Pair production from the vacuum by a weakly inhomogeneous space-dependent electric potential

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There exists a clear physical motivation for theoretical studies of the vacuum instability related to the production of electron-positron pairs from a vacuum due to strong external electric fields. Various nonperturbative (with respect to the external fields) calculation methods were developed. Some of these methods are based on possible exact solutions of the Dirac equation. Unfortunately, there are only few cases when such solutions are known. Recently, an approximate but still nonperturbative approach to treat the vacuum instability caused by slowly varying t-electric potential steps (time dependent external fields that vanish as $|t| \to \infty$), which does not depend on the existence of the corresponding exact solutions, was formulated in the reference [S. P. Gavrilov, D. M. Gitman, Phys. Rev. D **95**, 076013 (2017)]. Here, we present an approximate calculation method to treat nonperturbatively the vacuum instability in arbitrary weakly inhomogeneous x-electric potential steps (time-independent electric fields of a constant direction that are concentrated in restricted space areas, which means that the fields vanish as $|x| \to \infty$) in the absence of the corresponding exact solutions. Defining the weakly inhomogeneous regime in general terms, we demonstrate the universal character of the vacuum instability. This universality is associated with a large density of states excited from the vacuum by the electric field. Such a density appears in our approach as a large parameter. We derive universal representations for the total number and current density of the created particles. Relations of these representations with a locally constant field approximation for Schwinger's effective action are found.

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I. INTRODUCTION

Since the work of Schwinger pointing to the possible vacuum instability due to the pair production in strong external electriclike fields (the Schwinger effect) [1], this effect has always attracted the attention of physicists. At present, we know that astrophysical objects such as black holes and hot strange stars can generate huge electromagnetic fields in their vicinity (dozens of times higher than Schwinger's critical field, $E_c = m^2/e$); see, e.g., Refs. [2–5]. It can also be seen that in some situations in graphene and similar nanostructures the vacuum instability effects caused by strong (with respect to massless fermions) electric fields are of significant interest; see, e.g., Refs. [6–12] and references therein. Recent progress in laser physics allows one to hope that the particle creation effect will be experimentally observed in laboratory conditions in the near future, as the strong laser experimental community, for example, Center for Relativistic Laser Science (CoReLS), Extreme Light Infrastructure (ELI), and Exawatt Center for Extreme Light Studies (XCELS), is slowly approaching the critical field strengths for observable pair production (see Ref. [13] for a review). Thus, there exists a clear physical motivation for theoretical studies of the vacuum instability. Firstly, it seems necessary to us to mention theoretical works devoted to various nonperturbative (with respect to the external field) calculation methods. Some of these methods are formulated for time-dependent external fields that vanish as $|t| \to \infty$ (for t-electric potential steps in what follows) and are based on

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possible exact solutions of the Dirac equation; see, e.g., [6,14–17]. Some of the methods are based on the analysis of the Schwinger effective action (see [18] for a review). The so-called derivative expansion approximation method, being applied to the Schwinger effective action, allows one to treat effectively arbitrary slowly varying in time strong fields [19,20]. We note that the locally constant field approximation (LCFA), which is to limit oneself to leading contributions of the derivative expansion of the effective action, allows for reliable results for electromagnetic fields of arbitrary strength; see, e.g., Refs. [21,22]. An alternative approach to treat slowly varying t-electric potential steps, which does not depend on the existence of the corresponding exact solutions, was formulated in Ref. [23].

When achieving extreme field strengths, the inhomogeneity of realistic external fields becomes important. At the same time in astrophysics and in the physics of graphene, electric fields can be considered as time independent; see, e.g., references cited above. Thus, it is important to study the effect of pair creation in strong constant inhomogeneous fields and to develop corresponding nonperturbative methods. Here, it is natural to start with considering the vacuum instability caused by time-independent inhomogeneous electric fields of a constant direction that depend on only one coordinate x and are concentrated in restricted space areas, which means that the fields vanish as $|x| \to \infty$. The latter fields represent a kind of so-called x-electric potential steps for charged particles. Nonperturbative methods for treating quantum effects in t-electric potential steps with the help of exact solutions of the Dirac equation are not directly applicable to the x-electric potential steps. One of the main differences between these approaches is that unlike the case of t-electric potential steps, the magnitude of the corresponding x-potential step, ΔU, is crucial for pair creation from a vacuum regardless of a field intensity. Only critical steps, $\Delta U > 2m$ (m is electron mass), produce electron-positron pairs, and this production occurs only in a finite range of quantum numbers that is called the Klein zone. Depending on the localization of the constant field, a critical point (critical surface) exists in the space of inhomogeneous electric field configurations where the pair production probability vanishes. This critical surface separates the Klein zone from the adjacent ranges where the pair production is impossible. Note that the absence of the critical surface in the case of the t-electric potential step arises as a consequence of neglecting the fact that a realistic electric field occupies a finite space region.

An adequate nonperturbative technique for treating the vacuum instability in the x-electric potential steps was elaborated on in Ref. [24]. Similar to the case of t-electric potential steps, special sets of exact solutions of the relativistic wave equations with corresponding external fields are crucial in this formulation. This technique was effectively used to describe the particle creation effect in the Sauter field of the form $E(x) = E\cosh^{-2}(x/L_S)$, in a constant electric field between two capacitor plates and in exponential time-independent electric steps, where the corresponding exact solutions were available; see Refs. [24–26]. However, there are only a few cases when such exact solutions are known.

It was recently shown that near criticality, pair production exhibits universal properties similar to those of continuous phase transitions [27,28]. In our terminology, this corresponds to the situation when the Klein zone is relatively small. In the present article, we show that there also exists a completely different type of universality, believing that the Klein zone is quite extensive, so that the total number of created pairs itself can be considered as a large parameter. Here, we develop a nonperturbative approach that allows one to treat the vacuum instability effects for arbitrary weakly inhomogeneous x-electric potential steps in the absence of the corresponding exact solutions. The Schwinger effective action method shows that when the probability of the pair creation is exponentially small, there are certain relations between the probability and such total physical quantities as a mean number density, a current density, and an energy-momentum tensor of created pairs. However, for strong electric fields such relations were established only for t-electric potential steps slowly varying with time; see Ref. [23], and in a few existing exactly solvable cases; see the review [29]. The approach developed by us in the present article allows us to establish similar relations for arbitrary weakly inhomogeneous x-electric potential steps.

The article is organized as follows. In Sec. II, we give a definition of weakly inhomogeneous x-electric potential steps and present an overview of the vacuum instability due to such backgrounds for existing exactly solvable cases. We stress some universal features of the vacuum instability in all these examples. In Sec. III, we present the approximation of a weakly inhomogeneous field and derive universal forms for the flux density of created particles and the probability of a vacuum to remain a vacuum. We show that using these results it is possible to calculate the physical quantities for any slightly inhomogeneous but otherwise arbitrary constant electric field. In this way, we reproduce the results obtained with the help of existing exact solutions. Then we succeed to describe the vacuum instability in electric fields where no exact solution of the corresponding Dirac equation are known, namely, in the Gaussian peak $E(x) = E_0 \exp[-(x/L_G)^2]$ and a special inverse square field of a form $E(x) = E_0[1 + (2x/L_w)^2]^{-1}$. Finally, in Sec. IV, we derive a general expression for the current density vector of particles created from a vacuum and relate it to the flux density of created particles obtained in the approximation of a weakly inhomogeneous field.

II. WEAKLY INHOMOGENEOUS POTENTIAL STEPS: EXACTLY SOLVABLE CASES

Let us consider QED with a time-independent pure electric field[1] $\mathbf{E}(X) = \mathbf{E}(x) = (E(x), 0, \ldots, 0)$. The inhomogeneous electric field $E(x)$ has the form,[2]

$$E(x) = E = \text{const} > 0, \quad x \in S_{\text{int}} = (x_{\text{L}}, x_{\text{R}});$$
$$E(x) = 0, \quad x \in S_{\text{L}} = (-\infty, x_{\text{L}}], \quad x \in S_{\text{R}} = [x_{\text{R}}, \infty). \tag{1}$$

We assume that the basic Dirac particle is an electron with the mass m and the charge $-e$, $e > 0$, and the positron is its antiparticle. The electric field under consideration accelerates the electrons along the x axis in the negative direction and the positrons along the x axis in the positive direction. Potentials of the corresponding electromagnetic field $A^\mu(X)$ can be chosen as

$$A^\mu(X) = (A^0(x), A^j = 0, j = 1, 2, \ldots, D), \tag{2}$$

so that $E(x) = -\partial_x A_0(x)$.

We call the electric field $E(x)$ a weakly inhomogeneous electric field on a spatial interval Δl if the following condition holds true:

$$\left| \frac{\overline{\partial_x E(x)} \Delta l}{\overline{E(x)}} \right| \ll 1, \quad \Delta l / \Delta l_{\text{st}}^{\text{m}} \gg 1, \tag{3}$$

where $\overline{E(x)}$ and $\overline{\partial_x E(x)}$ are the mean values of $E(x)$ and $\partial_x E(x)$ on the spatial interval Δl, respectively, and Δl is significantly larger than the length scale $\Delta l_{\text{st}}^{\text{m}}$, which is

$$\Delta l_{\text{st}}^{\text{m}} = \Delta l_{\text{st}} \max\{1, m^2 / e\overline{E(x)}\}, \quad \Delta l_{\text{st}} = [e\overline{E(x)}]^{-1/2}. \tag{4}$$

Note that the length scale $\Delta l_{\text{st}}^{\text{m}}$ appears in Eq. (3) as the length scale when the perturbation theory with respect to the electric field breaks down and the Schwinger (nonperturbative) mechanism is primarily responsible for the pair creation. In what follows, we show that this condition is sufficient. We are primarily interested in strong electric fields, $m^2 / e\overline{E(x)} \lesssim 1$. In this case, the second inequality in Eq. (3) is simplified to the form $\Delta l / \Delta l_{\text{st}} \gg 1$, in which the mass m is absent. In such cases, the potential of the corresponding electric step hardly differs from the potential of a uniform electric field,

$$U(x) = -eA_0(x) \approx U_{\text{const}}(x) = e\overline{E(x)}x + U_0, \tag{5}$$

on the interval Δl, where U_0 is a given constant. We see this behavior for the fields of known exact solvable cases for x-electric potential steps, namely, the Sauter field, the L-constant electric field, and the exponential peak field.

The magnitude of the corresponding x-potential step is

$$\Delta U = U_{\text{R}} - U_{\text{L}} > 0, \qquad U_{\text{R}} = -eA_0(+\infty),$$
$$U_{\text{L}} = -eA_0(-\infty). \tag{6}$$

We are interested in electron-positron pair creation that exists for the critical steps, $\Delta U > 2m$; see Ref. [24] for details. The Dirac equation with an x-electric potential step has the form,

$$i\partial_0 \psi(X) = \hat{H}\psi(X), \ \hat{H} = \gamma^0(-i\gamma^j \partial_j + m) + U(x), \ j = 1, \ldots D,$$
$$\psi(X) = \exp(-ip_0 t + i\mathbf{p}_\perp \mathbf{r}_\perp)\psi_n(x), \ \mathbf{p}_\perp = (p^2, \ldots, p^D),$$
$$\psi_n(x) = \{\gamma^0[p_0 - U(x)] - \gamma^1 \hat{p}_x - \boldsymbol{\gamma}_\perp \mathbf{p}_\perp + m\}\varphi_n^{(\chi)}(x) v_\chi,$$
$$\gamma^0\gamma^1 v_\chi = \chi v_\chi, \ \chi = \pm 1, \ v^\dagger_{\chi', \sigma'} v_{\chi, \sigma} = \delta_{\chi'\chi}\delta_{\sigma'\sigma}. \tag{7}$$

Here, $\psi(x)$ is a $2^{[d/2]}$-component spinor, $[d/2]$ stands for the integer part of $d/2$, $m \neq 0$ is the electron mass, and γ^μ are the γ matrices in d dimensions. The complete set of solutions of such a Dirac equation is determined by the functions ${}_\zeta\varphi_n(x)$ and ${}^\zeta\varphi_n(x)$ with special right and left asymptotics ($\zeta = \pm$) at $x \in S_{\text{L}}$ and $x \in S_{\text{R}}$, respectively. These solutions are parametrized by the set of quantum numbers $n = (p_0, \mathbf{p}_\perp, \sigma)$, where p_0 stands for total energy, $\mathbf{p}_\perp$ is transversal momentum (the index $\perp$ stands for components of momentum that are perpendicular to the electric field), and σ is spin polarization. The solutions $\varphi_n^{(\chi)}(x)$, which only differ by the values of χ, are linearly dependent. Because of this, it suffices to work with solutions corresponding to one of the possible two values of χ; so here and in what follows, we omit the subscript χ in these solutions, implying that the spin quantum number χ is fixed in a certain way.

A critical step produces electron-positron pairs in the Klein zone Ω_3, defined by the double inequality,

$$\Omega_3 \colon \ U_{\text{L}} + \pi_\perp \leq p_0 \leq U_{\text{R}} - \pi_\perp, \quad 2\pi_\perp \leq \Delta U, \tag{8}$$

where $\pi_\perp = \sqrt{\mathbf{p}_\perp^2 + m^2}$. In this range, initial states are determined by functions ${}_-\varphi_n$ for a positron and ${}^-\varphi_n$ for an electron while final states are determined by functions ${}_+\varphi_n$ for a positron and ${}^+\varphi_n$ for an electron (see Ref. [24] for details). The latter functions satisfy the following asymptotic conditions:

[1]We recall that our system is placed in the $(d = D + 1)$-dimensional Minkowski spacetime parametrized by the coordinates $X = (X^\mu, \mu = 0, 1, \ldots, D) = (t, \mathbf{r})$, $X^0 = t$, $\mathbf{r} = (X^1, \ldots, X^D)$, $x = X^1$. It consists of a Dirac field $\psi(X)$ interacting with an external electromagnetic field $A^\mu(X)$ in the form of a x-electric potential step.

[2]We use the system of units, where $c = \hbar = 1$.

$$
\begin{aligned}
{}_\zeta\varphi_n(x) &= {}_\zeta\mathcal{N}\exp\left[ip^{\mathrm{L}}(x-x_{\mathrm{L}})\right], \quad x\in S_{\mathrm{L}},\\
{}^\zeta\varphi_n(x) &= {}^\zeta\mathcal{N}\exp\left[ip^{\mathrm{R}}(x-x_{\mathrm{R}})\right], \quad x\in S_{\mathrm{R}},\\
p^{\mathrm{L}} &= \zeta\sqrt{[\pi_0(\mathrm{L})]^2-\pi_\perp^2},\\
p^{\mathrm{R}} &= \zeta\sqrt{[\pi_0(\mathrm{R})]^2-\pi_\perp^2}, \quad \zeta=\pm,\\
\pi_0(\mathrm{L/R}) &= p_0-U_{\mathrm{L/R}}.
\end{aligned} \tag{9}
$$

The constants ${}^\zeta\mathcal{N}$ and ${}_\zeta\mathcal{N}$ are normalization factors with respect to the inner product on the x-constant hyperplane [24],

$$
\begin{aligned}
({}_\zeta\psi_n, {}_{\zeta'}\psi_{n'})_x &= \zeta\eta_{\mathrm{L}}\delta_{\zeta,\zeta'}\delta_{n,n'}, \quad \eta_{\mathrm{L}}=\mathrm{sgn}\pi_0(\mathrm{L}),\\
({}^\zeta\psi_n, {}^{\zeta'}\psi_{n'})_x &= \zeta\eta_{\mathrm{R}}\delta_{\zeta,\zeta'}\delta_{n,n'}, \quad \eta_{\mathrm{R}}=\mathrm{sgn}\pi_0(\mathrm{R}),\\
(\psi,\psi')_x &= \int \psi^\dagger(X)\gamma^0\gamma^1\psi'(X)dtd\mathbf{r}_\perp,
\end{aligned} \tag{10}
$$

having the form

$$
\begin{aligned}
&{}^\zeta\mathcal{N} = {}^\zeta CY, \quad {}_\zeta\mathcal{N} = {}_\zeta CY, \quad Y=(V_\perp T)^{-1/2},\\
&{}^\zeta C = [2|p^{\mathrm{R}}||\pi_0(\mathrm{R})-\chi p^{\mathrm{R}}|]^{-1/2},\\
&{}_\zeta C = [2|p^{\mathrm{L}}||\pi_0(\mathrm{L})-\chi p^{\mathrm{L}}|]^{-1/2},
\end{aligned} \tag{11}
$$

where $V_\perp$ is the spatial volume of the $(d-1)$-dimensional hypersurface orthogonal to the electric field direction x, and T is the time duration of the electric field. Both $V_\perp$ and T are macroscopically large. The functions ${}^\zeta\varphi_n(x)$ and ${}_\zeta\varphi_n(x)$ are connected by the decomposition,

$$
\begin{aligned}
{}^\zeta\varphi_n(x) &= {}_+\varphi_n(x)g({}_+|^\zeta) - {}_-\varphi_n(x)g({}_-|^\zeta),\\
({}_\pm\psi_n, {}^\zeta\psi_{n'})_x &= \delta_{n,n'}g({}_\pm|^\zeta)
\end{aligned} \tag{12}
$$

in the Klein zone.

Partial vacuum states for a given $n\notin\Omega_3$ are stable. The differential mean numbers of electrons and positrons from the electron-positron pairs created are nonzero for $n\in\Omega_3$ only and are defined as the average values,

$$
\begin{aligned}
N_n^a(\mathrm{out}) &= \langle 0,\mathrm{in}|{}^+a_n^\dagger(\mathrm{out}){}^+a_n(\mathrm{out})|0,\mathrm{in}\rangle,\\
N_n^b(\mathrm{out}) &= \langle 0,\mathrm{in}|{}_+b_n^\dagger(\mathrm{out}){}_+b_n(\mathrm{out})|0,\mathrm{in}\rangle,
\end{aligned} \tag{13}
$$

where ${}^+a_n^\dagger(\mathrm{out})$ and ${}^+a_n(\mathrm{out})$ are the creation and annihilation operators of final electrons while ${}_+b_n^\dagger(\mathrm{out})$ and ${}_+b_n(\mathrm{out})$ are the creation and annihilation operators of final positrons, respectively. We define two vacuum vectors $|0,\mathrm{in}\rangle$ and $|0,\mathrm{out}\rangle$; the first of which is the vacuum vector for all annihilation operators of initial particles, and the other is the vacuum vector for all annihilation operators of final particles. The numbers $N_n^a(\mathrm{out})$ and $N_n^b(\mathrm{out})$ are equal and represent the number of pairs created, N_n^{cr}, which can be expressed as

$$
N_n^b(\mathrm{out}) = N_n^a(\mathrm{out}) = N_n^{\mathrm{cr}} = |g({}_+|^-)|^{-2}, \quad n\in\Omega_3. \tag{14}
$$

The exact solutions of the Dirac equation are known for the following inhomogeneous electric fields.

(i) The Sauter electric field and its vector potential have the form,

$$
\begin{aligned}
E(x) &= E_0\cosh^{-2}(x/L_{\mathrm{S}}),\\
A_0(x) &= -L_{\mathrm{S}}E_0\tanh(x/L_{\mathrm{S}}), \quad L_{\mathrm{S}}>0,
\end{aligned} \tag{15}
$$

where the parameter L_{S} sets the scale. The corresponding solutions of the Dirac equation ${}_\zeta\varphi_n(x)$ and ${}^\zeta\varphi_n(x)$ and the number of particles created N_n^{cr} for this potential were found in Ref. [24]. This field is considered a weakly inhomogeneous one if the following condition holds true:

$$
eE_0L_{\mathrm{S}}^2 \gg \max\{1, m^2/eE_0\}. \tag{16}
$$

The leading contribution to the total number of pairs created is formed in the inner part of the Klein zone Ω_3,

$$
\begin{aligned}
&|p_0| < eE_0L_{\mathrm{S}} - K/L_{\mathrm{S}}, \qquad \pi_\perp < K_\perp/L_{\mathrm{S}},\\
&\pi_\perp^2 = m^2+\mathbf{p}_\perp^2,
\end{aligned} \tag{17}
$$

where K is

$$
K = L_{\mathrm{S}}\sqrt{(km)^2+\pi_\perp^2}, \tag{18}
$$

while $K_\perp$ and $k\gtrsim 1$ are given arbitrary numbers obeying the inequalities,

$$
km \ll eE_0L_{\mathrm{S}}, \quad mL_{\mathrm{S}} \ll K_\perp \ll eEL_{\mathrm{S}}^2. \tag{19}
$$

The differential mean number of particles created in this range is [24]

$$
\begin{aligned}
&N_n^{\mathrm{cr}} \approx N_n^{\mathrm{as}} = e^{-\pi\tau},\\
&\quad\tau = \exp\left[-\pi L_{\mathrm{S}}(2eE_0L_{\mathrm{S}} - |p^{\mathrm{R}}| - |p^{\mathrm{L}}|)\right],
\end{aligned} \tag{20}
$$

where left and right asymptotic momenta $|p^{\mathrm{R}}|$ and $|p^{\mathrm{L}}|$ are defined by Eq. (9). This distribution has its maximum at zero energy, $p_0=0$, which coincides with the number of particles created by a uniform electric field,

$$
N_n^0 = N_n^{\mathrm{cr}}|_{p_0=0} = \exp\left[-\pi\lambda\right], \quad \lambda=\pi_\perp^2/eE_0. \tag{21}
$$

(ii) The so-called L-constant field does not change within the spatial region L and is zero outside of it,

$$E(x) = \begin{cases} 0, & x \in S_{\mathrm{L}} \\ E_0 > 0, & x \in S_{\mathrm{int}} \\ 0, & x \in S_{\mathrm{R}} \end{cases} \Rightarrow A_0(x) = \begin{cases} -E_0 x_{\mathrm{L}}, & x \in S_{\mathrm{L}} \\ -E_0 x, & x \in S_{\mathrm{int}}, \\ -E_0 x_{\mathrm{R}}, & x \in S_{\mathrm{R}} \end{cases} \tag{22}$$

where the regions are $S_{\mathrm{L}} = (-\infty, x_{\mathrm{L}}]$, $S_{\mathrm{R}} = [x_{\mathrm{R}}, +\infty)$, $S_{\mathrm{int}} = (x_{\mathrm{L}}, x_{\mathrm{R}})$, and we chose that $x_{\mathrm{L}} = -L/2$, $x_{\mathrm{R}} = L/2$. The corresponding solutions of the Dirac equation ${}_{\zeta}\varphi_n(x)$ and ${}^{\zeta}\varphi_n(x)$ were found in Ref. [25]. The L-constant field can be considered a weakly inhomogeneous field if

$$\sqrt{eE_0}L \gg \max(1, m^2/eE_0). \tag{23}$$

The leading contribution to the number of particles created is formed in the inner part D of the Klein zone Ω_3,

$$\sqrt{\lambda} < K_\perp, \quad |p_0|/\sqrt{eE_0} \le \sqrt{eE_0}L/2 - K,$$
$$\sqrt{eE_0}L/2 \gg K \gg K_\perp^2 \gg \max\{1, m^2/eE_0\}. \tag{24}$$

The leading contribution to N_n^{cr} has the form (21).

(iii) An exponential peak electric field has the following structure. Its first part is increasing exponentially on the spatial interval $I = (-\infty, 0]$ and reaches its maximum value E_0 at $x = 0$. The other part decreases exponentially from the same value E_0 on the spatial interval $II = (0, +\infty)$. The potential $A_0(x)$ and the electric field $E(x)$ have the form,

$$E(x) = E_0 \begin{cases} e^{k_1 x}, & x \in I \\ e^{-k_2 x}, & x \in II \end{cases} \Rightarrow A_0(x)$$
$$= E_0 \begin{cases} k_1^{-1}(-e^{k_1 x} + 1), & x \in I \\ k_2^{-1}(e^{-k_2 x} - 1), & x \in II \end{cases}, \tag{25}$$

where k_1 and k_2 are some positive constants. The corresponding solutions of the Dirac equation ${}_{\zeta}\varphi_n(x)$ and ${}^{\zeta}\varphi_n(x)$ were found in Ref. [26].

The case of a weakly inhomogeneous exponential peak corresponds to small values of k_1 and k_2, and is characterized by the condition,

$$\min(h_1, h_2) \gg \max(1, m^2/eE_0), \quad h_{1,2} = 2eE_0/k_{1,2}^2. \tag{26}$$

The main contributions to the number of particles created N_n^{cr} are formed in the ranges of quantum numbers $\pi_\perp < \pi_0(\mathrm{L}) \le eE_0 k_1^{-1}$ and $-eE_0 k_2^{-1} \ge \pi_0(\mathrm{R}) > -\pi_\perp$, and have the following forms:

$$N_n^{\mathrm{cr}} \approx \exp\left[-\frac{2\pi}{k_1}(\pi_0(\mathrm{L}) - |p^{\mathrm{L}}|)\right], \quad \pi_\perp < \pi_0(\mathrm{L}) \le eE_0 k_1^{-1}, \tag{27a}$$

$$N_n^{\mathrm{cr}} \approx \exp\left[-\frac{2\pi}{k_2}(|\pi_0(\mathrm{R})| - |p^{\mathrm{R}}|)\right], \quad -eE_0 k_2^{-1} \ge \pi_0(\mathrm{R}) > -\pi_\perp. \tag{27b}$$

In the examples under discussion, the intervals of growth and decay are described by nearly the same functional form; that is, increasing and decreasing components of the fields are almost symmetric. One can consider a strongly asymmetric configuration of the peak field, when one of the parameters k, for example, k_1, is sufficiently large, so that

$$eE_0 k_1^{-2} \ll 1, \quad |p^{\mathrm{L}}|/k_1 \ll 1, \tag{28}$$

while the second one, $k_2 > 0$, is arbitrary. This field is a weakly inhomogeneous one if

$$h_2 \gg \max(1, m^2/eE_0). \tag{29}$$

The leading contribution to the number of particles created is formed in the range of quantum numbers $-eE_0 k_2^{-1} \ge \pi_0(\mathrm{R}) > -\pi_\perp$, and coincides with the Eq. (27b). Note that this situation can be easily transformed to the case with a large k_2 and arbitrary k_1 by a simultaneous change $k_1 \leftrightarrows k_2$ and $\pi_0(\mathrm{L}) \leftrightarrows -\pi_0(\mathrm{R})$.

For weakly inhomogeneous electric fields, the differential mean numbers of electron-positron pairs created from the vacuum are almost constant over the wide range of energies p_0 for any given transversal momenta $\mathbf{p}_\perp$, even if these distributions are different for different fields. Furthermore, for all exactly solvable cases, there are wide subranges where the distributions N_n^{cr} coincide with the corresponding distributions N_n^0 in a constant uniform electric field, given by Eq. (21). We call this phenomenon the stabilization of the particle creation effect. In these subranges of quantum numbers, N_n^{cr} hardly depend on the details of how the field grows and decays. We note that the similar effect takes place for slowly varying electric fields; see Ref. [23] for the details.

The total number of pairs N^{cr} created from vacuum by an x-electric potential step can be calculated by the summation over all possible quantum numbers in the Klein zone Ω_3,

$$N^{\mathrm{cr}} = \sum_{n \in \Omega_3} N_n^{\mathrm{cr}}. \tag{30}$$

It is proportional to the so-called transversal space-time volume $V_\perp T$, $N^{\mathrm{cr}} = V_\perp T n^{\mathrm{cr}}$, where d labels the space-time dimensions, and the corresponding densities n^{cr} have the form,

$$n^{\mathrm{cr}} = \frac{J_{(d)}}{(2\pi)^{d-1}} \int_{\Omega_3} dp_0 d\mathbf{p}_\perp N_n^{\mathrm{cr}}. \tag{31}$$

In fact, n^{cr} is the total flux density of created particles. In the latter expression, the summations over the energies and transversal momenta were transformed into integrals, and the summation over spin projections was fulfilled, $J_{(d)} = 2^{[d/2]-1}$ (the square brackets mean the integer part of $d/2$). In weakly inhomogeneous fields, the magnitude of a potential step ΔU is large and can be used as a large parameter. The integral on the right-hand side can be approximated by an integral over a subrange D that gives the dominant contribution with respect to the total increment to the flux density of created particles,

$$n^{\mathrm{cr}} \approx \tilde{n}^{\mathrm{cr}} = \frac{J_{(d)}}{(2\pi)^{d-1}} \int_D dp_0 d\mathbf{p}_\perp N_n^{\mathrm{cr}}. \tag{32}$$

The exact form of the subrange D for each particular field must be determined separately. The dominant contributions $\tilde{n}^{\mathrm{cr}}$ are proportional to the magnitude of potential steps (and then the maximum increments of a particle energy), which, in general, differs for different fields and, for example, has the following forms in the exactly solvable cases (i), (ii), and (iii):

$$\begin{aligned} &\text{(i)}\ \Delta U_S = 2eE_0 L_S \text{ for the Sauter field,}\\ &\text{(ii)}\ \Delta U_L = eE_0 L \text{ for the} L\text{-constant field,}\\ &\text{(iii)}\ \Delta U_P = eE_0(k_1^{-1} + k_2^{-1}) \text{ for the peak field.} \end{aligned} \tag{33}$$

We note that ΔU_P corresponds to the case of a strongly asymmetric exponential field configuration at $k_1^{-1} \to 0$ (or at $k_2^{-1} \to 0$).

In terms of the introduced quantities (33), the densities $\tilde{n}^{\mathrm{cr}}$ in the exactly solvable cases under consideration have the forms [24–26],

$$\begin{aligned} &\text{(i)}\ \tilde{n}^{\mathrm{cr}} = r^{\mathrm{cr}} \frac{\Delta U_S}{2eE_0} \delta \text{ for the Sauter field,}\\ &\text{(ii)}\ \tilde{n}^{\mathrm{cr}} = r^{\mathrm{cr}} \frac{\Delta U_L}{eE_0} \text{ for the } L\text{-constant field,}\\ &\text{(iii)}\ \tilde{n}^{\mathrm{cr}} = r^{\mathrm{cr}} \frac{\Delta U_P}{eE_0} G\left(\frac{d}{2}, \pi\frac{m^2}{eE_0}\right) \text{ for the peak field,} \end{aligned} \tag{34}$$

where

$$\begin{aligned} r^{\mathrm{cr}} &= \frac{J_{(d)}(eE_0)^{d/2}}{(2\pi)^{d-1}} \exp\left\{-\pi\frac{m^2}{eE_0}\right\},\\ G(\alpha, x) &= \int_1^\infty \frac{ds}{s^{\alpha+1}} e^{-x(s-1)} = e^x x^\alpha \Gamma(-\alpha, x),\\ \delta &= \int_0^\infty dt\, t^{-1/2}(t+1)^{-(d+1)/2} \exp\left(-t\pi\frac{m^2}{eE_0}\right)\\ &= \sqrt{\pi}\Psi\left(\frac{1}{2}, \frac{2-d}{2}; \pi\frac{m^2}{eE_0}\right). \end{aligned} \tag{35}$$

Here, the $\Gamma(-\alpha, x)$ is the incomplete gamma function, and $\Psi(a, b; x)$ is the confluent hypergeometric function.[3]

Equating the densities $\tilde{n}^{\mathrm{cr}}$ for a Sauter-like field (i) and for the peak field (iii) to the density n^{cr} for the L-constant field, we find effective lengths L_{eff} of the interval of the field action for both cases,

$$\begin{aligned} &\text{(i)}\ L_{\mathrm{eff}} = L_S \delta,\\ &\text{(iii)}\ L_{\mathrm{eff}} = (k_1^{-1} + k_2^{-1}) G\left(\frac{d}{2}, \pi\frac{m^2}{eE_0}\right). \end{aligned} \tag{36}$$

Note that the effective length L_{eff} for a strongly asymmetric exponential field configuration is given by the second line in Eq. (36) as $k_1^{-1} \to 0$ (or $k_2^{-1} \to 0$). It is obvious that $L_{\mathrm{eff}} = L$ for the L-constant field. One can say that the Sauter field, the peak electric field, and the asymmetric exponential field with the same L_{eff} are equivalent to the L-constant field with respect to the pair production. Note that the factors G and δ in Eq. (35) for weak ($m^2/eE_0 \gg 1$) and strong ($m^2/eE_0 \ll 1$) electric fields can be approximated as

$$\begin{aligned} &G\left(\frac{d}{2}, \pi\frac{m^2}{eE_0}\right) \approx \frac{eE_0}{\pi m^2}, \quad \delta \approx \frac{\sqrt{eE_0}}{m}, \quad \frac{m^2}{eE_0} \gg 1;\\ &G\left(\frac{d}{2}, \pi\frac{m^2}{eE_0}\right) \approx \frac{2}{d}, \quad \delta \approx \frac{\sqrt{\pi}\Gamma(d/2)}{\Gamma(d/2+1/2)}, \quad \frac{m^2}{eE_0} \ll 1. \end{aligned} \tag{37}$$

One can compare the scale lengths for the cases (i), (ii), and (iii), given by Eqs. (16), (23), and (28), for the same E_0 and energy increments $\Delta U_S = \Delta U_L = \Delta U_P$ (in this case, $2L_S = L = k_1^{-1} + k_2^{-1}$). The condition Eq. (23) is stronger than Eqs. (16) and (28) if the fields are weak, whereas they are equivalent if the fields are strong. For this reason, defining the scale $\Delta l_{\mathrm{st}}^{\mathrm{m}}$ in general terms, we choose the form (4).

Initial $|0, \mathrm{in}\rangle$ and final $|0, \mathrm{out}\rangle$ vacua do not coincide. In the general case, the probability for a vacuum to remain a vacuum $P_{\mathrm{v}} = |\langle 0, \mathrm{out}|0, \mathrm{in}\rangle|^2$ can be expressed via the distribution N_n^{cr} as

$$P_{\mathrm{v}} = \prod_{n \in \Omega_3} (1 - N_n^{\mathrm{cr}}), \tag{38}$$

whereas, for weakly inhomogeneous electric fields, it can be written

[3] In Ref. [24], the result for a Sauter field was obtained under an unnecessary assumption that $\lambda > 1$. However, the final form of $\delta = \sqrt{\pi}\Psi(\frac{1}{2}, \frac{2-d}{2}; \pi\frac{m^2}{eE})$ is given correctly for an arbitrary m^2/eE in Ref. [24]. We have to correct its derivation as follows. Note that for any λ, one can see that N_n^{as}, given by Eq. (20), is exponentially small if $km \sim \pi_\perp$, where k is any given number satisfying inequality $k \ll \pi m L_S/2$. Therefore, the range $\pi_\perp \ll km$ is of interest. In this range, the approximation $\tau \approx eEL_S^2\pi_\perp^2[(eEL_S)^2 - p_0^2]^{-1}$ holds true. Taking into account the relation between τ and p_0, one can find a correct form of δ that is presented in Eq. (35).

$$P_{\mathrm{v}} \approx \prod_{n\in D}(1-N_n^{\mathrm{cr}}), \tag{39}$$

where D is the same subrange that gives the dominant contribution to the flux density of created particles (32). The probability P_{v} is given by similar forms for the Sauter field (i), the L-constant field (ii), and the peak field (iii), respectively, with the corresponding N^{cr},

$$P_{\mathrm{v}} = \exp(-\mu V_\perp T\tilde{n}^{\mathrm{cr}}), \quad \mu = \sum_{l=0}^{\infty}\frac{\epsilon_{l+1}}{(l+1)^{d/2}}\exp\left(-l\pi\frac{m^2}{eE_0}\right),$$

$$\text{(i) } \epsilon_l = \epsilon_l^S = \delta^{-1}\sqrt{\pi}\Psi\left(\frac{1}{2},\frac{2-d}{2};l\pi\frac{m^2}{eE_0}\right),$$

$$\text{(ii) } \epsilon_l = \epsilon_l^L = 1,$$

$$\text{(iii) } \epsilon_l = \epsilon_l^P = G\left(\frac{d}{2},l\pi\frac{m^2}{eE_0}\right)\left[G\left(\frac{d}{2},\pi\frac{m^2}{eE_0}\right)\right]^{-1}. \tag{40}$$

In the case of a weak field ($m^2/eE_0 \gg 1$), $\epsilon_l^S \approx l^{-1/2}$ for the Sauter field, $\epsilon_l^P \approx l^{-1}$ for the peak field, and $\exp(-\pi m^2/eE_0) \ll 1$. Then $\mu \approx 1$ for all the cases in Eq. (40), and we have a universal relation $N^{\mathrm{cr}} \approx \ln P_{\mathrm{v}}^{-1}$. In the case of a strong field ($m^2/eE_0 \ll 1$), all the terms with the different ϵ_l^S and ϵ_l^P contribute significantly to the sum in Eq. (40) if $l\pi m^2/eE_0 \lesssim 1$, and the quantities μ for the Sauter and peak fields differ essentially from the case of the L-constant field. Consequently, in this situation, one cannot derive a universal relation between P_{v} and $\tilde{n}^{\mathrm{cr}}$ from particular cases given by Eq. (40). In addition, it should be noted that in the case of a strong field, when known semiclassical approaches are not applicable, the probability P_{v} (unlike the total number N^{cr}) no longer has a direct relation to the vacuum mean values of the physical quantities discussed above. Therefore, to study a universal behavior of the vacuum instability in weakly inhomogeneous fields, one should derive first a universal form for the total flux density $\tilde{n}^{\mathrm{cr}}$.

III. UNIVERSAL BEHAVIOR OF THE FLUX DENSITY OF CREATED PAIRS IN A STRONG ELECTRIC FIELD

Unlike the case of uniform time-dependent electric fields, in constant inhomogeneous electric fields, there is a critical surface in space of particle momenta, which separates the Klein zone Ω_3, defined by inequality (8), from the adjacent ranges Ω_2 and Ω_4 (we use notation defined in Ref. [24]). In the ranges Ω_2 and Ω_4, the work of an electric field is sufficient to ensure the total reflection for electrons and positrons, respectively, but is not sufficient to produce pairs from vacuum. Accordingly, it is expected that for any nonpathological field configuration, the pair creation vanishes close to this critical surface, so that $N_n^{\mathrm{cr}} \to 0$ if n tends to the boundary with either the range Ω_2 ($|p^{\mathrm{R}}| \to 0$) or the range Ω_4 ($|p^{\mathrm{L}}| \to 0$),

$$N_n^{\mathrm{cr}} \sim |p^{\mathrm{R}}| \to 0, \quad N_n^{\mathrm{cr}} \sim |p^{\mathrm{L}}| \to 0, \quad \forall\ \pi_\perp \neq 0. \tag{41}$$

This is exactly the behavior we see for all field configurations which are known as exact solvable models [24–26].

Absolute values of the asymptotic momenta $|p^{\mathrm{L}}|$ and $|p^{\mathrm{R}}|$ are determined by the quantum numbers p_0 and $p_\perp$; see Eq. (9). This fact imposes a certain relation between both quantities. In particular, one can see that $d|p^{\mathrm{L}}|/d|p^{\mathrm{R}}| < 0$, and at any given $p_\perp$, these quantities are restricted inside the range Ω_3,

$$0 \le |p^{\mathrm{R/L}}| \le p^{\max}, \quad p^{\max} = \sqrt{\Delta U(\Delta U - 2\pi_\perp)}. \tag{42}$$

It implies that

$$0 \le ||p^{\mathrm{L}}| - |p^{\mathrm{R}}|| \le p^{\max}. \tag{43}$$

Then for all p_0 and $p_\perp$ of the range Ω_3, the numbers N_n^{cr} are small and tend to zero if the Klein zone shrinks to zero,

$$N_n^{\mathrm{cr}} \sim |p^{\mathrm{R}}p^{\mathrm{L}}| \to 0 \quad \text{if } p^{\max} \to 0. \tag{44}$$

In this case, the probability of a pair creation with quantum numbers n, $P(+-|0)_{n,n}$, can be approximated by the mean number N_n^{cr} as

$$P(+-|0)_{n,n} = \frac{N_n^{\mathrm{cr}}}{1-N_n^{\mathrm{cr}}}P_{\mathrm{v}} \approx N_n^{\mathrm{cr}}. \tag{45}$$

The total number of created particles, N^{cr}, given by the sum (30) over such a tiny Klein zone, is small too. It tends to zero if the magnitude of the potential step tends to a critical point,

$$N^{\mathrm{cr}} \to 0 \quad \text{if } \Delta U \to 2m. \tag{46}$$

We see that $N^{\mathrm{cr}} \ll 1$ near the critical point, so the probability of a vacuum to remain a vacuum P_{v}, given by a general form (38), can be approximated by the total number of created particles as $P_{\mathrm{v}} \approx 1 - N^{\mathrm{cr}}$. On the other hand, this probability can be represented via the imaginary part of a one-loop effective action S by the seminal Schwinger formula,

$$P_{\mathrm{v}} = \exp(-2\mathrm{Im}S). \tag{47}$$

Taking into account that $P_{\mathrm{v}} \approx 1$–$2\mathrm{Im}S$, one finds a relation,

$$2\mathrm{Im}S \approx N^{\mathrm{cr}} \quad \text{if } N^{\mathrm{cr}} \ll 1. \tag{48}$$

Taking into account this relation, we can confirm the behavior (46) by the results [27,28] recently obtained for

ImS in inhomogeneous x-potential electric steps of an arbitrary configuration that decay asymptotically with a power law $\sim E_0(kx)^{-a}$, $a \geq 2$ or vanish at a finite point $\sim E_0(k|x - x_0|)^b$, $b > 0$ (by taking the limit $a \to \infty$ or $b \to \infty$, an exponentially decaying field is recovered), where E_0 is a characteristic field strength scale and k is a characteristic length scale of the inhomogeneous field. It was shown by using semiclassical worldline instanton methods [30] in the weak-field ($m^2/eE_0 > 1$) critical regime [27] and by analysis of solutions of the Klein-Gordon and Dirac equation in the immediate vicinity of the critical point for an arbitrary peak field strength [28] that near criticality pair production vanishes, exhibiting universal properties similar to those of continuous phase transitions.

In what follows, we consider a completely different type of universality, believing that the Klein zone is quite extensive, so that the total number of created pairs itself can be considered a large parameter. We face such situations in astrophysical and condensed matter problems where the electric field is strong. In these scenarios, the numbers of the pairs created can reach their limiting values, $N_n^{\mathrm{cr}} \to 1$, and the total number of pairs created, N^{cr}, is not a small value anymore. For the weakly inhomogeneous fields, this number is proportional to the large parameter $L_{\mathrm{eff}}/\Delta l_{\mathrm{st}}$. For an arbitrary weakly inhomogeneous strong electric field, one can derive in the leading-term approximation, a universal form for the total density of created pairs.

As it was explained above, the contributions to the total number of created particles due to the part of the Klein zone in the vicinity of the critical point are small and can be neglected in a following approximation. The main contribution to N^{cr} is formed in some inner subrange $D(x)$ of the Klein zone where transversal momentum $\pi_\perp$ and energy p_0 are small enough. This inner subrange $D(x)$ can be described as

$$D(x):\ |\pi_0(x)| \gg \pi_\perp, \quad \pi_0(x) = p_0 - U(x). \tag{49}$$

In $D(x)$, the effective particle energy is primarily determined by an increment of energy $U(x) - U_{\mathrm{L}}$ or $U_{\mathrm{R}} - U(x)$ on the spatial intervals $\Delta x = x - x_{\mathrm{L}}$ or $\Delta x = x_{\mathrm{R}} - x$, respectively. It should be noted that $D(x) \subset D(x')$ if $x' > x$.

Suppose that the electric field does not grow and decay abruptly at the edges of some final interval, that is, the field slowly weakens at $x \to \pm\infty$, and one of the points x_{L} or x_{R} or both are infinitely distant from the origin, $x_{\mathrm{L}} \to -\infty$ and $x_{\mathrm{R}} \to \infty$. In this case, the contributions to the flux density $\tilde{n}^{\mathrm{cr}}$, given by Eq. (32), from the regions $(x_{\mathrm{L}}, x_{\mathrm{eff}}^{\mathrm{in}}]$ and $(x_{\mathrm{eff}}^{\mathrm{out}}, x_{\mathrm{R}})$ are exponentially small and can be disregarded, since the electric field in these regions is very weak in comparison with the maximum value of the peak field E_0, $E(x_{\mathrm{eff}}^{\mathrm{in}}), E(x_{\mathrm{eff}}^{\mathrm{out}}) \ll E_0$. Therefore, in general, it is sufficient to consider only the finite interval $(x_{\mathrm{eff}}^{\mathrm{in}}, x_{\mathrm{eff}}^{\mathrm{out}}]$. We can divide this interval into M intervals,

$$\Delta l_i = x_{i+1} - x_i, \quad i = 1, \ldots, M,$$
$$\sum_{i=1}^{M} \Delta l_i = x_{\mathrm{eff}}^{\mathrm{out}} - x_{\mathrm{eff}}^{\mathrm{in}}, \quad x_1 = x_{\mathrm{eff}}^{\mathrm{in}}, \quad x_{M+1} = x_{\mathrm{eff}}^{\mathrm{out}}, \tag{50}$$

in such a way that Eqs. (3) and (4) hold true for each of these intervals. Let us show that this allows us to treat the electric field as approximately uniform in each interval Δl_i, $\overline{E(x)} \approx \bar{E}(x_i)$ for $x \in (x_i, x_{i+1}]$ despite the fact that at the beginning of each interval Δl_i, the electric field $E(x)$ changes abruptly. Note that it is possible to use, for example, sharp exponential steps for the regularization of rectangular steps (see the details in Ref. [26]) if the length of the interval where this change occurs is significantly smaller than the length of each corresponding interval Δl_i. However, as we can see, it is not necessary.

In the case of the strong L-constant field, $m^2 \lesssim eE_0$, and the large parameter $\sqrt{eE_0}L$, a rough estimation of the next-to-leading-term for the flux density of created pairs shows (see details in Ref. [25]) that it produces a small factor of the order of $(\sqrt{eE_0}L)^{-1}$, i.e.,

$$n^{\mathrm{cr}} = \tilde{n}^{\mathrm{cr}}\left[1 + \frac{O(K)}{\sqrt{eE_0}L}\right], \tag{51}$$

where $\tilde{n}^{\mathrm{cr}}$ is given by expression (ii) in Eq. (34). It is clear that the abrupt change of the L-constant field at $x_{\mathrm{L}} = -L/2$ and $x_{\mathrm{R}} = L/2$ entails considerable oscillations in the distributions. Comparing the case of the L-constant field with other examples of the exactly solvable cases [24,26], we see that it presents the roughest estimate of the neglected contributions for weakly inhomogeneous potential steps. In particular, considering dominant contributions to the flux density of pairs created by a very asymmetric exponential peak (a field that grows from zero to its maximum value very rapidly and then experiences a smooth decay with the large effective length L_{eff}), one can see that it does not depend on the details of the field growth for the case of a strong field [26]. Then we can conclude that this abrupt change cannot significantly influence the total value of $\tilde{n}^{\mathrm{cr}}$ as $N_n^{\mathrm{cr}} \leq 1$ for fermions.

In each interval Δl_i, $\overline{E(x)} \approx \bar{E}(x_i)$ for $x \in (x_i, x_{i+1}]$ assuming that $L = \Delta l_i$, we can approximate partial contribution to the flux density due to this interval, Δn_i^{cr}, as $\Delta n_i^{\mathrm{cr}} = \Delta \tilde{n}_i^{\mathrm{cr}} + O(K_i)$, where K_i is any given number satisfying the condition,

$$\sqrt{e\bar{E}(x_i)}\Delta l_i \gg K_i \gg \max\{1, m^2/e\bar{E}(x_i)\}.$$

Then, using Eqs. (33) and (34) for the L-constant field, we have for $\tilde{n}^{\mathrm{cr}}$ that

$$\tilde{n}^{\mathrm{cr}} = \sum_{i=1}^{M} \Delta\tilde{n}_i^{\mathrm{cr}}, \quad \Delta\tilde{n}_i^{\mathrm{cr}} \approx \frac{J_{(d)}}{(2\pi)^{d-1}} \int_{ex_i\bar{E}(x_i)}^{e(x_i+\Delta l_i)\bar{E}(x_i)} dp_0 \int_{\sqrt{\lambda_i}<K_\perp^{(i)}} d\mathbf{p}_\perp N_n^{(i)},$$
$$N_n^{(i)} = e^{-\pi\lambda_i}, \quad \lambda_i = \pi_\perp^2 / e\bar{E}(x_i), \tag{52}$$

where $K_\perp^{(i)}$ are any given numbers satisfying the conditions,

$$K_i \gg [K_\perp^{(i)}]^2 \gg \max\{1, m^2/e\bar{E}(x_i)\}.$$

We can formally represent the variable p_0 in the latter expression as

$$p_0 = U(x), \quad U(x) = \int_{x_\mathrm{L}}^{x} dx'\, eE(x') + U_\mathrm{L}, \quad dp_0 = eE(x)dx. \tag{53}$$

Then neglecting small contributions to the integral (52), we find the following universal form for the flux density of created pairs in the leading-term approximation for a weakly inhomogeneous, but otherwise arbitrary strong electric field,

$$\tilde{n}^{\mathrm{cr}} \approx \frac{J_{(d)}}{(2\pi)^{d-1}} \int_{x_\mathrm{L}}^{x_\mathrm{R}} dx\, eE(x) \int d\mathbf{p}_\perp N_n^{\mathrm{uni}}, \quad N_n^{\mathrm{uni}} = \exp\left[-\pi \frac{\pi_\perp^2}{eE(x)}\right]. \tag{54}$$

The quantity N_n^{uni} has a universal form which can be used to calculate any total characteristic of the pair creation effect. One can integrate the latter expression over $d\mathbf{p}_\perp$ to obtain the final form,

$$\tilde{n}^{\mathrm{cr}} \approx \frac{J_{(d)}}{(2\pi)^{d-1}} \int_{x_\mathrm{L}}^{x_\mathrm{R}} dx[eE(x)]^{d/2} \exp\left[-\pi \frac{m^2}{eE(x)}\right]. \tag{55}$$

These universal forms can be derived for bosons as well, if we are restricting them to the forms of external electric fields, namely, fields that have no abrupt variations of $E(x)$ that can produce significant growth of N_n^{cr} on a finite spatial interval; i.e., we have to include in the range D, the only subranges where $N_n^{\mathrm{cr}} \leq 1$. In this case, the universal forms for bosons are the same, Eqs. (54) and (55), with $J_{(d)} = 1$ for scalar particles and $J_{(d)} = 3$ for vector ones.

Using the identity $-\ln(1 - N_n^{\mathrm{cr}}) = N_n^{\mathrm{cr}} + (N_n^{\mathrm{cr}})^2 + \cdots$, in the same manner we can derive a universal form of the probability of a vacuum to remain a vacuum P_v defined for fermions by Eq. (38). First, we get

$$P_\mathrm{v} \approx \exp\left\{-\frac{V_\perp T J_{(d)}}{(2\pi)^{d-1}} \sum_{l=1}^{\infty} \int_{x_\mathrm{L}}^{x_\mathrm{R}} dx\, eE(x) \int d\mathbf{p}_\perp (N_n^{\mathrm{uni}})^l\right\}. \tag{56}$$

After integration over $\mathbf{p}_\perp$, we finally obtain

$$P_\mathrm{v} \approx \exp\left\{-\frac{V_\perp T J_{(d)}}{(2\pi)^{d-1}} \sum_{l=1}^{\infty} \int_{x_\mathrm{L}}^{x_\mathrm{R}} dx \frac{[eE(x)]^{d/2}}{l^{d/2}} \exp\left[-\pi \frac{lm^2}{eE(x)}\right]\right\}. \tag{57}$$

For bosons, we know that the vacuum-to-vacuum transition probability has the form,

$$P_\mathrm{v}^{(\mathrm{boson})} = \exp\left[-\sum_n \ln(1 + N_n^{\mathrm{cr}})\right], \tag{58}$$

so the universal form of the vacuum-to-vacuum transition probability for the Bose case is

$$P_\mathrm{v}^{(\mathrm{boson})} \approx \exp\left\{-\frac{V_\perp T J_{(d)}}{(2\pi)^{d-1}} \sum_{l=1}^{\infty} \int_{x_\mathrm{L}}^{x_\mathrm{R}} dx (-1)^{l-1} \frac{[eE(x)]^{d/2}}{l^{d/2}} \exp\left[-\pi \frac{lm^2}{eE(x)}\right]\right\}, \tag{59}$$

where $J_{(d)}$ is the number of boson spin degrees of freedom.

Using Eqs. (55) and (57), one can precisely reproduce expressions (34) and (40) that are found for the total densities and the vacuum-to-vacuum transition probabilities when directly adopting the weakly inhomogeneous field approximation to the exactly solvable cases. Comparing Eqs. (55) and (59) with the exact results obtained for bosons [24–26], one finds precise agreement too. Thus, we have a confirmation of the universal forms obtained above.

The representations (57) and (59) coincide with the vacuum-to-vacuum transition probabilities obtained from the imaginary part of a locally constant field approximation (LCFA) for the one-loop effective action in $d = 4$ dimensions [22,31]. In this approximation, the effective action S is expanded about the constant field case, in terms of derivatives of the background field strength $F_{\mu\nu}$,

$$S = S^{(0)}[F_{\mu\nu}] + S^{(2)}[F_{\mu\nu}, \partial_\mu F_{\nu\rho}] + \cdots, \tag{60}$$

where $S^{(0)}$ involves no derivatives of the background field strength $F_{\mu\nu}$ (that is, $S^{(0)}$ is a locally constant field approximation for S that has a form of the Heisenberg-Euler action), while the first correction $S^{(2)}$ involves two derivatives of the field strength, and so on; see Ref. [18] for a review. Using the representation (47), one finds the LCFA for the probability $P_{\rm v}$ as

$$P_{\rm v} \approx \exp\left(-2\mathrm{Im}S^{(0)}\right). \tag{61}$$

However, it should be stressed that unlike the representations obtained in Refs. [22,31], we derive Eqs. (57) and (59) in the framework of the general formulation of strong-field QED in the presence of x-electric potential steps [24], where $P_{\rm v}$ are defined by Eqs. (38) and (58), respectively. Therefore, we obtain Eqs. (57) and (59) independently from the derivative expansion approach, and the obtained result holds true for any strong field under consideration. It is known that for a general background field, it is extremely difficult to estimate and compare the magnitude of various terms in the derivative expansion. Only under the assumption $m^2/eE_0 > 1$, one can demonstrate that the derivative expansion is completely consistent with the semiclassical WKB analysis of the imaginary part of the effective action [19]. Thus, the representations (57) and (59) are proof that the imaginary part of the LCFA for the Heisenberg-Euler action is correct for an arbitrarily weakly inhomogeneous electric field of a constant direction. The universal forms (54) and (55) for the flux density of created pairs are completely new and present a LCFA for this physical quantity. It is a new kind of a LCFA obtained without any relation to the Heisenberg-Euler action.

One can see that the obtained universal forms have especially simple forms in two limited cases, for a weak electric field ($m^2/eE_0 \gg 1$), when the term $[eE(x)]^{d/2}$ can be approximated by its maximum value, $[eE_0]^{d/2}$, and a strong electric field ($m^2/eE_0 \ll 1$), when there exist spatial intervals where $m^2/eE(x) \ll 1$ and approximations of the type

$$\exp\left[-\frac{\pi l m^2}{eE(x)}\right] = 1 - \frac{\pi l m^2}{eE(x)} + \ldots \tag{62}$$

are available. For example, one can consider the case of a strong Gauss peak,

$$E(x) = E_0 \exp\left[-(x/L_{\rm G})^2\right], \tag{63}$$

with a large parameter $L_{\rm G} \to \infty$. In this case, we do not have an exact solution of the Dirac equation, and known semiclassical approximations (valid for a weak field) are not applicable. However, using approximation (62), we find from Eqs. (55) and (57), the leading term as

$$\tilde{n}^{\rm cr} \approx \frac{J_{(d)}(eE_0)^{d/2}L_{\rm G}}{d(2\pi)^{d-2}}, \quad P_{\rm v} \approx \exp\left(-V_\perp T \tilde{n}^{\rm cr} \sum_{l=1}^{\infty} l^{-d/2}\right). \tag{64}$$

As another example, we consider an inverse square electric field,

$$E(x) = E_0\left[1 + \left(\frac{2x}{L_{\rm w}}\right)^2\right]^{-1}, \quad A_0(x) = -\frac{L_{\rm w}}{2}E_0 \,\mathrm{arctg}\frac{2x}{L_{\rm w}}. \tag{65}$$

It is a particular case of an inhomogeneous field that was used to study the LCFA for the probability $P_{\rm v}$ in Ref. [31]. Using the Eqs. (55) and (57), we find in the leading order that

$$\tilde{n}^{\rm cr} \approx \frac{L_{\rm w}}{2} r^{\rm cr} \delta_{\rm w}, \tag{66}$$

$$P_{\rm v} \approx \exp\left\{-V_\perp T \tilde{n}^{\rm cr} \sum_{l=0}^{\infty} \frac{\epsilon_{l+1}}{(l+1)^{d/2}} \exp\left[-\frac{\pi l m^2}{eE_0}\right]\right\}, \tag{67}$$

where

$$\epsilon_l = \epsilon_l^{\rm w} = \sqrt{\pi}\Psi\left(\frac{1}{2}, \frac{3-d}{2}; \frac{\pi l m^2}{eE_0}\right)\delta_{\rm w}^{-1},$$
$$\delta_{\rm w} = \sqrt{\pi}\Psi\left(\frac{1}{2}, \frac{3-d}{2}; \frac{\pi m^2}{eE_0}\right). \tag{68}$$

Our result for the probability $P_{\rm v}$ (67) coincides with the one obtained in Ref. [31] for the particular case of $d = 4$.

For the case when the external background is given by a slowly varying in time electric field, the expressions for the total number of created particles and for the probability of vacuum-to-vacuum transition similar to Eqs. (55) and (57) were obtained in Ref. [23]. It is clear, however, that time-dependent fields and nonuniform fields describe physically distinct situations, and in the general case, the results have different forms.

IV. MEAN CURRENT

It is well known that in quantum field theory, measurable values and their mean values, generally speaking, are defined globally. This fact implies that those values are defined in some macroscopic volume at some fixed moment of time. Of course, the measurement procedure takes some macroscopic time itself. This is usually ignored under the assumption that the measured physical value does not change substantially during the measurement. If an external field does not violate vacuum stability, then it just causes vacuum polarization inside the area occupied by the field. This effect is a quasilocal one. In the case that is most interesting to us, i.e., when an external field is capable of violating vacuum stability, there is a global effect of the field due to an electron-positron pair creation. These pairs do not disappear when the field is turned off. The electrons and positrons created leave the area S_{int} occupied by the field, creating a constant flow of particles moving away from the field. For symmetry reasons, it is clear that this flow is aligned along the field direction (which is the axis x in our case) and creates a longitudinal electric current.

Created electrons and positrons leaving the field region S_{int} fly out into the regions S_{L} and S_{R}, respectively, and move away from the electric field at constant longitudinal velocities $-v^{\mathrm{L}}$ and v^{R}, where $v^{\mathrm{L}} = |p^{\mathrm{L}}/\pi_0(\mathrm{L})|$ and $v^{\mathrm{R}} = |p^{\mathrm{R}}/\pi_0(\mathrm{R})|$. This longitudinal current is coordinate independent and, therefore, can be determined by its value anywhere in S_{L} or S_{R}.

Using a general theory [24], we can derive the forms for the vacuum mean current. The charge operator $\hat{Q}$ is defined in Ref. [24] as a commutator,

$$\hat{Q} = -\frac{e}{2}\int [\hat{\Psi}^{\dagger}(X), \hat{\Psi}(X)]_{-} d\mathbf{r}. \tag{69}$$

The expression (69) for the charge operator suggests the definition for the current operator,

$$\hat{J}^{\mu} = -\frac{e}{2}[\hat{\Psi}^{\dagger}(X)\gamma^0\gamma^{\mu}, \hat{\Psi}(X)]_{-}. \tag{70}$$

These values can be expressed via the vacuum mean values of the electric current density of the Dirac field through the surface $x = \text{const}$, which are defined as follows:

$$\langle J^{\mu}(x)\rangle_{\mathrm{in}} = \langle 0,\mathrm{in}|\hat{J}^{\mu}|0,\mathrm{in}\rangle, \quad \langle J^{\mu}(x)\rangle_{\mathrm{out}} = \langle 0,\mathrm{out}|\hat{J}^{\mu}|0,\mathrm{out}\rangle, \tag{71}$$

where the Dirac Heisenberg operator $\hat{\Psi}(X)$ is assigned to the Dirac field $\psi(X)$. We stress the x coordinate dependence of mean values (71), which does exist due to the coordinate dependence of the external field. It should be noted that these densities depend on a vacuum definition and the structure of the electric field in the direction x. The renormalized vacuum mean value $\langle J^{\mu}(x)\rangle_{\mathrm{in}}$ is a source in the equations of motion for a mean electromagnetic field. The quantity $\langle J^{\mu}(x)\rangle_{\mathrm{out}}$ is needed to present the operator $\hat{J}^{\mu}$ in a normal ordering form with respect to the creation and annihilation operators of the final particles,

$$\hat{N}_{\mathrm{out}}(J^{\mu}) = \hat{J}^{\mu} - \langle J^{\mu}(x)\rangle_{\mathrm{out}}. \tag{72}$$

Mean values and probability amplitudes are described by the Feynman diagrams with two kinds of charged particle propagators in the external field under consideration, respectively. The probability amplitudes are calculated using the causal (Feynman) propagator $S^c(X,X')$ while mean values are found using the so-called in-in propagator $S^c_{\mathrm{in}}(X,X')$ and out-out propagator $S^c_{\mathrm{out}}(X,X')$,

$$\begin{aligned}
S^c(X,X') &= i\langle 0,\mathrm{out}|\hat{T}\,\hat{\Psi}(X)\hat{\Psi}^{\dagger}(X')\gamma^0|0,\mathrm{in}\rangle c_{\mathrm{v}}^{-1},\\
S^c_{\mathrm{in}}(X,X') &= i\langle 0,\mathrm{in}|\hat{T}\,\hat{\Psi}(X)\hat{\Psi}^{\dagger}(X')\gamma^0|0,\mathrm{in}\rangle,\\
S^c_{\mathrm{out}}(X,X') &= i\langle 0,\mathrm{out}|\hat{T}\,\hat{\Psi}(X)\hat{\Psi}^{\dagger}(X')\gamma^0|0,\mathrm{out}\rangle,
\end{aligned} \tag{73}$$

where $\hat{T}$ denotes the chronological ordering operation and c_{v} is the vacuum-to-vacuum transition amplitude, $c_{\mathrm{v}} = \langle 0,\mathrm{out}|0,\mathrm{in}\rangle$, $|c_{\mathrm{v}}|^2 = P_{\mathrm{v}}$. As usual, these propagators can be expressed via the following singular functions:

$$\begin{aligned}
S^c(X,X') &= \theta(t-t')S^{-}(x,x') - \theta(t'-t)S^{+}(x,x'),\\
S^c_{\mathrm{in/out}}(X,X') &= \theta(t-t')S^{-}_{\mathrm{in/out}}(X,X') - \theta(t'-t)S^{+}_{\mathrm{in/out}}(X,X').
\end{aligned} \tag{74}$$

The vacuum mean values (71) can be expressed via the propagators S^c_{in} and S^c_{out} while the causal propagator S^c determines the vacuum polarization contribution to current as

$$\begin{aligned}
\langle J^{\mu}(x)\rangle^c &= \langle 0,\mathrm{out}|\hat{J}^{\mu}|0,\mathrm{in}\rangle c_{\mathrm{v}}^{-1} = -ie\mathrm{tr}[\gamma^{\mu}S^c(X,X')]|_{X=X'},\\
\langle J^{\mu}(x)\rangle_{\mathrm{in/out}} &= -ie\mathrm{tr}[\gamma^{\mu}S^c_{\mathrm{in/out}}(X,X')]|_{X=X'}.
\end{aligned} \tag{75}$$

Using the explicit forms of these singular functions, given by in Ref. [24], we see that transversal components of these currents are equal to zero,

$$\langle J^k(x)\rangle_{\mathrm{in}} = \langle J^k(x)\rangle_{\mathrm{out}} = \langle J^k(x)\rangle^c = 0 \quad \text{if } k \neq 1, \tag{76}$$

due to the cylindrical symmetry of the problem.

Using a singular function,

$$S^p(X,X') = S^c_{\mathrm{in}}(X,X') - S^c(X,X'), \tag{77}$$

it is useful to introduce a current,

$$\langle J^{\mu}(x)\rangle^{p} = -ie\,\mathrm{tr}[\gamma^{\mu}S^{p}(X,X')]|_{X=X'}. \quad (78)$$

The explicit form of $S^{p}(X,X')$ is [24]

$$S^{p}(X,X') = i\sum_{n\in\Omega_3}\mathcal{M}_n^{-1}[g({}_-|^-)^*]^{-1} - \psi_n(X) - \bar{\psi}_n(X'),$$
$$\mathcal{M}_n = 2\frac{\tau^{(\mathrm{R})}}{T}|g({}_+|^-)|^2 = 2\frac{\tau^{(\mathrm{L})}}{T}|g({}_+|^-)|^2. \quad (79)$$

Note that S^p is formed in the range Ω_3 only and vanishes if there is no pair creation. Here, $\tau^{(\mathrm{L})}$ and $\tau^{(\mathrm{R})}$ are equal macroscopic times of motion for created particles in the regions S_{L} and S_{R}, respectively. It is assumed that the regions S_{L} and S_{R} are substantially wider than the region S_{int}, and it is possible to neglect the time period when the created particles are moving from the region S_{int} into regions S_{L} and S_{R}. We suppose that all the measurements are performed during a macroscopic time T when the external field can be considered as constant. In particular, a charge transport through planes $x = x_{\mathrm{L}}$ and $x = x_{\mathrm{R}}$ occurs during the time period T. In this case, the times $\tau^{(\mathrm{L})}$ and $\tau^{(\mathrm{R})}$ coincide with T, $\tau^{(\mathrm{L})} = \tau^{(\mathrm{R})} = T$, and we obtain that

$$\mathcal{M}_n^{-1} = \frac{1}{2}N_n^{\mathrm{cr}},$$

where N_n^{cr} is given by Eq. (14).

Thus, we have that

$$\langle J^{\mu}(x)\rangle_{\mathrm{in}} = \langle J^{\mu}(x)\rangle^{c} + \langle J^{\mu}(x)\rangle^{p}. \quad (80)$$

It is clear from Eq. (76) that only the components $\langle J^0(x)\rangle^p$ and $\langle J^1(x)\rangle^p$ are nonzero. Using representations (78) and (79) and decomposition (12), we find that

$$\langle J^0(x)\rangle^p = \begin{cases} -\bar{J}^0(\mathrm{L}) & \text{if } x\in S_{\mathrm{L}} \\ \bar{J}^0(\mathrm{R}) & \text{if } x\in S_{\mathrm{R}} \end{cases},$$
$$\bar{J}^0(\mathrm{L/R}) = \frac{e}{2}\sum_{n\in\Omega_3} j_n^0(\mathrm{L/R}), \quad j_n^0(\mathrm{L/R}) = j_n^1/v^{\mathrm{L/R}}; \quad (81)$$

$$\langle J^1(x)\rangle^p = \frac{e}{2}\sum_{n\in\Omega_3} j_n^1, \quad j_n^1 = N_n^{\mathrm{cr}}(TV_\perp)^{-1}. \quad (82)$$

Using Eqs. (9), we can present the singular functions S_{in}^c and S_{out}^c given by in Ref. [24], in an explicit form in the regions S_{L} and S_{R}, respectively. We find that

$$\langle J^1(x)\rangle_{\mathrm{in}} = -\langle J^1(x)\rangle_{\mathrm{out}} = \langle J^1(x)\rangle^p \ \text{ if } x\in S_{\mathrm{L}} \ \text{ or } \ S_{\mathrm{R}}. \quad (83)$$

Note that the values of electric current densities (83) [including each of the components of j_n^1, given by Eq. (82)] are conserved along the axis x. These expressions are defined for regions where an electric field is absent and do not contain contributions independent of an electric field. For example, if an electric field in the region S_{int} turns off,$E\to 0$, then the number of pairs created by the field vanishes, $N_n^{\mathrm{cr}}\to 0$. For this reason, the densities (83) are characteristics of real particles and cannot change after an electric field is turned off.

Taking into account the relations (72) and (83), we find that the longitudinal current of pairs created is

$$J_{\mathrm{cr}}^1 = \langle N_{\mathrm{out}}(J^1)\rangle_{\mathrm{in}} = 2\langle J^1(x)\rangle^p = e\sum_{n\in\Omega_3} j_n^1. \quad (84)$$

Here, j_n^1 is the flux density of particles created with a given n, and

$$\sum_{n\in\Omega_3} j_n^1 = n^{\mathrm{cr}} \quad (85)$$

is the total flux density of created particles, given by Eq. (31). This allows us to interpret the density $2\langle J^0(x)\rangle^p$ as the charge density of the particles created,

$$J_{\mathrm{cr}}^0(x) = 2\langle J^0(x)\rangle^p = \begin{cases} -e\sum_{n\in\Omega_3} j_n^0(\mathrm{L}) & \text{if } x\in S_{\mathrm{L}} \\ e\sum_{n\in\Omega_3} j_n^0(\mathrm{R}) & \text{if } x\in S_{\mathrm{R}} \end{cases}. \quad (86)$$

This interpretation also works for partial components of density (86). We see that created electrons with a given n move with a velocity v^{L} in a direction opposite to the direction of the axis x, i.e., in the direction opposite to the direction of the current density ej_n^1. During the time T that these electrons transport through the plane $x = x_{\mathrm{L}}$, the amount of charge per $V_\perp$ is equal to $ej_n^1 T$. Taking into account that this charge is distributed uniformly over the cylindrical volume with the length $v^{\mathrm{L}}T$, we obtain that the charge density of created electrons with a given n is equal to $ej_n^1/(-v^{\mathrm{L}}) = -ej_n^0(\mathrm{L})$, where $j_n^0(\mathrm{L})$ is given by Eq. (81). Created positrons with a given n move at a velocity v^{R} along axis x; i.e., the direction of their movement coincides with the direction of the current density ej_n^1. During the time T, positrons transport the same charge amount per $V_\perp$ as electrons, $ej_n^1 T$, through the plane $x = x_{\mathrm{R}}$ (in this case, it is uniformly distributed over a cylindrical volume with the length $v^{\mathrm{R}}T$). We find that the charge density of created positrons with a given n is $ej_n^1/v^{\mathrm{R}} = ej_n^0(\mathrm{R})$, where $j_n^0(\mathrm{R})$ is given by Eq. (81). We see that every pair ej_n^1 and $-ej_n^0(\mathrm{L})$ in S_{L} and ej_n^1 and $ej_n^0(\mathrm{R})$ in S_{R}, correspondingly, can be connected by a Lorentz boost and represents (nonzero) components of the same Lorentz vector. The number densities in both regions $x\in S_{\mathrm{L}}$ and $x\in S_{\mathrm{R}}$ are equal,

$$\sum_{n\in\Omega_3} j_n^0(\mathrm{L}) = \sum_{n\in\Omega_3} j_n^0(\mathrm{R}).$$

Thus, the charge densities of created electrons in $x\in S_{\mathrm{L}}$ and positrons in $x\in S_{\mathrm{R}}$ have the same value, but opposite sign. We see that the total charge of created particles is zero, and an electric field produces a charge polarization along axis x, just as one would expect.

The main contribution to the longitudinal current and flux density of the pair created is formed in the inner subrange $D(x)$ of the Klein zone, given by the inequality (49). In this subrange, $v^{\mathrm{L}} \simeq v^{\mathrm{R}} \simeq 1$, and we have that $j_n^0(\mathrm{L/R}) \simeq j_n^1$. Then, we can write that

$$J_{\mathrm{cr}}^1 = e\tilde{n}^{\mathrm{cr}}, \quad J_{\mathrm{cr}}^0(x) = \begin{cases} -e\tilde{n}^{\mathrm{cr}} & \text{if } x \in S_{\mathrm{L}} \\ e\tilde{n}^{\mathrm{cr}} & \text{if } x \in S_{\mathrm{R}} \end{cases}, \tag{87}$$

where $\tilde{n}^{\mathrm{cr}}$ is given by the universal forms (54) and (55).

V. CONCLUDING REMARKS

In the present article, we have presented the approximation that allows one to treat nonperturbatively the vacuum instability effects for arbitrary weakly inhomogeneous x-electric potential steps in the absence of the corresponding exact solutions. First, we have revised vacuum instability effects in three exactly solvable cases in QED with x-electric potential steps that have a real physical importance. These are the Sauter electric field, the so-called L-constant electric field, and the exponentially growing and decaying strong electric weakly inhomogeneous fields. Defining the conditions of a field being weakly inhomogeneous in general terms, we observed some universal features of vacuum effects caused by the strong electric fields. These universal features appear when the length of the external field is sufficiently large in comparison to the scale, $\Delta l_{\mathrm{st}} = [e\overline{E(x)}]^{-1/2}$. In this case, the scale of the variation for an external field and leading contributions to vacuum mean values are macroscopic. We found universal approximate representations for the flux density of created pairs (bosons and fermions) and the probability of the vacuum to remain a vacuum in the leading-term approximation for a weakly inhomogeneous, but otherwise arbitrary strong electric field. These representations do not require a knowledge of corresponding solutions of the Dirac equation; they have a form of simple functionals of a given weakly inhomogeneous electric field. The universal forms for the flux density of created pairs are completely new and present a LCFA for this physical quantity. It is a new kind of a LCFA obtained without any relation to the Heisenberg-Euler action. We established relations of these representations with leading term approximations of derivative expansion results. We have tested the obtained representations for cases of exactly solvable x-electric potential steps (based on using exact solutions). We have also considered two examples of x-electric potential steps where the exact solutions of the corresponding Dirac equation are not known, a Gauss peak and an inverse square electric field. We found the longitudinal current density and charge density of created electrons and positrons and related these densities to the corresponding flux density. In the regions S_{L} and S_{R}, where the electric field is absent (or negligible), leading vacuum characteristics are formed due to the real pair production. Thus, we have isolated global contributions that depend on the total history of an electric field from local contributions formed in the region S_{int}. The nonperturbative (with respect to the external field) technique elaborated on in Ref. [24] allows one to calculate all characteristics of zero-order processes and Feynman diagrams that describe all the characteristics of processes with an interaction between charged particles and photons. These diagrams formally have the usual form but contain special propagators. Using expressions for these propagators in terms of in- and out-solutions, presented in Ref. [24], our approximation method can be easily adapted to calculate one-loop and higher order contributions. The first step in developing the corresponding nonperturbative technique was recently done in Ref. [32], where a relation between the electron propagator in a constant electric field confined between two capacitor plates and the well-known Fock-Schwinger proper-time integral representation is established.

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