

ON THE IDENTIFICATION OF THE SOURCE OF EMISSION  
ON THE PLANE

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We consider the problem of identification of the position and the moment of the beginning of a radioactive source emission on the plane. The acts of emission constitute inhomogeneous Poisson processes and are registered by  $K$  detectors on the plane. We suppose that the moments of arriving of the signals at the detectors are measured with some small errors. Then, using these estimate, we construct the estimators of the position of source and the moment of the beginning of emission. We study the asymptotic properties of these estimators for large signals and prove their consistency.

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**Introduction.** There are given  $K$  detectors  $D_1, \dots, D_K$  placed at the points  $\vartheta_k = (x_k, y_k), k = 1, \dots, K$ , on the plane. We suppose that at an unknown location  $\vartheta_0 = (x_0, y_0) \in \Theta \subset \mathcal{R}^2$  at an unknown moment  $\tau_0$  a radioactive device starts emission. The detectors receive signals, and based on these detections, the statistician has to estimate the position of the source and the time  $\tau_0$  of the beginning of emission. We obtain a similar mathematical model of observations in the case of a weak optical source emitting photons. Note that in the problem of GPS-localization we have the same mathematical model for the inverse experiment. We have  $K$  emitters of signals  $D_1, \dots, D_K$  received by the device  $D_0$ , and using the observations of these signals, it is necessary to estimate the position of the device.

An example of such a model of observations is given in the Fig. 1, where  $D_0$  is the position of the source and  $D_1 - D_5$  are the detectors.

Due to importance of this problem in many applications there exists a large amount of literature on the identification of radioactive sources of engineering level (see [1–4]). To the best of our knowledge, the mathematical study of such problems

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is not yet sufficiently developed. This work is the continuation of the study started in papers [5–7].

We consider the case when the signals received by the detectors are inhomogeneous Poisson processes  $X^K = (X_1, \dots, X_K)$ , where the process  $X_k = (X_k(t), 0 \leq t \leq T)$  has intensity function

$$\lambda(\vartheta, t) = nS_k(t - \tau_k(\vartheta_0))\bar{\psi}(t - \tau_k(\vartheta_0)) + n\lambda_0, \quad 0 \leq t \leq T. \quad (1)$$

Here  $nS(\cdot)$  is a known positive continuous function (intensity of the signal),  $\tau_k(\vartheta_0)$  is the moment of arriving of the signal at the  $k$ -th detector and  $n\lambda_0$  is the background noise. The time of arriving  $\tau_k = \tau_k(\vartheta_0)$  can be written as follows

$$\tau_k = \tau_0 + v^{-1} \|\vartheta_k - \vartheta_0\| = \tau_0 + \tau_{k,0}, \quad (2)$$

where  $\tau_0 \in \mathcal{T}$  is the moment of beginning of emission,  $v > 0$  is the rate of propagation of the signals,  $\|\cdot\|$  is the Euclidean norm in  $\mathcal{R}^2$  and  $\|\vartheta_k - \vartheta_0\|$  is the distance between the point of emission and the  $k$ -th detector,  $\tau_{k,0}$  is the time to reach of the  $k$ -th detector after the beginning of emission [8].

The function  $\bar{\psi}(t) = 0$  for  $t < 0$  reflects the form of the signal at the moment of its arriving. We consider three different cases: smooth  $\psi_\delta(\cdot)$ , cusp-type  $\psi_{\delta,\kappa}$  and change-point type  $\psi(\cdot)$ , where

$$\begin{aligned} \psi_\delta(t) &= \frac{t}{\delta} \mathbb{I}_{\{0 \leq t \leq \delta\}} + \mathbb{I}_{\{t > \delta\}}, & \psi(t) &= \mathbb{I}_{\{t > 0\}}, \\ \psi_{\delta,\kappa}(t) &= \frac{1}{2} \left( 1 + \operatorname{sgn}(2t - \delta) \left| \frac{2t}{\delta} - 1 \right|^\kappa \right) \mathbb{I}_{\{0 \leq t \leq \delta\}} + \mathbb{I}_{\{t > \delta\}}, \end{aligned}$$

The parameter  $\delta > 0$  is known and small. In the cusp case  $\kappa \in \left(0, \frac{1}{2}\right)$ . The examples of such functions are given in the Fig. 1. The case b) in Fig. 1 corresponds to the function  $\psi_{\delta,\kappa}(\cdot)$  with the value  $\kappa = \frac{1}{2}$  and the case e) is obtained if in  $\psi_{\delta,\kappa}(\cdot)$  the parameter  $\kappa \in (-1, 0)$ .

Consider the problem of estimation of the position  $\vartheta_0 = (x_0, y_0)$  and the moment  $\tau_*$  by the observations  $X^K = (X_1, \dots, X_K)$ . The estimators of these quantities are studied in the asymptotics of large signals, i.e. as  $n \rightarrow \infty$ .

Recall that the cases a)–d) with  $\tau_0 = 0$ , i.e. a known moment of beginning of emission, were considered in [5–7]. It was shown that the Bayes estimators  $\tilde{\vartheta}_n$  of the parameter  $\vartheta_0$  have the following limits

$$\begin{aligned} \text{a) } \sqrt{n}(\tilde{\vartheta}_n - \vartheta_0) &\Longrightarrow \zeta_1, & \text{b) } \sqrt{n \ln n}(\tilde{\vartheta}_n - \vartheta_0) &\Longrightarrow \zeta_2, \\ \text{c) } n^{\frac{1}{2\kappa+1}}(\tilde{\vartheta}_n - \vartheta_0) &\Longrightarrow \zeta_3, & \text{d) } n(\tilde{\vartheta}_n - \vartheta_0) &\Longrightarrow \zeta_4, \end{aligned}$$

where  $\zeta_i$ ,  $i = 1, \dots, 4$ , are some random vectors all having polynomial moments (for details, see [5–7]).

It is possible to consider a different statement of the problem. Let us study  $K$  independent Poisson processes  $X_1, \dots, X_K$  with intensity functions (1) and estimate the parameters  $\tau_1, \dots, \tau_K$ . The corresponding MLE  $\hat{\tau}_{k,n}$ ,  $k = 1, \dots, K$ , and BE

$\tilde{\tau}_{k,n}$ ,  $k = 1, \dots, K$ , have the similar properties. For example,

$$\text{a) } \sqrt{n}(\hat{\tau}_{k,n} - \tau_k) \implies \xi_{k,1}, \quad \text{b) } \sqrt{n \ln n}(\hat{\tau}_{k,n} - \tau_k) \implies \xi_{k,2}, \quad (3)$$

$$\text{c) } n^{\frac{1}{2\kappa+1}}(\hat{\tau}_{k,n} - \tau_k) \implies \xi_{k,3}, \quad \text{d) } n(\hat{\tau}_{k,n} - \tau_k) \implies \xi_{k,4}, \quad (4)$$

$$\text{e) } n^{\frac{1}{\kappa+1}}(\hat{\tau}_{k,n} - \tau_k) \implies \xi_{k,5}. \quad (5)$$

For the cases a) and d) see [9], cases c) and e) (for  $\kappa \in (-1, 0)$ ) were studied in [10, 11], respectively. The case b) follows from the results presented in [5].

In this work we consider the estimation of parameters  $\tau_0, \vartheta_0$  in two steps. First, we estimate  $K$  moments  $\tau_1, \dots, \tau_K$ . Then having these estimators with properties (3)–(5) we estimate  $\tau_0, \vartheta_0$ .

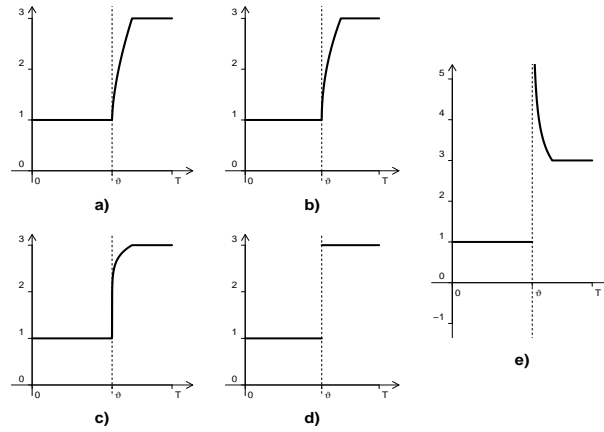


Fig. 1. Examples of functions  $\bar{\psi}(\cdot)$ . Here a) and b) smooth cases; c) cusp-type case; d) change-point case; e) non bounded intensity case.

**Geometrical View.** Consider the problem of estimation of location  $\vartheta_0$  and the moment of emission  $\tau_0$  in the situation, where there is no errors, i.e. the detectors measure  $\tau_k = \tau_k(\vartheta_0)$  exactly. If we know  $\tau_0$ , then 3 detectors are enough to find the location of the source (see, e.g., [5]). Now we want to see geometrically whether in the case of unknown  $\tau_0$  3 detectors are sufficient to find  $\vartheta_0$ . Suppose that we have 2 detectors  $D_1$  and  $D_2$  and we know exactly moments  $\tau_1$  and  $\tau_2$  when signals had arrived. We denote  $r_1 = \|\vartheta_1 - \vartheta_0\|$  the distance between our device and the detector  $D_1$ , in the same way  $r_2$  is the distance between device and the detector  $D_2$ .  $r_1$  and  $r_2$  are unknown, but we can calculate their difference. If  $r_1 = v(\tau_1 - \tau_0)$  and  $r_2 = v(\tau_2 - \tau_0)$ , then we have  $r_1 - r_2 = v(\tau_1 - \tau_2)$ .

We denote  $r = v(\tau_1 - \tau_2)$ , so the difference of distances is  $r$ . If we look all possible locations of the device it is a hyperbola branch with foci  $D_1$  and  $D_2$ . For every point  $(x, y)$  on this hyperbola we have  $(x - x_1)^2 + (y - y_1)^2 = r_1^2$ ,  $(x - x_2)^2 + (y - y_2)^2 = r_2^2$  and  $r_2^2 = (r_1 - r)^2$ , from this three equations we obtain the equation of our hyperbola:

$$(x_1 - x_2)(2x - x_1 - x_2) + (y_1 - y_2)(2y - y_1 - y_2) - r^2 - 2r\sqrt{(x - x_1)^2 + (y - y_1)^2} = 0.$$

Now if we add the third detector  $D_3$ , we can construct another hyperbola branch corresponding to the focuses  $D_3$  and  $D_2$  with the equation:

$$(x_2 - x_3)(2x - x_2 - x_3) + (y_2 - y_3)(2y - y_2 - y_3) - r'^2 - 2r'\sqrt{(x - x_2)^2 + (y - y_2)^2} = 0,$$

where  $r' = v(\tau_2 - \tau_3)$ . The device will be at the point of intersection of these two hyperbolas. We can see in Fig. 2 two points of intersection of hyperbolas. Thus, in general we can not identify the location of device with 3 detectors.

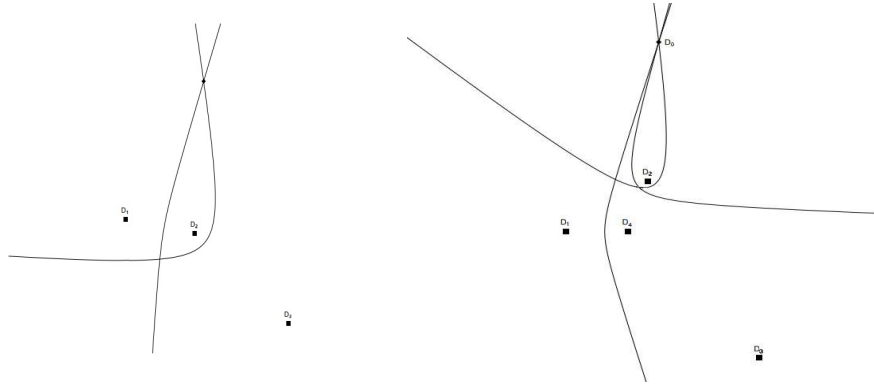


Fig. 2. Two hyperbolas with focuses  $D_1, D_2; D_2, D_3$ . Fig. 3. Detection with four detectors.

Now we will try to find out whether four detectors are enough to identify the exact location of the device. So we consider the signals of  $D_1 - D_3$  detectors that show us two possible points in the plane. Hence we have 2 intersection points of hyperbolas denoted by  $P_1$  and  $P_2$ . So we want to find a position  $D_4$  such that the hyperbola branch with focuses in  $D_4$  and at the location of one of the other detectors passes through only one of the points  $P_1$  and  $P_2$ . To this end we will find all possible focuses of hyperbola branch that passes by points  $P_1$  and  $P_2$ . Hence, if  $F_1$  and  $F_2$  are those focuses, we have

$$\rho(P_1, F_1) - \rho(P_1, F_2) = \rho(P_2, F_1) - \rho(P_2, F_2)$$

or

$$\rho(P_1, F_1) - \rho(P_1, F_2) = -\rho(P_2, F_1) + \rho(P_2, F_2).$$

So we have that  $\rho(P_1, F_1) - \rho(P_2, F_1) = \rho(P_1, F_2) - \rho(P_2, F_2)$  (or similar for the second equation), which means that all focuses of hyperbola branch passing by  $P_1$  and  $P_2$  are located on the other hyperbola branch with the focuses  $F_1$  and  $F_2$ . Thus to identify the location of device we need at least four detectors.

**Main Results.** We have  $K$  independent Poisson processes  $X^K = (X_1, \dots, X_K)$ , where the random process  $X_k = (X_k(t), 0 \leq t \leq T)$  has intensity function (1) and we have to estimate the parameters  $\tau_0, \vartheta_0$  by observations  $X^K$ . We will this problem solve in two steps. First we obtain  $K$  independent estimators  $\bar{\tau}_{k,n}$ ,  $k = 1, \dots, K$ , of the moments of signals arriving at the detectors. Then having these estimators we consider the problem of estimation of  $\tau_0, \vartheta_0$ . The advantage of this approach is its computational simplicity with respect to the traditional maximum likelihood

approach. Recall that in maximum likelihood approach all data have to come to one center of simultaneous treatment and then the estimators are obtained as a result of maximization of the likelihood ratio function of three variables

$$\ln L(\tau, \vartheta, X^K) = \sum_{k=1}^K \int_{\tau_k(\vartheta)}^T \ln \left( 1 + \frac{S_k(t - \tau_k(\vartheta)) \psi_\delta(t - \tau_k(\vartheta))}{\lambda_0} \right) dX_k(t) - n \sum_{k=1}^K \int_{\tau_k(\vartheta)}^T S_k(t - \tau_k(\vartheta)) \psi_\delta(t - \tau_k(\vartheta)) dt,$$

where  $\tau_k(\vartheta) = \tau_0 + v^{-1} \|\vartheta_k - \vartheta\|$ .

In our approach the estimators  $\hat{\tau}_{k,n}$  can be calculated in each detector and then transmitted to the center, where the problem of estimation is reduced to the solution of linear equations.

The convergences (3)–(5) can be summarized in the following representations

$$\hat{\tau}_{k,n} = \tau_0 + \tau_{k,0} + \varphi_n \eta_{k,n}, \quad \varphi_n \rightarrow 0, \quad (6)$$

where  $\eta_{k,n}$  converge in distribution to the corresponding random variable  $\xi_k$ .

The unknown parameters satisfy the following equations

$$v^2 (\tau_k - \tau_0)^2 = (x_k - x_0)^2 + (y_k - y_0)^2, \quad k = 1, \dots, K.$$

We have  $v^2 \tau_k^2 = x_k^2 + y_k^2 + x_0^2 + y_0^2 - v^2 \tau_0^2 - 2x_k x_0 - 2y_k y_0 + 2v^2 \tau_k \tau_0$ .

Let us denote

$$\gamma_1 = x_0, \quad \gamma_2 = y_0, \quad \gamma_3 = \tau_0, \quad \gamma_4 = \frac{1}{2} (x_0^2 + y_0^2 - v^2 \tau_0^2),$$

$$\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4), \quad z_k = \frac{1}{2} (x_k^2 + y_k^2 - v^2 \tau_k^2).$$

Then this equation can be written as follows

$$x_k \gamma_1 + y_k \gamma_2 - v^2 \tau_k \gamma_3 + \gamma_4 = z_k, \quad k = 1, \dots, K. \quad (7)$$

We have the “observations”  $\hat{\tau}_{k,n}$  and  $z_{k,n} = \frac{1}{2} (x_k^2 + y_k^2 - v^2 \hat{\tau}_{k,n}^2)$ . Therefore, we define

the estimator  $\gamma_n^* = (\gamma_{1,n}^*, \gamma_{2,n}^*, \gamma_{3,n}^*, \gamma_{4,n}^*)$  using the least squares approach as follows

$$\gamma_n^* = \arg \min_{\gamma} S_n(\gamma), \quad S_n(\gamma) = \sum_{k=1}^K [z_{k,n} - x_k \gamma_1 - y_k \gamma_2 + v^2 \hat{\tau}_{k,n} \gamma_3 - \gamma_4]^2.$$

It will be convenient to denote  $v^2 \hat{\tau}_{k,n} = -\hat{\rho}_{k,n}$ . Therefore, the least squares estimator  $\gamma_n^*$  is the solution of the equations

$$\frac{\partial S_n(\gamma)}{\partial \gamma_l} = 0, \quad l = 1, \dots, 4,$$

which can be written as

$$\begin{aligned} \sum_{k=1}^K x_k^2 \gamma_{1,n}^* + \sum_{k=1}^K x_k y_k \gamma_{2,n}^* + \sum_{k=1}^K x_k \hat{\rho}_{k,n} \gamma_{3,n}^* + \sum_{k=1}^K x_k \gamma_{4,n}^* &= \sum_{k=1}^K x_k z_{k,n}, \\ \sum_{k=1}^K x_k y_k \gamma_{1,n}^* + \sum_{k=1}^K y_k^2 \gamma_{2,n}^* + \sum_{k=1}^K y_k \hat{\rho}_{k,n} \gamma_{3,n}^* + \sum_{k=1}^K y_k \gamma_{4,n}^* &= \sum_{k=1}^K y_k z_{k,n}, \\ \sum_{k=1}^K x_k \hat{\rho}_{k,n} \gamma_{1,n}^* + \sum_{k=1}^K \hat{\rho}_{k,n} y_k \gamma_{2,n}^* + \sum_{k=1}^K \hat{\rho}_{k,n}^2 \gamma_{3,n}^* + \sum_{k=1}^K \hat{\rho}_{k,n} \gamma_{4,n}^* &= \sum_{k=1}^K \hat{\rho}_{k,n} z_{k,n}, \\ \sum_{k=1}^K x_k \gamma_{1,n}^* + \sum_{k=1}^K y_k \gamma_{2,n}^* + \sum_{k=1}^K \hat{\rho}_{k,n} \gamma_{3,n}^* + K \gamma_{4,n}^* &= \sum_{k=1}^K z_{k,n}, \end{aligned}$$

or in matrix form

$$\mathbb{A}_n \gamma_n^* = Z_n, \quad \gamma_n^* = \mathbb{A}_n^{-1} Z_n$$

with obvious notations. Since  $\hat{\tau}_{k,n} = \tau_0 + \tau_{k,0} + \varphi_n \eta_n \rightarrow \tau_k = \tau_0 + \tau_{k,0}$ , the matrix  $\mathbb{A}_n$  converges in probability  $\mathbb{A}_n = \mathbb{A}_0 + \varphi_n B_n \rightarrow \mathbb{A}_0$ , where the matrix

$$\mathbb{A}_0 = \begin{pmatrix} \|x\|_K^2, & \langle x, y \rangle_K, & +\langle x, \rho \rangle_K, & \langle x, 1 \rangle_K \\ \langle x, y \rangle_K, & \|y\|_K^2, & +\langle y, \rho \rangle_K, & \langle y, 1 \rangle_K \\ \langle x, \rho \rangle_K, & \langle y, \rho \rangle_K, & +\|\rho\|^2, & \langle \rho, 1 \rangle_K \\ \langle x, 1 \rangle_K, & \langle y, 1 \rangle_K, & +\langle 1, \rho \rangle_K, & K \end{pmatrix}.$$

Here  $\rho = (-v^2 \tau_1, \dots, -v^2 \tau_K)$  and

$$\|a\|_K^2 = \sum_{k=1}^K a_k^2, \quad \langle a, b \rangle_K = \sum_{k=1}^K a_k b_k.$$

Further we have convergence in probability

$$\begin{aligned} Z_{1,n} = \langle x, z_n \rangle_K &\rightarrow \langle x, z \rangle_K = Z_1, & Z_{2,n} = \langle y, z_n \rangle_K &\rightarrow \langle y, z \rangle_K = Z_2, \\ Z_{3,n} = \langle \hat{\tau}_n, z_n \rangle_K &\rightarrow \langle \tau, z \rangle_K = Z_3, & Z_{4,n} = \langle 1, z_n \rangle_K &\rightarrow \langle 1, z \rangle_K = Z_4, \end{aligned}$$

or  $Z_n \rightarrow Z$ , where the vector  $Z = (Z_1, \dots, Z_4)$ . Then we can write (7) as

$$\mathbb{A}_n \gamma_n^* = Z_n \quad \text{and} \quad \gamma_n^* = \mathbb{A}_n^{-1} Z_n.$$

We study the asymptotic ( $n \rightarrow \infty$ ) behavior of the estimator  $\gamma_n^*$ .

**Conditions  $\mathcal{C}$ .**

1. The set  $\Theta \subset \mathcal{R}^2$  is open, convex and bounded.
2. The set  $\mathcal{J} = (T_i, T_f)$  is such that  $\tau_k \in (0, T)$  for all  $\vartheta_0 \in \Theta$ .
3. The estimators  $\hat{\tau}_{k,n}$ ,  $k = 1, \dots, K$ , admit the representation (6), where  $\varphi_n \rightarrow 0$  and the random variables  $\eta_{k,n}$ ,  $k = 1, \dots, K$ , are bounded in probability.
4. There are at least four detectors, which are not on the same line and the matrix  $\mathbb{A}_0$  is non degenerate

$$\inf_{T_i \leq \tau_0 \leq T_f} \inf_{\|e\|_4=1} e^\top \mathbb{A}_0 e > 0,$$

where  $e \in \mathcal{R}^4$ .

Note that all these conditions in the case of known  $\tau_0$  are fulfilled in the problems considered in the works [5–7].

Therefore we proved the following result.

**Theorem .** *Let conditions  $\mathcal{C}$  be satisfied, then estimator  $\gamma_n^*$  is consistent.*

It can be verified that since the matrix  $\mathbb{A}_0$  is uniformly non-degenerate, we have

$$\mathbb{A}_n \rightarrow \mathbb{A}_0, \quad Z_n \rightarrow Z, \quad \mathbb{A}_n^{-1} \rightarrow \mathbb{A}_0^{-1}, \quad \gamma_n^* \rightarrow \mathbb{A}_0^{-1} Z = \gamma,$$

where  $\gamma = (x_0, y_0, \tau_0, 2^{-1}(x_0^2 + y_0^2 - v^2 \tau_0^2))$ .

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ՆԱՐԹՈՒԹՅԱՆ ՎՐԱ ՃԱՌԱԳԱՅԹՄԱՆ ԱՂԲՅՈՒՐԻ ՆԱՅՏՆԱԲԵՐՈՒՄԸ

Աշխատանքում դիտարկվում է հարթության վրա ռադիոակտիվ աղբյուրի դիրքի և ճառագայթման սկավելու պահի նույնականացման խնդիրը: Ճառագայթման ակտերը ոչ համասեռ պուասոնյան գործընթացներ են՝ գրանցված  $K$  դետեկտորների միջոցով: Մենք համարում ենք, որ ազդանշանները հասնելու պահերը որոշվում են փոքր սխալներով: Այնուհետև օգտագործելով այս չափումները մենք կառուցում ենք աղբյուրի դիրքի և ճառագայթման մեկնարկի պահի գնահատականներ: Մենք ուսումնասիրում ենք այդ գնահատականների հավելթյունները մեծ ազդանշանների սահմանում և ցույց ենք տալիս դրանց հերևողականությունը: