Horvat and J. Trampetic, Phys. Rev. D **79**, 087701 (2009); Phys. Lett. B **710**, 219 (2012); R. Horvat, D. Kekez, and J. Trampetic, Phys. Rev. D **83**, 065013 (2011); R. Horvat, D. Kekez, P. Schupp, J. Trampetic, and J. You, Phys. Rev. D **84**, 045004 (2011); I. Mocioiu, M. Pospelov, and R. Roiban, Phys. Lett. B **489**, 390 (2000).
[13] T. C. Adorno, D. M. Gitman, A. E. Shabad, and D. V. Vassilevich, Phys. Rev. D **84**, 085031 (2011); **84**, 065003 (2011); **86**, 027702 (2012).
[14] S. M. Carroll, J. A. Harvey, V. A. Kostelecky, C. D. Lane, and T. Okamoto, Phys. Rev. Lett. **87**, 141601 (2001).
[15] I. T. Drummond and S. J. Hathrell, Phys. Rev. D **22**, 343 (1980).
[16] A. E. Shabad and V. V. Usov, Phys. Rev. D **83**, 105006 (2011).
[17] A. D. Dolgov and I. D. Novikov, Phys. Lett. B **442**, 82 (1998).
[18] L. Brillouin, *Wave Propagation and Group Velocity* (Academic, New York, 1960).
[19] M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, Oxford, 1968).
[20] L. D. Landau and E. M. Lifshits, *Electrodynamics of Continuous Media* (GIF-ML, Moscow, 1959; Pergamon Press, Oxford, 1984).
[21] N. Seiberg and E. Witten, J. High Energy Phys. 09 (1999) 032.
[22] Z. Guralnik, R. Jackiw, S. Y. Pi, and A. P. Polychronakos, Phys. Lett. B **517**, 450 (2001).
[23] R. Cai, Phys. Lett. B **517**, 457 (2001).
[24] J. Plebanski, NORDITA, Copenhagen, (1970).
[25] A. A. Bichl, J. M. Grimstrup, L. Popp, M. Schweda, and R. Wulkenhaar, Int. J. Mod. Phys. A **17**, 2219 (2002).
[26] D. M. Gitman and A. E. Shabad, Phys. Rev. D **86**, 125028 (2012); C. V. Costa, D. M. Gitman, and A. E. Shabad, Phys. Rev. D **88**, 085026 (2013); T. C. Adorno, D. M. Gitman, and A. E. Shabad, Phys. Rev. D **89**, 047504 (2014); Eur. Phys. J. C **74**, 2838 (2014).
[27] H. Pérez Rojas and A. E. Shabad, Ann. Phys. (N.Y.) **121**, 432 (1979).

[28] A. E. Shabad, *Polarization of the Vacuum and a Quantum Relativistic Gas in an External Field* (Nova Science Publishers, New York, 1991), also in: Polarization Effects in External Gauge Fields, Trudy Fizicheskogo Instituta im. P.N. Lebedeva, edited by , **192**, 5 (1988)–in Russian.

[29] S. I. Kruglov, Phys. Lett. B **652**, 146 (2007).

[30] I. A. Batalin and A. E. Shabad, Zh. Eksp. Teor. Fiz. **60**, 894 (1971) [Sov. Phys. JETP **33**, 483 (1971)], A. E. Shabad, Ann. Phys. (N.Y.) **90**, 166 (1975).

[31] S. Villalba-Chavez and A. E. Shabad, Phys. Rev. D **86**, 105040 (2012).

[32] A. E. Shabad and V. V. Usov, Phys. Rev. D **81**, 125008 (2010).