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CHARACTERIZATIONS OF NON ASSOCIATIVE ORDERED SEMIGROUPS BY THE PROPERTIES OF THEIR FUZZY IDEALS WITH THRESHOLDS $(\alpha, \beta]$

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In this paper, we give the characterizations of regular (intra-regular, both regular and intra-regular) ordered AG-groupoids by the properties of fuzzy (left, right, quasi-, bi-, generalized bi-) ideals with thresholds $(\alpha, \beta]$.

Keywords: *fuzzy left (right, interior, quasi-, bi-, generalized bi-) ideals with thresholds $(\alpha, \beta]$, regular (intra-regular) ordered AG-groupoids.*

Introduction

In 1972, a generalization of commutative semigroups has been established by Kazim et al [1]. In ternary commutative law: $abc = cba$, they introduced the braces on the left side of this law and explored a new pseudo associative law, that is $(ab)c = (cb)a$. This law is called the left invertive law. A groupoid S is said to be a left almost semigroup (abbreviated as LA-semigroup) if it satisfies the left invertive law. This structure is also known as Abel — Grassmann's groupoid (abbreviated as AG-groupoid) [2]. An AG-groupoid is a midway structure between an abelian semigroup and a groupoid. Mushtaq et al [3] investigated the concept of ideals of AG-groupoids.

In [4] (resp. [5]), a groupoid S is said to be medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp. $(ab)(cd) = (db)(ca)$). In [1], an AG-groupoid is medial, but in general an AG-groupoid needs not to be paramedial. Every AG-groupoid with left identity is paramedial by Protic et al [2] and also satisfies $a(bc) = b(ac)$, $(ab)(cd) = (dc)(ba)$.

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and the like.

Although semigroups concentrate on theoretical aspects, they also include applications in error-correcting codes, control engineering, formal language, computer science and information science.

Algebraic structures especially ordered semigroups play a prominent role in mathematics with wide ranging applications in many disciplines such as control engineering, computer arithmetics, coding theory, sequential machines and formal languages.

In [6], if $(S, \cdot, \leqslant)$ is an ordered semigroup and $\varnothing \neq A \subseteq S$, we define a subset of S as follows: $(A] = \{s \in S : s \leqslant a \text{ for some } a \in A\}$. A non-empty subset A of S is called a subsemigroup of S if $A^2 \subseteq A$.

A non-empty subset A of S is called a left (resp. right) ideal of S if the following conditions hold: (1) $SA \subseteq A$ (resp. $AS \subseteq A$); (2) if $a \in A$ and $b \in S$ such that $b \leqslant a$ implies $b \in A$. Equivalent definition: A is called a left (resp. right) ideal of S if $(A] \subseteq A$ and $SA \subseteq A$ (resp. $AS \subseteq A$).

In [6, 7], an ordered semigroup S is said to be a regular if for every $a \in S$, there exists an element $x \in S$ such that $a \leqslant axa$. Equivalent definitions are as follows: (1) $A \subseteq (ASA]$

for every $A \subseteq S$; (2) $a \in (aSa]$ for every $a \in S$. In [7, 8], an ordered semigroup S is said to be an intra-regular if for every $a \in S$ there exist elements $x, y \in S$ such that $a \leqslant xa^2y$. Equivalent definitions are as follows: (1) $A \subseteq (SA^2S]$ for every $A \subseteq S$; (2) $a \in (Sa^2S]$ for every $a \in S$.

We will define the concept of fuzzy left (resp. right, interior, quasi-, bi-, generalized bi-) ideals with thresholds $(\alpha, \beta]$ of an ordered AG-groupoid S. We will establish a study by discussing the different properties of such ideals. We will also characterize regular (resp. intra-regular, both regular and intra-regular) ordered AG-groupoids by the properties of fuzzy left (right, quasi-, bi-, generalized bi-) ideals with thresholds $(\alpha, \beta]$.

1. Fuzzy ideals with thresholds $(\alpha, \beta]$ in ordered AG-groupoids

An ordered AG-groupoid S is a partially ordered set, at the same time an AG-groupoid such that $a \leqslant b$ implies $ac \leqslant bc$ and $ca \leqslant cb$ for all $a, b, c \in S$. Two conditions are equivalent to the one condition $(ca)d \leqslant (cb)d$ for all $a, b, c, d \in S$.

Let S be an ordered AG-groupoid and $\varnothing \neq A \subseteq S$, we define a subset $(A] = \{s \in S : s \leqslant a \text{ for some } a \in A\}$ of S, obviously $A \subseteq (A]$. If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. For $\varnothing \neq A, B \subseteq S$, then $AB = \{ab : a \in A, b \in B\}$, $((A]] = (A]$, $(A](B] \subseteq (AB]$, $((A](B]] = (AB]$, if $A \subseteq B$, then $(A] \subseteq (B]$, $(A \cap B] \neq (A] \cap (B]$ in general.

For $\varnothing \neq A \subseteq S$, A is called an ordered AG-subgroupoid of S if $A^2 \subseteq A$; A is called a left (resp. right) ideal of S if the following hold: (1) $SA \subseteq A$ (resp. $AS \subseteq A$); (2) if $a \in A$ and $b \in S$ such that $b \leqslant a$ implies $b \in A$. Equivalent definition: A is called a left (resp. right) ideal of S if $(A] \subseteq A$ and $SA \subseteq A$ (resp. $AS \subseteq A$). A is called an ideal of S if A is both a left ideal and a right ideal of S. In particular, if A and B are any types of ideals of S, then $(A \cap B] = (A] \cap (B]$.

We denote by $L(a), R(a), I(a)$ the left ideal, the right ideal and the ideal of S respectively, generated by a. We have $L(a) = \{s \in S : s \leqslant a \text{ or } s \leqslant xa \text{ for some } x \in S\} = (a \cup Sa]$, $S(a) = (a \cup aS]$, $I(a) = (a \cup Sa \cup aS \cup (Sa)S]$.

First time, Zadeh introduced the concept of fuzzy set in his classical paper [9] of 1965. This concept has provided a useful mathematical tool for describing the behavior of systems that are too complex to admit precise mathematical analysis by classical methods and tools. Extensive applications of fuzzy set theory have been found in various fields such as artificial intelligence, computer science, management science, expert systems, finite state machines, languages, robotics, coding theory and others.

Since then, many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, groupoids, semigroup, ordered semigroup, group theory, real analysis, measure theory, topology, etc.

Rosenfeld [10] was the first, who introduced the concept of fuzzy set in a group. The study of fuzzy set in semigroups was established by Kuroki [11, 12]. He studied fuzzy ideals and fuzzy interior (resp. quasi-, bi-, generalized bi-, semiprime, semiprime quasi-) ideals of semigroups. A systematic exposition of fuzzy semigroups appeared by Mordeson et al [13], where one can find the theoretical results on fuzzy semigroups and their use in fuzzy finite state machines and languages. Fuzzy sets in ordered semigroups/ordered groupoids were first explored by Kehayopulu et al [14, 15]. They also studied fuzzy ideals and fuzzy interior (resp. quasi-, bi-, generalized bi-) ideals in ordered semigroups.

By a fuzzy subset μ of an ordered AG-groupoid S, we mean a function $\mu : S \to [0, 1]$, the complement of μ is denoted by μ', is a fuzzy subset of S defined by $\mu'(x) = 1 - \mu(x)$ for all $x \in S$.

A fuzzy subset μ of S is called a fuzzy ordered AG-subgroupoid of S if $\mu(xy) \geqslant \mu(x) \wedge \mu(y)$ for all $x, y \in S$; μ is called a fuzzy left (resp. right) ideal of S if (1) $\mu(xy) \geqslant \mu(y)$ (resp. $\mu(xy) \geqslant \mu(x)$); (2) $x \leqslant y$ implies $\mu(x) \geqslant \mu(y)$ for all $x, y \in S$; μ is a fuzzy ideal of S if μ is both a fuzzy left and a fuzzy right ideal of S. Every fuzzy ideal (whether left, right, two-sided) is a fuzzy AG-subgroupoid of S but the converse is not true in general.

We denote by $F(S)$ the set of all fuzzy subsets of S. For $\varnothing \neq A \subseteq S$, the characteristic function of A is denoted by χ_A and defined by

$$\chi_A(a) = \begin{cases} 1, & \text{if } a \in A, \\ 0, & \text{if } a \notin A. \end{cases}$$

Let $\mu, \gamma \in F(S)$; by the symbols $\mu \wedge \gamma$ and $\mu \vee \gamma$ we mean the following fuzzy subsets:

$$(\mu \wedge \gamma)(x) = \min\{\mu(x), \gamma(x)\} \text{ and } (\mu \vee \gamma)(x) = \max\{\mu(x), \gamma(x)\}.$$

An ordered AG-groupoid S can be considered a fuzzy subset of itself and we write $S = \chi_S$, i.e., $S(x) = \chi_S(x) = 1$ for all $x \in S$. This implies that $S(x) = 1$ for all $x \in S$.

Let $x \in S$, we define a set $A_x = \{(y, z) \in S \times S : x \leqslant yz\}$. Let μ and γ be two fuzzy subsets of S, then product of μ and γ is denoted by $\mu \circ \gamma$ and defined by:

$$(\mu \circ \gamma)(x) = \begin{cases} \bigvee\limits_{(y,z) \in A_x} \min\{\mu(y), \gamma(z)\}, & \text{if } A_x \neq \varnothing, \\ 0, & \text{if } A_x = \varnothing. \end{cases}$$

A fuzzy subset μ of S is called a fuzzy interior ideal of S if: (1) $\mu((xy)z) \geqslant \mu(y)$; (2) $x \leqslant y$ implies $\mu(x) \geqslant \mu(y)$ for all $x, y, z \in S$. A fuzzy subset μ of S is called a fuzzy quasi-ideal of S if: (1) $(\mu \circ S) \cap (S \circ \mu) \subseteq \mu$; (2) $x \leqslant y$ implies $\mu(x) \geqslant \mu(y)$ for all $x, y \in S$. A fuzzy AG-subgroupoid μ of S is called a fuzzy bi-ideal of S if: (1) $\mu((xa)y) \geqslant \min\{\mu(x), \mu(y)\}$; (2) $x \leqslant y$ implies $\mu(x) \geqslant \mu(y)$ for all $x, a, y \in S$. A fuzzy subset μ of S is called a fuzzy generalized bi-ideal of S if: (1) $\mu((xa)y) \geqslant \min\{\mu(x), \mu(y)\}$; (2) $x \leqslant y$ implies $\mu(x) \geqslant \mu(y)$ for all $x, a, y \in S$.

Every fuzzy bi-ideal of S is a fuzzy generalized bi-ideal of S, but the converse is not true. A fuzzy ideal μ of S is called a fuzzy idempotent of S if $\mu \circ \mu = \mu$.

Now we define fuzzy ordered AG-subgroupoid with thresholds $(\alpha, \beta]$ and fuzzy left (resp. right, interior, quasi-, bi-, generalized bi-) ideals with thresholds $(\alpha, \beta]$ of an ordered AG-groupoid S.

A fuzzy subset μ of an ordered AG-groupoid S is called a fuzzy ordered AG-subgroupoid with thresholds $(\alpha, \beta]$ of S if $\max\{\mu(xy), \alpha\} \geqslant \min\{\mu(x), \mu(y), \beta\}$. A fuzzy subset μ of S is called a fuzzy left ideal with thresholds $(\alpha, \beta]$ of S if: (1) $\max\{\mu(xy), \alpha\} \geqslant \min\{\mu(y), \beta\}$; (2) $x \leqslant y$ implies $\mu(x) \geqslant \mu(y)$ for all $x, y \in S$ and $\alpha, \beta \in (0, 1]$ such that $\alpha < \beta$. A fuzzy subset μ of S is called a fuzzy right ideal with thresholds $(\alpha, \beta]$ of S if: (1) $\max\{\mu(xy), \alpha\} \geqslant \min\{\mu(x), \beta\}$; (2) $x \leqslant y$ implies $\mu(x) \geqslant \mu(y)$ for all $x, y \in S$ and $\alpha, \beta \in (0, 1]$ such that $\alpha < \beta$.

A fuzzy subset μ of S is called a fuzzy ideal with thresholds $(\alpha, \beta]$ of S if it is both a fuzzy left and a fuzzy right ideal with thresholds $(\alpha, \beta]$. Every fuzzy ideal (whether left, right, two-sided) with thresholds $(\alpha, \beta]$ is a fuzzy ordered AG-subgroupoid of S but the converse is not true in general.

A fuzzy subset μ of S is called a fuzzy interior ideal with thresholds $(\alpha, \beta]$ of S if: (1) $\max\{\mu((xy)z), \alpha\} \geqslant \min\{\mu(y), \beta\}$; (2) $x \leqslant y$ implies $\mu(x) \geqslant \mu(y)$ for all $x, y, z \in S$

and $\alpha, \beta \in (0,1]$ such that $\alpha < \beta$. A fuzzy subset μ of S is called a fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S if: (1) $\max\{\mu(x), \alpha\} \geqslant \min\{(\mu \circ S)(x), (S \circ \mu)(x), \beta\}$; (2) $x \leqslant y$ implies $\mu(x) \geqslant \mu(y)$ for all $x, y \in S$ and $\alpha, \beta \in (0,1]$ such that $\alpha < \beta$.

A fuzzy ordered AG-subgroupoid μ with thresholds $(\alpha, \beta]$ of S is called a fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of S if: (1) $\max\{\mu((xy)z), \alpha\} \geqslant \min\{\mu_A(x), \mu_A(z), \beta\}$; (2) $x \leqslant y$ implies $\mu(x) \geqslant \mu(y)$ for all $x, y, z \in S$ and $\alpha, \beta \in (0,1]$ such that $\alpha < \beta$.

A fuzzy subset μ of S is called a fuzzy generalized bi-ideal with thresholds $(\alpha, \beta]$ of S if: (1) $\max\{\mu((xy)z), \alpha\} \geqslant \min\{\mu(x), \mu(z), \beta\}$; (2) $x \leqslant y$ implies $\mu(x) \geqslant \mu(y)$ for all $x, y, z \in S$ and $\alpha, \beta \in (0,1]$ such that $\alpha < \beta$.

Every fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of S is a fuzzy generalized bi-ideal with thresholds $(\alpha, \beta]$ of S, but the converse is not true.

Let μ be a fuzzy set of an ordered AG-groupoid S and $\alpha, \beta \in (0,1]$ such that $\alpha < \beta$. We define a fuzzy subset μ_α^β of S as follow: $\mu_\alpha^\beta(x) = (\mu_A(x) \wedge \beta) \vee \alpha$ for all $x \in S$.

Let μ and γ be two fuzzy subsets of an ordered AG-groupoid S. We define fuzzy sets $\mu \wedge_\alpha^\beta \gamma$, $\mu \vee_\alpha^\beta \gamma$, $\mu \circ_\alpha^\beta \gamma$ and $\mu -_\alpha^\beta \gamma$ of S as follows:

$$(\mu_A \wedge_\alpha^\beta \mu_B)(x) = \{(\mu_A \wedge \mu_B)(x) \wedge \beta\} \vee \alpha,$$
$$(\mu_A \vee_\alpha^\beta \mu_B)(x) = \{(\mu_A \vee \mu_B)(x) \wedge \beta\} \vee \alpha,$$
$$(\mu_A \circ_\alpha^\beta \mu_B)(x) = \{(\mu_A \circ \mu_B)(x) \wedge \beta\} \vee \alpha,$$
$$(\mu_A -_\alpha^\beta \mu_B)(x) = \{(\mu_A - \mu_B)(x) \wedge \beta\} \vee \alpha$$

for all $x \in S$. Now we are giving the central properties of such ideals of an S, which will be very helpful for further sections.

Lemma 1. Let S be an ordered AG-groupoid. Then the following properties hold:
(1) $(\mu \circ_\alpha^\beta \gamma) \circ_\alpha^\beta \delta = (\delta \circ_\alpha^\beta \gamma) \circ_\alpha^\beta \mu$;
(2) $(\mu \circ_\alpha^\beta \gamma) \circ_\alpha^\beta (\delta \circ_\alpha^\beta \lambda) = (\mu \circ_\alpha^\beta \delta) \circ_\alpha^\beta (\gamma \circ_\alpha^\beta \lambda)$
for all fuzzy subsets μ, γ, δ and λ of S.

Proof. Let μ, γ and δ be fuzzy subsets of an ordered AG-groupoid S. We have to show that $(\mu \circ_\alpha^\beta \gamma) \circ_\alpha^\beta \delta = (\delta \circ_\alpha^\beta \gamma) \circ_\alpha^\beta \mu$. Now

$$((\mu \circ_\alpha^\beta \gamma) \circ_\alpha^\beta \delta)(x) = \{((\mu \circ \gamma) \circ \delta)(x) \wedge \beta\} \vee \alpha = \{((\delta \circ \gamma) \circ \mu)(x) \wedge \beta\} \vee \alpha = = ((\delta \circ_\alpha^\beta \gamma) \circ_\alpha^\beta \mu)(x).$$

In same lines, we can prove (2). ■

Proposition 1. Let S be an ordered AG-groupoid with left identity e. Then the following assertions hold:
(1) $\mu \circ_\alpha^\beta (\gamma \circ_\alpha^\beta \delta) = \gamma \circ_\alpha^\beta (\mu \circ_\alpha^\beta \delta)$;
(2) $(\mu \circ_\alpha^\beta \gamma) \circ_\alpha^\beta (\delta \circ_\alpha^\beta \lambda) = (\lambda \circ_\alpha^\beta \gamma) \circ_\alpha^\beta (\delta \circ_\alpha^\beta \mu)$;
(3) $(\mu \circ_\alpha^\beta \gamma) \circ_\alpha^\beta (\delta \circ_\alpha^\beta \lambda) = (\lambda \circ_\alpha^\beta \delta) \circ_\alpha^\beta (\gamma \circ_\alpha^\beta \mu)$
for all fuzzy subsets μ, γ, δ and λ of S.

Proof. Same as Lemma 1. ■

Theorem 1. Let A and B be two non-empty subsets of an ordered AG-groupoid S. Then the following conditions hold:
(1) $\chi_A \circ_\alpha^\beta \chi_B = (\chi_{(AB]})_\alpha^\beta$;
(2) $\chi_A \vee_\alpha^\beta \chi_B = (\chi_{A \cup B})_\alpha^\beta$;
(3) $\chi_A \wedge_\alpha^\beta \chi_B = (\chi_{A \cap B})_\alpha^\beta$.

Proof. Straight forward. ■

Theorem 2. Let A be a non-empty subset of an ordered AG-groupoid S. Then the following properties hold:

(1) A is an AG-subgroupoid of S if and only if χ_A is a fuzzy AG-subgroupoid with thresholds $(\alpha, \beta]$ of S;

(2) A is a left (resp. right, two-sided) ideal of S if and only if χ_A is a fuzzy left (resp. right, two-sided) ideal with thresholds $(\alpha, \beta]$ of S.

Proof.

(1) Let A be an AG-subgroupoid of an ordered AG-groupoid S and $x, y \in S$. If $x, y \notin A$, then by definition of characteristic function $\chi_A(x) = 0 = \chi_A(y)$. Thus

$$\chi_A(xy) \geqslant \min\{\chi_A(x), \chi_A(y)\} = \min\{\chi_A(x), \chi_A(y), \beta\} \Rightarrow$$
$$\Rightarrow \chi_A(xy) \geqslant \min\{\chi_A(x), \chi_A(y), \beta\} \Rightarrow \max\{\chi_A(xy), \alpha\} \geqslant \min\{\chi_A(x), \chi_A(y), \beta\}.$$

In same lines, we have $\max\{\chi_A(xy), \alpha\} \geqslant \min\{\chi_A(x), \chi_A(y), \beta\}$, when $x, y \in A$. Hence the characteristic function χ_A of A is a fuzzy AG-subgroupoid with thresholds $(\alpha, \beta]$of S.

Conversely, suppose that the characteristic function $\chi_A$of A is a fuzzy AG-subgroupoid with thresholds $(\alpha, \beta]$ of an ordered AG-groupoid S. Let $x, y \in A$, then by definition $\chi_A(x) = 1 = \chi_A(y)$. Since $\max\{\chi_A(xy), \alpha\} \geqslant \min\{\chi_A(x), \chi_A(y), \beta\} = \beta$, χ_A being a fuzzy AG-subgroupoid with thresholds $(\alpha, \beta]$ of S. Thus $\max\{\chi_A(xy), \alpha\} \geqslant \beta$, this implies that $\chi_A(xy) = 1$, i.e., $xy \in A$. Hence A is an AG-subgroupoid of S.

(2) Let A be a left ideal of an ordered AG-groupoid S and $x, y \in S$. If $y \notin A$, then by definition of characteristic function $\chi_A(y) = 0$. Thus

$$\chi_A(xy) \geqslant \chi_A(y) = \min\{\chi_A(y), \beta\} \Rightarrow$$
$$\Rightarrow \chi_A(xy) \geqslant \min\{\chi_A(y), \beta\} \Rightarrow \max\{\chi_A(xy), \alpha\} \geqslant \min\{\chi_A(y), \beta\}.$$

Similarly, we have $\max\{\chi_A(xy), \alpha\} \geqslant \min\{\chi_A(y), \beta\}$, when $y \in A$. Therefore the characteristic function χ_A of A is a fuzzy left ideal with thresholds $(\alpha, \beta]$ of S.

Conversely, assume that the characteristic function $\chi_A$of A is a fuzzy left ideal with thresholds $(\alpha, \beta]$ of an ordered AG-groupoid S. Let $y \in A$and $z \in S$, then by definition $\chi_A(y) = 1$.Since $\max\{\chi_A(zy), \alpha\} \geqslant \min\{\chi_A(y), \beta\} = \beta$, χ_A being a fuzzy left ideal with thresholds $(\alpha, \beta]$ of S. Thus $\max\{\chi_A(zy), \alpha\} \geqslant \beta$. This implies that $\chi_A(zy) = 1$, i.e., $zy \in A$. Therefore A is a left ideal of S. ■

Theorem 3. Let μ be a fuzzy subset of an ordered AG-groupoid S. Then the following assertions hold:

(1) μ is a fuzzy ordered AG-subgroupoid with thresholds $(\alpha, \beta]$ of S if and only if $\mu \circ_\alpha^\beta \mu \subseteq \mu_\alpha^\beta$;

(2) μ is a fuzzy left (resp. right) ideal with thresholds $(\alpha, \beta]$ of S if and only if $S \circ_\alpha^\beta \mu \subseteq \mu_\alpha^\beta$ (resp. $\mu \circ_\alpha^\beta S \subseteq \mu_\alpha^\beta$).

Proof.

(1) Suppose that μ is a fuzzy AG-subgroupoid with thresholds $(\alpha, \beta]$ of an ordered AG-groupoid S and $x \in S$. If $(\mu \circ_\alpha^\beta \mu)(x) = 0$, then obvious $\mu \circ_\alpha^\beta \mu \subseteq \mu_\alpha^\beta$, otherwise we have

$$(\mu \circ_\alpha^\beta \mu)(x) = \{(\mu \circ \mu)(x) \wedge \beta\} \vee \alpha = \Big\{\Big(\bigvee_{(y,z) \in A_x} \min\{\mu(y), \mu(z)\}\Big) \wedge \beta\Big\} \vee \alpha \leqslant$$
$$\leqslant \Big\{\bigvee_{(y,z) \in A_x} \min\{\mu(yz)\} \wedge \beta\Big\} \vee \alpha = \{\mu(x) \wedge \beta\} \vee \alpha = \mu_\alpha^\beta(x) \Rightarrow \mu \circ_\alpha^\beta \mu \subseteq \mu_\alpha^\beta.$$

Conversely, assume that $\mu \circ_\alpha^\beta \mu \subseteq \mu_\alpha^\beta$. Let $x, y \in S$ such that $a \leqslant xy$. Now

$$\max\{\mu(xy), \alpha\} \geqslant \max\{\mu(a), \alpha\} = \max\{\min\{\mu(a), \beta\}, \alpha\} = \mu_\alpha^\beta(a) \geqslant (\mu \circ_\alpha^\beta \mu)(a) =$$
$$= \{(\mu \circ \mu)(a) \wedge \beta\} \vee \alpha = \Big\{\Big(\bigvee_{(s,t)\in A_a} \min\{\mu(s), \mu(t)\}\Big) \wedge \beta\Big\} \vee \alpha \geqslant$$
$$\geqslant \{\mu(x) \wedge \mu(y) \wedge \beta\} \vee \alpha = \mu(x) \wedge \mu(y) \wedge \beta = \min\{\mu(x), \mu(y), \beta\} \Rightarrow$$
$$\Rightarrow \max\{\mu(xy), \alpha\} \geqslant \min\{\mu(x), \mu(y), \beta\}.$$

Hence μ is a fuzzy AG-subgroupoid with thresholds $(\alpha, \beta]$ of S.

(2) Suppose that μ is a fuzzy left ideal with thresholds $(\alpha, \beta]$ of an ordered AG-groupoid S and $x \in S$. If $(S \circ_\alpha^\beta \mu)(x) = 0$, then obvious $S \circ_\alpha^\beta \mu \subseteq \mu_\alpha^\beta$, otherwise we have

$$(S \circ_\alpha^\beta \mu)(x) = \{(S \circ \mu)(x) \wedge \beta\} \vee \alpha = \Big\{\Big(\bigvee_{(y,z)\in A_x} \min\{S(y), \mu(z)\}\Big) \wedge \beta\Big\} \vee \alpha =$$
$$= \Big\{\Big(\bigvee_{(y,z)\in A_x} \min\{1, \mu(z)\}\Big) \wedge \beta\Big\} \vee \alpha = \Big\{\Big(\bigvee_{(y,z)\in A_x} \min\{\mu(z)\}\Big) \wedge \beta\Big\} \vee \alpha \leqslant$$
$$\leqslant \Big\{\Big(\bigvee_{(y,z)\in A_x} \min\{\mu(yz)\}\Big) \wedge \beta\Big\} \vee \alpha = (\mu(x) \wedge \beta) \vee \alpha = \mu_\alpha^\beta(x) \Rightarrow S \circ_\alpha^\beta \mu \subseteq \mu_\alpha^\beta.$$

Conversely, assume that $S \circ_\alpha^\beta \mu \subseteq \mu_\alpha^\beta$. Let $y, z \in S$ such that $x \leqslant yz$. Now

$$\max\{\mu(yz), \alpha\} \geqslant \max\{\mu(x), \alpha\} = \max\{\min\{\mu(x), \beta\}, \alpha\} = \mu_\alpha^\beta(x) \geqslant (S \circ_\alpha^\beta \mu)(x) =$$
$$= \{(S \circ \mu)(x) \wedge \beta\} \vee \alpha = \Big\{\Big(\bigvee_{(s,t)\in A_x} \min\{S(s), \mu(t)\}\Big) \wedge \beta\Big\} \vee \alpha \geqslant ((S(y) \wedge \mu(z)) \wedge \beta) \vee \alpha =$$
$$= (1 \wedge \mu(z)) \wedge \beta = \min\{\mu(z), \beta\} \Rightarrow \max\{\mu(yz), \alpha\} \geqslant \min\{\mu(z), \beta\}.$$

Therefore μ is a fuzzy left ideal with thresholds $(\alpha, \beta]$ of S. ■

Lemma 2. If μ and γ are two fuzzy AG-subgroupoid (resp. (left, right, two-sided) ideals) with thresholds $(\alpha, \beta]$ of an ordered AG-groupoid S, then $\mu \wedge_\alpha^\beta \gamma$ is also a fuzzy AG-subgroupoid (resp. (left, right, two-sided) ideal) with thresholds $(\alpha, \beta]$ of S.

Proof. Let μ and γ be two fuzzy AG-subgroupoids with thresholds $(\alpha, \beta]$ of an ordered AG-groupoid S. We have to show that $\mu \wedge_\alpha^\beta \gamma$ is also a fuzzy AG-subgroupoid with thresholds $(\alpha, \beta]$ of S. Now

$$\max\{(\mu \wedge_\alpha^\beta \gamma)(xy), \alpha\} = \max\{\{\{(\mu \wedge \gamma)(xy) \wedge \beta\} \vee \alpha\}, \alpha\} = \{(\mu \wedge \gamma)(xy) \wedge \beta\} \vee \alpha =$$
$$= \{\mu(xy) \wedge \gamma(xy) \wedge \beta\} \vee \alpha \geqslant \{\mu(x) \wedge \mu(y) \wedge \gamma(x) \wedge \gamma(y) \wedge \beta\} \vee \alpha =$$
$$= \{\mu(x) \wedge \gamma(x) \wedge \mu(y) \wedge \gamma(y) \wedge \beta\} \vee \alpha = \{(\mu \wedge \gamma)(x) \wedge (\mu \wedge \gamma)(y) \wedge \beta \wedge \beta \wedge \beta\} \vee \alpha =$$
$$= \{((\mu \wedge \gamma)(x) \wedge \beta) \wedge ((\mu \wedge \gamma)(y) \wedge \beta) \wedge \beta\} \vee \alpha =$$
$$= (\{(\mu \wedge \gamma)(x) \wedge \beta\} \vee \alpha) \wedge (\{(\mu \wedge \gamma)(y) \wedge \beta\} \vee \alpha) \wedge (\beta \vee \alpha) =$$
$$= (\mu \wedge_\alpha^\beta \gamma)(x) \wedge (\mu \wedge_\alpha^\beta \gamma)(y) \wedge \beta = \min\{(\mu \wedge_\alpha^\beta \gamma)(x), (\mu \wedge_\alpha^\beta \gamma)(y), \beta\}.$$

Thus $\max\{(\mu \wedge_\alpha^\beta \gamma)(xy), \alpha\} \geqslant \min\{(\mu \wedge_\alpha^\beta \gamma)(x), (\mu \wedge_\alpha^\beta \gamma)(y), \beta\}$. Hence $\mu \wedge_\alpha^\beta \gamma$ is a fuzzy AG-subgroupoid with thresholds $(\alpha, \beta]$ of S. ■

Lemma 3. If μ and γ are two fuzzy AG-subgroupoids with thresholds $(\alpha, \beta]$ of an ordered AG-groupoid S, then $\mu \circ_\alpha^\beta \gamma$ is also a fuzzy AG-subgroupoid with thresholds $(\alpha, \beta]$ of S.

Proof. Suppose that μ and γ are two fuzzy AG-subgroupoids with thresholds $(\alpha,\beta]$ of an ordered AG-groupoid S. We have to show that $\mu\circ_{\alpha}^{\beta}\gamma$ is also a fuzzy AG-subgroupoid with thresholds $(\alpha,\beta]$ of S. Now

$$(\mu\circ_{\alpha}^{\beta}\gamma)^2=(\mu\circ_{\alpha}^{\beta}\gamma)\circ_{\alpha}^{\beta}(\mu\circ_{\alpha}^{\beta}\gamma)=(\mu\circ_{\alpha}^{\beta}\mu)\circ_{\alpha}^{\beta}(\gamma\circ_{\alpha}^{\beta}\gamma)\subseteq\mu_{\alpha}^{\beta}\circ_{\alpha}^{\beta}\gamma_{\alpha}^{\beta}=\mu\circ_{\alpha}^{\beta}\gamma.$$

Therefore $\mu\circ_{\alpha}^{\beta}\gamma$ is a fuzzy AG-subgroupoid with thresholds $(\alpha,\beta]$ of S. ■

Remark 1. If μ is a fuzzy AG-subgroupoid with thresholds $(\alpha,\beta]$ of an ordered AG-groupoid S, then $\mu\circ_{\alpha}^{\beta}\mu$ is also a fuzzy AG-subgroupoid with thresholds $(\alpha,\beta]$ of S.

Lemma 4. Let S be an ordered AG-groupoid with left identity e. Then every fuzzy right ideal with thresholds $(\alpha,\beta]$ of S is a fuzzy ideal with thresholds $(\alpha,\beta]$ of S.

Proof. Suppose that μ is a fuzzy right ideal with thresholds $(\alpha,\beta]$ of an ordered AG-subgroupoid S and $x,y\in S$. Thus

$$\begin{aligned}\max\{\mu\left(xy\right),\alpha\}=\max\{\mu\left(\left(ex\right)y\right),\alpha\}=\max\{\mu\left(\left(yx\right)e\right),\alpha\}\geqslant\\ \geqslant\min\{\mu\left(yx\right),\beta\}\geqslant\min\{\mu\left(y\right),\beta\}.\end{aligned}$$

Therefore μ is a fuzzy ideal with thresholds $(\alpha,\beta]$ of S. ■

Lemma 5. If μ and γ are two fuzzy left (resp. right) ideals with thresholds $(\alpha,\beta]$ of an ordered AG-groupoid S with left identity e, then $\mu\circ_{\alpha}^{\beta}\gamma$ is also a fuzzy left (resp. right) ideal with thresholds $(\alpha,\beta]$ of S.

Proof. Let μ and γ be two fuzzy left ideals with thresholds $(\alpha,\beta]$ of an ordered AG-groupoid S. We have to show that $\mu\circ_{\alpha}^{\beta}\gamma$ is also a fuzzy left ideal with thresholds $(\alpha,\beta]$ of S. Now

$$S\circ_{\alpha}^{\beta}(\mu\circ_{\alpha}^{\beta}\gamma)=(S\circ_{\alpha}^{\beta}S)\circ_{\alpha}^{\beta}(\mu\circ_{\alpha}^{\beta}\gamma)=(S\circ_{\alpha}^{\beta}\mu)\circ_{\alpha}^{\beta}(S\circ_{\alpha}^{\beta}\gamma)\subseteq\mu\circ_{\alpha}^{\beta}\gamma.$$

Hence $\mu\circ_{\alpha}^{\beta}\gamma$ is a fuzzy left ideal with thresholds $(\alpha,\beta]$ of S. Similarly, we can prove for right ideals. ■

Remark 2. If μ is a fuzzy left (resp. right) ideal with thresholds $(\alpha,\beta]$ of an ordered AG-groupoid S with left identity e, then $\mu\circ_{\alpha}^{\beta}\mu$ is a fuzzy ideal with thresholds $(\alpha,\beta]$ of S.

Lemma 6. If μ and γ are two fuzzy ideals with thresholds $(\alpha,\beta]$ of an ordered AG-groupoid S, then $\mu\circ_{\alpha}^{\beta}\gamma\subseteq\mu\wedge_{\alpha}^{\beta}\gamma$.

Proof. Let μ and γ be two fuzzy ideals with thresholds $(\alpha,\beta]$ of an ordered AG-groupoid S and $x\in S$. If $(\mu\circ_{\alpha}^{\beta}\gamma)(x)=0$, then obvious $\mu\circ_{\alpha}^{\beta}\gamma\subseteq\mu\wedge_{\alpha}^{\beta}\gamma$, otherwise we have

$$\begin{aligned}(\mu\circ_{\alpha}^{\beta}\gamma)(x)=\{(\mu\circ\gamma)(x)\wedge\beta\}\vee\alpha=\Big\{\Big(\bigvee_{(y,z)\in A_x}\min\{\mu(y),\gamma(z)\}\Big)\wedge\beta\Big\}\vee\alpha\leqslant\\ \leqslant\Big\{\Big(\bigvee_{(y,z)\in A_x}\min\{\mu(yz),\gamma(yz)\}\Big)\wedge\beta\Big\}\vee\alpha=\Big\{\Big(\bigvee_{(y,z)\in A_x}\min\{(\mu\wedge\gamma)(yz)\}\Big)\wedge\beta\Big\}\vee\alpha=\\ =\{(\mu\wedge\gamma)\left(x\right)\wedge\beta\}\vee\alpha=(\mu\wedge_{\alpha}^{\beta}\gamma)(x).\end{aligned}$$

Therefore $\mu\circ_{\alpha}^{\beta}\gamma\subseteq\mu\wedge_{\alpha}^{\beta}\gamma$. ■

Remark 3. If μ is a fuzzy ideal with thresholds $(\alpha,\beta]$ of an ordered AG-groupoid S, then $\mu\circ_{\alpha}^{\beta}\mu\subseteq\mu_{\alpha}^{\beta}$.

Lemma 7. Let S be an ordered AG-groupoid. Then $\mu \circ_\alpha^\beta \gamma \subseteq \mu \wedge_\alpha^\beta \gamma$ for every fuzzy right ideal μ with thresholds $(\alpha, \beta]$ and for every fuzzy left ideal γ with thresholds $(\alpha, \beta]$ of S.

Proof. Same as Lemma 6. ■

Theorem 4. Let A be a non-empty subset of an ordered AG-groupoid S. Then the following conditions are true:

(1) A is an interior ideal of S if and only if χ_A is a fuzzy interior ideal with thresholds $(\alpha, \beta]$ of S;
(2) A is a quasi-ideal of S if and only if χ_A is a fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S;
(3) A is a bi-ideal of S if and only if χ_A is a fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of S;
(4) A is a generalized bi-ideal of S if and only if χ_A is a fuzzy generalized bi-ideal with thresholds $(\alpha, \beta]$ of S.

Proof.

(1) Let A be an interior ideal of an ordered AG-groupoid S. Let $x, y, a \in S$. If $a \notin A$, then by definition of characteristic function $\chi_A(a) = 0$. Thus

$$\chi_A((xa)y) \geqslant \chi_A(a) = \min\{\chi_A(a), \beta\} \Rightarrow \chi_A((xa)y) \geqslant \min\{\chi_A(a), \beta\} \Rightarrow$$
$$\Rightarrow \max\{\chi_A((xa)y), \alpha\} \geqslant \min\{\chi_A(a), \beta\}.$$

Similarly, we have $\max\{\chi_A((xa)y), \alpha\} \geqslant \min\{\chi_A(a), \beta\}$, when $a \in A$. Hence the characteristic function χ_A of A is a fuzzy interior ideal with thresholds $(\alpha, \beta]$ of S.

Conversely, suppose that the characteristic function $\chi_A$of A is a fuzzy interior ideal with thresholds $(\alpha, \beta]$ of S. Let $t \in (SA)S$, so $t = (xa)y$, where $a \in A$ and $x, y \in S$. Then by definition $\chi_A(a) = 1$. Since $\max\{\chi_A((xa)y), \alpha\} \geqslant \min\{\chi_A(a), \beta\} = \beta$, χ_A being a fuzzy interior ideal with thresholds $(\alpha, \beta]$ of S. This implies that $\chi_A((xa)y) \geqslant \beta$, thus $\chi_A((xa)y) = 1$, i.e., $(xa)y \in A$. Hence A is an interior ideal of S.

(2) Let A be a quasi-ideal of S. Let $x \in S$ and $x \notin A$, then $x \notin SA$ or $x \notin AS$. If $x \notin SA$, then by definition of characteristic function $(S \circ \chi_A)(x) = 0$. Thus $\max\{\chi_A(x), \alpha\} \geqslant 0 = \min\{(\chi_A \circ S)(x), (S \circ \chi_A)(x), \beta\}$. If $x \in A$, then $\max\{\chi_A(x), \alpha\} = 1 \geqslant \min\{(\chi_A \circ S)(x), S \circ \chi_A(x), \beta\}$. Therefore the characteristic function χ_A of A is a fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S.

Conversely, assume that the characteristic function χ_A of A is a fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S. Let x be an element of $AS \cap SA$, this means that $x \in AS$ and SA. Since

$$\max\{\chi_A(x), \alpha\} \geqslant \min\{(\chi_A \circ S)(x), (S \circ \chi_A)(x), \beta\} = \min\{(\chi_A \circ \chi_S)(x), (\chi_S \circ \chi_A)(x), \beta\} =$$
$$= \min\{\chi_{AS}(x), \chi_{SA}(x), \beta\} = \beta \Rightarrow \max\{\chi_A(x), \alpha\} \geqslant \beta.$$

Thus $\chi_A(x) = 1$, i.e., $x \in A$. Therefore A is a quasi-ideal of S.

(3) Let A be a bi-ideal of S. Let $x, y, a \in S$. If $x, y \notin A$, then by definition of characteristic function $\chi_A(x) = \chi_A(y) = 0$. Thus

$$\chi_A((xa)y) \geqslant \chi_A(x) \wedge \chi_A(y) = \min\{\chi_A(x), \chi_A(y), \beta\} \Rightarrow \chi_A((xa)y) \geqslant \min\{\chi_A(x), \chi_A(y), \beta\} \Rightarrow$$
$$\Rightarrow \max\{\chi_A((xa)y), \alpha\} \geqslant \min\{\chi_A(x), \chi_A(y), \beta\}.$$

Similarly, we have $\max\{\chi_A((xa)y), \alpha\} \geqslant \min\{\chi_A(x), \chi_A(y), \beta\}$, when $x, y \in A$. Hence the characteristic function χ_A of A is a fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of S.

Conversely, suppose that the characteristic function $\chi_A$of A is a fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of S. Let $t \in (AS)A$, so $t = (xa)y$, where $x, y \in A$ and $a \in S$. Then by definition $\chi_A(x) = \chi_A(y) = 1$. As $\max\{\chi_A((xa)y), \alpha\} \geqslant \min\{\chi_A(x), \chi_A(y), \beta\} = \beta$, χ_A being a fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of S. This implies that $\chi_A((xa)y) \geqslant \beta$, thus $\chi_A((xa)y) = 1$, i.e., $(xa)y \in A$. Hence A is bi-ideal of S. Similarly, we can prove (4). ■

Theorem 5. Let μ be a fuzzy subset of an ordered AG-groupoid S. Then μ is a fuzzy interior ideal with thresholds $(\alpha, \beta]$ of S if and only if $(S \circ_\alpha^\beta \mu) \circ_\alpha^\beta S \subseteq \mu_\alpha^\beta$.

Proof. Suppose that μ is a fuzzy interior ideal with thresholds $(\alpha, \beta]$ of an ordered AG-groupoid S and $x \in S$. If $((S \circ_\alpha^\beta \mu) \circ_\alpha^\beta S)(x) = 0$, then obvious $(S \circ_\alpha^\beta \mu) \circ_\alpha^\beta S \subseteq \mu_\alpha^\beta$, otherwise there exist $a, b, c, d \in S$ such that $x \leqslant ab$ and $a \leqslant cd$. Since μ is a fuzzy interior ideal with thresholds $(\alpha, \beta]$ of S, this implies that $\max\{\mu((cd)b), \alpha\} \geqslant \min\{\mu(d), \beta\}$. Now

$$((S \circ_\alpha^\beta \mu) \circ_\alpha^\beta S)(x) = \{((S \circ \mu) \circ S)(x) \wedge \beta\} \vee \alpha =$$
$$= \Big\{\Big(\bigvee_{(a,b)\in A_x} \min\{(S \circ \mu)(a), S(b)\}\Big) \wedge \beta\Big\} \vee \alpha = \Big\{\Big(\bigvee_{(a,b)\in A_x} \min\{(S \circ \mu)(a), 1\}\Big) \wedge \beta\Big\} \vee \alpha =$$
$$= \Big\{\Big(\bigvee_{(a,b)\in A_x} \min\{(S \circ \mu)(a)\}\Big) \wedge \beta\Big\} \vee \alpha =$$
$$= \Big\{\Big(\bigvee_{(a,b)\in A_x} \min\Big\{\Big(\bigvee_{(c,d)\in A_a} \min\{S(c), \mu(d)\}\Big)\Big\}\Big) \wedge \beta\Big\} \vee \alpha =$$
$$= \Big\{\Big(\bigvee_{(a,b)\in A_x} \min\Big\{\Big(\bigvee_{(c,d)\in A_a} \min\{1, \mu(d)\}\Big)\Big\}\Big) \wedge \beta\Big\} \vee \alpha =$$
$$= \Big\{\Big(\bigvee_{(a,b)\in A_x} \min\Big\{\Big(\bigvee_{(c,d)\in A_a} \min\{\mu(d)\}\Big)\Big\}\Big) \wedge \beta\Big\} \vee \alpha = \Big\{\Big(\bigvee_{((c,d),b)\in A_x} \min\{\mu(d)\}\Big) \wedge \beta\Big\} \vee \alpha \leqslant$$
$$\leqslant \Big\{\Big(\bigvee_{((c,d),b)\in A_x} \min\{\mu((cd)b)\}\Big) \wedge \beta\Big\} \vee \alpha = \{\mu(x) \wedge \beta\} \vee \alpha = \mu_\alpha^\beta(x) \Rightarrow (S \circ_\alpha^\beta \mu) \circ_\alpha^\beta S \subseteq \mu_\alpha^\beta.$$

Conversely, assume that $(S \circ_\alpha^\beta \mu) \circ_\alpha^\beta S \subseteq \mu_\alpha^\beta$ and $x, y, z \in S$ such that $a \leqslant (xy)z$. Now

$$\max\{\mu((xy)z), \alpha\} = \max\{\min\{\mu((xy)z), \beta\}, \alpha\} \geqslant \max\{\min\{\mu(a), \beta\}, \alpha\} = \mu_\alpha^\beta(a) \geqslant$$
$$\geqslant ((S \circ_\alpha^\beta \mu) \circ_\alpha^\beta S)(a) = \{((S \circ \mu) \circ S)(a) \wedge \beta\} \vee \alpha =$$
$$= \Big\{\Big(\bigvee_{(s,t)\in A_a} \min\{(S \circ \mu)(s), S(t)\}\Big) \wedge \beta\Big\} \vee \alpha \geqslant \{((S \circ \mu)(xy) \wedge S(z)) \wedge \beta\} \vee \alpha =$$
$$= \{((S \circ \mu)(xy) \wedge 1) \wedge \beta\} \vee \alpha = \{(S \circ \mu)(xy) \wedge \beta\} \vee \alpha =$$
$$= \Big\{\Big(\bigvee_{(m,n)\in A_{xy}} \min\{S(m), \mu(n)\}\Big) \wedge \beta\Big\} \vee \alpha \geqslant \{(S(x) \wedge \mu(y)) \wedge \beta\} \vee \alpha =$$
$$= \{(1 \wedge \mu(y)) \wedge \beta\} \vee \alpha = \mu(y) \wedge \beta = \min\{\mu(y), \beta\} \Rightarrow \max\{\mu((xy)z), \alpha\} \geqslant \min\{\mu(y), \beta\}.$$

Therefore μ is a fuzzy interior ideal with thresholds $(\alpha, \beta]$ of S. ■

Theorem 6. Let μ be a fuzzy ordered AG-subgroupoid with thresholds $(\alpha, \beta]$ of an ordered AG-groupoid S. Then μ is a fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of S if and only if $(\mu \circ_\alpha^\beta S) \circ_\alpha^\beta \mu \subseteq \mu_\alpha^\beta$.

Proof. Same as Theorem 5. ■

Theorem 7. Let μ be a fuzzy subset of an ordered AG-groupoid S. Then μ is a fuzzy generalized bi-ideal with thresholds $(\alpha, \beta]$ of S if and only if $(\mu \circ_\alpha^\beta S) \circ_\alpha^\beta \mu \subseteq \mu_\alpha^\beta$.

Proof. Same as Theorem 5. ■

Lemma 8. If μ and γ are two fuzzy bi- (resp. generalized bi-, quasi-, interior) ideals with thresholds $(\alpha,\beta]$ of an ordered AG-groupoid S, then $\mu\wedge_{\alpha}^{\beta}\gamma$ is also a fuzzy bi- (resp. generalized bi-, quasi-, interior) ideal with thresholds $(\alpha,\beta]$ of S.

Proof. Let μ and γ be two fuzzy bi-ideals with thresholds $(\alpha,\beta]$ of an ordered AG-groupoid S. We have to show that $\mu\wedge_{\alpha}^{\beta}\gamma$ is also a fuzzy bi-ideal with thresholds $(\alpha,\beta]$ of S. Since μ and γ are fuzzy AG-subgroupoids with thresholds $(\alpha,\beta]$ of S then $\mu\wedge_{\alpha}^{\beta}\gamma$ is also a fuzzy AG-subgroupoid with thresholds $(\alpha,\beta]$ of S by the Lemma 2. We have to show that $\max\{(\mu\wedge_{\alpha}^{\beta}\gamma)((xa)y),\alpha\}\geqslant\min\{(\mu\wedge_{\alpha}^{\beta}\gamma)(x),(\mu\wedge_{\alpha}^{\beta}\gamma)(y),\beta\}$. Now

$$\begin{gathered}\max\{(\mu\wedge_{\alpha}^{\beta}\gamma)((xa)y),\alpha\}=\max\{\{\{(\mu\wedge\gamma)((xa)y)\wedge\beta\}\vee\alpha\},\alpha\}=\\=\{(\mu\wedge\gamma)((xa)y)\wedge\beta\}\vee\alpha=\{\mu((xa)y)\wedge\gamma((xa)y)\wedge\beta\}\vee\alpha\geqslant\\\geqslant\{\mu(x)\wedge\mu(y)\wedge\gamma(x)\wedge\gamma(y)\wedge\beta\}\vee\alpha=\{\mu(x)\wedge\gamma(x)\wedge\mu(y)\wedge\gamma(y)\wedge\beta\}\vee\alpha=\\=\{(\mu\wedge\gamma)(x)\wedge(\mu\wedge\gamma)(y)\wedge\beta\wedge\beta\wedge\beta\}\vee\alpha=\{((\mu\wedge\gamma)(x)\wedge\beta)\wedge((\mu\wedge\gamma)(y)\wedge\beta)\wedge\beta\}\vee\alpha=\\=(\{(\mu\wedge\gamma)(x)\wedge\beta\}\vee\alpha)\wedge(\{(\mu\wedge\gamma)(y)\wedge\beta\}\vee\alpha)\wedge(\beta\vee\alpha)=\\=(\mu\wedge_{\alpha}^{\beta}\gamma)(x)\wedge(\mu\wedge_{\alpha}^{\beta}\gamma)(y)\wedge\beta=\min\{(\mu\wedge_{\alpha}^{\beta}\gamma)(x),(\mu\wedge_{\alpha}^{\beta}\gamma)(y),\beta\}.\end{gathered}$$

Thus $\max\{(\mu\wedge_{\alpha}^{\beta}\gamma)((xa)y),\alpha\}\geqslant\min\{(\mu\wedge_{\alpha}^{\beta}\gamma)(x),(\mu\wedge_{\alpha}^{\beta}\gamma)(y),\beta\}$. Hence $\mu\wedge_{\alpha}^{\beta}\gamma$ is a fuzzy bi-ideal with thresholds $(\alpha,\beta]$ of S. ■

Lemma 9. If μ and γ are two fuzzy bi- (resp. generalized bi-, interior) ideals with thresholds $(\alpha,\beta]$ of an ordered AG-groupoid S with left identity e, then $\mu\circ_{\alpha}^{\beta}\gamma$ is also a fuzzy bi- (resp. generalized bi-, interior) ideal with thresholds $(\alpha,\beta]$ of S.

Proof. Let μ and γ be two fuzzy bi-ideals with thresholds $(\alpha,\beta]$ of an ordered AG-groupoid S. We have to show that $\mu\circ_{\alpha}^{\beta}\gamma$ is also a fuzzy bi-ideal with thresholds $(\alpha,\beta]$ of S. Since μ and γ are fuzzy AG-subgroupoids with thresholds $(\alpha,\beta]$ of S, then $\mu\circ_{\alpha}^{\beta}\gamma$ is also a fuzzy AG-subgroupoid with thresholds $(\alpha,\beta]$ of S by the Lemma 3. Now

$$\begin{gathered}((\mu\circ_{\alpha}^{\beta}\gamma)\circ_{\alpha}^{\beta}S)\circ_{\alpha}^{\beta}(\mu\circ_{\alpha}^{\beta}\gamma)=((\mu\circ_{\alpha}^{\beta}\gamma)\circ_{\alpha}^{\beta}(S\circ_{\alpha}^{\beta}S))\circ_{\alpha}^{\beta}(\mu\circ_{\alpha}^{\beta}\gamma)=\\=((\mu\circ_{\alpha}^{\beta}S)\circ_{\alpha}^{\beta}(\gamma\circ_{\alpha}^{\beta}S))\circ_{\alpha}^{\beta}(\mu\circ_{\alpha}^{\beta}\gamma)=((\mu\circ_{\alpha}^{\beta}S)\circ_{\alpha}^{\beta}\mu)\circ_{\alpha}^{\beta}((\gamma\circ_{\alpha}^{\beta}S)\circ_{\alpha}^{\beta}\gamma)\subseteq\mu_{\alpha}^{\beta}\circ_{\alpha}^{\beta}\gamma_{\alpha}^{\beta}=\mu\circ_{\alpha}^{\beta}\gamma.\end{gathered}$$

Therefore $\mu\circ_{\alpha}^{\beta}\gamma$ is a fuzzy bi-ideal with thresholds $(\alpha,\beta]$ of S. ■

Lemma 10. Every fuzzy ideal with thresholds $(\alpha,\beta]$ of an ordered AG-groupoid S is a fuzzy interior ideal with thresholds $(\alpha,\beta]$ of S. The converse is not true in general.

Proof. Let μ be a fuzzy ideal with thresholds $(\alpha,\beta]$ of an ordered AG-groupoid S and $x,y,z\in S$. Thus $\max\{\mu((xy)z),\alpha\}\geqslant\min\{\mu(xy),\beta\}\geqslant\min\{\mu(y),\beta\}$. Hence μ is a fuzzy interior ideal with thresholds $(\alpha,\beta]$ of S. ■

Proposition 2. Let μ be a fuzzy subset of an ordered AG-groupoid S with left identity e. Then μ is a fuzzy ideal with thresholds $(\alpha,\beta]$ of S if and only if μ is a fuzzy interior ideal with thresholds $(\alpha,\beta]$ of S.

Proof. Suppose that μ is a fuzzy interior ideal with thresholds $(\alpha,\beta]$ of an ordered AG-groupoid S and $x,y\in S$. Thus $\max\{\mu(xy),\alpha\}=\max\{\mu((ex)y),\alpha\}\geqslant\min\{\mu(x),\beta\}$. So μ is a fuzzy right ideal with thresholds $(\alpha,\beta]$ of S. Therefore μ is a fuzzy ideal with thresholds $(\alpha,\beta]$ of S by the Lemma 4. Converse is true by the Lemma 10. ■

Lemma 11. Every fuzzy left (resp. right, two-sided) ideal with thresholds $(\alpha,\beta]$ of an ordered AG-groupoid S is a fuzzy bi-ideal with thresholds $(\alpha,\beta]$ of S. The converse is not true in general.

Proof. Assume that μ is a fuzzy right ideal with thresholds $(\alpha, \beta]$ of an ordered AG-groupoid S and $x, y, z \in S$. Thus $\max\{\mu((xy)z), \alpha\} \geqslant \min\{\mu(xy), \beta\} \geqslant \min\{\mu(x), \beta\}$ and $\max\{\mu((xy)z), \alpha\} = \max\{\mu((zy)x), \alpha\} \geqslant \min\{\mu(zy), \beta\} \geqslant \min\{\mu(z), \beta\}$.

This implies that $\max\{\mu((xy)z), \alpha\} \geqslant \min\{\mu(x), \mu(z), \beta\}$. So μ is a fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of S. ■

Lemma 12. Every fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of an ordered AG-groupoid S is a fuzzy generalized bi-ideal with thresholds $(\alpha, \beta]$ of S. The converse is not true in general.

Proof. Obvious. ■

Lemma 13. Every fuzzy left (resp. right, two-sided) ideal with thresholds $(\alpha, \beta]$ of an ordered AG-groupoid S is a fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S. The converse is not true in general.

Proof. Let μ be a fuzzy left ideal with thresholds $(\alpha, \beta]$ of an ordered AG-groupoid S. Thus

$$\max\{\mu(x), \alpha\} \geqslant \min\{(S \circ \mu)(x), \beta\} \geqslant \min\{(\mu \circ S)(x), (S \circ \mu)(x), \beta\}.$$

Hence μ is a fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S. ■

Proposition 3. Every fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of an ordered AG-groupoid S is a fuzzy AG-subgroupoid with thresholds $(\alpha, \beta]$ of S.

Proof. Suppose that μ is a fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of an ordered AG-groupoid S. Since $\mu \circ_\alpha^\beta \mu \subseteq \mu \circ_\alpha^\beta S$ and $\mu \circ_\alpha^\beta \mu \subseteq S \circ_\alpha^\beta \mu$, this implies that $\mu \circ_\alpha^\beta \mu \subseteq \mu \circ_\alpha^\beta S \wedge S \circ_\alpha^\beta \mu \subseteq \mu_\alpha^\beta$. Therefore μ is a fuzzy AG-subgroupoid with thresholds $(\alpha, \beta]$ of S. ■

Proposition 4. Let μ be a fuzzy right ideal with thresholds $(\alpha, \beta]$ and γ be a fuzzy left ideal with thresholds $(\alpha, \beta]$ of an ordered AG-groupoid S, respectively. Then $\mu \wedge_\alpha^\beta \gamma$ is a fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S.

Proof. We have to show that $\mu \wedge_\alpha^\beta \gamma$ is a fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of an ordered AG-subgroupoid S. Now

$$((\mu \wedge_\alpha^\beta \gamma) \circ_\alpha^\beta S) \wedge (S \circ_\alpha^\beta (\mu \wedge_\alpha^\beta \gamma)) \subseteq (\mu \circ_\alpha^\beta S) \wedge (S \circ_\alpha^\beta \gamma) \subseteq \mu_\alpha^\beta \wedge \gamma_\alpha^\beta = \mu \wedge_\alpha^\beta \gamma.$$

Thus $\mu \wedge_\alpha^\beta \gamma$ is a fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S. ■

Lemma 14. Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then every fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S is a fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of S.

Proof. Assume that μ is a fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of an ordered AG-subgroupoid S. This implies that μ is a fuzzy AG-subgroupoid with thresholds $(\alpha, \beta]$ of S. We have to show that $(\mu \circ_\alpha^\beta S) \circ_\alpha^\beta \mu \subseteq \mu_\alpha^\beta$. Now

$$(\mu \circ_\alpha^\beta S) \circ_\alpha^\beta \mu \subseteq (S \circ_\alpha^\beta S) \circ_\alpha^\beta \mu \subseteq S \circ_\alpha^\beta \mu$$
$$\text{and } (\mu \circ_\alpha^\beta S) \circ_\alpha^\beta \mu \subseteq (\mu \circ_\alpha^\beta S) \circ_\alpha^\beta S = (\mu \circ_\alpha^\beta S) \circ_\alpha^\beta (e \circ_\alpha^\beta S) = (\mu \circ_\alpha^\beta e) \circ_\alpha^\beta (S \circ_\alpha^\beta S) \subseteq$$
$$\subseteq (\mu \circ_\alpha^\beta e) \circ_\alpha^\beta S_\alpha^\beta = \mu_\alpha^\beta \circ_\alpha^\beta S_\alpha^\beta = \mu \circ_\alpha^\beta S \Rightarrow (\mu \circ_\alpha^\beta S) \circ_\alpha^\beta \mu \subseteq \mu \circ_\alpha^\beta S \wedge S \circ_\alpha^\beta \mu \subseteq \mu_\alpha^\beta.$$

So μ is a fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of S. ■

Proposition 5. If μ and γ are two fuzzy quasi-ideals with thresholds $(\alpha, \beta]$ of an ordered AG-groupoid S with left identity e, such that $(xe)S = xS$ for all $x \in S$, then $\mu \circ_\alpha^\beta \gamma$ is a fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of S.

Proof. Let μ and γ be two fuzzy quasi-ideals with thresholds $(\alpha,\beta]$ of an ordered AG-groupoid S, this implies that μ and γ be two fuzzy bi-ideals with thresholds $(\alpha,\beta]$ of S, by the Lemma 14. Then $\mu \circ_{\alpha}^{\beta} \gamma$ is also a fuzzy bi-ideal with thresholds $(\alpha,\beta]$ of S by the Lemma 9. ■

2. Regular Ordered AG-groupoids

An ordered AG-groupoid S will be called a regular if, for every $x \in S$, there exists an element $a \in S$ such that $x \leqslant (xa)x$. Equivalent definitions are as follows:

(1) $A \subseteq ((AS)A]$ for every $A \subseteq S$;

(2) $x \in ((xS)x]$ for every $x \in S$.

In this section, we give the characterizations of regular ordered AG-groupoids by the properties of fuzzy left (right, quasi-, bi-, generalized bi-) ideals with thresholds $(\alpha,\beta]$.

Lemma 15. Every fuzzy right ideal with thresholds $(\alpha,\beta]$ of a regular ordered AG-groupoid S is a fuzzy ideal with thresholds $(\alpha,\beta]$ of S.

Proof. Suppose that μ is a fuzzy right ideal with thresholds $(\alpha,\beta]$ of S. Let $x,y \in S$, this implies that there exists $a \in S$, such that $x \leqslant (xa)x$. Thus

$$\max\{\mu(xy),\alpha\} \geqslant \max\{\mu(((xa)x)y),\alpha\} = \max\{\mu((yx)(xa)),\alpha\} \geqslant$$
$$\geqslant \min\{\mu(yx),\beta\} \geqslant \min\{\mu(y),\beta\}.$$

Hence μ is a fuzzy ideal with thresholds $(\alpha,\beta]$ of S. ■

Lemma 16. Every fuzzy ideal with thresholds $(\alpha,\beta]$ of a regular ordered AG-groupoid S is a fuzzy idempotent with thresholds $(\alpha,\beta]$.

Proof. Assume that μ is a fuzzy ideal with thresholds $(\alpha,\beta]$ of S and $\mu \circ_{\alpha}^{\beta} \mu \subseteq \mu_{\alpha}^{\beta}$. We have to show that $\mu_{\alpha}^{\beta} \subseteq \mu \circ_{\alpha}^{\beta} \mu$. Let $x \in S$, this means that there exists $a \in S$ such that $x \leqslant (xa)x$. Thus

$$(\mu \circ_{\alpha}^{\beta} \mu)(x) = \{(\mu \circ \mu)(x) \wedge \beta\} \vee \alpha = \Big\{\Big(\bigvee_{(y,z)\in A_x} \min\{\mu(y),\mu(z)\}\Big) \wedge \beta\Big\} \vee \alpha \geqslant$$
$$\geqslant \{\mu(xa) \wedge \mu(x) \wedge \beta\} \vee \alpha = (\mu(xa) \vee \alpha) \wedge (\mu(x) \vee \alpha) \wedge (\beta \vee \alpha) \geqslant$$
$$\geqslant (\mu(x) \wedge \beta) \wedge \mu(x) \wedge \beta = \mu(x) \wedge \beta = (\mu(x) \wedge \beta) \vee \alpha = \mu_{\alpha}^{\beta}(x) \Rightarrow \mu_{\alpha}^{\beta} \subseteq \mu \circ_{\alpha}^{\beta} \mu.$$

Therefore $\mu_{\alpha}^{\beta} = \mu \circ_{\alpha}^{\beta} \mu$. ■

Remark 4. Every fuzzy right ideal with thresholds $(\alpha,\beta]$ of a regular ordered AG-groupoid S is a fuzzy idempotent with thresholds $(\alpha,\beta]$.

Proposition 6. Let μ be a fuzzy subset of a regular ordered AG-groupoid S. Then μ is a fuzzy ideal with thresholds $(\alpha,\beta]$ of S if and only if μ is a fuzzy interior ideal with thresholds $(\alpha,\beta]$ of S.

Proof. Consider that μ is a fuzzy interior ideal with thresholds $(\alpha,\beta]$ of S. Let $x,y \in S$, then there exists an element $a \in S$, such that $x \leqslant (xa)x$. Thus

$$\max\{\mu(xy),\alpha\} \geqslant \max\{\mu(((xa)x)y),\alpha\} = \max\{\mu((yx)(xa)),\alpha\} \geqslant \min\{\mu(x),\beta\}.$$

Consequently μ is a fuzzy right ideal with thresholds $(\alpha,\beta]$ of S. So μ is a fuzzy ideal with thresholds $(\alpha,\beta]$ of S by the Lemma 15. Converse is true by the Lemma 10. ■

Remark 5. The concept of fuzzy (two-sided, interior) ideals with thresholds $(\alpha,\beta]$ coincides in regular ordered AG-groupoids.

Proposition 7. Let S be a regular ordered AG-groupoid. Then $(\mu\circ_\alpha^\beta S)\wedge(S\circ_\alpha^\beta\mu) = \mu_\alpha^\beta$ for every fuzzy right ideal μ with thresholds $(\alpha, \beta]$ of S.

Proof. Suppose that μ is a fuzzy right ideal with thresholds $(\alpha, \beta]$ of S. This implies that $(\mu \circ_\alpha^\beta S) \wedge (S \circ_\alpha^\beta \mu) \subseteq \mu_\alpha^\beta$, because every fuzzy right ideal with thresholds $(\alpha, \beta]$ of S is a fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S by the Lemma 13. Let $x \in S$, this implies that there exists $a \in S$, such that $x \leqslant (xa)x$. Thus

$$(\mu \circ_\alpha^\beta S)(x) = \{(\mu \circ S)(x) \wedge \beta\} \vee \alpha = \left\{\Big(\bigvee_{(y,z)\in A_x} \min\{\mu(y), S(z)\}\Big) \wedge \beta\right\} \vee \alpha \geqslant$$

$$\geqslant \{\mu\,(xa) \wedge S\,(x) \wedge \beta\} \vee \alpha = \{\mu\,(xa) \wedge \beta\} \vee \alpha = (\mu\,(xa) \vee \alpha) \wedge (\beta \vee \alpha) \geqslant (\mu\,(x) \wedge \beta) \wedge \beta =$$
$$= \mu\,(x) \wedge \beta = (\mu\,(x) \wedge \beta) \vee \alpha = \mu_\alpha^\beta(x) \Rightarrow \mu_\alpha^\beta \subseteq \mu \circ_\alpha^\beta S.$$

Similarly, we have $\mu_\alpha^\beta \subseteq S\circ_\alpha^\beta\mu$, i.e., $\mu_\alpha^\beta \subseteq (\mu\circ_\alpha^\beta S)\wedge(S\circ_\alpha^\beta\mu)$. Hence $(\mu\circ_\alpha^\beta S)\wedge(S\circ_\alpha^\beta\mu) = \mu_\alpha^\beta$. ■

Lemma 17. Let S be a regular ordered AG-groupoid. Then $\mu \circ_\alpha^\beta \gamma = \mu \wedge_\alpha^\beta \gamma$ for every fuzzy right ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy left ideal γ with thresholds $(\alpha, \beta]$ of S.

Proof. Since $\mu \circ_\alpha^\beta \gamma \subseteq \mu \wedge_\alpha^\beta \gamma$, for every fuzzy right ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy left ideal γ with thresholds $(\alpha, \beta]$ of S by the Lemma 7. Let $x \in S$, this means that there exists $a \in S$ such that $x \leqslant (xa)x$. Thus

$$(\mu \circ_\alpha^\beta \gamma)(x) = \{(\mu \circ \gamma)(x) \wedge \beta\} \vee \alpha = \left\{\Big(\bigvee_{(y,z)\in A_x} \min\{\mu(y), \gamma(z)\}\Big) \wedge \beta\right\} \vee \alpha \geqslant$$
$$\geqslant \{\mu\,(xa) \wedge \gamma\,(x) \wedge \beta\} \vee \alpha = (\mu\,(xa) \vee \alpha) \wedge (\gamma\,(x) \vee \alpha) \wedge (\beta \vee \alpha) \geqslant$$
$$\geqslant (\mu\,(x) \wedge \beta) \wedge \gamma\,(x) \wedge \beta = \mu\,(x) \wedge \gamma\,(x) \wedge \beta = (\mu \wedge \gamma)\,(x) \wedge \beta =$$
$$= \{(\mu \wedge \gamma)\,(x) \wedge \beta\} \vee \alpha = (\mu \wedge_\alpha^\beta \gamma)(x) \Rightarrow \mu \wedge_\alpha^\beta \gamma \subseteq \mu \circ_\alpha^\beta \gamma.$$

Therefore $\mu \circ_\alpha^\beta \gamma = \mu \wedge_\alpha^\beta \gamma$. ■

Lemma 18. Let S be an ordered AG-groupoid with left identity e. Then Sa is a smallest left ideal of S containing a.

Proof. Let $x \in Sa$ and $s \in S$, this implies that $x = s_1a$, $s_1 \in S$. Now

$$sx = s(s_1a) = (es)(s_1a) = ((s_1a)s)e = ((s_1a)(es))e =$$
$$= ((s_1e)(as))e = (e(as))(s_1e) = (as)(s_1e) = ((s_1e)s)a \in Sa.$$

Hence $sx \in Sa$ and $(Sa] \subseteq Sa$. Now $a = ea \in Sa$, so Sa is a left ideal of S containing a. Let I be another left ideal of S containing a. Since $sa \in I$, because I is a left ideal of S. But $sa \in Sa$, this means that $Sa \subseteq I$. Therefore Sa is a smallest left ideal of S containing a. ■

Lemma 19. Let S be an ordered AG-groupoid with left identity e. Then aS is a left ideal of S.

Proof. Straight forward. ■

Proposition 8. Let S be an ordered AG-groupoid with left identity e. Then $aS \cup Sa$ is a smallest right ideal of S containing a.

Proof. We have to show that $aS \cup Sa$ is a smallest right ideal of S containing a. Now

$$(aS \cup Sa)S = (aS)S \cup (Sa)S = (SS)a \cup (Sa)(eS) \subseteq Sa \cup (Se)(aS) = Sa \cup S(aS) =$$
$$= Sa \cup a(SS) \subseteq Sa \cup aS = aS \cup Sa.$$

Thus $(aS \cup Sa)S \subseteq aS \cup Sa$ and also $(aS \cup Sa] \subseteq aS \cup Sa$. Therefore $aS \cup Sa$ is a right ideal of S. Since $a \in Sa$, i.e., $a \in aS \cup Sa$. Let I be another right ideal of S containing a. Now $aS \in IS \subseteq I$ and $Sa = (SS)a = (aS)S \in (IS)S \subseteq IS \subseteq I$, i.e., $aS \cup Sa \subseteq I$. Hence $aS \cup Sa$ is a smallest right ideal of S containing a. ■

Theorem 8. Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent:

(1) S is a regular;

(2) $\mu \wedge_\alpha^\beta \gamma = \mu \circ_\alpha^\beta \gamma$ for every fuzzy right ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy left ideal γ with thresholds $(\alpha, \beta]$ of S;

(3) $\delta_\alpha^\beta = (\delta \circ_\alpha^\beta S) \circ \delta_\alpha^\beta$ for every fuzzy quasi-ideal δ with thresholds $(\alpha, \beta]$ of S.

Proof. Consider that (1) holds and δ be a fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S. This implies that $(\delta \circ_\alpha^\beta S) \circ_\alpha^\beta \delta \subseteq \delta_\alpha^\beta$, because every fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S is a fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of S by the Lemma 14. Let $x \in S$, then there exists an element $a \in S$ such that $x \leqslant (xa)x$. Thus

$$((\delta \circ_\alpha^\beta S) \circ_\alpha^\beta \delta)(x) = \{((\delta \circ S) \circ \delta)(x) \wedge \beta\} \vee \alpha = \left\{\Big(\bigvee_{(y,z)\in A_x} \in \{(\delta \circ S)(y), \delta(z)\}\Big) \wedge \beta\right\} \vee \alpha \geqslant$$
$$\geqslant \{(\delta \circ S)(xa) \wedge \delta(x) \wedge \beta\} \vee \alpha = ((\delta \circ S)(xa) \vee \alpha) \wedge (\delta(x) \vee \alpha) \wedge (\beta \vee \alpha) =$$
$$= ((\delta \circ S)(xa) \vee \alpha) \wedge \delta(x) \wedge \beta = \Big(\Big(\bigvee_{(s,t)\in A_{xa}} \min\{\delta(s), S(t)\}\Big) \vee \alpha\Big) \wedge \delta(x) \wedge \beta \geqslant$$
$$\geqslant (\{\delta(x) \wedge S(a)\} \vee \alpha) \wedge \delta(x) \wedge \beta = (\{\delta(x) \wedge 1\} \vee \alpha) \wedge \delta(x) \wedge \beta = (\delta(x) \vee \alpha) \wedge \delta(x) \wedge \beta =$$
$$= \delta(x) \wedge \beta = (\delta(x) \wedge \beta) \vee \alpha = \delta_\alpha^\beta(x) \Rightarrow \delta_\alpha^\beta \subseteq (\delta \circ_\alpha^\beta S) \circ_\alpha^\beta \delta.$$

So $\delta_\alpha^\beta = (\delta \circ_\alpha^\beta S) \circ_\alpha^\beta \delta$, i.e., (1) implies (3). Suppose that (3) holds. Let μ be a fuzzy right ideal with thresholds $(\alpha, \beta]$ and γ be a fuzzy left ideal with thresholds $(\alpha, \beta]$ of S. This implies that μ and γ be fuzzy quasi-ideals with thresholds $(\alpha, \beta]$ of S by the Lemma 13, so $\mu \wedge_\alpha^\beta \gamma$ be also a fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S. Then by our supposition, $\mu \wedge_\alpha^\beta \gamma = ((\mu \wedge_\alpha^\beta \gamma) \circ_\alpha^\beta S) \circ_\alpha^\beta (\mu \wedge_\alpha^\beta \gamma) \subseteq (\mu \circ_\alpha^\beta S) \circ_\alpha^\beta \gamma \subseteq \mu \circ_\alpha^\beta \gamma$, i.e., $\mu \wedge_\alpha^\beta \gamma \subseteq \mu \circ_\alpha^\beta \gamma$. Since $\mu \circ_\alpha^\beta \gamma \subseteq \mu \wedge_\alpha^\beta \gamma$, so $\mu \circ_\alpha^\beta \gamma = \mu \wedge_\alpha^\beta \gamma$, i.e., (3) $\Rightarrow$ (2). Assume that (2) is true and $a \in S$. Then Sa is a left ideal of S containing a by the Lemma 18 and $aS \cup Sa$ is a right ideal of S containing a by the Proposition 8. This means that χ_{Sa} is a fuzzy left ideal with thresholds $(\alpha, \beta]$ and $\chi_{aS\cup Sa}$ is a fuzzy right ideal with thresholds $(\alpha, \beta]$ of S by the Theorem 2. Then by our assumption $\chi_{aS\cup Sa} \wedge_\alpha^\beta \chi_{Sa} = \chi_{aS\cup Sa} \circ_\alpha^\beta \chi_{Sa}$, i.e., $(\chi_{(aS\cup Sa)\cap Sa})_\alpha^\beta = (\chi_{((aS\cup Sa)Sa]})_\alpha^\beta$ by the Theorem 1. Thus $(aS \cup Sa) \cap Sa = ((aS \cup Sa)Sa]$. Since $a \in (aS \cup Sa) \cap Sa$, i.e., $a \in ((aS \cup Sa)Sa]$, so $a \in ((aS)(Sa) \cup (Sa)(Sa)]$. Now $(Sa)(Sa) = ((Se)a)(Sa) = ((ae)S)(Sa) = (aS)(Sa)$. This implies that

$$((aS)(Sa) \cup (Sa)(Sa)] = ((aS)(Sa) \cup (aS)(Sa)] = ((aS)(Sa)].$$

Thus $a \in ((aS)(Sa)]$. Then

$$a \leqslant (ax)(ya) = ((ya)x)a = (((ey)a)x)a = (((ay)e)x)a =$$
$$= ((xe)(ay))a = (a((xe)y))a \in (aS)a, \text{ for any } x, y \in S.$$

This means that $a \in ((aS)a]$, i.e., a is regular. Hence S is a regular, i.e., (2) $\Rightarrow$ (1). ■

Theorem 9. Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. the following conditions are equivalent:

(1) S is a regular;
(2) $\mu_\alpha^\beta = (\mu \circ_\alpha^\beta S) \circ_\alpha^\beta \mu$ for every fuzzy quasi-ideal μ with thresholds $(\alpha, \beta]$ of S;
(3) $\gamma_\alpha^\beta = (\gamma \circ_\alpha^\beta S) \circ_\alpha^\beta \gamma$ for every fuzzy bi-ideal γ with thresholds $(\alpha, \beta]$ of S;
(4) $\delta_\alpha^\beta = (\delta \circ_\alpha^\beta S) \circ_\alpha^\beta \delta$ for every fuzzy generalized bi-ideal δ with thresholds $(\alpha, \beta]$ of S.

Proof.
(1) $\Rightarrow$ (4) is obvious.
(4) $\Rightarrow$ (3), since every fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of S is a fuzzy generalized bi-ideal with thresholds $(\alpha, \beta]$ of S by the Lemma 12.
(3) $\Rightarrow$ (2), since every fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S is a fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of S by the Lemma 14.
(2) $\Rightarrow$ (1) by the Theorem 8. ■

Theorem 10. Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent:

(1) S is a regular;
(2) $\mu \wedge_\alpha^\beta \nu = (\mu \circ_\alpha^\beta \nu) \circ_\alpha^\beta \mu$ for every fuzzy quasi-ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy ideal ν with thresholds $(\alpha, \beta]$ of S;
(3) $\gamma \wedge_\alpha^\beta \nu = (\gamma \circ_\alpha^\beta \nu) \circ_\alpha^\beta \gamma$ for every fuzzy bi-ideal γ with thresholds $(\alpha, \beta]$ and every fuzzy ideal ν with thresholds $(\alpha, \beta]$ of S;
(4) $\delta \wedge_\alpha^\beta \nu = (\delta \circ_\alpha^\beta \nu) \circ_\alpha^\beta \delta$ for every fuzzy generalized bi-ideal δ with thresholds $(\alpha, \beta]$ and every fuzzy ideal ν with thresholds $(\alpha, \beta]$ of S.

Proof. Suppose that (1) holds. Let δ be a fuzzy generalized bi-ideal with thresholds $(\alpha, \beta]$ and ν be a fuzzy ideal with thresholds $(\alpha, \beta]$ of S. Now $(\delta \circ_\alpha^\beta \nu) \circ_\alpha^\beta \delta \subseteq (S \circ_\alpha^\beta \nu) \circ_\alpha^\beta S \subseteq \nu \circ_\alpha^\beta S \subseteq \nu_\alpha^\beta$ and $(\delta \circ_\alpha^\beta \nu) \circ_\alpha^\beta \delta \subseteq (\delta \circ_\alpha^\beta S) \circ_\alpha^\beta \delta \subseteq \delta_\alpha^\beta$, i.e., $(\delta \circ_\alpha^\beta \nu) \circ_\alpha^\beta \delta \subseteq \delta_\alpha^\beta \wedge \nu_\alpha^\beta = \delta \wedge_\alpha^\beta \nu$. Let $x \in S$, this implies that there exists $a \in S$ such that $x \leqslant (xa)x$. Now $xa \leqslant ((xa)x)a = (ax)(xa) = x((ax)a)$. Thus

$$((\delta \circ_\alpha^\beta \nu) \circ_\alpha^\beta \delta)(x) = \{((\delta \circ \nu) \circ \delta)(x) \wedge \beta\} \vee \alpha = \left\{\Big(\bigvee_{(y,z)\in A_x} \min\{(\delta \circ \nu)(y), \delta(z)\}\Big) \wedge \beta\right\} \vee \alpha \geqslant$$
$$\geqslant \{(\delta \circ \nu)(xa) \wedge \delta(x) \wedge \beta\} \vee \alpha = ((\delta \circ \nu)(xa) \vee \alpha) \wedge (\delta(x) \vee \alpha) \wedge (\beta \vee \alpha) =$$
$$= ((\delta \circ \nu)(xa) \vee \alpha) \wedge \delta(x) \wedge \beta = \Big(\Big(\bigvee_{(s,t)\in A_{xa}} \min\{\delta(s), \nu(t)\}\Big) \vee \alpha\Big) \wedge \delta(x) \wedge \beta \geqslant$$
$$\geqslant (\{\delta(x) \wedge \nu((ax)a)\} \vee \alpha) \wedge \delta(x) \wedge \beta = (\delta(x) \vee \alpha) \wedge (\nu((ax)a) \vee \alpha) \wedge \delta(x) \wedge \beta \geqslant$$
$$\geqslant \delta(x) \wedge (\nu(x) \wedge \beta) \wedge \delta(x) \wedge \beta = \delta(x) \wedge \nu(x) \wedge \beta = (\delta \wedge \nu)(x) \wedge \beta =$$
$$= \{(\delta \wedge \nu)(x) \wedge \beta\} \vee \alpha = (\delta \wedge_\alpha^\beta \nu)(x) \Rightarrow \delta \wedge_\alpha^\beta \nu \subseteq (\delta \circ_\alpha^\beta \nu) \circ_\alpha^\beta \delta.$$

Therefore $\delta \wedge_\alpha^\beta \nu = (\delta \circ_\alpha^\beta \nu) \circ_\alpha^\beta \delta$, i.e., (1) $\Rightarrow$ (4). Since (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2). Assume that (2) holds. Then $\mu \wedge_\alpha^\beta S = (\mu \circ_\alpha^\beta S) \circ_\alpha^\beta \mu$, where S itself is a fuzzy two-sided ideal with thresholds $(\alpha, \beta]$ of S, i.e., $\mu_\alpha^\beta = (\mu \circ_\alpha^\beta S) \circ \mu_\alpha^\beta$. Hence S is a regular by the Theorem 8, i.e., (2) $\Rightarrow$ (1). ■

Theorem 11. Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent:

(1) S is a regular;
(2) $\mu \wedge_\alpha^\beta \lambda \subseteq \lambda \circ_\alpha^\beta \mu$ for every fuzzy quasi-ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy right ideal λ with thresholds $(\alpha, \beta]$ of S;
(3) $\gamma \wedge_\alpha^\beta \lambda \subseteq \lambda \circ_\alpha^\beta \gamma$ for every fuzzy bi-ideal γ with thresholds $(\alpha, \beta]$ and every fuzzy right ideal λ with thresholds $(\alpha, \beta]$ of S;

(4) $\delta \wedge_\alpha^\beta \lambda \subseteq \lambda \circ_\alpha^\beta \delta$ for every fuzzy generalized bi-ideal δ with thresholds $(\alpha, \beta]$ and every fuzzy right ideal λ with thresholds $(\alpha, \beta]$ of S.

Proof. (1) $\Rightarrow$ (4) is obvious. It is clear that (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2). Assume that (2) holds, this means that $\lambda \wedge_\alpha^\beta \mu = \mu \wedge_\alpha^\beta \lambda \subseteq \lambda \circ_\alpha^\beta \mu$, where μ is a fuzzy left ideal with thresholds $(\alpha, \beta]$ of S. Since $\lambda \circ_\alpha^\beta \mu \subseteq \lambda \wedge_\alpha^\beta \mu$, so $\lambda \wedge_\alpha^\beta \mu = \lambda \circ_\alpha^\beta \mu$. Therefore S is a regular by the Theorem 8, i.e., (2) $\Rightarrow$ (1). ■

Theorem 12. Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent:

(1) S is a regular;

(2) $\mu \wedge_\alpha^\beta \lambda \wedge_\alpha^\beta \psi \subseteq (\mu \circ_\alpha^\beta \lambda) \circ_\alpha^\beta \psi$ for every fuzzy quasi-ideal μ with thresholds $(\alpha, \beta]$, every fuzzy right ideal λ with thresholds $(\alpha, \beta]$ and every fuzzy left ideal ψ with thresholds $(\alpha, \beta]$ of S;

(3) $\gamma \wedge_\alpha^\beta \lambda \wedge_\alpha^\beta \psi \subseteq (\gamma \circ_\alpha^\beta \lambda) \circ_\alpha^\beta \psi$ for every fuzzy bi-ideal γ with thresholds $(\alpha, \beta]$, every fuzzy right ideal λ with thresholds $(\alpha, \beta]$ and every fuzzy left ideal ψ with thresholds $(\alpha, \beta]$ of S;

(4) $\delta \wedge_\alpha^\beta \lambda \wedge_\alpha^\beta \psi \subseteq (\delta \circ_\alpha^\beta \lambda) \circ_\alpha^\beta \psi$ for every fuzzy generalized bi-ideal δ with thresholds $(\alpha, \beta]$, every fuzzy right ideal λ with thresholds $(\alpha, \beta]$ and every fuzzy left ideal ψ with thresholds $(\alpha, \beta]$ of S.

Proof. Consider that (1) holds. Let δ be a fuzzy generalized bi-ideal with thresholds $(\alpha, \beta]$, ψ be a fuzzy left ideal with thresholds $(\alpha, \beta]$ and λ be a fuzzy right ideal with thresholds $(\alpha, \beta]$ of S. Let $x \in S$, then there exists an element $a \in S$ such that $x \leqslant (xa)x$. Now

$$\begin{gathered} x \leqslant (xa)x; \\ xa \leqslant ((xa)x)a = (ax)(xa) = x((ax)a); \\ (ax)a \leqslant (a((xa)x))a = ((xa)(ax))a = (a(ax))(xa) = x((a(ax))a) = x(((ea)(ax))a) = \\ = x(((xa)(ae))a) = x((((ae)a)x)a) = x((nx)a) = x((nx)(ea)) = x((ae)(xn)) = \\ = x(x((ae)n)) = x(xm) \Rightarrow xa = x((ax)a) = x(x(xm)) = (ex)(x(xm)) = ((xm)x)(xe). \end{gathered}$$

Thus

$$\begin{gathered} ((\delta \circ_\alpha^\beta \lambda) \circ_\alpha^\beta \psi)(x) = \{((\delta \circ \lambda) \circ \psi)(x) \wedge \beta\} \vee \alpha = \\ = \Big\{ \Big(\bigvee_{(y,z) \in A_x} \min\{(\delta \circ \lambda)(y), \psi(z)\} \Big) \wedge \beta \Big\} \vee \alpha \geqslant \{(\delta \circ \lambda)(xa) \wedge \psi(x) \wedge \beta\} \vee \alpha = \\ = ((\delta \circ \lambda)(xa) \vee \alpha) \wedge (\psi(x) \vee \alpha) \wedge (\beta \vee \alpha) = ((\delta \circ \lambda)(xa) \vee \alpha) \wedge \psi(x) \wedge \beta = \\ = \Big(\Big(\bigvee_{(s,t) \in A_{xa}} \min\{\delta(s), \lambda(t)\} \Big) \vee \alpha \Big) \wedge \psi(x) \wedge \beta \geqslant (\{\delta((xm)x) \wedge \lambda(xe)\} \vee \alpha) \wedge \psi(x) \wedge \beta = \\ = (\delta((xm)x) \vee \alpha) \wedge (\lambda(xe) \vee \alpha) \wedge \psi(x) \wedge \beta \geqslant (\delta(x) \wedge \delta(x) \wedge \beta) \wedge (\lambda(x) \wedge \beta) \wedge \psi(x) \wedge \beta = \\ = \delta(x) \wedge \lambda(x) \wedge \psi(x) \wedge \beta = (\delta(x) \wedge \lambda(x) \wedge \psi(x) \wedge \beta) \vee \alpha = (\delta \wedge_\alpha^\beta \lambda \wedge_\alpha^\beta \psi)(x). \end{gathered}$$

Hence $\delta \wedge_\alpha^\beta \lambda \wedge_\alpha^\beta \psi \subseteq (\delta \circ_\alpha^\beta \lambda) \circ_\alpha^\beta \psi$, i.e., (1) $\Rightarrow$ (4). It is clear that (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2). Assume that (2) holds. Then $\mu \wedge_\alpha^\beta S \wedge_\alpha^\beta \psi \subseteq (\mu \circ_\alpha^\beta S) \circ_\alpha^\beta \psi$, where μ is a fuzzy right ideal with thresholds $(\alpha, \beta]$ of S, i.e., $\mu \wedge_\alpha^\beta \psi \subseteq \mu \circ_\alpha^\beta \psi$. Since $\mu \circ_\alpha^\beta \psi \subseteq \mu \wedge_\alpha^\beta \psi$, thus $\mu \circ_\alpha^\beta \psi = \mu \wedge_\alpha^\beta \psi$. So S is a regular by the Theorem 8, i.e., (2) $\Rightarrow$ (1). ■

3. Intra-regular ordered AG-groupoids

An ordered AG-groupoid S will be called an intra-regular ordered AG-groupoid if for every $x \in S$ there exist elements $a, b \in S$ such that $x \leqslant (ax^2)b$. Equivalent definitions are as follows:

(1) $A \subseteq ((SA^2)S]$ for every $A \subseteq S$;
(2) $x \in ((Sx^2)S]$ for every $x \in S$.

In this section, we characterize intra-regular ordered AG-groupoids in terms of fuzzy left (right, quasi-, bi-, generalized bi-) ideals with thresholds $(\alpha, \beta]$.

Lemma 20. Every fuzzy left (right) ideal with thresholds $(\alpha, \beta]$ of an intra-regular ordered AG-groupoid S is a fuzzy ideal with thresholds $(\alpha, \beta]$ of S.

Proof. Suppose that μ is a fuzzy left ideal with thresholds $(\alpha, \beta]$ of S. Let $x, y \in S$, this implies that there exist $a, b \in S$, such that $x \leqslant (ax^2)b$. Thus

$$\max\{\mu(xy), \alpha\} \geqslant \max\{\mu(((ax^2)b)y), \alpha\} = \max\{\mu((yb)(ax^2)), \alpha\} \geqslant \\ \geqslant \min\{\mu(a(xx)), \beta\} \geqslant \min\{\mu(xx), \beta\} \geqslant \min\{\mu(x), \beta\}.$$

Hence μ is a fuzzy ideal with thresholds $(\alpha, \beta]$ of S. ■

Lemma 21. Let S be an intra-regular ordered AG-groupoid with left identity e. Then every fuzzy ideal with thresholds $(\alpha, \beta]$ of S is a fuzzy idempotent with thresholds $(\alpha, \beta]$.

Proof. Assume that μ is a fuzzy ideal with thresholds $(\alpha, \beta]$ of S and $\mu \circ_\alpha^\beta \mu \subseteq \mu_\alpha^\beta$. Let $x \in S$, this means that there exist $a, b \in S$, such that $x \leqslant (ax^2)b$. Now

$$x \leqslant (ax^2)b = (a(xx))b = (x(ax))b = (x(ax))(eb) = (xe)((ax)b) = (ax)((xe)b).$$

Thus

$$(\mu \circ_\alpha^\beta \mu)(x) = \{(\mu \circ \mu)(x) \wedge \beta\} \vee \alpha = \Big\{\Big(\bigvee_{(y,z)\in A_x} \min\{\mu(y), \mu(z)\}\Big) \wedge \beta\Big\} \vee \alpha \geqslant \\ \geqslant \{\mu\,(ax) \wedge \mu\,((xe)b) \wedge \beta\} \vee \alpha = (\mu\,(ax) \vee \alpha) \wedge (\mu\,((xe)b) \vee \alpha) \wedge (\beta \vee \alpha) \geqslant \\ \geqslant (\mu\,(x) \wedge \beta) \wedge (\mu\,(x) \wedge \beta) \wedge \beta = \mu\,(x) \wedge \beta = (\mu\,(x) \wedge \beta) \vee \alpha = \mu_\alpha^\beta(x) \Rightarrow \mu_\alpha^\beta \subseteq \mu \circ_\alpha^\beta \mu.$$

Therefore $\mu_\alpha^\beta = \mu \circ_\alpha^\beta \mu$. ■

Proposition 9. Let μ be a fuzzy subset of an intra-regular ordered AG-groupoid S with left identity e. Then μ is a fuzzy ideal with thresholds $(\alpha, \beta]$ of S if and only if μ is a fuzzy interior ideal with thresholds $(\alpha, \beta]$ of S.

Proof. Consider that μ is a fuzzy interior ideal with thresholds $(\alpha, \beta]$ of S. Let $x, y \in S$, then there exist elements $a, b \in S$, such that $x \leqslant (ax^2)b$. Thus

$$\max\{\mu(xy), \alpha\} \geqslant \max\{\mu(((ax^2)b)y), \alpha\} = \max\{\mu((yb)(ax^2)), \alpha\} = \\ = \max\{\mu((yb)(a(xx))), \alpha\} = \max\{\mu((yb)(x(ax))), \alpha\} = \max\{\mu((yx)(b(ax))), \alpha\} \geqslant \\ \geqslant \min\{\mu(x), \beta\} \Rightarrow \max\{\mu(xy), \alpha\} \geqslant \min\{\mu(x), \beta\}.$$

Therefore μ is a fuzzy right ideal with thresholds $(\alpha, \beta]$ of S. So μ is a fuzzy ideal with thresholds $(\alpha, \beta]$ of S by the Lemma 20. Converse is true by the Lemma 10. ■

Remark 6. The concept of fuzzy (two-sided, interior) ideals with thresholds $(\alpha, \beta]$ coincides in intra-regular ordered AG-groupoids with left identity.

Lemma 22. Let S be an intra-regular ordered AG-groupoid with left identity e. Then $\gamma \wedge_{\alpha}^{\beta} \mu \subseteq \mu \circ_{\alpha}^{\beta} \gamma$ for every fuzzy left ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy right ideal γ with thresholds $(\alpha, \beta]$ of S.

Proof. Suppose that μ is a fuzzy left ideal with thresholds $(\alpha, \beta]$ and γ is a fuzzy right ideal with thresholds $(\alpha, \beta]$ of S. Let $x \in S$, this implies that there exist $a, b \in S$ such that $x \leqslant (ax^2)b$. Now

$$x \leqslant (ax^2)b = (a(xx))b = (x(ax))b = (x(ax))(eb) = (xe)((ax)b) = (ax)((xe)b).$$

Thus

$$\begin{aligned}(\mu \circ_{\alpha}^{\beta} \gamma)(x) = \{(\mu \circ \gamma)(x) \wedge \beta\} \vee \alpha = \Big\{\Big(\bigvee_{(y,z)\in A_x} \min\{\mu(y), \gamma(z)\}\Big) \wedge \beta\Big\} \vee \alpha \geqslant \\ \geqslant \{\mu(ax) \wedge \gamma((xe)b) \wedge \beta\} \vee \alpha = (\mu(ax) \vee \alpha) \wedge (\gamma((xe)b) \vee \alpha) \wedge (\beta \vee \alpha) \geqslant \\ \geqslant (\mu(x) \wedge \beta) \wedge (\gamma(x) \wedge \beta) \wedge \beta = \mu(x) \wedge \gamma(x) \wedge \beta = \gamma(x) \wedge \mu(x) \wedge \beta = \\ = (\gamma \wedge \mu)(x) \wedge \beta = \{(\gamma \wedge \mu)(x) \wedge \beta\} \vee \alpha = (\gamma \wedge_{\alpha}^{\beta} \mu)(x).\end{aligned}$$

Hence $\gamma \wedge_{\alpha}^{\beta} \mu \subseteq \mu \circ_{\alpha}^{\beta} \gamma$. ■

Theorem 13. Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent:

(1) S is an intra-regular;
(2) $\gamma \wedge_{\alpha}^{\beta} \mu \subseteq \mu \circ_{\alpha}^{\beta} \gamma$ for every fuzzy left ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy right ideal γ with thresholds $(\alpha, \beta]$ of S.

Proof. (1) $\Rightarrow$ (2) is true by the Lemma 22. Assume that (2) holds and $a \in S$. Then Sa is a left ideal of S containing a by the Lemma 18 and $aS \cup Sa$ is a right ideal of S containing a by the Proposition 8. This means that χ_{Sa} is a fuzzy left ideal with thresholds $(\alpha, \beta]$ and $\chi_{aS\cup Sa}$ is a fuzzy right ideal with thresholds $(\alpha, \beta]$ of S by the Theorem 2. By our assumption $\chi_{aS\cup Sa} \wedge_{\alpha}^{\beta} \chi_{Sa} \subseteq \chi_{Sa} \circ_{\alpha}^{\beta} \chi_{aS\cup Sa}$, i.e., $(\chi_{(aS\cup Sa)\cap Sa})_{\alpha}^{\beta} \subseteq (\chi_{((Sa)(aS\cup Sa)]})_{\alpha}^{\beta}$ by the Theorem 1. Thus $(aS \cup Sa) \cap Sa \subseteq (Sa(aS \cup Sa)]$. Since $a \in (aS \cup Sa) \cap Sa$, i.e., $a \in (Sa(aS \cup Sa)] = ((Sa)(aS) \cup (Sa)(Sa)]$. Now

$$\begin{aligned}(Sa)(aS) = (Sa)((ea)(SS)) = (Sa)((SS)(ae)) = (Sa)(((ae)S)S) = (Sa)((aS)S) = \\ = (Sa)((SS)a) = (Sa)(Sa).\end{aligned}$$

This implies that

$$\begin{aligned}((Sa)(aS) \cup (Sa)(Sa)] = ((Sa)(Sa) \cup (Sa)(Sa)] = ((Sa)(Sa)] = ((Sa)a)S] = \\ = (((Sa)(ea))S] = (((Se)(aa))S] = ((Sa^2)S].\end{aligned}$$

Thus $a \in (Sa^2)S$, i.e., a is an intra-regular. Hence S is an intra-regular, i.e., (2) $\Rightarrow$ (1). ■

Theorem 14. Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent:

(1) S is an intra-regular;
(2) $\mu \wedge_{\alpha}^{\beta} \nu = (\mu \circ_{\alpha}^{\beta} \nu) \circ_{\alpha}^{\beta} \mu$ for every fuzzy quasi-ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy ideal ν with thresholds $(\alpha, \beta]$ of S;
(3) $\gamma \wedge_{\alpha}^{\beta} \nu = (\gamma \circ_{\alpha}^{\beta} \nu) \circ_{\alpha}^{\beta} \gamma$ for every fuzzy bi-ideal γ with thresholds $(\alpha, \beta]$ and every fuzzy ideal ν with thresholds $(\alpha, \beta]$ of S;

(4) $\delta \wedge_{\alpha}^{\beta} \nu = (\delta \circ_{\alpha}^{\beta} \nu) \circ_{\alpha}^{\beta} \delta$ for every fuzzy generalized bi-ideal δ with thresholds $(\alpha, \beta]$ and every fuzzy ideal ν with thresholds $(\alpha, \beta]$ of S.

Proof. Consider that (1) holds. Let δ be a fuzzy generalized bi-ideal with thresholds $(\alpha, \beta]$ and ν be a fuzzy ideal with thresholds $(\alpha, \beta]$ of S. Now $(\delta \circ_{\alpha}^{\beta} \nu) \circ_{\alpha}^{\beta} \delta \subseteq (S \circ_{\alpha}^{\beta} \nu) \circ_{\alpha}^{\beta} S \subseteq \subseteq \nu \circ_{\alpha}^{\beta} S \subseteq \nu_{\alpha}^{\beta}$ and $(\delta \circ_{\alpha}^{\beta} \nu) \circ_{\alpha}^{\beta} \delta \subseteq (\delta \circ_{\alpha}^{\beta} S) \circ_{\alpha}^{\beta} \delta \subseteq \delta_{\alpha}^{\beta}$, thus $(\delta \circ \nu) \circ_{\alpha}^{\beta} \delta \subseteq \delta_{\alpha}^{\beta} \wedge \nu_{\alpha}^{\beta} = \delta \wedge_{\alpha}^{\beta} \nu$. Let $x \in S$, then there exist elements $a, b \in S$ such that $x \leqslant (ax^2)b$. Now

$$x \leqslant (ax^2)b = (a(xx))b = (x(ax))b = (b(ax))x;$$
$$b(ax) \leqslant b(a((ax^2)b)) = b((ax^2)(ab)) = b((ax^2)c) = (ax^2)(bc) = (ax^2)d = (ax^2)(ed) =$$
$$= (de)(x^2a) = m(x^2a) = x^2(ma) = (xx)l = (lx)x = (lx)(ex) = (xe)(xl) = x((xe)l).$$

Thus

$$((\delta \circ_{\alpha}^{\beta} \nu) \circ_{\alpha}^{\beta} \delta)(x) = \{((\delta \circ \nu) \circ \delta)(x) \wedge \beta\} \vee \alpha =$$
$$= \Big\{\Big(\bigvee_{(y,z) \in A_x} \min\{(\delta \circ \nu)(y), \delta(z)\}\Big) \wedge \beta\Big\} \vee \alpha \geqslant \{(\delta \circ \nu)(b(ax)) \wedge \delta(x) \wedge \beta\} \vee \alpha =$$
$$= ((\delta \circ \nu)(b(ax)) \vee \alpha) \wedge (\delta(x) \vee \alpha) \wedge (\beta \vee \alpha) = ((\delta \circ \nu)(b(ax)) \vee \alpha) \wedge \delta(x) \wedge \beta =$$
$$= \Big(\Big(\bigvee_{(s,t) \in A_{b(ax)}} \min\{\delta(s), \nu(t)\}\Big) \vee \alpha\Big) \wedge \delta(x) \wedge \beta \geqslant (\{\delta(x) \wedge \nu((xe)l)\} \vee \alpha) \wedge \delta(x) \wedge \beta =$$
$$= (\delta(x) \vee \alpha) \wedge (\nu((xe)l) \vee \alpha) \wedge \delta(x) \wedge \beta \geqslant \delta(x) \wedge (\nu(x) \wedge \beta) \wedge \delta(x) \wedge \beta =$$
$$= \delta(x) \wedge \nu(x) \wedge \beta = (\delta \wedge \nu)(x) \wedge \beta = \{(\delta \wedge \nu)(x) \wedge \beta\} \vee \alpha = (\delta \wedge_{\alpha}^{\beta} \nu)(x) \Rightarrow$$
$$\Rightarrow \delta \wedge_{\alpha}^{\beta} \nu \subseteq (\delta \circ_{\alpha}^{\beta} \nu) \circ_{\alpha}^{\beta} \delta.$$

Hence $\delta \wedge_{\alpha}^{\beta} \nu = (\delta \circ_{\alpha}^{\beta} \nu) \circ_{\alpha}^{\beta} \delta$, i.e., (1) implies (4). It is clear that (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2). Suppose that (2) holds. Let μ be a fuzzy right ideal with thresholds $(\alpha, \beta]$ and ν be a fuzzy two-sided ideal with thresholds $(\alpha, \beta]$ of S. Since every fuzzy right ideal with thresholds $(\alpha, \beta]$ of S is a fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S by the Lemma 13, this implies that μ is a fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S. By our supposition, $\mu \wedge_{\alpha}^{\beta} \nu = = (\mu \circ_{\alpha}^{\beta} \nu) \circ_{\alpha}^{\beta} \mu \subseteq (S \circ_{\alpha}^{\beta} \nu) \circ_{\alpha}^{\beta} \mu \subseteq \nu \circ_{\alpha}^{\beta} \mu$, i.e., $\mu \wedge_{\alpha}^{\beta} \nu \subseteq \nu \circ_{\alpha}^{\beta} \mu$. So S is an intra-regular by the Theorem 13, i.e., (2) $\Rightarrow$ (1). ■

Theorem 15. Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = = xS$ for all $x \in S$. Then the following conditions are equivalent:

(1) S is an intra-regular;

(2) $\mu \wedge_{\alpha}^{\beta} \psi \subseteq \psi \circ_{\alpha}^{\beta} \mu$ for every fuzzy quasi-ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy left ideal ψ with thresholds $(\alpha, \beta]$ of S;

(3) $\gamma \wedge_{\alpha}^{\beta} \psi \subseteq \psi \circ_{\alpha}^{\beta} \gamma$ for every fuzzy bi-ideal γ with thresholds $(\alpha, \beta]$ and every fuzzy left ideal ψ with thresholds $(\alpha, \beta]$ of S;

(4) $\delta \wedge_{\alpha}^{\beta} \psi \subseteq \psi \circ_{\alpha}^{\beta} \delta$ for every fuzzy generalized bi-ideal δ with thresholds $(\alpha, \beta]$ and every fuzzy left ideal ψ with thresholds $(\alpha, \beta]$ of S.

Proof. Suppose that (1) holds. Let δ be a fuzzy generalized bi-ideal with thresholds $(\alpha, \beta]$ and ψ be a fuzzy left ideal with thresholds $(\alpha, \beta]$ of S. Let $x \in S$, this implies that there exist $a, b \in S$ such that $x \leqslant (ax^2)b$. Now $x \leqslant (a(xx))b = (x(ax))b = (b(ax))x$. Thus

$$(\psi \circ_{\alpha}^{\beta} \delta)(x) = \{(\psi \circ \delta)(x) \wedge \beta\} \vee \alpha = \Big\{\Big(\bigvee_{(y,z) \in A_x} \min\{\psi(y), \delta(z)\}\Big) \wedge \beta\Big\} \vee \alpha \geqslant$$
$$\geqslant \{\{\psi(b(ax)) \wedge \delta(x)\} \wedge \beta\} \vee \alpha = (\psi(b(ax)) \vee \alpha) \wedge (\delta(x) \vee \alpha) \wedge (\beta \vee \alpha) \geqslant$$

$$\geqslant (\psi(x) \wedge \beta) \wedge \delta(x) \wedge \beta = \psi(x) \wedge \delta(x) \wedge \beta = \delta(x) \wedge \psi(x) \wedge \beta = (\delta \wedge \psi)(x) \wedge \beta =$$
$$= \{(\delta \wedge \psi)(x) \wedge \beta\} \vee \alpha = (\delta \wedge_\alpha^\beta \psi)(x) \Rightarrow \delta \wedge_\alpha^\beta \psi \subseteq \psi \circ_\alpha^\beta \delta.$$

Hence (1) $\Rightarrow$ (4). It is clear that (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2). Assume that (2) holds. Let μ be a fuzzy right ideal with thresholds $(\alpha, \beta]$ and ψ be a fuzzy left ideal with thresholds $(\alpha, \beta]$ of S. Since every fuzzy right ideal with thresholds $(\alpha, \beta]$ of S is a fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S, this means that μ is a fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S. By our assumption, $\mu \wedge_\alpha^\beta \psi \subseteq \psi \circ_\alpha^\beta \mu$. Therefore S is an intra-regular by the Theorem 13, i.e., (2) $\Rightarrow$ (1). ■

Theorem 16. Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent:

(1) S is an intra-regular;
(2) $\mu \wedge_\alpha^\beta \psi \wedge_\alpha^\beta \lambda \subseteq (\psi \circ_\alpha^\beta \mu) \circ_\alpha^\beta \lambda$ for every fuzzy quasi-ideal μ with thresholds $(\alpha, \beta]$, every fuzzy left ideal ψ with thresholds $(\alpha, \beta]$, and every fuzzy right ideal λ with thresholds $(\alpha, \beta]$ of S;
(3) $\gamma \wedge_\alpha^\beta \psi \wedge_\alpha^\beta \lambda \subseteq (\psi \circ_\alpha^\beta \gamma) \circ_\alpha^\beta \lambda$ for every fuzzy bi-ideal γ with thresholds $(\alpha, \beta]$, every fuzzy left ideal ψ with thresholds $(\alpha, \beta]$, and every fuzzy right ideal λ with thresholds $(\alpha, \beta]$ of S;
(4) $\delta \wedge_\alpha^\beta \psi \wedge_\alpha^\beta \lambda \subseteq (\psi \circ_\alpha^\beta \delta) \circ_\alpha^\beta \lambda$ for every fuzzy generalized bi-ideal δ with thresholds $(\alpha, \beta]$, every fuzzy left ideal ψ with thresholds $(\alpha, \beta]$, and every fuzzy right ideal λ with thresholds $(\alpha, \beta]$ of S.

Proof. Assume that (1) holds. Let δ be a fuzzy generalized bi-ideal with thresholds $(\alpha, \beta]$, ψ be a fuzzy left ideal with thresholds $(\alpha, \beta]$ and λ be a fuzzy right ideal with thresholds $(\alpha, \beta]$ of S. Let $x \in S$, this means that there exist $a, b \in S$ such that $x \leqslant (ax^2)b$. Now

$$x \leqslant (a(xx))b = (x(ax))b = (b(ax))x;$$
$$b(ax) \leqslant b(a((ax^2)b)) = b((ax^2)(ab)) = b((ax^2)c) = (ax^2)(bc) = (ax^2)d =$$
$$= (a(xx))d = (x(ax))d = (d(ax))x.$$

Thus

$$((\psi \circ_\alpha^\beta \delta) \circ_\alpha^\beta \lambda)(x) = \{((\psi \circ \delta) \circ \lambda)(x) \wedge \beta\} \vee \alpha =$$
$$= \left\{\Big(\bigvee_{(y,z) \in A_x} \min\{(\psi \circ \delta)(y), \lambda(z)\}\Big) \wedge \beta\right\} \vee \alpha \geqslant \{(\psi \circ \delta)(b(ax)) \wedge \lambda(x) \wedge \beta\} \vee \alpha =$$
$$= ((\psi \circ \delta)(b(ax)) \vee \alpha) \wedge (\lambda(x) \vee \alpha) \wedge (\beta \vee \alpha) \geqslant ((\psi \circ \delta)(b(ax)) \vee \alpha) \wedge \lambda(x) \wedge \beta =$$
$$= \Big(\Big(\bigvee_{(s,t) \in A_{b(ax)}} \min\{\psi(s), \delta(t)\}\Big) \vee \alpha\Big) \wedge \lambda(x) \wedge \beta \geqslant (\{\psi(d(ax)) \wedge \delta(t)\} \vee \alpha) \wedge \lambda(x) \wedge \beta =$$
$$= (\psi(d(ax)) \vee \alpha) \wedge (\delta(x) \vee \alpha) \wedge \lambda(x) \wedge \beta \geqslant (\psi(x) \wedge \beta) \wedge \delta(x) \wedge \lambda(x) \wedge \beta =$$
$$= \psi(x) \wedge \delta(x) \wedge \lambda(x) \wedge \beta = (\psi \wedge \delta \wedge \lambda)(x) \wedge \beta = \{(\delta \wedge \psi \wedge \lambda)(x) \wedge \beta\} \vee \alpha =$$
$$= (\delta \wedge_\alpha^\beta \psi \wedge_\alpha^\beta \lambda)(x) \Rightarrow \delta \wedge_\alpha^\beta \psi \wedge_\alpha^\beta \lambda \subseteq (\psi \circ_\alpha^\beta \delta) \circ_\alpha^\beta \lambda.$$

Therefore (1) $\Rightarrow$ (4). Since (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2). Suppose that (2) holds. Then $\psi \wedge_\alpha^\beta S \wedge_\alpha^\beta \delta \subseteq (S \circ_\alpha^\beta \psi) \circ_\alpha^\beta \delta$, where ψ is a fuzzy left ideal with thresholds $(\alpha, \beta]$ of S, i.e., $\psi \wedge_\alpha^\beta \delta \subseteq \psi \circ_\alpha^\beta \delta$. Hence S is an intra-regular, i.e., (2) $\Rightarrow$ (1). ■

4. Regular and Intra-regular Ordered AG-groupoids

In this section, we give the characterizations of both regular and intra-regular ordered AG-groupoid in terms of fuzzy left (right, quasi-, bi-, generalized bi-) ideals with thresholds $(\alpha, \beta]$.

Theorem 17. Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent:

(1) S is both a regular and an intra-regular;

(2) Every fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S is a fuzzy idempotent with thresholds $(\alpha, \beta]$.

Proof. Suppose that S is both a regular and an intra-regular. Let μ be a fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S. Then μ be a fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of S and $\mu \circ_\alpha^\beta \mu \subseteq \mu_\alpha^\beta$. Let $x \in S$, this implies that there exists an element $a \in S$ such that $x \leqslant (xa)x$, and also there exist elements $a, b \in S$ such that $x \leqslant (ax^2)b$. Now

$$x \leqslant (xa)x;$$
$$xa \leqslant ((ax^2)b)a = (ab)(ax^2) = c(a(xx)) = c(x(ax)) = x(c(ax)) = x((ec)(ax)) = \\ = x((xa)(ce)) = x((xa)d) = x((da)x) = x(lx) = l(xx) = (el)(xx) = (xx)(le) = \\ = (xx)m = (mx)x;$$
$$mx \leqslant m((ax^2)b) = (ax^2)(mb) = (a(xx))n = (x(ax))n = (x(ax))(en) = (xe)((ax)n) = \\ = (xe)((ax)(en)) = (xe)((ae)(xn)) = (xe)(x((ae)n)) = (xe)(xu) = x((xe)u) = xw;$$
$$xa \leqslant (mx)x = (xw)x.$$

Thus

$$(\mu \circ_\alpha^\beta \mu)(x) = \{(\mu \circ \mu)(x) \wedge \beta\} \vee \alpha = \Big\{\Big(\bigvee_{(y,z)\in A_x} \min\{\mu(y), \mu(z)\}\Big) \wedge \beta\Big\} \vee \alpha \geqslant$$
$$\geqslant \{\mu((xw)x) \wedge \mu(x) \wedge \beta\} \vee \alpha = (\mu((xw)x) \vee \alpha) \wedge (\mu(x) \vee \alpha) \wedge (\beta \vee \alpha) \geqslant$$
$$\geqslant (\mu(x) \wedge \mu(x) \wedge \beta) \wedge \mu(x) \wedge \beta = \mu(x) \wedge \beta = (\mu(x) \wedge \beta) \vee \alpha = \mu_\alpha^\beta(x) \Rightarrow \mu_\alpha^\beta \subseteq \mu \circ_\alpha^\beta \mu.$$

Hence $\mu_\alpha^\beta = \mu \circ_\alpha^\beta \mu$. Conversely, assume that every fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S is a fuzzy idempotent with thresholds $(\alpha, \beta]$ of S. Let $a \in S$, then Sa is a left ideal of S containing a by the Lemma 18. This means that Sa is a quasi-ideal of S, so χ_{Sa} is a fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S by the Theorem 4. By our assumption, $(\chi_{Sa})_\alpha^\beta = \chi_{Sa} \circ_\alpha^\beta \chi_{Sa} = (\chi_{((Sa)(Sa)]})_\alpha^\beta$, i.e., $Sa = ((Sa)(Sa)]$. Since $a \in Sa$, i.e., $a \in ((Sa)(Sa)]$. Thus a is both a regular and an intra-regular by the Theorems 8 and 13, respectively. Therefore S is both a regular and an intra-regular, i.e., (2) $\Rightarrow$ (1). ■

Theorem 18. Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent:

(1) S is both a regular and an intra-regular;

(2) $\mu \wedge_\alpha^\beta \gamma \subseteq \mu \circ_\alpha^\beta \gamma$ for all fuzzy quasi-ideals μ and γ with thresholds $(\alpha, \beta]$ of S;

(3) $\mu \wedge_\alpha^\beta \gamma \subseteq \mu \circ_\alpha^\beta \gamma$ for every fuzzy quasi-ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy bi-ideal γ with thresholds $(\alpha, \beta]$ of S;

(4) $\mu \wedge_\alpha^\beta \gamma \subseteq \mu \circ_\alpha^\beta \gamma$ for every fuzzy bi-ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy quasi-ideal γ with thresholds $(\alpha, \beta]$ of S;

(5) $\mu \wedge_\alpha^\beta \gamma \subseteq \mu \circ_\alpha^\beta \gamma$ for all fuzzy bi-ideals μ and γ with thresholds $(\alpha, \beta]$ of S;

(6) $\mu \wedge_\alpha^\beta \gamma \subseteq \mu \circ_\alpha^\beta \gamma$ for every fuzzy bi-ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy generalized bi-ideal γ with thresholds $(\alpha, \beta]$ of S;

(7) $\mu \wedge_{\alpha}^{\beta} \gamma \subseteq \mu \circ_{\alpha}^{\beta} \gamma$ for every fuzzy generalized bi-ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy quasi-ideal γ with thresholds $(\alpha, \beta]$ of S;
(8) $\mu \wedge_{\alpha}^{\beta} \gamma \subseteq \mu \circ_{\alpha}^{\beta} \gamma$ for every fuzzy generalized bi-ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy bi-ideal γ with thresholds $(\alpha, \beta]$ of S;
(9) $\mu \wedge_{\alpha}^{\beta} \mu \subseteq \mu \circ_{\alpha}^{\beta} \gamma$ for all fuzzy generalized bi-ideals μ and γ with thresholds $(\alpha, \beta]$ of S.

Proof. Assume that (1) holds. Let μ and γ be two fuzzy generalized bi-ideals with thresholds $(\alpha, \beta]$ of S. Let $x \in S$, then means that there exists an element $a \in S$ such that $x = (xa)x$, and also there exist elements $a, b \in S$ such that $x \leqslant (ax^2)b$. Since $x = (xa)x = ((xw)x)x$ by the Theorem 17. Thus

$$(\mu \circ_{\alpha}^{\beta} \gamma)(x) = \{(\mu_A \circ \gamma)(x) \wedge \beta\} \vee \alpha = \left\{\Big(\bigvee_{(y,z)\in A_x} \min\{\mu(y), \gamma(z)\}\Big) \wedge \beta\right\} \vee \alpha \geqslant$$
$$\geqslant \{\mu\left((xw)x\right) \wedge \gamma\left(x\right) \wedge \beta\} \vee \alpha = \left(\mu\left((xw)x\right) \vee \alpha\right) \wedge \left(\gamma\left(x\right) \vee \alpha\right) \wedge \left(\beta \vee \alpha\right) \geqslant$$
$$\geqslant \left(\mu\left(x\right) \wedge \mu\left(x\right) \wedge \beta\right) \wedge \gamma\left(x\right) \wedge \beta = \mu\left(x\right) \wedge \gamma\left(x\right) \wedge \beta = \left(\mu \wedge \gamma\right)\left(x\right) \wedge \beta =$$
$$= \{\left(\mu \wedge \gamma\right)\left(x\right) \wedge \beta\} \vee \alpha = (\mu \wedge_{\alpha}^{\beta} \gamma)(x) \Rightarrow \mu \wedge_{\alpha}^{\beta} \gamma \subseteq \mu \circ_{\alpha}^{\beta} \gamma.$$

Hence (1) $\Rightarrow$ (9). It is clear that (9) $\Rightarrow$ (8) $\Rightarrow$ (7) $\Rightarrow$ (4) $\Rightarrow$ (2) and (9) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (3). Suppose that (2) holds. Let μ be a fuzzy right ideal with thresholds $(\alpha, \beta]$ and γ be a fuzzy left ideal with thresholds $(\alpha, \beta]$ of S. Since every fuzzy right ideal with thresholds $(\alpha, \beta]$ and every fuzzy left ideal with thresholds $(\alpha, \beta]$ of S is a fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of S by the Lemma 13. By our supposition, $\mu \wedge_{\alpha}^{\beta} \gamma \subseteq \mu \circ_{\alpha}^{\beta} \gamma$. Since $\mu \circ_{\alpha}^{\beta} \gamma \subseteq \mu \wedge_{\alpha}^{\beta} \gamma$, so $\mu \wedge_{\alpha}^{\beta} \gamma = \mu \circ_{\alpha}^{\beta} \gamma$, i.e., S is a regular. Again by our supposition, $\mu \wedge_{\alpha}^{\beta} \gamma = \gamma \wedge_{\alpha}^{\beta} \mu \subseteq \gamma \circ_{\alpha}^{\beta} \mu$, i.e., S is an intra-regular. Therefore S is both a regular and an intra-regular, i.e., (2) $\Rightarrow$ (1). In similar way, we can prove that (3) $\Rightarrow$ (1). ∎

Theorem 19. Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent:

(1) S is both a regular and an intra-regular;
(2) $\mu \wedge_{\alpha}^{\beta} \gamma \subseteq (\mu \circ_{\alpha}^{\beta} \gamma) \wedge (\gamma \circ_{\alpha}^{\beta} \mu)$ for every fuzzy right ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy left ideal γ with thresholds $(\alpha, \beta]$ of S;
(3) $\mu \wedge_{\alpha}^{\beta} \gamma \subseteq (\mu \circ_{\alpha}^{\beta} \gamma) \wedge (\gamma \circ_{\alpha}^{\beta} \mu)$ for every fuzzy right ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy quasi-ideal γ with thresholds $(\alpha, \beta]$ of S;
(4) $\mu \wedge_{\alpha}^{\beta} \gamma \subseteq (\mu \circ_{\alpha}^{\beta} \gamma) \wedge (\gamma \circ_{\alpha}^{\beta} \mu)$ for every fuzzy right ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy bi-ideal γ with thresholds $(\alpha, \beta]$ of S;
(5) $\mu \wedge_{\alpha}^{\beta} \gamma \subseteq (\mu \circ_{\alpha}^{\beta} \gamma) \wedge (\gamma \circ_{\alpha}^{\beta} \mu)$ for every fuzzy right ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy generalized bi-ideal γ with thresholds $(\alpha, \beta]$ of S;
(6) $\mu \wedge_{\alpha}^{\beta} \gamma \subseteq (\mu \circ_{\alpha}^{\beta} \gamma) \wedge (\gamma \circ \mu)$ for every fuzzy left ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy quasi-ideal γ with thresholds $(\alpha, \beta]$ of S;
(7) $\mu \wedge_{\alpha}^{\beta} \gamma \subseteq (\mu \circ_{\alpha}^{\beta} \gamma) \wedge (\gamma \circ_{\alpha}^{\beta} \mu)$ for every fuzzy left ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy bi-ideal γ with thresholds $(\alpha, \beta]$ of S;
(8) $\mu \wedge_{\alpha}^{\beta} \gamma \subseteq (\mu \circ_{\alpha}^{\beta} \gamma) \wedge (\gamma \circ_{\alpha}^{\beta} \mu)$ for every fuzzy left ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy generalized bi-ideal γ with thresholds $(\alpha, \beta]$ of S;
(9) $\mu \wedge_{\alpha}^{\beta} \gamma \subseteq (\mu \circ_{\alpha}^{\beta} \gamma) \wedge (\gamma \circ_{\alpha}^{\beta} \mu)$ for all fuzzy quasi-ideals μ and γ with thresholds $(\alpha, \beta]$ of S;
(10) $\mu \wedge_{\alpha}^{\beta} \gamma \subseteq (\mu \circ_{\alpha}^{\beta} \gamma) \wedge (\gamma \circ_{\alpha}^{\beta} \mu)$ for every fuzzy quasi-ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy bi-ideal γ with thresholds $(\alpha, \beta]$ of S;
(11) $\mu \wedge_{\alpha}^{\beta} \gamma \subseteq (\mu \circ_{\alpha}^{\beta} \gamma) \wedge (\mu \circ_{\alpha}^{\beta} \gamma)$ for every fuzzy quasi-ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy generalized bi-ideal γ with thresholds $(\alpha, \beta]$ of S;

(12) $\mu \wedge_\alpha^\beta \gamma \subseteq (\mu \circ_\alpha^\beta \gamma) \wedge (\gamma \circ_\alpha^\beta \mu)$ for all fuzzy bi-ideals μ and γ with thresholds $(\alpha, \beta]$ of S;

(13) $\mu \wedge_\alpha^\beta \gamma \subseteq (\mu \circ_\alpha^\beta \gamma) \wedge (\gamma \circ_\alpha^\beta \mu)$ for every fuzzy bi-ideal μ with thresholds $(\alpha, \beta]$ and every fuzzy generalized bi-ideal γ with thresholds $(\alpha, \beta]$ of S;

(14) $\mu \wedge_\alpha^\beta \gamma \subseteq (\mu \circ_\alpha^\beta \gamma) \wedge (\gamma \circ_\alpha^\beta \mu)$ for all fuzzy generalized bi-ideals μ and γ with thresholds $(\alpha, \beta]$ of S.

Proof. Consider that (1) holds. Since $\mu \wedge_\alpha^\beta \gamma \subseteq \mu \circ_\alpha^\beta \gamma$ and $\mu \wedge_\alpha^\beta \gamma \subseteq \gamma \circ_\alpha^\beta \mu$ for all fuzzy generalized bi-ideals μ and γ with thresholds $(\alpha, \beta]$ of S by the Theorem 18. Hence $\mu \wedge_\alpha^\beta \gamma \subseteq$ $\subseteq (\mu \circ_\alpha^\beta \gamma) \wedge (\gamma \circ_\alpha^\beta \mu)$, i.e., (1) $\Rightarrow$ (14). It is clear that (14) $\Rightarrow$ (13) $\Rightarrow$ (12) $\Rightarrow$ (9) $\Rightarrow$ (6) $\Rightarrow$ (2), (14) $\Rightarrow$ (11) $\Rightarrow$ (10) $\Rightarrow$ (9), (14) $\Rightarrow$ (8) $\Rightarrow$ (7) $\Rightarrow$ (6) and (14) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2). Suppose that (2) holds. Let μ be a fuzzy right ideal with thresholds $(\alpha, \beta]$ and γ be a fuzzy left ideal with thresholds $(\alpha, \beta]$ of S. By our supposition, $\mu \wedge_\alpha^\beta \gamma \subseteq (\mu \circ_\alpha^\beta \gamma) \wedge (\gamma \circ_\alpha^\beta \mu) \subseteq \gamma \circ_\alpha^\beta \mu$, i.e., S is an intra-regular. Again $\mu \wedge_\alpha^\beta \gamma \subseteq (\mu \circ_\alpha^\beta \gamma) \wedge (\gamma \circ_\alpha^\beta \mu) \subseteq \mu \circ_\alpha^\beta \gamma$. Since $\mu \circ_\alpha^\beta \gamma \subseteq \mu \wedge_\alpha^\beta \gamma$, so $\mu \wedge_\alpha^\beta \gamma = \mu \circ_\alpha^\beta \gamma$, i.e., S is a regular. Hence S is both a regular and an intra-regular, i.e., (2) $\Rightarrow$ (1). ■

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