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ON THE ONE EDGE ALGORITHM FOR THE ORTHOGONAL DOUBLE COVERS

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The existing problem of the orthogonal double covers of the graphs is well-known in the theory of combinatorial designs. In this paper, a new technique called the one edge algorithm for constructing the orthogonal double covers of the complete bipartite graphs by copies of a graph is introduced. The advantage of this algorithm is that it is accessible to discrete mathematicians not intimately familiar with the theory of the orthogonal double covers.

Keywords: *graph decomposition, symmetric starter, orthogonal double covers.*

Nomenclature:

K_m	complete graph on m vertices;
$K_{m,n}$	complete bipartite graph with independent sets of sizes m and n;
mG	m disjoint copies of G ;
$C_k(n_1, n_2, \ldots, n_k)$	caterpillar (tree) obtained from the path $P_k = x_1x_2\ldots x_k$ by joining vertex x_i to n_i new vertices where $k \geqslant 1$, $n_1, n_2, \ldots, n_k$ are positive integers, $n_1, n_k \geqslant 1$ and $n_i \geqslant 0$ for $i \in \{2, 3, \ldots, k-1\}$.

Introduction

Graphs serve as a mathematical model to solve many real-world problems successfully. Some problems in chemistry, physics, computer technology, communication science, psychology, genetics, linguistics, and sociology can be formulated as problems in graph theory. Also, many branches of mathematics, such as topology, probability, group theory, and matrix theory, have close connections with the graph theory. Some puzzles of a practical nature have been instrumental in the development of various topics in graph theory. The cyclic graphs theory was developed for solving many problems of electrical networks, and the study of "trees" is considered a helping tool for enumerating isomers of organic compounds.

An *orthogonal double cover* (ODC) of H by G is a collection $\mathcal{G} = \{\pi(x) : x \in V(H)\}$ of isomorphic subgraphs (to G) of H (called pages) such that (i) every edge of H is contained in exactly two pages of $\mathcal{G}$ and (ii) $\pi(a)$ and $\pi(b)$share an edge if and only if a and b are adjacent in H.

The existence problem of ODC has attracted much attention during the last few years. The problem is known to be hard in general as it includes some long-standing open problems like the existing problems of biplanes. The ODC problem originally stems from problems in database optimization, statistical design of experiments, and design theory. In [1, 2], J. Demetrovics et al. have inspected the key of the Armstrong databases of minimum size. The ODC of K_n whose elements consist of distinct cliques is equivalent to Armstrong database of size n. Also, ODCs are related to several graph decomposition problems [3, 4]. The initial interest was concerned with the complete graphs, but the specialists have solved the ODC problem for several graphs such as the Cayley graphs [5] and the complete bipartite graphs (e.g., [6, 7]).

The labeling of the vertices of the complete bipartite graph $K_{n,n}$ is defined by the bijective mapping $\phi : V(K_{n,n}) \to \mathbb{Z}_n \times \mathbb{Z}_2$. The product $(v, j) \in \mathbb{Z}_n \times \mathbb{Z}_2$ will be written as v_j referring to the corresponding vertex and the edge $(c_\gamma, d_\delta) \in E(K_{n,n})$ if $\gamma \neq \delta$ for all $c, d \in \mathbb{Z}_n$ and $\gamma, \delta \in \mathbb{Z}_2$. We shall denote by (c, d) the edge between the vertices c_0 and d_1. If G is a subgraph of $K_{n,n}$ and $i \in \mathbb{Z}_n$, then $G + i$ is called i-translate of G. The edges of $G + i$ are obtained from G by rotating the edges of G, i.e., by mapping the edge (c, d) in G to the edge $(c + i, d + i)$ in $G + i$ (with calculations modulo n). If $e = (p, q) \in E(G)$, then it has a length defined by $d(e) = q - p$ (with calculations modulo n). For the graph G, if $|E(G)| = n$ and the lengths of all edges in G are mutually distinct and equal to $\mathbb{Z}_n$, then G is said to be a half starter w.r.t. $\mathbb{Z}_n$. R. El-Shanawany et al. [8] have proved the following three results.

A. The union of all translates of G forms an edge decomposition of $K_{n,n}$, i.e., $\bigcup\limits_{z \in \mathbb{Z}_n} E(G+z) =$ $= E(K_{n,n})$, if and only if G is a half starter.

In what follows, a half starter G will be represented by the vector $v(G) = (v_0, v_1, \ldots, v_{n-1}) \in \mathbb{Z}_n^n$ where $v_i, i \in \mathbb{Z}_n$ and $(v_i)_0$ is the unique vertex $((v_i, 0) \in \mathbb{Z}_n \times \{0\})$ that belongs to the unique edge of length i in G. The two half starter vectors $v(G_0)$ and $v(G_1)$ are said to be orthogonal if $\{v_i(G_0) - v_i(G_1) : i \in \mathbb{Z}_n\} = \mathbb{Z}_n$.

B. If $v(G_0)$ and $v(G_1)$ are orthogonal two half starter vectors, then $\mathcal{G} = \{G_{z,l} : (z, l) \in$ $\in \mathbb{Z}_n \times \mathbb{Z}_2\}$ with $G_{z,l} = G_l + z$ is considered the ODC of $K_{n,n}$.

If the graph G_s is a subgraph of $K_{n,n}$ with $E(G_s) = \{(p_0, q_1) : (q_0, p_1) \in E(G)\}$, then G_s is the symmetric graph of G. It is easy to prove that if G is a half starter, then G_s is also a half starter. Also, if $v(G)$ and $v(G_s)$ are orthogonal, then the half starter G is called a symmetric starter w.r.t. $\mathbb{Z}_n$.

C. If n is any positive integer and G is a half starter with $v(G) = (v_0, v_1, \ldots, v_{n-1})$, then G is symmetric starter iff $\{v_i - v_{-i} + i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n$.

For illustrative purposes, Fig. 1 exhibits the graph $K_{4,4}$ and its ODC by $K_{1,4}$.

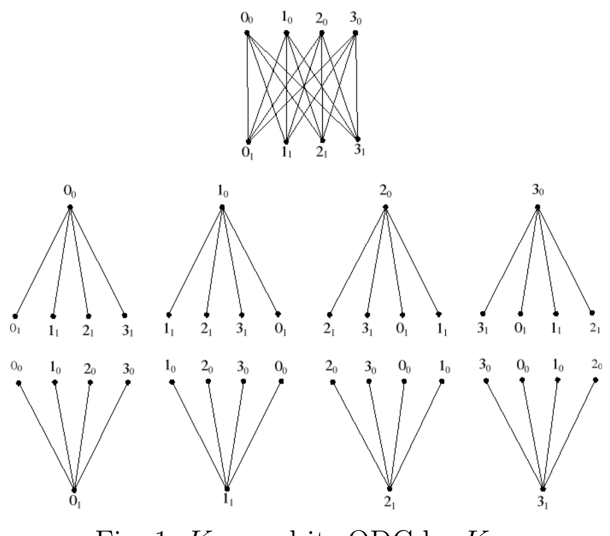

Fig. 1. $K_{4,4}$ and its ODC by $K_{1,4}$

The aim of this paper is to present a new algorithm which is called the one edge algorithm. This algorithm is a helping tool for constructing the ODCs of the complete bipartite graphs. In addition, we present a number of new results as a direct application of this algorithm. The difference between our paper and [9] is as follows. In [9], El-Serafi et al. constructed the orthogonal double cover of complete bipartite graphs by a complete bipartite graph and the disjoint union of complete bipartite graphs with the cartesian product of symmetric starter vectors. For several results of ODCs by different graph classes, see [10–12].

1. Main results

In this section, we construct the orthogonal double covers of the complete bipartite graphs by copies of stars, copies of a caterpillar, and by copies of a complete bipartite graph.

Definition 1. For $k \geqslant 2$, a caterpillar graph $C_k(m_1, m_2, \ldots, m_k)$ is obtained from a path $P_k = v_1 v_2 \ldots v_k$ by attaching $n_i \geqslant 0$ pendant vertices $u_{i,j}$ $(1 \leqslant j \leqslant m_i)$ to each v_i.

Definition 2. A complete bipartite graph of the form $K_{1,n}$ is called a star.

In what follows, we prove the theorems on existence of ODCs for particular classes of complete bipartite graphs $K_{mn,mn}$ and $K_{2mn,2mn}$ with $n > 0$ and $m \equiv 1, 5 \pmod 6$. Really, our effort was concentrated on these classes and in the future we hope to solve the problem for the other classes to be used in the design theory. The proofs of these theorems base on the direct constructions of ODCs. The main strengths of represented algorithms is that they are clear for discrete mathematicians not intimately familiar with the theory of the orthogonal double covers.

Theorem 1. Let m and n be positive integers with $m \equiv 1, 5 \pmod 6$. Then, there is an ODC of $K_{mn,mn}$ by $mK_{1,n}$.

Proof. Algorithm 1 proves the Theorem 1. ■

Algorithm 1.	ODC of $K_{mn,mn}$ by $mK_{1,n}$
1: [Inauguration.]	Choose the values $m, n \in \mathbb{N}$, $m \equiv 1, 5 \pmod 6$.
2: [Initial edge.]	Construct the edge $(0, 0)$.
3: [Complementary edges.]	Add $(0, -\beta)$ to $(0, 0)$ where $\beta \in \mathbb{Z}_n \setminus \{0\}$.
4: [Initial graph $\cong G^0$.]	The union of the edge set in steps 2, 3 gives the edge set of $K_{1,n}$.
5: [Complementary graphs $\cong \bigcup_{\alpha=1}^{m-1} G^\alpha$.]	Add $(\alpha n, 2\alpha n)$ to the edge set of step 4 where $\alpha \in \mathbb{Z}_m \setminus \{0\}$.
6: [Symmetric starter $\cong \bigcup_{\alpha=0}^{m-1} G^\alpha$.]	The union of the edge set in steps 4, 5 gives the edge set of $G \cong mK_{1,n}$.
7: [Symmetric graph.]	If the edge (a, b) belongs to the edge set of step 6, then (b, a) is an edge of the graph $G_s \cong mK_{1,n}$.
8: [Translation.]	Add (γ, γ) to the edge set in steps 6, 7 where $\gamma \in \mathbb{Z}_{mn}$.
9: [Done.]	Output the edge set of step 8 which represents the ODC of $K_{mn,mn}$ by $mK_{1,n}$.

For more illustration to Theorem 1, let $m = 5$ and $n = 3$, then there is an ODC of $K_{15,15}$ by $5K_{1,3}$, see Fig. 2.

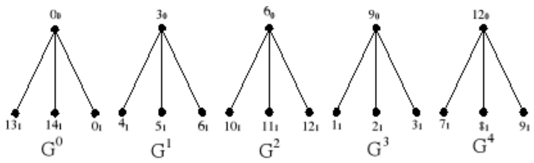

Fig. 2. Symmetric starter of an ODC of $K_{15,15}$ by $5K_{1,3}$

Theorem 2. Let m and $n \geqslant 5$ be positive integers with $m \equiv 1, 5 \pmod 6$. Then, there is an ODC of $K_{mn,mn}$ by $2mK_2 \cup mK_{1,n-2}$.

Proof. Algorithm 2 proves the Theorem 2. ■

Algorithm 2.	ODC of $K_{mn,mn}$ by $2mK_2 \cup mK_{1,n-2}$
1: [Inauguration.]	Choose the values of m and n where m and $n \geqslant 5$ are positive integers with $m \equiv 1, 5 \pmod 6$.
2: [Initial edge.]	Construct the edge $(0, 0)$.
3: [Complementary edges.]	Add $(1 + n, 2n - 1), (2n - 1, 3n - 2)$, and $(0, -2\beta)$ to $(0, 0)$ where $\beta \in \mathbb{Z}_n \setminus \mathbb{Z}_3$.
4: [Initial graph $\cong G^0$.]	The union of the edge set in steps 2, 3 gives the edge set of $2K_2 \cup K_{1,n-2}$.
5: [Complementary graphs $\cong \bigcup_{\alpha=1}^{m-1} G^{\alpha}$.]	Add $(\alpha n, 2\alpha n)$ to the edge set of step 4 where $\alpha \in \mathbb{Z}_m \setminus \{0\}$.
6: [Symmetric starter of $G \cong \bigcup_{\alpha=0}^{m-1} G^{\alpha}$.]	The union of the edge set in steps 4, 5 gives the edge set $\cong 2mK_2 \cup mK_{1,n-2}$.
7: [Symmetric graph.]	If the edge (a, b) belongs to the edge set of step 6, then (b, a) is an edge of the graph $G_s \cong 2mK_2 \cup mK_{1,n-2}$.
8: [Translation.]	Add (γ, γ) to the edge set in steps 6, 7 where $\gamma \in \mathbb{Z}_{mn}$.
9: [Done.]	Output the edge set of step 8 which represents the ODC of $K_{mn,mn}$ by $2mK_2 \cup mK_{1,n-2}$.

For more illustration to Theorem 2, let $m = 5$ and $n = 5$, then there is an ODC of $K_{25,25}$ by $5K_{1,3} \cup 10K_2$, see Fig. 3.

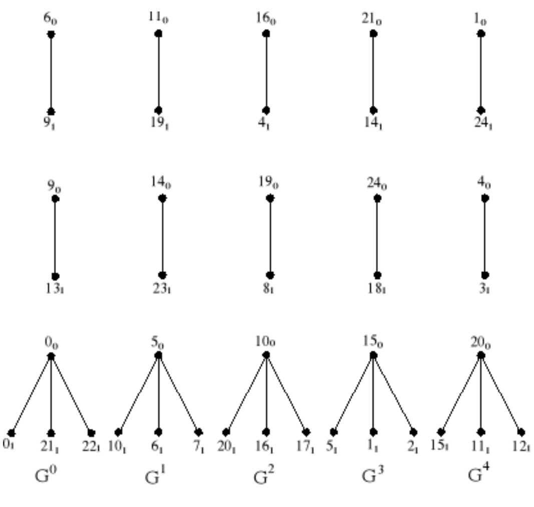

Fig. 3. Symmetric starter of an ODC of $K_{25,25}$ by $5K_{1,3} \cup 10K_2$

Theorem 3. Let m and $n \geqslant 4$ be positive integers with $m \equiv 1, 5 \pmod 6$. Then, there is an ODC of $K_{mn,mn}$ by $mC_2(1, n-2)$.

Proof. Algorithm 3 proves the Theorem 3. ■

Algorithm 3.	ODC of $K_{mn,mn}$ by $mC_2(1, n-2)$
1: [Inauguration.]	Choose the values of m and n where m and $n \geqslant 4$ are positive integers with $m \equiv 1, 5 \pmod 6$.
2: [Initial edge.]	Construct the edge $(0, 0)$.
3: [Complementary edges.]	Add $(1, 1+\beta-n)$ to $(0, 0)$ where $\beta \in \mathbb{Z}_n \setminus \{0\}$.
4: [Initial graph $\cong G^0$.]	The union of the edge set in steps 2, 3 gives the edge set of $C_2(1, n-2)$.
5: [Complementary graphs $\cong \bigcup_{\alpha=1}^{m-1} G^\alpha$.]	Add $(\alpha n, 2\alpha n)$ to the edge set of step 4 where $\alpha \in \mathbb{Z}_m \setminus \{0\}$.
6: [Symmetric starter $\cong \bigcup_{\alpha=0}^{m-1} G^\alpha$.]	The union of the edge set in steps 4, 5 gives the edge set of $G \cong mC_2(1, n-2)$.
7: [Symmetric graph.]	If the edge (a, b) belongs to the edge set of step 6, then (b, a) is an edge of the graph $G_s \cong mC_2(1, n-2)$.
8: [Translation.]	Add (γ, γ) to the edge set in steps 6, 7 where $\gamma \in \mathbb{Z}_{mn}$.
9: [Done.]	Output the edge set of step 8 which represents the ODC of $K_{mn,mn}$ by $mC_2(1, n-2)$.

For more illustration to Theorem 3, let $m = 5$ and $n = 4$, then there is an ODC of $K_{20,20}$ by $5C_2(1, 2)$, see Fig. 4.

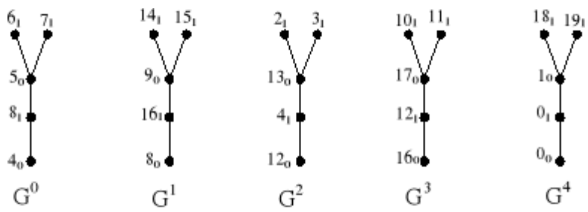

Fig. 4. Symmetric starter of an ODC of $K_{20,20}$ by $5C_2(1, 2)$

Theorem 4. Let m and $n \geqslant 2$ be positive integers with $m \equiv 1, 5 \pmod 6$. Then, there is an ODC of $K_{2mn,2mn}$ by $mK_{2,n}$.

Proof. Algorithm 4 proves the Theorem 4. ■

Algorithm 4.	ODC of $K_{2mn,2mn}$ by $mK_{2,n}$
1: [Inauguration.]	Choose the values of m and n where m and $n \geqslant 2$ are positive integers with $m \equiv 1, 5 \pmod 6$.
2: [Initial edge.]	Construct the edge $(0, 0)$.
3: [Complementary edges.]	Add $(0, -2\beta)$ and $(1, -2\beta)$ to $(0, 0)$ where $\beta \in \mathbb{Z}_n \setminus \{0\}$.
4: [Initial graph $\cong G^0$.]	The union of the edge set in steps 2, 3 gives the edge set of $K_{2,n}$.
5: [Complementary graphs $\cong \bigcup_{\alpha=1}^{m-1} G^\alpha$.]	Add $(2\alpha n, 4\alpha n)$ to the edge set of step 4 where $\alpha \in \mathbb{Z}_m \setminus \{0\}$.
6: [Symmetric starter $\cong \bigcup_{\alpha=0}^{m-1} G^\alpha$.]	The union of the edge set in steps 4, 5 gives the edge set of $G \cong mK_{2,n}$.
7: [Symmetric graph.]	If the edge (a, b) belongs to the edge set of step 6, then (b, a) is an edge of the graph $G_s \cong mK_{2,n}$.

8: [Translation.]	Add (γ, γ) to the edge set in steps 6, 7 where $\gamma \in \mathbb{Z}_{2mn}$.
9: [Done.]	Output the edge set of step 8 which represents the ODC of $K_{2mn,2mn}$ by $mK_{2,n}$.

For more illustration to Theorem 4, let $m = 5$ and $n = 3$, then there is an ODC of $K_{30,30}$ by $5K_{2,3}$, see Fig. 5.

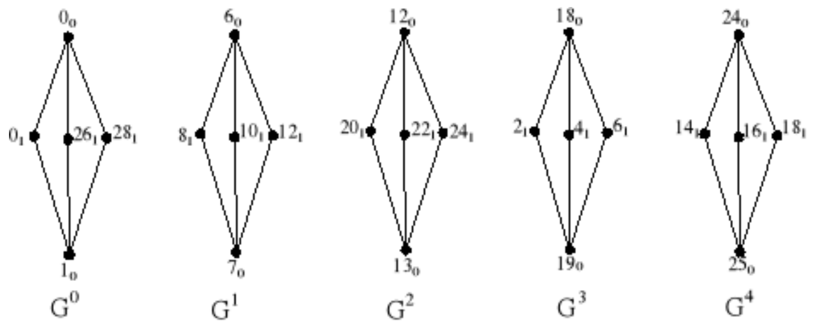

Fig. 5. Symmetric starter of an ODC of $K_{30,30}$ by $5K_{2,3}$

Theorem 5. Let m and $n \geqslant 5$ be positive integers with $m \equiv 1, 5 \pmod 6$. Then, there is an ODC of $K_{mn,mn}$ by $mC_5(1, 0, 0, 0, n-5)$.

Proof. Algorithm 5 proves the Theorem 5. ■

Algorithm 5.	ODC of $K_{mn,mn}$ by $mC_5(1, 0, 0, 0, n-5)$.
1: [Inauguration.]	Choose the values of m and n where m and $n \geqslant 5$ are positive integers with $m \equiv 1, 5 \pmod 6$.
2: [Initial edge.]	Construct the edge $(0, 0)$.
3: [Complementary edges.]	Add $(0, 2-n), (2, 2-n+\beta), ((m+7)n/2, \gamma - n)$ to $(0, 0)$ where $\beta \in \mathbb{Z}_n \setminus \{0, 2, n-2, n-1\}$ and $\gamma \in \{2, 3\}$.
4: [Initial graph $\cong G^0$.]	The union of the edge set in steps 2, 3 gives the edge set of $C_5(1, 0, 0, 0, n-5)$.
5: [Complementary graphs $\cong \bigcup_{\alpha=1}^{m-1} G^\alpha$.]	Add $(\alpha n, 2\alpha n)$ to the edge set of step 4 where $\alpha \in \mathbb{Z}_m \setminus \{0\}$.
6: [Symmetric starter $\cong \bigcup_{\alpha=0}^{m-1} G^\alpha$.]	The union of the edge set in steps 4, 5 gives the edge set of $G \cong mC_5(1, 0, 0, 0, n-5)$.
7: [Symmetric graph.]	If the edge (a, b) belongs to the edge set of step 6, then (b, a) is an edge of the graph $G_s \cong mC_5(1, 0, 0, 0, n-5)$.
8: [Translation.]	Add (γ, γ) to the edge set in steps 6, 7 where $\gamma \in \mathbb{Z}_{mn}$.
9: [Done.]	Output the edge set of step 8 which represents the ODC of $K_{mn,mn}$ by $mC_5(1, 0, 0, 0, n-5)$.

For more illustration to Theorem 5, let $m = 5$ and $n = 7$, then there is an ODC of $K_{35,35}$ by $5C_5(1, 0, 0, 0, 2)$, see Fig. 6.

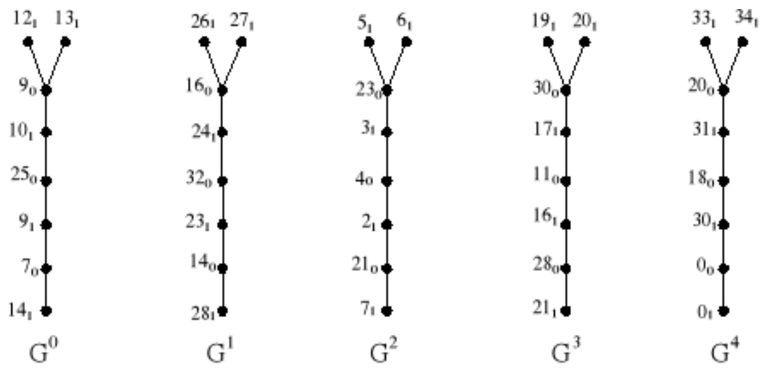

Fig. 6. Symmetric starter of an ODC of $K_{35,35}$ by $5C_5(1,0,0,0,2)$

Conclusion

In this paper, we study the orthogonal double covers of the complete bipartite graphs by algorithms for generating the orthogonal double covers by one edge.

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