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Spectral estimates of the p-Laplace Neumann operator and Brennan's conjecture

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Abstract In this paper we obtain lower estimates for the first non-trivial eigenvalue of the p-Laplace Neumann operator in bounded simply connected planar domains $\Omega \subset \mathbb{R}^2$. This study is based on a quasiconformal version of the universal two-weight Poincaré–Sobolev inequalities obtained in our previous papers for conformal weights and its non weighted version for so-called K-quasiconformal α-regular domains. The main technical tool is the geometric theory of composition operators in relation with the Brennan's conjecture for (quasi)conformal mappings.

Keywords Elliptic equations · Sobolev spaces · Quasiconformal mappings

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1 Introduction

In this paper we obtain lower estimates for the first non-trivial eigenvalue of the p-Laplace operator with the Neumann boundary condition

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$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \mu_p |u|^{p-2}u & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

in bounded simply connected planar domains $\Omega \subset \mathbb{R}^2$. The weak statement of this spectral problem is as follows: a function u solves the previous problem iff $u \in W^1_p(\Omega)$ and

$$\int\limits_{\Omega} (|\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x))\, dx = \mu_p \int\limits_{\Omega} |u(x)|^{p-2}u(x)v(x)\, dx$$

for all $v \in W^1_p(\Omega)$.

We demonstrate that integrability of Jacobians of quasiconformal mappings with the exponent greater than one permit us to obtain lower estimates of the first non-trivial eigenvalue $\mu_p^{(1)}(\Omega)$ in terms of Sobolev norms of quasiconformal mappings of the unit disc $\mathbb{D}$ onto Ω. So, we can conclude that $\mu_p^{(1)}(\Omega)$ depends on the quasiconformal geometry of Ω only:

Theorem A *Let $\Omega \subset \mathbb{R}^2$ be a K-quasiconformal α-regular domain and $\varphi : \Omega \to \mathbb{D}$ be a corresponding K-quasiconformal mapping. Suppose that the Brennan's conjecture holds. Then for every*

$$p \in \left(\max\left\{\frac{4K}{2K+1}, \frac{\alpha(2K-1)+2}{\alpha K}\right\}, 2\right)$$

the following estimate

$$\frac{1}{\mu_p^{(1)}(\Omega)} \leq K \| J_{\varphi^{-1}} \mid L_{\frac{\alpha}{2}}(\mathbb{D})\| \inf_{q\in I} \left\{ B^p_{\frac{\alpha p}{\alpha-2},q}(\mathbb{D}) \| |D\varphi^{-1}|^{p-2} \mid L_{\frac{q}{p-q}}(\mathbb{D})\| \right\}$$

holds, where $I = [1, 2p/(4K-(2K-1)p))$.

Here $B_{r,q}(\mathbb{D})$ is the best constant in the (r,q)-Poincaré–Sobolev inequality in the unit disc $\mathbb{D}$ for $r = \alpha p/(\alpha-2)$ which satisfies [11,18]:

$$B_{r,q}(\mathbb{D}) \leq \frac{2}{\pi^{\nu}} \left(\frac{1-\nu}{1/2-\nu}\right)^{1-\nu}, \quad \nu = 1/q - 1/r.$$

Remark 1 In Theorem A the unit disc $\mathbb{D}$ can be replaced by any quasiisometrical image of $\mathbb{D}$.

As an example consider lower estimates of the first non-trivial eigenvalue of p-Laplace operator in non-convex star-shaped domains $\Omega^*_{\varepsilon(k)}$ (Example C), where $\Omega^*_{\varepsilon(k)}$ is the image of the square

$$\mathbb{Q} := \left\{(x,y) \in \mathbb{R}^2 : -\frac{\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} < y < \frac{\sqrt{2}}{2}\right\}$$

under the $(k+1)$-quasiconformal mapping

$$w = |z|^k z, \qquad z = x + iy, \quad k > 0.$$

If $p = 3/2$ (this operator arises in study of porous media flows [35]), then

$$\frac{1}{\mu_{3/2}^{(1)}(\Omega^*_{\varepsilon(k)})} \leq 16\sqrt{\frac{(k+1)^3}{2-k}}, \quad 0 \leq k < 2.$$

More examples of lower estimates of first-non-trivial eigenvalues of the p-Laplace operator will be given in Sect. 5.

Let us give few detailed comments to the theorem:

1.1 K-quasiconformal α-regular domains

Recall that a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ between planar domains is called K-quasiconformal if it preserves orientation, belongs to the Sobolev class $W^1_{2,\mathrm{loc}}(\Omega)$ and its directional derivatives ∂_ξ satisfy the distortion inequality

$$\max_\xi |\partial_\xi \varphi| \leq K \min_\xi |\partial_\xi \varphi| \quad \text{a.e. in } \Omega.$$

The notion of conformal regular domains was introduced in [8] and was used for the study of conformal spectral stability of the Laplace operator. In the present work we introduce a more general notion of quasiconformal regular domains.

Definition 1 A simply connected planar domain $\Omega \subset \mathbb{R}^2$ is called a K-quasiconformal α-regular domain if there exists a K-quasiconformal mapping $\varphi : \Omega \to \mathbb{D}$ such that

$$\int_{\mathbb{D}} |J(y, \varphi^{-1})|^{\frac{\alpha}{2}}\, dy < \infty \quad \text{for some } \alpha > 2.$$

The domain $\Omega \subset \mathbb{R}^2$ is called a K-quasiconformal regular domain if it is a K-quasiconformal α-regular domain for some $\alpha > 2$.

Note that the class of quasiconformal regular domains includes the class of Gehring domains [1] and can be described in terms of quasihyperbolic geometry [23].

Remark 2 The notion of quasiconformal α-regular domain is more general then the notion of conformal α-regular domain. Consider, for example, the unit square $\mathbb{Q} \subset \mathbb{R}^2$. Then $\mathbb{Q}$ is a conformal α-regular domain for $2 < \alpha \leq 4$ [18] and is a quasiconformal α-regular domain for all $2 < \alpha \leq \infty$ because the unit square $\mathbb{Q}$ is quasiisometrically equivalent to the unit disc $\mathbb{D}$. It implies, in particular, that in Theorem A (its more detailed version is Theorem 8) the unit ball $\mathbb{D}$ can be replaced by the square $\mathbb{Q}$ (or by another domain quasiisometrically equivalent to the unit disc $\mathbb{D}$).

Remark 3 Because $\varphi : \Omega \to \mathbb{D}$ is a quasiconformal mapping, then integrability of the derivative is equivalent to integrability of the Jacobian:

$$\int_{\mathbb{D}} |J(y, \varphi^{-1})|^{\frac{\alpha}{2}}\, dy \leq \int_{\mathbb{D}} |D\varphi^{-1}(y)|^{\alpha}\, dy \leq K^{\frac{\alpha}{2}} \int_{\mathbb{D}} |J(y, \varphi^{-1})|^{\frac{\alpha}{2}}\, dy.$$

1.2 Brennan's conjecture

We can conclude from Theorem A that Brennan's conjecture leads to the spectral estimates of the p-Laplace operator in quasiconformal α-regular domains $\Omega \subset \mathbb{R}^2$.

Generalized Brennan's conjecture for quasiconformal mappings [22] states that

$$\int_{\Omega} |D\varphi(x)|^{\beta}\, dx < +\infty, \quad \text{for all } \frac{4K}{2K+1} < \beta < \frac{2K\beta_0}{(K-1)\beta_0 + 2}. \tag{1}$$

If $K = 1$ we have Brennan's conjecture for conformal mappings [7] which is proved for $\beta \in (4/3, \beta_0)$, where $\beta_0 = 3.752$ [20], and conjectured for $\beta_0 = 4$ [7].

1.3 Historical sketch

In 1961 Pólya [27] obtained upper estimates for eigenvalues of Neumann–Laplace operator in so-called plane-covering domains. Namely, for the first eigenvalue:

$$\mu_2^{(1)}(\Omega) \leq 4\pi |\Omega|^{-1}.$$

The lower estimates for the $\mu_p^{(1)}(\Omega)$ were originally established only for convex domains. In the classical work [26] it was proved that if Ω is convex with diameter $d(\Omega)$ (see, also [9,10,29]), then

$$\mu_2^{(1)}(\Omega) \geq \frac{\pi^2}{d(\Omega)^2}. \tag{2}$$

In [9] it was proved that if $\Omega \subset \mathbb{R}^n$ is a bounded convex domain having diameter d then for $p \geq 2$

$$\mu_p^{(1)}(\Omega) \geq \left(\frac{\pi_p}{d(\Omega)}\right)^p$$

where

$$\pi_p = 2 \int\limits_0^{(p-1)^{\frac{1}{p}}} \frac{dt}{(1 - t^p/(p-1))^{\frac{1}{p}}} = 2\pi \frac{(p-1)^{\frac{1}{p}}}{p \sin(\pi/p)}.$$

In the case of non-convex domains in [19] it was proved that if $\Omega \subset \mathbb{R}^2$ is a conformal α-regular domain then for every $p \in (\max\{4/3, (\alpha+2)/\alpha\}, 2)$ the following inequality holds

$$\frac{1}{\mu_p^{(1)}(\Omega)} \leq \inf_{q \in [1, 2p/(4-p))} \left\{ B^p_{\frac{\alpha p}{\alpha - 2}, q}(\mathbb{D}) \cdot \| D\varphi^{-1}|^{p-2} \mid L_{\frac{q}{p-q}}(\mathbb{D})\| \right\} \|D\varphi^{-1}| \mid L_\alpha(\mathbb{D})\|^2,$$

where $\varphi : \Omega \to \mathbb{D}$ is a conformal mapping.

The lower estimates in terms of isoperimetric constants relative to Ω were obtained in [5,6].

The first non-trivial eigenvalue of the Neumann boundary problem for the p-Laplace operator $\mu_p^{(1)}(\Omega)^{-\frac{1}{p}}$ is equal to the best constant $B_{p,p}(\Omega)$ (see, for example, [25]) in the p-Poincaré–Sobolev inequality

$$\inf_{c \in \mathbb{R}} \|f - c \mid L_p(\Omega)\| \leq B_{p,p}(\Omega) \|\nabla f \mid L_p(\Omega)\|, \quad f \in W_p^1(\Omega).$$

1.4 Methods

The suggested method is based on the geometric theory of composition operators in relation with Brennan's conjecture that allows us to obtain universal two-weight Poincaré–Sobolev inequalities in any simply connected domain $\Omega \neq \mathbb{R}^2$

$$\inf_{c \in \mathbb{R}} \left(\int\limits_\Omega |f(x) - c|^r h(x) dx \right)^{\frac{1}{r}} \leq B_{r,p}(\Omega, h) \left(\int\limits_\Omega |\nabla f(x)|^p dx \right)^{\frac{1}{p}}, \quad f \in W_p^1(\Omega),$$

where quasiconformal measures $h(x)dx := |J(x,\varphi)|dx$ are generated by quasiconformal homeomorphisms $\varphi : \Omega \to \mathbb{D}$ and dx is the Lebesgue measure (Theorem 5). In quasiconformal regular domains these two-weight inequalities imply non-weight Poincaré–Sobolev inequalities. This method is also based on the theory of composition operators [28,33] and its applications to the Sobolev type embedding theorems [13,14].

The following diagram illustrates the main idea:

$$\begin{array}{ccc} W^1_p(\Omega) & \overset{(\varphi^{-1})^*}{\longrightarrow} & W^1_q(\mathbb{D}) \\ \downarrow & & \downarrow \\ L_s(\Omega) & \overset{\varphi^*}{\longleftarrow} & L_r(\mathbb{D}). \end{array}$$

Here the operator $(\varphi^{-1})^*$ defined by the composition rule $(\varphi^{-1})^*(f) = f \circ \varphi^{-1}$ is a bounded composition operator on Sobolev spaces induced by a homeomorphism φ of Ω onto $\mathbb{D}$ and the operator φ^* defined by the composition rule $\varphi^*(f) = f \circ \varphi$. This operator is a bounded composition operator on Lebesgue spaces. This combination of methods allows us to transfer Poincaré–Sobolev inequalities from regular domains (for example, from the unit disc $\mathbb{D}$) to Ω.

Theorem B *Let $\Omega \subset \mathbb{R}^2$ be a K-quasiconformal α-regular domain and $\varphi : \Omega \to \mathbb{D}$ be a corresponding K-quasiconformal mapping. Suppose that the Brennan's conjecture holds. Then for every*

$$p \in \left(\max\left\{\frac{4K}{2K+1}, \frac{\alpha(2K-1)+2}{\alpha K}\right\}, 2\right)$$

the p-Poincaré–Sobolev inequality

$$\inf_{c\in\mathbb{R}} \|f - c \mid L_p(\Omega)\| \leq B_{p,p}(\Omega)\|\nabla f \mid L_p(\Omega)\|, \quad f \in W^1_p(\Omega),$$

holds with the constant

$$B^p_{p,p}(\Omega) \leq \inf_{q\in I}\left\{B^p_{\frac{\alpha p}{\alpha-2},q}(\mathbb{D}) \cdot K \left\| |D\varphi^{-1}|^{p-2} \mid L_{\frac{q}{p-q}}(\mathbb{D})\right\| \cdot \left\|J_{\varphi^{-1}} \mid L_{\frac{\alpha}{2}}(\mathbb{D})\right\|\right\},$$

where $I = [1, 2p/(4K-(2K-1)p))$.

Remark 4 In Sect. 1 we formulated the main result under the assumptions that the Brennanś conjecture holds true i.e. $4K/(2K+1) < \beta < 4K/(2K-1)$. In the main part of the paper we will prove main results for $4K/(2K+1) < \beta < 2K\beta_0/(\beta_0(K-1)+2)$ for β_0 that is a recent known value for which the Brennanś conjecture is correct.

The next main theorem establish a connection between Brennan's conjecture and composition operators on Sobolev spaces:

Theorem C *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain. Generalized Brennan's conjecture holds for a number $\beta \in (4K/(2K+1),\ 4K/(2K-1))$ if and only if any K-quasiconformal homeomorphism $\varphi : \Omega \to \mathbb{D}$ induces a bounded composition operator*

$$\varphi^* : L^1_p(\mathbb{D}) \to L^1_q(\Omega)$$

for any $p \in (2, +\infty)$ and $q = p\beta/(p+\beta-2)$.

Remark 5 Theorem C states equivalence of the integrability of Jacobians of quasiconformal mappings and boundedness of compositions operators on Sobolev spaces. The proof of the necessity is based on the work [28] (see, also [33]).

In the recent works we studied composition operators on Sobolev spaces defined on simply connected planar domains in connection with the conformal mappings theory [15]. This connection leads to two-weight Sobolev embeddings [16,17] with universal conformal weights. Another application of conformal composition operators to spectral stability in conformal regular domains was given in [8].

2 Composition operators and quasiconformal mappings

In this section we recall basic facts about composition operators on Lebesgue and Sobolev spaces and the quasiconformal mapping theory. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a n-dimensional Euclidean domain. For any $1 \leq p < \infty$ we consider the Lebesgue space $L_p(\Omega)$ of measurable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid L_p(\Omega)\| = \left(\int\limits_{\Omega} |f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty.$$

The following theorem about composition operators on Lebesgue spaces is well known (see, for example [33]):

Theorem 1 *Let $\varphi : \Omega \to \widetilde{\Omega}$ be a weakly differentiable homeomorphism between two domains Ω and $\widetilde{\Omega}$. Then the composition operator*

$$\varphi^* : L_r(\widetilde{\Omega}) \to L_s(\Omega), \quad 1 \leq s \leq r < \infty,$$

is bounded, if and only if φ^{-1} possesses the Luzin N-property and

$$\left(\int\limits_{\widetilde{\Omega}} \left|J(y, \varphi^{-1})\right|^{\frac{r}{r-s}} \, dy\right)^{\frac{r-s}{rs}} = K < \infty, \quad 1 \leq s < r < \infty,$$

$$\operatorname*{ess\,sup}_{y \in \widetilde{\Omega}} \left|J(y, \varphi^{-1})\right|^{\frac{1}{s}} = K < \infty, \quad 1 \leq s = r < \infty.$$

The norm of the composition operator $\|\varphi^*\| = K$.

We consider the Sobolev space $W^1_p(\Omega)$, $1 \leq p < \infty$, as a Banach space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid W^1_p(\Omega)\| = \left(\int\limits_{\Omega} |f(x)|^p \, dx\right)^{\frac{1}{p}} + \left(\int\limits_{\Omega} |\nabla f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

Recall that the Sobolev space $W^1_p(\Omega)$ coincides with the closure of the space of smooth functions $C^\infty(\Omega)$ in the norm of $W^1_p(\Omega)$.

We consider also the homogeneous seminormed Sobolev space $L^1_p(\Omega)$, $1 \leq p < \infty$, of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following seminorm:

$$\|f \mid L^1_p(\Omega)\| = \left(\int\limits_\Omega |\nabla f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

Recall that the embedding operator $i : L^1_p(\Omega) \to L_{1,\mathrm{loc}}(\Omega)$ is bounded.

By the standard definition functions of $L^1_p(\Omega)$ are defined only up to a set of measure zero, but they can be redefined quasieverywhere i.e. up to a set of p-capacity zero. Indeed, every function $u \in L^1_p(\Omega)$ has a unique quasicontinuous representation $\tilde{u} \in L^1_p(\Omega)$. A function $\tilde{u}$ is termed quasicontinuous if for any $\varepsilon > 0$ there is an open set U_ε such that the p-capacity of U_ε is less then ε and on the set $\Omega \setminus U_\varepsilon$ the function $\tilde{u}$ is continuous (see, for example [21,25]).

Let Ω and $\widetilde{\Omega}$ be domains in $\mathbb{R}^n$. We say that a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ induces a bounded composition operator

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \quad 1 \leq q \leq p \leq \infty,$$

by the composition rule $\varphi^*(f) = f \circ \varphi$, if the composition $\varphi^*(f) \in L^1_q(\Omega)$ is defined quasi-everywhere in Ω and there exists a constant $K_{p,q}(\Omega) < \infty$ such that

$$\|\varphi^*(f) \mid L^1_q(\Omega)\| \leq K_{p,q}(\Omega) \|f \mid L^1_p(\widetilde{\Omega})\|$$

for any function $f \in L^1_p(\widetilde{\Omega})$ [34].

Let $\Omega \subset \mathbb{R}^n$ be an open set. A mapping $\varphi : \Omega \to \mathbb{R}^n$ belongs to $L^1_{p,\mathrm{loc}}(\Omega)$, $1 \leq p \leq \infty$, if its coordinate functions φ_j belong to $L^1_{p,\mathrm{loc}}(\Omega)$, $j = 1, \ldots, n$. In this case the formal Jacobi matrix $D\varphi(x) = (\frac{\partial \varphi_i}{\partial x_j}(x))$, $i, j = 1, \ldots, n$, and its determinant (Jacobian) $J(x, \varphi) = \det D\varphi(x)$ are well defined at almost all points $x \in \Omega$. The norm $|D\varphi(x)|$ of the matrix $D\varphi(x)$ is the norm of the corresponding linear operator $D\varphi(x) : \mathbb{R}^n \to \mathbb{R}^n$ defined by the matrix $D\varphi(x)$.

Let $\varphi : \Omega \to \widetilde{\Omega}$ be weakly differentiable in Ω. The mapping φ is the mapping of finite distortion if $|D\varphi(z)| = 0$ for almost all $x \in Z = \{z \in \Omega : J(x, \varphi) = 0\}$.

A mapping $\varphi : \Omega \to \mathbb{R}^n$ possesses the Luzin N-property if a image of any set of measure zero has measure zero. Mote that any Lipschitz mapping possesses the Luzin N-property.

The following theorem gives an analytic description of composition operators on Sobolev spaces:

Theorem 2 [28,33] *A homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ between two domains Ω and $\widetilde{\Omega}$ induces a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \quad 1 \leq q < p < \infty,$$

if and only if $\varphi \in W^1_{1,\mathrm{loc}}(\Omega)$, has finite distortion, and

$$K_{p,q}(\Omega) = \left(\int\limits_\Omega \left(\frac{|D\varphi(x)|^p}{|J(x,\varphi)|}\right)^{\frac{q}{p-q}} dx\right)^{\frac{p-q}{pq}} < \infty.$$

Recall that a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ is called a K-quasiconformal mapping if $\varphi \in W^1_{n,\mathrm{loc}}(\Omega)$ and there exists a constant $1 \leq K < \infty$ such that

$$|D\varphi(x)|^n \leq K|J(x,\varphi)| \quad \text{for almost all } x \in \Omega.$$

Quasiconformal mappings have a finite distortion, i.e. $D\varphi(x) = 0$ for almost all points x that belongs to set $Z = \{x \in \Omega : J(x,\varphi) = 0\}$ and any quasiconformal mapping possesses Luzin N-property. A mapping which is inverse to a quasiconformal mapping is also quasiconformal.

If $\varphi : \Omega \to \widetilde{\Omega}$ is a K-quasiconformal mapping then φ is differentiable almost everywhere in Ω and

$$|J(x,\varphi)| = J_{\varphi}(x) := \lim_{r\to 0} \frac{|\varphi(B(x,r))|}{|B(x,r)|} \quad \text{for almost all } x \in \Omega.$$

Note, that a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ is a K-quasiconformal mapping if and only if φ generates by the composition rule $\varphi^*(f) = f \circ \varphi$ an isomorphism of Sobolev spaces $L^1_n(\Omega)$ and $L^1_n(\widetilde{\Omega})$:

$$K^{-\frac{1}{n}} \|f \mid L^1_n(\widetilde{\Omega})\| \leq \|\varphi^* f \mid L^1_n(\Omega)\| \leq K^{\frac{1}{n}} \|f \mid L^1_n(\widetilde{\Omega})\|$$

for any $f \in L^1_n(\widetilde{\Omega})$ [30].

Boundedness of composition operators generated by quasiconformal mappings in the case $p \neq n$ was considered also in [24].

For any planar K-quasiconformal homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$, the following sharp results is known: $J(x,\varphi) \in L_{\alpha^*,\mathrm{loc}}(\widetilde{\Omega})$ for any $\alpha^* < K/(K-1)$ [2,12].

If $K \equiv 1$ then 1-quasiconformal homeomorphisms are conformal mappings and in the space $\mathbb{R}^n$, $n \geq 3$, are exhausted by Möbius transformations.

3 Composition operators and Brennan's conjecture

Brennan's conjecture [7] is that if $\varphi : \Omega \to \mathbb{D}$ is a conformal mappings of a simply connected planar domain Ω, $\Omega \neq \mathbb{R}^2$, onto the unit disc $\mathbb{D}$ then

$$\int_{\Omega} |\varphi'(x)|^{\beta}\, dx < +\infty, \quad \text{for all } \frac{4}{3} < \beta < 4. \tag{3}$$

For $4/3 < s < 3$, it is a comparatively easy consequence of the Koebe distortion theorem (see, for example, [4]). Brennan [7] (1978) extended this range to $4/3 < s < 3 + \delta$, where $\delta > 0$, and conjectured it to hold for $4/3 < s < 4$. The example of $\Omega = \mathbb{C} \setminus (-\infty, -1/4]$ shows that this range of s cannot be extended.

Brennan's conjecture has been established for $\beta \in (4/3, \beta_0)$, where $\beta_0 = 3.752$ [20]. Brennan's conjecture for quasiconformal mappings was considered in [22]. In [22] it was proved that, if $\varphi : \Omega \to \mathbb{D}$ be a K-quasiconformal mapping, then

$$\int_{\Omega} |D\varphi(x)|^{\beta}\, dx < +\infty, \quad \text{for all } \frac{4K}{2K+1} < \beta < \frac{2K\beta_0}{(K-1)\beta_0 + 2}. \tag{4}$$

Here β_0 is the proved upper bound for Brennan's conjecture.

Now we prove that Generalized Brennan's conjecture leads to boundedness of composition operators on Sobolev spaces generates by quasiconformal mappings.

Theorem C *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain. Generalized Brennan's conjecture holds for a number $\beta \in (4K/(2K+1), 2K\beta_0/(\beta_0(K-1)+2))$ if and only if any K-quasiconformal homeomorphism $\varphi : \Omega \to \mathbb{D}$ induces a bounded composition operator*

$$\varphi^* : L^1_p(\mathbb{D}) \to L^1_q(\Omega)$$

for any $p \in (2, +\infty)$ *and* $q = p\beta/(p + \beta - 2)$.

Proof By the composition theorem [28,33] a homeomorphism $\varphi : \Omega \to \mathbb{D}$ induces a bounded composition operator

$$\varphi^* : L^1_p(\mathbb{D}) \to L^1_q(\Omega), \quad 1 \leq q < p < \infty.$$

if and only if $\varphi \in W^1_{1,\mathrm{loc}}(\Omega)$, has finite distortion and

$$K_{p,q}(\Omega) = \left(\int_\Omega \left(\frac{|D\varphi(x)|^p}{|J(x,\varphi)|} \right)^{\frac{q}{p-q}} dx \right)^{\frac{p-q}{pq}} < \infty.$$

Because φ is a quasiconformal mapping, then $\varphi \in W^1_{n,\mathrm{loc}}(\Omega)$ and Jacobian $J(x,\varphi) \neq 0$ for almost all $x \in \Omega$. Hence the p-dilatation

$$K_p(x) = \frac{|D\varphi(x)|^p}{|J(x,\varphi)|}$$

is well defined for almost all $x \in \Omega$ and so φ is a mapping of finite distortion. By Brennan's conjecture

$$\int_\Omega |D\varphi(x)|^\beta \, dx < +\infty, \quad \text{for all } \frac{4K}{2K+1} < \beta < \frac{2K\beta_0}{(K-1)\beta_0 + 2}.$$

Then

$$K_{p,q}^{\frac{pq}{p-q}}(\Omega) = \int_\Omega \left(\frac{|D\varphi(x)|^p}{|J(x,\varphi)|} \right)^{\frac{q}{p-q}} dx = \int_\Omega \left(\frac{|D\varphi(x)|^2}{|J(x,\varphi)|} |D\varphi(x)|^{p-2} \right)^{\frac{q}{p-q}} dx$$
$$\leq K^{\frac{q}{p-q}} \int_\Omega \left(|D\varphi(x)|^{p-2} \right)^{\frac{q}{p-q}} dx = K^{\frac{q}{p-q}} \int_\Omega |D\varphi(x)|^\beta \, dx < \infty,$$

for $\beta = (p-2)q/(p-q)$. Hence we have a bounded composition operator

$$\varphi^* : L^1_p(\mathbb{D}) \to L^1_q(\Omega)$$

for any $p \in (2, +\infty)$ and $q = p\beta/(p + \beta - 2)$.

Let us check that $q < p$. Because $p > 2$ we have that $p + \beta - 2 > \beta > 0$ and so $\beta/(p + \beta - 2) < 1$. Hence we obtain $q < p$.

Now, let the composition operator

$$\varphi^* : L^1_p(\mathbb{D}) \to L^1_q(\Omega), \quad q < p,$$

is bounded for any $p \in (2, +\infty)$ and $q = p\beta/(p + \beta - 2)$. Then, using the Hadamard inequality:

$$|J(x,\varphi)| \leq |D\varphi(x)|^2 \quad \text{for almost all } x \in \Omega,$$

and Theorem 2, we have

$$\int_\Omega |D\varphi(x)|^\beta \, dx = \int_\Omega |D\varphi(x)|^{\frac{(p-2)q}{p-q}} dx \leq \int_\Omega \left(\frac{|D\varphi(x)|^p}{|J(x,\varphi)|} \right)^{\frac{q}{p-q}} dx < +\infty.$$

The suggested approach to the Poincaré–Sobolev type inequalities in bounded planar domains $\Omega \subset \mathbb{R}^2$ is based on translation of these inequalities from the unit disc $\mathbb{D}$ to Ω. On this way we use the following duality [32]:

Theorem 3 *If a homeomorphism $\varphi : \Omega \to \Omega'$, $\Omega, \Omega' \subset \mathbb{R}^2$, generates by the composition rule $\varphi^*(f) = f \circ \varphi$ a bounded composition operator*

$$\varphi^* : L^1_p(\Omega') \to L^1_q(\Omega), \quad 1 < q \le p < \infty,$$

then the inverse mapping $\varphi^{-1} : \Omega' \to \Omega$ generates by the composition rule $(\varphi^{-1})^*(g) = g \circ \varphi^{-1}$ a bounded composition operator

$$(\varphi^{-1})^* : L^1_{q'}(\Omega) \to L^1_{p'}(\Omega'), \quad \frac{1}{q} + \frac{1}{q'} = 1, \frac{1}{p} + \frac{1}{p'} = 1.$$

From Theorem C and Theorem 3 we immediately obtain Theorem 4:

Theorem 4 *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain and $\varphi : \Omega \to \mathbb{D}$ be a K-quasiconformal homeomorphism. Suppose that $p \in (2K\beta_0/((K+1)\beta_0 - 2), 2)$.*

Then the inverse mapping φ^{-1} induces a bounded composition operator

$$(\varphi^{-1})^* : L^1_p(\Omega) \to L^1_q(\mathbb{D})$$

for any q such that

$$1 \le q \le \frac{(2\beta_0 - 4)p}{2K\beta_0 - ((K-1)\beta_0 + 2)p} < \frac{2p}{4K - (2K-1)p}.$$

The inequality

$$\|(\varphi^{-1})^* f \mid L^1_q(\mathbb{D})\| \le K^{\frac{1}{p}} \left(\int\limits_{\mathbb{D}} |D\varphi^{-1}(y)|^{\frac{(p-2)q}{p-q}} \, dy \right)^{\frac{p-q}{pq}} \|f \mid L^1_p(\Omega)\| \tag{5}$$

holds for any function $f \in L^1_p(\Omega)$.

Proof By Theorem C we have a bounded composition operator

$$\varphi^* : L^1_{q'}(\mathbb{D}) \to L^1_{p'}(\Omega)$$

for $q' \in (2, +\infty)$ and $p' = q'\beta/(q' + \beta - 2)$.

Then, by Brennan's conjecture, $p' \in (2, 2K\beta_0/((K-1)\beta_0 + 2))$. Now using Theorem 3 we have

$$p = \frac{p'}{p'-1} \in \left(\frac{2K\beta_0}{(K+1)\beta_0 - 2}, 2 \right).$$

Since

$$p = \frac{p'}{p'-1} = \frac{q'\beta}{q'\beta - (q' + \beta + 2)},$$

we obtain by direct calculations that

$$q' = \frac{(4 - 2\beta_0)p}{2K\beta_0 - ((K+1)\beta_0 - 2)p}.$$

By Theorem 3 $q = q'/(q'-1)$ and $q \leq p$. By elementary calculations

$$1 \leq q \leq \frac{(2\beta_0 - 4)p}{2K\beta_0 - ((K-1)\beta_0 + 2)p} < \frac{2p}{4K - (2K-1)p}.$$

Now we prove the inequality (5). Let $f \in L^1_p(\Omega) \cap C^\infty(\Omega)$. Then the composition $g = (\varphi^{-1})^*(f) \in L^1_{1,\mathrm{loc}}(\mathbb{D})$ [31]. Hence, using Theorem 2 we obtain

$$\begin{aligned}
\|g \mid L^1_q(\mathbb{D})\| &\leq \left(\int\limits_{\mathbb{D}} \left(\frac{|D\varphi^{-1}(y)|^p}{|J(y,\varphi^{-1})|} \right)^{\frac{q}{p-q}} dy \right)^{\frac{p-q}{pq}} \|f \mid L^1_p(\Omega)\| \\
&= \left(\int\limits_{\mathbb{D}} \left(\frac{|D\varphi^{-1}(y)|^2 \cdot |D\varphi^{-1}(y)|^{p-2}}{|J(y,\varphi^{-1})|} \right)^{\frac{q}{p-q}} dy \right)^{\frac{p-q}{pq}} \|f \mid L^1_p(\Omega)\| \\
&\leq K^{\frac{1}{p}} \left(\int\limits_{\mathbb{D}} |D\varphi^{-1}(y)|^{\frac{(p-2)q}{p-q}} dy \right)^{\frac{p-q}{pq}} \|f \mid L^1_p(\Omega)\|.
\end{aligned}$$

Approximating an arbitrary function $f \in L^1_p(\Omega)$ by smooth functions [32,33], we obtain the required inequality.

4 Poincaré–Sobolev inequalities

Two-weight Poincaré–Sobolev inequalities Let $\Omega \subset \mathbb{R}^2$ be a planar domain and let $v : \Omega \to \mathbb{R}$ be a real valued function, $v > 0$ a.e. in Ω. We consider the weighted Lebesgue space $L_p(\Omega, v)$, $1 \leq p < \infty$, of measurable functions $f : \Omega \to \mathbb{R}$ with the finite norm

$$\|f \mid L_p(\Omega, v)\| := \left(\int\limits_{\Omega} |f(x)|^p v(x) dx \right)^{\frac{1}{p}} < \infty.$$

It is a Banach space for the norm $\|f \mid L_p(\Omega, v)\|$.

In the monograph [21] one-weight Poincaré–Sobolev inequalities with weights generates by Jacobians of quasiconformal mappings were studied. Using Theorem 4 we prove existence of universal two-weight Poincaré–Sobolev inequalities in any simply connected bounded domain $\Omega \subset \mathbb{R}^2$.

Theorem 5 *Suppose that $\Omega \subset \mathbb{R}^2$ is a simply connected domain and $h(x) = |J(x,\varphi)|$ is the quasiconformal weight defined by a K-quasiconformal homeomorphism $\varphi : \Omega \to \mathbb{D}$. Then for every $p \in (2K\beta_0/((K+1)\beta_0 - 2), 2)$ and every function $f \in W^1_p(\Omega)$, the inequality*

$$\inf_{c \in \mathbb{R}} \left(\int\limits_{\Omega} |f(x) - c|^r h(x) dx \right)^{\frac{1}{r}} \leq B_{r,p}(\Omega, h) \left(\int\limits_{\Omega} |\nabla f(x)|^p dx \right)^{\frac{1}{p}}$$

holds for any r such that

$$1 \leq r < \frac{p}{2-p} \cdot \frac{2\beta_0 - 4}{K\beta_0}$$

with the constant

$$B_{r,p}(\Omega,h) \le \inf_{q\in[1,2p/(4K-(2K-1)p))} \{\widetilde{K}_{p,q}(\mathbb{D})\cdot B_{r,q}(\mathbb{D})\}.$$

Here $B_{r,q}(\mathbb{D})$ is the best constant in the (unweighted) Poincaré–Sobolev inequality in the unit disc $\mathbb{D}\subset\mathbb{R}^2$ and

$$\widetilde{K}_{p,q}(\mathbb{D}) = K^{\frac{1}{p}}\left(\int\limits_{\mathbb{D}} |D\varphi^{-1}(y)|^{\frac{(p-2)q}{p-q}}\,dy\right)^{\frac{p-q}{pq}}.$$

Proof By conditions of the theorem for a K-quasiconformal mapping $\varphi:\Omega\to\mathbb{D}$

$$\int\limits_{\Omega} |D\varphi(x)|^{\beta}\,dx < +\infty, \quad \text{for all } \frac{4K}{2K+1} < \beta < \frac{2K\beta_0}{(K-1)\beta_0+2}.$$

By Theorem 4, if

$$1 \le q \le \frac{(2\beta_0-4)p}{2K\beta_0-((K-1)\beta_0+2)p} < \frac{2p}{4K-(2K-1)p} \tag{6}$$

then the inequality

$$\|\nabla(f\circ\varphi^{-1}) \mid L_q(\mathbb{D})\| \le \widetilde{K}_{p,q}(\mathbb{D})\|\nabla f \mid L_p(\Omega)\| \tag{7}$$

holds for every function $f\in L^1_p(\Omega)$.

Let $f\in L^1_p(\Omega)\cap C^1(\Omega)$. Then the function $g=f\circ\varphi^{-1}$ is defined almost everywhere in $\mathbb{D}$ and belongs to the Sobolev space $L^1_q(\mathbb{D})$ [31]. Hence, by the Sobolev embedding theorem $g=f\circ\varphi^{-1}\in W^{1,q}(\mathbb{D})$ [25] and the classical Poincaré–Sobolev inequality,

$$\inf_{c\in\mathbb{R}} \|f\circ\varphi^{-1}-c \mid L_r(\mathbb{D})\| \le B_{r,q}(\mathbb{D})\|\nabla(f\circ\varphi^{-1}) \mid L_q(\mathbb{D})\| \tag{8}$$

holds for any r such that

$$1 \le r \le \frac{2q}{2-q}. \tag{9}$$

By elementary calculations from the inequality (6), it follows that

$$\frac{2q}{2-q} \le \frac{\beta_0-2}{K\beta_0}\cdot\frac{2p}{2-p} < \frac{1}{K}\cdot\frac{p}{2-p}. \tag{10}$$

Combining inequalities (9) and (10) we conclude that the inequality (8) holds for any r such that

$$1 \le r \le \frac{\beta_0-2}{K\beta_0}\cdot\frac{2p}{2-p} < \frac{1}{K}\cdot\frac{p}{2-p}.$$

Using the change of variable formula for quasiconformal mappings [31], the classical Poincaré–Sobolev inequality for the unit disc

$$\inf_{c\in\mathbb{R}}\left(\int\limits_{\mathbb{D}} |g(y)-c|^r\,dy\right)^{\frac{1}{r}} \le B_{r,q}(\mathbb{D})\left(\int\limits_{\mathbb{D}} |\nabla g(y)|^q\,dy\right)^{\frac{1}{q}}$$

and inequality (7), we finally infer

$$\inf_{c\in\mathbb{R}}\left(\int_{\Omega}|f(x)-c|^r h(x)dx\right)^{\frac{1}{r}}=\inf_{c\in\mathbb{R}}\left(\int_{\Omega}|f(x)-c|^r|J(x,\varphi)|dx\right)^{\frac{1}{r}}$$
$$=\inf_{c\in\mathbb{R}}\left(\int_{\mathbb{D}}|g(y)-c|^r dy\right)^{\frac{1}{r}}\leq B_{r,q}(\mathbb{D})\left(\int_{\mathbb{D}}|\nabla g(y)|^q dy\right)^{\frac{1}{q}}$$
$$\leq \widetilde{K}_{p,q}(\mathbb{D})B_{r,q}(\mathbb{D})\left(\int_{\Omega}|\nabla f(x)|^p dx\right)^{\frac{1}{p}}.$$

Approximating an arbitrary function $f\in W^1_p(\Omega)$ by smooth functions we have

$$\inf_{c\in\mathbb{R}}\left(\int_{\Omega}|f(x)-c|^r h(x)dx\right)^{\frac{1}{r}}\leq B_{r,p}(\Omega,h)\left(\int_{\Omega}|\nabla f(x)|^p dx\right)^{\frac{1}{p}}$$

with the constant

$$B_{r,p}(\Omega,h)\leq \inf_{q\in[1,2p/(4K-(2K-1)p))}\{\widetilde{K}_{p,q}(\mathbb{D})\cdot B_{r,q}(\mathbb{D})\}.$$

The property of the K-quasiconformal α-regularity implies the integrability of a Jacobian of quasiconformal mappings and therefore for any K-quasiconformal α-regular domain we have the embedding of weighted Lebesgue spaces $L_r(\Omega,h)$ into non-weight Lebesgue spaces $L_s(\Omega)$ for $s=\frac{\alpha-2}{\alpha}r$:

Lemma 1 *Let Ω be a K-quasiconformal α-regular domain. Then for any function $f\in L_r(\Omega,h)$, $\alpha/(\alpha-2)\leq r<\infty$, the inequality*

$$\|f\mid L_s(\Omega)\|\leq\left(\int_{\mathbb{D}}\left|J(y,\varphi^{-1})\right|^{\frac{\alpha}{2}}dy\right)^{\frac{2}{\alpha}\cdot\frac{1}{s}}\|f\mid L_r(\Omega,h)\|$$

holds for $s=\frac{\alpha-2}{\alpha}r$.

Proof By the assumptions of the lemma these exists a K-quasiconformal mapping $\varphi:\Omega\to\mathbb{D}$ such that

$$\int_{\mathbb{D}}\left|J(y,\varphi^{-1})\right|^{\frac{\alpha}{2}}dy<+\infty.$$

Let $s=\frac{\alpha-2}{\alpha}r$. Then using the change of variable formula for quasiconformal mappings [31], Hölder's inequality with exponents $(r, rs/(r-s))$ and equality $|J(x,\varphi)|=h(x)$, we obtain

$$
\begin{aligned}
\|f \mid L_s(\Omega)\| &= \left(\int\limits_{\Omega} |f(x)|^s dx\right)^{\frac{1}{s}} = \left(\int\limits_{\Omega} |f(x)|^s \left|J(x,\varphi)\right|^{\frac{s}{r}} \left|J(x,\varphi)\right|^{-\frac{s}{r}}\, dx\right)^{\frac{1}{s}} \\
&\leq \left(\int\limits_{\Omega} |f(x)|^r |J(x,\varphi)| dx\right)^{\frac{1}{r}} \left(\int\limits_{\Omega} \left|J(x,\varphi)\right|^{-\frac{s}{r-s}}\, dx\right)^{\frac{r-s}{rs}} \\
&= \left(\int\limits_{\Omega} |f(x)|^r h(x)\, dx\right)^{\frac{1}{r}} \left(\int\limits_{\mathbb{D}} \left|J(y,\varphi^{-1})\right|^{\frac{r}{r-s}}\, dy\right)^{\frac{r-s}{rs}} \\
&= \left(\int\limits_{\Omega} |f(x)|^r h(x)\, dx\right)^{\frac{1}{r}} \left(\int\limits_{\mathbb{D}} \left|J(y,\varphi^{-1})\right|^{\frac{\alpha}{2}}\, dy\right)^{\frac{2}{\alpha}\cdot\frac{1}{s}}.
\end{aligned}
$$

The following theorem gives an upper estimate of the Poincaré constant and follows from Theorem 5 and Lemma 1:

Theorem 6 *Suppose that $\Omega \subset \mathbb{C}$ is a K-quasiconformal α-regular domain. Then for every*

$$
p \in \left(\max\left\{\frac{4K}{2K+1}, \frac{2K\alpha\beta_0}{(K+2)\alpha\beta_0 - 4(\alpha+\beta_0-2)}\right\}, 2\right),
$$

every

$$
s \in \left[1,\ \frac{(\alpha-2)(\beta_0-2)}{K\alpha\beta_0}\cdot\frac{2p}{2-p}\right]
$$

and every function $f \in W^1_p(\Omega)$, the inequality

$$
\inf_{c\in\mathbb{R}} \left(\int\limits_{\Omega} |f(x)-c|^s dx\right)^{\frac{1}{s}} \leq B_{s,p}(\Omega) \left(\int\limits_{\Omega} |\nabla f(x)|^p dx\right)^{\frac{1}{p}}
$$

holds with the constant

$$
\begin{aligned}
B_{s,p}(\Omega) &\leq \left(\int\limits_{\mathbb{D}} \left|J(y,\varphi^{-1})\right|^{\frac{\alpha}{2}}\, dy\right)^{\frac{2}{\alpha}\cdot\frac{1}{s}} B_{r,p}(\Omega,h) \\
&\leq \inf_{q\in I}\left\{ B_{\frac{\alpha s}{\alpha-2},q}(\mathbb{D}) \cdot \widetilde{K}_{p,q}(\mathbb{D}) \cdot \left(\int\limits_{\mathbb{D}} \left|J(y,\varphi^{-1})\right|^{\frac{\alpha}{2}}\, dy\right)^{\frac{2}{\alpha}\cdot\frac{1}{s}} \right\},
\end{aligned}
$$

where $I = [1, 2p/(4K-(2K-1)p))$.

Proof Let $f \in W^1_p(\Omega)$. Then by Theorem 5 and Lemma 1 we obtain

$$\inf_{c\in\mathbb{R}}\left(\int_\Omega |f(x)-c|^s dx\right)^{\frac{1}{s}} \leq \left(\int_{\mathbb{D}} \left|J(y,\varphi^{-1})\right|^{\frac{\alpha}{2}}\, dy\right)^{\frac{2}{\alpha}\cdot\frac{1}{s}} \inf_{c\in\mathbb{R}}\left(\int_\Omega |f(x)-c|^r h(x) dx\right)^{\frac{1}{r}}$$
$$\leq B_{r,p}(\Omega,h)\left(\int_{\mathbb{D}} \left|J(y,\varphi^{-1})\right|^{\frac{\alpha}{2}}\, dy\right)^{\frac{2}{\alpha}\cdot\frac{1}{s}} \left(\int_\Omega |\nabla f(x)|^p dx\right)^{\frac{1}{p}}$$

for $s \geq 1$.

Because by Lemma 1 $s = \frac{\alpha-2}{\alpha} r$ and by Theorem 5

$$1 \leq r \leq \frac{\beta_0 - 2}{K\beta_0}\cdot\frac{2p}{2-p} < \frac{1}{K}\cdot\frac{p}{2-p},$$

then

$$1 \leq s \leq \frac{(\alpha-2)(\beta_0-2)}{K\alpha\beta_0}\cdot\frac{2p}{2-p} < \frac{\alpha-2}{K\alpha}\cdot\frac{p}{2-p}.$$

Hence, by direct calculations, we obtain that

$$p \geq \frac{2K\alpha\beta_0}{(K+2)\alpha\beta_0 - 4(\alpha+\beta_0-2)}.$$

If Brennan's conjecture holds, i.e. $\beta_0 = 4$ then we can take that

$$p > \frac{2K\alpha}{(K+1)\alpha - 2}.$$

In the case of (p,p)-Poincaré–Sobolev inequalities we have:

Theorem 7 *Let $\Omega \subset \mathbb{R}^2$ be a K-quasiconformal α-regular domain and $\varphi : \Omega \to \mathbb{D}$ be a K-quasiconformal mapping. Then for every*

$$p \in \left(\max\left\{\frac{4K}{2K+1}, \frac{2(K-1)\alpha\beta_0 + 4(\alpha+\beta_0-2)}{K\alpha\beta_0}\right\}, 2\right)$$

the p-Poincaré–Sobolev inequality

$$\inf_{c\in\mathbb{R}} \|f - c \mid L_p(\Omega)\| \leq B_{p,p}(\Omega)\|\nabla f \mid L_p(\Omega)\|, \quad f \in W^1_p(\Omega),$$

holds with the constant

$$B^p_{p,p}(\Omega) \leq \inf_{q\in I}\left\{B^p_{\frac{\alpha p}{\alpha-2},q}(\mathbb{D})\cdot \widetilde{K}^p_{p,q}(\mathbb{D})\cdot\left(\int_{\mathbb{D}} \left|J(y,\varphi^{-1})\right|^{\frac{\alpha}{2}}\, dy\right)^{\frac{2}{\alpha}}\right\},$$

where $I = [1, 2p/(4K-(2K-1)p))$.

Proof By Lemma 1 $p = \frac{\alpha-2}{\alpha} r$ and by Theorem 5

$$1 \leq r \leq \frac{\beta_0 - 2}{K\beta_0}\cdot\frac{2p}{2-p} < \frac{1}{K}\cdot\frac{p}{2-p}.$$

Hence

$$\frac{\alpha}{\alpha-2} \le \frac{\beta_0-2}{K\beta_0}\cdot\frac{2}{2-p} < \frac{1}{K}\cdot\frac{1}{2-p}.$$

By elementary calculations

$$p \ge \frac{2(K-1)\alpha\beta_0+4(\alpha+\beta_0-2)}{K\alpha\beta_0}.$$

If Brennan's conjecture holds, i.e. $\beta_0=4$ then we can take that

$$p > \frac{(2K-1)\alpha+2}{K\alpha}.$$

Theorem 7 implies the lower estimates of the first non-trivial eigenvalue $\mu_p^{(1)}(\Omega)$:

Theorem 8 *Let $\varphi : \Omega \to \mathbb{D}$ be a K-quasiconformal homeomorphism from a K-quasiconformal α-regular domain Ω to the unit disc $\mathbb{D}$. Then for every*

$$p \in \left(\max\left\{\frac{4K}{2K+1}, \frac{2(K-1)\alpha\beta_0+4(\alpha+\beta_0-2)}{K\alpha\beta_0}\right\}, 2\right)$$

the following inequality holds

$$\frac{1}{\mu_p^{(1)}(\Omega)} \le \inf_{q\in I}\left\{B^p_{\frac{\alpha p}{\alpha-2},q}(\mathbb{D})\cdot \widetilde{K}^p_{p,q}(\mathbb{D})\cdot\left(\int\limits_{\mathbb{D}}\left|J(y,\varphi^{-1})\right|^{\frac{\alpha}{2}}\,dy\right)^{\frac{2}{\alpha}}\right\},$$

where $I=[1,2p/(4K-(2K-1)p))$.

In the case $\alpha=\infty$ we have the following assertion:

Corollary 1 *Let $\varphi : \Omega \to \mathbb{D}$ be a K-quasiconformal homeomorphism from a K-quasiconformal ∞-regular domain Ω to the unit disc $\mathbb{D}$. Then for every*

$$p \in \left(\max\left\{\frac{4K}{2K+1}, \frac{2(K-1)\beta_0+4}{K\beta_0}\right\}, 2\right)$$

the following inequality holds

$$\frac{1}{\mu_p^{(1)}(\Omega)} \le \inf_{q\in I}\left\{B^p_{p,q}(\mathbb{D})\cdot \widetilde{K}^p_{p,q}(\mathbb{D})\right\}\cdot \|J_{\varphi^{-1}} \mid L_\infty(\mathbb{D})\|,$$

where $I=[1,2p/(4K-(2K-1)p))$.

5 Examples

Example A The homeomorphism

$$w = Az + B\overline{z}, \qquad z = x+iy, \quad A > B \ge 0,$$

is K-quasiconformal with $K=\frac{A+B}{A-B}$ and maps the unit disc $\mathbb{D}$ onto the interior of the ellipse

$$\Omega_e = \left\{(x,y)\in\mathbb{R}^2 : \frac{x^2}{(A+B)^2}+\frac{y^2}{(A-B)^2}=1\right\}.$$

We calculate the norm of the derivative of mapping w by the formula [3]

$$|Dw| = |w_z| + |w_{\overline{z}}|$$

and the Jacobian of mapping w by the formula [3]

$$J(z,w) = |w_z|^2 - |w_{\overline{z}}|^2.$$

Here

$$w_z = \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right) \quad \text{and} \quad w_{\overline{z}} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right).$$

By elementary calculations

$$w_z = A \quad \text{and} \quad w_{\overline{z}} = B.$$

Hence

$$|Dw| = A + B \quad \text{and} \quad J(z,w) = A^2 - B^2.$$

Then by Theorem 8 we have

$$\frac{1}{\mu_p^{(1)}(\Omega_e)} \le \inf_{q\in I}\left(\frac{2}{\pi^\nu}\left(\frac{1-\nu}{1/2-\nu}\right)^{1-\nu}\right)^p$$
$$\times \frac{A+B}{A-B}\left(\int\limits_{\mathbb{D}} (A^2-B^2)^{\frac{\alpha}{2}}\,dy\right)^{\frac{2}{\alpha}}\left(\int\limits_{\mathbb{D}} (A+B)^{\frac{(p-2)q}{p-q}}\,dy\right)^{\frac{p-q}{q}}$$
$$= (A+B)^p \inf_{q\in I}\left(\frac{2}{\pi^\nu}\left(\frac{1-\nu}{1/2-\nu}\right)^{1-\nu}\right)^p \pi^{\frac{2q+\alpha(p-q)}{\alpha q}},$$

where $I = [1, 2p/(4K-(2K-1)p))$ and $\nu = 1/q - (\alpha-2)/\alpha p$.

Let $p = 3/2$. Then, taking $q = 1$ and $\alpha = \infty$, we get

$$\frac{1}{\mu_{3/2}^{(1)}(\Omega_e)} \le 8\sqrt{2}(A+B)^{\frac{3}{2}}.$$

Example B The homeomorphism

$$w = (|z|^{k-1}z + 1)^2, \qquad z = x + iy, \quad k \ge 1,$$

is k-quasiconformal and maps the unit disc $\mathbb{D}$ onto the interior of the cardioid

$$\Omega_c = \left\{(x,y) \in \mathbb{R}^2 : (x^2+y^2-2x)^2 - 4(x^2+y^2) = 0\right\}.$$

We calculate the partial derivatives of the mapping w

$$w_z = (k+1)|z|^{k-1}(|z|^{k-1}z+1) \quad \text{and} \quad w_{\overline{z}} = (k-1)|z|^{k-3}z^2(|z|^{k-1}z+1).$$

Hence

$$|Dw| = 2k|z|^{k-1}\sqrt{|z|^{2k} + |z|^{k-1}(z+\overline{z}) + 1}$$

and

$$J(z,w) = 4k|z|^{2k-2}(|z|^{2k} + |z|^{k-1}(z+\overline{z}) + 1).$$

Then by Corollary 1 we have

$$\frac{1}{\mu_p^{(1)}(\Omega_c)} \leq 4(2k)^p \inf_{q\in I}\left(\frac{2}{\pi^\nu}\left(\frac{1-\nu}{1/2-\nu}\right)^{1-\nu}\right)^p$$
$$\times\left(\int_0^{2\pi}\left(\int_0^1\left(\rho^{k-1}\sqrt{\rho^{2k}+2\rho^k\cos\psi+1}\right)^{\frac{(p-2)q}{p-q}}\rho\,d\rho\right)d\psi\right)^{\frac{p-q}{q}}. \quad (11)$$

Here $I=[1,2p/(4k-(2k-1)p))$ and $\nu=1/q-1/p$.

Let us consider the case of porous media flows ($p=3/2$). Then, taking $q=1$ and $k=3/2$ we obtain,

$$\frac{1}{\mu_{3/2}^{(1)}(\Omega_c)} \leq \frac{96\sqrt{30}}{\sqrt{\pi}} \approx 297.$$

In the case when $p=8/5$ we use the following estimate of the integral (11):

$$\frac{1}{\mu_p^{(1)}(\Omega_c)} \leq 4(2k)^p \inf_{q\in I}\left(\frac{2}{\pi^\nu}\left(\frac{1-\nu}{1/2-\nu}\right)^{1-\nu}\right)^p$$
$$\times\left(2\pi\int_0^1\left(\rho^{k-1}\left(\rho^k-1\right)\right)^{\frac{(p-2)q}{p-q}}\rho\,d\rho\right)^{\frac{p-q}{q}}.$$

Taking $q=1$ and $k=2$ we obtain

$$\frac{1}{\mu_{8/5}^{(1)}(\Omega_c)} \leq 20\cdot 2^{\frac{27}{5}}\left(\int_0^1\frac{\rho^{\frac{1}{3}}}{(1-\rho^2)^{\frac{2}{3}}}\,d\rho\right)^{\frac{3}{5}} = 20\cdot 2^{\frac{27}{5}}\left(\frac{\pi}{\sqrt{3}}\right)^{\frac{3}{5}} \approx 1268.$$

Example C The homeomorphism

$$w=|z|^k z, \qquad z=x+iy, \quad k\geq 0,$$

is $(k+1)$-quasiconformal and maps the square

$$\mathbb{Q} := \left\{(x,y)\in\mathbb{R}^2 : -\frac{\sqrt{2}}{2}<x<\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}<y<\frac{\sqrt{2}}{2}\right\}$$

onto k-star-shaped domains $\Omega^*_{\varepsilon(k)}$ with vertices $(\pm\sqrt{2}/2,\ \pm\sqrt{2}/2)$, $(\pm\varepsilon,\ 0)$ and $(0,\ \pm\varepsilon)$, where $\varepsilon(k)=(\sqrt{2}/2)^{k+1}$ (Fig. 1).

We calculate the partial derivatives of the mapping w

$$w_z=\left(\frac{k}{2}+1\right)|z|^k \quad\text{and}\quad w_{\overline{z}}=\frac{k}{2}|z|^{k-2}z^2.$$

Thus

$$|Dw|=(k+1)|z|^k \quad\text{and}\quad J(z,w)=(k+1)|z|^{2k}.$$

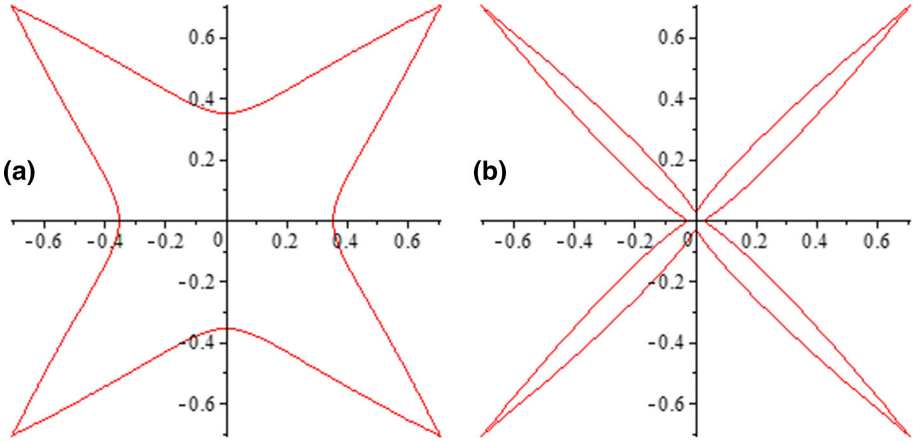

Fig. 1 Domains $\Omega^*_{\varepsilon(k)}$ under $\varepsilon = \frac{1}{2\sqrt{2}}$ and $\varepsilon = \frac{1}{32}$

Then by Corollary 1 we have

$$\frac{1}{\mu_p^{(1)}(\Omega^*_{\varepsilon(k)})} \leq (k+1)^p \inf_{q\in I} \left(\frac{2}{\pi^\nu}\left(\frac{1-\nu}{1/2-\nu}\right)^{1-\nu}\right)^p \times \left(\int\limits_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \left(\int\limits_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \left(x^2+y^2\right)^{\frac{(p-2)kq}{2(p-q)}}\, dy\right) dx\right)^{\frac{p-q}{q}}$$

$$\leq (k+1)^p \inf_{q\in I} \left(\frac{2}{\pi^\nu}\left(\frac{1-\nu}{1/2-\nu}\right)^{1-\nu}\right)^p \times \left(\frac{2\pi(p-q)}{(p-2)kq+2(p-q)}\right)^{\frac{p-q}{q}},$$

where $I = [1, 2p/(4K-(2K-1)p))$ and $\nu = 1/q - 1/p$.

In the case of porous media flows ($p = 3/2$), taking $q = 1$, we have

$$\frac{1}{\mu_{3/2}^{(1)}(\Omega^*_{\varepsilon(k)})} \leq 16\sqrt{\frac{(k+1)^3}{2-k}}, \quad 0 \leq k < 2.$$

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