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GUARANTEED PARAMETER ESTIMATION OF ARARCH(1,1) WITH DRIFTING PARAMETER

In this paper we solve the estimation problem of the drifting parameter of ARARCH(1,1) model with guaranteed accuracy in the mean square sense by sample of fixed size. Cases of known and unknown noise variances are considered. For the first case we apply the truncated sequential estimation method and for the second one we use the truncated estimation method. It is shown that the obtained results can be applied to more general models.

Keywords: *parameter estimation, ARARCH model, drifting parameter, truncated sequential estimators, guaranteed accuracy.*

Nowadays mathematical statistics is turned to development of data processing methods by dependent sample of finite size. One of such possibilities gives a well-known sequential estimation method, which was successfully applied to parametric and non-parametric problems, see, e.g. [1, 2].

To obtain sequential estimators with an arbitrary accuracy one needs to have a sample of unbounded size. However, in practice the observation time of a system is usually not only finite but fixed. One of the possibilities for finding estimators with the guaranteed accuracy of inference using a sample of fixed size is provided by the approach of truncated sequential estimation. The truncated sequential estimation method was developed in [3–5] and others for parameter estimation problems in discrete-time dynamic models. In these papers, estimators of dynamic system parameters with known variance by sample of fixed size were constructed. Another but very similar approach was proposed in [6], where truncated estimation method was developed for estimation of ratio type functionals.

It is known that nonlinear stochastic systems are being widely used for describing real processes in economics, technics, medicine etc. We consider as a model the combination of two well-known models – AR with drifting parameters and ARARCH model. Such model can describe more accurately the real processes with complicated behavior.

The estimation problem of drifting parameter in ARARCH(1,1) model is considered. In the case of known noise variances truncated sequential estimation method developed in [3, 4] is applied. It makes possible to solve the estimation problem even for unstable observable process, while the truncated estimation method proposed in [6] works in the case of unknown variances and can be applied for more complicated multivariate models.

1. Problem statement

Consider one-dimensional process ARARCH(1,1) with drifting parameter

$$x_n = (\lambda + s_{n-1})x_{n-1} + \sqrt{\sigma_0^2 + \sigma_1^2 \cdot x_{n-1}^2} \cdot \xi_n, \qquad (1)$$

with the initial zero mean variable x_0 having the forth moment; (s_n) and (ξ_n) are sequences of i.i.d. zero mean random variables such that x_0, (s_n) and (ξ_n) are mutually independent and $\sigma_s^2 = Es_n^2$, $Es_n^4 < \infty$, $E\xi_n^2 = 1$, $E\xi_n^4 < \infty$.

Our aim is to estimate with guaranteed accuracy the unknown parameter λ. Cases of known and unknown noise variances will be considered.

2. Case of known noise variances

We suppose that the parameters $\sigma_0^2, \sigma_1^2, \sigma_s^2$ are known and the noise ξ_n satisfies the following additional condition $D\tilde{\alpha}_1 = E(\tilde{\alpha}_1 - m)^2 < \infty$, where $\tilde{\alpha}_1 == \dfrac{\xi_1^2}{\tilde{\sigma}_s^2 \xi_1^2 + 1}$, $m = E\tilde{\alpha}_1$. Moreover, noises (ξ_n) have central symmetric density $f_\xi(x)$ such that the set $\{x : f_\xi(x) \ge u\}$ is convex for every u.

Similar to general estimation procedure in [5] the truncated sequential estimator of parameter λ has the form

$$\tilde{\lambda}_N = \frac{1}{H_N}\sum_{n=1}^{\tau_N}\frac{\beta_n x_n \cdot x_{n-1}}{\sigma_0^2+(\sigma_1^2+\sigma_s^2)x_{n-1}^2}\chi\left[\sum_{n=1}^{N}\frac{x_{n-1}^2}{\sigma_0^2+(\sigma_1^2+\sigma_s^2)x_{n-1}^2}\ge H_N\right], \quad (2)$$

where $H_N = hN, h\in(0,m)$, the stopping time τ_N and weights β_n have respectively the following forms:

$$\tau_N = \begin{cases} \inf\left\{k\in[1,N]: \sum_{n=1}^{k}\frac{x_{n-1}^2}{\sigma_0^2+(\sigma_1^2+\sigma_s^2)x_{n-1}^2}\ge H_N\right\}, & \sum_{n=1}^{N}\frac{x_{n-1}^2}{\sigma_0^2+(\sigma_1^2+\sigma_s^2)x_{n-1}^2}\ge H_N, \\ N, & \sum_{n=1}^{N}\frac{x_{n-1}^2}{\sigma_0^2+(\sigma_1^2+\sigma_s^2)x_{n-1}^2}< H_N, \end{cases} \quad (3)$$

$$\beta_n = \begin{cases} 1, \ 1\le n<\tau_N, \\ 1, \ n=\tau_N, \ \sum_{n=1}^{N}\frac{x_{n-1}^2}{\sigma_0^2+(\sigma_1^2+\sigma_s^2)x_{n-1}^2}< H_N, \\ \overline{\alpha}_N, \ n=\tau_N, \ \sum_{n=1}^{N}\frac{x_{n-1}^2}{\sigma_0^2+(\sigma_1^2+\sigma_s^2)x_{n-1}^2}\ge H_N, \end{cases}$$

$$\overline{\alpha}_N = \left(H_N - \sum_{n=1}^{\tau_N-1}\frac{x_{n-1}^2}{\sigma_0^2+(\sigma_1^2+\sigma_s^2)x_{n-1}^2}\right)\Bigg/\frac{x_{\tau_N-1}^2}{\sigma_0^2+(\sigma_1^2+\sigma_s^2)x_{n-1}^2}.$$

According to Theorem 1 from [5] we have

$$E_\lambda(\tilde{\lambda}_N-\lambda)^2 \le \frac{C}{N}+\lambda^2\cdot P_\lambda\left(\sum_{n=1}^{N}\frac{x_{n-1}^2}{\sigma_0^2+(\sigma_1^2+\sigma_s^2)x_{n-1}^2}< H_N\right).$$

Now we estimate the probability in the right-hand side of the inequality. Denote $\tilde{\sigma}_s^2=\sigma_1^2+\sigma_s^2$, $\alpha_{n-1}=\frac{x_{n-1}^2}{\tilde{\sigma}_s^2 x_{n-1}^2+\sigma_0^2}$, $F_N=\sigma(x_0,s_0,\ldots,s_N,\xi_1,\ldots,\xi_N)$, $\tilde{H}_{N-2}=H_N-\sum_{n=1}^{N-1}\alpha_{n-1}$ and similar to [2, 4], using the conditions on the density $f_\xi(.)$, estimate the conditional probability:

$$P\left(\sum_{n=1}^{N}\frac{x_{n-1}^2}{\sigma_0^2+(\sigma_1^2+\sigma_s^2)x_{n-1}^2}< H_N \mid F_{N-2}\right) = P\left(\sum_{n=1}^{N}\alpha_{n-1}< H_N \mid F_{N-2}\right) = P(\alpha_{N-1}<\tilde{H}_{N-2}\mid F_{N-2}) =$$

$$= P\left(\frac{x_{N-1}^2}{\tilde{\sigma}_s^2 x_{N-1}^2+\sigma_0^2}<\tilde{H}_{N-2}\mid F_{N-2}\right) \le P\left(\frac{(\sigma_0^2+\sigma_1^2\cdot x_{N-2}^2)\xi_{N-1}^2}{\tilde{\sigma}_s^2(\sigma_0^2+\sigma_1^2\cdot x_{N-2}^2)\xi_{N-1}^2+\sigma_0^2}<\tilde{H}_{N-2}\mid F_{N-2}\right)\le$$

$$\le P\left(\frac{\xi_{N-1}^2}{\tilde{\sigma}_s^2\xi_{N-1}^2+1}<\tilde{H}_{N-2}\mid F_{N-2}\right)\le P\left(\tilde{\alpha}_{N-1}+\sum_{n=2}^{N-1}\alpha_{n-1}<H_N\mid F_{N-2}\right),$$

where
$$\tilde{\alpha}_N=\frac{\xi_N^2}{\tilde{\sigma}_s^2\xi_N^2+1}.$$

Analogously, we obtain

$$P\left(\tilde{\alpha}_{N-1}+\sum_{n=2}^{N-1}\alpha_{n-1}<H_N\mid F_{N-3}\right)\le P\left(\tilde{\alpha}_{N-1}+\tilde{\alpha}_{N-2}+\sum_{n=2}^{N-2}\alpha_{n-1}<H_N\mid F_{N-3}\right)$$

etc. So that finally we get

$$P\left(\sum_{n=1}^{N}\alpha_{n-1}<H_N\right)\le P\left(\sum_{n=2}^{N}\tilde{\alpha}_{n-1}<H_N\right)=P\left(\sum_{n=2}^{N}(\tilde{\alpha}_{n-1}-m)<H_N-m(N-1)\right)\le$$

$$\le P\left(\left|\sum_{n=2}^{N}(\tilde{\alpha}_{n-1}-m)\right|>(m-h)N-m\right)\le\frac{1}{[(m-h)N-m]^2}E\left[\sum_{n=2}^{N}(\tilde{\alpha}_{n-1}-m)\right]^2=\frac{(N-1)D\tilde{\alpha}_1}{[(m-h)N-m]^2}\le\frac{C}{N}.$$

Thus, it was proved that for every L $\sup\limits_{|\lambda|\le L}E_\lambda(\tilde{\lambda}_N-\lambda)^2\le\frac{C}{N}.$

Similar result can be obtained if the noise variances are unknown but their upper bounds are known $\sigma_s^2\le\bar{\sigma}_s^2$, $\sigma_0^2\le\bar{\sigma}_0^2$, $\sigma_1^2\le\bar{\sigma}_1^2$ and $\tilde{\sigma}_s^2\le\bar{\sigma}^2$. In this case the condition $D\tilde{\alpha}_1<\infty$ must be changed to the fol-lowing one:

$$D\left(\frac{\sigma_0^2\cdot\xi_1^2}{\sigma_0^2\cdot\bar{\sigma}^2\cdot\xi_1^2+\bar{\sigma}_0^2}\right)<\infty.$$

3. Case of unknown noise variances

When the noise variances as well as their upper bounds are unknown the truncated estimation method in the represented form cannot be applied. That is why we use the truncated estimation method presented in [6].

In addition to general conditions on the noises and parameters of the observable process we suppose that there exists the known lower bound for the parameter $\sigma_0^2\ge\sigma_*^2>0$ and the following stability condition

$$E(\lambda+s_1)^4+6\sigma_s^2\sigma_1^2+(\sigma_1^2)^2E\xi_1^4<1 \qquad (4)$$

holds. Define the set Θ of vectors $\theta=(\lambda,\sigma_s^2,\sigma_0^2,\sigma_1^2)$ satisfying the condition (4).

It is easy to verify that under this condition the process (x_n) has the following property

$$\sup_{n\ge1}Ex_n^4<\infty$$

and positive limits exist

$$m_2=\lim_{n\to\infty}Ex_n^2,\quad m_4=\lim_{n\to\infty}x_n^4.$$

According to the general estimation procedure in [6] we define the truncated estimator as follows:

$$\hat{\lambda}_N=\frac{1}{\sum\limits_{n=1}^{N}x_{n-1}^2}\sum_{n=1}^{N}x_n\cdot x_{n-1}\cdot\chi\left[\sum_{n=1}^{N}x_{n-1}^2\ge H_N\right],$$

where $H_N=\sigma_*^2\cdot N.$

From (1) it follows that the deviation of the estimator $\hat{\lambda}_N$ has the form

$$\hat{\lambda}_N-\lambda=\frac{1}{\sum\limits_{n=1}^{N}x_{n-1}^2}\sum_{n=1}^{N}\left[x_{n-1}s_{n-1}+\sqrt{\sigma_0^2+\sigma_1^2x_{n-1}^2}\cdot\xi_n\right]x_{n-1}\cdot\chi\left[\sum_{n=1}^{N}x_{n-1}^2\ge H_N\right]-\lambda\cdot\chi\left[\sum_{n=1}^{N}x_{n-1}^2<H_N\right].$$

Using this representation similar to [6] we can prove for every $\theta\in\Theta$ the following property

$$E_\theta(\hat{\lambda}_N-\lambda)^2\le\frac{C}{N},$$

where the number C depends on m_2 and m_4.

Remark. Similar results can be obtained using the truncated estimation method for ARARCH(p,q) model with drifting parameters of the form

$$x_n=(\lambda_1+s_1(n-1))\cdot x_{n-1}+\ldots+(\lambda_p+s_p(n-1))\cdot x_{n-p}+\sqrt{\sigma_0^2+\sigma_1^2\cdot x_{n-1}^2+\ldots+\sigma_q^2\cdot x_{n-q}^2}\cdot\xi_n$$

and using the truncated sequential estimation method for ARARCH (1, q).

4. Simulations

To confirm the convergence of the truncated sequential estimators we made the simulations of the esti-mator $\tilde{\lambda}_N$ defined in (2), (3) (Tables 1, 2) with $H_N = h \cdot N$ in case of known $\sigma_0^2 = 1, \sigma_1^2 = 0.1, \sigma_s^2 = 0.05$.

For this purpose we used the software package MATLAB. In Tables 1, 2 the average

$$\tilde{\lambda}(N) = \frac{1}{100} \sum_{k=1}^{100} \tilde{\lambda}_N(k)$$

of estimators (2) for the *k*-th realization $x^{(k)} = \left(x_n^{(k)}\right)$, $k = 1...100$ of the process (1) and their quality char-acteri-stics

$$S_\lambda^2(N) = \frac{1}{100} \sum_{k=1}^{100} \left(\tilde{\lambda}_N(k) - \lambda\right)^2$$

for different N are given.

Table 1

Estimation of the parameter λ with *h* = 0.2

λ	$N = 100$		$N = 200$		$N = 500$	
	$\tilde{\lambda}(N)$	$S_\lambda(N)$	$\tilde{\lambda}(N)$	$S_\lambda(N)$	$\tilde{\lambda}(N)$	$S_\lambda(N)$
0.2	0.2136	0.1630	0.2118	0.0871	0.2057	0.0393
–0.2	–0.2004	0.1722	–0.2373	0.0999	–0.2010	0.0589
0.9	0.8876	0.2494	0.9199	0.1164	0.9156	0.0413
–0.9	–0.8886	0.2781	–0.8726	0.1307	–0.8908	0.0589
1	1.0128	0.2392	0.9820	0.1312	1.0341	0.0577
–1	–0.9811	0.2065	–1.0105	0.1246	–1.0004	0.0542
4	3.9711	0.2661	4.0862	0.2083	4.0575	0.0868
–4	–4.0118	0.2901	–3.9984	0.1512	–4.0366	0.0533

Table 2

Estimation of the parameter λ with *h* = 0.6

λ	$N = 100$		$N = 200$		$N = 500$	
	$\tilde{\lambda}(N)$	$S_\lambda(N)$	$\tilde{\lambda}(N)$	$S_\lambda(N)$	$\tilde{\lambda}(N)$	$S_\lambda(N)$
0.2	0.2234	0.0721	0.1966	0.0249	0.2103	0.0149
–0.2	–0.1918	0.0613	–0.1837	0.0389	–0.2058	0.0123
0.9	0.8958	0.0665	0.9182	0.0341	0.8920	0.0191
–0.9	–0.8884	0.0520	–0.8829	0.0350	–0.8914	0.0229
1	1.0234	0.0714	1.0076	0.0539	1.0050	0.0212
–1	–1.0159	0.0755	–1.0318	0.0453	–1.0064	0.0338
4	4.0079	0.1023	4.0451	0.0544	4.0165	0.0202
–4	–4.0581	0.1098	–3.9939	0.0724	–3.9829	0.0184

Looking at the simulation results we can say that the deviation becomes less with growth of the sample size. It means that the estimator's value becomes closer to the true meaning of the parameter. This fact proves that these estimation procedures are quite effective. Moreover, the considered estimator works in the unstable case as well.

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