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MHD flow in a vertical channel under the effect of temperature dependent physical parameters

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ABSTRACT

Mixed convective flow in a vertical channel filled with electrically conducting viscous fluid with isothermal wall conditions is investigated for variable properties. The combined effects of temperature dependent viscosity and temperature dependent thermal conductivity are analyzed. The solutions are obtained both analytically by perturbation method and numerically by Runge–Kutta method with shooting technique. The dimensionless governing parameters affecting velocity and temperature fields are variable viscosity parameter ($-0.5 \le b_\nu \le 0.5$), variable thermal conductivity parameter ($-0.5 \le b_k \le 0.5$), Hartmann number ($1 \le M \le 3$), applied electric field parameter ($E_0 = \pm 1, 0$), wall temperature ratio parameter ($-2 \le m \le 2$) and buoyancy parameter ($0 < N \le 1.5$). For some limiting cases, the obtained results are validated by comparing with those available from the existing literature. Correlations for skin friction and Nusselt number in terms of governing parameters are developed.

1. Introduction

When a conductive fluid moves through a magnetic field, an ionized gas is electrically conductive, the fluid may be influenced by the magnetic field. The birth of magnetohydrodynamic (MHD) phenomenon may be identified with the first experiments by Faraday who attempted to measure the electric potential induced between the opposite banks of the Thames River by the motion of the (weakly) conducting water in the Earth's magnetic field [1]. The principle behind Faraday's (unsuccessful) experiment is the same which underlies modern MHD flow meters. About in the same period, Ritchie developed a rudimentary electro-magnetic pumping device, although the first working MHD pump was presented only much later [2]. MHD natural convection heat transfer flow is of considerable interest in the technical field due to its frequent occurrence in industrial technology and geothermal application, high-temperature plasmas applicable to nuclear fusion energy conversion, liquid metal fluids, and MHD power generation systems. Roming [3] studied the effect of electric and magnetic fields on the heat transfer of electrically conducting fluids. Hartmann [4] carried out the pioneer work on the study of steady MHD channel flow of a conducting fluid under a uniform magnetic field transverse to an electrically insulated channel wall. Later Osterle and Young [5] investigated the effect of viscous and Joule dissipations on hydromagnetic free convection flows and heat transfer between two vertical plates with transverse magnetic field under short circuit condition. Garandet et al. [6] analyzed the buoyancy driven convection in a rectangular enclosure with a transverse magnetic field. The analysis of MHD mixed convection interaction with thermal radiation and higher order chemical reaction is carried out by Makinde [7]. Mahian et al. [8] discussed entropy generation between two vertical cylinders in the presence of MHD flow subjected to constant wall temperature. Irreversibility analysis of a vertical annulus using TiO_2/water nanofluid with MHD flow

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Nomenclature

B_0	uniform magnetic field
b	width of the channel
$\tilde{b}$	empirical constant for the thermal conductivity
b_ν	viscosity variation parameter
b_k	conductivity variation parameter
E	electric field loading parameter
E_0	applied electric field
g	acceleration due to gravity
K	thermal conductivity of the fluid
K_0	thermal conductivity at temperature T_0
M	Hartmann number
m	wall temperature ratio
N	buoyancy parameter
Nu_1, Nu_2	dimensionless Nusselt numbers
p	pressure
P	dimensionless pressure
T_0	reference temperature
T	fluid temperature
T_1, T_2	wall temperatures
U	velocity
u	dimensionless velocity
Y	dimensional coordinate axis
y	dimensionless coordinate axis

Greek symbols

β	thermal expansion coefficient
μ	dynamic viscosity
μ_0	dynamic viscosity at temperature T_0
θ	dimensionless temperature
ρ	density of the fluid
ρ_0	static density
σ_e	electrical conductivity
τ_1, τ_2	dimensionless skin friction
ν_0	kinematic viscosity
ΔT	temperature difference

effects was also studied by Mahian et al. [9]. Aziz [10] theoretically examined a similarity solution for a laminar thermal boundary layer over a flat plate with a convective surface boundary condition. He found an interesting result that a similarity solution is possible if the convective heat transfer along with the hot fluid on the lower surface of the plate is inversely proportional to the square root of the axial distance. Öztop et al. [11] studied MHD natural convection in an enclosure from two semi-circular heaters on the bottom wall. Effects of moving lid direction on MHD mixed convection in a linearly heated cavity were analyzed by Al-Salem et al. [12]. The combined effects of an exponentially decaying internal heat generation and a convective boundary condition on the thermal boundary layer over a flat plate are investigated by Olanrewaju et al. [13]. Malashetty et al. [14] studied the magneto-hydrodynamic two fluid flow and heat transfer in an inclined channel. Prathap Kumar et al. [15] analyzed the MHD mixed convection flow of viscous fluid in a vertical channel. Umavathi et al. [16] and Umavathi and Liu [17] studied the laminar MHD convective flow in a vertical channel in the presence of heat generation and heat absorption. Recently Selimefendigil and Öztop [18] worked on the analysis of MHD mixed convection in a flexible walled and nanofluids filled lid-driven cavity with volumetric heat generation. The same authors Selimefendigil and Öztop [19] also analyzed MHD mixed convection and entropy generation of power law fluids in a cavity with a partial heater under the effect of a rotating cylinder. Soid et al. studied numerically the problem of unsteady MHD stagnation point flow over a stretching/shrinking sheet in a viscous fluid with viscous dissipation and ohmic heating [20] and steady MHD flow past a radially stretching or shrinking disk [21]. Analysis was performed using the similarity technique and Matlab software. Impact of magnetic field was examined in detail.

The use of electrically conducting fluids under the influence of magnetic fields in various industries has lead to the renewed interest in investigating flow structures and heat transfer in different geometries. For example, Sparrow and Cess [22] and Umavathi [23] studied MHD convective heat transfer in vertical channel in the presence of electric field. Umavathi et al. [24] numerically studied fully developed magnetoconvection flow in a vertical rectangular duct.

In recent years, much attention has been devoted to the study of MHD effects on natural and mixed convection flows [25–35]. Indeed, convective flows in presence of magnetic fields occur in many technical applications, for instance, the optimization of industrial casting of metals [33]. In particular, an analytical solution for the natural convection in a two-dimensional rectangular cavity in the presence of a vertical magnetic field has been determined by Garandet et al. [25]. Pan and Li [26] has studied the mixed convection in a vertical plane channel with a horizontal magnetic field, in conditions of micro gravity with a gravitational acceleration that oscillates in time with a sinusoidal law (g-jitter effect). The mixed convection flow in a horizontal circular duct in the presence of a uniform vertical magnetic field has been studied numerically in [27]. An experimental study of the natural convection of $Na^{22}K^{78}$ alloy in a cavity with a rectangular section under the vertical magnetic field effect, has been presented by Burr and Muller [28]. These authors have shown that the magnetic field produces a symmetric reduction of heat fluxes in the fluid. Bondareva and Sheremet [29,30] have numerically analyzed the effect of uniform magnetic field on natural convection melting in a square [29] and cubical [30] cavities with a local heater. It has been found that uniform magnetic field allows to essentially homogenize the melting inside the cavity, while the melt rate changes insignificantly. Mixed convection in a vertical channel under the effects of viscous dissipation and Joule heating has been studied by Umavathi and Malashetty [31]. They determined the velocity and temperature distributions both analytically, by means of a perturbation expansion and numerically, by the finite difference method. Sposito and Ciofalo [32] have obtained analytical solutions of the local balance equations for fully developed mixed convection in a vertical plane channel, by considering isothermal walls and several electric boundary conditions. Sheikholeslami and Rokni [34] have examined MHD nanofluid flow and heat transfer with melting thermal transmission using the Buongiorno's two-component model. Effects of governing parameters on the average Nusselt number have been described. Sheikholeslami and Oztop [35] have investigated an influence of external magnetic source on ferrofluid convection in a cavity with sinusoidal outer cylinder. It has been shown that an impact of adding nanoparticles is more effective in high Hartmann number.

All the above mentioned studies continued their discussion by assuming the uniform fluid viscosity and thermal conductivity. However, it is known that the physical properties of fluid may be changed significantly with temperature [36–42]. The variations of properties with temperature have several practical applications in the field of metallurgy and chemical engineering. The temperature increase leads to a local growth of the transport phenomena by reducing the viscosity across the momentum boundary layer and the heat transfer rate at the wall is also affected. Therefore, to predict the flow behavior accurately it is necessary to take into account the viscosity variation for incompressible fluids. Gary et al. [43] and Mehta and Sood [44] showed that, when this effect is included the flow characteristic may be changed substantially compared to constant viscosity assumption. For lubricating fluids the heat generated by internal friction and the corresponding rise in the temperature affect the viscosity of the fluid and so the fluid viscosity is no longer to be constant.

Liquid metals have high thermal conductivity and they are used as coolants, moreover they have high electrically conductivity hence they are susceptible to transverse magnetic field. The Prandtl number of liquid metals is low and generally it is of order 0.01–0.1, e.g. Bismuth ($Pr = 0.011$), Mercury ($Pr = 0.023$), PbBi ($Pr = 0.18$) etc. Kay [45] reported that thermal conductivity of liquids with low Prandtl number varies linearly with temperature in range of 0°F–400°F. Arunachalam and Rajappa [46] considered forced convection in liquid metals (fluids with low Prandtl numbers) with variable thermal conductivity and heat capacity in potential flow and they derived explicit closed form of analytical solution. Chaim [47] studied the heat transfer of low Prandtl number fluid with variable thermal conductivity, induced due to stretching sheet and he compared the numerical results with the results obtained by perturbation technique. Van den Berg et al. [48] and Van den Berg and Yuen [49] showed that variable thermal conductivity can delay the secular cooling of the mantle with a constant viscosity model. El-Aziz [50] analyzed the effect of temperature dependent viscosity and thermal conductivity on MHD flow over stretching sheet in the presence of Ohmic heating. Elgazery and Elazem [51] studied the effect of variable viscosity and thermal conductivity on the heat and mass transfer in MHD natural convective flow. Sharma and Singh [52] obtained the similarity solution of natural convective flow taking into account temperature dependent viscosity and thermal conductivity in the presence of transverse magnetic field and exponentially decaying heat source.

Aim of the present paper is to investigate the combined effects of exponential varying viscosity and thermal conductivity on the mixed convection flow of viscous incompressible electrically conducting fluid in a vertical channel in the presence of viscous dissipation, Ohmic heating, transverse magnetic field and applied electric field. The non-dimensional governing equations are solved analytically using the perturbation method and numerically by Runge–Kutta method with shooting technique. The combined effect of viscosity and thermal conductivity variation on the flow variables such as velocity, temperature, skin friction, and Nusselt number is studied. The solutions obtained by the Runge–Kutta method with shooting technique are justified by comparing with solutions obtained by perturbation method.

2. Mathematical formulation

Fully developed flow of Newtonian fluid in a vertical channel is studied (see Fig. 1). The vertical channel walls are kept at constant temperatures T_1 and T_2. The domain of interest is defined by Y-coordinate, where $-b \le Y \le b$. It should be noted that fluid flow is defined by the effect of buoyancy force and electromagnetic force. Along the rigid walls one can find a zero velocity and these walls are infinite in the X-direction. The latter characterizes that the problem is essentially one-dimensional. The fluid is heat-

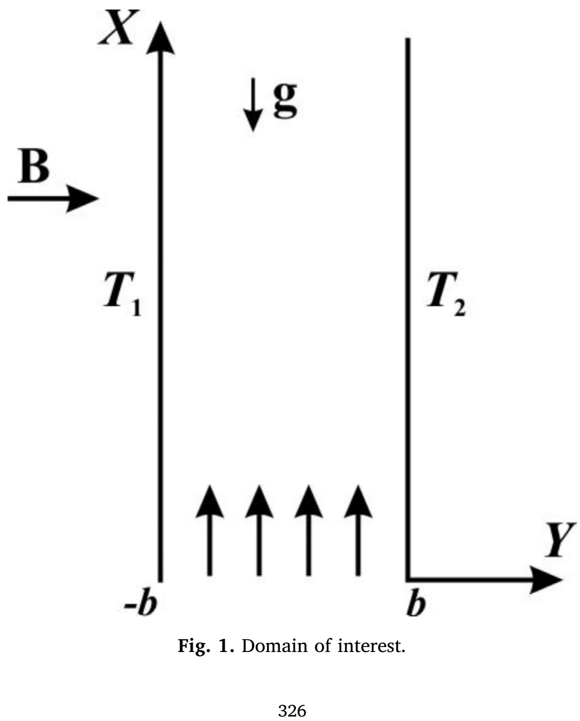

Fig. 1. Domain of interest.

conducting, viscous and the Boussinesq approximation is valid. The fluid flow and heat transfer are described taking into account the temperature dependent viscosity and thermal conductivity.

The governing equations in the presence of viscous dissipation, Ohmic heating, transverse magnetic field and applied electric field in the case of variable physical properties can be defined as follows

$$\frac{d}{dY}\left(\mu\frac{dU}{dY}\right)+\rho_0 g\beta(T-T_0)-\sigma_e B_0 E_0-\sigma_e B_0^2 U-\frac{\partial p}{\partial X}=0 \tag{1}$$

$$K_0\frac{d^2T}{dY^2}+\mu\left(\frac{dU}{dY}\right)^2+\sigma_e(E_0+B_0U)^2=0 \tag{2}$$

The boundary conditions for the velocity and temperature fields are given as

$$U=0 \text{ at } Y=\pm b \tag{3}$$

$$T=T_1 \text{ at } Y=-b,\quad T=T_2 \text{ at } Y=b \tag{4}$$

The fluid viscosity μ is considered as follows [53,54]

$$\mu=\mu_0 e^{-a(T-T_0)} \tag{5}$$

where the subscript “0” denotes the reference state and a is an empirical constant.

The thermal conductivity of the fluid is assumed as [53,54]

$$K=K_0 e^{-\tilde{b}(T-T_0)}=K_0(1+\tilde{b}(T_0-T)) \tag{6}$$

Eqs. (1) and (2) can be written in a non-dimensional form using the following parameters

$$\begin{aligned} u &= \frac{\mu_0}{\rho_0 g\beta b^2\Delta T}U,\ \theta=\frac{T-T_0}{\Delta T},\ y=\frac{Y}{b},\ N=\frac{{\rho_0}^2 b^4 g^2\beta^2\Delta T}{\mu_0 K_0},\\ M^2 &= \frac{\sigma_e B_0^2 b^2}{\mu_0},\ E=\frac{\nu_0 E_0}{b^2 B_0 g\beta\Delta T},\ P=\frac{b^2}{\mu_0\bar{u}}\frac{\partial p}{\partial X} \end{aligned} \tag{7}$$

Using temperature dependences for viscosity (5) and thermal conductivity (6) the governing Eqs. (1)–(4) can be reduced to

$$\frac{d^2u}{dy^2}-b_v\frac{d\theta}{dy}\frac{du}{dy}+\theta+b_v\theta^2-M^2(E+U)-(1+b_v\theta)P=0 \tag{8}$$

$$\frac{d^2\theta}{dy^2}-b_k\left(\frac{d\theta}{dy}\right)^2+N\left(\frac{du}{dy}\right)^2+(b_k-b_v)N\theta\left(\frac{du}{dy}\right)^2-b_v b_k N\theta^2\left(\frac{du}{dy}\right)^2+NM^2(1-b_v\theta)(1+b_k\theta)(E+u)^2=0 \tag{9}$$

with corresponding boundary conditions

$$u=0 \text{ at } y=\pm 1 \tag{10}$$

$$\theta=1+m \text{ at } y=-1,\quad \theta=1 \text{ at } y=1 \tag{11}$$

where $\Delta T=T_2-T_0,\ b_v=a\Delta T$ is the variable viscosity parameter, $m=\frac{T_1-T_2}{\Delta T}$ is the wall temperature ratio and N is the buoyancy parameter.

3. Solutions

The solutions of the governing equations of motion are found using perturbation method and numerical method based on the Runge–Kutta algorithm with shooting technique.

3.1. Perturbation method

Eqs. (8) and (9) are coupled nonlinear equations due to variable physical parameters, viscous dissipation, Ohmic heating, transverse magnetic field and applied electric field and it is difficult, in general, to solve analytically. When neglecting the viscous dissipative heating ($N=0$), Eqs. (8) and (9) become linear and solutions can be easily obtained. In many practical applications cited above, N can not be zero ($N\neq 0$), but in many situations it can take small values. For example, for mercury in a channel of half-width 2 cm, and with $T_1-T_0=20$°C, N takes the value of 0.128. Small values of $N(<1)$ facilitate the finding analytical solutions of Eqs. (9) and (10) in the form

$$u=u_0+Nu_1+\ldots\ldots \tag{12}$$

$$\theta=\theta_0+N\theta_1+\ldots\ldots \tag{13}$$

where the second and higher order terms on the right-hand side give a correction to u_0, θ_0 accounting for the dissipative effects. Substituting Eqs. (12) and (13) into Eqs. (8) and (9), and equating like powers of N to zero, we obtain the following equations.

Zeroth order equations are

$$\frac{d^2u_0}{dy^2} - b_v\frac{d\theta_0}{dy}\frac{du_0}{dy} + \theta_0 + b_v{\theta_0}^2 - M^2(E + u_0) - (1 + b_v\theta_0)P = 0 \tag{14}$$

$$\frac{d^2\theta_0}{dy^2} - b_k\left(\frac{d\theta_0}{dy}\right)^2 = 0 \tag{15}$$

with corresponding boundary conditions

$$u_0 = 0 \text{ at } y = \pm 1 \tag{16}$$

$$\theta_0 = 1 + m \text{ at } y = -1,\ \ \theta_0 = 1 \ \text{ at } y = 1 \tag{17}$$

First order equations are

$$\frac{d^2u_1}{dy^2} - b_v\frac{d\theta_0}{dy}\frac{du_1}{dy} - b_v\frac{d\theta_1}{dy}\frac{du_0}{dy} + \theta_1 + 2b_v\theta_0\theta_1 - M^2u_1 - b_v\theta_1P = 0 \tag{18}$$

$$\frac{d^2\theta_1}{dy^2} - 2b_k\frac{d\theta_0}{dy}\frac{d\theta_1}{dy} + \left(\frac{du_0}{dy}\right)^2 + (b_k - b_v)\theta_0\left(\frac{du_0}{dy}\right)^2 - b_kb_v\theta_0^2\left(\frac{du_0}{dy}\right)^2 + M^2(1 - b_v\theta_0)(1 + b_k\theta_0)(E + u_0)^2 = 0 \tag{19}$$

with corresponding boundary conditions

$$u_1 = 0 \text{ at } y = \pm 1 \tag{20}$$

$$\theta_1 = 0 \text{ at } y = \pm 1 \tag{21}$$

We shall further perform a perturbation analysis of the Eqs. (14), (15), (18) and (19) considering variable conductivity parameter as a perturbation parameter. The solutions of Eqs. (14), (15), (18) and (19) are assumed as follows

$$u_0 = u_{00} + b_ku_{01} \tag{22}$$

$$\theta_0 = \theta_{00} + b_k\theta_{01} \tag{23}$$

$$u_1 = u_{10} + b_ku_{11} \tag{24}$$

$$\theta_1 = \theta_{10} + b_k\theta_{11} \tag{25}$$

Substituting Eqs. (22) and (23) into Eqs. (14) and (15) and considering powers of b_k, we obtain the following boundary value problems.

Zeroth order equations are

$$\frac{d^2u_{00}}{dy^2} - b_v\frac{d\theta_{00}}{dy}\frac{du_{00}}{dy} + \theta_{00} + b_v\theta_{00}^2 - M^2(E + u_{00}) - (1 + b_v\theta_{00})P = 0 \tag{26}$$

$$\frac{d^2\theta_{00}}{dy^2} = 0 \tag{27}$$

Substituting Eqs. (24) and (25) into Eqs. (18) and (19) and considering powers of b_k, we obtain the following boundary value problems.

First order equations are

$$\frac{d^2u_{01}}{dy^2} - b_v\frac{d\theta_{00}}{dy}\frac{du_{01}}{dy} - b_v\frac{d\theta_{01}}{dy}\frac{du_{00}}{dy} + \theta_{01} + 2b_v\theta_{00}\theta_{01} - M^2u_{01} - b_v\theta_{01}P = 0 \tag{28}$$

$$\frac{d^2\theta_{01}}{dy^2} - \left(\frac{d\theta_{00}}{dy}\right)^2 = 0 \tag{29}$$

with corresponding boundary conditions

$$\begin{aligned}
&u_{00} = 0 \text{ at } y = \pm 1,\\
&\theta_{00} = 1 + m \text{ at } y = -1,\ \theta_{00} = 1 \ \text{ at } y = 1,\\
&u_{01} = 0 \text{ at } y = \pm 1,\ \theta_{01} = 0 \text{ at } y = \pm 1,\\
&u_{10} = 0 \text{ at } y = \pm 1,\ \theta_{10} = 0 \text{ at } y = \pm 1,\\
&u_{11} = 0 \text{ at } y = \pm 1,\ \theta_{11} = 0 \text{ at } y = \pm 1
\end{aligned} \tag{30}$$

The solutions of Eqs. (26)–(29) can be obtained easily and hence they are not represented here. The results are presented in graphs and tabular form.

3.2. Numerical solutions

The analytical solutions obtained in the above section are valid for small values of perturbation parameters. Further it is seen in the above section that it is not possible to find solutions of even the first order. Hence we used Runge–Kutta method with shooting technique (RKSM) for numerical solution to the formulated problem. The validity of RKSM is justified by comparing the solutions with the results obtained by the perturbation method and the values are displayed in Tables. The perturbation method and RKSM solutions agree very well in the absence of perturbation parameter.

Also the used RKSM has been validated using the data of Crepeau and Clarksean [55] presented in Table 1.

3.3. Skin friction and Nusselt number

During the present problem we also defined the following parameters: the dimensionless skin friction

$$\tau_1 = e^{-b_\nu(1+m)}\left(\frac{du}{dy}\right)_{y=-1} \quad \text{and} \quad \tau_2 = e^{-b_\nu}\left(\frac{du}{dy}\right)_{y=1} \tag{31}$$

and Nusselt numbers

$$Nu_1 = -(1 + b_k(1+m))\left(\frac{d\theta}{dy}\right)_{y=-1} \quad \text{and} \quad Nu_2 = -(1 + b_k)\left(\frac{d\theta}{dy}\right)_{y=+1} \tag{32}$$

4. Results and discussion

To the study the behavior of solutions, numerical calculations for several values of variable viscosity parameter b_ν, thermal conductivity parameter b_k, Hartmann number M, wall temperature ratio m and buoyancy parameter N for open ($E = \pm 1$) and short ($E = 0$) circuits have been carried out and displayed graphically. Figs. 2–11 show the effect of these governing parameters on fluid flow and heat transfer.

The effects of variable viscosity parameter b_ν on the velocity and temperature profiles for open and short circuits can be found in Figs. 2 and 3. An increase in b_ν leads to a growth of the temperature profiles for open and short circuit. While in the case of velocity profiles, it is possible to conclude that for $E = 1$ one can find a formation of descending flow where an increase in b_ν leads to a reduction of the absolute velocity value. At the same time one can find an asymmetrical distribution of velocity inside the channel due to a presence of variable physical properties. Further one can also observe that for short circuit $E = 0$ and $E = -1$, as b_ν increases, the velocity increases and the profiles for constant viscosity ($b_\nu = 0$) lies above $b_\nu < 0$ and below $b_\nu > 0$. For open and short circuit, the effect of variable viscosity parameter b_ν on the velocity was the similar to results observed by Attia [56] for MHD channel flow of dusty fluids. For constant viscosity ($b_\nu = 0$), the results agree well with data of Umavathi and Malashetty [31].

The effects of variable conductivity parameter b_k on the velocity and temperature fields for open and short circuits are observed in Figs. 4 and 5. The effect of thermal conductivity parameter b_k shows the opposite results on the velocity and temperature fields when compared with variable viscosity parameter (Figs. 2 and 3). That is to say that an increase in the thermal conductivity parameter leads to a reduction of temperature. While velocity decreases for $E = -1$ and 0 and it increases for $E = 1$, where a descending flow formed. For open and short circuits, the effect of b_k is similar to results obtained by Attia [57] and Palani and Kim [58]. For constant thermal conductivity ($b_k = 0$), the results agree well with data of Umavathi and Malashetty [31].

Figs. 6 and 7 show the effect of Hartmann number M. It is seen that an increase in M leads to an attenuation of the flow for short circuit ($E = 0$), whereas for open circuits ($E = \pm 1$) one can find an acceleration of the fluid flow. The nature of velocity profiles for $E = -$ and for $E = 1$ is in the opposite directions exhibiting the tunneling effect, namely, boundary layer nature near the boundaries and a flattening nature in the middle of the channel. This is a classical Hartmann result. This is also a similar result observed by Umavathi and Malashetty [31] for constant properties. The effect of Hartmann number M on temperature field is not very effective. However one can see from Fig. 7 that as M increases the temperature decreases for both open and short circuits, the magnitude of temperature is low for $E = 0$ and high for $E = 1$.

Figs. 8 and 9 illustrate the plots of u and θ for variations of wall temperature ratio m. As the wall temperature ratio m increases the velocity and temperature are enhanced for both open and short circuits. It is an expected result because the boundary conditions for temperature are defined as $\theta = 1 + m$ at the left wall and $\theta = 1$ at the right wall. For values of $m < 0$ the temperature of the left wall is less then the temperature of the right wall, for $m = 0$ both walls are maintained at equal temperatures, and for values of $m > 0$ the left wall is at higher temperature. Therefore as m increases, rate of heat transfer also increases which in turn results in the

Table 1
Comparison of $d\theta/d\eta|_{\eta=0}$ for the similarity equations for a constant temperature plate.

Pr	0.1	1.0	10.0
Data of Crepeau and Clarksean [55]	−0.2302	−0.5671	−1.169
Obtained data	−0.230142	−0.567147	−1.16933

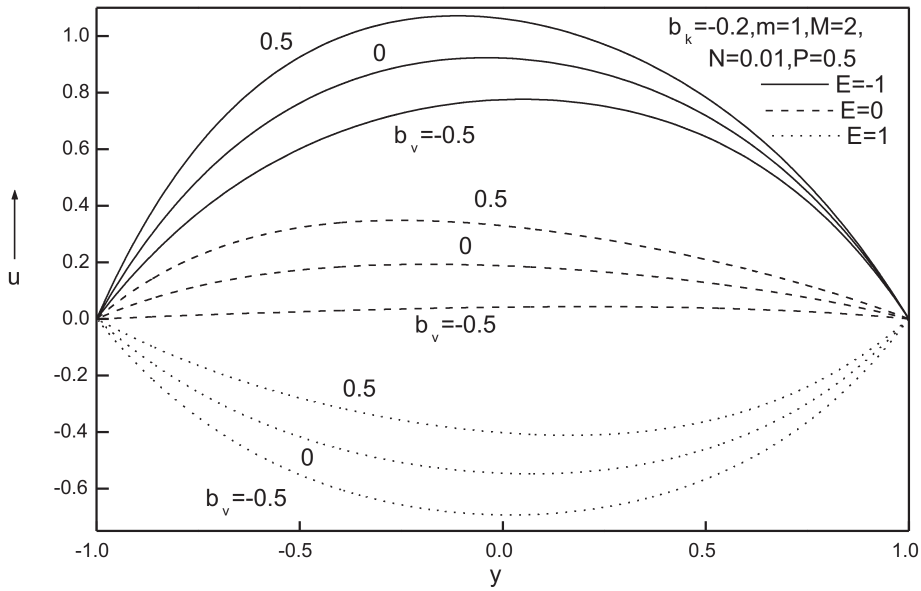

Fig. 2. Velocity profile of combined effect for different values of b_v.

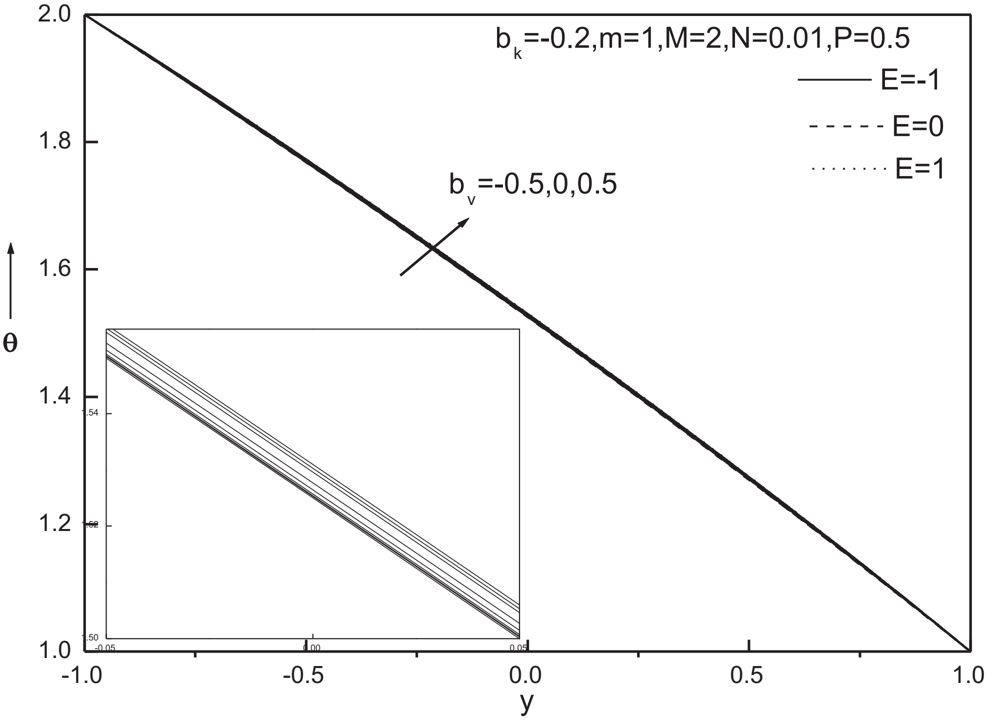

Fig. 3. Temperature profile of combined effect for different values of b_v.

enhancement of velocity profiles.

Figs. 10 and 11 show the fluid flow intensification and heat transfer enhancement with buoyancy parameter N for both open and short circuits. From Fig. 10, it is seen that the direction of velocity for $E > 0$ is opposite to that for $E < 0$ and hence present results can be used effectively for the flow reversal situation required in many practical problems. Fig. 11 display that as the buoyancy parameter N increases, temperature increases for both open ($E = \pm 1$) and short circuits ($E = 0$). The increase in temperature is higher for $E = 1$. The effect of N is to enhance the velocity and temperature fields. Physically, increase in the buoyancy parameter implies that N acts as the driving force in the momentum equation, the velocity and/or velocity gradient increases and therefore the effect of dissipation increases, which results in the enhancement of temperature field also. The effect of N on the flow for variable properties is

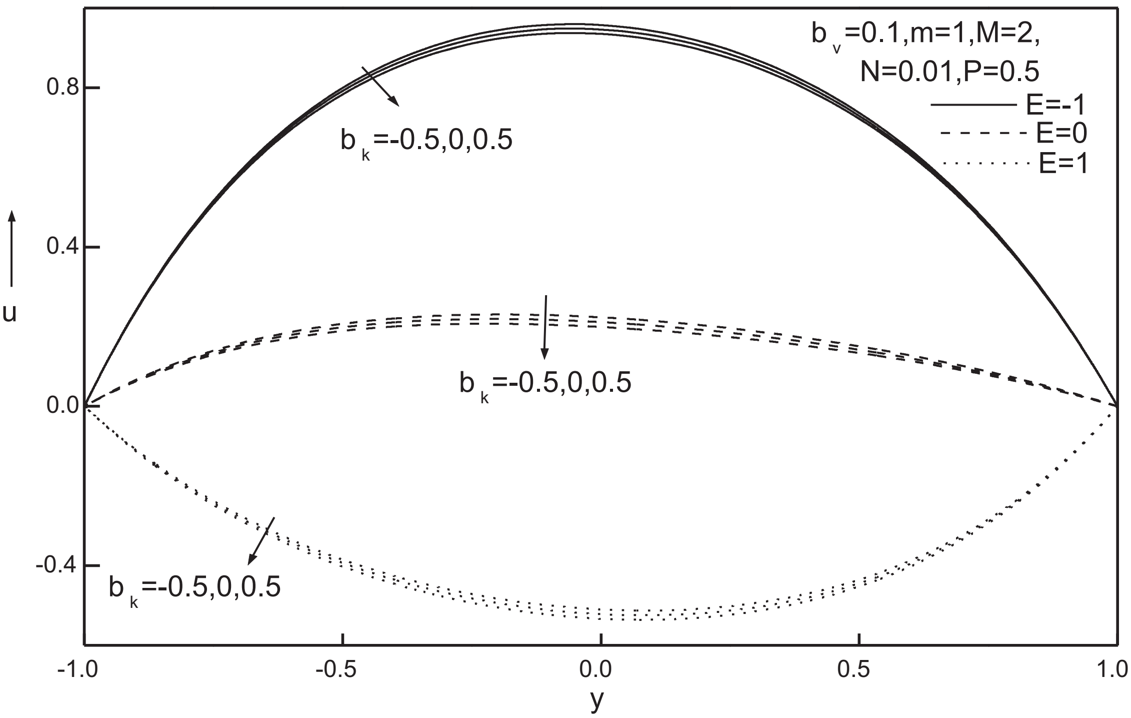

Fig. 4. Velocity profile of combined effect for different values of b_k.

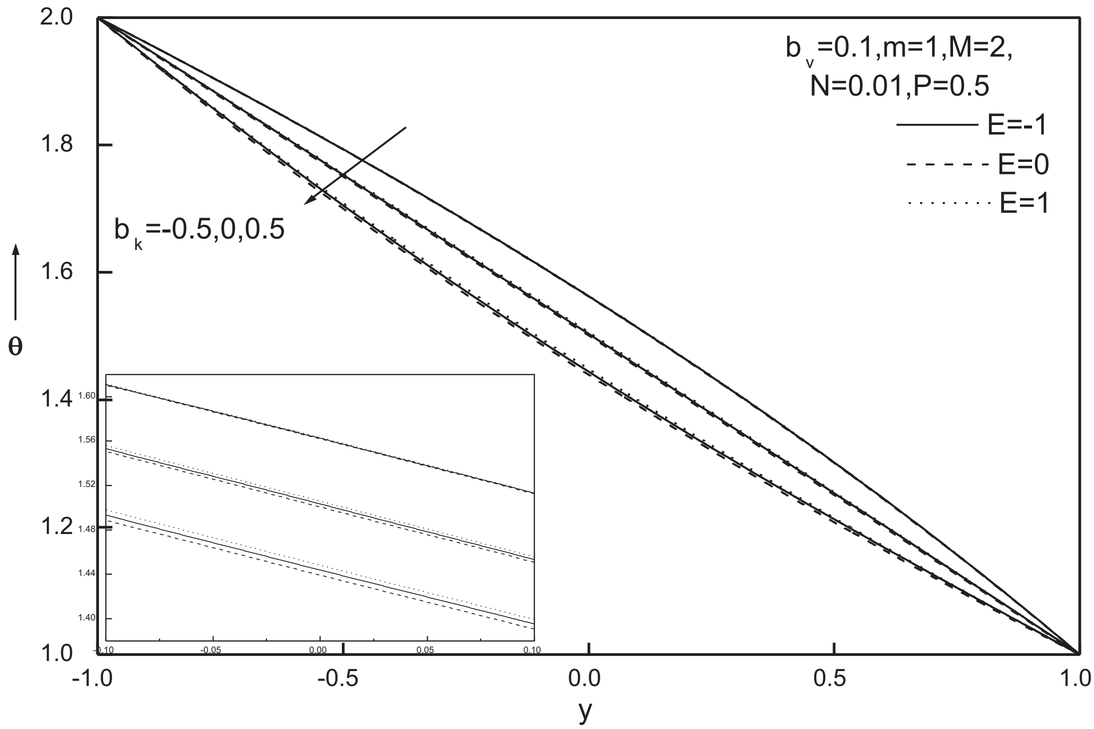

Fig. 5. Temperature profile of combined effect for different values of b_k.

the similar result observed by Umavathi and Malashetty [31] for constant properties.

Table 2 shows the values of skin friction and Nusselt number for different values of variable viscosity parameter b_ν, variable thermal conductivity parameter b_k, Hartmann number M, wall temperature ratio m and buoyancy parameter N for open and short

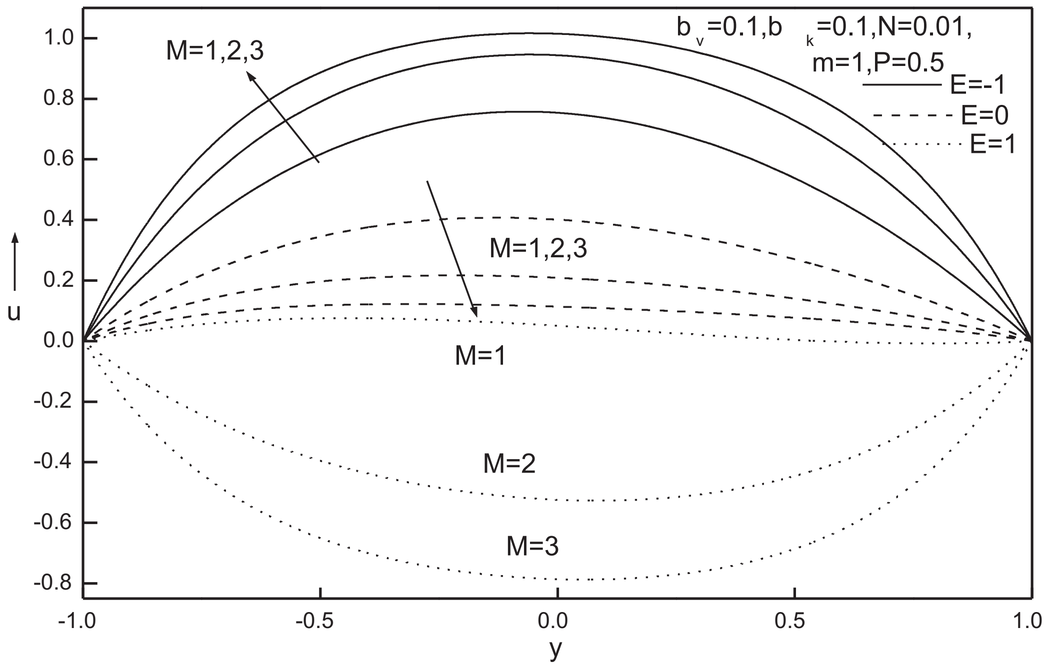

Fig. 6. Velocity profile of combined effect for different values of M.

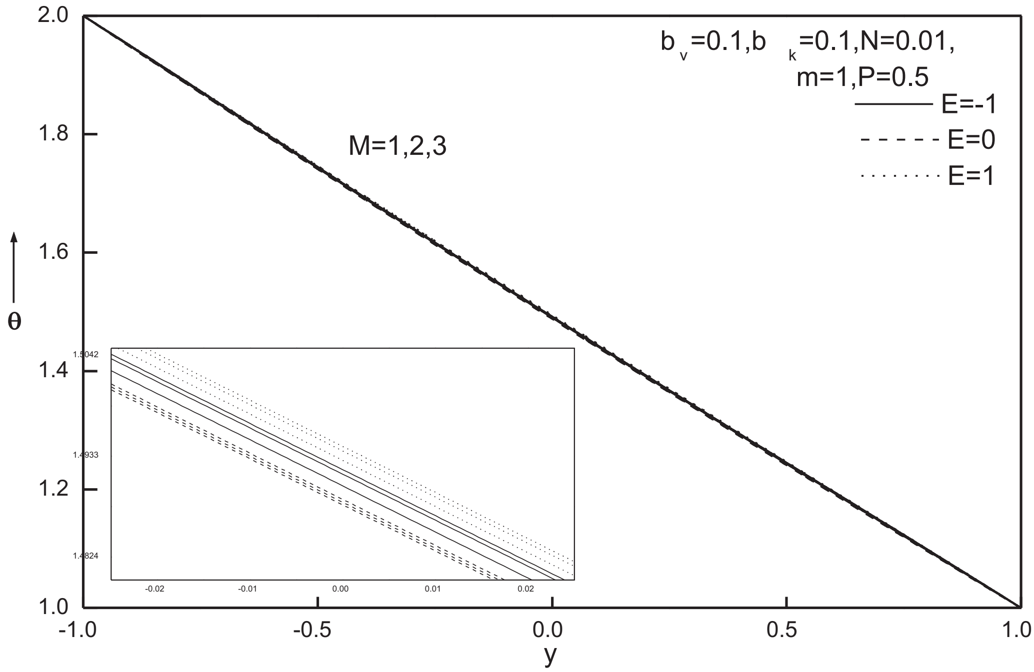

Fig. 7. Temperature profile of combined effect for different values of M.

circuits. One can find that as b_ν increases the absolute value of skin friction decreases at both walls for open circuits, while for short circuits the maximum absolute values of τ_1 and τ_2 are at $b_\nu = 0$. At the same time, Nusselt numbers increase at the left wall and decrease at the right wall for open circuit whereas for short circuit the effect is nonlinear. Fixing $b_\nu = 0.1$, a growth of b_k leads to a decrease in the absolute values of skin friction at both walls for $E = -1$ and 0, while the absolute values of skin friction increase with b_k for $E = 1$. At the same time Nusselt numbers at both walls are increasing functions of b_k for open and short circuits. As the Hartmann number M increases the absolute values of skin friction increase for $E = -1$ and 1 and these values decrease for $E = 0$. While the Nusselt number decreases at the left wall and increases at the right wall with Hartmann number for $E = -1$ and 1 and for

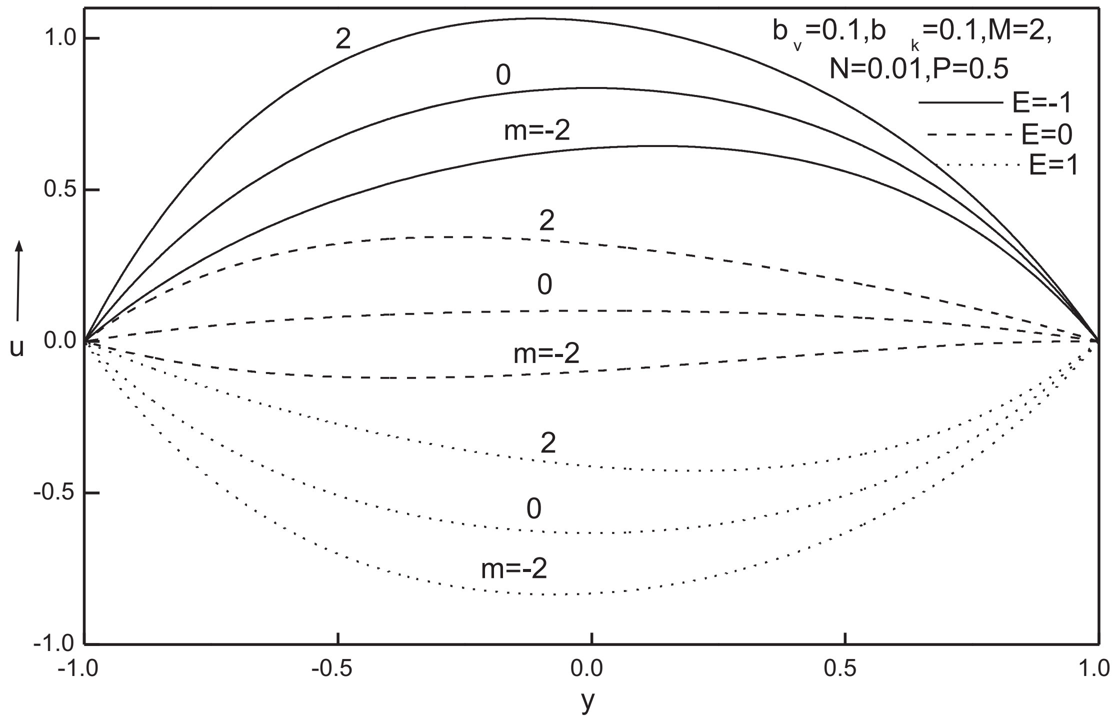

Fig. 8. Velocity profile of combined effect for different values of *m*.

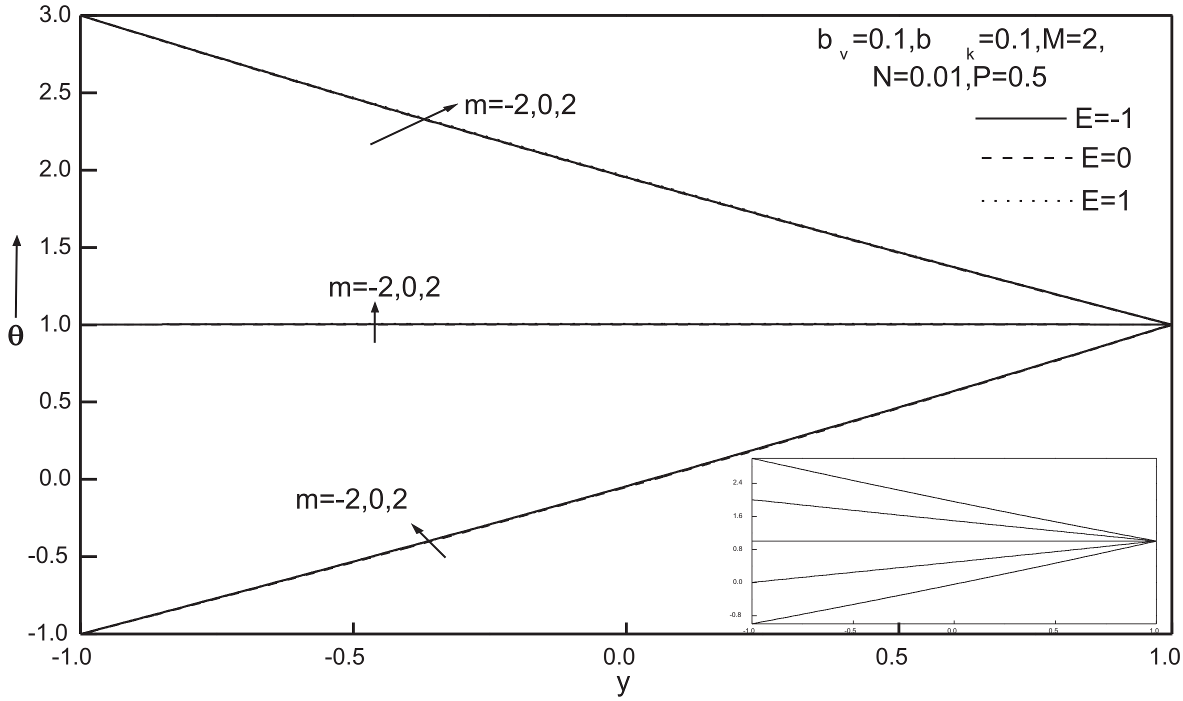

Fig. 9. Temperature profile of combined effect for different values of *m*.

$E = 0$ the behavior is opposite. When the wall temperature ratio m rises, the absolute values of skin friction increase at the left and right walls for $E = -1$ and 0 and these values decrease at both walls for $E = 1$. At the same time the Nusselt number increases at both walls for open and short circuits. A growth of the buoyancy parameter N increases the skin friction at the left wall and decreases it at the right wall for both open and short circuits. Nusselt number is decreased at the left wall and it is increased at the right wall as

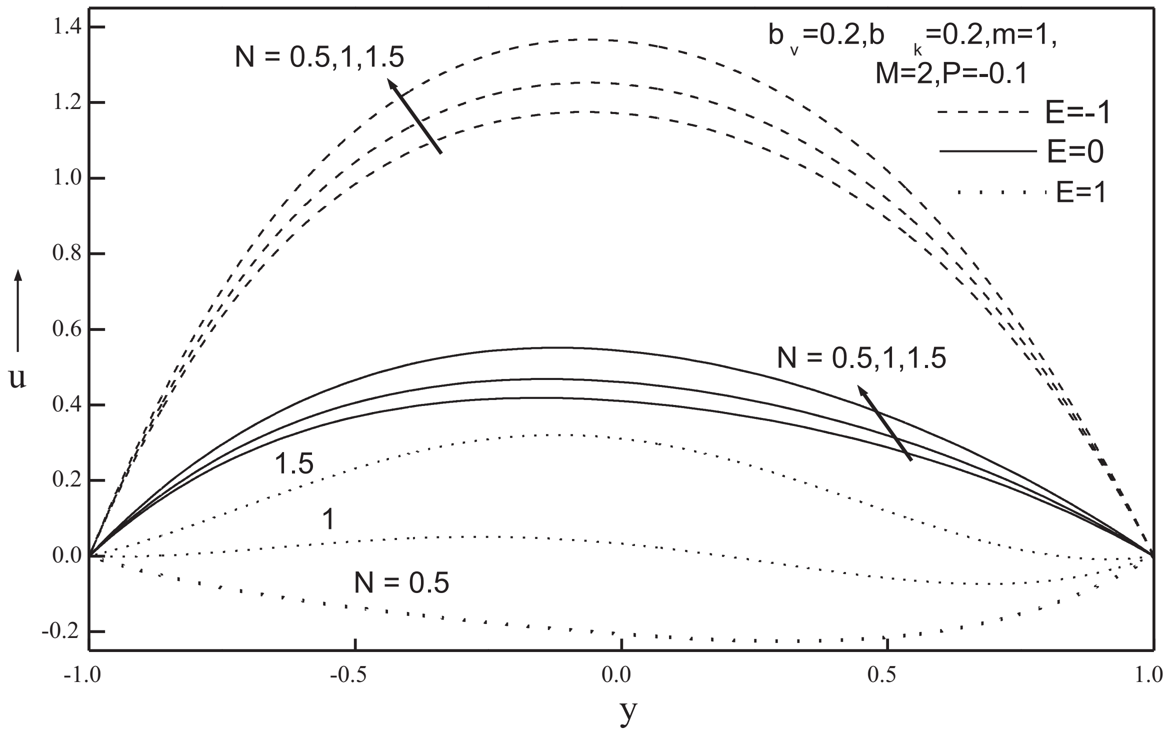

Fig. 10. Velocity profile of combined effect for different values of *N*.

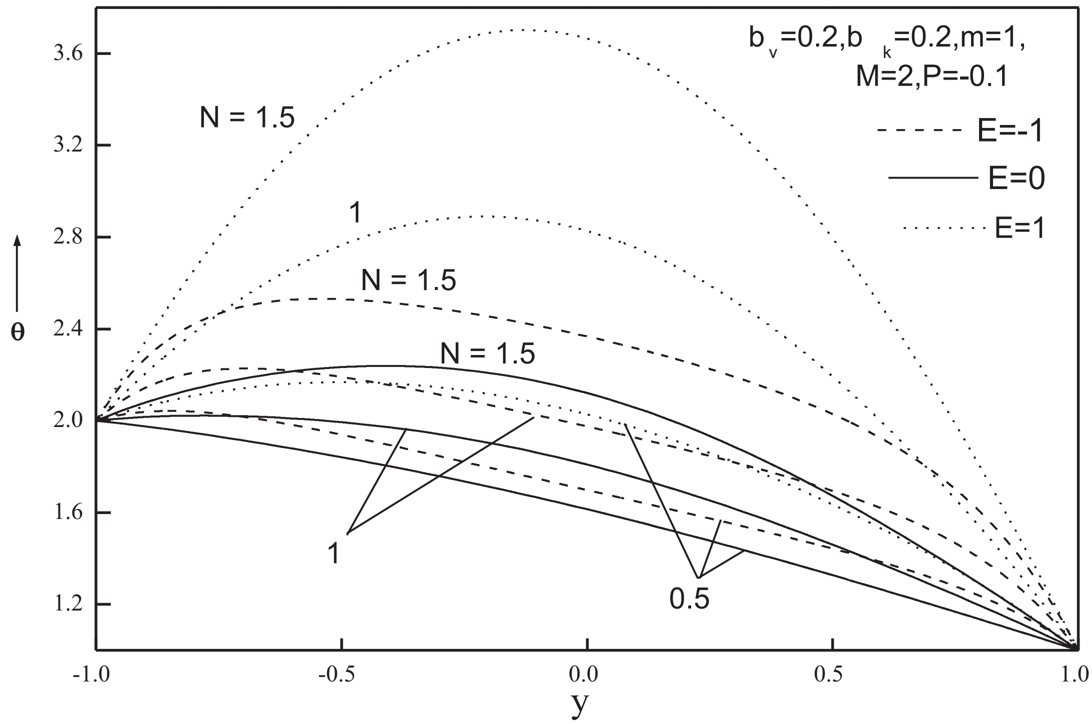

Fig. 11. Temperature profile of combined effect for different values of *N*.

buoyancy parameter *N* increases for both open and short circuits.

The analytical solutions obtained by regular perturbation method are valid only for small values of perturbation parameter. To overcome this restriction, the governing equations have been solved numerically using the Runge–Kutta method with shooting

Table 2
Computations showing the effect of governing parameters on skin frictions and Nusselt numbers.

	τ_1	τ_2	Nu_1	Nu_2
b_ν	$b_k = -0.2,\ m = 1, M = 2, N = 0.01, P = 0.5, E = -1$			
−0.5	5.12485	−3.59855	0.258501	0.46259
0	2.55161	−2.28339	0.264175	0.455602
0.5	1.20202	−1.42732	0.271253	0.448532
b_ν	$b_k = -0.2,\ m = 1, M = 2, N = 0.01, P = 0.5, E = 0$			
−0.5	0.140698	−0.219221	0.271869	0.442887
0	0.622773	−0.354437	0.271307	0.443532
0.5	0.456137	−0.326862	0.271655	0.443431
b_ν	$b_k = -0.2,\ m = 1, M = 2, N = 0.01, P = 0.5, E = 1$			
−0.5	−4.84775	3.160200	0.258415	0.462437
0	−1.30410	1.572330	0.264277	0.455822
0.5	−0.28940	0.772945	0.270752	0.448054
b_k	$b_\nu = 0.1,\ m = 1, M = 2, N = 0.01, P = 0.5, E = -1$			
−0.5	2.20944	−2.08838	0.000000	0.328313
0	2.19844	−2.07598	0.482244	0.518034
0.5	2.18629	−2.06468	1.223810	0.629572
b_k	$b_\nu = 0.1,\ m = 1, M = 2, N = 0.01, P = 0.5, E = 0$			
−0.5	0.616629	−0.367296	0.0000	0.3246
0	0.601713	−0.349951	0.498247	0.501263
0.5	0.584880	−0.33453	1.29057	0.592599
b_k	$b_\nu = 0.1,\ m = 1, M = 2, N = 0.01, P = 0.5, E = 1$			
−0.5	−0.975782	1.35300	0.000000	0.328068
0	−0.992439	1.37335	0.482611	0.518427
0.5	−1.011700	1.39125	1.223650	0.629914
M	$b_\nu = 0.1, b_k = 0.1,\ m = 1, N = 0.01, P = 0.5, E = -1$			
1	1.52272	−1.28547	0.617333	0.534105
2	2.19609	−2.07363	0.605427	0.545157
3	2.89706	−2.89909	0.594193	0.555951
M	$b_\nu = 0.1, b_k = 0.1,\ m = 1, N = 0.01, P = 0.5, E = 0$			
1	0.888787	−0.605220	0.626599	0.526202
2	0.598493	−0.346706	0.628535	0.524884
3	0.431879	−0.218634	0.629551	0.524230
M	$b_\nu = 0.1, b_k = 0.1,\ m = 1, N = 0.01, P = 0.5, E = 1$			
1	0.257739	0.0720698	0.617582	0.534361
2	−0.99609	1.3771400	0.605921	0.545614
3	−2.03108	2.4595600	0.594670	0.556378
m	$b_\nu = 0.1, b_k = 0.1,\ N = 0.01, M = 2, P = 0.5, E = -1$			
−2	1.56060	−1.80565	−0.8325220	−1.194770
0	1.98544	−1.98544	−0.0214239	0.0214239

(*continued on next page*)

Table 2 (*continued*)

	τ_1	τ_2	Nu_1	Nu_2
2	2.39597	−2.15978	1.40730000	1.0198100
m	$b_v = 0.1,\ b_k = 0.1,\ N = 0.01,\ M = 2,\ P = 0.5,\ E = 0$			
−2	−0.528263	−0.0175782	−0.8163050	−1.2173600
0	0.2399270	−0.2399270	−0.0004268	0.0004268
2	0.9324300	−0.4498430	1.43210000	1.0001800
m	$b_v = 0.1,\ b_k = 0.1,\ N = 0.01,\ M = 2,\ P = 0.5,\ E = 1$			
−2	−2.614290	1.76763	−0.8332680	−1.195920
0	−1.502550	1.50255	−0.0214246	0.0214246
2	−0.528163	1.25700	1.40810000	1.0206000
N	$b_v = -0.1,\ b_k = 0.2,\ m = 1,\ M = 2,\ P = 0.5,\ E = -1$			
0.1	2.96011	−2.51178	0.294366	0.864145
0.5	3.07973	−2.61609	−1.69180	2.188390
1	3.24026	−2.75574	−4.34601	3.960940
N	$b_v = -0.1,\ b_k = 0.2,\ m = 1,\ M = 2,\ P = 0.5,\ E = 0$			
0.1	0.606163	−0.335355	0.750355	0.556560
0.5	0.613481	−0.341525	0.647516	0.610361
1	0.623583	−0.350078	0.507077	0.684830
N	$b_v = -0.1,\ b_k = 0.2,\ m = 1,\ M = 2,\ P = 0.5,\ E = 1$			
0.1	−1.70534	1.80222	0.300400	0.870547
0.5	−1.63005	1.72840	−1.69499	2.242880
1	−1.51543	1.61259	−4.51823	4.184910

technique. The validity of the Runge–Kutta shooting method is justified by Table 1 and comparing with the results obtained by perturbation method in the absence of variable thermal conductivity parameter b_k and displayed in Table 3. It is viewed from Table 3 that the analytical and numerical solutions are exact for $b_k = 0$ and the error increases as variable thermal conductivity parameter b_k increases.

5. Conclusions

The problem of mixed convective flow of an electrically conducting fluid in a vertical channel under the effects of viscous dissipation, Ohmic heating, transverse magnetic field and applied electric field was studied analytically and numerically. The analytical solutions were defined by the perturbation method valid for small values of perturbation parameter, while numerical solutions were found by Runge–Kutta shooting method valid for any values of governing parameters. The Runge–Kutta shooting method and perturbation method show good agreement in the absence of buoyancy parameter. The following results were revealed:

1. For the combined effect of variable viscosity and thermal conductivity, an increase in the variable viscosity parameter enhances the flow and heat transfer whereas the increase in the variable thermal conductivity parameter suppresses the flow and heat transfer for $E = -1$ and 0. Behavior is opposite for $E = 1$. The Hartmann number accelerates the flow for open circuit and suppresses the flow for short circuit.
2. The wall temperature ratio and buoyancy parameter enhance the flow for the combined effect of variable viscosity parameter and variable thermal conductivity parameter for both open and short circuits.
3. The solutions obtained by Runge–Kutta shooting method and perturbation method are exact in the absence of variable thermal conductivity parameter and the error increases as the variable thermal conductivity parameter increases.

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Table 3
Comparison of velocity and temperature for $b_v = 0.1$, $m = 1$, $M = 2$, $N = 0$, $E = -1$, $P = 0.5$ between analytical (perturbation method) and numerical (Runge–Kutta method with shooting technique) methods.

Velocity				
y	$b_k = 0$		$b_k = 0.5$	
	Analytical	Numerical	Analytical	Numerical
1	0.00000000	0.00000000	0.00000000	0.00000001
0.8	0.38006000	0.38005698	0.37691000	0.37710024
0.6	0.63577000	0.63577350	0.62935000	0.62969547
0.4	0.80276000	0.80276379	0.79357000	0.79396178
0.2	0.90246000	0.90245991	0.89140000	0.89170550
0.0	0.94546000	0.94546167	0.93366000	0.93378275
−0.2	0.93316000	0.93315514	0.92183000	0.92174648
−0.4	0.85785000	0.85784590	0.84822000	0.84797175
−0.6	0.70141000	0.70141391	0.69455000	0.69425999
−0.8	0.43226000	0.43225593	0.42885000	0.42866742
−1	0.00000000	0.00000000	0.00000000	0.00000000
Temperature				
y	$b_k = 0$		$b_k = 0.5$	
	Analytical	Numerical	Analytical	Numerical
1	1.0000000	1.00000000	1.00000000	0.99999999
0.8	1.1000000	1.10000000	1.07750000	1.08028389
0.6	1.2000000	1.20000000	1.16000000	1.16392580
0.4	1.3000000	1.30000000	1.24750000	1.25121896
0.2	1.4000000	1.40000000	1.34000000	1.34249674
0.0	1.5000000	1.50000000	1.43750000	1.43814039
−0.2	1.6000000	1.60000000	1.54000000	1.53858861
−0.4	1.7000000	1.70000000	1.64750000	1.64434977
−0.6	1.8000000	1.80000000	1.76000000	1.75601743
−0.8	1.9000000	1.90000000	1.87750000	1.87429055
−1	2.0000000	2.00000000	2.00000000	2.00000000

and suggestions.

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