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Prethermalization after a sudden quench in a weakly interacting Bose system

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Chapter 1

Introduction

Recent progresses in both experimental and theoretical studies of quantum nonequilibrium systems [1, 2] lead to the expectation that nonintegrable systems driven out of equilibrium tend to thermalize [3] whereas systems modeled by an integrable Hamiltonian should be described in the long time limit by a Generalized Gibbs Ensemble (GGE) taking into account the conserved quantities of the model [4]. The dynamics of a quantum many-body system towards thermal equilibration is in turn a fundamental subject that still lacks a complete understanding [2–6]. One possibility is that there is a single timescale governing the process so that during equilibration all observables relax simultaneously towards their long-time limit. Alternatively some quantities may reach rapidly a quasi-stationary state, while true thermalization, if any, should occur only later on a longer timescale. This first relaxation towards a quasi-stationary state is called *prethermalization*. The prethermalized quasi-stationary state is a dephased state in which some physical observables may take a value corresponding to what they would have at equilibrium at an effective temperature T_{eff}, hence appearing "thermal". However, a prethermalized state is truly different from a thermal equilibrium one as it can be evinced from its non-thermal quasi-particle distribution.

Prethermalization was initially proposed in the context of high energy physics [7] to understand the fact that, in heavy-ions collision experiments, integrated quantities (e.g. kinetic energy) reach their final value much earlier than mode-dependent observables. Meanwhile, the astounding progress in the manipulation of ultracold atoms allowed the realization of isolated quantum systems which are the ideal playground for experimental quantum non-equilibrium physics [1]. In this context, it is possible to prepare a system in a specific state and study its unitary time evolution without the dephasing due to the interaction with the environment. This possibility, together with the continuous development in measurement accuracy and precision, had the effect of shifting the interest in the dynamics of non-equilibrium many-body physics from the theoretical ground to the experimental one. The study of prethermalization for cold atomic systems [8, 9], has shown some intimate connection between this phenomenon and concepts that emerged in the physics of thermalization, for example, the expectation value of observables in the prethermalization plateau of an almost integrable system can be computed as the GGE expectation value built with some quantities that are nearly conserved under the time evolution

of the model and the asymptotic state of an integrable system should be considered as a prethermalized state that never decays [10]. In addition prethermalization was predicted for a split one-dimensional quasi-condensate loaded on an elongated microtrap on an atom chip [11] and it was later actually observed experimentally for that same setup [12, 13].

The idea that prethermalization might emerge for almost integrable systems is very appealing; indeed, integrable systems are a useful but only ideal construction: integrability breaking effects, however small, necessarily appear in experimental implementations [14]. This unavoidable departure from integrability does not necessarily spoil the original features of the model: a typical example is the "Quantum Newton's Cradle" experiment [15] (two quasi one-dimensional bosonic clouds counter-oscillating in a parabolic potential) in which the realized system is so close to its integrable model [16] that the integrability breaking effects are not enough to produce visible thermalization, at least on the timescale of the experiment (although one would expect the system to eventually thermalize). A setup equivalent to their experiment, but in three dimensions was predicted to thermalize already after three collisions [17, 18]. If energy redistribution between the modes is that fast then, whatever the kind of relaxation dynamics followed by the system, no experiment can resolve the two stages of the process and the prethermalization plateau: the system would reach too early the final thermal state to show interesting dynamical features. If, however, the same system were taken out of equilibrium without subjecting it to a large disturbance such as the one induced by a split process, is prethermalization expected despite the non-integrability of the model?

We choose to adress this question by focusing on a small *quantum quench* in the interaction strength g between bosonic atoms: the system is prepared in the ground state of an Hamiltonian $\hat{H}(g_0)$, at a given time the parameter is suddenly changed to a new value g_1 and the system evolves under this new Hamiltonian $\hat{H}(g_1)$, and the initial state of the system should not be one of its eigenstate, otherwise the physics would be trivial. It may be useful to explain what we mean by "sudden" from an experimental point of view, since any parameter variation cannot be really instantaneous and real suddenness is just a theoretical idealisation. Let us call τ_r the relaxation time of the sytem, i.e. the time-scale that governs the response of the system to external perturbations; if g is the Hamiltonian parameter we are willing to change, then we can define $\tau_q = g/\dot{g}$ as the time-scale of our experiment. If both τ_q and the time during which the variation of the parameters takes place are much smaller than τ_r, then we can say we are effectively performing a quench: in this scheme the system is not able to respond immediately to the perturbation we induce and lags behind. This framework is basically the opposite of the adiabatic protocol which prescribes the system to be almost at equilibrium at any time during the process. Despite the conceptual simplicity of the protocol, quantum quenches offer diverse and important opportunities to investigate

non-equilibrium processes and have been deeply studied both theoretically and experimentally [19–22].

In this work we study a system of weakly interacting condensed bosons after a small quench in the interaction strength. We will analyse this scenario in one and three dimensions, restricting our focus on a set of important observables, namely the many-body local correlations g_p. We define

$$g_p = \frac{1}{\rho^p} \left\langle (\psi^\dagger)^p (\psi)^p \right\rangle$$

where $\rho = \langle \psi^\dagger \psi \rangle$ is the particle density, in particular g_3 is proportional to the tree-body recombination rate [23]. We will also perform calculations for the non-local observable

$$g_2(x) = \frac{1}{\rho^2} \left\langle \psi^\dagger(x)\psi^\dagger(0)\psi(0)\psi(x) \right\rangle,$$

which gives the probability of finding two particles at a distance x apart and therefore gives the characteristic length scale of decay for density–density fluctuations. The choice of this setup and of these observables is also inspired by the recent calculation of $g_2(x)$ in the limits of small and large interaction and the exact calculation of g_3 for a degenerate 1D Bose system [24–29].

In Chapter 2 we recall some basic concepts about cold atom physics necessary for the following calculations. In particular, since the weakness of the interaction between the particles will allow us to use Bogoliubov approximation, we discuss how the latter works and what are its limits of applicability; then employing Bogoliubov rotation we compute the equilibrium behaviour of the quantities presented above. These results are mainly obtained from the literature but are somehow generalized and will be used later to characterize the state of the system after the quench. Chapter 3 contains a general review on the subjects of thermalization and prethermalization. Thermalization is described in its relationship with ergodicity and lack of integrability in the classical and quantum realm respectively. We also introduce the concept of prethermalization and show how to calculate prethermalized expectation values from the almost-conserved quantities of the model, moreover we describe the experimental setup in which prethermalization was actually found. The original results of the work are presented in Chapter 4 where we use the Bogoliubov approximation and related tools to calculate the consequences of a small interaction quench: a stationary state for the local and non-local correlation functions is found, whose features lead us to claim it is actually a prethermal state; in addition we obtain that the dynamic behaviour of $g_p(t)$ follows a power law whose exponent is $D + 1$, where D is the dimensionality of the system. In the same chapter we show that this result can be obtained also in a different way, namely with the Keldysh technique. Keldysh contour integration is described and employed to derive the equations that govern the

dynamics of the model; moreover we show how this field theory approach is a good starting point to generalise the present results and as an example in this direction we calculate the effect of an additional Hartree–Fock interaction term in the Hamiltonian. Subsequently we investigate the possibility and meaningfulness of the definition of an effective temperature for the system after the quench and we conclude that, in agreement with Ref. [10], the correct and meaningful description is indeed a GGE. Then, considered that the 3D system should finally thermalize, we investigate the evolution towards thermal equilibrium by introducing in the Hamiltonian an interaction term between the quasi-particles responsible for two-body decay. The dynamics generated by this decay channel is extracted from a linearized Boltzmann equation and the approach to equilibrium is computed starting from the prethermal plateau previously determined. The outcome is that the dynamics is again described by a power-law behaviour but for this second stage $g_p(t) \sim t^{-3}$ so we conclude that in this system prethermalization might actually emerge as a crossover phenomenon rather than a well defined plateau.

Chapter 2

Cold atoms

One of the most astonishing manifestations of the quantum nature of matter is the phenomenon of *Bose–Einstein condensation*, i.e. the macroscopic occupation of a quantum state by a system of bosonic particles when the temperature falls below a critical value [30]. The formation of Bose–Einstein condensates for spatially homogeneous systems at low dimensionality at finite temperature is excluded by the Mermin–Wagner theorem [31]. Indeed phase fluctuations are too large to allow a finite coherence length across the sample, but density fluctuations can still be limited by a strong enough repulsive interaction. To describe this situation characterized by strong phase fluctuations but negligible density variations, Popov introduced the concept of *quasi-condensate* [32], whose study is complicated by the fact that if one tries to naïvely introduce the usual density-phase formalism, divergencies appear. Anyway it is possible to extend the Bogoliubov approach to treat quasi-condensates of weakly interacting one-dimensional Bose gases [33] by a method that requires to discretize the real space in small cells in order to allow for a consistent definition of phase and density operators: two small parameters are then introduced: the relative density fluctuation inside a cell and the phase gradient across different cells. The usual interesting observables can be expanded in terms of these parameters and one obtains divergenceless results that reduce to the known Bogoliubov predictions in the limit of an ordinary 3D condensate. Even though this treatment gives the correct theoretical foundation for the study of quasi-condensates in the weak interaction limit, we found more convenient to rely on the procedure employed in Ref. [28] to obtain simpler expressions for the observables we are interested in, without the need of any discretization of the real space. For this reason in the following we proceed with the usual Bogoliubov approximation, explaining when needed when a formally rigorous construction is necessary.

2.1 Bogoliubov approximation

One of the simplest but still meaningful ways to make calculations for condensate systems is to employ the *Bogoliubov approximation* which is anyway powerful enough to provide at least qualitative understanding of the condensate behaviour. In order to set the notation we describe now how it works on a D-dimensional system of N bosonic particles interacting via a generic

potential V:

$$\hat{H} = -\int \mathrm{d}^D\mathbf{x}\, \frac{1}{2m}\hat{\psi}^\dagger(\mathbf{x})\nabla^2\hat{\psi}(\mathbf{x}) + \frac{1}{2}\int \mathrm{d}^D\mathbf{x}\, d^D\mathbf{x}'\, \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{x}')V(\mathbf{x},\mathbf{x}')\hat{\psi}(\mathbf{x}')\hat{\psi}(\mathbf{x}).$$

If the system is uniform and spans a volume $\mathcal{V} = L^D$, we can move to Fourier space and rewrite

$$\hat{\psi}(\mathbf{x}) = \frac{1}{\sqrt{L^D}}\sum_{\mathbf{k}} \hat{a}_{\mathbf{k}}\ \mathrm{e}^{\imath\mathbf{k}\cdot\mathbf{x}}.$$

The potentials we will consider in the following are of the form $V(\mathbf{x},\mathbf{x}') = V(|\mathbf{x}-\mathbf{x}'|)$, so it is easy to write in momentum space:

$$\hat{H} = \sum_{\mathbf{k}} \frac{k^2}{2m}\ \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{1}{2L^D}\sum_{\mathbf{k}\,\mathbf{p}\,\mathbf{q}} \widetilde{V}(q)\ \hat{a}^\dagger_{\mathbf{k}+\mathbf{q}}\hat{a}^\dagger_{\mathbf{p}\text{-}\mathbf{q}}\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{p}}$$

where

$$\widetilde{V}(q) = \int \mathrm{d}^D\mathbf{r}\ V(r)\ \mathrm{e}^{\imath\mathbf{q}\cdot\mathbf{r}}$$

is the Fourier transform of the potential times the volume.

Up to now the Hamiltonian is still exact, but the quartic term impedes simple analytic calculations; moreover we want to employ our knowledge that a large part of the particles that constitute the system is in the condensate fraction [34], indeed the full system can be thought to behave as the sum of two interacting sub-systems: a large condensed fraction and a small amount of particles outside it, populating higher energy levels. We can start our investigation by disregarding the interaction between these more energetic particles; formally this amounts to a truncation of the Hamiltonian. Operatively this means that we must discard, in the quartic part, terms with more than two $\mathbf{k}$ different from zero. Subsequently we focus on $\hat{a}_{\mathbf{0}}$, the operator that destroys condensate particles and notice that this object, as well as its hermitian conjugate, only appears in the form $\hat{\psi}_0 = \hat{a}_{\mathbf{0}}/\sqrt{\mathcal{V}}$ or $\hat{\psi}_0^\dagger = \hat{a}^\dagger_{\mathbf{0}}/\sqrt{\mathcal{V}}$ in the Hamiltonian and these combinations commute in thermodynamic limit

$$\left[\hat{\psi}_0, \hat{\psi}_0^\dagger\right] = \frac{1}{\mathcal{V}} \xrightarrow[\mathcal{V}\to\infty]{} 0$$

without being trivial when acting on the ground state

$$\langle\psi_0|\, \frac{\hat{a}^\dagger_{\mathbf{0}}}{\sqrt{\mathcal{V}}}\ \frac{\hat{a}_{\mathbf{0}}}{\sqrt{\mathcal{V}}}\, |\psi_0\rangle = \frac{N_0}{\mathcal{V}} = \rho,$$

so they can be replaced by a c-number, in particular Bogoliubov approximation dictates $\hat{a}_{\mathbf{0}} = \sqrt{N_0} \approx \sqrt{N}$ where N_0 counts only the particles in the condensate fraction. In the last relation we relied on the fact that the large majority of the bosons is actually in the condensate. As we will see later on, this choice limits the validity of the Bogoliubov approximation.

The combination of this two approximations (the truncation of the Hamiltonian and the replacement of some operators with c-numbers) gives a quadratic Hamiltonian as a result. Since however

$$\hat{a}^\dagger_{\mathbf{0}}\hat{a}_{\mathbf{0}} + \sum_{\mathbf{p}\neq\mathbf{0}} \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} = N,$$

we need a final manipulation in order to work consistently at second order in the field operators. In particular we need to fix

$$\hat{a}^\dagger_{\mathbf{0}}\hat{a}^\dagger_{\mathbf{0}}\hat{a}_{\mathbf{0}}\hat{a}_{\mathbf{0}} = N^2 - 2N\sum_{\mathbf{p}\neq\mathbf{0}} \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}.$$

The Hamiltonian we finally get is

$$\hat{H} = \frac{N\rho}{2}\widetilde{V}(0) + \sum_{\mathbf{k}} \frac{k^2}{2m}\ \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{0}} + \frac{\rho}{2}\sum_{\mathbf{k}\neq\mathbf{0}} \widetilde{V}(k)\left(2\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\text{-}\mathbf{k}} + \hat{a}_{\mathbf{k}}\hat{a}_{\text{-}\mathbf{k}}\right) \quad (2.1)$$

where $\rho = N/L^D$ is the particle density. This Hamiltonian is quadratic and diagonalizable though quite different from the original one due to the approximations we performed. For example the establishment of an order parameter with $U(1)$ symmetry makes the effective quasi-particle Hamiltonian no more number conserving. However, a number-conserving version of the Bogoliubov approximation can be defined and requires some subtleties in the derivation [34], though the details are not important for the computations we are going to perform.

The Hamiltonian (2.1) can be diagonalized via the Bogoliubov rotation. This amounts to the introduction of a new set of bosonic operators $\hat{b}_{\mathbf{k}}$ which are linearly related to the original ones:

$$\hat{a}_{\mathbf{k}} = u_k\hat{b}_{\mathbf{k}} + v_{-k}\hat{b}^\dagger_{\text{-}\mathbf{k}}$$
$$\hat{a}^\dagger_{\mathbf{k}} = u^*_k\hat{b}^\dagger_{\mathbf{k}} + v^*_{-k}\hat{b}_{\text{-}\mathbf{k}}$$

with $u_k = \cosh\xi_k$ and $v_k = \sinh\xi_k$ in order to preserve canonical commutation relations. After substitution into (2.1), we get a new expression containing both a diagonal and a non-diagonal part. The coefficient of the non-diagonal part becomes zero if we choose

$$\tanh 2\xi_k = \frac{\rho\widetilde{V}(k)}{E(k) + \rho\widetilde{V}(k)}$$

where $E(k) = k^2/2m$. This condition defines the angle of the Bogoliubov rotation. In the end the Hamiltonian is diagonal in the $\hat{b}_{\mathbf{k}}$'s

$$\hat{H} = E_0 + \sum_{\mathbf{k}} \epsilon(k)\ \hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}$$

with a constant shift in the energy

$$E_0 = \frac{\rho N}{2}\widetilde{V}(0) + \frac{1}{2}\sum_{\mathbf{k}\neq\mathbf{0}} \epsilon(k) - \rho\widetilde{V}(k) - E(k),$$

and the energy of the quasiparticles is given by the famous Bogoliubov dispersion relation

$$\epsilon(k) = \sqrt{E(k)^2 + 2\rho\widetilde{V}(k)E(k)}\ ,$$

whereas the rotation coefficients are

$$u_k, v_k = \frac{E(k) \pm \epsilon(k)}{2\sqrt{\epsilon(k)E(k)}}\ .$$

2.2 Equilibrium behaviour

As stated in the Introduction, in the following we will focus our attention on a set of interesting observables, namely the many-body correlations

$$g_p = \frac{1}{\rho^p}\left\langle (\psi^\dagger)^p (\psi)^p \right\rangle$$

and

$$g_2(x) = \frac{1}{\rho^2}\left\langle \psi^\dagger(x)\psi^\dagger(0)\psi(0)\psi(x) \right\rangle .$$

While setting $V_1(r) = 2g\delta(r)$ corresponding to $\widetilde{V}_1(k) \sim g$ independent of k, allows the extraction of all correlation functions in one dimension, in three dimension this potential needs a regularisation in order to avoid divergencies: we will therefore use in three dimensions a Yukawa potential

$$V_3(r) = \frac{g}{r}\ \mathrm{e}^{-\mu r}.$$

With these choices the transformed potential can be written in the generale form

$$\widetilde{V}_D(k) = \frac{g\ \Omega_D}{(k^2+\mu^2)^{\frac{D-1}{2}}}$$

where Ω_D is the unit solid angle in D dimensions($\Omega_1 = 2$ and $\Omega_3 = 4\pi$). We also define the dimensionless coupling constant

$$\gamma = \frac{mg}{\mu^{D-1}}\rho^{1-\frac{2}{D}},$$

which corresponds in one dimension to the usual choice for the Lieb-Liniger parameter [16].Correspondingly we have the degeneracy temperature

$$T_d = \frac{\rho^{\frac{2}{D}}}{2m}\ .$$

For non-local quantities at large separation the physics is dominated by the small-momentum part of the potential which is completely described by the pseudopotential approximation via a single measurable parameter: the scattering length. One important point is that this quantity can be related to γ, indeed in one dimension

$$a_{1D} = -\frac{2}{mg} = -\frac{2}{\rho\gamma},$$

as can be found in Ref. [35]; and for the 3D Yukawa potential

$$a_{3D} = \frac{2mg}{\mu^2} = 2\gamma\rho^{-\frac{1}{3}}$$

as is computed in Appendix A.1. This means we expect that, at least for non-local quantities, our results would be expressed only in terms of γ and ρ.

The validity of the Bogoliubov approximation is established by checking its internal consistency. In other words, since this formalism is used to describe the situation of negligible depletion, i.e. of small occupation of the higher nergy levels, we require this quantity to be truly small. For the 3D Bose gas the depletion is proportional to $(\rho a_{3D}^3)^{1/2}$ [36] therefore we can employ Bogoliubov expression as long as $\gamma \ll 1$. For a 1D quasicondensate the applicability of the construction developed in [33] requires $(\rho/mg)^{1/2} \gg 1$ which means again $\gamma \ll 1$. This condition is consistent with the calculation of the "depletion":

$$\begin{aligned}\mathcal{F} = \frac{n_{ex}}{\rho} &= \frac{1}{\rho L}\sum_{\mathbf{k}>0}|v_k|^2 \\ &= \frac{1}{2\rho}\int_\lambda^\infty \mathrm{d}k\left[\frac{E(k)+\rho g}{\epsilon(k)} - 1\right] \\ &\approx \sqrt{\gamma}\left[\ln\left(\sqrt{\gamma}\frac{\rho}{\lambda}\right) - 1\right]\end{aligned}$$

and again the occupation of the excited levels is kept low by the condition $\gamma \ll 1$. The infrared cutoff λ is required for the convergence of the integral.

The equilibrium value of g_2 at finite temperature in one dimension has already been computed both at $x = 0$ and for finite separation [24, 28]. The same has been done for local g_3 [24]. In the following we derive again these results and generalize them to higher dimensionality. In the spirit of the Bogoliubov approximation we write $\hat{\psi} = \psi_0 + \delta\hat{\psi}$ where ψ_0 is a macroscopic component containing contributions from modes up to some k_0 whereas $\delta\hat{\psi}$ is a small component including larger k contributions: later we will be able to set $k_0 = 0$, relying on the convergence properties of the integrals. Notice that $\delta\hat{\psi}$ is still an operator while ψ_0 became a c-number. Neglecting fluctuations beyond quadratic order in $\delta\hat{\psi}$ we get

$$|\psi_0|^{2p} \simeq \rho^p - p\ \rho^{p-1}\delta\hat{\psi}^\dagger\delta\hat{\psi},$$

which implies

$$g_p = \frac{1}{\rho^p}\left\langle (\psi^\dagger)^p(\psi)^p \right\rangle \simeq 1 + \frac{p(p-1)}{\rho}\left(\left\langle \delta\hat{\psi}^\dagger\delta\hat{\psi}\right\rangle + \Re\left\langle \delta\hat{\psi}\delta\hat{\psi}\right\rangle\right)$$

where we have employed $\langle\delta\hat{\psi}\rangle = \langle\delta\hat{\psi}^\dagger\rangle = 0$. This quantity can be computed by means of a Bogoliubov rotation of the type we have already seen, i.e. we can expand

$$\delta\hat{\psi} = \frac{1}{\sqrt{L^D}}\sum_{\mathbf{k}}\left(u_k\hat{b}_{\mathbf{k}} + v_k\hat{b}^\dagger_{-\mathbf{k}}\right)\mathrm{e}^{\imath\mathbf{k}\cdot\mathbf{x}},$$

and write the expectation values we need in terms of the Bogoliubov coefficients:

$$\left\langle \delta\hat{\psi}^\dagger\delta\hat{\psi}\right\rangle = \frac{1}{L}\sum_{\mathbf{k}} |u_k|^2 n_k + |v_k|^2(1+n_k)$$

$$\left\langle \delta\hat{\psi}\delta\hat{\psi}\right\rangle = \frac{1}{L}\sum_{\mathbf{k}} u_k v_k(2n_k+1)$$

where $n_k = \langle\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}\rangle = (\mathrm{e}^{\beta\epsilon(k)}-1)^{-1}$ is the occupation number of the k level at temperature $T=\beta^{-1}$. In termodynamic limit, after substitution of the known form of the coefficients we get

$$g_p = 1 + \frac{p(p-1)}{2\rho}\frac{\Omega_D}{(2\pi)^D}\int_0^\infty \mathrm{d}k\; k^{D-1}\left[\frac{E(k)}{\epsilon(k)}\left(1+2n_k\right) - 1\right]. \tag{2.2}$$

For a one-dimensional quasi-condensate at zero temperature we get

$$g_p = 1 - \frac{p(p-1)}{\pi}\sqrt{2\gamma}$$

which is a generalization of the result found in Ref. [25], whereas in three dimensions

$$g_p \approx 1 - \frac{p(p-1)\mu^2}{\rho^{\frac{1}{3}}}\frac{\gamma}{\sqrt{\mu^2 - 16\pi\rho^{\frac{2}{3}}\gamma} + 4\rho^{\frac{1}{3}}\sqrt{\pi\gamma}},$$

as can be seen in Fig. 2.1.

Calculations at finite temperature are difficult to perform analitically but we can extract at least some information from the thermal part ($g_p^{(T)}$) of the integrals defined above, i.e. from the component containing n_k. At low temperature we can approximate $\epsilon(k)$ with its low energy expression $\epsilon(k) \simeq (2\rho E(k)\widetilde{V}(0))^{1/2} = c\,k$, so the integral becomes tractable giving

$$\begin{aligned} g_p^{(T)} &= \frac{p(p-1)\Omega_D}{\rho\;(2\pi)^D}\int_0^\infty \mathrm{d}k\; k^{D-1}\frac{E(k)}{\epsilon(k)}\left(\mathrm{e}^{\frac{\epsilon(k)}{T}} - 1\right)^{-1} \\ &\approx \frac{p(p-1)}{4\;(4\pi)^D}\frac{D!\;\zeta(D+1)}{\gamma^{\frac{D}{2}+1}\Omega_D^{\frac{D}{2}}}\left(\frac{T}{T_d}\right)^{D+1}. \end{aligned} \tag{2.3}$$

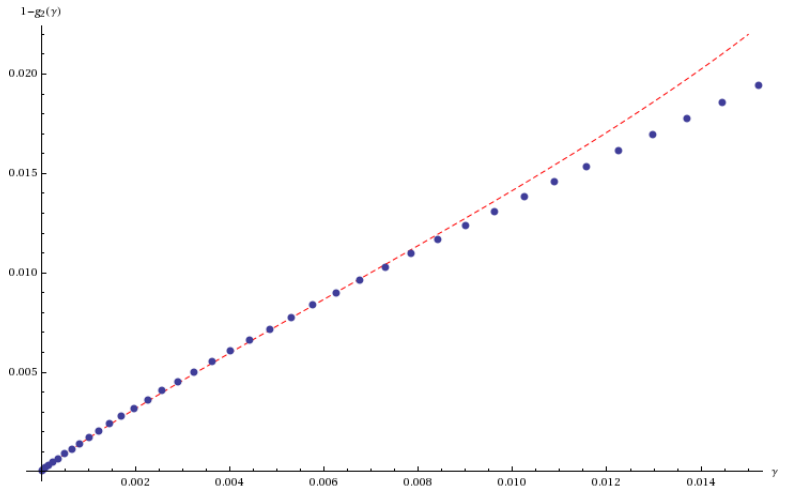


Figure 2.1: Equilibrium value of g_2 as function of γ in 3D. The blue dots corrspond to numerical exact interation and the red line comes from Eq. 2.2 which is a good approximation for small interaction.

A similar computation can be also performed for the non-local correlator $g_2(x)$

$$\begin{aligned} g_2(x) &= \frac{1}{\rho^2}\left\langle \psi^\dagger(x)\psi^\dagger(0)\psi(0)\psi(x)\right\rangle \\ &= 1+\frac{1}{\rho}\left\langle \delta\psi^\dagger(x)\delta\psi(0)+h.c.\right\rangle + \frac{1}{\rho}\left\langle \delta\psi^\dagger(x)\delta\psi^\dagger(0)+h.c.\right\rangle. \end{aligned}$$

Using the usual Bogoliubov rotation, taking the thermodynamic limit and noticing that we can reshuffle all sums in $\mathbf{k}$ since all functions only depend on $|\mathbf{k}|$, the result is that non-local functions can be computed multiplying the integrand of the corresponding local one by $\mathrm{e}^{\imath\mathbf{k}\cdot\mathbf{x}}$.

$$g_2(x) = 1+\frac{1}{(2\pi)^D\rho}\int \mathrm{d}\Omega \mathrm{d}k\; k^{D-1}\left(\frac{E(k)}{\epsilon(k)}\left(1+2n_k\right)-1\right)\mathrm{e}^{\imath\mathbf{k}\cdot\mathbf{x}},$$

where the additional angular integration in three dimensions can be immediately performed and gives

$$\int \mathrm{d}\Omega\; \mathrm{e}^{\imath\mathbf{k}\cdot\mathbf{x}} = \int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\phi\; \mathrm{e}^{\imath kx\cos(\theta)} = 4\pi\frac{\sin(kx)}{kx}.$$

Let us look at the large-x dependence at zero temperature; to do this we can again substitute $\epsilon(k)$ with its linear part and add an ultraviolet regularisation:

$$g_2(x) \simeq 1+\frac{1}{(2\pi)^D\rho}\int \mathrm{d}\Omega \mathrm{d}k\; k^{D-1}\left(\frac{k}{2n^{\frac{1}{D}}\sqrt{\Omega_D\gamma}}-1\right)\mathrm{e}^{\imath\mathbf{k}\cdot\mathbf{x}}\mathrm{e}^{-\alpha k},$$

at the end we can send $\alpha \to 0$ and get

$$g_2(x) = 1 - \frac{1}{2\pi x^{D+1}\rho^{1+\frac{1}{D}}\sqrt{\Omega_D \gamma}} \times \begin{cases} 1 & 1D \\ \frac{1}{\pi} & 3D \end{cases}.$$

Subsequently we may investigate the $T \neq 0$ behaviour for the non-local correlation, to do this we must remember that the regime we are interested in is characterized by both small temperature ($T \ll T_d$) and large x, thus we will fix the separation and look at the leading order in the temperature expansion. The one-dimensional result is already known [25]:

$$g_2^{(T)}(x) \approx \frac{1}{2\pi\rho^2 x^2\sqrt{2\gamma}} - \frac{\pi}{4\sqrt{2}\gamma^{\frac{3}{2}}}\left(\frac{T}{T_d}\right)^2 \exp\left[-\frac{\pi x\rho}{\sqrt{2\gamma}}\frac{T}{T_d}\right],$$

whereas in the the 3D case

$$g_2^{(T)}(x) \approx \frac{1}{4\pi^{\frac{5}{2}}\rho^{\frac{4}{3}}\sqrt{\gamma}x^4} - \frac{1}{2^6\rho^{\frac{1}{3}}\pi x\gamma^2}\left(\frac{T}{T_d}\right)^3 \exp\left[-\frac{\rho^{\frac{1}{3}}\pi x}{\sqrt{4\pi\gamma}}\frac{T}{T_d}\right].$$

In both cases the large-x expansion has a term that cancels the power law behaviour at zero temperature and we are left with an exponential decay. The details of both calculations can be found in Appendix B.1.

Chapter 3

Thermalization and prethermalization

In this chapter we recall some basic facts about the concept of thermalization both in classical and quantum physics, connecting it to system properties like ergodicity and integrability and describing some important experiments that gave a particular insight on the subject. We also introduce the idea of prethermalization presenting some relevant theoretical result, and describe an experiment that was actually able to observe this phenomenon.

3.1 Ergodicity and equilibration in classical statistical mechanics

In the classical realm the central concept to understand in order to study the process of equilibration of a system out of equilibrium is *ergodicity*. Let us consider a classical system characterized by the Hamiltonian $\mathcal{H}$ which is a function of a set of variables collectively called $x \in \Gamma$, where Γ is the phase space of our system; the value of this x uniquely defines the state of the system during its time evolution. A generic experiment should give us acces to some observable $\mathcal{O}$, which is itself a function of the dynamical state of the system: $\mathcal{O}(x(t))$. However any realistic setup will measure the value of the desired observable as an average over a discrete amount of time, which is large as compared to the microscopic timescales of the system,so large that we can pretend to average over an infinitely long interval of time. Formally, the time average of an observable is defined as:

$$\overline{\mathcal{O}} := \lim_{\tau\to\infty} \frac{1}{\tau} \int_{t_0}^{t_0+\tau} \mathrm{d}t \; \mathcal{O}(x(t)).$$

Despite the conceptual simplicity of this definition, its usage is far from trivial. Indeed, it requires the solution of the corresponding equations of motion (which are as many as the degrees of freedom, so of the order of the Avogadro number for a macroscopic system), task that may be hard to achieve. Moreover the determination of the initial conditions for the dynamics with the necessary accuracy can be affected by chaoticity: any small error in the evaluation of the starting point gets exponentially amplified along the evolution so that two

infinitesimally close initial conditions give rise to dynamics that are arbitrarily distant in the phase-space. This implies that the knowledge of the exact dynamics may be extremely difficult to obtain, making it necessary to look for a different approach.

Statistical mechanics gives us this possibility; it suggests to take infinitely many copies of a system (an*ensemble*) somehow distributed throughout the phase space. The ensemble is thus defined by the probability density $\rho(x)$ that the system is in the state x; this probability density will obey the usual positivity and unitarity conditions. The choice of the ensemble and of the corresponding ρ depends on some features of the system and of our knowledge about it. If, for example, we study an isolated system, we know that the total energy must be the same in any of its microscopical realizations so the probability density is nonzero only on the region of the phase space with the correct energy value, moreover if we do not assue any other feature of its behaviour, then ρ must be uniform in that subregion. This particular probability density is widely employed in statistical physics and is known as the microcanonical distribution ρ_{mc}. Once we have introduced the ensemble construction, the outcome of the measure of $\mathcal{O}$ needs to be a weighted averarge over the outcomes in the different realisations of the system: i.e. an ensemble average.

$$\langle \mathcal{O} \rangle_\rho := \int_\Gamma \mathrm{d}x \ \rho(x)\mathcal{O}(x).$$

An important advantage of the statistical approach is that instead of knowing exactly the initial state of the system (which is quite an utopic aim), we need to deal just with a small set of its properties, usually easily controlled by experimental means (e.g. volume, temperature, etc.).

The connection between these complementary approaches is given by the concept of *ergodicity*. The system is said to be ergodic if, for any $\mathcal{O}$ and for almost any initial state x_0, the time average is equal to the microcanonical average, in formulas

$$\overline{\mathcal{O}} \stackrel{?}{=} \langle \mathcal{O} \rangle_{\rho_{mc}} \ .$$

This is equivalent to the statement that the dynamical evolution of the system drives it arbitrarily close to any state compatible with energy conservation. If a system is ergodic we can then compute the long time average of any observable as an ensemble average. Not every system is ergodic: important exceptions are *integrable* system. By definition a classical system with f degrees of freedom is integrable if there are f independent integrals of motion which are Poisson-commuting; this means that the equations of motion can be exactly integrated via action-angle variables and the solutions are forced to move periodically on tori in the phase space. Thus integrability can confine the dynamics of the system in a subset of the phase space and the evolution cannot cover it densely enough; the consequence is that the long-timeaverage is different from the microcanonical one.

A requirement stronger than ergodicity is *mixing*: it requires that the long-time limit of the expectation value for any observable coincides with the corresponding microcanonical average, this means that if we put the system out of equilibrium it will evolve towards a stationary state and the observables computed there will have exactly the value given by the microcanonical average. The difference with respect to ergodicity is that no time average is involved so if this condition is satisfied it implies ergodicity.

3.2 Thermalization in quantum statistical physics

Dealing with quantum mechanical systems, the counterpart of the state x is the wavefunction $|\Psi\rangle \in H$ where H is a vector space called the Hilbert space and, as in the classical case, we hide in this single function all the information about the system. Any observable is a function of the state $|\Psi\rangle$, so the result of a measurement is the expectation value of the operator corresponding to the observable computed over the state of the system,

$$\langle \hat{O} \rangle_{\Psi_0} = \langle \Psi_0 | \hat{O} | \Psi_0 \rangle .$$

The time evolution of this state is governed by the Hamiltonian operator $\hat{H}$ that generates the equation of motion.

$$\imath \frac{\partial}{\partial t} |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle$$

The linearity of the Schrödinger equation implies that if we work in the basis $|n\rangle$ of the Hamiltonian, so that $\hat{H}|n\rangle = E_n |n\rangle$, decomposing the state as

$$|\Psi_0\rangle = \sum_n c_n |n\rangle ,$$

then the time evolution looks like:

$$|\Psi(t)\rangle = \sum_n c_n \mathrm{e}^{-\imath E_n t} |n\rangle .$$

The number of the c_n's grows with the dimension of the Hilbert space, i.e. exponentially with the system size so, even in the quantum case, the exact determination of the initial state of the system and the calculation of its evolution is practically forbidden for any macroscopic size; therefore we look again for an alternative statistical description.

The concept of ensemble can be defined also in quantum mechanics by introducing the density matrix $\hat{\rho}$: for example the quantum equivalent of the microcanonical ensemble is the density matrix $\hat{\rho}_{mc}$

$$\hat{\rho}_{mc}(E) = \frac{1}{\mathcal{N}} \sum_{n \in S_E} |n\rangle \langle n| ,$$

where S_E is the set of eigenstates $|n\rangle$ of $\hat{H}$ such that $E \leq E_n \leq E + \delta E$ with infinitesimal δE and $\mathcal{N}$ is the cardinality of S_E. The expectation value of an observable over this ensemble is

$$\langle \hat{O} \rangle_{mc} = \mathrm{Tr}\left[\hat{\rho}_{mc}\, \hat{O}\right] = \frac{1}{\mathcal{N}} \sum_{n \in S_E} \langle n| \hat{O} |n\rangle .$$

A naïve extension of the ergodicity condition would require to inspect if the microcanonical density matrix were equal to the long-time average of the density matrix, but if we look at this last quantity

$$\overline{|\Psi(t)\rangle \langle \Psi(t)|} = \sum_n |c_n|^2 |n\rangle \langle n|$$

we deduce that the only way to satisfy this tentative ergodicity requirement is to impose the condition $c_n = \mathcal{N}^{-1/2}$ which is rather special and cannot be satisfied for a generic initial state [37]. Another important constrain follows from the unitarity of the quantum evolution: if we take a pure state as a starting point, its density matrix is $\hat{\rho} = |\Psi_0\rangle \langle \Psi_0|$; the trace of its square is identically one and remains constant under the temporal evolution. But a thermal state has a peculiar density matrix

$$\hat{\rho}_{th} = \frac{1}{Z} \mathrm{e}^{-\beta \hat{H}},$$

which does not satisfy that unitarity condition. This means that true thermalization in a closed system is forbidden and we must necessarily understand it in an effective sense [2].

A possible sensible definition of thermalization can be proposed if we restrict our interest to a subset Λ of the original system, so that the remaining part $\overline{\Lambda}$ will act effectively as a bath and can lead the subsystem to thermalization; if we adopt this point of view, the key quantity to look at is the reduced density matrix

$$\hat{\rho}_\Lambda = \mathrm{Tr}_{\overline{\Lambda}}[\hat{\rho}]$$

that is obtained from the original one tracing over the degrees of freedom of the bath. A meaningful statement about thermalization can now be stated: we can ask if in the thermodynamic limit, for some subsystem, the long-time average of the reduced density matrix equals the appropriate density matrix defined for some statistical ensemble, in formulas

$$\overline{\hat{\rho}_\Lambda(t)} \stackrel{?}{=} \hat{\rho}_{\Lambda,mc}.$$

If this is the case, then the outcome of the measurement of some observable can be determined equivalently either as the long-time average of the dynamical expectation value or as the expectation value over the correct ensemble

$$\overline{\langle \Psi(t)| \hat{O} |\Psi(t)\rangle} = \mathrm{Tr}\left[\hat{O}\, \hat{\rho}_{\Lambda,mc}\right].$$

Analogously, we would like to define a quantum counterpart of integrability, but this is not trivial at all [38]: first, it is not clear which should be the number of degrees of freedom (the size of the system or the dimension of the Hilbert space?), moreover the projectors on the eigenstates of the Hamiltonian constitute formally a commuting set of conserved operators thus satisfying a straightforward extension of the classical definition, but this is true for any Hamiltonian and thus gives no way to discriminate between systems with different "integrability" behaviour. A sound definition may be the one proposed by Sutherland [39]: a quantum system is integrable if any scattering is factorizable in a series of binary collisions; this implies that quasi-particles can scatter only elastically and their identity is preserved upon collisions. In addition to the conservation of energy, such systems show other non-trivial conservation laws that allow an exact solution of the model and constrain the dynamics, therefore they are not expected to show thermalization [3, 4], instead in the long-time limit their behaviour is usually well described by the *Generalized Gibbs Ensemble* (GGE) which is defined by the density matrix that has the highest entropy and is still compatible with the aforementioned constrains. Formally, if $\{\hat{I}_n\}_n$ is the set of the independent conserved quantities, then

$$\left[\hat{I}_n\ , \hat{I}_m\right] = 0 \quad \text{and} \quad \left[\hat{I}_n\ , \hat{H}\right] = 0 \quad \forall n, m$$

and the GGE density matrix is defined as

$$\hat{\rho}_{GGE} := \frac{1}{\mathcal{Z}_{GGE}} \mathrm{e}^{-\sum_n \lambda_n \hat{I}_n}$$

where the partition function is

$$\mathcal{Z}_{GGE} = \mathrm{Tr}\left[\mathrm{e}^{-\sum_n \lambda_n \hat{I}_n}\right]$$

and the Lagrange multipliers λ_n are fixed by the consistency of the expectation values

$$\langle \hat{I}_n \rangle_{GGE} := \mathrm{Tr}\left[\hat{I}_n\ \hat{\rho}_{GGE}\right] \stackrel{!}{=} \langle \Psi_0 |\, \hat{I}_n \,| \Psi_0 \rangle \quad \forall n.$$

Even with this tentative definition of integrability, we still lack a groundend connection to thermalization like in the classical realm. A proposed answer to this problem has been the *Eigenstate Thermalization Hypothesis* (ETH) [3, 5]. It states that thermalization happens at the level of individual energy eigenstates so, in order to compute thermal expectation values, it is enough to average over a single many-body energy level in the appropriate energy window. In this perspective the time evolution has just the effect of dephasing.

The idea that some observables might reach a prethermal steady state along the process of thermal equilibration was first mentioned in the context of high-energy physics [7]: the study of experimental data coming from relativistic heavy-ions collisions suggested the existence of a two-stage relaxation

process. Initially (on a time-scale τ_{pt}) a dephasing mechanism leads to the equipartition of energy between kinetic and potential component and the establishment of an approximately time-independent "equation of state" (i. e. the functional relation between pressure and energy density); this process is independent of the details of the interaction and is very rapid. Inelastic collisions are responsible for the existence of a second, larger time-scale τ_{damp} which characterize the relaxation of mode occupation numbers (see e.g. Fig. 3.1); the most part of the dependence on the initial conditions is already lost at this stage, even though true equilibration happens only later (for $t \sim \tau_{eq}$). In particular, for a linear sigma model, they extract the dynamics of integrated and mode dependenst observables to compute the aforementioned time-scales; when these are expressed in terms of a reference mass m they find: $\tau_{pt} \sim m^{-1}$, $\tau_{damp} \sim 25 \div 28 m^{-1}$ and $\tau_{eq} \sim 95 m^{-1}$, so the prethermalization time-scale is two orders of magnitude smaller than the equilibration one.

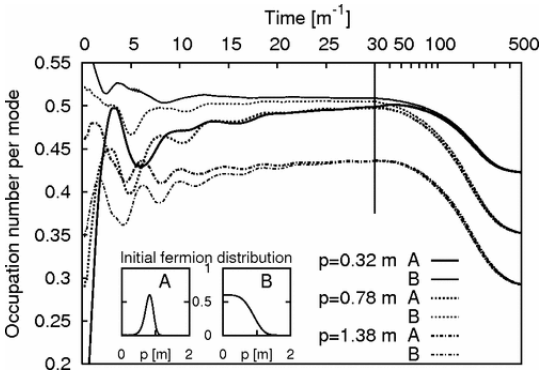


Figure 3.1: Fermion occupation number as a function of time for three different momentum modes. Notice that the second, slower, stage of relaxation is represented in logarithmic scale.

The first important investigation about prethermalization in a condensed matter framework is the study of the non-equilibrium dynamics of a Fermi–Hubbard model at half filling in more than one dimension after a sudden interaction switch-on [8, 40]. The system Hamiltonian contains a uniform hopping term between nearest neighbours and a on-site interaction, in formulas it reads

$$\hat{H} = -h \sum_{\substack{<ij> \\ \sigma=\uparrow,\downarrow}} \hat{c}^{\dagger}_{i\sigma} \hat{c}_{j\sigma} + \theta(t)\, U \sum_{i} \hat{n}_{i\uparrow} \hat{n}_{i\downarrow}$$

where $\hat{n}_{i\sigma} = \hat{c}^{\dagger}_{i\sigma} \hat{c}_{i\sigma}$, and $U \ll h$ so that the system is always in the Fermi liquid regime. The system is initially prepared in the ground state of the $U = 0$ Hamiltonian then, as the Heaviside theta function indicates, at $t = 0$ the interaction is suddenly turned on bringing the system out of equilibrium. The

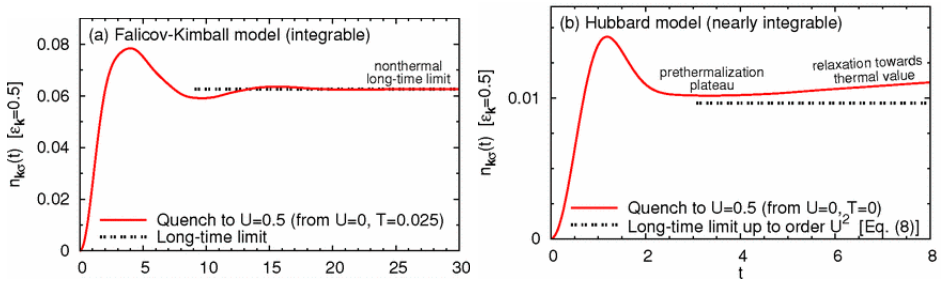


Figure 3.2: Relaxation dynamics of the momentum occupation number after an interaction quench. For an integrable model (right) a non-thermal long-time limit is observed, whose value is predicted by the GGE; for an almost integrable model (left) a non-thermal quasi-stationary state is followed by a second relaxation towards thermal equilibrium, in this case the prethermal plateau is well approximated by a GGE built with the approximate conserved quantities of the model.

time evolution is analytically computed using the flow equation method [41] in the limit of infinite dimensions focusing on three observables: the kinetic energy, the potential energy and the momentum distribution functions. At $t = 0$ both kinetic and potential energy vanish: their sum is kept constant by the quench but is unevenly parted between the two component and this redistribution takes place already at an early stage of the time evolution of the system, when the momentum distribution is still far from its asymptotic value. Indeed the dynamics of this latter mode-dependent observable is far richer and can be divided in three stages: initially, for $0 < t \lesssim \rho_F^{-1} U^{-2}$ where $\rho_F \sim h^{-1}$ is the density of states at the Fermi energy, there is a short phase during which correlations between particles are built with the effect that a quasi-particle description of the system becomes useful; this description remains effective during the second stage, that appears as a quasi-stationary regime for the observable under inspection. Later on, for $t \gtrsim \rho_F^{-3} U^{-4}$, thermalization occurs and can be effectively described by a quantum Boltzmann equation: the final state of the system is a Fermi-Dirac distribution with temperature $T \sim U$. This evolution, therefore, is characterized by two time-scales and the quasi-stationary stage between the two is a prethermalization plateau. Notice that also in this setup the "integrated" observables reach their asymptotic value much earlier than the mode-dependent ones just like in the high-energy framework described above. Using DMFT it was possible to confirm this results [42] and determine the existence of a second prethermalization process in the limit of strong interaction ($U \gg h$) [43].

Another important result for the understanding of the prethermalization phenomenon is the establishment of a direct link between GGE predictions

and prethermal quasi-stationary states: GGE stationary states for integrable system could be seen as prethermalization plateaus that never decay and, conversely, the prethermal intermediate state of nearly integrable system can be computed as if it was a GGE asymptotic state constructed with specific quasi-conserved quantities [10]. Since we will employ this relationship in the following let us understand it in a more formal way; to do this we consider the Hamiltonian of a system close to integrability

$$\hat{H} = \hat{H}_0 + g\hat{H}_1$$

where $\hat{H}_0$ is integrable and has a set $\left\{\hat{I}_\alpha\right\}_\alpha$ of constants of motion so that

$$\hat{H}_0 = \sum_\alpha \epsilon_\alpha \hat{I}_\alpha$$

and $\hat{I}_\alpha |\mathbf{n}\rangle = n_\alpha |\mathbf{n}\rangle$. Application of ordinary perturbation theory to study the evolution after a quanch will lead to the appearance of secular terms that grow polinomially in time, so it is better to rely on unitary perturbation theory: we absorb the perturbation term of the Hamiltonian by means of a canonical transformation generated by an anti-Hermitian operator $\hat{S}$, then we can safely evolve in time and, at the end of the evolution, transform back to get the desired quantities; some details about the implementation of this feature can be found in Appendix A.2. By means of this technique we can construct the expectation value for an observable at leading order in the integrability breaking parameter g. The long-time average gives

$$\langle \hat{A} \rangle_{preth} = \overline{\langle \hat{A} \rangle_t} = 2\langle \hat{A} \rangle_{\widetilde{0}} - \langle \hat{A} \rangle_0 + O(g^3),$$

where the averages $\langle \cdot \rangle_0$ and $\langle \cdot \rangle_{\widetilde{0}}$ are taken respectively on the initial state and on its transformed. The $t \to \infty$ limit in the time average must be understood here in the sense that $g^{-1} \ll t \ll g^{-2}$, i.e. it is large enough to enable us to average, but small enough to prevent the onset of the higher order processes leading the system to thermalization; this is possible as long as g is so small that the scales g^{-1} and g^{-2} are well separated. It should be also pointed out that this asymptotic expectation value is generally different from one corresponding to a thermal ensemble (be it canonical or microcanonical) so if true thermalization occurs it is due to the $O(g^3)$ terms.

Since the Hamiltonian we are dealing with is almost integrable one may expect that the conservation laws possessed by $\hat{H}_0$ are not totally lost; indeed a set of approximate integrals of motion $\widetilde{I}_\alpha$ for the full theory can be constructed and the Hamiltonian can be written as

$$\hat{H} = \sum_\alpha \epsilon_\alpha \widetilde{I}_\alpha + \sum_{\widetilde{\mathbf{n}}} |\widetilde{\mathbf{n}}\rangle \left(g E^{(1)}_{\mathbf{n}} + g^2 E^{(2)}_{\mathbf{n}}\right) \langle \widetilde{\mathbf{n}}| + O(g^3)$$

where the meaning of the terms is explained in the appendix. Once we have these quantities it is straightforward to build the corresponding GGE

$$\hat{\rho}_{\widetilde{G}} = \frac{1}{\mathcal{Z}} \exp\left[- \sum_{\alpha} \lambda_{\alpha} \widetilde{I}_{\alpha} \right]$$

with the constrain

$$\langle \widetilde{I}_{\alpha} \rangle_{\widetilde{G}} := \mathrm{Tr}\left[\hat{\rho}_{\widetilde{G}} \widetilde{I}_{\alpha} \right] \stackrel{!}{=} \langle \widetilde{I}_{\alpha} \rangle_0 .$$

The main claim of Ref. [10] is

$$\langle \hat{A} \rangle_{preth} = \langle \hat{A} \rangle_{\widetilde{G}} + O(g^3)$$

and its proof is outlined in appendix. This result shows that GGE predictions are valid also away form integrability and can be employed to compute the value of observables in the prethermalization quasi-stationary state.

3.3 An experimental perspective on thermalization

Recent experimental progresses in cold atoms physics shifted the point of view on quantum many-body physics from academic to practical. The rising interest in cold atoms is due to some striking properties of this kind of systems: first of all, their diluteness and coldness imply that they can survive and be manipulated for a time which is many orders of magnitude longer than the usual scales of equilibration, thus we are able to inspect dynamical features that would be otherwise inaccessible. Moreover their interaction can be tuned via the Feshbach resonances method [44]: Feshbach resonances are low-energy scattering resonances in the spectrum of cold atomic systems, they can be tuned with the aid of an external magnetic field in order to give a strong mixing between two neighbouring energy levels. The final effect of this procedure is the variation of the effective scattering length and consequently a change in the effective interaction strength among the atoms. Experimentalists also developed a great control over the realisation of optical lattices and magnetic traps in which the cold atomic clouds could be loaded, this means that it is possible to effectively engineer some peculiar condensed matter Hamiltonians by shaping external potentials and optical lattices in the appropriate fashion: in particular this allows to build low dimensional models and subsequently study the interplay between dimensionality and interaction.

As a first example of this kind of experiment we must cite the work of Kinoshita *et al.* [15], who investigated the role of integrability for thermalization in a system of 1D bosons. They put two one-dimensional clouds of interacting atoms out of equilibrium in a cigar-shaped confinement and let them collide, sampling the momentum distribution at given intervals: they found no evidence of thermalization even after many collisions as can be seen in Fig. 3.3.

Indeed the system under study is very close to an integrable model, apart from small deviations due to defects in the confinement, so we may expect its thermalization to be prevented (or at least delayed) by the presence of many conserved quantities.

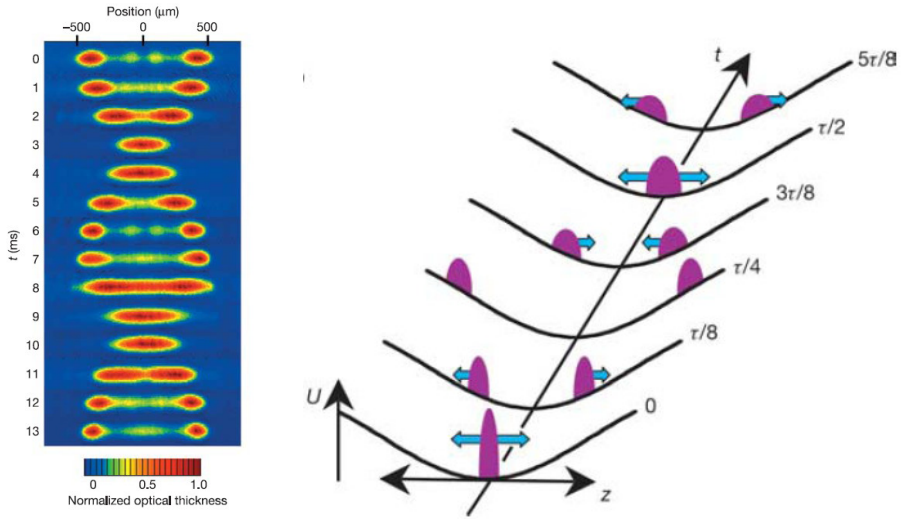


Figure 3.3: Left: a sketch of the experimental setup of Ref. [15]. Right: absorption images for two quasi-one-dimensional clouds of interacting bosons. The particles scatter at any oscillation, still no appartent sign of thermalization is visible on the timescale of the experiment.

In turn an important result in the search for the prethermalization plateau was achieved by the Schmiedmayer group [12] who found strong evidences for the presence of a non-thermal steady state in a quasi-one-dimensional Bose system. that was later suddenly split along the longitudinal axis, producing two uncoupled Bose systems. Although they are indeed uncoupled, they are not independent since they retain the memory of their common origin from a single quasi-condensate: they share an almost identical longitudinal phase profile. The observation of how this memory of the initial state evolves in time can be followed looking at the phase difference $\phi_s(x,t) = \phi_1(x,t) - \phi_2(x,t)$ and gives important informations on the relaxation of the system and its eventual approach towards an equilibrium state.

To investigate the evolution they let the system evolve for some time t_e, then they released the trap and let the two clouds interfere. They did not measure directly the local phase difference between the two Bose gases along the longitudinal direction, instead they looked at the interference pattern along the transverse direction, integrated it over a variable length L and extracted

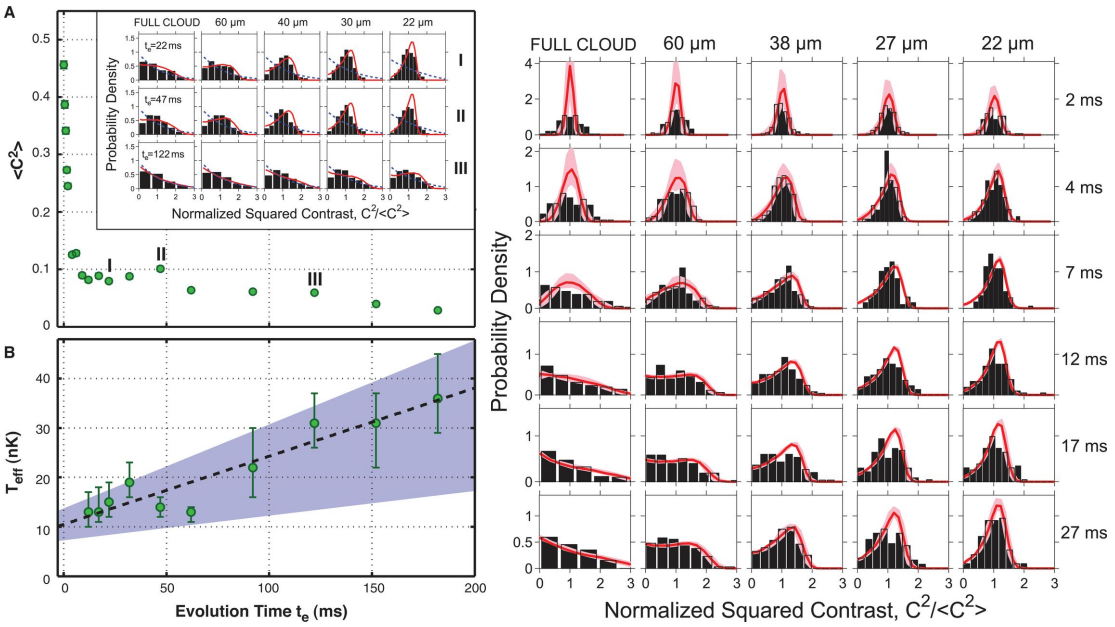


Figure 3.4: Experimental results for the experiment described in Ref. [12]. Left: A) Evolution of $\langle C^2 \rangle$ integrated over the entire length. After a rapid decay, a slower further evolution sets on. Inset: experimental distributions of the squared contrast for three different values of t_e: red lines are the best fit with equilibrium distributions from which T_{eff} is extracted, blue dashed lines are the equilibrium distribution at the actual setup temperature B) Evolution of T_{eff}, the shaded area represents the heating of the trap during the experiment. Right: short time detail of the evolution of $\langle C^2 \rangle$. When integration is performed on the full cloud the distribution rapidly becomes exponential, whereas for the shortest integration length the distribution keeps memory of the initial condition much longer.

the contrast $C(L)$,

$$C(L) = \frac{1}{L}\left|\int_{-L/2}^{L/2} \mathrm{d}x\ \mathrm{e}^{\imath\phi_s(x,t)}\right|^2$$

i.e. the amplitude of the inteference profile wich is a direct measure of the relative phase fluctuations. If this procedure is performed repeatedly for different evolution times t_e it is possible to obtain an image of the evolution of the system during the relaxation process. Initially, since the phase difference is negligible, the contrast is large and independendent of the integration amplitude L; as the evolution proceeds stochastic fluctuations increase the phase difference between the two parts of the system and this imply that the contrast is reduced with respect to the initial state, beginning with large L. As a comparison, two *independent* 1D clouds separately prepared at equilibrium would show a very small constrast as a consequence of their independent phases. When this procedure was applied to the given system they found an initial rapid decay of $\langle C^2\rangle$ on the scale of $\sim$ 10 ms, the emergence of a quasy stationary state followed by a much slower further evolution. It is apparent that it is crucial to characterize the intermediate state in order to understand the features of the relaxation process: to this aim they computed from the measured data the full distribution of the contrast $P(C^2)\mathrm{d}C^2$ in the steady state and compared it to the distribution predicted at thermal equilibrium and found that the experimental observation was remarkably well descripted by an equilibrium distribution at some effective temperature T_{eff} which was a factor of five smaller than the actual temperature: they therefore deduced that this quasi-stationary state could not be the true thermal equilibrium of the system. To state more clearly the result it must be observed that for times smaller than those corresponding to the quasy-stationary state ($\sim$ 10 ms) the distribution of the contrast could not be described by a thermal equilibrium state at any effective temperature. Moreover they found that as the system further relaxes, T_{eff} increases in a way that is consistent with the heating of the trap: this would suggest that if true thermalization is ever reached by the system, it must be a very slow process. These features of the examined experimental system are well described by an integrable model based on the Tomonaga–Luttinger liquid formalism which also correctly predicts the dependence of the computed effective temperature on the particle density and on the actual temperature of the clouds.

The authors claim that the observed quasi-stationary state is actually a prethermal state as predicted for almost-integrable system [10]. Indeed the procedures inherent to the realization of a one-dimensional system starting from a threedimensional cloud necessarily spoil integrability [14]. We therefore expect that the system under study should be modeled by an Hamiltonian which is only close to an integrable point and will eventually thermalize on some very long time scale. A sensible question in this context regards the

persistence of the prethermalization plateau: how long does it last? On which time-scale does subsequent true thermalization begin? The importance of this concept can be also inferred by its paradigmatic character. Indeed after the idea of prethermalization was introduced, another experiment by the Schmiedmayer group [45] performed five years before had to be reinterpreted accordingly. The setup of the two experiments is identical, but in the older one the measurement was performed for a shorter time and only the coherence factor was investigated. This quantity is related to the mean interference contrast but could not offer the same knowledge as the measurement of both the full probability distribution of phase and contrast of the matter-wave interference pattern. Moreover the short experimental time prevented them from realizing the true quasi-stationary character of the prethermal plaeau. The conclusion of the former paper was that full decoherence and thermalization were found, but actually in hindsight the authors recognise the outcome of the experiment must be reinterpreted as a clue of the same fast "integrable" dephasing of the relative modes in the split of the 1D Bose system. In retrospect we know that this fast dephasing is followed by a slower true equilibration whose features were unfortunately invisible so far.

Chapter 4

Prethermalization in a weakly interacting Bose system

The main subject of the thesis is the effect of a small interaction quench on a system of degenerate bosons in one and three dimensions. We will begin by performing calculations in the Bogoliubov approximation at zero temperature using techniques closely related to the one used in equilibrium and we will look at the long time average of correlation functions. For the aforementioned desire to perturb only slightly the system, we are interested in small quenches $\gamma_0 \to \gamma_1$, therefore we will usually fix γ_1 and expand the results in $\delta\gamma = \gamma_0 - \gamma_1$. In a subsequent section we will define of a set of Green functions suitable to describe the evolution of the condensate and its thermal cloud, we will introduce the Keldysh contour integration in order to describe the evolution of these functions out of equilibrium and show that this procedure gives the same result of the previous one though it is in principle the ideal framework to move beyond the Bogoliubov approximation. As an example, we will look at the effect of the inclusion of a Hartree–Fock term in the interaction Hamiltonian.

The stationary state found with the previous techniques is a prethermalized one, as we will show that it is not possible to describe it in terms of an effective temperature, but rather in terms of a GGE. Investigation of the dynamics towards pretermalisation and thermalisation is also performed in order to compare the two processes in the perspective of being able to distinguish them experimentally.

4.1 Interaction quench

In Section 2.2 we computed the equilibrium expectation value of the correlation functions in the Bogoliubov approximation. In this section we will be interested in the evolution of the same quantities after a sudden change in the interaction parameter. In the following the density of the condensate fraction will be considered as a constant parameter of system; this assumption can be relaxed, as shown in Appendix A.4, even though for realistic values of the system parameters the effect of the variation of the density is negligible. The technique used to compute these time-dependent expectation values is analogous to the one used in equilibrium provided we use consistently the fact that

both the initial and the final Hamiltonian are diagonalizable by means of a Bogoliubov rotation, though with different parameters. If we still call $\hat{b}_{\mathbf{k}}$ and $\hat{b}^\dagger_{\mathbf{k}}$ the operators that diagonalize the *final* Hamiltonian and u_k and v_k the parameters of the corresponding rotation, for the initial Hamiltonian we write

$$\hat{H}(\gamma_0) = E_0(\gamma_0) + \sum_{\mathbf{k}} \epsilon(k,\gamma_0)\ \hat{c}^\dagger_{\mathbf{k}}\hat{c}_{\mathbf{k}} \tag{4.1}$$

and

$$\hat{a}_{\mathbf{k}} = r_k \hat{c}_{\mathbf{k}} + s_k \hat{c}^\dagger_{\text{-}\mathbf{k}}. \tag{4.2}$$

This last relation implies that it is also possible to write the final $\hat{b}_{\mathbf{k}}$'s in terms of the initial $\hat{c}_{\mathbf{k}}$'s; formally

$$\hat{b}_{\mathbf{k}} = \chi_k \hat{c}_{\mathbf{k}} + \omega_k \hat{c}^\dagger_{\text{-}\mathbf{k}} \tag{4.3}$$

where

$$\begin{aligned} \chi_k &= u_k r_k - v_k s_k = \frac{\epsilon(k,g_1)+\epsilon(k,g_0)}{2\sqrt{\epsilon(k,g_1)\epsilon(k,g_0)}} \\ \omega_k &= u_k s_k - v_k r_k = \frac{\epsilon(k,g_1)-\epsilon(k,g_0)}{2\sqrt{\epsilon(k,g_1)\epsilon(k,g_0)}}. \end{aligned} \tag{4.4}$$

Once these new quantities are defined we can proceed with the computation in a way closely related to the equilibrium one

$$g_p(t) = \frac{1}{\rho^2}\langle\phi_0|\left(\hat{\psi}^\dagger(r,t)\right)^p\left(\hat{\psi}(r,t)\right)^p|\phi_0\rangle \tag{4.5}$$

$$= 1 + \frac{p(p-1)}{\rho L^D}\sum_{\mathbf{k}}\left[u_k v_k(\chi_k^2+\omega_k^2) + u_k^2\omega_k^2 + v_k^2\chi_k^2 + \right. \tag{4.6}$$

$$\left. + \chi_k\omega_k(u_k+v_k)^2\cos(2\epsilon(k,\gamma_1)t)\right], \tag{4.7}$$

where $|\phi_0\rangle$ is the state of the system before the quench, i.e. the ground state of $\hat{H}(\gamma_0)$. The calculation above is shown in greater detail in Appendix C.1. The asymptotic long-time limit of the latter expression, denoted in the following with an overline can be obtained computing the time average. After substitution of the coefficients and some algebra, in thermodinamic limit we get

$$\overline{g}_p = 1 + \frac{p(p-1)}{2\rho}\frac{\Omega_D}{(2\pi)^D}\int_0^\infty \mathrm{d}k\ k^{D-1}\left[E(k)\frac{\epsilon(k,\gamma_0)^2+\epsilon(k,\gamma_1)^2}{2\epsilon(k,\gamma_0)\epsilon(k,\gamma_1)^2} - 1\right]. \tag{4.8}$$

For the one-dimensional case this can be computed analytically

$$\overline{g}_p = 1 - \frac{p(p-1)\sqrt{2}}{\pi}\sqrt{\gamma_0} + \begin{cases} \frac{p(p-1)}{\sqrt{2}\pi}\sqrt{\gamma_0-\gamma_1}\ \operatorname{arccosh}\left[\sqrt{\frac{\gamma_0}{\gamma_1}}\right] & \text{if } \gamma_0 > \gamma_1 \\ -\frac{p(p-1)}{\sqrt{2}\pi}\sqrt{\gamma_1-\gamma_0}\arccos\left[\sqrt{\frac{\gamma_0}{\gamma_1}}\right] & \text{if } \gamma_0 < \gamma_1 \end{cases}, \tag{4.9}$$

but in three dimensions calculations are not that straightforward using however the smallness of the quench we can expand the integrand with the substitution $\gamma_0 = \gamma_1 + \delta\gamma$:

$$\begin{aligned}\epsilon(k,\gamma_0) &= \sqrt{E(k)^2 + 2\rho E(k)\widetilde{V}(k,\gamma_1+\delta\gamma)} \\ &= \sqrt{E(k)^2 + 2\rho E(k)\widetilde{V}(k,\gamma_1) + 2\rho E(k)\widetilde{V}(k,\delta\gamma)} \\ &= \epsilon(k,\gamma_1)\left[1 + \frac{\rho E(k)\partial_\gamma \widetilde{V}(k,\gamma)}{\epsilon(k,\gamma_1)^2}\delta\gamma - \frac{1}{2}\left(\frac{\rho E(k)\partial_\gamma \widetilde{V}(k,\gamma)}{\epsilon(k,\gamma_1)^2}\right)^2 (\delta\gamma)^2\right],\end{aligned} \tag{4.10}$$

$$\frac{1}{\epsilon(k,\gamma_0)} = \frac{1}{\epsilon(k,\gamma_1)}\left[1 - \frac{\rho E(k)\partial_\gamma \widetilde{V}(k,\gamma)}{\epsilon(k,\gamma_1)^2}\delta\gamma + \frac{3}{2}\left(\frac{\rho E(k)\partial_\gamma \widetilde{V}(k,\gamma)}{\epsilon(k,\gamma_1)^2}\right)^2 (\delta\gamma)^2\right].$$

We may insert this expansion in the expression for g_p and it is just a matter of algebraic manipulations to see that terms linear in $\delta\gamma$ cancel out and we are left with

$$\begin{aligned}\overline{g}_p &= 1 + \frac{p(p-1)}{2\rho}\frac{\Omega_D}{(2\pi)^D}\int_0^\infty \mathrm{d}k\ k^{D-1}\left[E(k)\frac{\epsilon(k,\gamma_0)^2 + \epsilon(k,\gamma_1)^2}{2\epsilon(k,\gamma_0)\epsilon(k,\gamma_1)^2} - 1\right] \\ &\approx 1 + \frac{p(p-1)}{2\rho}\frac{\Omega_D}{(2\pi)^D}\left[\int_0^\infty \mathrm{d}k\ k^{D-1}\left(\frac{E(k)}{\epsilon(k,\gamma_1)} - 1\right) + \right. \\ &\qquad \left. + \frac{\rho^{\frac{4}{D}}\Omega_D^2}{m^2\mu^{2(1-D)}}(\delta\gamma)^2\int_0^\infty \mathrm{d}k\ \frac{k^{D-1}}{(k^2+\mu^2)^{D-1}}\frac{E(k)^3}{\epsilon(k,\gamma_1)^5}\right].\end{aligned} \tag{4.11}$$

One may notice that the first two terms give just the equilibrium value of g_p corresponding to the final value of the interaction whereas the dependence on the quench amplitude is contained in the last term.

To understand how this long-time limit is reached we should look at the dynamics of the system by computing $g_p^{osc}(t)$, the time dependent term of Eq. 4.1:

$$g_p^{osc}(t) = \frac{p(p-1)\Omega_D}{4\rho(2\pi)^D}\int_0^\infty \mathrm{d}k\ k^{D-1}E(k)\ \frac{\epsilon(k,\gamma_1)^2 - \epsilon(k,\gamma_0)^2}{\epsilon(k,\gamma_0)\,\epsilon(k,\gamma_1)^2}\cos(2\epsilon(k,\gamma_1)t). \tag{4.12}$$

Despite the fact that hese integrals cannot be exactly computed, we can evaluate the long-time behaviour for a small quench. Since the numerator in the integrand is linear in $\delta\gamma$, to keep the calculation at first order in this parameter we retain just the zeroth order expansion in the denominator; moreover we use

for $\epsilon(k)$ its low energy approximation:

$$\begin{aligned} g_p^{osc}(t) &\approx -\frac{p(p-1)\Omega_D^2}{(2\pi)^D}\frac{\mu^{D-1}\rho^{\frac{2}{D}}}{2m\rho}\ \delta\gamma\ \int_0^\infty \mathrm{d}k\ k^{D-1}\frac{E(k)^2}{\epsilon(k,\gamma_1)^3}\frac{\cos(2\epsilon(k,\gamma_1)t)}{(k^2+\mu^2)^{\frac{D-1}{2}}} \\ &\approx -\frac{p(p-1)\Omega_D^2}{(2\pi)^D}\frac{\rho^{\frac{2}{D}-1}}{2m}\ \delta\gamma\ \int_0^\infty \mathrm{d}k\ k^{D-1}\frac{E(k)^2}{\left(\frac{k\rho^{\frac{1}{D}}}{m}\sqrt{\Omega_D\gamma_1}\right)^3}\cos\left(2\frac{k\rho^{\frac{1}{D}}}{m}\sqrt{\Omega_D\gamma_1}\ t\right) \\ &= -\frac{p(p-1)}{(2\pi)^D\rho^{2+\frac{2}{D}}}\frac{m^{D+1}}{2^{D+4}}\frac{1}{\sqrt{\Omega_{\mathrm{d}}^D\gamma_1^{D+4}}}\ \frac{\delta\gamma}{t^{D+1}}\ \int_0^\infty \mathrm{d}\lambda\ \lambda^D\cos(\lambda) \\ &= -\frac{p(p-1)}{(2\pi)^D\rho^{2+\frac{2}{D}}}\frac{m^{D+1}}{2^{D+4}}\frac{1}{\sqrt{\Omega_{\mathrm{d}}^D\gamma_1^{D+4}}}\ \frac{\delta\gamma}{t^{D+1}}\sin\left(\frac{D\pi}{2}\right)\frac{\Gamma(D+1)}{t^{D+1}}. \end{aligned} \tag{4.13}$$

At large times g_p should decay as t^{-2} in one dimension and as t^{-4} in three dimensions. These behaviors are compatible with the numerics as can be seen in Fig. 4.1.

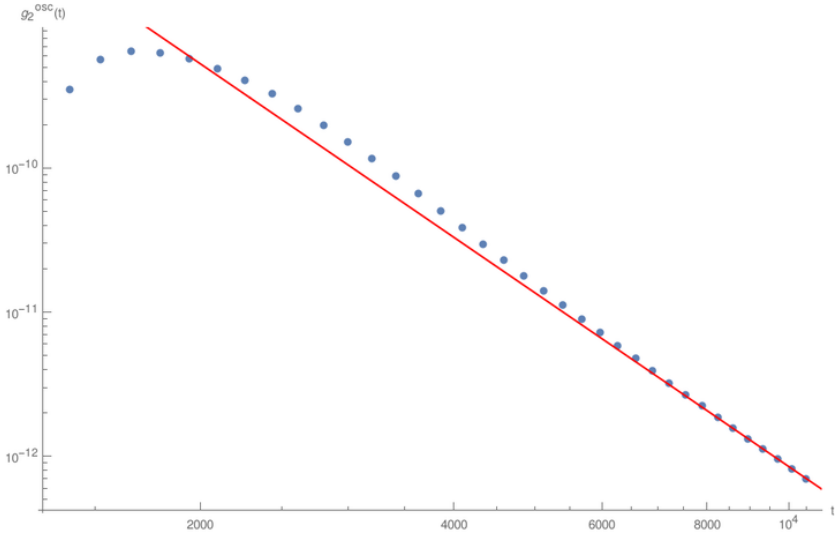


Figure 4.1: Logarithmic plot of $g_2^{osc}(t)$. The red line corresponds to 4.13.

At fixed finite separation x and for late times in 1D we obtain:

$$\begin{aligned} g_2^{osc}(x,t) &\approx -\frac{\rho}{\pi m}\ \delta\gamma\ \int_0^\infty \mathrm{d}k\ \frac{E(k)^2}{\left(\frac{k\rho}{m}\sqrt{2\gamma_1}\right)^3}\cos\left(2\frac{k\rho}{m}\sqrt{2\gamma_1}\ t\right)\cos(kx) \\ &= -\frac{1}{8\pi}\ \frac{\delta\gamma}{\sqrt{2\gamma_1}}\ \frac{8\widetilde{t}^2+\widetilde{x}^2}{(8\widetilde{t}^2-\widetilde{x}^2)^2}, \end{aligned} \tag{4.14}$$

where $\widetilde{t} = t\ \frac{\rho^2\gamma_1}{m}$ and $\widetilde{x} = x\rho\sqrt{\gamma_1}$. In 3D we get

$$g_2^{osc}(x,t) \approx -\frac{1}{\pi^2}\frac{\rho^{-\frac{1}{3}}}{2m}\ \delta\gamma\ \int_0^\infty \mathrm{d}k\ k^2 \frac{E(k)^2}{\left(\frac{k\rho^{\frac{1}{3}}}{m}\sqrt{4\pi\gamma_1}\right)^3}\cos\left(2\frac{k\rho^{\frac{1}{3}}}{m}\sqrt{4\pi\gamma_1}\ t\right)\frac{\sin(kx)}{kx} \tag{4.15}$$

$$= \frac{1}{32\pi^{\frac{7}{2}}}\ \sqrt{\gamma}\ \delta\gamma\ \frac{3\widetilde{t}^2+\widetilde{x}^2}{(\widetilde{t}^2-\widetilde{x}^2)^3}. \tag{4.16}$$

In the long time limit for negligible separation we recover the $t^{-(D+1)}$ asymptotic behaviour.

A study of $g_2(x,t)$ within the time-dependent Bogoliubov approximation was previously performed in Ref. [46], where, however, only two particular quenches were considered: from a finite value of interaction to the non-interacting gas and viceversa (notice that these particular quenches lay outside the domain of validity of our approach since we need $\delta\gamma \ll \gamma$ to hold both for the initial and final interaction strength γ). It was found that for the quench from finite to zero interaction one obtains a power law decay $g_2^{osc}(x,t) \sim t^{-1}$, whereas for the reverse quench the approach to the stationary state is asymptotically fast

$$g_2^{osc}(t) \sim \mathrm{e}^{-g\rho t}. \tag{4.17}$$

This was attributed to the different low-energy behaviour of the final Hamiltonian. Indeed when the final Hamiltonian is non-interacting the group velocity of the quasi-particles is proportional to the momentum and the wave-packet spreads as it moves. On the contrary when the final Hamiltonian is interacting we have a linear dispersion relation at low energy so the group velocity is constant meaning that a wave-packet would not spread. This latter behaviour would give a faster decay of correlation. Notice however that their argument is not consistent with the result we obtained since according to it we would expect the same exponential decay also for our quench between two finite values of the interaction, while we obtained a mundane power law decay.

4.2 Field theoretic approach

The time evolution of correlation functions can be obtained also more directly in a field theoretical context and in particular within the Keldysh approach, whose fundamentals are explained in Appendix A.3. This method has also the great advantage of being a good starting point to relax some assumptions of the Bogoliubov approximation and construct a more detailed picture of the system, let us define the reducible Green function as

$$G_{red}(x,y) = \langle T[\psi(x)\psi^\dagger(y)]\rangle \tag{4.18}$$

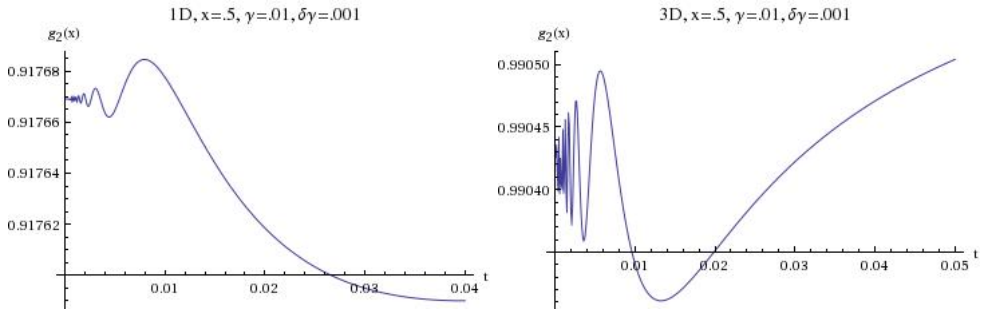


Figure 4.2: Time evolution of $g_2(x)$ for the two dimensionalities for typical values of the parameters

and separate the contribution of the condensate from that of the thermal cloud so as to obtain

$$G_{red}(x,y) = |\psi_0|^2 + \langle T[\delta\psi(x)\delta\psi^\dagger(y)]\rangle. \tag{4.19}$$

In the following we will always discuss the irreducible functions, obtained once the expectation value of the condensate field is subtracted.

We can describe the bosonic propagator, the presence of a condensate, its thermal cloud, and the interaction between them in a diagrammatic way, as done in Fig. 4.3, where solid lines represent quasi-particles, wavy lines stand for the interaction and dots are condensed particles.

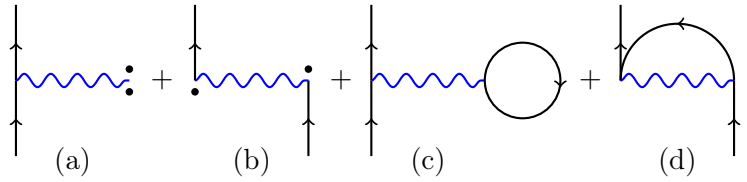


Figure 4.3: Diagrammatic stucture of interaction between condensate and thermal cloud at first order in the interaction strength. ∿ is the interaction, ↠ is a quasi-particle, and • is a condensate particle.

The Bogoliubov approximation amounts to considering only the first two diagrams (a) and (b), while diagrams (c) and (d)are needed in the Bogoliubov-Hartree-Fock approximation. Let us start with the first two terms. The Hamiltonian can be written as

$$\hat{H} = \sum_{\mathbf{k}\neq 0}\left[E(k) + \rho\widetilde{V}(k)\right]\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{\rho}{2}\sum_{\mathbf{k}\neq 0}\widetilde{V}(k)\left[\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\text{-}\mathbf{k}} + \hat{a}_{\mathbf{k}}\hat{a}_{\text{-}\mathbf{k}}\right]. \tag{4.20}$$

Let us now define the "lesser" and "greater" Green functions:

$$G^<(\mathbf{p},t;\mathbf{p}',t') = -\imath\langle a^\dagger_{\mathbf{p}'}(t')a_{\mathbf{p}}(t)\rangle \quad G^>(\mathbf{p},t;\mathbf{p}',t') = -\imath\langle a_{\mathbf{p}}(t)a^\dagger_{\mathbf{p}'}(t')\rangle. \tag{4.21}$$

The same can be done for the anomalous functions

$$F(\mathbf{p},t;\mathbf{p}',t') = -\imath\langle a_{-\mathbf{p}'}(t')a_{\mathbf{p}}(t)\rangle \quad F^\dagger(\mathbf{p},t;\mathbf{p}',t') = \imath\langle a^\dagger_{\mathbf{p}}(t)a^\dagger_{-\mathbf{p}'}(t')\rangle. \quad (4.22)$$

Equations of motion for this functions are computed by commuting them with the Hamiltonian:

$$\begin{aligned}\imath\frac{\partial}{\partial t}G^<(\mathbf{p},t;\mathbf{p}',t') &= -\imath\langle a^\dagger_{\mathbf{p}'}(t')\left[a_{\mathbf{p}}(t),\hat{H}\right]\rangle \\ &= \left[E(p)+\rho\widetilde{V}(p)\right]G^<(\mathbf{p},t;\mathbf{p}',t') - \rho\widetilde{V}(p)F^\dagger(\mathbf{p}',t';\mathbf{p},t)\end{aligned} \quad (4.23)$$

and similarly for the derivative with respect to t'

$$\begin{aligned}\imath\frac{\partial}{\partial t'}G^<(\mathbf{p},t;\mathbf{p}',t') &= -\imath\langle\left[a^\dagger_{\mathbf{p}'}(t'),\hat{H}\right]a_{\mathbf{p}}(t)\rangle \\ &= -\left[E(p')+\rho\widetilde{V}(p')\right]G^<(\mathbf{p},t;\mathbf{p}',t') - \rho\widetilde{V}(p')F(\mathbf{p},t;\mathbf{p}',t').\end{aligned} \quad (4.24)$$

Summing on both sides and restricting to equal time and momentum we get

$$\imath\frac{\partial}{\partial t}G^<_{\mathbf{p},t} = -\rho\widetilde{V}(p)\left[F_{\mathbf{p},t}+F^\dagger_{\mathbf{p},t}\right]. \quad (4.25)$$

With a similar procedure one can find the equations of motion for all the Green functions and put them together in a compact form

$$\imath\partial_t\begin{pmatrix}F+F^\dagger\\ F-F^\dagger\\ G^>+G^<\end{pmatrix}_{\mathbf{p},t} = \Lambda(p)\begin{pmatrix}F+F^\dagger\\ F-F^\dagger\\ G^>+G^<\end{pmatrix}_{\mathbf{p},t} \quad (4.26)$$

where the evolution matrix Λ is

$$\Lambda(p) = 2\begin{pmatrix}0 & E(p)+\rho\widetilde{V}(p) & \rho\widetilde{V}(p)\\ E(p)+\rho\widetilde{V}(p) & 0 & 0\\ -\rho\widetilde{V}(p) & 0 & 0\end{pmatrix}. \quad (4.27)$$

Notice that the eigenvalues of the evolution matrix are $\{0,\pm 2\epsilon_1(p)\}$.

This system of equations can be solved exactly. The initial condition can be fixed by writing the Green functions in terms of the operators that diagonalize the initial Hamiltonian

$$\begin{pmatrix}F+F^\dagger\\ F-F^\dagger\\ G^>+G^<\end{pmatrix}_{\mathbf{p},0} = \frac{\imath}{\epsilon_0(p)}\begin{pmatrix}0\\ \rho\widetilde{V}_0(p)\\ -E(p)+\rho\widetilde{V}_0(p)\end{pmatrix}. \quad (4.28)$$

The solution of the equations of motion is

$$\begin{pmatrix} F+F^\dagger \\ F-F^\dagger \\ G^>+G^< \end{pmatrix}_{\mathbf{p},t} = \imath E(p)\frac{E(p)+\rho(\widetilde{V}_1+\widetilde{V}_0)(p)}{\epsilon_0\epsilon_1^2}\begin{pmatrix} 0 \\ \rho\widetilde{V}_1(p) \\ -(E(p)+\rho\widetilde{V}_1(p)) \end{pmatrix} + \\ + \imath\rho E(p)\frac{(\widetilde{V}_0-\widetilde{V}_1)(p)}{\epsilon_0\epsilon_1^2}\begin{pmatrix} \epsilon_1\sin(2\epsilon_1 t) \\ (E(p)+\rho\widetilde{V}_1(p))\cos(2\epsilon_1 t) \\ -\rho\widetilde{V}_1(p)\cos(2\epsilon_1 t) \end{pmatrix}. \tag{4.29}$$

The correlation functions can be written in terms of these Green functions

$$g_2(x) = 1 + \imath\frac{p(p-1)}{\rho}\int\frac{\mathrm{d}^D\mathbf{k}}{(2\pi)^D}\left[\imath + \left(G^>+G^<\right)_{\mathbf{k},t} + (F-F^\dagger)_{\mathbf{k},t}\right]\mathrm{e}^{\imath\mathbf{k}\cdot\mathbf{x}}. \tag{4.30}$$

Substituting Eq. (4.29) in Eq. (4.30) one gets the results obtained previously, Eq. (4.1).

For the sake of completeness let us discuss the insertion of the remaining two first-order diagrams containing the Hartree–Fock part of the interaction, even though their use for the characterization of the dynamics is postponed to future studies. At the operatorial level they correspond to the terms obtained from the original quartic part of the Hamiltonian once we have contracted all possible couple of operators.

$$\hat{H}_{HF} = \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{y}\; V(r)\Big[2\langle\delta\psi_x^\dagger\delta\psi_x\rangle\delta\psi_y^\dagger\delta\psi_y + 2\langle\delta\psi_x^\dagger\delta\psi_y\rangle\delta\psi_y^\dagger\delta\psi_x + \\ + \langle\delta\psi_x^\dagger\delta\psi_y^\dagger\rangle\delta\psi_y\delta\psi_x + \langle\delta\psi_x\delta\psi_y\rangle\delta\psi_y^\dagger\delta\psi_x^\dagger\Big] \\ = \frac{1}{2}\int \mathrm{d}^3\mathbf{r}\; V(r)\sum_{\mathbf{k}}[2(n_0+n_r)a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + (\rho+\alpha_r)a_{\mathbf{k}}^\dagger a_{\text{-}\mathbf{k}}^\dagger + (\rho+\alpha_r^\dagger)a_{\mathbf{k}}a_{\text{-}\mathbf{k}}] \tag{4.31}$$

where

$$n_r = n(r) = \langle\delta\psi_x^\dagger\delta\psi_{x\pm r}\rangle = \imath\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}G_{kt}^<\;\mathrm{e}^{-\imath\mathbf{k}\cdot\mathbf{r}}, \tag{4.32}$$

and

$$\alpha_r = \alpha(r) = \langle\delta\psi_x\delta\psi_{x\pm r}\rangle = \imath\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}F_{kt}\;\mathrm{e}^{\imath\mathbf{k}\cdot\mathbf{r}}. \tag{4.33}$$

The physical mean-field quantities are just the Fourier transformed Green functions. The equations of motion for the Green functions are computed in the same way as in the Bogoliubov case

$$\imath\partial_t G_{pt}^< = -\int\mathrm{d}^3\mathbf{r}\; V(r)[(\rho+\alpha_r^\dagger)F_{pt} + (\rho+\alpha_r)F_{pt}^\dagger]\mathrm{e}^{\imath\mathbf{p}\cdot\mathbf{r}} \\ \imath\partial_t F_{pt} = 2E(p)F_{pt} + \int\mathrm{d}^3\mathbf{r}\; V(r)[2(\rho+n_0+n_r)F_{pt} + (\rho+\alpha_r)(G_{pt}^< + G_{-pt}^>)]\mathrm{e}^{\imath\mathbf{p}\cdot\mathbf{r}} \tag{4.34}$$

and by taking the Fourier transform we get the equations of motion for the integrated quantities

$$\begin{aligned}
\imath\partial_t n(x) &= -\int \mathrm{d}^3\mathbf{r}\; V(r)[(\rho+\alpha_r^\dagger)\alpha_{r-x} - (\rho+\alpha_r)\alpha_{r-x}^\dagger \\
\imath\partial_t \alpha(x) &= 2\imath\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} E(k) F_{kt} \mathrm{e}^{-\imath\mathbf{k}\cdot\mathbf{x}} + \\
&\quad + \int \mathrm{d}^3\mathbf{r}\; V(r)[2(\rho+n_0+n_r)\alpha_{r-x} + (\rho+\alpha_r)(2n_{r-x}+\delta^3(\mathbf{r}-\mathbf{x}))].
\end{aligned} \tag{4.35}$$

In particular the equation for the depletion is:

$$\imath\partial_t n(0) = \rho\int \mathrm{d}^3\mathbf{r}\, V(r)(\alpha_r^\dagger - \alpha_r). \tag{4.36}$$

With the Keldysh machinery we can also study the evolution of the condensate wavefunction $\psi_x = \psi(x,t)$ which must be considered a dynamical quantity just like the propagators. The full diagramatic representation of the condensate at first order is shown in Fig. 4.4.

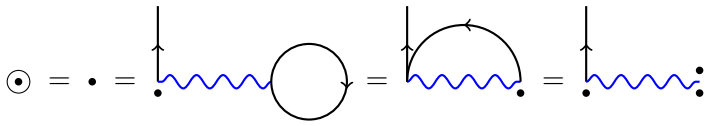


Figure 4.4: Diagrammatic representation of the condensate particle. The meaning of the symbols is the same as in Fig. 4.3.

that translates into formulas as

$$\begin{aligned}
\langle\psi_x\rangle = \langle\psi_x\rangle^0 + \int_C \mathrm{d}\tau \int \mathrm{d}^3x_1\, \mathrm{d}^3x_2\; V(|x_1-x_2|)\, G^0(t,x;\tau,x_1)\times \\
\times [\imath G^0(\tau,x_2;\tau,x_2)\langle\psi_{x_1}\rangle^0 + \imath G^0(\tau,x_1;\tau,x_2)\langle\psi_{x_2}\rangle^0 + \langle\psi_{x_1}\rangle^0\langle\psi_{x_2}\rangle^0\langle\psi_{x_2}^\dagger\rangle^0]
\end{aligned} \tag{4.37}$$

When we evaluate the time integral on the Keldysh contour depicted in Fig. 4.5, substituting the bare condensate functions with the full ones inside the integrals, we get three terms on which we should apply the operator $(\imath\partial_t + \nabla^2/2m)$ in order to obtain the equation of motion. The first term is evaluated

as:

$$
\begin{aligned}
&\imath \int \mathrm{d}^3x_1\mathrm{d}^3x_2 V_{12} \int_{-\infty}^{t} \mathrm{d}\tau \Big[G^{0>}(x,t;x_1,\tau)\; G^{0^T}(x_2,\tau;x_2,\tau) + \\
&\qquad\qquad - G^{0<}(x,t;x_1,\tau)\; G^{0^{\widetilde{T}}}(x_2,\tau;x_2,\tau) \Big] \langle \psi_{x_1} \rangle \\
&\imath \int \mathrm{d}^3x_1\mathrm{d}^3x_2 V_{12} \int_{-\infty}^{t} \mathrm{d}\tau \Big[G^{0>} - G^{0<} \Big](x,t;x_1,\tau)\; G^{0^T}(x_2,\tau;x_2,\tau) \langle \psi_{x_1} \rangle \\
&\downarrow \qquad \text{apply } (\imath\partial_t + \frac{\nabla^2}{2m}) \\
&\imath \int \mathrm{d}^3x_1\mathrm{d}^3x_2 V_{12} (-\imath)\delta^3(x-x_1)\; G^{0^T}(t,x_2;t,x_2) \langle \psi_{x_1} \rangle \\
&\langle \psi_x \rangle \int \mathrm{d}^3x_2 V(x_2)\; G^{0^T}(x_2,t;x_2,t) \\
&\langle \psi_x \rangle \int \mathrm{d}^3x_2 V(x_2) \int \mathrm{d}^3y G^{0^T}(x_2,t;y,t)\delta^3(y-x_2) \\
&\langle \psi_x \rangle \int \mathrm{d}^3x_2 V(x_2) \int \mathrm{d}^3y (-\frac{\imath}{2})[\delta^3(y-x_2)]^2 \\
&-\frac{\imath}{2}\frac{\langle \psi_x \rangle}{L^3} \int \mathrm{d}^3x_2 V(x_2) \\
&-\frac{\imath}{2L^3}\langle \psi_x \rangle \widetilde{V}(0)
\end{aligned}
\tag{4.38}
$$

an identical term comes from the second part of the integrand, whereas the third part gives a different result

$$
\begin{aligned}
&\imath \int \mathrm{d}^3x_1\mathrm{d}^3x_2 V_{12} \int_{-\infty}^{t} \mathrm{d}\tau \Big[G^{0>} - G^{0<} \Big](x,t;x_1,\tau) \langle \psi_{x_1}(t) \rangle \langle \psi_{x_2}(t) \rangle \langle \psi^\dagger_{x_2}(t) \rangle \\
&\downarrow \qquad \text{apply } (\imath\partial_t + \frac{\nabla^2}{2m}) \\
&\imath \int \mathrm{d}^3x_1\mathrm{d}^3x_2 V_{12} \Big[G^{0>} - G^{0<} \Big](x,t;x_1,t) \langle \psi_{x_1}(t) \rangle \langle \psi_{x_2}(t) \rangle \langle \psi^\dagger_{x_2}(t) \rangle \\
&\int \mathrm{d}^3x_1\mathrm{d}^3x_2 V_{12} \delta^3(x-x_1) \langle \psi_{x_1}(t) \rangle \left| \langle \psi_{x_2}(t) \rangle \right|^2 \\
&\approx \widetilde{V}(0) \langle \psi_x(t) \rangle \left| \langle \psi_x(t) \rangle \right|^2
\end{aligned}
\tag{4.39}
$$

Collecting all the pieces and disregarding the $O(L^{-3})$ terms whose contribution is negligible in thermodynamic limit we get

$$
\left(\imath\partial_t + \frac{\nabla^2}{2m} \right) \langle \psi(x,t) \rangle = \widetilde{V}(0) \langle \psi(x,t) \rangle \left| \langle \psi(x,t) \rangle \right|^2 , \tag{4.40}
$$

which is a Gross–Pitaevski equation.

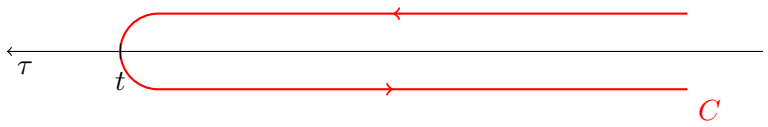


Figure 4.5: Keldysh contour for the condensate wave-function

4.3 Effective temperature or GGE?

The correlation functions we calculated within the Bogoliubov approximation always relax to a finite value in the long time limit. This relaxation is related to inhomogeneous dephasing of noninteracting Bogoliubov quasi-particles: we therefore expect these stationary states not to be thermal, hence not described by a single effective temperature. In order to see this we study the relation between the value of g_2 after a quench and the value at equilibrium at an effective temperature T_{eff} corresponding to the energy E_Q injected in the quench.

Let us start by computing the energy injected in the quench. We have

$$\begin{aligned} E_Q &= E_0(\gamma_1) + \sum_{\mathbf{k}} \epsilon(k,\gamma_1) \langle \phi_0 | b_k^\dagger b_k | \phi_0 \rangle \\ &= E_0(\gamma_1) + \sum_{\mathbf{k}} \epsilon(k,\gamma_1) |\omega_k|^2 \\ &= E_0(\gamma_1) + \frac{\Omega_D \, L^D}{4(2\pi)^D} \int_0^\infty \mathrm{d}k \; k^{D-1} \frac{[\epsilon(k,\gamma_1) - \epsilon(k,\gamma_0)]^2}{\epsilon(k,\gamma_0)}. \end{aligned} \tag{4.41}$$

In 1D, for a small quench we get

$$E_Q = E_0(\gamma_1) + \frac{\Omega^{\frac{1}{D}}}{\sqrt{2}\pi} \frac{\rho^3}{m\sqrt{\gamma_1}} (\delta\gamma)^2, \tag{4.42}$$

which is to be compared to the energy at a given (small) temperature E_T

$$E_T = E_0(\gamma_1) + \Omega^{\frac{1}{D}} \frac{\sqrt{2}\pi m}{12\rho\sqrt{\gamma_1}} T^2. \tag{4.43}$$

Inverting this expression we find the relation $T_{eff} = (2\sqrt{6}/\pi)\, T_d \; \delta\gamma$. If the state of the system would be thermal substituting T_{eff} into the thermal g_2 we should obtain in the stationary state

$$g_2(T_{eff}) = 1 - \frac{2\sqrt{2}}{\pi}\sqrt{\gamma} + \frac{1}{2\sqrt{2}\pi} \gamma_1^{-\frac{3}{2}} (\delta\gamma)^2. \tag{4.44}$$

Instead we have

$$\overline{g}_2 = 1 - \frac{2\sqrt{2}}{\pi}\sqrt{\gamma} + \frac{1}{6\sqrt{2}\pi} \gamma_1^{-\frac{3}{2}} (\delta\gamma)^2. \tag{4.45}$$

From this simple exercise, we see that despite being parametrically similar, the value after the quench is constantly smaller than the corresponding thermal one.

We can attempt the same analysis in 3D despite the fact that the integrals cannot be solved analytically. Let us by focusing on a particular quench, namely from zero to finite interaction. We can calculate for different observables the equilibrium value at given γ and T and compare with the long-time average of the same quantity after the quench. By comparison one should be able to get a value for the effective interaction and temperature corresponding to the quench. The hope is that the outcomes from different quantities are consistent. As an example let us consider the 3D case with delta interaction

$$\begin{aligned} g_2(x) &= 1 - \frac{1}{2\pi^2\rho^{\frac{4}{3}}x^4\sqrt{\gamma}} + \frac{1}{\rho\pi^2}\int_0^\infty dk k^2 \frac{E(k)}{\epsilon(k)} \frac{1}{\mathrm{e}^{\frac{\epsilon(k)}{T}}-1}\frac{\sin(kx)}{kx} \\ &\approx 1 - \frac{1}{2\pi^2\rho^{\frac{4}{3}}x^4\sqrt{\gamma}} + \frac{1}{\rho\pi^2}\int_0^\infty dk k^2 \frac{k^2}{2m}\frac{1}{ck} \frac{1}{\mathrm{e}^{\frac{ck}{T}}-1}\frac{\sin(kx)}{kx} \\ &= 1 - \frac{\pi T^3}{2m\rho x c^4}\coth(\frac{\pi T x}{c})\cosh(\frac{\pi T x}{c})^{-2} \\ &\approx 1 - \frac{\pi\tau^3}{4\rho^{\frac{1}{3}}x\gamma^2}\mathrm{e}^{-\frac{\pi\rho^{\frac{1}{3}}x\tau}{\sqrt{\gamma}}}, \end{aligned} \tag{4.46}$$

where $\tau = T/T_d$. Here we used twice the large x limit, once to expand $\epsilon(k)$ and keep only its linear part with coefficent c, and then to express the hyperbolic functions in terms of a single exponential. Let us compare this to the case of a quench from zero to finite interaction $\widetilde{\gamma}$

$$\overline{g}_2(x) = 1 - \frac{\widetilde{\gamma}}{2\pi\rho^{\frac{1}{3}}x}\mathrm{e}^{-2\rho^{\frac{1}{3}}x\sqrt{\widetilde{\gamma}}}, \tag{4.47}$$

One might be tempted to define both effective temperature and interaction as

$$\tau = \frac{8}{\pi^2}\widetilde{\gamma} \qquad \gamma = \frac{16}{\pi^2}\widetilde{\gamma}. \tag{4.48}$$

in order to put the two expressions in direct correspondence; the prefactors depend on the scale we choose to define the temperature, but the qualitative behavior is fixed and is consistent with the expression for the depletion. Indeed at equilibrium at small temperature and interaction we have from Ref. [47]

$$n_{ex} = \frac{1}{3\pi^2}\gamma^{\frac{3}{2}}\left[1 + (2\pi)^2\frac{\tau^2}{\gamma^2}\right] \tag{4.49}$$

and after the quench

$$\overline{n}_{ex} = \frac{1}{4\pi}\widetilde{\gamma}^{\frac{3}{2}}. \tag{4.50}$$

Notice however that the procedure above is far from being general; indeed, if we instead perform a quench between two finite values of the interaction ($\gamma_0 \to \gamma_1$) the result is radically different. The depletion can be exactly computed to be

$$\overline{n}_{ex} = \frac{1}{3\pi^2}\gamma_0^{\frac{3}{2}} - \frac{1}{2\pi^2}\gamma_1\sqrt{\gamma_1-\gamma_0}\arccos\left(\sqrt{\frac{\gamma_0}{\gamma_1}}\right) \tag{4.51}$$

and the non-local correlation function for large separation is well approximated by

$$\overline{g}_2(x) = 1 - \frac{1}{4\pi^2\rho^{\frac{4}{3}}x^4}\frac{\gamma_0+\gamma_1}{\gamma_1\sqrt{\gamma_0}}. \tag{4.52}$$

For a small quench we can expand $\gamma_1 = \gamma_0 + \delta\gamma$. Notice that this expansion is different from the one used before and will give rise to terms linear in $\delta\gamma$,

$$\overline{n}_{ex} = \frac{1}{3\pi^2}\gamma_0^{\frac{3}{2}} - \frac{1}{2\pi^2}\sqrt{\gamma_0}\,\delta\gamma \qquad \overline{g}_2(x) = 1 - \frac{1}{2\pi^2\rho^{\frac{4}{3}}x^4\sqrt{\gamma_0}} + \frac{\delta\gamma}{4\pi^2\rho^{\frac{4}{3}}x^4\gamma_0^{\frac{3}{2}}}. \tag{4.53}$$

At the same time we know that at equilibrium at zero temperature we have

$$n_{ex} = \frac{1}{3\pi^2}\gamma^{\frac{3}{2}} \qquad g_2(x) = 1 - \frac{1}{2\pi^2\rho^{\frac{4}{3}}x^4\sqrt{\gamma}}, \tag{4.54}$$

and if we expand this γ as $\gamma = \gamma_0 + \widetilde{\delta\gamma}$ we get expressions identical to those coming from the quench with $\delta\gamma = \widetilde{\delta\gamma}$. In this case an effective temperature can hardly be defined.

While in $1D$ the system under analysis is a simple limit of a Lieb–Liniger model, thus integrable and never thermalizing, in higher dimensions we expect the interaction between the quasi-particles to lead to the thermalization of the system and our observables should reach their equilibrium value corresponding to the final equilibrium temperature. The stationary state found here should therefore be only a prethermalized one that should be described by an effective GGE.

Since the description in terms of a thermal ensemble does not work let us now attempt a description in terms of a GGE. After the quench the Hamiltonian is

$$\hat{H}(\gamma_1) = E_0(\gamma_1) + \sum_{\mathbf{k}} \hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} = E_0(\gamma_1) + \sum_{\mathbf{k}} \hat{n}_1(\mathbf{k}), \tag{4.55}$$

so the occupation numbers for any level $\mathbf{k}$ are conserved quantities and we can build the corresponding Generalized Gibbs Ensemble

$$\hat{\rho}_{GGE} = \frac{1}{\mathcal{Z}}\mathrm{e}^{-\sum_{\mathbf{k}}\lambda_k\hat{n}_1(\mathbf{k})} = \frac{1}{\mathcal{Z}}\prod_{\mathbf{k}}\mathrm{e}^{-\lambda_k\hat{n}_1(\mathbf{k})}. \tag{4.56}$$

From the unitarity of the trace it follows that

$$1 \equiv \mathrm{Tr}[\hat{\rho}_{GGE}] = \frac{1}{\mathcal{Z}}\sum_{\{\alpha\}}\langle\alpha|\prod_{\mathbf{k}}\mathrm{e}^{-\lambda_k\hat{n}_1(\mathbf{k})}|\alpha\rangle = \frac{1}{\mathcal{Z}}\prod_{\mathbf{k}}\sum_n \mathrm{e}^{-\lambda_k n} = \frac{1}{\mathcal{Z}}\prod_{\mathbf{k}}\frac{1}{1-\mathrm{e}^{\lambda_k}}$$

where α is an eigenstate of $\bigotimes_{\mathbf{k}} \hat{n}_1(\mathbf{k})$, and so

$$\ln \mathcal{Z} = -\sum_{\mathbf{k}} \ln\left(1 - \mathrm{e}^{\lambda_k}\right). \tag{4.57}$$

As we explained earlier, the Lagrange multipliers λ_k are fixed by imposing the correct expectation for the conserved quantities on the initial state, in this case if we apply the operator on the state of the system before the quench we get

$$\langle\phi_0|\, \hat{n}_1(\mathbf{k})\, |\phi_0\rangle = \langle\phi_0|\, \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}}\, |\phi_0\rangle = \omega_k^2\, \langle\phi_0|\, \hat{c}^\dagger_{\mathbf{k}} \hat{c}_{\mathbf{k}}\, |\phi_0\rangle = \omega_k^2. \tag{4.58}$$

This quantity is showed in Fig. 4.6, computing its average in the *GGE* ensemble we obtain

$$\begin{aligned}
\mathrm{Tr}[\hat{n}_1(\mathbf{q})\hat{\rho}_{GGE}] &= \frac{1}{\mathcal{Z}} \sum_{\{\alpha\}} \langle\alpha|\, \hat{n}_1(\mathbf{q}) \prod_{\mathbf{k}} \mathrm{e}^{-\lambda_k \hat{n}_1(\mathbf{k})}\, |\alpha\rangle \\
&= \frac{1}{\mathcal{Z}} \prod_{\mathbf{k}} \sum_{n_k} n_q \mathrm{e}^{-\lambda_k n_k} \\
&= -\partial_{\lambda_q} \ln \mathcal{Z} \\
&= 1 + \frac{1}{\mathrm{e}^{\lambda_q} - 1}.
\end{aligned} \tag{4.59}$$

Comparison between these two quantities gives

$$\ln \lambda_q = \frac{\omega_q^2}{\omega_q^2 - 1}. \tag{4.60}$$

We might use this knowledge to construct the expectation for g_2: since we performed all the calculation in the Bogoliubov approximation, i.e. at quadratic order, we keep the same level of approximation in the following and obtain

$$\begin{aligned}
\mathrm{Tr}\left[\hat{g}_2\ \hat{\rho}_{GGE}\right] &= \frac{1}{\rho^2} \mathrm{Tr}\left[\hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \hat{\rho}_{GGE}\right] \\
&\approx \frac{1}{\rho^2} \mathrm{Tr}\left[\left(\psi_0^4 + \psi_0^2(\delta\hat{\psi}\delta\hat{\psi} + \delta\hat{\psi}^\dagger\delta\hat{\psi}^\dagger + 4\delta\hat{\psi}^\dagger\delta\hat{\psi})\right) \hat{\rho}_{GGE}\right] \\
&= 1 + \frac{2}{\rho L^D} \sum_{\mathbf{k}} \left[(u_k + v_k)^2 \omega_k^2 + v_k(u_k + v_k)\right] \\
&= 1 + \frac{1}{\rho} \frac{\Omega_D}{(2\pi)^D} \int_0^\infty \mathrm{d}k\ k^{D-1} \left[E(k) \frac{\epsilon(k,\gamma_0)^2 + \epsilon(k,\gamma_1)^2}{2\ \epsilon(k,\gamma_0)\epsilon(k,\gamma_1)^2} - 1\right],
\end{aligned} \tag{4.61}$$

which gives back equation (4.1), thus suggesting that, at quadratic level, the GGE prediction is correct. More details on the above calculation can be found in Appendix C.2.

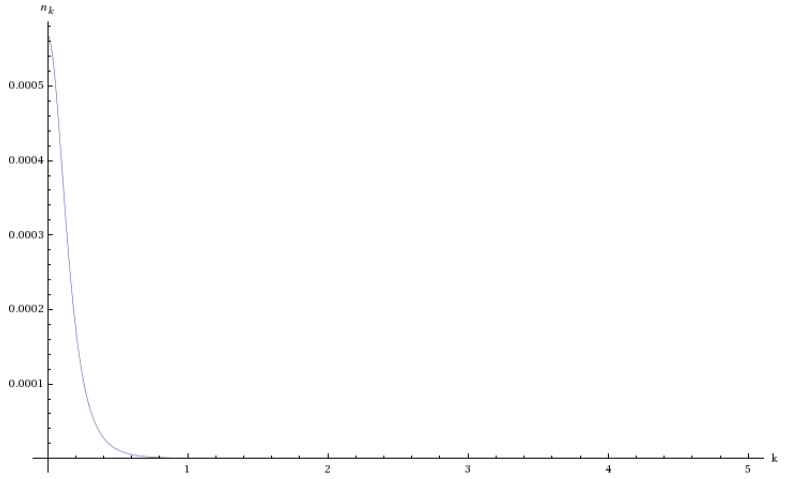


Figure 4.6: Momentum occupation as a function of k. The occupation rapidly decays as k^{-1}.

What happens however if we consider the two-particle correlator computing that via Wick theorem?

$$\left\langle \delta\hat{\psi}^\dagger\delta\hat{\psi}^\dagger\delta\hat{\psi}\delta\hat{\psi}\right\rangle_{GGE} \sim \left\langle \delta\hat{\psi}^\dagger\delta\hat{\psi}^\dagger\right\rangle_{GGE}\left\langle \delta\hat{\psi}\delta\hat{\psi}\right\rangle_{GGE} + 2\left\langle \delta\hat{\psi}^\dagger\delta\hat{\psi}\right\rangle^2_{GGE}$$
$$= \frac{1}{L^{2D}}\left[\left(\sum_{\mathbf{k}} u_k v_k(2\omega_k^2+1)\right)^2 + 2\left(\sum_{\mathbf{k}}\left[(u_k^2+v_k^2)\omega_k^2+v_k^2\right]\right)^2\right]. \tag{4.62}$$

The time average of the expectation value, instead, is computed from the decomposition

$$\left\langle \delta\hat{\psi}^\dagger(t)\delta\hat{\psi}^\dagger(t)\delta\hat{\psi}(t)\delta\hat{\psi}(t)\right\rangle \sim \left\langle \delta\hat{\psi}^\dagger(t)\delta\hat{\psi}^\dagger(t)\right\rangle\left\langle \delta\hat{\psi}(t)\delta\hat{\psi}(t)\right\rangle + 2\left\langle \delta\hat{\psi}^\dagger(t)\delta\hat{\psi}(t)\right\rangle^2,$$

where now the averages are performed on the state of the system before the quench and the different terms have already been computed in Appendix C.1. Substituting and averaging over time gives

$$\overline{\left\langle \delta\hat{\psi}^\dagger\delta\hat{\psi}^\dagger\delta\hat{\psi}\delta\hat{\psi}\right\rangle} = \frac{1}{L^{2D}}\left[\left(\sum_{\mathbf{k}} u_k v_k(2\omega_k^2+1)\right)^2 + 2\left(\sum_{\mathbf{k}}\left[(u_k^2+v_k^2)\omega_k^2+v_k^2\right]\right)^2 + \right.$$
$$\left. + \sum_{\mathbf{k}} \omega_k^2\chi_k^2(u_k^2+v_k^2)^2\right]. \tag{4.63}$$

The two expressions are different: the additional component in the long time average comes from terms whose oscillating part cancel out when they are

multiplied, so the discrepancy comes from the fact that the time average of the product is not the product of the time averages: the GGE prediction cannot be relied on already at this level. Notice however that terms of this type, while being in principle present in the calculation of g_2, have to be consistently neglected in the Bogoliubov approximation and represent correction of order L^{-2D}, unimportant for our purposes.

4.4 Boltzmann Equation

In three dimensions the interaction among Bogoliubov quasi-particles (scattering, as well as decay) is expected to lead to the thermalization of the system. In order to study the decay from the prethermal plateau to a full thermal state let us first consider quasi-particle decay. Indeed, if the dispersion relation satisfies $\partial^2\epsilon(p) > 0$ then quasi-particles can decay in two other quasi-particles [36, 48], thus we are compelled to consider this three-particles process before the four-particles scattering.

To understand the phonon decay one should start from the hydrodinamic description of the superfluid. In this framework the expression for the energy is

$$\hat{H} = \int \mathrm{d}^3 r \left[\frac{m}{2}\nabla\phi\rho\nabla\phi + e(\rho)\right], \tag{4.64}$$

where ρ and ϕ are canonically conjugate and represent the density and the phase of the superfluid wavefunction respectively, and the function $e(\rho)$ is the internal energy per particle.

We promote the observables to operators and define $\hat{\rho}' = \hat{\rho} - \rho_0$ as the fluctuation around the average density. Expand in $\hat{\rho}'$ and $\hat{v} = \nabla\hat{\phi}$

$$\hat{H}^{(0)} = \int \mathrm{d}^3 r \; e(\rho_0) \qquad \hat{H}^{(1)} = \int \mathrm{d}^3 r \;\; \hat{\rho}'\partial_\rho e(\rho_0)\big|_{\rho_0} = 0, \tag{4.65}$$

since $\hat{\rho}'$ is just the fluctuation above the average and thus must integrate to zero. Then

$$\hat{H}^{(2)} = \int \mathrm{d}^3 r \;\; \left[\frac{m\rho_0}{2}\hat{v}^2 + \frac{mu^2}{2\rho_0}\hat{\rho}'^2\right] \tag{4.66}$$

$$\hat{H}^{(3)} = \int \mathrm{d}^3 r \;\; \left[\frac{m}{2}\hat{v}\hat{\rho}'\hat{v} + \frac{m}{6}\partial_{\rho_0}\left(\frac{u^2}{\rho_0}\right)\hat{\rho}'^3\right], \tag{4.67}$$

where $u^2 = \partial_\rho^2 \, e(\rho)\big|_{\rho_0}$ is the squared sound velocity. We write $\hat{\rho}'$ and $\hat{\phi}$ in quantized form

$$\begin{aligned}
\hat{\phi}(r,t) &= L^{-\frac{3}{2}} \sum_{\mathbf{k}} \sqrt{\frac{u}{2\rho_0 m \; k}} \left[\hat{b}_{\mathbf{k}} \mathrm{e}^{\imath(\mathbf{k}\cdot\mathbf{r} - ukt)} + \hat{b}^\dagger_{\mathbf{k}} \mathrm{e}^{-\imath(\mathbf{k}\cdot\mathbf{r} - ukt)}\right] \\
\hat{\rho}'(r,t) &= L^{-\frac{3}{2}} \imath \sum_{\mathbf{k}} \sqrt{\frac{\rho_0 \; k}{2mu}} \left[\hat{b}_{\mathbf{k}} \mathrm{e}^{\imath(\mathbf{k}\cdot\mathbf{r} - ukt)} - \hat{b}^\dagger_{\mathbf{k}} \mathrm{e}^{-\imath(\mathbf{k}\cdot\mathbf{r} - ukt)}\right],
\end{aligned} \tag{4.68}$$

whose coefficients are fixed by the continuity equation $\partial_t \hat{\rho}' = -\rho_0 \nabla \cdot \nabla \hat{\phi}$ and the commutation relation $\left[\hat{\phi}(\mathbf{x}), \hat{\rho}'(\mathbf{y})\right] = -\imath\, m^{-1} \delta(\mathbf{x}-\mathbf{y})$.

The second order term, upon substitution of the operators gives

$$\hat{H}^{(2)} = \sum_{\mathbf{k}} uk(\hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}} + \frac{1}{2}), \tag{4.69}$$

validating the interpretation of u as the sound velocity, the third order terms give rise to phonon decay. To calculate the decay rate we apply Fermi Golden rule:

$$\mathrm{d}w = 2\pi |V_{if}|^2 \delta(E_f - E_i) \frac{L^6}{(2\pi)^6}\, \mathrm{d}^3 p_2\, \mathrm{d}^3 p_3. \tag{4.70}$$

The matrix element V_{if} is computed on states compatible with the decay of a phonon of momentum $\mathbf{p}_1$ in two other of momenta $\mathbf{p}_2$ and $\mathbf{p}_3$ or with the reverse process. For example for the decay the only surviving terms in $\hat{H}^{(3)}$ are those proportional to $\hat{b}_{\mathbf{p}_1} \hat{b}^\dagger_{\mathbf{p}_2} \hat{b}^\dagger_{\mathbf{p}_3}$, and there are six terms of this kind:

$$\begin{aligned} V_{if} &= \langle f | \hat{H}^{(3)} | i \rangle \\ &= -\imath \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \frac{3!(2\pi)^3}{2(2L^3)^{\frac{3}{2}}} \sqrt{\frac{u}{m\rho_0} p_1 p_2 p_3} \left[1 + \frac{\rho_0^2}{3u^2} \partial_{\rho_0} \frac{u^2}{\rho_0}\right] \mathrm{e}^{-\imath u(p_1 - p_2 - p_3)t} \end{aligned} \tag{4.71}$$

Since all the momenta are small by construction, the product of the three, inside the matrix element, guarantees the applicability of perturbation theory. The square of the matrix element would actually contain the square of a delta function which must be replaced with the substitution

$$[\delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)]^2 = \left(\frac{L}{2\pi}\right)^3 \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3), \tag{4.72}$$

which comes directly from the integral representation of the delta function. Integration on the product momenta gives the following expression for the decay probability per unit time:

$$w = \frac{3p_1^5}{320\pi m \rho_0} \left[1 + \frac{\rho_0^2}{3u^2} \partial_{\rho_0} \frac{u^2}{\rho_0}\right]^2. \tag{4.73}$$

As mentioned before, quasi-particles follow Bose–Einstein statistics so we should actually include the non-thermal occupation of final states. If this modification is performed it turns out that for realistic values of the parameters and for small quenches the result is unchanged, indeed the n_p terms are of order 10^{-3} after the quench (as a reference, $\rho \sim 10^3\ \mu\mathrm{m}^{-3}$, $\gamma \sim 10^{-2}$ and $\delta\gamma \sim 10^{-4}$).

To determine the dynamics towards thermalization we now write the Boltzmann equation for the unknown distribution $f(r, p; t)$

$$\left(\partial_t + \frac{p}{m}\nabla_r + F\nabla_p\right) f(r,p;t) = \left(\frac{\partial f}{\partial t}\right)_{coll}. \tag{4.74}$$

Since we work in an homogeneous system, the space derivative is zero and the diffusive term is thus disregarded, we neglect also the external force field although the particles are supposed to be somehow confined. Thus we are left with the collision term in the Boltzmann equation: for the specific process we are interested in the kernel comes from the diagrams shown in Fig. 4.7

$$\left(\frac{\partial f}{\partial t}(p_1;t)\right)^B_{coll} = \frac{L^6}{(2\pi)^5}\int \mathrm{d}^3p_2\mathrm{d}^3p_3\ [(1+f_1)f_2f_3 - f_1(1+f_2)(1+f_3)]\times$$
$$\times\, |V_{if}|^2\delta(E_f - E_i), \tag{4.75}$$

where the superscript B labels the "Beljaev" term and we simplified the expression with the shorthand notation$f_1 = f(r, p_1; t)$. This kernel correctly becomes zero if f is replaced with $f^0(p) = (\mathrm{e}^{\beta\epsilon(p)} - 1)^{-1}$ which is just the Bose–Einstein distribution that we expect at equilibrium where the time derivative vanishes.

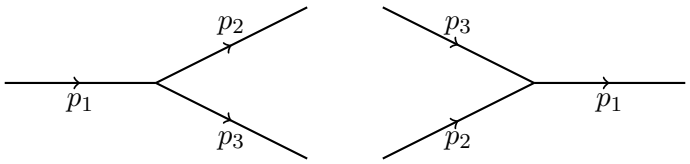


Figure 4.7: "Beljaev" diagrams

To proceed further we linearize the equation around the equilibrium distribution, i.e. we expand it at first order in $\delta f(p;t) = f(p;t) - f^0(p)$; to simplify the following calculations it turns out that it is better to linearize in terms of an auxiliary variable $\varphi(p,t)$ defined as

$$f(p;t) = \frac{1}{\mathrm{e}^{\beta\epsilon(p)-\varphi(p,t)} - 1}, \tag{4.76}$$

so that when the variation is performed in the last expression we get

$$\delta f(p;t) = f^0(p)(1+f^0(p))\ \varphi(p,t). \tag{4.77}$$

After substitution into the collision kernel we get the linearized Boltzmann equation

$$\partial_t\varphi(p_1,t)^B = \frac{V^2}{(2\pi)^5}\int \mathrm{d}^3p_2\,\mathrm{d}^3p_3\ \frac{f_2^0 f_3^0}{f_1^0}[-\varphi(p_1,t) + \varphi(p_2,t) + \varphi(p_3,t)]\times$$
$$\times\, |V_{if}|^2\delta(E_f - E_i). \tag{4.78}$$

This collision kernel describes only the decay of the given phonon and the reverse process of recombination of two phonons into it as shown in picture 4.7.

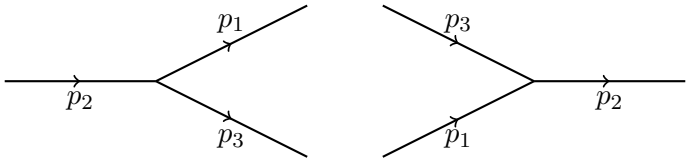


Figure 4.8: "Non-Beljaev" diagrams

In order to be consistent, however, we need to take into account all the diagram of the same order, this means we must consider the process of combination of our initial phonon with a second one into a third phonon and the inverse decay process as depicted in Fig. 4.8. The squared matrix element $|V_{if}|^2$ is the same as before because it is invariant under permutation of the initial and final momenta but the second collision kernel, now labeled nB for "non-Beljaev" is

$$\left(\frac{\partial f}{\partial t}(p_1;t)\right)_{coll}^{nB} = \frac{V^2}{(2\pi)^5}\int \mathrm{d}^3p_2\,\mathrm{d}^3p_3\ [(1+f_1)f_2(1+f_3) - f_1(1+f_2)f_3]\times$$
$$\times\,|V_{if}|^2\delta(E_f - E_i), \tag{4.79}$$

which should also be linearized in the vicinity of the equilibrium distribution to give

$$\partial_t\varphi(p_1,t)^{nB} = \frac{V^2}{(2\pi)^5}\int \mathrm{d}^3p_2\mathrm{d}^3p_3\ \frac{f_2^0(f_3^0+1)}{f_1^0}[-\varphi(p_1,t)+\varphi(p_2,t)-\varphi(p_3,t)]\times$$
$$\times\,|V_{if}|^2\delta(E_f - E_i). \tag{4.80}$$

Since the labels of the phonons in the two processes are different, also the delta functions will be different and this will be reflected in the integration extrema too. As a last simplification we notice that the second term in square brackets in equation (4.71) is negligible if realistic values of the parameters are plugged in, so

$$\begin{aligned}\partial_t\varphi_1 = {}& \frac{9}{2^3(2\pi)^2}\frac{u}{m\rho_0}p_1\times \\ &\times\int \mathrm{d}^3p_2\mathrm{d}^3p_3\ p_2p_3\left[\frac{f_2^0 f_3^0}{f_1^0}\delta(\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_3)\delta(\epsilon_{p_1}-\epsilon_{p_2}-\epsilon_{p_3})[-\varphi_1+\varphi_2+\varphi_3]\right. \\ &\left. +\ \frac{f_2^0(f_3^0+1)}{f_1^0}\delta(\mathbf{p}_2-\mathbf{p}_1-\mathbf{p}_3)\delta(\epsilon_{p_2}-\epsilon_{p_1}-\epsilon_{p_3})[-\varphi_1+\varphi_2-\varphi_3]\right].\end{aligned} \tag{4.81}$$

First we use the delta over momenta to eliminate one of the integration variables, then we approximate the energy as purely phononic (thus linear in

the momentum) to express the second delta as

$$\delta(\epsilon_{p_1} - \epsilon_{p_2} - \epsilon_{p_3}) \simeq \frac{1}{u}\delta(p_1 - p_2 - |\mathbf{p}_1 - \mathbf{p}_2|) = \frac{|\mathbf{p}_1 - \mathbf{p}_2|}{u\ p_1 p_2}\delta(1 - \cos\theta) \quad (4.82)$$

in order to integrate over the angles of the remaining variable. The last passage assumes the produced phonons are almost collinear with the initial one. Finally, realizing that $f^0_q + 1 = -f^0_{-q}$ and renaming dummy variables, we have

$$\begin{aligned}\partial_t \varphi_p = &\frac{9}{2^3(2\pi)m\rho_0}\times \\ &\left[\int_0^p \mathrm{d}q\, q^2(p-q)^2\ \frac{f^0_q f^0_{p-q}}{f^0_p}\ [2\varphi_q - \varphi_p] + \int_p^\infty \mathrm{d}q\ q^2(p-q)^2\ \frac{f^0_q f^0_{p-q}}{f^0_p}\ [\varphi_p - \varphi_q] + \right. \\ &\left. - \int_0^\infty \mathrm{d}q\ q^2(p+q)^2\ \frac{f^0_{-q} f^0_{p+q}}{f^0_p}\ \varphi_q \right]. \end{aligned} \quad (4.83)$$

Once we dicretise momentum and turn integrals into sums, the RHS of the last equation can be seen as as the action of a linear operator

$$\mathcal{L}(p,q) = D(p)\delta_{pq} + K(p,q) \quad (4.84)$$

so that

$$\partial_t \varphi(p,t) = \sum_q \mathcal{L}(p,q)\ \varphi(q,t) = D(p)\ \varphi(p,t) + \sum_q K(p,q)\ \varphi(q,t), \quad (4.85)$$

and the problem reduces to the diagonalization of the $\mathcal{L}(p,q)$ operator.

The off-diagonal elements of this operators are negligible with respect to those on the diagonal so the eigenvalues are well approximated by $D(p)$. It can be written in terms of polylogarithmic functions but this gives no insight in the behaviour of such terms, instead since we have already employed the small momentum condition and we are anyway relying on the properties of the phononic part of the energy spectrum we can safely extract just the first terms in p.

$$\begin{aligned} D(p) &= \frac{9}{2^3(2\pi)m\rho_0}\left[\int_p^\infty - \int_0^p\right] dq\ q^2(p-q)^2\ \frac{f^0_q f^0_{p-q}}{f^0_p} \\ &= \frac{9}{2^3(2\pi)m\rho_0}(\mathrm{e}^{\beta u p} - 1)\left[\int_p^\infty - \int_0^p\right] dq \frac{q^2(p-q)^2}{(\mathrm{e}^{\beta u q} - 1)(\mathrm{e}^{\beta u (p-q)} - 1)} \\ &\approx -\frac{9\beta u p}{2^3(2\pi)m\rho_0}\int_0^\infty dq \frac{q^4 \mathrm{e}^{\beta u q}}{(\mathrm{e}^{\beta u q} - 1)^2} \\ &= -\frac{3\pi^3}{20}\frac{p}{m\rho_0\beta^4 u^4} = -\mathcal{D}p. \end{aligned} \quad (4.86)$$

Where for later convenience we collect in $\mathcal{D}$ all numerical and parametrical prefactors. We recall here that β is fixed by the quench amplitude and $u^2 = \rho_0 \widetilde{V}_0 m^{-1}$.

So $\partial_t \varphi(p,t) = D(p)\varphi(p,t)$ and from the chosen way to perturb around equilibrium follows that

$$\begin{aligned} n_p(t) - f^0(p) &= (n_{preth}(p) - f^0(p))\mathrm{e}^{D(p)t} \\ &= \left[\frac{(\epsilon(p,\gamma_0) - \epsilon(p,\gamma_1))^2}{4\epsilon(p,\gamma_0)\epsilon(p,\gamma_1)} - \frac{1}{\mathrm{e}^{\beta up} - 1}\right]\mathrm{e}^{D(p)t} \\ &\approx \left[\frac{1}{4\mu^2}\frac{\gamma_0+\gamma_1}{\sqrt{\gamma_0\gamma_1}} - \frac{1}{2} - \frac{1}{\beta up}\right]\mathrm{e}^{-\mathcal{D}pt} \end{aligned} \tag{4.87}$$

Where $n_{preth}(p) = |\omega_p|^2$ is the level occupation at the prethermal plateau as computed earlier.

We can express the correlations in terms of the occupation numbers

$$g_p(t) = 1 + \frac{p(p-1)}{2\pi^2\rho}\int_0^\infty dk\ k^2 \left[\frac{E(k)}{\epsilon(k,\gamma_1)}(2n_k(t) - 1)\right] \tag{4.88}$$

The $f^0(p)$ part of $n_k(t)$ gives just the equilibrium component of $g_p(t)$, so we focus on the truly time-dependent part of the observable: we can approximae the dispersion relation with its phononic part and consider only the leading $O(p^{-1})$ contribution in $n_p(t)$. this gives:

$$\begin{aligned} g_p(t) - g_p(eq) &= \frac{p(p-1)}{2\pi^2\rho}\int_0^\infty dk\ k^2 \frac{E(k)}{\epsilon(k,\gamma_1)}\left[n_{preth}(k) - f^0(k)\right]\mathrm{e}^{D(k)t} \\ &\approx \frac{p(p-1)}{2\pi^2\rho}\int_0^\infty dk\ \frac{k^3}{2mu}\left(-\frac{1}{\beta uk}\right)\mathrm{e}^{-\mathcal{D}pt} \\ &= -\frac{p(p-1)}{2\pi^2}\frac{1}{m\rho\beta u^2}\frac{1}{\mathcal{D}^3 t^3} \\ &= -\frac{p(p-1)20^3}{54\pi^{29}}\frac{m^2\rho^2\beta^{11}u^{10}}{t^3} \end{aligned} \tag{4.89}$$

The result for $g_p(t)$ is a power law decay as t^{-3} to be compared with the t^{-4} one in the prethermalization stage.

In both cases, therefore, the dynamics follows a power law and this implies that the natural time scales we would extract from an exponential decay are not present in the system under examination. This absence of scale separation suggests a different scenario with respect to Ref. [8, 40], where a clear scale separation was met by the presence of an apparent prethermalization plateau; instead for our relaxation the absence of natural scales and the difference between the two power-law exponents induce us to propose that for threedimensional weakly interacting bosons prethermalization may appear as a crossover between two different stages of relaxation and not as a clearly defined plateau, but anyway an experiment may be able to discriminate them.

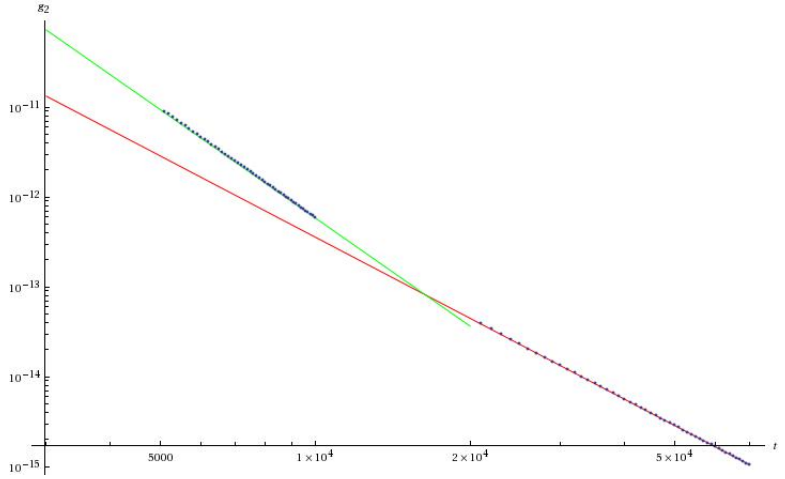


Figure 4.9: Logarithmic plot of $g_2(t)$ in the two relaxation stages. The lines line corresponds to Eqs. (4.13) and (4.89). The experiments may be able to discriminate the crossover between the two regimes.

Chapter 5

Conclusions

This PhD thesis deals with the fascinating subject of the dynamics of a system after it has been driven out of equilibrium, focusing the attention in particular on *prethermalization*. The idea that some observables may reach a non-thermal quasi-stationary state during the time evolution towards thermalization was proposed for the first time to describe some experimental evidences emerging from relativistic heavy-ions collisions but proved to be useful also in condensed matter physics to address some peculiar features of the relaxation process of cold atomic systems. Indeed many important results on this subject were produced in the last years, for example prethermalization was theoretically predicted for a Fermi–Hubbard system in more than one dimension after a sudden interaction switch-on, as confirmed by DMRG calculations, and was experimentally found for a split quasi-one-dimensional condensate.

An interesting candidate to look for prethermalization are those close to integrability; this is due to the fact that it was proved that this kind of system possess a set of "almost conserved" integrals of motion by means of which a Generalized Gibbs Ensemble can be built; the expectation values of many observables on this ensemble corresponds (up to third order in the integrability breaking parameter) to the value measured on the prethermal quasy-stationary state. Actually any experimental setup aimed at realizing an integrable model will necessarily lead to an almost integrable system since the presence of integrability breaking effects, however small, is unavoidable. For example also the famous "Quantum Newton's Cradle" setup will contain some term that spoils the integrability of the original model, even though on the timescale of the experiment its effect its not apparent and the system does not reach neither thermalization nor prethermalization. The threedimensional version of that model, instead, is expected to thermalize so quickly that any eventual presence of a two-stage process would be blurred by a single rapid dephasing. We may wonder, however, if this rapid energy redistribution between the modes can be avoided in order to let the richer features of the prethermalization appear. A possibility in this sense is a different way to bring the system out of equilibrium: instead of a split (which is expected to inject a lot of energy) one may perform a small quantum quench and change infinitesimally the interaction between the atoms. Following this idea, in this work we investigated the effect of a small interaction quench on a system of threedimensional weakly interacting bosons. Inspired by the recent exact calculation of two and

three-body correlation functions for one-dimensional systems, we chose to focus on these observables for our study of prethermalization; these observables are physically relevant since the local three-body correlation is proportional to the rate of three-body losses and non-local two-body correlation is related to the characteristic length scale of decay for density–density correlations: this choice proved to be correct.

In chapter 2 we introduced the Bogoliubov approximation and employed Bogoliubov rotation to compute the equilibrium behaviour of local and non-local correlation functions. While part of these results were known, in particular for one-dimensional systems,we generalized the to higher dimemsions and used them to characterize the state of the system after the quench. In chapter 3 we presented a brief explanation of the topics of thermalization and prethermalization and the main results obtained so far: we described the relationship between thermalization, ergodicity and integrability both in the classical and quantum world. Then we focused on prethermalization: we gave an overview of its meaning when it was originally introduced and of some recent results. In particular we showed how to construct prethermal expectation values from the almost conserved quantities of the model and described the first experiment in which a prethermalization plateau was found. The original part of the work is mainly contained in Chapter 4, where the consequences of a small interaction quench are computed in the Bogoliubov approximation. A stationary state for the local and non-local correlation functions was found, whose features led us to claim it is actually a prethermal state; we also computed the dynamic behaviour of $g_p(t)$: it follows a power law whose exponent is $D+1$, where D is the dimensionality of the system. As expected it was not possible to define an effective temperature for this state, instead the value of the correlation functions is well approximated by a GGE prediction. In the same chapter we showed that this result could be also obtained via Keldysh contour integration and how this field theory approach is a good starting point to generalise the present results, for example by introducing additional terms in the Hamiltonian like we did with an Hartree–Fock interaction. Since the system should eventually thermalize, we studied the evolution towards thermal equilibrium by introducing in the Hamiltonian an interaction term between the quasi-particles that allows two-body decay whose dynamics was computed via a linearized Boltzmann equation and is again described by a power-law behaviour, with the relevant difference that for this second stage $g_p(t) \sim t^{-3}$ so we concluded that in this system prethermalization might actually emerge as a crossover phenomenon rather than a well defined plateau.

The work presented here can be considered a starting point for a deeper analysis of the prethermalization behaviour of threedimensional bosons: for example one could try to employ the Keldysh technique to add higher order terms to the interaction Hamiltonian; indeed we worked only at the leading order, thanks to the smallness of the quench, but it may be interesting to look,

for example, for subleading effects in the behaviour of the condensate cloud and find a richer description than that given by the Gross–Pitaevskii equation. Antother possibility is the search for the presence of a second prethermal plateau in the case of a small quench but for strongly interacting bosons, as in the case of the Fermi–Hubbard model; in this case the Bogoliubov approximation can not be used anymore but one may still rely on Keldysh technique or computational methods like DMRG to predict the time evolution. From a more experimental perspective, a possible additional line of research may focus on the effect of this quench on the actual measurement of g_3, the three-body local correlation function. Since it is proportional to the rate of three body losses, it is usually extracted from experiments by fitting the number of atoms that leave the condensate as time goes on; if this quantity is itself a time-dependent variable we may wonder how can one measure it in an independent way.

Appendix A

A.1 Scattering length

The experimentally measurable quantities are the density ρ and the scattering length a, so we would like to express all our observables in terms of only these two observables. In this appendix we show how the scattering length for the Yukawa potential is computed,w e start from the Schrödinger equation: since the potential is central we can separate the radial part of the wave function $R(r)$ from the spherical harmonics. We define $\chi(r) = rR(r)$ and determine the equation for this reduced function

$$\nabla^2\chi(r) + \left[2m(E-V) - \frac{l(l+1)}{r^2}\right]\chi(r) = 0$$

If the potential is absent we have a similar equation for χ^0:

$$\nabla^2\chi^0(r) + \left[2mE - \frac{l(l+1)}{r^2}\right]\chi^0(r) = 0.$$

Multiplying the first by $\chi^0(r)$, the second by $\chi(r)$, subtracting both sides and integrating on r with the condition that $\chi(0) = 0$ we get

$$\nabla\chi(r)\chi^0(r) - \chi(r)\nabla\chi^0(r) = 2m\int_0^r \mathrm{d}r'\, V(r')\,\chi(r')\,\chi^0(r')$$

For $V = 0$, the differential equation for the radial part can be solved exactly for any value of l and E: $\chi_l^{(0)}(r) = 2rkj_l(kr) \approx 2\sin(kr - \pi l/2)$ where the last approximate identity is valid at great distance. When the potential is present, in the phase-shift formalism we write $\chi_l(r) \approx 2\sin(kr - \pi l/2 + \delta_l)$. On the LHS of the last equation we found we put the asymptotic expressions whereas in the RHS, in a Born-approximation perspective, we assume we can treat the potential as a perturbation so we set $\chi \approx \chi^{(0)}$ and we use the exact expression. In the end

$$\sin\delta_l = -2mk\int_0^\infty \mathrm{d}r\, V(r)\, r^2\, j_l^2(kr).$$

In particular for $l = 0$ for the Yukawa potential $V(r) = g\, r^{-1}\, \mathrm{e}^{-\mu r}$ we get

$$\begin{aligned}
\sin\delta_0 &= -2mkg \int_0^\infty \mathrm{d}r\, \frac{\mathrm{e}^{-\mu r}}{r} \left(\frac{\sin(kr)}{kr}\right)^2 r^2 \\
&= -\frac{2mg}{k} \int_0^\infty \mathrm{d}\lambda\, \frac{\mathrm{e}^{-\frac{\mu\lambda}{k}} \sin^2\lambda}{\lambda} \\
&= -\frac{mg}{2k}\ \ln\left(1 + 4\frac{k^2}{\mu^2}\right) \\
&\approx -\frac{2mkg}{\mu^2}\left(1 - 2\frac{k^2}{\mu^2}\right)
\end{aligned}$$

the scattering length a_s is defined as

$$-a_s^{-1} = \lim_{k\to 0} k \cot\delta_0.$$

and from the expression of $\sin\delta_0$ we get

$$a_s = \frac{2mg}{\mu^2}.$$

A.2 Unitary perturbation theory

In this appendix we set the notation and prove some identities that are necessary to understand the connection between prethermalization plateaus and GGE expectation values for nearly-integrable systems that we want to establish in Section 3.2. We start with the Hamiltonian of a system close to integrability

$$\hat{H} = \hat{H}_0 + g\hat{H}_1$$

where $\hat{H}_0$ is integrable and has a set $\{\hat{I}_\alpha\}_\alpha$ of constants of motion so that

$$\hat{H}_0 = \sum_\alpha \epsilon_\alpha \hat{I}_\alpha.$$

In addition we define an anti-Hermitian operator $\hat{S}$ that will generate the required canonical transformations and can be expanded in powers of g as

$$\hat{S} = g\,\hat{S}_1 + \frac{g^2}{2}\hat{S}_2 + O(g^3)$$

so that the canonical transformation of $\hat{H}$ is

$$\begin{aligned}
\hat{\tilde{H}} &:= \mathrm{e}^{\hat{S}}\hat{H}\mathrm{e}^{-\hat{S}} \\
&= \hat{H}_0 + g\,\hat{H}^{(1)} + g^2\hat{H}^{(2)} + O(g^3) \\
&= \hat{H}_0 + g\left(\hat{H}_1 + [\hat{S}_1, \hat{H}_0]\right) + g^2\left(\frac{1}{2}[\hat{S}_2, \hat{H}_0] + [\hat{S}_1, \hat{H}_1] + \frac{1}{2}[\hat{S}_1, [\hat{S}_1, \hat{H}_0]]\right) + O(g^3)
\end{aligned}$$

The transformed Hamiltonian should retain the $\hat{I}_\alpha$ as constants of motions, so we impose $[\mathrm{e}^{\hat{S}}\hat{H}\mathrm{e}^{-\hat{S}}, \hat{I}_\alpha] = 0$ at any order in g for all α to compute

$$\langle \mathbf{n}| \hat{S}_1 |\mathbf{m}\rangle = \frac{\langle \mathbf{n}| \hat{H}_1 |\mathbf{m}\rangle}{E_\mathbf{n} - E_\mathbf{m}}, \quad \langle \mathbf{n}| \hat{S}_2 |\mathbf{m}\rangle = \frac{\langle \mathbf{n}| [\hat{S}_1, \hat{H}_1 + \hat{H}^{(1)}] |\mathbf{m}\rangle}{E_\mathbf{n} - E_\mathbf{m}},$$
$$\hat{H}^{(i)} = \sum_\mathbf{n} |\mathbf{n}\rangle E^{(i)}_\mathbf{n} \langle \mathbf{n}|,$$
$$E^{(1)}_\mathbf{n} = \langle \mathbf{n}| \hat{H}_1 |\mathbf{n}\rangle, \quad E^{(2)}_\mathbf{n} = \sum_{\mathbf{m}\neq\mathbf{n}} \frac{|\langle \mathbf{m}| \hat{H}_1 |\mathbf{n}\rangle|^2}{E_\mathbf{n} - E_\mathbf{m}},$$

where $|\mathbf{n}\rangle$ are eigenstates of the integrals of motion with $\hat{I}_\alpha |\mathbf{n}\rangle = n_\alpha |\mathbf{n}\rangle$.

The time-evolution of an observable $\hat{A}$ that commutes with all the constants of motion can be perturbatively computed by means of this canonical transformation as

$$\begin{aligned}
\langle \hat{A}\rangle_t &= \langle \psi_0| \mathrm{e}^{\imath\hat{H}t}\hat{A}\mathrm{e}^{-\imath\hat{H}t} |\psi_0\rangle \\
&= \langle \psi_0| \mathrm{e}^{-\hat{S}}\mathrm{e}^{\hat{S}(t)}\hat{A}\mathrm{e}^{-\hat{S}(t)}\mathrm{e}^{\hat{S}} |\psi_0\rangle \\
&= \langle \psi_0| \hat{A} + [\hat{S}(t) - \hat{S}, \hat{A}] - \frac{1}{2}[\hat{S}, [2\hat{S}(t) - \hat{S}, \hat{A}]] + \frac{1}{2}[\hat{S}(t), [\hat{S}(t), \hat{A}]] |\psi_0\rangle + O(g^3) \\
&= -2 \langle \psi_0| \hat{S}\hat{A}\hat{S} |\psi_0\rangle + 2\Re \langle \psi_0| \hat{S}\hat{A}\hat{S}(t) |\psi_0\rangle + O(g^3)
\end{aligned}$$

where $\hat{S}(t) := \mathrm{e}^{\imath\tilde{\hat{H}}t}\hat{S}\mathrm{e}^{-\imath\tilde{\hat{H}}t}$ and the initial state $|\psi_0\rangle$ is an eigenstate of both $\hat{H}_0$ and $\hat{I}_\alpha$ and is annihilated by $\hat{A}$. The individual terms in the last row are computed expanding $\hat{S}$ in powers of g and inserting resolutions of the identity in order to give

$$\langle \hat{A}\rangle_t = \langle \hat{A}\rangle_0 + 4g^2 \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{J(\omega)}{\omega^2} \sin^2\left(\frac{\omega t}{2}\right) + O(g^3)$$

where

$$J(\omega) = \left\langle \hat{H}_1 \left(\hat{A} - \langle \hat{A}\rangle_0\right) \, \delta\Big(\hat{H}_0 - \langle \hat{H}_0\rangle - \omega\Big) \, \hat{H}_1 \right\rangle_0 .$$

Therefore the long-time average gives

$$\begin{aligned}
\langle \hat{A}\rangle_{preth} := \overline{\langle \hat{A}\rangle_t} &= \langle \hat{A}\rangle_0 + 2g^2 \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{J(\omega)}{\omega^2} + O(g^3) \\
&= 2\langle \hat{A}\rangle_{\widetilde{0}} - \langle \hat{A}\rangle_0 + O(g^3).
\end{aligned}$$

Let us move to the computation of the approximate constants of motion: they are just the transformed of the original ones under the canonical transformation defined above:

$$\begin{aligned}
\widetilde{I}_\alpha &= \mathrm{e}^{-\hat{S}}\hat{I}_\alpha\mathrm{e}^{\hat{S}} \\
&= \hat{I}_\alpha - [\hat{S}, \hat{I}_\alpha] + [\hat{S}, [\hat{S}, \hat{I}_\alpha]] + O(g^3),
\end{aligned}$$

and the full Hamiltonian can be written as

$$\hat{H} = \sum_{\alpha} \epsilon_\alpha \widetilde{I}_\alpha + \sum_{\widetilde{\mathbf{n}}} |\widetilde{\mathbf{n}}\rangle \left(g E^{(1)}_{\mathbf{n}} + g^2 E^{(2)}_{\mathbf{n}}\right) \langle\widetilde{\mathbf{n}}| + O(g^3)$$

where $|\widetilde{\mathbf{n}}\rangle = \mathrm{e}^{-\hat{S}} |\mathbf{n}\rangle$, $\hat{H}\,|\widetilde{\mathbf{n}}\rangle = \widetilde{E}_{\mathbf{n}}\,|\widetilde{\mathbf{n}}\rangle$ and $\widetilde{E}_{\mathbf{n}} = E_{\mathbf{n}} + g E^{(1)}_{\mathbf{n}} + g^2 E^{(2)}_{\mathbf{n}} + O(g^3)$. The statement that $\{\widetilde{I}_\alpha\}_\alpha$ are approximately conserved means that

$$\left[\hat{H}, \widetilde{I}_\alpha\right] = O(g^3) \quad \forall\alpha.$$

The corresponding GGE is

$$\hat{\rho}_{\widetilde{G}} = \frac{1}{\mathcal{Z}} \exp\left[-\sum_\alpha \lambda_\alpha \widetilde{I}_\alpha\right]$$

with the constrain

$$\langle \widetilde{I}_\alpha \rangle_{\widetilde{G}} := \mathrm{Tr}\left[\hat{\rho}_{\widetilde{G}} \widetilde{I}_\alpha\right] \stackrel{!}{=} \langle \widetilde{I}_\alpha \rangle_0.$$

The expectation value of an observable $\hat{A}$ on the GGE is

$$\begin{aligned}
\langle \hat{A} \rangle_{\widetilde{G}} &:= \frac{\mathrm{Tr}[\hat{A}\ \mathrm{e}^{-\sum_\alpha \lambda_\alpha \widetilde{I}_\alpha}]}{\mathrm{Tr}[\mathrm{e}^{-\sum_\alpha \lambda_\alpha \widetilde{I}_\alpha}]} \\
&= \frac{\mathrm{Tr}[\mathrm{e}^{-\hat{S}} \hat{A} \mathrm{e}^{-\hat{S}}\ \mathrm{e}^{-\sum_\alpha \lambda_\alpha \hat{I}_\alpha}]}{\mathrm{Tr}[\mathrm{e}^{-\sum_\alpha \lambda_\alpha \hat{I}_\alpha}]} \\
&= \langle \hat{A} + [\hat{S}, \hat{A}] + \frac{1}{2}[\hat{S}, [\hat{S}, \hat{A}]] \rangle_G
\end{aligned}$$

where the $\langle\cdot\rangle_G$ average is performed on a GGE with the original conserved quantities $\hat{I}_\alpha$ but with the λ_α still constrained as in the transformed average. Let us assume that the observable can be written as a product of some of the original conserved quantities,

$$\hat{A} = \prod_{i=1}^{n} \hat{I}_{\alpha_i},$$

Then the second term on the last row vanishes, whereas the other two give

$$\begin{aligned}
\langle \hat{A} \rangle_G = \langle \prod_{i=1}^{n} \hat{I}_{\alpha_i} \rangle_G &= \prod_{i=1}^{n} \langle \hat{I}_{\alpha_i} \rangle_G = \prod_{i=1}^{n} \langle \widetilde{I}_{\alpha_i} \rangle_{\widetilde{G}} \\
&= \prod_{i=1}^{n} \langle \widetilde{I}_{\alpha_i} \rangle_0 = \prod_{i=1}^{n} \langle \hat{I}_{\alpha_i} \rangle_{\widetilde{0}} + O(g^3) \\
\frac{1}{2}\langle [\hat{S}, [\hat{S}, \hat{A}]] \rangle_G = \frac{g^2}{2} \langle [\hat{S}_1, [\hat{S}_1, \hat{A}]] \rangle_0 + O(g^3) &= \langle \hat{A} \rangle_{\widetilde{0}} - \langle \hat{A} \rangle_0 + O(g^3) \\
&= \langle \prod_{i=1}^{n} \hat{I}_{\alpha_i} \rangle_{\widetilde{0}} - \prod_{i=1}^{n} \langle \hat{I}_{\alpha_i} \rangle_0 + O(g^3).
\end{aligned}$$

If $\langle \prod_{i=1}^{n} \hat{I}_{\alpha_i} \rangle_{\widetilde{0}} = \prod_{i=1}^{n} \langle \hat{I}_{\alpha_i} \rangle_{\widetilde{0}}$ then it follows that

$$\langle \hat{A} \rangle_{preth} = \langle \hat{A} \rangle_{\widetilde{G}} + O(g^3).$$

A.3 Keldysh contour

Field-theoretical methods proved to be very useful to solve problems in many-body physics, indeed they are a general and versatile tool to address questions concerning these systems; in particular they are ideal to investigate the emergence of collective phenomena, a subject otherwise difficult to approach. In this appendix we briefly explain the equilibrium situation before describing the true Keldysh formalism needed for non-equilibrium systems.

Let us say we have an Hamiltonian $\hat{H}$ which can be written as:

$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

where $\hat{H}_0$ is the free part and is exactly solvable and $\hat{H}_1$ is time independent and contains interaction terms. We call ψ the quantum field our theory deals with, assuming it's bosonic just for simplicity. We define the two-time Green function as the time-ordered two-point correlation function:

$$G(t,t') = -\imath \left\langle T\{\psi_{\hat{H}}(t)\psi^{\dagger}_{\hat{H}}(t')\} \right\rangle_0$$

where the $\langle . \rangle_0$ means we are averaging on ψ_0, the ground state of the full Hamiltonian. This expression can be recast in a more transparent form by moving to the interaction representation and defining the time evolution operator as:

$$U(t) = \mathrm{e}^{\imath \hat{H}_0 t} \mathrm{e}^{-\imath \hat{H} t}$$

and

$$S(t,t') = U(t)U^{\dagger}(t') = T\mathrm{e}^{-\imath \int_{t'}^{t} d\tau \hat{H}_1(\tau)}.$$

Performing the evolution and employing the properties of the evolution matrices we get

$$G(t,t') = -\imath \langle \psi_0 | \, T\{S(0,t)\psi_{\hat{H}_0}(t)S(t,t')\psi^{\dagger}_{\hat{H}_0}(t')S(t',0)\} \, |\psi_0\rangle \, .$$

In this representation the fields are evolved via the (simpler) free Hamiltonian instead of the more difficult complete one but we still do not know how to take averages on the ground state of the theory. To address this problem we employ the Gell-Mann–Low Theorem: it states that the ground state of the full theory at finite time can be constructed by evolving adiabatically the free ground state ϕ_0 from $t = -\infty$ and that if we continue to evolve it adiabatically

to $t = +\infty$ we get the original free ground state except for a phase factor. In formulas this means that

$$|\psi_0\rangle = S(0, -\infty)\,|\phi_0\rangle$$
$$\mathrm{e}^{\imath L}\,|\phi_0\rangle = S(+\infty, -\infty)\,|\phi_0\rangle\,.$$

Thanks to this theorem we can write our Green function as

$$G(t,t') = -\imath \frac{\langle\phi_0|\,T\{S(+\infty,-\infty)\psi_{\hat{H}_0}(t)\psi^\dagger_{\hat{H}_0}(t')\}\,|\phi_0\rangle}{\langle\phi_0|\,T\{S(+\infty,-\infty)\}\,|\phi_0\rangle}$$

where the denominator comes from the phase factor and cancels out the disconnected diagrams of the theory.

Unfortunately the second statement of the Gell-Mann–Low Theorem is not valid out of equilibrium: indeed if dissipation is present during the temporal evolution we are no more guaranteed to go back to the original ground state. Actually we can say nothing about the long-time asymptotic state and we should rely only on the knowledge of the initial one. The idea behind the Keldysh formalism is to avoid directly to evolve the system to $t = +\infty$ but, instead, build a time contour that starts at $t = -\infty$, runs up to a fixed time t_M (whose exact value is irrelevant) and then goes back to $t = -\infty$: this is known as the Keldysh contour and is shown in Figure A.1.

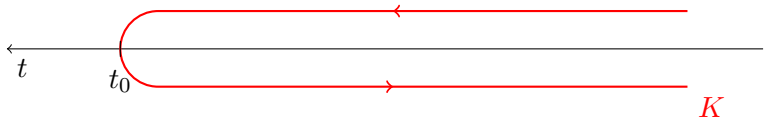


Figure A.1: Keldysh contour

On this contour the usual time ordering can still be defined, as well as the usual evolution operators: this allows us to construct averages in a way really similar to those we saw when working at equilibrium. To be clear, let us say that our original system is taken out-of-equilibrium because of the presence of a time dependent term in the Hamiltonian, namely

$$\hat{H} = \hat{H}_0 + \hat{H}_1 + \hat{H}'(t).$$

Again it is useful to move to interaction representation to define the evolution operator

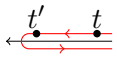
$$S^0_K = T_K\{\mathrm{e}^{-\imath \int_K d\tau (H_1 + H')^0(\tau)}\}$$

where the K subscript means we are integrating along the keldysh contour and the 0 superscript means the two Hamiltonian terms are defined in the

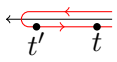
interaction representation. The expectation value of any observable on the initial state may be written as:

$$\left\langle \hat{O}(t) \right\rangle_0 = \frac{\langle \phi_0 |\, T_K\{S_K^0\ \hat{O}^0(t)\}\, |\phi_0\rangle}{\langle \phi_0 |\, T_K\{S_K^0\}\, |\phi_0\rangle}.$$

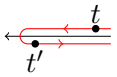
For observables with two time arguments (like the correlation functions we are interested in) we face the problem of the relative position of the two instants of time along the contour. Are they both on the upper or lower half? Or maybe one of them is above and the other below? This ambiguity is resolved by defining a set of four different Green functions



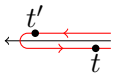
$$G^T(t,t') = -\imath\theta(t-t')\left\langle \psi(t)\psi^\dagger(t')\right\rangle + \imath\theta(t'-t)\left\langle \psi^\dagger(t')\psi(t)\right\rangle$$



$$G^{\widetilde{T}}(t,t') = -\imath\theta(t'-t)\left\langle \psi(t)\psi^\dagger(t')\right\rangle + \imath\theta(t-t')\left\langle \psi^\dagger(t')\psi(t)\right\rangle$$



$$G^<(t,t') = -\imath\left\langle \psi^\dagger(t')\psi(t)\right\rangle$$



$$G^>(t,t') = -\imath\left\langle \psi(t)\psi^\dagger(t')\right\rangle.$$

They are called the time-ordered, anti time-ordered, lesser and greater Green funcion respectively. At equibrium they are just the same function but this identity is no more valid in our non-equilibrium setting. Anyway they are not independent all since

$$G^T + G^{\widetilde{T}} = G^< + G^>.$$

Sometimes it is better to introduce the so-called advanced and retarded Green functions:

$$G^a(t,t') = -\imath\theta(t-t')\left\langle \left[\psi(t),\psi^\dagger(t')\right]\right\rangle = \theta(t-t')(G^<(t,t') - G^>(t,t'))$$
$$G^r(t,t') = \imath\theta(t'-t)\left\langle \left[\psi(t),\psi^\dagger(t')\right]\right\rangle = \theta(t-t')(G^<(t,t') - G^>(t,t'))$$

so that

$$G^r - G^a = G^> - G^<.$$

A.4 Evolution of the condensate density

In Section 4.1 correlators were computed under the assumption that the condensate density was constant across the quench and it was simply a parameter of the Hamiltonian. This assumption can be dropped and the computation can be repeated taking into account the variation of the condensate density after the quench: we will thus show that its effect is negligible for realistic values of

the parameters. Following [49] we can generalize the Bogoliubov rotation to a time-dependent scenario as

$$u_k(t), v_k(t) = \sqrt{\frac{1}{2}\left(\frac{E(k)+g_1\rho(t)}{\epsilon(k,t)} \pm 1\right)}$$

with a time dependent dispersion relation $\epsilon(k,t) = \sqrt{E(k)(E(k)+2g_1\rho(t))}$.

The time evolution for the rotated operators comes from the Heisenberg equation and gives

$$\frac{d\hat{c}_{\mathbf{k}}}{dt} = -\imath\epsilon_k(t)\hat{c}_{\mathbf{k}} + g_1\dot{\rho}(t)\frac{E(k)}{2\epsilon^2(k,t)}\hat{c}^\dagger_{-\mathbf{k}}.$$

Again $\hat{c}_{\mathbf{k}}$ and $\hat{c}^\dagger_{-\mathbf{k}}$ can be written in term of the $\hat{b}_{\mathbf{k}}$ via the coefficients χ_k and ω_k whose equations of motion are now

$$\frac{d\chi_k}{dt} = -\imath\epsilon(k,t)\chi_k - \frac{E(k)g_1}{2\epsilon^2(k,t)}\dot{\rho}(t)\omega_k$$
$$\frac{d\omega_k}{dt} = \imath\epsilon(k,t)\omega_k - \frac{E(k)g_1}{2\epsilon^2(k,t)}\dot{\rho}(t)\chi_k$$

In addition we have a self-consistency condition on ρ:

$$\dot{\rho}(t) = 2\rho(t)\frac{g_1}{L^D}\sum_{\mathbf{k}\neq 0}\Im\left[\chi^*_k(t)\omega_k(t)\right]$$

This set of equations allows a stationary state for $t \to \infty$. Indeed let us assume that $\dot{\rho} \approx 0$ then $\chi_k(t) \approx \chi^s_k \mathrm{e}^{-\imath\epsilon^s(k)t}$ and an analogous form for ω_k, where s labels the value in the stationary state. Substitution into the last equation of the set gives in the thermodinamic limit

$$\dot{\rho}(t) \propto \int \mathrm{d}k \frac{k^2}{2\pi^2}\Im\left[\chi^{s*}_k(t)\omega^s_k(t)\mathrm{e}^{2\imath\epsilon^s_k t}\right]$$

In the asymptotic limit this integral vanishes for well-behaved $k^2\chi^{s*}_k(t)\omega^s_k(t)$ due to destructive interference and this means that the stationary state characterized by $\dot{\rho}(t) = 0$ is compatible with the equations.

To obtain the stationary state values we proceed iteratively. First we compute $\dot{\rho}$ using the initial values for ω_k and χ_k, then we obtain a better form of these coefficients from the first approximation of the condensate density and in the end we get a second approximation for this quantity. All these steps need to be performed numerically, in particular we employed a IV-order Runge-Kutta algorithm to integrate the differential equations for the coefficients ω_k and χ_k. We did this for different initial states and for different values of the quench amplitude, obtaining that $\dot{\rho}$ decays to zero exponentially, with a time scale τ that is inversely proportional both to g_0 and to δg. If $\dot{\rho} = \alpha\mathrm{e}^{-t/\tau}$, then the

asymptotic value of the condensate density is $\rho(\infty) = \rho(0) - \alpha\tau$ so our previous assumption of constant condensate density is correct in the limit of the product $\alpha\tau$ much smaller than the initial density. In our numerical computation such a product appears to be several orders of magnitude smaller than $\rho(0)$.

Appendix B

Calculations for Chapter 2

B.1 Finite temperature behaviour at equilibrium

In 1D

$$
\begin{aligned}
g_2^{(T)}(x) &= \frac{2}{\rho\pi}\int_0^\infty \mathrm{d}k\, \frac{E(k)}{\epsilon(k)}\frac{1}{\mathrm{e}^{\frac{\epsilon(k)}{T}}-1}\cos(kx) \\
&= \frac{2}{\rho\pi}\int_0^\infty \mathrm{d}k\, \frac{k}{\sqrt{k^2+8\rho^2\gamma}}\left(\mathrm{e}^{\frac{T_d}{T}\frac{k}{\rho^2}\sqrt{k^2+8\rho^2\gamma}}-1\right)^{-1}\cos(kx) \\
&= \frac{2}{\rho\pi x}\int_0^\infty \mathrm{d}\lambda\, \frac{\lambda}{\sqrt{\lambda^2+8\rho^2x^2\gamma}}\left(\mathrm{e}^{\frac{T_d}{T}\frac{\lambda}{x^2\rho^2}\sqrt{\lambda^2+8\rho^2x^2\gamma}}-1\right)^{-1}\cos(\lambda) \\
&\approx \frac{1}{\pi\rho^2x^2\sqrt{2\gamma}}\int_0^\infty \mathrm{d}\lambda\,\lambda\left(\mathrm{e}^{\frac{T_d}{T}\frac{\lambda 2\sqrt{2\gamma}}{x\rho}}-1\right)^{-1}\cos(\lambda) \\
&= \frac{1}{\pi\rho^2x^2\sqrt{2\gamma}}\left[\frac{1}{2}-\frac{\pi^2x^2n^2}{16\gamma}\left(\frac{T}{T_d}\right)^2\left(\sinh(\frac{\pi x\rho T}{2\sqrt{2\gamma}T_d})\right)^{-2}\right] \\
&\approx \frac{1}{2\pi\rho^2x^2\sqrt{2\gamma}}-\frac{\pi}{4\sqrt{2}\gamma^{\frac{3}{2}}}\left(\frac{T}{T_d}\right)^2\exp\left[-\frac{\pi x\rho T}{\sqrt{2\gamma}T_d}\right].
\end{aligned}
$$

In 3D

$$
\begin{aligned}
g_2^{(T)}(x) &= \frac{1}{\rho\pi^2}\int_0^\infty \mathrm{d}k\; k^2\frac{E(k)}{\epsilon(k)}\frac{1}{\mathrm{e}^{\frac{\epsilon(k)}{T}}-1}\frac{\sin(kx)}{kx} \\
&\approx \frac{1}{\rho\pi^2}\frac{1}{2\rho^{\frac{4}{3}}x\sqrt{4\pi\gamma}}\left(\frac{Tm}{\sqrt{4\pi\gamma}}\right)^3\int_0^\infty \mathrm{d}k\,\frac{k^2}{\mathrm{e}^\lambda-1}\sin\left(\frac{Txm}{\rho^{\frac{1}{3}}\sqrt{4\pi\gamma}\lambda}\right) \\
&= \frac{1}{4\pi^{\frac{5}{2}}\rho^{\frac{4}{3}}\sqrt{\gamma}x^4}-\frac{1}{2^8\rho^{\frac{1}{3}}\pi x\gamma^2}\left(\frac{T}{T_d}\right)^3\coth\left(\frac{\rho^{\frac{1}{3}}\pi x}{2\sqrt{4\pi\gamma}}\frac{T}{T_d}\right)\sinh\left(\frac{\rho^{\frac{1}{3}}\pi x}{2\sqrt{4\pi\gamma}}\frac{T}{T_d}\right)^{-2} \\
&\approx \frac{1}{4\pi^{\frac{5}{2}}\rho^{\frac{4}{3}}\sqrt{\gamma}x^4}-\frac{1}{2^6\rho^{\frac{1}{3}}\pi x\gamma^2}\left(\frac{T}{T_d}\right)^3\exp\left(-\frac{\rho^{\frac{1}{3}}\pi x}{\sqrt{4\pi\gamma}}\frac{T}{T_d}\right).
\end{aligned}
$$

Appendix C

Calculations for Chapter 4

C.1 Bogoliubov form of the correlation functions

If the Bogoliubov approximation is performed, the form of the correlation functions that we want to compute is reduced to

$$\begin{aligned} g_p(t) &= \frac{1}{\rho^2} \langle\phi_0| \left(\hat{\psi}^\dagger(r,t)\right)^p \left(\hat{\psi}(r,t)\right)^p |\phi_0\rangle \\ &= 1 + \frac{p(p-1)}{\rho} \left[\langle\phi_0| \delta\hat{\psi}^\dagger(r,t)\delta\hat{\psi}(r,t) |\phi_0\rangle + \Re \langle\phi_0| \delta\hat{\psi}(r,t)\delta\hat{\psi}(r,t) |\phi_0\rangle\right] \end{aligned}$$

Let us compute each term separately

$$\begin{aligned} &\langle\phi_0| \delta\hat{\psi}^\dagger(r,t)\delta\hat{\psi}(r,t) |\phi_0\rangle = \langle\phi_0| \mathrm{e}^{\imath\hat{H}(\gamma_1)t}\ \delta\hat{\psi}^\dagger(r)\delta\hat{\psi}(r)\ \mathrm{e}^{-\imath\hat{H}(\gamma_1)t} |\phi_0\rangle \\ &= \frac{1}{L^D} \sum_{\mathbf{k},\mathbf{k}'} \langle\phi_0|\ \mathrm{e}^{\imath\hat{H}(\gamma_1)t}(u_k\hat{b}^\dagger_{\mathbf{k}} + v_k\hat{b}_{-\mathbf{k}})(u_{k'}\hat{b}_{\mathbf{k}'} + v_{k'}\hat{b}^\dagger_{-\mathbf{k}'})\mathrm{e}^{-\imath\hat{H}(\gamma_1)t} |\phi_0\rangle\ \mathrm{e}^{\imath(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}} \\ &= \frac{1}{L^D} \sum_{\mathbf{k},\mathbf{k}'} \langle\phi_0| (u_k\hat{b}^\dagger_{\mathbf{k}}\mathrm{e}^{\imath\epsilon(k,\gamma_1)t} + v_k\hat{b}_{-\mathbf{k}}\mathrm{e}^{-\imath\epsilon(k,\gamma_1)t})\times \\ &\quad \times (u_{k'}\hat{b}_{\mathbf{k}'}\mathrm{e}^{-\imath\epsilon(k',\gamma_1)t} + v_{k'}\hat{b}^\dagger_{-\mathbf{k}'}\mathrm{e}^{\imath\epsilon(k',\gamma_1)t}) |\phi_0\rangle\, \mathrm{e}^{\imath(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}} \\ &= \frac{1}{L^D} \sum_{\mathbf{k}} \Big[(u_k^2\chi_k^2 + v_k^2\omega_k^2) \langle\phi_0| \hat{c}^\dagger_{\mathbf{k}}\hat{c}_{\mathbf{k}} |\phi_0\rangle + (u_k^2\omega_k^2 + v_k^2\chi_k^2) \langle\phi_0| \hat{c}_{\mathbf{k}}\hat{c}^\dagger_{\mathbf{k}} |\phi_0\rangle + \\ &\quad + 2u_kv_k\chi_k\omega_k \cos(2\epsilon(k,\gamma_1)t) \langle\phi_0| \hat{c}_{\mathbf{k}}\hat{c}^\dagger_{\mathbf{k}} + \hat{c}^\dagger_{\mathbf{k}}\hat{c}_{\mathbf{k}} |\phi_0\rangle\Big] \\ &= \frac{1}{L^D} \sum_{\mathbf{k}} \big[\big((u_k^2 + v_k^2)(\chi_k^2 + \omega_k^2) + 4u_kv_k\chi_k\omega_k \cos(2\epsilon(k,\gamma_1)t)\big)\, n_k(\gamma_0) + \\ &\quad + u_k^2\omega_k^2 + v_k^2\chi_k^2 + 2u_kv_k\chi_k\omega_k \cos(2\epsilon(k,\gamma_1)t)\big] \end{aligned}$$

where

$$n_k(\gamma_0) = \langle\phi_0| \hat{c}^\dagger_{\mathbf{k}}\hat{c}_{\mathbf{k}} |\phi_0\rangle$$

is the occupancy of the $\mathbf{k}$ level in the initial state. We will usually quench the ground state of the initial Hamiltonian, i.e. the vacuum of the quasi-particles so all our calculations will be done for $n_k(\gamma_0) = 0$.

Similarly

$$\begin{aligned}
&\langle\phi_0|\,\delta\hat{\psi}(r,t)\delta\hat{\psi}(r,t)\,|\phi_0\rangle = \langle\phi_0|\,\mathrm{e}^{\imath\hat{H}(\gamma_1)t}\;\delta\hat{\psi}(r)\delta\hat{\psi}(r)\;\mathrm{e}^{-\imath\hat{H}(\gamma_1)t}\,|\phi_0\rangle\\
&= \frac{1}{L^D}\sum_{\mathbf{k},\mathbf{k}'}\langle\phi_0|\;\mathrm{e}^{\imath\hat{H}(\gamma_1)t}(u_k\hat{b}_{\mathbf{k}}+v_k\hat{b}^\dagger_{-\mathbf{k}})(u_{k'}\hat{b}_{\mathbf{k}'}+v_{k'}\hat{b}^\dagger_{-\mathbf{k}'})\mathrm{e}^{-\imath\hat{H}(\gamma_1)t}\,|\phi_0\rangle\;\;\mathrm{e}^{\imath(\mathbf{k}'+\mathbf{k})\cdot\mathbf{r}}\\
&= \frac{1}{L^D}\sum_{\mathbf{k},\mathbf{k}'}\langle\phi_0|\,(u_k\hat{b}_{\mathbf{k}}\mathrm{e}^{-\imath\epsilon(k,\gamma_1)t}+v_k\hat{b}^\dagger_{\mathbf{k}}\mathrm{e}^{-\imath\epsilon(k,\gamma_1)t})\times\\
&\quad\times(u_{k'}\hat{b}_{\mathbf{k}'}\mathrm{e}^{-\imath\epsilon(k',\gamma_1)t}+v_{k'}\hat{b}^\dagger_{-\mathbf{k}'}\mathrm{e}^{\imath\epsilon(k',\gamma_1)t})\,|\phi_0\rangle\,\mathrm{e}^{\imath(\mathbf{k}'+\mathbf{k})\cdot\mathbf{r}}\\
&= \frac{1}{L^D}\sum_{\mathbf{k}}\Big[\omega_k\chi_k(u_k^2\mathrm{e}^{-2\imath\epsilon(k,\gamma_1)t}+v_k^2\mathrm{e}^{2\imath\epsilon(k,\gamma_1)t})+u_kv_k(\chi_k^2+\omega_k^2)\Big]\,(2n_k(\gamma_0)+1).
\end{aligned}$$

From this we deduce:

$$\begin{aligned}
&\langle\phi_0|\,\delta\hat{\psi}^\dagger(r,t)\delta\hat{\psi}^\dagger(r,t)\,|\phi_0\rangle =\\
&= \frac{1}{L^D}\sum_{\mathbf{k}}\Big[\omega_k\chi_k(u_k^2\mathrm{e}^{2\imath\epsilon(k,\gamma_1)t}+v_k^2\mathrm{e}^{-2\imath\epsilon(k,\gamma_1)t})+u_kv_k(\chi_k^2+\omega_k^2)\Big]\,(2n_k(\gamma_0)+1).
\end{aligned}$$

In the end the expression for the correlation functions in the Bogoliubov approximation is

$$\begin{aligned}
g_p(t) = 1+\frac{p(p-1)}{\rho L^D}\sum_{\mathbf{k}}\big\{&u_kv_k(\chi_k^2+\omega_k^2)+u_k^2\omega_k^2+v_k^2\chi_k^2+\\
&+\chi_k\omega_k(u_k+v_k)^2\cos(2\epsilon(k,\gamma_1)t)+\\
&+(u_k+v_k)^2\left[(\chi_k^2+\omega_k^2)+2\chi_k\omega_k\cos(2\epsilon(k,\gamma_1)t)\right]n_k(\gamma_0)\big\}\,.
\end{aligned}$$

The first line contains the time independent term, which gives the asymptotic expectation after the quench; the second line is an oscillating term whose time average is zero. The third line contains both time dependent and independent terms but is different from zero only for a quench whose initial state has excited levels already populated. Since we do not work in this scenario we can disregard this lats term.

C.2 GGE calculations

At quadratic order in the $\delta\hat{\psi}$ operators the GGE prediction for g_2 is given by:

$$
\begin{aligned}
\langle \hat{g}_2 \rangle_{GGE} &= \mathrm{Tr}[\hat{g}_2 \hat{\rho}_{GGE}] = \frac{1}{\rho^2}\mathrm{Tr}[\hat{\psi}^\dagger\hat{\psi}^\dagger\hat{\psi}\hat{\psi}\hat{\rho}_{GGE}] \\
&\approx \frac{1}{\rho^2}\mathrm{Tr}[(\psi_0^4 + \psi_0^2(\delta\hat{\psi}\delta\hat{\psi} + \delta\hat{\psi}^\dagger\delta\hat{\psi}^\dagger + 4\delta\hat{\psi}^\dagger\delta\hat{\psi}))\hat{\rho}_{GGE}] \\
&= 1 + \frac{1}{\rho}\mathrm{Tr}[(\delta\hat{\psi}\delta\hat{\psi} + \delta\hat{\psi}^\dagger\delta\hat{\psi}^\dagger + 2\delta\hat{\psi}^\dagger\delta\hat{\psi})\hat{\rho}_{GGE}] \\
&= 1 + \frac{1}{\rho L^D}\sum_{\mathbf{k},\mathbf{p}} \mathrm{Tr}[(\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{p}} + 2\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{p}})\hat{\rho}_{GGE}] \\
&= 1 + \frac{1}{\mathcal{Z}}\frac{2}{\rho L^D}\sum_{\mathbf{k}}\sum_{\{\alpha\}} \langle\alpha|\,(u_k+v_k)^2 b^\dagger_{\mathbf{k}} b_{\mathbf{k}} + 2v_k(u_k+v_k)\,|\alpha\rangle \prod_{\mathbf{q}} \mathrm{e}^{-\lambda_q n_q} \\
&= 1 + \frac{1}{\mathcal{Z}}\frac{2}{\rho L^D}\sum_{\mathbf{k}} \big((u_k+v_k)^2(-\partial_{\lambda_k}) + v_k(u_k+v_k)\big)\prod_{\mathbf{q}} \mathrm{e}^{-\lambda_q n_q} \\
&= 1 + \frac{2}{\rho L^D}\sum_{\mathbf{k}} \big((u_k+v_k)^2\omega_k^2 + v_k(u_k+v_k)\big) \\
&= 1 + \frac{1}{\rho}\frac{\Omega_D}{(2\pi)^D}\int_0^\infty \mathrm{d}k k^{D-1}\left[E(k)\frac{\epsilon(k,\gamma_0)^2 + \epsilon(k,\gamma_1)^2}{2\;\epsilon(k,\gamma_0)\epsilon(k,\gamma_1)^2} - 1\right].
\end{aligned}
$$

At the next order we perform all possible contractions

$$
\left\langle \delta\hat{\psi}^\dagger\delta\hat{\psi}^\dagger\delta\hat{\psi}\delta\hat{\psi}\right\rangle_{GGE} \sim \left\langle \delta\hat{\psi}^\dagger\delta\hat{\psi}^\dagger\right\rangle_{GGE}\left\langle \delta\hat{\psi}\delta\hat{\psi}\right\rangle_{GGE} + 2\left\langle \delta\hat{\psi}^\dagger\delta\hat{\psi}\right\rangle^2_{GGE}
$$

The anomalous term is

$$
\begin{aligned}
\left\langle \delta\hat{\psi}^\dagger\delta\hat{\psi}^\dagger\right\rangle_{GGE} &= \frac{1}{L^D}\sum_{\mathbf{k},\mathbf{p}}\left\langle (u_k b^\dagger_{\mathbf{k}} + v_k b_{-\mathbf{k}})(u_p b^\dagger_{\mathbf{p}} + v_p b_{-\mathbf{p}})\right\rangle_{GGE} \\
&= \frac{1}{L^D}\sum_{\mathbf{k}} u_k v_k \left\langle 2\hat{n}_1(\mathbf{k}) + 1\right\rangle_{GGE} \\
&= \frac{1}{L^D}\sum_{\mathbf{k}} u_k v_k (2\omega_k^2 + 1)
\end{aligned}
$$

and since it's real we get an identical expression for $\left\langle \delta\hat{\psi}\delta\hat{\psi}\right\rangle_{GGE}$. For the conventional average instead:

$$
\begin{aligned}
\left\langle \delta\hat{\psi}^\dagger\delta\hat{\psi}\right\rangle_{GGE} &= \frac{1}{L^D}\sum_{\mathbf{k},\mathbf{p}}\left\langle (u_k b^\dagger_{\mathbf{k}} + v_k b_{-\mathbf{k}})(u_p b_{\mathbf{p}} + v_p b^\dagger_{-\mathbf{p}})\right\rangle_{GGE} \\
&= \frac{1}{L^D}\sum_{\mathbf{k}} \left[(u_k^2 + v_k^2)\left\langle \hat{n}_1(\mathbf{k})\right\rangle_{GGE} + v_k^2\right] \\
&= \frac{1}{L^D}\sum_{\mathbf{k}} \left[(u_k^2 + v_k^2)\omega_k^2 + v_k^2\right]
\end{aligned}
$$

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