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Solutions of Nonlocal Nonlinear Diffusion Equations in Data Filtering Problems

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Abstract. We explore methods of signal filtering using solutions of diffusion-type nonlinear and nonlocal model equations as filter kernels. Basic feature of the considered filtering is replacement of commonly used Gaussian filter by a filter based on solutions of diffusion-type equations.

INTRODUCTION

Data filtering belongs to one of important theoretical and practical problems in signal processing and data mining. One of the promising application is connected with chemometric analysis of spectral profile of breath air [1-3]. The well-known example provides the Gaussian filter (GF) which is the convolution of an input signal with the Gaussian kernel (filter). The normalized Gaussian kernel reads:

$$G(x,t|x_0) = (2\pi Dt)^{-\frac{n}{2}} exp\left(-\frac{(x-x_0)^2}{2Dt}\right), \quad t \in R, x, x_0 \in R^n, x = (x_1, \dots, x_n), x^2 = \sum_i x_i^2 \tag{1}$$

and the convolution of the input signal $f(x)$ and the kernel $G(x,t)$ is the output signal $F(x,t)$,

$$F(x,t) = G(x,t) * f(x) = \int G(x,y,t)\, f(y)\, dy \tag{2}$$

The Gaussian filters are widely used in image processing and in 1D signal processing because of their simplicity and some useful properties, e.g. their support in the time domain is equal to their support in the frequency domain. This comes about from the Gaussian being its own Fourier Transform. On the other hand, GFs have some essential disadvantages. One of them is that the GF does not have a sharp cutoff at some pass band frequency beyond which all higher frequencies are removed. It means that if a signal has edges or peaks as high frequency components, the GF will remove them, and this manifests itself as the edges/peaks becoming more 'smudged' together with noise component in the signal.

To overcome such disadvantages, a number of different modifications and generalizations of the GFs have been developed based on the idea that the Gaussian kernel is the Green function of the diffusion (heat) equation

$$\partial_t u(x,t) = \frac{D}{2} \Delta u(x,t)\, , \Delta = \sum_i \partial_{x_i x_i} \tag{3}$$

with the diffusion coefficient D.

In this paper we discuss possible future perspectives for diffusion filtering using results of [4, 5] for exact and approximate integration of nonlocal and nonlinear diffusion type equations.

The diffusion-induced filtering methods considered here are of interest in a variety of medical applications including analysis of MRI data (see the review paper [6]). Digital signal processing can involve linear or nonlinear operations. Nonlinear signal processing is closely related to nonlinear system identification and can be implemented in the time, frequency, and spatial-temporal domains.

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In recent paper [7] a novel method of non-local filtering has been proposed whose average weights are related to both the image FBP (filtered back projection) reconstructed from restored sinogram data and the image directly FBP reconstructed from noisy sinogram data.

THE ORNSTEIN-UHLENBECK PROCESS

Consider a generalization of the Gaussian filter using solutions of the Fokker-Planck equation for the Ornstein-Uhlenbeck process (FPOU equation). An input signal obtained experimentally, as a rule, does not have any specific properties convenient to extract the significant information contained in the signal. This information is usually represented as a feature vector of the object. The GF endows the input signal with the properties which have the kernel of the filter, in particular, the scale invariance [8].

Note that the Gaussian kernel (1), being the solution of the diffusion equation (3), inherits some of the symmetries of the equation. To study symmetry properties, effective analytic methods have been developed such as the group analysis [9-10]. This allows to construct filters starting from a modification of the diffusion equation (3) endowed with the appropriate symmetry properties and using then solutions of the modified equation as the filter kernels.

The Ornstein–Uhlenbeck process is described by the Fokker-Planck equation which in the 1D case is

$$\partial_t u(x,t) = \partial_x \left(\lambda\, x u(x,t) + \frac{D}{2} \partial_x u(x,t) \right) \tag{4}$$

where λ is the additional parameter compared to the diffusion equation (3). The Green function of (4) reads:

$$G(x,t|x_0,t_0) = \frac{\sqrt{\lambda}}{\sqrt{\pi D[1-exp(-2\lambda(t-t_0))]}} exp\left\{-\frac{\lambda[x-x_0 exp(-\lambda(t-t_0))]^2}{\pi D[1-exp(-2\lambda(t-t_0))]}\right\} \tag{5}$$

In [4] we explored the linear filter (2) with the kernel (5) as a component of the SIFT algorithm and stability of the SIFT algorithm was shown to increase with a special choice of the parameters.

Dependence of the Green function (5) on the parameters t and λ is more complex compared with the usual Gaussian function (1). We call the filter with the kernel (5) the Ornstein–Uhlenbeck filter (OU-filter). Applications of the GF and the OU-filter to different input signals show that results of filtering are similar. But, if we do the parameters in (5) dependent on the spatial coordinates, we get a new filter.

Note that since the variance indicates the degree of blur, a change in the scale parameter t in the Gaussian (1) can vary the amount of blur. To compare properties of (1) and (5) in more details, we present dependence of the variance of the kernels (1) and (5) on the parameter t for the Gaussian kernel (1) and on t and λ for the kernel (5) provided that $t \in [0,5]$; $\lambda \in [0,10]$; $x_0 = 0$; $D = 0.01$ (see Fig. 1).

If $x \in 1..n$ then variance of a function $F(x)$ can be calculated by the formula

$$.\, variance = \frac{\sum_x F(x)^2 - \frac{1}{n}(\sum_x F(x))^2}{n-1} \tag{6}$$

As an illustration, consider the OU-filter with the kernel (5) in which the parameter λ depends on the spatial coordinates. To do this, we put

$$,\qquad \lambda = |a * \cos(x_0)| \tag{7}$$

where $a = 4$; $t = 1$; $x_0 \in [-5,5]$; $D = 0.01$. The kernel (5), (7) takes the form shown in Fig. 2.

It can be seen that in the convolution of an input function with the kernel shown in Fig. 2, the filtered signal at a given point involves the values of the input signal not only in the neighbourhood of this point but also at more distant points. We illustrate kernels on Fig. 3: the Gaussian function (1) and the Green's function (5) (7) for $x_0 \in [-5,5]$, $D = 0.01$, $a = 4$, $t = 0.25$.

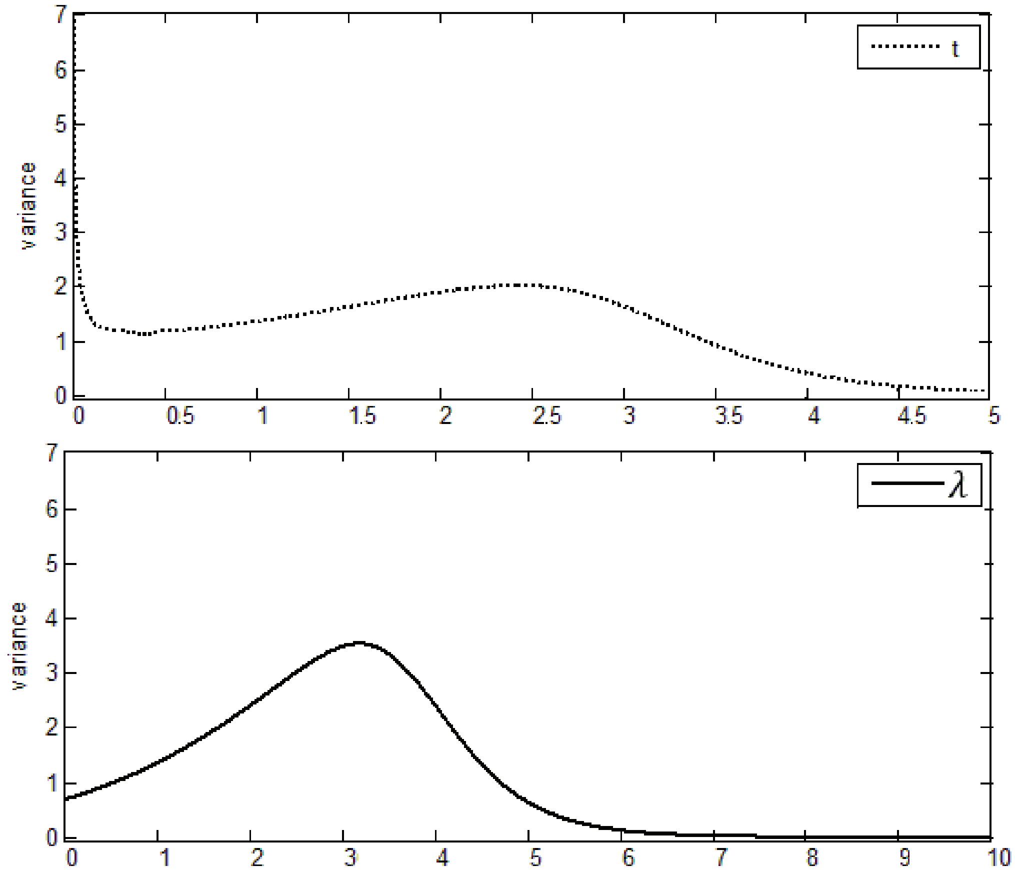

FIGURE 1. Dependence of the variance (6) of the Green function (5) on the parameters t (a) and λ (b)

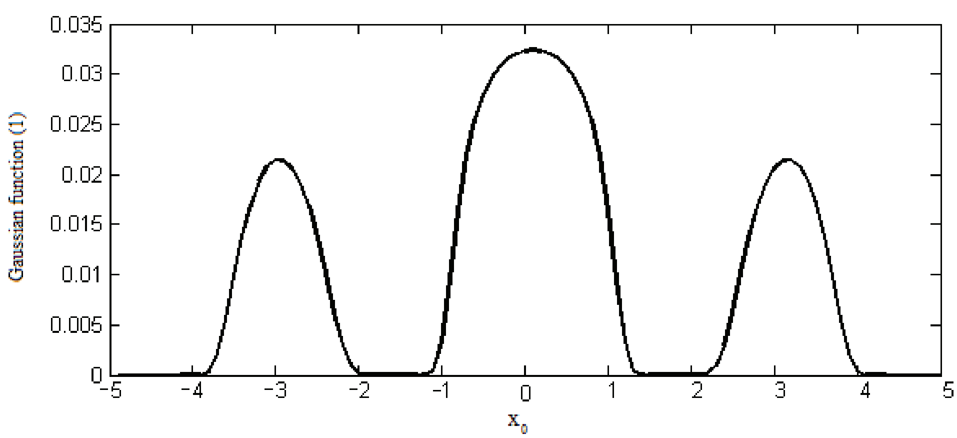

FIGURE 2. The function (5), (7) for $a = 4, t = 1, D = 0.01$ $x_0 \in [-5,5]$

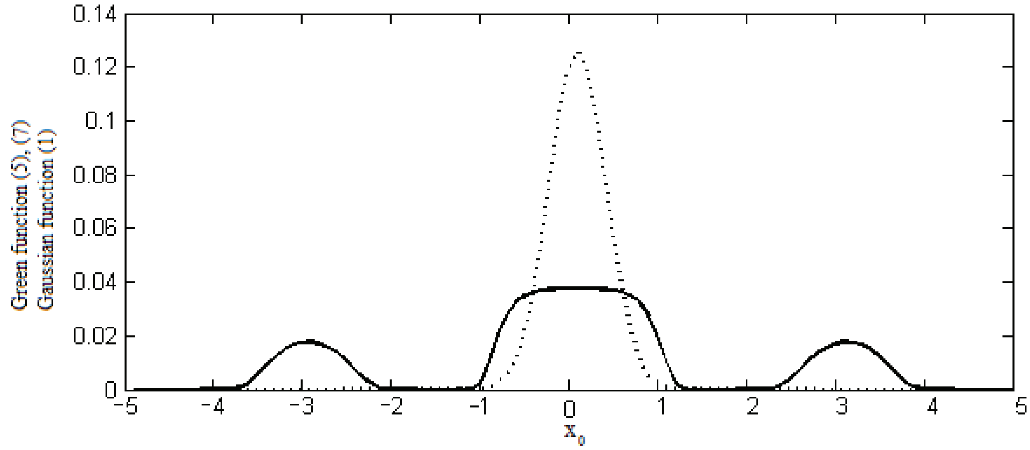

FIGURE 3. The Green function (5), (7) and the Gaussian function (1) for $x_0 \in [-5,5]$, $D = 0.01$, $a = 4$, t = 0.25

To compare the GF and the OU-filter with the kernel given by (5) and (7), we apply these filters to an input signal which is a model profile of an absorption spectrum of breath air for healthy people and for patients with chronic obstructive pulmonary disease (COPD). The spectrum profile had been obtained on the basis of HITRAN-2008 database data for the following conditions. Water vapor concentration in mixtures was 0.9 и 1 % regardless from disease presence. Carbon dioxide concentration was 4–6 % for healthy patients' probes and 2–4 % for patients with COPD probes. Methane concentration in all probes matches it's concentration in ambient air 1.7 ppm. NO was not include in gas probes because it's expected concentration is not more than 0.1ppm10, that matches contribution of 2 – 3% of summary contribution, caused by water vapor presence in whole studied spectral range (900 cm^{-1}–4000 cm^{-1}). CO and N2O concentrations match medians, maximal and minimal values for adult healthy patient and COPD ones in acute stage11. Besides, when calculating absorption spectrum of two probes of COPD patients an absorption was added, caused by ethane and hydrogen peroxide presence (20 ppm for both gases).

Results of filtering are shown in Fig. 4.

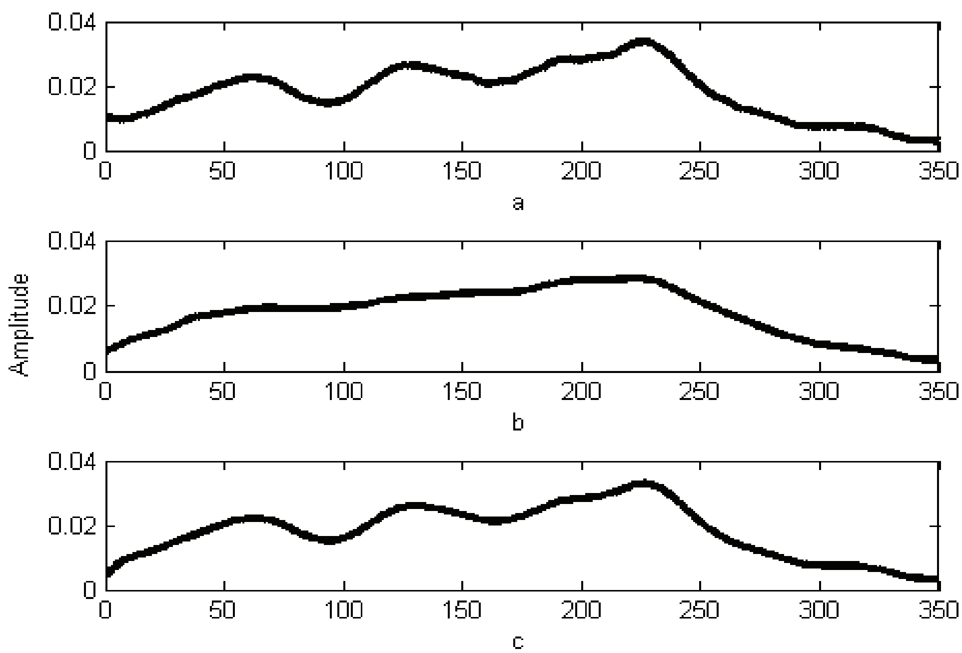

FIGURE 4. Filtering of the model profile of an absorption spectrum using GF and the OU-filter under the condition (7)

An input signal, the model profile of an absorption spectrum, is plotted in Fig. 4a. The input signal processed by the Gaussian filter is given in Fig. 4b; the input signal processed by the OU-filter with the condition (7) is plotted in Fig. 4c.

This example shows that even a small change in the Gaussian kernel can significantly affect the outcome. One example of such a filter can be the bilateral filter [11] the convolution of the input signal $f(x)$ and is the output signal $F(x,t)$:

$$F(x,t) = \int G(x,y,t)H\big(f(x),f(y)\big)f(y)dy \qquad (8)$$

where f is an input signal, G – gaussian, H – function depends on input signal. This filter has more complex structure as the kernel (8) depends on the input data.

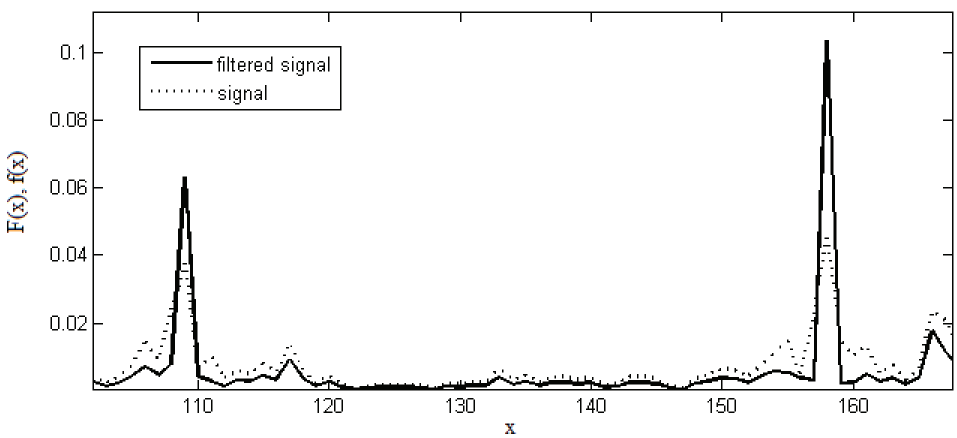

FIGURE 5. The input and filtered signal using a bilateral filter

Figure 5 depicts filtered signal and the original signal. This example shows that the bilateral filter allows identifying peaks, but filtering out unnecessary emissions.

SUMMARY

In this paper we consider possible methods of signal filtering based on diffusion-type equations, the Fokker-Plank equation for the Ornstein-Uhlenbeck process.

The Green function of the FPOU equation contains an additional parameter and it is more general then the Green function of the diffusion equation (the kernel of the GF). Therefore, the filter with the Green function of the FPOU as a kernel (the OU-filter) is the generalization of the GF. On the other hand, the OU-filter has small differences from the GF, respectively, filtering outputs obtained with the OU-filter and with the GF are similar. To get a new filter from the OU-filter, we can assume that the additional parameter in the Green function of the FPOU equation depends on the spatial variables. The example considered shows that such a modification of the OU-filter has some prospects. In particular, the diffusion filtering facilitates the transition to the bilateral filtering and other nonlinear and nonlocal filtering. Note also, that the FPOU equation possesses symmetries and the Green function inherits some symmetries of the equation. This feature allows endowing the output signal by certain symmetry properties.

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