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КАФЕДРА ВЫСШЕЙ МАТЕМАТИКИ

Integrals and Differential Equations

методические указания на английском языке
по дисциплине **«Математика»**

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Данные методические указания адресованы преподавателям и студентам технических ВУЗов для проведения аудиторных занятий и организации самостоятельной работы студентов при изучении материала из рассматриваемых разделов. Методические указания на английском языке «Integrals and Differential Equations» содержат необходимый материал по темам «Интегральное исчисление функции одной переменной», «Обыкновенные дифференциальные уравнения и системы», «Кратные интегралы», «Криволинейные интегралы», изучаемые студентами БрГТУ технических специальностей в курсе дисциплины «Математика». Теоретический материал сопровождается рассмотрением достаточного количества примеров и задач, при необходимости приводятся соответствующие иллюстрации. Для удобства пользования каждая тема разделена на три части: краткие теоретические сведения (определения, основные теоремы, формулы для расчетов); задания для аудиторной работы и задания для индивидуальной работы.

Данные методические указания являются продолжением серии методических разработок на английском языке коллектива авторов [1]-[5]. Практика использования разработок данной серии показала целесообразность её применения в процессе обучения студентов не только технических, но и экономических специальностей. Также были получены положительные отзывы об упомянутой серии от иностранных студентов.

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CONTENT

I INDEFINITE INTEGRAL

1.1 Indefinite Integrals

Definition A function $F(x)$ is called an ***antiderivative*** of $f(x)$ on an interval I if $F'(x) = f(x)$ for all x in I.

For instance, let $f(x) = x^2$. If $F(x) = 1/3 \cdot x^3$, then $F'(x) = x^2 = f(x)$. But the function $G(x) = 1/3 \cdot x^3 + \pi$ also satisfies $G'(x) = x^2$. Therefore, both $F(x)$ and $G(x)$ are antiderivatives of $f(x)$. Indeed, any function of the form $H(x) = 1/3 \cdot x^3 + C$, where C is a constant, is an antiderivative of $f(x)$.

Theorem If $F(x)$ is an antiderivative of $f(x)$ on an interval I, then the most general antiderivative of $f(x)$ on I is $F(x) + C$, where C is an arbitrary constant.

Definition The set of antiderivatives of $f(x)$ on an interval I is called an ***indefinite integral*** and is denoted by $\int f(x)dx = F(x) + C,\ C = const$.

Finding an indefinite integral of a function is called *integrating* a given function. This operation is the inverse of differentiation. So we can regard an indefinite integral as representing an entire *family* of functions (one antiderivative for each value of the constant *C*).

Basic integration rules

1. $d\left(\int f(x)dx\right) = f(x)dx$ or $\left(\int f(x)dx\right)' = f(x)$.
2. $\int dF(x) = F(x) + C$.
3. $\int c \cdot f(x)dx = c\int f(x)dx$.
4. $\int (f(x) \pm g(x))dx = F(x) \pm G(x) + C$.
5. If $\int f(x)dx = F(x) + C$ and $u = \varphi(x)$ is a differentiable function, then
$$\int f(u)du = F(u) + C.$$
6. $\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$.

Every differentiation formula, when read from right to left, gives rise to an antidifferentiation formula. We therefore determine the Table of Indefinite Integrals.

Any formula can be verified by differentiating the function on the right side and obtaining the integrand.

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$	2. $\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C$
3. $\int a^x dx = \frac{a^x}{\ln a} + C$	4. $\int e^x dx = e^x + C$
5. $\int \cos x dx = \sin x + C$	6. $\int \sin x dx = -\cos x + C$
7. $\int \frac{dx}{\cos^2 x} = tgx + C$	8. $\int \frac{dx}{\sin^2 x} = -ctgx + C$
9. $\int \frac{dx}{x} = \ln\lvert x \rvert + C$	10. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left\lvert \frac{x-a}{x+a} \right\rvert + C$
11. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} arctg \frac{x}{a} + C$	12. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C$
13. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left\lvert x + \sqrt{x^2 - a^2} \right\rvert + C$	14. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left\lvert x + \sqrt{x^2 + a^2} \right\rvert + C$
15. $\int chx\, dx = shx + C$	16. $\int shx\, dx = chx + C$
17. $\int \frac{dx}{ch^2 x} = thx + C$	18. $\int \frac{dx}{sh^2 x} = -cthx + C,\ x \neq 0$

Note 1 All integral formulas of the table of integrals remain valid if we substitute some differentiable function $\varphi(x)$ for the variable x in them.

The Simplest Methods of Integration

The simplest integration methods include finding indefinite integrals using the basic integration rules and a table of integrals, integrating by introducing a derivative under the sign of differential.

All integral formulas remain valid, if we substitute the variable x a certain differentiated function from x in them. The case of reducing the integral to tabular integral in question sometimes it sufficient to represent dx with one of the following formulas:

1. $dx = d(x+a)$ 2. $dx = \frac{1}{a} d(ax)$ 3. $dx = \frac{1}{a} d(ax+b)$.

Example 1 Find

a) $\int \left(4x^3 - 2\sqrt[3]{x^2} + \frac{2}{x^3} + 1 \right) dx$ *b)* $\int \frac{dx}{\sqrt{16x^2 + 9}}$ *c)* $\int \frac{dx}{(2x-1)^5} dx$

d) $\int \frac{dx}{\sin^2 6x}$ *e)* $\int \frac{dx}{x^2 + 2x + 3}$ *f)* $\int (4x^3 + 1)\cos(x^4 + x) dx$

Solution

a) $\int\left(4x^3-2\sqrt[3]{x^2}+\frac{2}{x^3}+1\right)dx=$(we will use the table of indefinite integrals)=

$$=4\cdot\int x^3dx-2\cdot\int x^{\frac{2}{3}}dx+2\cdot\int x^{-3}dx+\int dx==4\cdot\frac{x^4}{4}-2\cdot\frac{3}{5}x^{\frac{5}{3}}+2\cdot\frac{x^{-2}}{-2}+x+C=$$

$$=x^4-\frac{6}{5}x^{\frac{5}{3}}-\frac{1}{x^2}+x+C.$$

b) $\int\frac{dx}{\sqrt{16x^2+9}}=\int\frac{dx}{\sqrt{(4x)^2+3^2}}=\frac{1}{4}\int\frac{d(4x)}{\sqrt{(4x)^2+3^2}}=\frac{1}{4}\ln|4x+\sqrt{16x^2+9}|+C.$

c) $\int\frac{dx}{(2x-1)^5}dx=\int(2x-1)^{-5}dx=\frac{1}{2}\int(2x-1)^{-5}d(2x-1)=$

$$=\frac{1}{-8}(2x-1)^{-4}+C=-\frac{1}{8(2x-1)^4}+C.$$

d) $\int\frac{dx}{\sin^2 6x}=\frac{1}{6}\int\frac{d(6x)}{\sin^2 6x}=-\frac{1}{6}ctg6x+C.$

e) $\int\frac{dx}{x^2+2x+3}=\int\frac{d(x+1)}{(x+1)^2+2}=\frac{1}{\sqrt{2}}arctg\frac{x+1}{\sqrt{2}}+C.$

f) $\int(4x^3+1)\cos(x^4+x)dx=\int\cos(x^4+x)d(x^4+x)=\sin(x^4+x)+C$

In the following examples the method of introduction of derivative under the sign of differential will be used. It is based on the use of the formula $\phi'(x)dx=d(\phi(x))$, from which, in particular, it follows that

$$xdx=\frac{1}{2}\left(x^2\right)'dx=\frac{1}{2}d\left(x^2\right);\qquad x^2dx=\frac{1}{3}\left(x^3\right)'dx=\frac{1}{3}d\left(x^3\right);$$

$$\frac{dx}{x}=(\ln x)'dx=d(\ln x);\qquad \cos xdx=(\sin x)'dx=d(\sin x);$$

$$\sin xdx=-(\cos x)'dx=-d(\cos x);\qquad e^xdx=\left(e^x\right)'dx=d\left(e^x\right);$$

$$\frac{dx}{\cos^2 x}=(tgx)'dx=d(tgx);\qquad \frac{dx}{\sin^2 x}=-(ctgx)'dx=-d(ctgx);$$

$$\frac{dx}{1+x^2}=(arctgx)'dx=d(arctgx).$$

Example 2 Find

a) $\int x^2\sqrt{4+x^3}dx$ *b)* $\int\frac{dx}{(x+1)\ln(x+1)}$ *c)* $\int\frac{dx}{\arcsin x\sqrt{1-x^2}}$ *d)* $\int\frac{x-arctgx}{1+x^2}dx$

Solution

a) $\int x^2\sqrt{4+x^3}dx=\frac{1}{3}\int\left(4+x^3\right)^{\frac{1}{2}}\left(4+x^3\right)'dx=\frac{1}{3}\int\left(4+x^3\right)^{\frac{1}{2}}d(4+x^3)=$

$$=\frac{2}{9}\left(4+x^3\right)^{\frac{3}{2}}+C=\frac{2}{9}\sqrt{(4+x^3)^3}+C.$$

b) $\displaystyle\int\frac{dx}{(x+1)\ln(x+1)}=\int\frac{(\ln(x+1))'dx}{\ln(x+1)}=\int\frac{d(\ln(x+1))}{\ln(x+1)}=\ln|\ln(x+1)|+C$.

c) $\displaystyle\int\frac{dx}{\arcsin x\sqrt{1-x^2}}=\int\frac{(\arcsin x)'dx}{\arcsin x}=\int\frac{d(\arcsin x)}{\arcsin x}=\ln|\arcsin x|+C.$

d) $\displaystyle\int\frac{x-arctgx}{1+x^2}dx=\int\frac{xdx}{1+x^2}-\int\frac{arctgx}{1+x^2}dx=\frac{1}{2}\int\frac{d(x^2+1)}{x^2+1}-\int arctgx\,d(arctgx)=$

$$=\frac{1}{2}\ln\left(x^2+1\right)-\frac{1}{2}arctg^2x+C.$$

Exercise Set 1.1

In exercises 1 to 12 evaluate the integral.

$\displaystyle\int\frac{dx}{\sqrt{x^2-5}}$ $\displaystyle\int\frac{dx}{x^2-9}$ $\displaystyle\int\frac{dx}{\sqrt{3-3x^2}}$

$\displaystyle\int\frac{dx}{\sqrt{x^2+5}}$ $\displaystyle\int\frac{dx}{x+5}$ $\displaystyle\int\sqrt{x+15}\,dx$

$\displaystyle\int\frac{1}{2x+3}dx$ $\displaystyle\int\frac{dx}{3x^2+7}$ $\displaystyle\int\frac{dx}{\sqrt{4+x^2}}$

$\displaystyle\int\frac{dx}{\sqrt{8-x^2}}$ $\displaystyle\int\frac{dx}{\sqrt{5x-2}}$ $\displaystyle\int\frac{3\cdot2^x-2\cdot3^x}{5^x}dx$

In exercises 13 to 24 evaluate the integral.

13. $\displaystyle\int\left(5x^7-3\sqrt[5]{x^3}+\frac{3}{x^4}\right)dx$

14. $\displaystyle\int\left((1-2x)^2+2^x\cdot e^x\right)dx$

15. $\displaystyle\int\frac{\cos2x}{\cos^2x\cdot\sin^2x}dx$

16. $\displaystyle\int\left(\sqrt{x}+1\right)\left(x-\sqrt{x}+1\right)dx$

17. $\displaystyle\int\left(2\sin(1-6x)+5e^{7x+6}\right)dx$

18. $\displaystyle\int\left(\frac{3}{\sqrt{5x^2+4}}-\frac{1}{3x^2-4}\right)dx$

19. $\displaystyle\int\left(\frac{6}{(2+4x)^6}-5(3x+7)^7\right)dx$

20. $\displaystyle\int\left(3\sin x-2^x\cdot3^{2x}+\frac{3}{9+x^2}\right)dx$

21. $\displaystyle\int\left(2\sqrt{x^9}-\frac{10}{x^6}+\frac{4x^4}{\sqrt{x^3}}\right)dx$

22. $\displaystyle\int\left(9\sqrt{x^5}-\frac{1}{2x}-\frac{8\sqrt{x}}{x^2}\right)dx$

23. $\displaystyle\int\left(2\sqrt{x^5}-\frac{5}{x^4}+\frac{7x}{\sqrt{x^3}}\right)dx$

24. $\displaystyle\int\left(4\sqrt{x^3}-\frac{5}{x^3}-\frac{2x^2}{\sqrt{x^3}}\right)dx$

In exercises 25 to 38 evaluate the integral.

25. $\int \sqrt{\sin x} \cdot \cos x\, dx$

26. $\int 2x\sqrt{x^2+8}\, dx$

27. $\int \frac{x^4}{\sqrt{4+x^5}} dx$

28. $\int \frac{(arctg\, x)^2}{1+x^2} dx$

29. $\int e^x \sin e^x\, dx$

30. $\int e^{\sin^2 x} \sin 2x dx$

31. $\int \frac{\sqrt{1+\ln x}}{x} dx$

32. $\int \frac{3x-1}{x^2+9} dx$

33. $\int \frac{2x-3}{x^2-3x+8} dx$

34. $\int \frac{3x-1}{x-2} dx$

35. $\int \frac{x}{x+9} dx$

36. $\int \frac{(1+x)^2}{x^2+1} dx$

37. $\int \frac{x^4}{1-x} dx$

38. $\int \frac{xdx}{\sqrt{1+x^4}}$

Individual Tasks 1.1

1-10. Evaluate the integral.

	II.
1. $\int \frac{dx}{x^2+7}$	1. $\int \frac{dx}{x^2-10}$
2. $\int \left(\frac{3}{\sqrt{2x^2+4}} - \frac{1}{3x^2+4} \right) dx$	2. $\int \left(\frac{2}{\sqrt{4-x^2}} - \frac{1}{3x^2+1} \right) dx$
3. $\int \left(2\sin(1-6x) + 4e^{3+5x} \right) dx$	3. $\int \left(2\sin(1-8x) + 6e^{3+4x} \right) dx$
4. $\int \frac{2x}{x-3} dx$	4. $\int \frac{4x-3}{x-2} dx$
5. $\int \frac{dx}{x^2+4x+13}$	5. $\int \frac{dx}{x^2-2x+10}$
6. $\int \frac{2x-1}{x^2-3x+4} dx$	6. $\int \frac{2x+5}{x^2-2x-8} dx$
7. $\int \frac{x+3}{\sqrt{x^2-4}} dx$	7. $\int \frac{xdx}{2x^2+3}$
8. $\int \frac{2x-5arctg^2 x}{1+x^2} dx$	8. $\int \frac{arctg^3 4x}{1+16x^2} dx$
9. $\int \sqrt{\arcsin 2x} \cdot \frac{dx}{\sqrt{1-4x^2}}$	9. $\int \arcsin^3 2x \cdot \frac{dx}{\sqrt{1-4x^2}}$
10. $\int \frac{x^2}{1+x^6} dx$	10. $\int \frac{e^x}{2e^x-1} dx$

1.2 Replacement of Variable in the Indefinite Integral

The method of integration by the replacement of a variable is based on the use of the formula

$$\int f(x)dx = \int g(\varphi(x))\varphi'(x)dx = \int g(t)dt .$$

Note 1 The above considered operation of introducing the derivative $\varphi'(x)$ under the sign of the differential in the integral $\int g\left(\varphi(x)\right)\varphi'(x)dx$ is equivalent to replacing the variable $\varphi(x) = t$.

Example 1 Find

$$\int x\sqrt{x-1}\,dx \qquad \int \left(1+\sin x\right)^{\frac{1}{3}}\cos x\,dx \qquad \int \frac{\sqrt[4]{x+1}+2}{\sqrt{x+1}}dx$$

Solution

a) $\displaystyle\int x\sqrt{x-1}\,dx = 2\int\left(t^2\ +1\right)t^2\,dt = 2\int(t^4+t^2)dt = 2\int t^4dt + 2\int t^2dt = \frac{2}{5}t^5+\frac{2}{3}t^3+C =$

$\displaystyle= \frac{2}{15}t^3(3t^2+5)+C = \frac{2}{15}\sqrt{(x-1)^3}\,(3x+2)+C,$

where $x-1=t^2$; $dx = d(t^2+1) = 2tdt$.

b) $\displaystyle\int (1+\sin x)^{\frac{1}{3}}\cos x dx = \int t^{\frac{1}{3}}dt = \frac{3}{4}t^{\frac{4}{3}}+C = \frac{3}{4}(1+\sin x)^{\frac{4}{3}}+C,$

where $t = 1+\sin x$.

c) $\displaystyle\int \frac{\sqrt[4]{x+1}+2}{\sqrt{x+1}}dx = \left|\begin{matrix} x+1=t^4, & dx=4t^3dt \\ x=t^4-1. & \end{matrix}\right| =$

$\displaystyle= \int\frac{t+2}{t^2}4t^3dt = 4\int(t^2+2t)dt = \frac{4}{3}t^3+4t^2+C = \frac{4}{3}(x+1)^{\frac{3}{4}}+4\sqrt{x+1}+C.$

Consider the usage of the replacement of a variable when some functions containing a square trinomial must be integrated. To find the integrals of the form $\displaystyle\int\frac{Ax+B}{ax^2+bx+c}dx,\ \int\frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$ the square trinomial in the denominator of the integrand can be written in the following form:

$$ax^2+bx+c = a\left(x^2+2x\frac{b}{2a}+\frac{b^2}{4a^2}\right)+c-\frac{b^2}{4a} = a\left(x+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a}.$$

Then the replacement of the variable $x+\dfrac{b}{2a}=t,\ x=t-\dfrac{b}{2a},\ dx=dt$ can be made.

Example 2 Find

$$a)\ \int\frac{xdx}{2x^2+2x+5} \qquad b)\ \int\frac{3x-1}{\sqrt{x^2-4x+8}}dx$$

Solution

a) $\displaystyle\int\frac{xdx}{2x^2+2x+5}=\frac{1}{2}\int\frac{xdx}{x^2+x+\frac{5}{2}}=\frac{1}{2}\int\frac{xdx}{\left(x+\frac{1}{2}\right)^2+\frac{9}{4}}=$

$$=\left[\begin{matrix} x+\frac{1}{2}=t, & x=t-\frac{1}{2} \\ dx=dt & \end{matrix}\right]=\frac{1}{2}\int\frac{t-\frac{1}{2}}{t^2+\frac{9}{4}}dt.$$

We divide the obtained integral by the algebraic sum of two integrals. The first integral is found by introducing a derivative under the sign of the differential, and the second integral is a tabular one.

$$\frac{1}{2}\int\frac{t-\frac{1}{2}}{t^2+\frac{9}{4}}dt=\frac{1}{2}\int\frac{tdt}{t^2+\frac{9}{4}}-\frac{1}{4}\int\frac{dt}{t^2+\frac{9}{4}}=\frac{1}{4}\left(\int\frac{d\left(t^2+\frac{9}{4}\right)}{t^2+\frac{9}{4}}-\int\frac{dt}{t^2+\frac{9}{4}}\right)=$$

$$=\frac{1}{4}\left(\ln\left(t^2+\frac{9}{4}\right)-\frac{2}{3}arctg\frac{2t}{3}\right)+C=\frac{1}{4}\ln\left(x^2+x+\frac{5}{2}\right)-\frac{1}{6}arctg\frac{2x+1}{3}+C.$$

b) $\displaystyle\int\frac{3x-1}{\sqrt{x^2-4x+8}}dx=\int\frac{3x-1}{\sqrt{(x-2)^2+4}}dx=\left[\begin{matrix} x-2=t, & x=t+2 \\ dx=dt & \end{matrix}\right]=$

$$=\int\frac{3t+5}{\sqrt{t^2+4}}dt=3\int\frac{tdt}{\sqrt{t^2+4}}+5\int\frac{dt}{\sqrt{t^2+4}}=\frac{3}{2}\int\frac{d\left(t^2+4\right)}{\sqrt{t^2+4}}+5\int\frac{dt}{\sqrt{t^2+4}}=$$

$$=3\sqrt{t^2+4}+5\ln|t+\sqrt{t^2+4}|+C=3\sqrt{x^2-4x+8}+5\ln|x-2+\sqrt{x^2-4x+8}|+C.$$

Exercise Set 1.2

In exercises 1 to 15 evaluate the integral.

1. $\displaystyle\int\frac{dx}{1+\sqrt{x+3}}$
2. $\displaystyle\int e^{\sqrt{4-3x}}\frac{dx}{\sqrt{4-3x}}$
3. $\displaystyle\int\frac{\sqrt[3]{x}+2}{\sqrt{x}}dx$
4. $\displaystyle\int\frac{x^3dx}{\sqrt{x-1}}$
5. $\displaystyle\int\frac{\sqrt{1+\ln x}}{x\ln x}dx$
6. $\displaystyle\int\frac{dx}{x\sqrt{2x+1}}$
7. $\displaystyle\int\frac{x^2dx}{\sqrt{4-x^2}}$
8. $\displaystyle\int\frac{dx}{x^2+2x+3}$
9. $\displaystyle\int\frac{dx}{4x^2+4x+5}$
10. $\displaystyle\int\frac{(x+2)dx}{2x^2+6x+4}$
11. $\displaystyle\int\frac{dx}{\sqrt{1-(2x+3)^2}}$
12. $\displaystyle\int\frac{dx}{\sqrt{4x-3-x^2}}$
13. $\displaystyle\int\frac{(2x+1)dx}{\sqrt{2x^2+4x+6}}$
14. $\displaystyle\int\frac{(3x-5)dx}{\sqrt{2x^2-8x+6}}$
15. $\displaystyle\int\frac{xdx}{\sqrt{x^2+8x+5}}$

Individual Tasks 1.2

1-8. Evaluate the integral.

	II.
1. $\int e^{\sqrt{5+x}} \cdot \frac{dx}{\sqrt{5+x}}$	1. $\int \cos\sqrt{3-5x} \cdot \frac{dx}{\sqrt{3-5x}}$
2. $\int x\sqrt{1-2x}dx$	2. $\int x\sqrt{6-7x}dx$
3. $\int \frac{dx}{e^x+1}$	3. $\int \frac{dx}{\sqrt{e^x-1}}$
4. $\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}dx$	4. $\int \frac{1}{\sqrt{x}+\sqrt{\sqrt{x}}}dx$
5. $\int \frac{dx}{x^2-8x-9}$	5. $\int \frac{dx}{x^2+4x+6}$
6. $\int \frac{xdx}{x^2+6x+5}$	6. $\int \frac{(x-1)dx}{4x^2+6x+4}$
7. $\int \frac{dx}{\sqrt{x^2+4x+20}}$	7. $\int \frac{dx}{\sqrt{-x^2+8x+5}}$
8. $\int \frac{(2x-1)dx}{\sqrt{-2x^2+8x+6}}$	8. $\int \frac{(4x+1)dx}{\sqrt{x^2-2x+4}}$

1.3 Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called the rule for *integration by parts.*

The method of integration by parts is based on the use of the formula

$$\int u(x)dv(x) = u(x)v(x) - \int v(x)du(x) \quad or \quad \int udv = uv - \int vdu,$$

where $u = u(x)$, $v = v(x)$ are continuously differentiated functions.

The application of a formula is expedient, when under the integral sign there is a function work of different classes. In certain cases it is necessary to use the formula of integration by parts several times.

Example 1 Find

d) $\int (2x+1)\cos 3xdx$ *e)* $\int x^2 \sin xdx$

f) $\int \ln xdx$ *g)* $\int 2x\, arctgx\, dx$

Solution

a) $$\int (2x+1)\cos 3xdx = \left| \begin{matrix} u = 2x+1, & du = 2dx \\ dv = \cos 3xdx, & v = \frac{1}{3}\sin 3x \end{matrix} \right| = \frac{2x+1}{3}\sin 3x - \frac{2}{3}\int \sin 3xdx =$$

$$= \frac{2x+1}{3}\sin 3x + \frac{2}{9}\cos 3x + C.$$

b) $\int x^2 \sin x\,dx = \begin{vmatrix} u = x^2, & du = 2x\,dx \\ dv = \sin x\,dx, & v = -\cos x \end{vmatrix} = -x^2\cos x + 2\int x\cos x\,dx =$

$= \begin{vmatrix} u = x, & du = dx \\ dv = \cos x dx, & v = \sin x \end{vmatrix} = -x^2\cos x + 2\left(x\sin x - \int \sin x\,dx\right) =$

$= -x^2\cos x + 2x\sin x + 2\cos x + C$.

c) $\int \ln x dx = \begin{vmatrix} u = \ln x, & du = \dfrac{dx}{x} \\ dv = dx, & v = x \end{vmatrix} = x\ln x - \int x\dfrac{dx}{x} = x\ln x - \int dx =$

$= x\ln x - x + C = x(\ln x - 1) + C$.

d) $\int 2x\,arctgx\,dx = \begin{vmatrix} u = arctgx, & du = \dfrac{dx}{1+x^2} \\ dv = 2xdx, & v = x^2 \end{vmatrix} = x^2 arctgx - \int \dfrac{x^2 dx}{1+x^2} =$

$= x^2 arctgx - \int \dfrac{(x^2+1)-1}{1+x^2}dx = x^2 arctgx - \int dx + \int \dfrac{dx}{1+x^2} =$

$= x^2 arctgx - x + arctgx + C$.

Exercise Set 1.3

In exercises 1 to 15 evaluate the integral.

1. $\int xe^{-7x}dx$	*2.* $\int \left(x^2 - 2x + 5\right)e^{-x}\,dx$	*3.* $\int x\cos 3x\,dx$
4. $\int x\cos\left(3x+1\right)dx$	*5.* $\int (1-x^2)\sin x dx$	*6.* $\int (x^2 - 4x)\cos x dx$
7. $\int \arccos x dx$	*8.* $\int (1-3x)\ln(4x)\,dx$	*9.* $\int \ln\left(x-3\right)dx$
10. $\int x\,arctg\,2x\,dx$	*11.* $\int \cos(\ln x)dx$	*12.* $\int x^3 e^{-x^2}dx$
13. $\int \dfrac{\arcsin x dx}{x^2}$	*14.* $\int e^{\sqrt{x}}\,dx$	*15.* $\int \dfrac{x\cos x}{\sin^2 x}dx$

Individual Tasks 1.3

1-7. Evaluate the integral.

	II.
1. $\int (3-x)\sin 4x dx$	*1.* $\int (x-2)\cos 5x dx$
2. $\int (x^2+3)e^{-2x}dx$	*2.* $\int (x^2-4)e^{3x}dx$
3. $\int x\ln(2x)dx$	*3.* $\int (x-3)\ln(5x)dx$
4. $\int \arcsin 2x dx$	*4.* $\int \arccos 3x dx$
5. $\int x^3 e^{-x^2}dx$	*5.* $\int \ln\left(1+x^2\right)dx$

6. $\int \frac{xdx}{\sin^2 x}$	6. $\int \frac{x^2 arctgx}{1+x^2} dx$
7. $\int \sqrt{x^2+169}\, dx$	7. $\int e^x \sin x\, dx$

1.4 Integration of Rational Functions

The relation of two polynomials is called ***the rational function*** (*rational fraction*), i.e., the fraction of the form $\frac{P_n(x)}{Q_m(x)}$ where $P_n(x)$ is the polynomial of the degree n, $Q_m(x)$ is the polynomial of the degree m. If $n \geq m$, then that rational fraction is called *incorrect*, if $n < m$, then that rational fraction is called *correct*.

Theorem Any incorrect rational fraction can be uniquely represented in the form of the sum of a polynomial and correct rational fraction

$$\frac{P_n(x)}{Q_m(x)} = M_{n-m}(x) + \frac{R_s(x)}{Q_m(x)}.$$

Example 1 The rational fraction of the form $\frac{x^5-3x^4+5x^3-1}{x^3-2x}$ is incorrect.

Solution Since the degree of numerator $(n=5)$ is more than the degree of denominator $(m=3)$, then the given rational fraction is incorrect. We divide the polynomial of the numerator "by long division" into the polynomial of denominator. Then in the quotient we obtain a polynomial $M(x)$, and in the remainder-a polynomial $R(x)$.

$$\frac{x^5-3x^4+5x^3-1}{x^3-2x} = x^2-3x+7-\frac{6x^2-14x+1}{x^3-2x}.$$

The rational fractions of the following forms are called *the simplest rational fractions*:

1. $\frac{A}{x-a}$.	2. $\frac{A}{(x-a)^m}$, $m>1$, $m \in N$.
3. $\frac{Ax+B}{x^2+px+q}$, $D=p^2-4q<0$.	4. $\frac{Ax+B}{(x^2+px+q)^m}$, $D<0$, $m>1, m \in N$.

The integration of such functions:

1. $\int \frac{A}{x-a} dx = A \ln|x-a| + C$.

2. $\int \frac{A}{(x-a)^m} dx = A\int (x-a)^{-m} d(x-a) = \frac{A}{-m+1}(x-a)^{-m+1} + C$.

3. $\int \frac{Ax+B}{ax^2+bx+c} dx$.

It is necessary to isolate the complete square in the denominator of integrand in a square trinomial

$$ax^2+bx+c=a\left(x^2+2x\frac{b}{2a}+\frac{b^2}{4a^2}\right)+c-\frac{b^2}{4a}=a\left(x+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a}.$$

The replacement variable $x+\dfrac{b}{2a}=t,\ x=t-\dfrac{b}{2a},\ dx=dt$ can be made (see chapter 1.2).

4. $\dfrac{Ax+B}{(x^2+px+q)^m},\ D<0,\ m>1,\ m\in N$. This rational fraction can be evaluated by using the recurrent formula.

Theorem It is possible to uniquely represent each correct rational function $\dfrac{P_n(x)}{Q_m(x)}$ in the form of the sums of the simplest rational functions.

We factor the denominator as $Q(x)=(x-a)^k(x-b)(x^2+px+q)(x^2+px+q)^m$.

Then the rational function can be represented in the form

$$\frac{P(x)}{Q(x)}=\frac{A_1}{(x-a)^k}+\frac{A_2}{(x-a)^{k-1}}+\cdots+\frac{A_k}{x-a}+\frac{B}{x-b}+\frac{C_1x+D_1}{x^2+px+q}+\frac{C_2x+D_2}{(x^2+px+q)^2}+$$
$$+\frac{C_{m-1}x+D_{m-1}}{(x^2+px+q)^{m-1}}+\cdots+\frac{C_mx+D_m}{\left(x^2+px+q\right)^m},$$

where $A_1,A_2,\cdots,A_k,B,C,D,C_1,D_1,\cdots,C_m,D_m$ are real numbers, which must be determined.

In the obtained decomposition we reduce both parts to the common denominator. We make level the numerators. The obtained equation is correct for any x. We find unknown coefficients either by *the method of particular values* or *equalizing coefficients with the identical degrees* x, or combining these two methods.

Example 2 Find

a) $I_1=\displaystyle\int\frac{2x^2-x+3}{x^3+x^2-2x}dx$ *b)* $I_2=\displaystyle\int\frac{x^2+4}{x^3(x+1)^2}dx$ *c)* $I_3=\displaystyle\int\frac{2x^2-3x+1}{x^3+1}dx$

Solution

a) We factor the denominator as

$$Q(x)=x^3+x^2-2x=x(x^2+x-2)=x(x-1)(x+2).$$

The partial fraction decomposition of the integrand has the form:

$$\frac{2x^2-x+3}{x^3+x^2-2x}=\frac{2x^2-x+3}{x(x-1)(x+2)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+2}=$$
$$=\frac{A(x-1)(x+2)+Bx(x+2)+Cx(x-1)}{x(x-1)(x+2)}$$

We make level the numerators

$$2x^2-x+3=A(x-1)(x+2)+Bx(x+2)+Cx(x-1).$$

The polynomials in the last equation are identical, so their coefficients must be equal. Let's choose values of x that simplify the equation.

$$x=0: \quad 3=A(-1)2, \qquad A=-3/2$$

$$x=1: \quad 2-1+3=3B, \qquad B=4/3$$

$$x=-2: \quad 8+2+3=C(-2)(-3), \qquad C=13/6$$

The expansion of the rational function into partial fractions was obtained

$$I_1=\int\frac{2x^2-x+3}{x^3+x^2-2x}dx=\int\left(-\frac{3}{2x}+\frac{4}{3(x-1)}+\frac{13}{6(x+2)}\right)dx=$$

$$=-\frac{3}{2}\ln|x|+\frac{4}{3}\ln|x-1|+\frac{13}{6}\ln|x+2|+C.$$

b) The partial fraction decomposition of the integrand has the form:

$$\frac{x^2+4}{x^3(x+1)^2}=\frac{A}{x^3}+\frac{B}{x^2}+\frac{C}{x}+\frac{D}{(x+1)^2}+\frac{E}{x+1}=\frac{(A+Bx+Cx^2)(x+1)^2+(D+E(x+1))x^3}{x^3(x+1)^2}$$

We make level the numerators

$$x^2+4=(A+Bx+Cx^2)(x^2+2x+1)+(D+Ex+E)x^3,$$

$$x^2+4=(C+E)x^4+(B+2C+D+E)x^3+(A+2B+C)x^2+(B+2A)x+A.$$

The polynomials in the last equation are identical, so their coefficients must be equal. The coefficients of polynomials are equal and the constant terms are equal. This gives the following system of equations for A, B and C.

$$x^4: \quad C+E=0, \qquad E=-C=-13.$$

$$x^3: \quad B+2C+D+E=0, \qquad D=-B-2C-E=8-26+13=-5.$$

$$x^2: \quad A+2B+C=1, \qquad C=1-A-2B=1-4+16=13.$$

$$x^1: \quad B+2A=0, \qquad B=-2A=-8.$$

$$x^0: \quad A=4. \qquad A=4.$$

The expansion of the rational function into partial fractions was obtained:

$$\frac{x^2+4}{x^3(x+1)^2}=\frac{4}{x^3}-\frac{8}{x^2}+\frac{13}{x}-\frac{5}{(x+1)^2}-\frac{13}{x+1}$$

$$I_2=-\frac{2}{x^2}+\frac{8}{x}+13\ln|x|+\frac{5}{x+1}-13\ln|x+1|+C.$$

c) The partial fraction decomposition of the integrand has the form

$$\frac{2x^2-3x+1}{x^3+1}=\frac{2x^2-3x+1}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}.$$

The polynomials in the last equation are identical, so their coefficients must be equal. The coefficients of polynomials are equal and the constant terms are equal. This gives the following system of equations for A, B and C.

$$\begin{cases} A+B=2 \\ -A+B+C=-3 \\ A+C=1 \end{cases} \Leftrightarrow \begin{cases} B=2-A \\ C=1-A \\ -A+2-A+1-A=-3 \end{cases} \Leftrightarrow \begin{cases} B=2-A \\ C=1-A \\ -3A=-6 \end{cases} \Leftrightarrow \begin{cases} A=2 \\ B=0 \\ C=-1 \end{cases}$$

The expansion of the rational function into the partial fractions was obtained:

$$I_3=\int\frac{2x^2-3x+1}{x^3+1}dx=\int\left(\frac{2dx}{x+1}-\frac{dx}{x^2-x+1}\right)=2\ln|x+1|-\int\frac{d\left(x-\frac{1}{2}\right)}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}=$$

$$=2\ln|x+1|-\frac{2}{\sqrt{3}}arctg\frac{2x-1}{\sqrt{3}}+C.$$

Exercise Set 1.4

In exercises 1 to 12 present rational fractions as the sum of the simplest fractions, without calculating the coefficients.

1. $\frac{4x^3-2x^2+6x-1}{(x^2-1)^2}$

2. $\frac{2x^2-7}{(x^3+16x)(x-3)}$

3. $\frac{6x^2-3x+7}{(x+2)(x^2+4)^2}$

4. $\frac{7x^3+x^2-4x+5}{(x^2-4)^2}$

5. $\frac{3x^2-5x+7}{(x+2)(x^2+1)}$

6. $\frac{2x^2-4}{(x+1)^2(x+3)^3}$

7. $\frac{4x^3-x^2+5x-1}{(x^2-4x+3)^2(x-5)}$

8. $\frac{x^3+9}{x^3-5x^2+6x}$

9. $\frac{3x-11}{\left(x^3+4x\right)\left(x^2+8x+18\right)}$

10. $\frac{2x^2-4x+6}{\left(x+2\right)^2\left(x^2-1\right)}$

11. $\frac{3x+5}{x^4-x^2}$

12. $\frac{4x+1}{(x-3)^2(x^2+2x+5)}$

In exercises 13 to 24 evaluate the integral.

13. $\int\frac{x-4}{x^2-5x+6}dx$

14. $\int\frac{x^5+x^4-8}{x^3-4x}dx$

15. $\int\frac{x^3+1}{x^3-x^2}dx$

16. $\int\frac{x^2-2x+3}{\left(x-1\right)\left(x^3-4x^2+3x\right)}dx$

17. $\int\frac{x^2}{x^4-1}dx$

18. $\int\frac{x^4-4x^3+5x^2+10x-10}{x^3-3x^2+x+5}dx$

19. $\int\frac{2x}{\left(x+1\right)\left(x^2+1\right)^2}dx$

20. $\int\frac{dx}{\left(x-1\right)\left(x+2\right)\left(x+3\right)}$

21. $\int \frac{2x^2-3x-3}{(x-1)(x^2-2x+5)}dx$ 22. $\int \frac{4dx}{x\left(x^2+4\right)}$

23. $\int \frac{x^2-3x+2}{x\left(x^2+2x+1\right)}dx$ 24. $\int \frac{x^3+1}{x^3-9x}dx$

Individual Tasks 1.4

1-6. Evaluate the integral.

	II.
1. $\int \frac{x^2+x-8}{x^3-4x}dx$	1. $\int \frac{2x^2+6x+7}{x^3-1}dx$
2. $\int \frac{2x-3}{(x^2-1)(x+2)}dx$	2. $\int \frac{6x^2-12x+6}{(x-2)^3}dx$
3. $\int \frac{2x^2+x+3}{x^3+2x^2-3x}dx$	3. $\int \frac{x-1}{x^3+2x^2-8x}dx$
4. $\int \frac{x^2-2x+6}{(x+1)(x-2)^3}dx$	4. $\int \frac{3x^2-6x+7}{\left(x+1\right)^2\left(x-2\right)}dx$
5. $\int \frac{3x+31}{\left(x^3+3x^2+3x+1\right)x}dx$	5. $\int \frac{x^4+1}{x^3-x^2+x-1}dx$
6. $\int \frac{x-1}{x^4-13x^2+36}dx$	6. $\int \frac{2x+1}{x^4-5x^2+4}dx$

1.5 Integration of Nonrational Functions

Let us examine such nonrational functions, whose integration is reduced with the aid of the specific replacement of the variable of integration to the integration of some rational functions.

1) If $\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$.

It is necessary to isolate the complete square in the denominator of integrand in the square trinomial

$$ax^2+bx+c=a\left(x^2+2x\frac{b}{2a}+\frac{b^2}{4a^2}\right)+c-\frac{b^2}{4a}=a\left(x+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a}.$$

The replacement variable $x+\frac{b}{2a}=t,\ x=t-\frac{b}{2a},\ dx=dt$ can be made (see chapter 1.2).

Note 1 If $\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx$, where $P_n(x)$ is the polynomial of the degree n, then the following formula can be used:

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx=Q_{n-1}(x)\cdot\sqrt{ax^2+bx+c}+\lambda\int \frac{dx}{\sqrt{ax^2+bx+c}},$$

where $Q_{n-1}(x)$ is the polynomial of the degree $n-1$ with unknown coefficients and λ is some unknown coefficient whose must be determined by the method of particular values or equalizing coefficients with the identical degrees x. Differentiating both sides of the equality, the identity from which the unknown coefficients are found will be obtained:

$$\frac{P_n(x)}{\sqrt{ax^2+bx+c}} = \left(Q_{n-1}(x)\cdot\sqrt{ax^2+bx+c}\right)' + \frac{\lambda}{\sqrt{ax^2+bx+c}}.$$

2) If $\int \frac{dx}{(x-\alpha)^n\sqrt{ax^2+bx+c}}$, the substitution $\frac{1}{x-\alpha}=t$ can be used.

3) If $\int R(x,\sqrt[k]{x},\sqrt[m]{x},\cdots,\sqrt[s]{x})dx$, where R is the rational function of its arguments, the substitution $x=t^n$ can be used (n- all least common multiple indices $k,m,\cdots,s$).

4) If $\int R(x,\sqrt[n]{ax+b})dx$, the substitution $ax+b=t^n$ can be used.

5) If $\int R(\sqrt{x^2-a^2},x)dx;\ \int R(\sqrt{x^2+a^2},x)dx;\ \int R(\sqrt{a^2-x^2},x)dx$, where R is the rational function of its arguments, we must use trigonometric substitution.

In the following table we list trigonometric substitutions that are effective for the given radical expressions because of the specified trigonometric identities.

Table of trigonometric substitutions

Expression	*Substitution*	*Identity*
$\sqrt{a^2-x^2}$	$x=a\sin\theta,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$	$1-\sin^2\theta=\cos^2\theta$
$\sqrt{a^2+x^2}$	$x=atg\theta,\ -\frac{\pi}{2}<\theta<\frac{\pi}{2}$	$1+tg^2\theta=\frac{1}{\cos^2\theta}$
$\sqrt{x^2-a^2}$	$x=\frac{a}{\cos^2\theta},\ 0\le\theta<\frac{\pi}{2}$	$1+tg^2\theta=\frac{1}{\cos^2\theta}$

In each case the restriction on θ is imposed to ensure that the function that defines the substitution is one-to-one.

6) If $\int x^m\left(a+bx^n\right)^p dx$, where $m,n,p\in Q;\ a,b\in R$, then the integrand is called ***differential binomial***. The integral of it is reduced to the integrals of rational functions in the following three cases:

a) If $p\in Z$, then the substitution $x=t^N$ can be used (N is the common denominator of m and n).

b) If $p\notin Z,\ \frac{m+1}{n}\in Z$, then the substitution $a+bx^n=t^N$ can be used (N is the denominator of p).

c) If $p \notin Z,\ \frac{m+1}{n} \notin Z,\left(\frac{m+1}{n}+p\right) \in Z$, then the substitution $a+bx^n = x^n t^N$ can be used (N is the denominator of p).

If $p \notin Z,\ \frac{m+1}{n} \notin Z,\left(\frac{m+1}{n}+p\right) \notin Z$, then the integrals of this type cannot be expressed by a finite combination of elementary functions.

Example 1 Find

a) $\int \frac{3x-1}{\sqrt{x^2-4x+8}}dx$ *b)* $\int \frac{\sqrt{x}dx}{x-\sqrt[3]{x^2}}$

c) $\int \frac{xdx}{\sqrt[3]{(x+1)^4}-\sqrt[6]{(x+1)^5}}$ *d)* $\int \frac{x^3dx}{\sqrt{2-x^2}}$

e) $\int \frac{x^2}{\sqrt{1-2x-x^2}}dx$ *f)* $\int \frac{\sqrt[3]{\sqrt[4]{x}+1}dx}{\sqrt{x}}$

Solution

a) $$\int \frac{3x-1}{\sqrt{x^2-4x+8}}dx = \int \frac{3x-1}{\sqrt{(x-2)^2+4}}dx = \left| \begin{matrix} x-2=t, & dx=dt \\ x=t+2. & \end{matrix} \right| = \int \frac{3t+5}{\sqrt{t^2+4}}dt =$$

$$= 3\int \frac{tdt}{\sqrt{t^2+4}} + 5\int \frac{dt}{\sqrt{t^2+4}} = \frac{3}{2}\int \frac{d(t^2+4)}{\sqrt{t^2+4}} + 5\int \frac{dt}{\sqrt{t^2+4}} = 3\sqrt{t^2+4} + 5\ln|t+\sqrt{t^2+4}| + C =$$

$$= 3\sqrt{x^2-4x+8} + 5\ln|x-2+\sqrt{x^2-4x+8}| + C.$$

b) $$\int \frac{\sqrt{x}dx}{x-\sqrt[3]{x^2}} = \left| \begin{matrix} x=t^6, & dx=6t^5dt \\ \sqrt{x}=t^3, & \sqrt[3]{x}=t^2 \end{matrix} \right| = \int \frac{t^3 6t^5dt}{t^6-t^4} = 6\int \frac{t^8}{t^4(t^2-1)}dt = 6\int \frac{t^4}{t^2-1}dt =$$

$$= 6\int \frac{(t^4-1)+1}{t^2-1}dt = 6\int \left(t^2+1+\frac{1}{t^2-1}\right)dt = 2t^3+6t+3\ln\left|\frac{t-1}{t+1}\right| + C =$$

$$= 2\sqrt{x} + 6\sqrt[6]{x} + 3\ln\left|\frac{\sqrt[6]{x}-1}{\sqrt[6]{x}+1}\right| + C.$$

c) $$\int \frac{xdx}{\sqrt[3]{(x+1)^4}-\sqrt[6]{(x+1)^5}} = \left| \begin{matrix} x+1=t^6, & dx=6t^5dt \\ \sqrt[6]{x+1}=t, & \sqrt[3]{x+1}=t^2 \end{matrix} \right| = \int \frac{(t^6-1)6t^5dt}{t^8-t^5} =$$

$$= 6\int \frac{(t^6-1)t^5dt}{t^5(t^3-1)} = 6\int (t^3+1)dt = 6\left(\frac{t^4}{4}+t\right) + C = \frac{3}{2}\sqrt[3]{(x+1)^2} + 6\sqrt[6]{x+1} + C.$$

d) We must use trigonometric substitution $x=\sqrt{2}\sin t$.

$$\int \frac{x^3dx}{\sqrt{2-x^2}} = \left| \begin{matrix} x=\sqrt{2}\sin t, \\ dx=\sqrt{2}\cos t\,dt, \\ \sqrt{2-x^2}=\sqrt{2}\cos t \end{matrix} \right| = \int \frac{\left(\sqrt{2}\right)^3 \sin^3 t\cdot\sqrt{2}\cos t}{\sqrt{2}\cos t}dt = 2\sqrt{2}\int \sin^3 t\,dt =$$

$$= -2\sqrt{2}\int (1-\cos^2 t)\,d(\cos t) = -2\sqrt{2}\left(\cos t + \frac{\cos^3 t}{3}\right) + C =$$

$$= -2\sqrt{2-x^2} + \frac{\sqrt{(2-x^2)^3}}{3} + C.$$

e) $I = \int \frac{x^2}{\sqrt{1-2x-x^2}}dx = (Ax+B)\sqrt{1-2x-x^2} + \lambda\int \frac{dx}{\sqrt{1-2x-x^2}}.$

Differentiating both sides of the equality, the following identity will be obtained

$$\frac{x^2}{\sqrt{1-2x-x^2}} = A\sqrt{1-2x-x^2} + (Ax+b)\frac{-2-2x}{2\sqrt{1-2x-x^2}} + \frac{\lambda}{\sqrt{1-2x-x^2}}, \text{ i.e.}$$

$$x^2 \equiv A\left(1-2x-x^2\right) + \left(Ax+B\right)\left(-1-x\right) + \lambda;$$

$$x^2 \equiv A - 2Ax - Ax^2 - Ax - B - Ax^2 - Bx + \lambda.$$

Compare coefficients with the same degrees

$$x^2:\quad -A-A=1, \qquad A=-\frac{1}{2}.$$

$$x^1:\quad -2A-A-B=0, \qquad B=\frac{3}{2}.$$

$$x^0:\quad A-B+\lambda=0. \qquad \lambda=2.$$

Therefore $I = \int \frac{x^2}{\sqrt{1-2x-x^2}}dx = \left(-\frac{1}{2}x+\frac{3}{2}\right)\sqrt{1-2x-x^2} + 2\int \frac{dx}{\sqrt{2-\left(x+1\right)^2}} =$

$$= \left(-\frac{1}{2}x+\frac{3}{2}\right)\sqrt{1-2x-x^2} + 2\arcsin\frac{x+1}{\sqrt{2}} + C.$$

f) Since $I = \int \frac{\sqrt[3]{\sqrt[4]{x}+1}\,dx}{\sqrt{x}} = \int x^{-\frac{1}{2}}\cdot\left(1+x^{\frac{1}{4}}\right)^{\frac{1}{3}}dx,$

then $m=-\frac{1}{2}, n=\frac{1}{4}, p=\frac{1}{3}, \frac{m+1}{n}=2$. The following substitution must be used: $\sqrt[4]{x}+1=t^3$. Therefore $x=\left(t^3-1\right)^4, dx=4\left(t^3-1\right)3t^2dt, t=\sqrt[3]{\sqrt[4]{x}+1}$ and

$$I = \int \frac{\sqrt[3]{\sqrt[4]{x}+1}\,dx}{\sqrt{x}} = \int \frac{t}{\left(t^3-1\right)^2}\cdot 12t^2\left(t^3-1\right)^3dt = 12\int\left(t^6-t^3\right)dt =$$

$$= 12\cdot\frac{t^7}{7} - 12\cdot\frac{t^4}{4} + C = \frac{12}{7}\left(\sqrt[4]{x}+1\right)^{\frac{7}{3}} - 3\left(\sqrt[4]{x}+1\right)^{\frac{4}{3}} + C.$$

Exercise Set 1.5

In exercises 1 to 15 evaluate the integral.

1. $\int \frac{2x+7}{\sqrt{x^2+5x-4}}dx$ *2.* $\int \frac{3x+7}{\sqrt{-x^2+4x+8}}dx$ *3.* $\int \frac{x^2-2}{\sqrt{x^2+2x-3}}dx$

4. $\int \frac{x^2-x+7}{\sqrt{-x^2+2x+3}}dx$ *5.* $\int \frac{dx}{\sqrt{x}+\sqrt[4]{x}}$ *6.* $\int \frac{\sqrt{x}}{\sqrt[3]{x^2}-\sqrt[4]{x}}dx$

7. $\int \sqrt{\frac{1-x}{1+x}}\frac{dx}{x}$ *8.* $\int x^5\sqrt[3]{\left(1+x^3\right)^2}dx$ *9.* $\int \frac{\sqrt{1-x^2}}{x}dx$

10. $\int \frac{dx}{x^2\sqrt{x^2-1}}$ *11.* $\int \frac{\sqrt{x^2+4}}{x^2}dx$ *12.* $\int \frac{x^3}{\sqrt{x^2+2}}dx$

13. $\int \frac{x^3}{\sqrt{x-1}}dx$ *14.* $\int x^3\sqrt{9-x^2}dx$ *15.* $\int \frac{1+\sqrt[4]{x}}{1+\sqrt{x}}dx$

Individual Tasks 1.5

1-7. Evaluate the integral.

	II.
1. $\int \frac{7x-1}{\sqrt{2-3x-x^2}}dx$	*1.* $\int \frac{2x+7}{\sqrt{x^2+x-6}}dx$
2. $\int \frac{x^2-1}{\sqrt{2-3x-x^2}}dx$	*2.* $\int \frac{x^2+x}{\sqrt{x^2+2x-3}}dx$
3. $\int \frac{dx}{\sqrt{x+2}+\sqrt[3]{x+2}}$	*3.* $\int \frac{\sqrt{x+1}+2}{(x+1)^2-\sqrt{x+1}}dx$
4. $\int \frac{\sqrt{x}+\sqrt[3]{x}}{\sqrt[4]{x^5}-\sqrt[6]{x^7}}dx$	*4.* $\int \frac{dx}{\sqrt[4]{x}\left(\sqrt[12]{x^5}-\sqrt[3]{x}\right)}$
5. $\int \frac{\sqrt{x^2+9}}{x^4}dx$	*5.* $\int \frac{dx}{x^2\sqrt{9-x^2}}$
6. $\int \sqrt{x}\left(1+\sqrt[3]{x}\right)^2dx$	*6.* $\int x^5\sqrt[3]{\left(1+x^3\right)^2}dx$
7. $\int \frac{dx}{x^2\sqrt{x^2-1}}$	*7.* $\int \frac{\sqrt{9-x^2}}{x^4}dx$

1.6 Trigonometric Integrals

Trigonometric identities can be used to integrate certain combinations of trigonometric functions. We start with powers of sine and cosine.

1) If $\int \sin^{2m}x\cos^{2n}xdx,\ m>0, n>0$, element of integration must be converted with the aid of the formulas of reduction in the degree

$$\cos^2 x=\frac{1+\cos 2x}{2}, \qquad \sin^2 x=\frac{1-\cos 2x}{2}.$$

2) If $\int \sin mx \cos nx dx,\ \int \sin mx \sin nx dx,\ \int \cos mx \cos nx dx$, the multiplication of trigonometric functions should be replaced with the sum

$$\sin mx \cos nx = \tfrac{1}{2}\left(\sin(m+n)x + \sin(m-n)x\right)$$

$$\sin mx \sin nx = \tfrac{1}{2}\left(\cos(m-n)x - \cos(m+n)x\right)$$

$$\cos mx \cos nx = \tfrac{1}{2}\left(\cos(m-n)x + \cos(m+n)x\right).$$

3) If $\int R(\sin x, \cos x)dx$, where R is the rational function of its arguments, the universal trigonometric substitution $t = tg\frac{x}{2}$ can be used to obtain the integral of the rational function variable t.

$$\int R(\sin x, \cos x)dx = \left| \begin{matrix} tg\frac{x}{2} = t, & \sin x = \frac{2t}{1+t^2} \\ \cos x = \frac{1-t^2}{1+t^2}, & dx = \frac{2dt}{1+t^2} \end{matrix} \right| = \int R_1(t)dt.$$

4) If $R(-\sin x, \cos x) = -R(\sin x, \cos x)$, the substitution $t = \cos x$ can be used.

5) If $R(\sin x, -\cos x) = -R(\sin x, \cos x)$, the substitution $t = \sin x$ can be used.

6) If $R(-\sin x, -\cos x) = R(\sin x, \cos x)$, the substitution $t = tgx$ can be used.

Example 1 Find

a) $\int \frac{ctg^6 3x}{\sin^2 3x}dx$ *b)* $\int \sin^3 2x \cos^4 2x dx$ *c)* $\int \sin^2 x \cos^2 x dx$

d) $\int tg^5 dx$ *e)* $\int \sin x \sin 3x \sin 2x dx$ *f)* $\int \frac{dx}{2\sin x + 3\cos x + 5}$

Solution

a) $$\int \frac{ctg^6 3x}{\sin^2 3x}dx = \left| \begin{matrix} t = ctg3x, & dt = -\frac{3dx}{\sin^2 3x} \\ \frac{dx}{\sin^2 3x} = -\frac{dt}{3} & \end{matrix} \right| = -\frac{1}{3}\int t^6 dt =$$

$$= -\frac{1}{21}t^7 + C = -\frac{ctg^7 3x}{21} + C.$$

b) $$\int \sin^3 2x \cos^4 2x dx = \int \sin^2 2x \cos^4 2x \sin 2x dx =$$

$$= \left| \begin{matrix} t = \cos 2x, & dt = -2\sin 2x dx \\ \sin^2 2x = 1 - t^2, & \sin 2x dx = -\frac{1}{2}dt \end{matrix} \right| = -\frac{1}{2}\int (1-t^2)t^4 dt =$$

$$=\frac{1}{2}\int(t^6-t^4)dt=\frac{t^7}{14}-\frac{t^5}{10}+C=\frac{\cos^7 2x}{14}-\frac{\cos^5 2x}{10}+C.$$

c) $$\int\sin^2 x\cos^2 xdx=\frac{1}{4}\int(2\sin x\cos x)^2 dx=\frac{1}{4}\int\sin^2 2xdx=\frac{1}{4}\int\frac{1-\cos 4x}{2}dx=$$

$$=\frac{1}{8}\int(1-\cos 4x)dx=\frac{1}{8}\left(x-\frac{1}{4}\sin 4x\right)+C=\frac{1}{8}x-\frac{1}{32}\sin 4x+C.$$

d) $$\int tg^5 dx=\left|\begin{matrix} t=tgx, & x=arctgt \\ dx=\dfrac{dt}{1+t^2}. & \end{matrix}\right|=\int\frac{t^5dt}{1+t^2}=\int\frac{t^4}{1+t^2}tdt=\frac{1}{2}\int\frac{t^4}{1+t^2}d(t^2)=$$

$$=\left|t^2=z\right|=\frac{1}{2}\int\frac{z^2}{1+z}dz=\frac{1}{2}\int\frac{(z^2-1)+1}{z+1}dz=\frac{1}{2}\int\left(z-1+\frac{1}{z+1}\right)dz=$$

$$=\frac{1}{2}\left(\frac{z^2}{2}-z+\ln|z+1|\right)+C=\frac{1}{4}\left(tg^4x-2tg^2x+2\ln(1+tg^2x)\right)+C.$$

e) $$\int\sin x\sin 3x\sin 2xdx=\frac{1}{2}\int(\cos 2x-\cos 4x)\sin 2xdx=$$

$$=\frac{1}{2}\int\cos 2x\sin 2xdx-\frac{1}{2}\int\cos 4x\sin 2xdx=\frac{1}{4}\int\sin 4xdx-\frac{1}{4}\int(\sin 6x-\sin 2x)dx=$$

$$=-\frac{\cos 4x}{16}+\frac{\cos 6x}{24}-\frac{\cos 2x}{8}+C.$$

f) $$\int\frac{dx}{2\sin x+3\cos x+5}=\left|\begin{matrix} tg\dfrac{x}{2}=t, & dx=\dfrac{2dt}{1+t^2} \\ \cos x=\dfrac{1-t^2}{1+t^2}, & \sin x=\dfrac{2t}{1+t^2} \end{matrix}\right|=\int\frac{2dt}{\left(1+t^2\right)\left(\dfrac{4t}{1+t^2}+3\dfrac{1-t^2}{1+t^2}+5\right)}=$$

$$=\int\frac{2dt}{4t+3-3t^2+5+5t^2}=\int\frac{dt}{t^2+2t+4}=\int\frac{dt}{(t+1)^2+3}=\frac{1}{\sqrt{3}}arctg\frac{t+1}{\sqrt{3}}+C=$$

$$=\frac{1}{\sqrt{3}}arctg\frac{1+tg\,x/2}{\sqrt{3}}+C.$$

Example 2 Find $\int\frac{dx}{1+\sin^2 x}$.

Solution $$\int\frac{dx}{1+\sin^2 x}=\int\frac{dx}{2\sin^2 x+\cos^2 x}=\int\frac{dx}{\sin^2 x(2+ctg^2x)}=-\int\frac{d(ctgx)}{ctg^2x+2}=$$

$$=-\frac{1}{\sqrt{2}}arctg\frac{ctgx}{\sqrt{2}}+C.$$

Exercise Set 1.6

In exercises 1 to 16 evaluate the integral.

1. $\int \frac{\sin x\,dx}{(1-\cos x)^2}$

2. $\int \frac{\sin 4x\,dx}{\cos x}$

3. $\int \sin^2 x \cos^3 x dx$

4. $\int \cos^4 2x\,dx$

5. $\int \frac{dx}{\sqrt{3}\cos x + \sin x}$

6. $\int \sin 3x \cos 5x\,dx$

7. $\int \frac{dx}{\sin^4 x \cos^2 x}$

8. $\int \frac{dx}{3\sin x + \cos x + 1}$

9. $\int \frac{dx}{16\sin^2 x + \cos^2 x}$

10. $\int \frac{\sin 8x}{16 - \cos^2 8x} dx$

11. $\int \sqrt[9]{\cos^7 x} \sin 2x dx$

12. $\int \frac{\cos^6 x}{\sin^3 x} dx$

13. $\int \sin^4 2x \cos^2 2x dx$

14. $\int \cos^6 3x dx$

15. $\int tg^2 5x dx$

16. $\int \sin\frac{x}{3} \cos\frac{5x}{3} dx$

Individual Tasks 1.6

1-6. Evaluate the integral.

	I.	II.
1.	$\int \sin^7 x \cos^5 x\,dx$	$\int \sqrt[7]{\cos^2 x} \sin^3 x\,dx$
2.	$\int \sin^2 2x \cos^2 2x dx$	$\int \sin^4 3x\,dx$
3.	$\int \frac{\sin x dx}{4 + \cos^2 x}$	$\int \frac{dx}{\sqrt[4]{tg^3 2x} \cos^2 2x}$
4.	$\int \frac{dx}{3\cos x + \sin x}$	$\int \frac{dx}{5 - 4\sin x + 6\cos x}$
5.	$\int \frac{dx}{\sin^2 x - 2\sin x \cos x - 3\cos^2 x}$	$\int \frac{dx}{4\sin^2 x + \cos^2 x + 6}$
6.	$\int \sin x \cos 9x\,dx$	$\int \cos 6x \cos 8x\,dx$

Additional Tasks 1

Evaluate the integral.

1. $\int \frac{dx}{\sin^2 5x}$	2. $\int \frac{dx}{5-2x}$	3. $\int tg 3x\,dx$
4. $\int \frac{tgx dx}{\cos^2 x}$	5. $\int ctg\left(e^x\right) e^x\,dx$	6. $\int \frac{\cos x dx}{\sin^2 x}$
7. $\int \frac{\cos 2x dx}{(\sin 2x + 3)^3}$	8. $\int \frac{dx}{x \ln x}$	9. $\int 2x\left(x^2+1\right)^6 dx$

10. $\int \frac{e^{2x}dx}{2+e^{2x}}$	11. $\int \frac{e^{x}dx}{1+e^{2x}}$	12. $\int \frac{x^2dx}{16-x^6}$
13. $\int \frac{xdx}{x^4+a^4}$	14. $\int \frac{xdx}{\sqrt{x^2-a^2}}$	15. $\int \frac{dx}{\sqrt{3-5x^2}}$
16. $\int \frac{(\arcsin x - x)dx}{\sqrt{1-x^2}}$	17. $\int \left(e^{2x}\right)^3 dx$	18. $\int \frac{\sqrt[3]{tg^2 x}dx}{\cos^2 x}$
19. $\int \frac{dx}{\arcsin x\sqrt{1-x^2}}$	20. $\int e^{x^2+4x+3}(x+2)dx$	21. $\int x\cos 3x\,dx$
22. $\int xarctgx\,dx$	23. $\int x^3 \ln x\,dx$	24. $\int \frac{x\arcsin x dx}{\sqrt{1-x^2}}$
25. $\int \ln\left(x+\sqrt{1+x^2}\right)dx$	26. $\int (5x-4)\cos 8x\,dx$	27. $\int \frac{(6x-7)dx}{3x^2-12x+15}$
28. $\int \frac{(3x-2)dx}{x^2+8x+20}$	29. $\int \frac{dx}{\sqrt{3-4x-x^2}}$	30. $\int \frac{(x+3)dx}{\sqrt{4x^2+4x+3}}$
31. $\int \frac{x^5+x^4-8}{x^3-4x}dx$	32. $\int \frac{dx}{(x-1)^2(x+2)}$	33. $\int \frac{3x-7}{x^3+x^2+4x+4}dx$
34. $\int \frac{\sqrt{x}dx}{\sqrt[4]{x^3}+1}$	35. $\int \frac{dx}{x\sqrt{x^2+1}}$	36. $\int \frac{x^3dx}{\sqrt{2-x^2}}$
37. $\int \frac{\sqrt{x^2+1}}{x}dx$	38. $\int \frac{x^2dx}{\sqrt{1-x^2}}$	39. $\int \sin^3 x\cos^4 x\,dx$
40. $\int \cos^4 x\,dx$	41. $\int tg^3 x\,dx$	42. $\int \frac{dx}{\sin^4 x}$
43. $\int \sin x\sin 3x\,dx$	44. $\int \frac{dx}{4-5\sin x}$	45. $\int \frac{\sin^2 xdx}{1+\cos^2 x}$
46. $\int \frac{1+tgx}{1-tgx}dx$	47. $\int \frac{dx}{8-4\sin x+7\cos x}$	48. $\int \sin^3 x\,dx$

II DEFINITE INTEGRAL

There is a connection between integral calculus and differential calculus. The *Fundamental Theorem of Calculus* relates the integral to the derivative, and we will see in this chapter that it greatly simplifies the solution of many problems. In this chapter we discover that in trying to find the area under a curve, the arc length or the volume of the solid.

2.1 The Definite Integral

We start with the solving the *area problem:* find the area of the region that lies under the curve $y = f(x)$ from a to b. This means that the region, illustrated in

Figure 1, is bounded by the graph of a continuous function $f(x)$ (where $f(x) \geq 0$), the vertical lines $x = a$ and $x = b$, and the x axis.

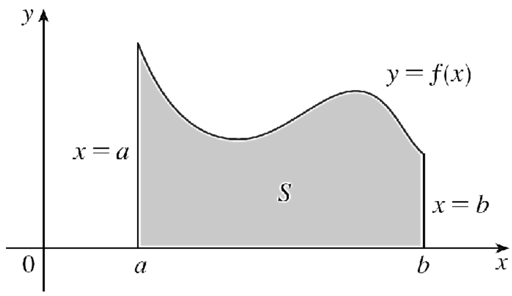


Figure 1

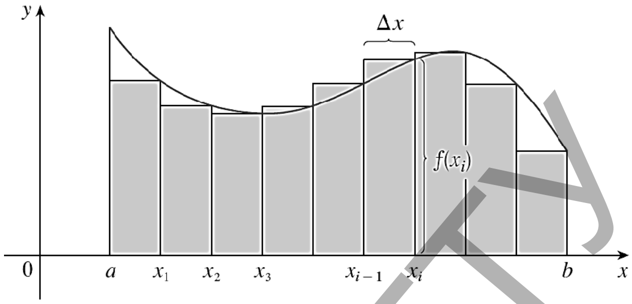


Figure 2

We all have an intuitive idea of what the area of a region is. But a part of the area problem is to make this intuitive idea precise by giving an exact definition of the area. We first approximate the region by rectangles and then we take the limit of the areas of these rectangles as we increase the number of rectangles (Figure 2).

The width of the interval $[a,b]$ is $b-a$, so the width of each of the n strips is $\Delta x = \dfrac{b-a}{n}$. These strips divide the interval $[a,b]$ into n subintervals $[a,x_1]$, $[x_1,x_2]$, $[x_2,x_3]$,..., $[x_{n-1},b]$. Let's approximate the i-th strip by A_i, a rectangle with the width Δx and the height $h = f(x_i)$, which is the value of $f(x)$ at the right endpoint (see Figure 2). Then the area of the rectangle is $S_i = f(x_i)\Delta x_i$. What we think of intuitively as the area of the region is approximated by the sum of the areas of these rectangles, which is

$$R_n = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + ... + f(x_n)\Delta x_n .$$

Definition The *area* S of the region A that lies under the graph of the continuous function $f(x)$ is the limit of the sum of the areas of approximating rectangles:

$$S = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i .$$

It can be proved that the limit in definition always exists, since we assume that $f(x)$ is continuous.

The same type of limit occurs in a wide variety of situations even when the function is not necessarily positive. We therefore give a special name and notation to this type of a limit.

Definition (Definite Integral) If $f(x)$ is a function defined for $a \leq x \leq b$, we divide the interval $[a,b]$ into n subintervals of equal width $\Delta x = \dfrac{b-a}{n}$. We let $a = x_0$, x_1, x_2,..., $x_n = b$ be the endpoints of these subintervals and we let x_1^*, x_2^*,..., x_n^* be any *sample points* in these subintervals, so x_i^* lies in the i-th subinterval $[x_{i-1}, x_i]$. Then the ***definite integral of*** $f(x)$ ***from a to b*** is

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

provided that this limit exists. If it does exist, we say that $f(x)$ is *integrable* on $[a,b]$.

Note 1 In the notation $\int_a^b f(x)dx$ $f(x)$ is called the *integrand*, a and b are called the *limits of integration*; a is the *lower limit* and b is the *upper limit*. The *dx* simply indicates that the independent variable is *x*. The procedure of calculating an integral is called *integration*.

Note 2 The definite integral $\int_a^b f(x)dx$ is a number; it does not depend on x. In fact, we could use any letter in place of x without changing the value of the integral:

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du .$$

Note 3 The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$ that occurs in the last Definition is called a *Riemann sum* after the German mathematician Bernhard Riemann (1826–1866).

Theorem If $f(x)$ is continuous on $[a,b]$, or if $f(x)$ has only a finite number of jump discontinuities, then $f(x)$ is integrable on $[a,b]$; that is, the definite integral $\int_a^b f(x)dx$ exists.

Properties of the Integral

1) $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

2) $\int_a^a f(x)dx = 0$.

3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

4) $\int_a^b \left(f_1(x) \pm f_2(x)\right)dx = \int_a^b f_1(x)dx \pm \int_a^b f_2(x)dx$.

5) $\int_a^b cf(x)dx = c\int_a^b f(x)dx,\ c = const$.

6) If $f(x) \ge 0$ $(f(x) \le 0)$ on $[a,b]$, then $\int_a^b f(x)dx \ge 0$ $\left(\int_a^b f(x)dx \le 0\right)$.

7) If $f(x) \ge \varphi(x),\ a \le x \le b$, then $\int_a^b f(x)dx \ge \int_a^b \varphi(x)dx$.

8) If $m \le f(x) \le M$, then $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$.

The Fundamental Theorem of Calculus (Newton and Leibniz Theorem) Suppose $f(x)$ is continuous on $[a,b]$.

1. If $F(x) = \int_a^x f(t)dt$, then $F'(x) = f(x)$.

2. $\int_a^b f(x)dx = F(b) - F(a)$, where $F(x)$ is any antiderivative of $f(x)$.

The Fundamental Theorem of Calculus says that differentiation and integration are inverse processes. The Fundamental Theorem of Calculus is the most important theorem in calculus.

Rules of the Calculation of the Definite Integral

1. The formula of Newton – Leibniz. If $f(x)$ is continuous on $[a,b]$, then $\int_a^b f(x)dx = F(b) - F(a)$.

2. The replacement of a variable in the definite integral. If $f(x)$ is continuous on $[a,b]$, the function $x = \varphi(t)$ is differentiated in the section $[\alpha, \beta]$ and $t \in [\alpha, \beta]$, $\varphi(t) \in [a,b]$, $\varphi(\alpha) = a$, $\varphi(\beta) = b$, then $\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$.

3. Evaluate definite integrals by parts $\int_a^b u(x)dv(x) = u(x)v(x)\Big|_a^b - \int_a^b v(x)du(x)$.

4. $\int_{-a}^a f(x)dx = 0,\ if\ \ f(-x) = -f(x)$;

$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$, $if\ \ f(-x) = f(x)$.

Example 1 Find

$$\int_1^8 \left(\sqrt[3]{x}-1\right)dx \qquad \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{\sqrt{1-x^2}}{x^2}dx$$

$$\int_e^{e^2} x\ln x dx \qquad \int_0^5 \frac{dx}{2+\sqrt{3x+1}}$$

Solution

a) Using the formula of Newton – Leibniz for this integral, however, we have

$$\int_1^8 (\sqrt[3]{x}-1)dx = \int_1^8 \left(x^{\frac{1}{3}}-1\right)dx = \left(\frac{3}{4}x^{\frac{4}{3}}-x\right)\Bigg|_1^8 = \left(\frac{3}{4}\cdot 2^4-8\right)-\left(\frac{3}{4}-1\right)=$$

$$=12-8-0.75+1=4.25.$$

b) $$\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{\sqrt{1-x^2}}{x^2}dx = \left| \begin{matrix} x=\sin t, & dx=\cos t dt \\ x=\frac{\sqrt{2}}{2}, t=\frac{\pi}{4}, & x=\frac{\sqrt{3}}{2}, t=\frac{\pi}{3} \end{matrix} \right| = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos^2 t dt}{\sin^2 t} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1-\sin^2 t}{\sin^2 t}dt =$$

$$= (-ctgt-t)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = -\left(ctg\frac{\pi}{3}+\frac{\pi}{3}\right)+\left(ctg\frac{\pi}{4}+\frac{\pi}{4}\right) = -\left(\frac{1}{\sqrt{3}}+\frac{\pi}{3}\right)+\left(1+\frac{\pi}{4}\right)=$$

$$=1-\frac{\pi}{12}-\frac{1}{\sqrt{3}}\approx 0.161.$$

c) Using the formula for integration by parts we get

$$I = \left| \begin{matrix} u=\ln x, & du=\frac{dx}{x} \\ dv=xdx, & v=\frac{x^2}{2} \end{matrix} \right| = \left(\frac{x^2}{2}\ln x\right)\Bigg|_e^{e^2} - \int_e^{e^2} \frac{x^2 dx}{2x} = \frac{e^4}{2}\cdot 2 - \frac{e^2}{2} - \left(\frac{x^2}{4}\right)\Bigg|_e^{e^2} = e^4-\frac{e^2}{2}-$$

$$-\frac{e^4}{4}+\frac{e^2}{4}=\frac{1}{4}(3e^4-e^2)\approx 39.10.$$

d) $$\int_0^5 \frac{dx}{2+\sqrt{3x+1}} = \left| \begin{matrix} \sqrt{3x+1}=t, & x=0 \Rightarrow t=1 \\ x=\frac{1}{3}(t^2-1), & x=5 \Rightarrow t=4 \\ dx=\frac{2}{3}t\,dt & \end{matrix} \right| = \int_1^4 \frac{\frac{2}{3}t}{2+t}dt =$$

$$=\frac{2}{3}\cdot\int_1^4 \frac{t}{2+t}dt = \frac{2}{3}\cdot\int_1^4 \frac{t+2-2}{2+t}dt = \frac{2}{3}\cdot\int_1^4 \left(1-\frac{2}{t+2}\right)dt = \frac{2}{3}\cdot\left(t+2\ln|t+2|\right)\Big|_1^4 =$$

$$=\frac{8}{3}-\frac{2}{3}+\frac{4}{3}\ln 6-\frac{4}{3}\ln 3=2+\frac{4}{3}\ln 2\approx 2.924\,.$$

Exercise Set 2.1

In exercises 1 to 15 calculate

1. $\int_1^2\left(2x^2+\frac{2}{x^4}\right)dx$ *2.* $\int_0^1\sqrt{1+x}\,dx$ *3.* $\int_0^{16}\frac{dx}{\sqrt{x+9}-\sqrt{x}}$

4. $\int_1^{e^3}\frac{dx}{x\sqrt{1+\ln x}}$ *5.* $\int_0^1\frac{dx}{x^2+4x+5}$ *6.* $\int_0^4\frac{dx}{1+\sqrt{2x+1}}$

7. $\int_{-\pi/2}^{\pi/2}\sqrt{\cos x-\cos^3 x}\,dx$ *8.* $\int_0^{\pi/2}x\cos x\,dx$ *9.* $\int_0^{e-1}\ln(x+1)\,dx$

10. $\int_1^3\frac{dx}{x\sqrt{x^2+5x+1}}$ *11.* $\int_3^4\frac{x^2-x+2}{x^4-5x^2+4}dx$ *12.* $\int_{\pi/6}^{\pi/3}\frac{xdx}{\cos^2 x}$

13. $\int_0^{\sqrt{3}}x\sqrt[3]{1+x^2}\,dx$ *14.* $\int_3^6\frac{\sqrt{x^2-9}}{x^4}dx$ *15.* $\int_{2\sqrt{3}}^{2\sqrt{8}}\frac{x^3dx}{\sqrt{4+x^2}}$

Individual Tasks 2.1

1-8. Calculate

	II.
1. $\int_0^1\frac{x}{\left(x^2+1\right)^2}dx$	*1.* $\int_0^{\ln 2}\frac{dx}{e^x\sqrt{1-e^{-2x}}}$
2. $\int_1^5\frac{dx}{x+\sqrt{2x-1}}$	*2.* $\int_0^5\frac{dx}{2+\sqrt{3x+1}}$
3. $\int_2^3 y\ln\left(y-1\right)dy$	*3.* $\int_{-1/2}^{1/2}\arccos 2x\,dx$
4. $\int_{-1/3}^{-2/3}xe^{-3x}dx$	*4.* $\int_0^{\pi}x\sin x\,dx$
5. $\int_0^{\pi/2}\frac{dx}{2+\cos x}$	*5.* $\int_{-\pi/2}^{\pi/2}\frac{dx}{1+\cos x}$
6. $\int_2^3\frac{3x-2}{x^2-4x+5}dx$	*6.* $\int_1^3\frac{dx}{x\sqrt{x^2+5x+1}}$
7. $\int_0^{\sqrt{3}}x\sqrt[3]{1+x^2}\,dx$	*7.* $\int_0^{\sqrt{3}}x^5\sqrt{1+x^2}\,dx$

8. $\int_{-\pi/2}^{-\pi/4} \frac{\cos^3 x dx}{\sqrt[3]{\sin x}}$	8. $\int_{\pi/2}^{\pi} \frac{\sin x}{(1-\cos x)^3} dx$

2.2 Improper Integrals

In defining a definite integral $\int_a^b f(x)dx$ we dealt with the function $f(x)$ defined on a finite interval $[a,b]$, and we assumed it does not have an infinite discontinuity. In this section we extend the concept of a definite integral to the case where the interval is infinite and also to the case where $f(x)$ has an infinite discontinuity in $[a,b]$. In either case the integral is called an *improper* integral.

Type 1: Infinite Intervals

Definition (Improper Integral of Type 1)

(a) If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then

$$\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty} \int_a^t f(x)dx \quad (1)$$

provided this limit exists (as a finite number) and is called an *improper integral on an infinite interval* $[a,+\infty)$.

(b) If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx \quad (2)$$

provided this limit exists (as a finite number) and is called an *improper integral on an infinite interval* $(-\infty,b]$.

Definition The improper integrals $\int_a^{+\infty} f(x)dx$ and $\int_{-\infty}^{b} f(x)dx$ are called *convergent* if the corresponding limit exists and *divergent* if the limit does not exist.

If both $\int_a^{+\infty} f(x)dx$ and $\int_{-\infty}^{b} f(x)dx$ are convergent, then we define

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{+\infty} f(x)dx = \lim_{t\to-\infty} \int_t^a f(x)dx + \lim_{t\to+\infty} \int_a^t f(x)dx \quad (3)$$

In the formula (3) any real number can be used.

Any of the improper integrals in Definition 1 can be interpreted as an area provided that $f(x)$ is a positive function. For instance, in case (1) if $f(x) \geq 0$ and the integral $\int_{a}^{+\infty} f(x)dx$ is convergent, then we define the area of the region $A=\{(x,y)\,|\,x\ge a,\ 0\le y\le f(x)\}$ in Figure 3 to be $S(A)=\int_{a}^{+\infty} f(x)dx$.

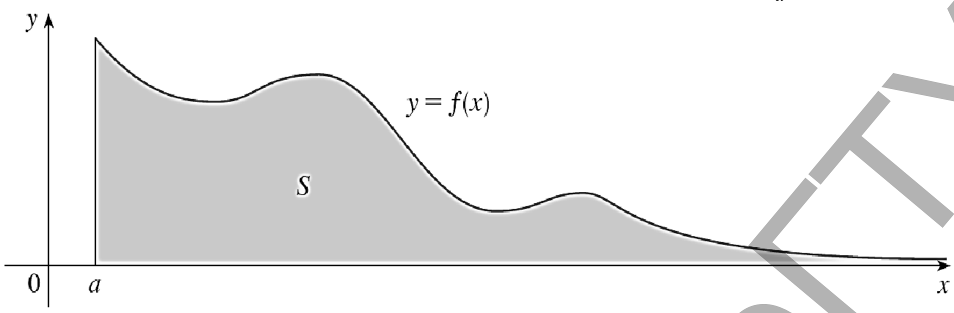


Figure 3

Example 1 Evaluate $\int_{0}^{+\infty} xe^{-x^2}dx$.

Solution

$$\int_{0}^{+\infty} xe^{-x^2}dx=-\frac{1}{2}\int_{0}^{+\infty} e^{-x^2}d(-x^2)=-\frac{1}{2}\lim_{b\to+\infty}\int_{0}^{b} e^{-x^2}d(-x^2)=-\frac{1}{2}\lim_{b\to+\infty}(e^{-x^2})\Big|_0^b=$$

$$=-\frac{1}{2}\lim_{b\to+\infty}\frac{1}{e^{b^2}}+\frac{1}{2}e^0=\frac{1}{2}.$$

Example 2 Evaluate $\int_{1}^{\infty}\frac{2}{x\cdot\left(9+\ln^2 x\right)}dx$.

Solution

$$\int_{1}^{\infty}\frac{2}{x\cdot\left(9+\ln^2 x\right)}dx=\lim_{b\to+\infty}\int_{1}^{b}\frac{2}{x\cdot\left(9+\ln^2 x\right)}dx=\lim_{b\to+\infty}\int_{1}^{b}\frac{2\,d(\ln x)}{9+\ln^2 x}=$$

$$=\lim_{b\to+\infty}2\cdot\frac{1}{3}arctg\left(\ln|x|\right)\Big|_1^b=\frac{2}{3}\cdot\lim_{b\to+\infty}\left(arctg(\ln b)-arctg(\ln 1)\right)=\frac{2}{3}\cdot\frac{\pi}{2}=\frac{\pi}{3}.$$

Type 2: Discontinuous Integrands

Definition (Improper Integral of Type 2)

(a) If $f(x)$ is continuous on $[a,b)$ and is discontinuous at b, then

$$\int_{a}^{b} f(x)dx=\lim_{\varepsilon\to 0}\int_{a}^{b-\varepsilon} f(x)dx \qquad (4)$$

provided this limit exists (as a finite number) and is called an *improper integral of discontinuous at $x=b$ function $f(x)$*.

(b) If $f(x)$ is continuous on $(a,b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{\delta \to 0} \int_{a+\delta}^b f(x)dx \qquad (5)$$

provided this limit exists (as a finite number) and is called an *improper integral of discontinuous at* $x = a$ *function* $f(x)$.

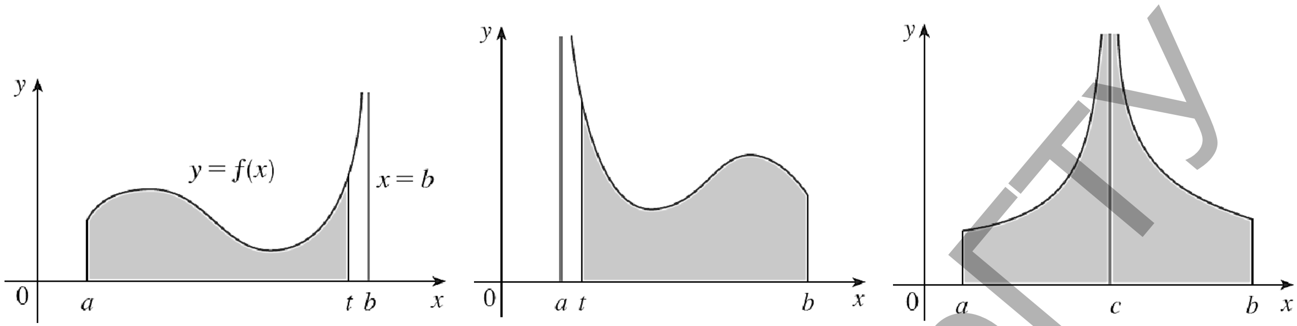


Figure 4

If $f(x)$ has a discontinuity at $x = c$, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)dx + \lim_{\delta \to 0} \int_{c+\delta}^b f(x)dx \qquad (6)$$

Parts of Definition are illustrated in Figure 4 for the case where $f(x) \geq 0$ and has vertical asymptotes at a, b and c respectively.

Example 3 Evaluate $\int_0^2 \frac{dx}{\sqrt{2-x}}$.

Solution

$$\int_0^2 \frac{dx}{\sqrt{2-x}} = -\lim_{\varepsilon \to 0} \int_0^{2-\varepsilon} (2-x)^{-\frac{1}{2}} d(2-x) = -\lim_{\varepsilon \to 0} 2\sqrt{2-x}\Big|_0^{2-\varepsilon} = -2\lim_{\varepsilon \to 0} \sqrt{\varepsilon} + 2\sqrt{2} = 2\sqrt{2}.$$

Example 4 Evaluate $\int_1^4 \frac{dx}{x^2 - 6x + 9}$.

Solution

$$\int_1^4 \frac{dx}{x^2 - 6x + 9} = \int_1^4 \frac{dx}{(x-3)^2} = \int_1^3 \frac{dx}{(x-3)^2} + \int_3^4 \frac{dx}{(x-3)^2} \; ;$$

$$\lim_{\alpha \to +0} \int_1^{3-\alpha} \frac{dx}{(x-3)^2} = -\lim_{\alpha \to +0} \frac{1}{x-3}\Bigg|_1^{3-\alpha} = -\lim_{\alpha \to +0} \left(\frac{1}{3-\alpha-3} - \frac{1}{1-3} \right) =$$

$$= -\lim_{\alpha\to+0}\left(\frac{1}{-\alpha}-\frac{1}{-2}\right)=\lim_{\alpha\to+0}\left(\frac{1}{\alpha}\right)-\frac{1}{2}=+\infty-\frac{1}{2}=+\infty;$$

$$\lim_{\beta\to+0}\int\limits_{3+\beta}^{4}\frac{dx}{(x-3)^2}=-\lim_{\beta\to+0}\frac{1}{x-3}\Bigg|_{3+\beta}^{4}=-\lim_{\beta\to+0}\left(\frac{1}{4-3}-\frac{1}{3+\beta-3}\right)=$$

$$=-\lim_{\beta\to+0}\left(1-\frac{1}{+\beta}\right)=-1+\lim_{\beta\to+0}\left(\frac{1}{\beta}\right)=-1+\infty=+\infty.$$

Since limits are equal to infinity, then improper integral diverges.

Comparison Test for Improper Integrals

Sometimes it is impossible to find the exact value of an improper integral and yet it is important to know whether it is convergent or divergent. In such cases the following theorem is useful. Although we state it for Type 1 integrals, a similar theorem is true for Type 2 integrals.

Theorem (Comparison Theorem) Suppose $f(x)$ and $g(x)$ are continuous functions with $f(x)\ge g(x)\ge 0$ for $x\ge a$.

a) If $\int\limits_{a}^{+\infty} f(x)dx$ is convergent, then $\int\limits_{a}^{+\infty} g(x)dx$ is convergent.

b) If $\int\limits_{a}^{+\infty} g(x)dx$ is divergent, then $\int\limits_{a}^{+\infty} f(x)dx$ is divergent.

Geometrical Interpretation If the area under the top curve $y=f(x)$ is finite, then so is the area under the bottom curve $y=g(x)$. And, if the area under $y=g(x)$ is infinite, then so is the area under $y=f(x)$ (see Figure 5).

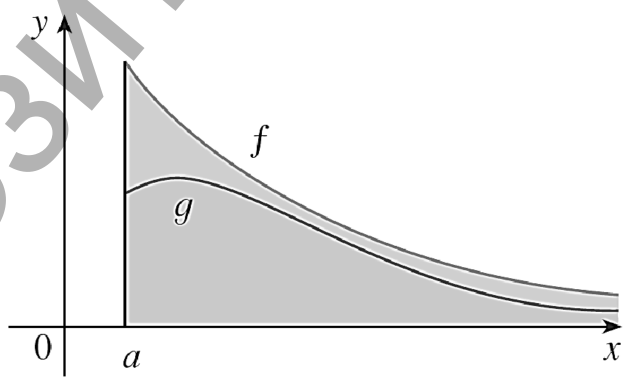


Figure 5

Example 5 Evaluate $\int\limits_{1}^{+\infty}\frac{2+\sin x}{\sqrt{x}}dx$.

Solution Let us estimate the integrand for all x from the space of integration, we will obtain the inequality

$$\frac{1}{\sqrt{x}}\le\frac{2+\sin x}{\sqrt{x}}\le\frac{3}{\sqrt{x}}.$$

$$\int_1^{\infty}\frac{dx}{\sqrt{x}}=\lim_{b\to\infty}\int_1^{b}\frac{dx}{\sqrt{x}}=2\lim_{b\to\infty}\sqrt{x}\Big|_1^b=2\lim_{b\to\infty}\sqrt{b}-2=\infty .$$

It means that the given integral is divergent.

Exercise Set 2.2

In exercises 1 to 15 calculate improper integrals or establish their divergence.

1. $\int_2^{+\infty}\frac{dx}{x^2}$ *2.* $\int_0^{+\infty}\sin 2x\,dx$ *3.* $\int_0^{+\infty}\frac{dx}{x^2+4}$

4. $\int_{-\infty}^{\infty}\frac{dx}{x^2+2x+2}$ *5.* $\int_0^{1}\frac{xdx}{\sqrt{1-x^2}}$ *6.* $\int_1^{e}\frac{dx}{x\ln x}$

7. $\int_1^{2}\frac{xdx}{\sqrt{x-1}}$ *8.* $\int_0^{2,5}\frac{dx}{x^2-5x+6}$ *9.* $\int_1^{\infty}\frac{dx}{x}$

10. $\int_0^{\infty}\frac{arctgx}{x^2+1}dx$ *11.* $\int_e^{\infty}\frac{dx}{x\ln^2 x}$ *12.* $\int_{-\infty}^{0}\frac{dx}{x^2+8x+17}$

13. $\int_{\frac{\pi}{2}}^{\infty}\frac{\sin x}{x^2}dx$ *14.* $\int_0^{1}\frac{dx}{e^x-\cos x}$ *15.* $\int_0^{2}\frac{x^3dx}{\sqrt[5]{(x^5+3x^4)^7}}$

Individual Tasks 2.2

1-7. Calculate improper integrals or establish their divergence.

	II.
1. $\int_{-\infty}^{0}\frac{dx}{x^2+8x+17}$	*1.* $\int_{-\infty}^{1}\frac{dx}{x^2+2x+10}$
2. $\int_3^{+\infty}\frac{dx}{x^3-4x^2}$	*2.* $\int_0^{+\infty}\frac{dx}{x^3+1}$
3. $\int_1^{3}\frac{dx}{x^2-6x+8}$	*3.* $\int_0^{3}\frac{dx}{x^2-8x+7}$
4. $\int_0^{\infty}x^3e^{-x^2}dx$	*4.* $\int_0^{1}x\ln x\,dx$
5. $\int_{-1}^{1}\frac{dx}{(2-x)\sqrt{1-x^2}}$	*5.* $\int_0^{1}\frac{dx}{(3-x)\sqrt{1-x^2}}$
6. $\int_0^{1/2}\frac{dx}{x\ln^2 x}$	*6.* $\int_1^{2}\frac{xdx}{\sqrt{x-1}}$

7. $\int_{2}^{+\infty} \frac{dx}{(4+x^2)\sqrt{arctg 0,5x}}$	7. $\int_{e}^{\infty} \frac{dx}{x\ln^2 x}$

2.3 Geometrical Applications of Integration

In this chapter we explore some of the applications of the definite integral by using it to compute areas between curves, volumes of solids, and arc length. The common theme is the following general method, which is similar to the one we used to find areas under curves: we break up a quantity Q into a large number of small parts. Next we approximate each small part by a quantity of the form $f(x_i^*)\Delta x_i$ and thus approximate Q by a Riemann sum. Then we take the limit and express Q as an integral. Finally, we evaluate the integral using the Fundamental Theorem of Calculus.

Areas between Curves

Integrals can be used to find areas of regions that lie between the graphs of two functions. Consider the region A that lies between two curves $y = f(x)$, $y = g(x)$ and between the vertical lines $x = a$, $x = b$, where $y = f(x)$ and $y = g(x)$ are continuous functions and $f(x) \ge g(x)$ for all $x \in [a,b]$ (See Figure 6).

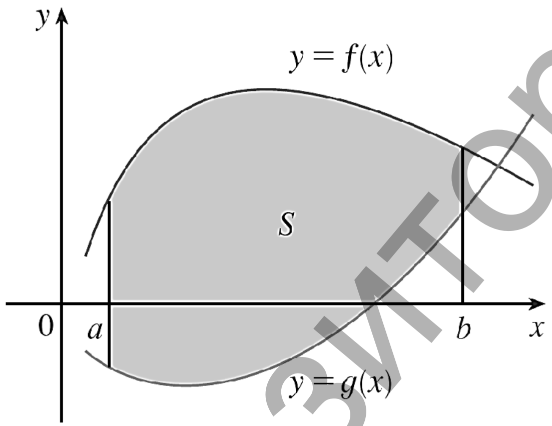


Figure 6

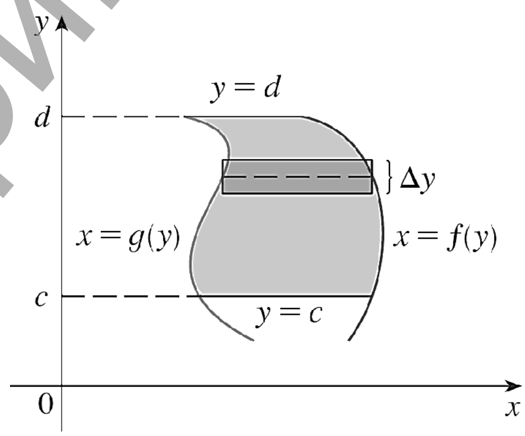


Figure 7

The area S of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where $y = f(x)$ and $y = g(x)$ are continuous and $f(x) \ge g(x)$ for all $x \in [a,b]$ can be calculated by the formula

$$S = \int_a^b \left(f(x) - g(x)\right)dx \qquad (1)$$

Note 1 If $y = g(x) = 0$, then $S = \int_a^b f(x)dx$.

Note 2 Some regions are best treated by regarding x as a function of y. If a region is bounded by curves with the equations $x = f(y)$, $x = g(y)$, $y = c$, $y = d$, where $x = f(y)$ and $x = g(y)$ are continuous and $f(y) \ge g(y)$ for $y \in [c,d]$ (see Figure 7), then its area is

$$S = \int_c^d \big(f(y) - g(y)\big)dy \qquad (2)$$

Note 3 If we are asked to find the area between the curves $y = f(x)$ and $y = g(x)$ where $f(x) \ge g(x)$ for some values of x, but $f(x) \le g(x)$ for other values of x, then we split the given region into several regions $A_1, A_2, A_3, \ldots$ with areas $S_1, S_2, S_3, \ldots$ as shown in Figure 8. We then define the area of the region A to be the sum of the areas of the smaller regions $A_1, A_2, A_3, \ldots$, that is $S = S_1 + S_2 + \ldots$.

The area between the curves $y = f(x)$ and $y = g(x)$, between $x = a$ and $x = b$ is

$$S = \int_a^b \left| f(x) - g(x) \right| dx \qquad (3)$$

Note 4 If the curve is assigned by the parametric equations $x = x(t), y = y(t)$, the area A of the region bounded by this curve is

$$S = \int_\alpha^\beta y(t) \cdot x'(t)dt, \quad where \ \ a = x(\alpha), \ \ b = x(\beta) \qquad (4)$$

Note 5 The area of figures in the polar coordinates (see Figure 9 and Figure 10) is

$$S = \frac{1}{2}\int_\alpha^\beta r^2(\theta)d\theta \quad \text{or} \quad S = \frac{1}{2}\int_\alpha^\beta \big(f^2(\theta) - g^2(\theta)\big)d\theta \qquad (5)$$

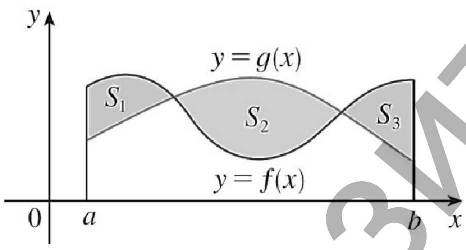


Figure 8

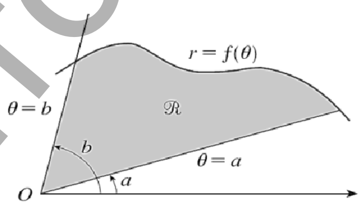


Figure 9

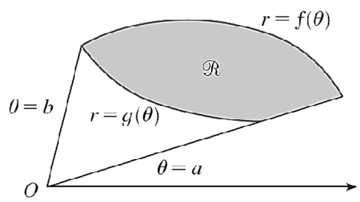


Figure 10

Example 1 Find the area enclosed by the line $g(x) = -x + 1$ and the parabola $f(x) = -x^2 + 2x + 5$.

Solution By solving the two equations we find that the points of intersection are $(-1;2)$, $(4;-3)$:

$$f(x) = g(x) \Rightarrow -x + 1 = -x^2 + 2x + 5 \Rightarrow x^2 - 3x - 4 = 0,$$

$$x_1 = -1; \ x_2 = 4,$$

$$y_1 = 2; \ y_2 = -3.$$

Thus

$$S=\int_{-1}^{4}\left(f(x)-g(x)\right)dx=\int_{-1}^{4}\left(-x^2+2x+5-(-x+1)\right)dx=\int_{-1}^{4}\left(-x^2+2x+5+x-1\right)dx=$$

$$=\int_{-1}^{4}\left(-x^2+3x+4\right)dx=\left(-\frac{x^3}{3}+3\cdot\frac{x^2}{2}+4x\right)\Bigg|_{-1}^{4}=\left(-\frac{4^3}{3}+3\cdot\frac{4^2}{2}+4\cdot4\right)-$$

$$-\left(-\frac{(-1)^3}{3}+3\cdot\frac{(-1)^2}{2}+4\cdot(-1)\right)=20\frac{5}{6}.$$

Example 2 Find the area enclosed by the curve $\begin{cases}x=2\cos^3 t\\ y=2\sin^3 t\end{cases}$.

Solution These equations determine an astroid. Since the figure is symmetrical relative to coordinate axes, then let us find $1/4$ of the area, which lies in the first quadrant.

$x'(t)=2\cdot3\cdot\cos^2 t\cdot\left(-\sin t\right)=-6\cdot\cos^2 t\cdot\sin t$.

If $x\left(t_1\right)=0\Rightarrow t_1=\frac{\pi}{2}$; $x\left(t_2\right)=2\Rightarrow t_2=0$.

Thus $S=\int_{t_1}^{t_2} y(t)\cdot x'(t)\,dt$:

$$S=\int_{\frac{\pi}{2}}^{0}2\sin^3 t\cdot\left(-6\cos^2 t\cdot\sin t\right)dt=12\int_{0}^{\frac{\pi}{2}}\sin^4 t\cdot\cos^2 t\,dt=3\int_{0}^{\frac{\pi}{2}}\sin^2 2t\cdot\sin^2 t\,dt=$$

$$=3\int_{0}^{\frac{\pi}{2}}\sin^2 2t\cdot\frac{1-\cos 2t}{2}dt=\frac{3}{2}\int_{0}^{\frac{\pi}{2}}\sin^2 2t\,dt-\frac{3}{2}\int_{0}^{\frac{\pi}{2}}\sin^2 2t\cdot\cos 2t\,dt=$$

$$=\frac{3}{2}\int_{0}^{\frac{\pi}{2}}\frac{1-\cos 4t}{2}dt-\frac{3}{4}\int_{0}^{\frac{\pi}{2}}\sin^2 2t\,d(\sin 2t)=\frac{3}{4}t\Big|_{0}^{\frac{\pi}{2}}-\frac{3}{16}\sin 4t\Big|_{0}^{\frac{\pi}{2}}-\frac{1}{4}\sin^3 2t\Big|_{0}^{\frac{\pi}{2}}=\frac{3\pi}{8}.$$

After multiplying the obtained area to 4, we will obtain the area of the entire of the astroid $S_{astroid}=4\cdot S=4\cdot\frac{3\pi}{8}=\frac{3\pi}{2}\approx 4.712$.

Volumes

Trying to find the volume of a solid we face the same type of problem as in finding areas. We have an intuitive idea of what the volume means, but we must make this idea precise by using calculus to give an exact definition of the volume.

For a solid S that is not a cylinder we first "cut" S into pieces and approximate each piece by a cylinder. We estimate the volume of S by adding the volumes of the cylinders. We arrive at the exact volume of S through a limiting process in which the number of pieces becomes large.

We start by intersecting S with a plane and obtaining a plane region that is called a *cross-section* of S.

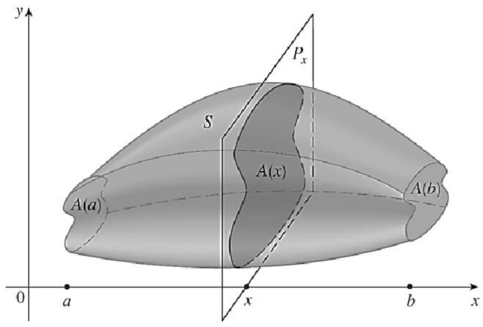


Figure 11

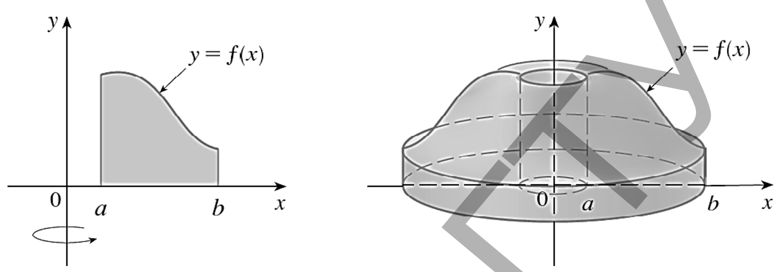


Figure 12

Let $A(x)$ be the area of the cross-section of S in a plane P_x perpendicular to the $x-$ axis and passing through the point x, where $a \le x \le b$ (see Figure 11). Think of slicing S with a knife through x and computing the area of this slice. The cross-sectional area $A(x)$ will vary x as increases from a to b.

Definition Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where $A(x)$ is a continuous function, then the ***volume*** of S is

$$V = \int_a^b A(x)dx \qquad (6)$$

Note 1 The volume of the solid in Figure 12, obtained by rotating the region under the curve $y = f(x)$ from a to b about the y-axis, is

$$V_x = \pi \int_a^b f^2(x)dx \qquad \left(V_y = 2\pi \int_a^b x \cdot f(x)dx \right) \qquad (7)$$

Note 2 If a curvilinear sector revolves around the polar axis, then the volume of the body of revolution is found by the formula

$$V = \frac{2}{3}\pi \int_\alpha^\beta r^3(\theta) \cdot \sin\theta \, d\theta \qquad (8)$$

Example 3 Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

Solution Figure 13 shows the region and a cylindrical shell formed by the rotation about the line $x = 2$. It has radius $2 - x$, circumference $2\pi \cdot (2 - x)$, and height $h = x - x^2$.

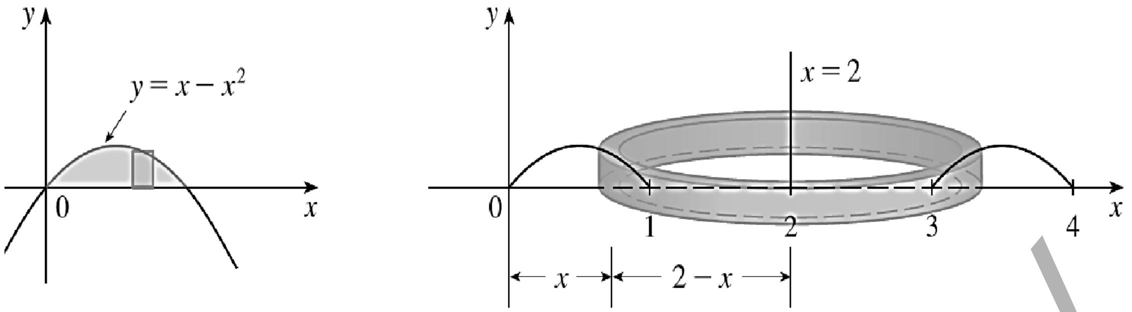


Figure 13

The volume of the given solid is

$$V=\int_0^1 2\pi\cdot(2-x)(x-x^2)dx=2\pi\cdot\int_0^1\left(x^3-3x^2+2x\right)dx=2\pi\cdot\left(\frac{x^4}{4}-x^3+x^2\right)\Bigg|_0^1=\frac{\pi}{2}.$$

Arc Length

Suppose that a curve C is defined by the equation $y=f(x)$, where $f(x)$ is continuous and $a\le x\le b$. We obtain a polygonal approximation to C by dividing the interval $[a,b]$ into n subintervals with endpoints $a=x_0, x_1, x_2,..., x_n=b$ and equal width Δx. If $y_i=f(x_i)$, then the point $P_i(x_i;y_i)$ lies on C and the polygon with vertices $P_1, P_2, P_3,...$ illustrated in Figure 14, is an approximation to C.

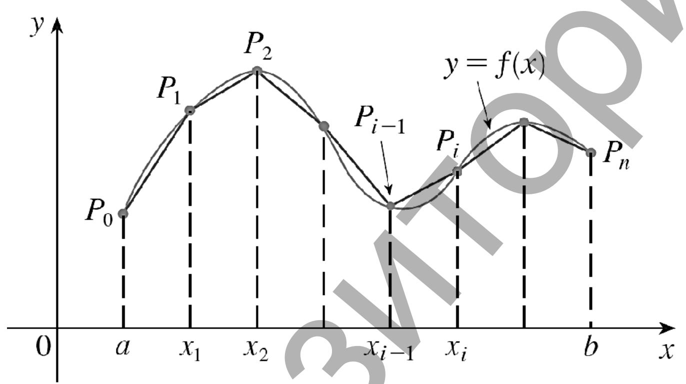


Figure 14

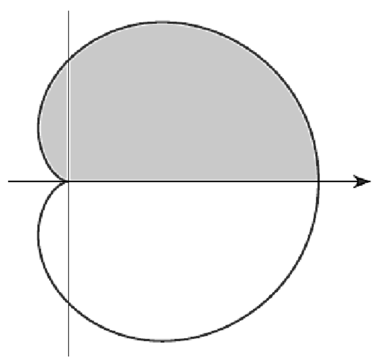


Figure 15

The length L of C is approximately the length of this polygon and the approximation gets better as we let n increase.

The Arc Length Formula If $y=f(x)$ is continuous on $[a,b]$, then the length of the curve $y=f(x)$, $a\le x\le b$ is

$$l=\int_a^b\sqrt{1+\left(f'(x)\right)^2}dx \qquad (9)$$

Note 1 If a curve has the equation $x=\varphi(y)$, $c\le y\le d$, and $x=\varphi(y)$ is a continuous function, the following formula can be used:

$$l=\int_c^d \sqrt{1+\left(\varphi'(y)\right)^2}\,dy \qquad (10)$$

Note 2 If a curve is assigned by the parametric equations $x=x(t),\ y=y(t),\ t\in[\alpha,\beta]$ the following formula can be used:

$$l=\int_\alpha^\beta \sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2}\,dt \qquad (11)$$

Note 3 If it is known that the polar equation of arc AB is $r=r(\theta),\ \theta\in[\alpha,\beta]$ the length of its arc is equal to:

$$l=\int_\alpha^\beta \sqrt{r^2(\theta)+\left(r'(\theta)\right)^2}\,d\theta \qquad (12)$$

Example 4 Find the arc length function for the curve $\begin{cases} y^2=(x+1)^3, \\ -1\le x\le 0. \end{cases}$

Solution The given line can be described by the formula $y=(x+1)^{\frac{3}{2}}$ and it is symmetrical relative to x-axis. Therefore we will search for the length of the line lying at the second quater. Let us calculate the derivative $y'=\frac{3}{2}\cdot(x+1)^{\frac{1}{2}}$.

Then $l=\int_{-1}^{0}\sqrt{1+\left(\frac{3}{2}(x+1)^{\frac{1}{2}}\right)^2}\,dx=\int_{-1}^{0}\sqrt{1+\frac{9}{4}(x+1)}\,dx=\frac{1}{2}\cdot\int_{-1}^{0}\sqrt{9x+13}\,dx=$

$$=\frac{1}{2}\cdot\frac{1}{9}\cdot\frac{(9x+13)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_{-1}^{0}=\frac{1}{27}\left(\sqrt{13^3}-8\right)=\frac{13\sqrt{13}-8}{27}.$$

$$L=2\cdot l=2\cdot\frac{13\sqrt{13}-8}{27}=\frac{26\sqrt{13}-16}{27}\approx 2.879.$$

Example 5 Find the arc length function for the curve $r=3(1+\cos\theta)$.

Solution Cardioid is a curve symmetrical relative to polar axis. Let us calculate the length of the arc lying above the polar axis (See Figure 15).

For calculating the arc length of this line we will use the formula $l=\int_\alpha^\beta\sqrt{r^2+(r')^2}\,d\theta$. Let us calculate the derivative $r'=-3\sin\theta$. Then

$$l=\int_0^\pi\sqrt{9(1+\cos\theta)^2+9\sin^2\theta}\,d\theta=3\int_0^\pi\sqrt{2+2\cos\theta}\,d\theta=6\int_0^\pi\cos\frac{\theta}{2}\,d\theta=12\sin\frac{\theta}{2}\Bigg|_0^\pi=12.$$

Consequently, the length of the entire cardioid is equal $L=2\cdot l=2\cdot 12=24$.

Exercise Set 2.3

In exercises 1 to 15 calculate the areas of the figures, bounded by the assigned curves.

1. $y=x^2+4,\ y=x+4$	*2.* $y=\ln x, x=e, x=e^2, y=0$	*3.* $y^3=x, y=1, x=5$
4. $y^2=9x,\ y=3x$	*5.* $y^2=x^2-x^4$	*6.* $y^2=1-x,\ x=-3$
7. $\begin{cases} x=a(t-\sin t) \\ y=a(1-\cos t) \end{cases}, y=0$	*8.* $y=x^2-6x+9,$ $4x-y=12$	*9.* $y^2+8x=16,$ $y^2-24x=48$
10. $\begin{cases} x=3\cos t \\ y=5\sin t \end{cases}$	*11.* $\begin{cases} x=3\cos^3 t \\ y=3\sin^3 t \end{cases}$	*12.* $\begin{cases} x=3t^2 \\ y=4t-t^3 \end{cases}$
13. $r=5(1+\cos\theta)$	*14.* $r=3\cos 2\theta$	*15.* $r^2=a^2\sin 2\theta$

In exercises 16 to 24 find the volume of the body, formed by the rotation of the figure bounded by the assigned curves around the indicated axis.

16. $y=4x-x^2, y=x, OY$	*17.* $y=x^2, 4x-y=0,$ OY	*18.* $y=x^3, x=2,\ y=0, OY$
19. $y=0{,}5x^2-2x+2,$ $y=2, OY$	*20.* $y^2=16x,\ x=4, OY$	*21.* $y=x^2, y=\sqrt{x}, OX$
22. $y=x^3, x=2,\ y=0,$ OX	*23.* $x^2-y^2=9,\ x=6,$ OX	*24.* $xy=4,\ 2x+y-6=0$ OX

In exercises 25 to 36 find the length of the arc of the curve.

25. $y=\frac{2}{3}\sqrt{(x-1)^3}, x_1=1,$ $x_2=9$	*26.* $y=\sqrt{2x-x^2}-1,$ $x_1=0{,}25,\ x_2=1$	*27.* $y=\ln\left(1-x^2\right),$ $x_1=0, x_2=0{,}5$
28. $x=\ln\cos y, y_1=0,$ $y_2=\pi/3$	*29.* $x=0{,}25y^2-0{,}5\ln y$ $y_1=1, y_2=e$	*30.* $\begin{cases} x=3\cos t \\ y=3\sin t \end{cases}$ $0\le t\le \pi/2$
31. $\begin{cases} x=a(t-\sin t) \\ y=a(1-\cos t) \end{cases}$ $0\le t\le 2\pi$	*32.* $\begin{cases} x=3\cos^3 t \\ y=3\sin^3 t \end{cases}$ $0\le t\le \pi$	*33.* $\begin{cases} x=e^t\cos t \\ y=e^t\sin t \end{cases}$ $0\le t\le \ln\pi$
34. $r=4(1+\cos\theta)$	*35.* $r=2\cos^3(\theta/3)$	*36.* $r^2=a^2\sin 2\theta$

Individual Tasks 2.3

1-3. Calculate the areas of the figures, bounded by the assigned curves.

4-5. Find the length of the arc of the curve.

6. Find the volume of the body, formed by the rotation of the figure bounded by the assigned curves around the indicated axis

	II.
1. $y=x^2, y=2-x^2$	1. $y^2=4x,\ x^2=4y$
2. $r=7(1+\sin\theta)$	2. $r=9(1-\sin\theta)$
3. $\begin{cases} x=5\cos t \\ y=5\sin t \end{cases}$	3. $\begin{cases} x=4\cos t \\ y=3\sin t \end{cases}$
4. $y=1-\ln\cos x, x_1=0, x_2=\pi/6$	4. $y=\ln\sin x, x_1=\pi/3, x_2=\pi/2$
5. $r=4\cos\theta$	5. $r=5\sin\theta$
6. $y=x^2, x=y^2, OX$	6. $x^2=16y,\ y=4, OX$

2.4 Applications to Physics and Engineering

Work

In physics the term *work* has a technical meaning that depends on the idea of a *force.* Intuitively, you can think of a force as describing a push or pull on an object.

Let the material point move along x-axis under the action of force $F(s)$. The work of this force in the section of way $[a,b]$ is determined by the formula

$$A=\int_a^b F(s)ds \tag{1}$$

Moments and Centers of Mass

Among many applications of integral calculus to physics and engineering, we consider one here: centers of mass. As with our previous applications to geometry (areas, volumes, and lengths) and to work, our strategy is to break up the physical quantity into a large number of small parts, approximate each small part, add the results, take the limit, and then evaluate the resulting integral.

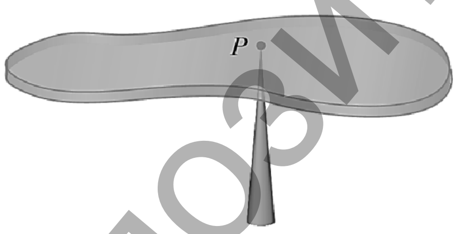


Figure 16

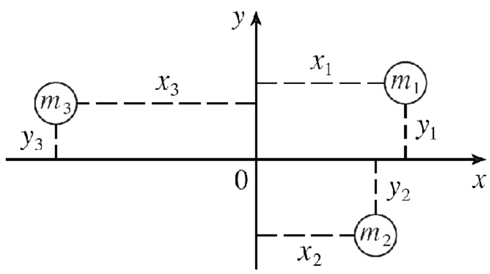


Figure 17

Our main objective here is to find the point on which a thin plate of any given shape balances horizontally as in Figure 16. This point is called the ***center of mass*** (or *center of gravity*) of the plate.

If we have a system of particles with masses $m_1, m_2, ..., m_n$, located at the points $x_1, x_2, ..., x_n$ on the x-axis, it can be shown similarly that the center of mass of the system is located at

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \qquad (2)$$

The sum of the individual moments $M = \sum_{i=1}^{n} m_i x_i$ is called the *moment of the system about the origin***.** Then Equation 2 could be rewritten as $m \cdot \bar{x} = M$, which says that if the total mass was considered to be concentrated at the center of mass *x*, then its moment would be the same as the moment of the system.

Now we consider a system of particles with masses $m_1, m_2, ..., m_n$, located at the points $(x_1; y_1)$, $(x_2; y_2)$,..., $(x_n; y_n)$ in the xy-plane as shown in Figure 17.

By analogy with the one-dimensional case, we define the ***moment of the system about the y-axis*** as

$$M_y = \sum_{i=1}^{n} m_i x_i \qquad (3)$$

and the ***moment of the system about the x-axis*** as

$$M_x = \sum_{i=1}^{n} m_i y_i \qquad (4)$$

Then M_y measures the tendency of the system to rotate about the y-axis and M_x measures the tendency to rotate about the x-axis.

As in the one-dimensional case, the coordinates of the center of mass are given in terms of the moments by the formulas

$$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M} \qquad (5)$$

Next we consider a flat plate (called a *lamina*) with a uniform density ρ that occupies a region $\Re$ of the plane. We wish to locate the center of mass of the plate, which is called the ***centroid*** of $\Re$. In doing so we use the following physical principles:

The ***symmetry principle*** says that if $\Re$ is symmetric about a line l, then the centroid of $\Re$ lies on l. (If $\Re$ is reflected about l, then $\Re$ remains the same, so its centroid remains fixed. But the only fixed points lie on l). Thus *the centroid of a rectangle is its center*.

Moments should be defined so that if the entire mass of a region is concentrated at the center of mass, then its moments remain unchanged. Also, the moment of the union of two nonoverlapping regions should be the sum of the moments of the individual regions.

Suppose that the region $\Re$ is of the type shown in Figure 18; that is, $\Re$ lies between the lines $x = a$ and $x = b$, above the x-axis, and beneath the graph of $f(x)$, where $y = f(x)$ is a continuous function. We divide the interval $[a,b]$ into n subintervals with endpoints $a = x_0, x_1, x_2, ..., x_n = b$ and the equal width Δx. We choose the sample point x_i^* to be the midpoint $\overline{x_i}$ of the i-th subinterval, that is $\overline{x_i} = (x_i + x_{i+1})/2$. This determines the polygonal approximation to $\Re$ shown in Figure 19. The centroid of the i-th approximating rectangle R_i is its center $C_i\left(\overline{x_i}; \frac{1}{2} f\left(\overline{x_i}\right)\right)$. Its area is $f(\overline{x_i})\Delta x_i$, so its mass is $m_i = \rho \cdot f(\overline{x_i})\Delta x_i$.

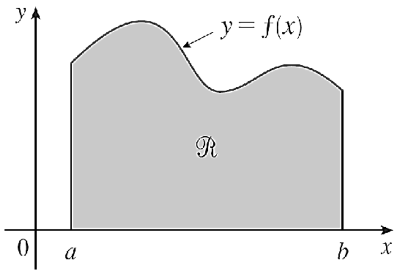


Figure 18

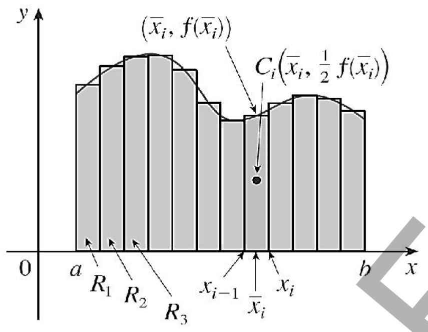


Figure 19

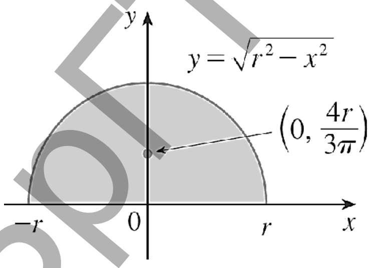


Figure 20

Adding these moments, we obtain the moment of the polygonal approximation to $\Re$, and then by taking the limit as $n \to \infty$ we obtain the moment of $\Re$ itself about the $y-$ axis:

$$M_y = \rho \int_a^b x \cdot f(x) dx \qquad (6)$$

Again we add these moments and take the limit to obtain the moment of $\Re$ about the x- axis:

$$M_x = \frac{1}{2} \rho \int_a^b f^2(x) dx \qquad (7)$$

The mass of the plate is the product of its density and its area: $M = \rho \int_a^b f(x) dx$.

Note 1 If the plane figure is limited by the lines $y = f_1(x),\ y = f_2(x),\ x \in [a,b]$ and $\rho = \rho(x)$ is the surface density of figure, then

$$M = \int_a^b \rho(x)\left(f_2(x) - f_1(x)\right) dx,$$

$$M_y = \int_a^b x \cdot \rho(x)\left(f_2(x) - f_1(x)\right) dx, \qquad (8)$$

$$M_x = \frac{1}{2} \int_a^b \rho(x)\left(f_2^2(x) - f_1^2(x)\right) dx$$

Note 2 The static moments of the material arc, assigned by the equation $y = f(x),\ x \in [a,b]$ relative to coordinate axes are found by the formulas

$$M = \int_a^b \rho(x)\sqrt{1+\left(f'(x)\right)^2}\,dx,$$

$$M_x = \int_a^b \rho(x)\cdot f(x)\sqrt{1+\left(f'(x)\right)^2}\,dx, \qquad (9)$$

$$M_y = \int_a^b \rho(x)\cdot x\sqrt{1+\left(f'(x)\right)^2}\,dx.$$

Example 1 Find the center of mass of a semicircular plate of radius r.

Solution We place the semicircle as in Figure 20 so that $f(x) = \sqrt{r^2 - x^2}$ and $a = -r,\ b = r$.

Here there is no need to use the formula to calculate $\overline{x}$ because, by the symmetry principle, the center of mass must lie on the y-axis, so $\overline{x} = 0$. The area of the semicircle is $S = \frac{1}{2}\pi r^2$, so

$$\overline{y} = \frac{1}{\frac{1}{2}\pi r^2}\cdot\frac{1}{2}\int_{-r}^{r}\left(\sqrt{r^2-x^2}\right)^2 dx = \frac{2}{\pi r^2}\cdot\int_0^r\left(r^2-x^2\right)dx = \frac{2}{\pi r^2}\left[r^2x-\frac{x^3}{3}\right]_0^r =$$

$$= \frac{2}{\pi r^2}\cdot\frac{2r^3}{3} = \frac{4r}{3\pi}.$$

The center of mass is located at the point $\left(0;\frac{4r}{3\pi}\right)$.

Exercise Set 2.4

In exercises 1 to 6 find the coordinates of the center of the masses of a flat uniform figure (Φ) bounded by the assigned curves.

1. $\begin{cases} y = x^2 \\ y = 2 - x \end{cases}$ *2.* $\begin{cases} y^2 = 2x - 2 \\ y = x - 1 \end{cases}$ *3.* $\begin{cases} x^2 + 4y - 16 = 0 \\ y = 0 \end{cases}$

4. $\begin{cases} y^2 = 4 - x \\ y = 0 \end{cases}$ *5.* $\begin{cases} y = x^2 \\ y = \sqrt{x} \end{cases}$ *6.* $r = 1 + \cos\theta$

In exercises 7 to 12 find the coordinates of the center of the masses of a flat uniform curve (L).

7. $r = 2\cos\theta,$ $0 \le \varphi \le \pi / 4$

8. $r = 3\sin\theta,$ $\pi / 6 \le \varphi \le \pi / 4$

9. $r = 1 + \cos\theta,\ 0 \le \varphi \le \pi$

10. $\begin{cases} x = 5\cos t \\ y = 5\sin t \end{cases},$ $0 \le t \le \pi / 2$

11. $\begin{cases} x = 3\cos^3 t \\ y = 3\sin^3 t \end{cases},$ $0 \le t \le \pi / 2$

12. $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases},$ $0 \le t \le 2\pi$

Individual Tasks 2.4

1-2. Find the coordinates of the center of the masses of a flat uniform figure (Φ) bounded by the assigned curves.

3. Find the coordinates of the center of the masses of a flat uniform curve (L).

	II.
1. $\begin{cases} x = y^2 \\ y = x^2 \end{cases}$	1. $\begin{cases} y = x^2 \\ y = x + 2 \end{cases}$
2. $\begin{cases} y = 0 \\ x = \pi \\ y = \sin x \end{cases}$	2. $\begin{cases} y = 0, x = 0 \\ x = \pi / 2 \\ y = \cos x \end{cases}$
3. $r = 2\sqrt{3}\cos\theta,\ 0 \le \varphi \le \pi / 4$	3. $r = 2\sin\theta,\ 0 \le \varphi \le \pi / 4$

Additional Tasks 2

1. Calculate the areas of the figures, bounded by the assigned curves.

a) $y^2 = 4x,\ x^2 = 4y$ *b)* $r = 2(1 + \sin\theta)$ *c)* $\begin{cases} x = 4\cos t \\ y = 3\sin t \end{cases}$

2. Find the arc length function for the curve.

a) $y = 1 - \ln\cos x$, $x_1 = 0$, $x_2 = \pi / 6$ *b)* $r = 5\sin\theta$

3. Find the volume of the body, formed by the rotation of the figure bounded by the assigned curves around the indicated axis.

$x^2 = 16y,\ y = 4, OX$ $r = 2(1 + \cos\theta)$, polar axis

4. Calculate the areas of the figures, bounded by the first and second loops of the Archimedes spiral $r = a\theta$.

5. Find the distance traveled by the material point to a stop, if the velocity of movement is given by $v = te^{-0.01t}$.

6. A circular swimming pool has a diameter of 24 ft, the sides are 5 ft high, and the depth of the water is 4 ft. How much work is required to pump all of the water out over the side?

III DIFFERENTIAL EQUATIONS

The one of the most important of all the applications of calculus is the differential equations. When physical scientists or social scientists use calculus, they analyze a differential equation that has arisen in the process of modeling some phenomenon that they are studying.

3.1 General Differential Equations. Separable Equations

Definition A ***differential equation*** is an equation that contains an unknown function and one or more of its derivatives. The ***order*** of a differential equation is the order of the highest derivative that occurs in the equation.

Thus, equation $y' = xy$ is a *first-order equation*. In this equation the independent variable is called x. A differential equation of the first order is $F(x, y, y') = 0$ or $y' = f(x, y)$.

Definition A function $f(x)$ is called a ***solution*** of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation.

When we are asked to *solve* a differential equation we are expected to find all possible solutions of the equation.

Definition A ***general solution*** of the first-order differential equation is a function $y = \varphi(x, C)$, $(C = const)$ such that:

1) $y = \varphi(x, C)$ is a solution of this equation for any value of C;

2) for any admissible initial condition $y(x_0) = y_0$ there is a value $C = C_0$ at which the function $y = \varphi(x, C_0)$ satisfies the given initial condition.

Definition A ***particular solution*** of a differential equation is the solution obtained from the general solution for a specific value of the constant C.

When applying differential equations, we are usually not as interested in finding a family of solutions (*the general solution*) as in finding a solution that satisfies some additional requirement. In many physical problems we need to find the particular solution that satisfies a condition of the form $y(x_0) = y_0$. This is called an *initial condition*, and the problem of finding a solution of the differential equation that satisfies the initial condition is called an ***initial-value problem***.

Geometrically, when we impose an initial condition, we look at the family of solution curves and pick the one that passes through the point (x_0, y_0). Physically, this corresponds to measuring the state of a system at time t_0 and using the solution of the initial-value problem to predict the future behavior of the system.

Modeling with Differential Equations

In describing the process of modeling, we talked about formulating a mathematical model of a real-world problem either through intuitive reasoning about the phenomenon or from a physical law based on evidence from experiments. The mathematical model often takes the form of a differential equation, that is, an equation that contains an unknown function and some of its derivatives.

Models of Population Growth

One model for the growth of a population is based on the assumption that the population grows at a rate proportional to the size of the population. That is a reasonable assumption for a population of bacteria or animals under ideal conditions (unlimited environment, adequate nutrition, absence of predators, immunity from disease).

Let's identify and name the variables in this model: $t =$ time (the independent variable); $P =$ the number of individuals in the population (the dependent variable).

The rate of growth of the population is the derivative dP/dt. So our assumption that the rate of growth of the population is proportional to the population size is written as the equation

$$\frac{dP}{dt} = kP \qquad (1)$$

where k is the proportionality constant. Equation 1 is our first model for population growth; it is a differential equation because it contains an unknown function $P(t)$ and its derivative dP/dt.

Having formulated a model, let's look at its consequences. If we rule out a population of 0, then $P(t) > 0$ for all values of t. So, if $k > 0$, then Equation 1 shows, that $P'(t) > 0$ for all t.

This means that the population is always increasing. In fact, as $P(t)$ increases, Equation 1 shows that dP/dt becomes larger. In other words, the growth rate increases as the population increases.

Equation 1 asks us to find a function whose derivative is a constant multiple of itself. We know that exponential functions have that property. In fact, if we let $P(t) = Ce^{kt}$, then $P'(t) = (Ce^{kt})' = Cke^{kt} = k(Ce^{kt}) = kP(t)$.

Thus any exponential function of the form $P(t) = Ce^{kt}$ is a solution of Equation 1. Allowing C to vary through all the real numbers, we get the *family* of solutions $P(t) = Ce^{kt}$ whose graphs are shown in Figure 21. But populations have only positive values and so we are interested only in the solutions with $C > 0$. And we are probably concerned only with the values of t greater than the initial time $t = 0$. Figure 22 shows the physically meaningful solutions. Putting $t = 0$, we get $P(0) = Ce^{0k} = C$, so the constant C turns out to be the initial population $P(0)$.

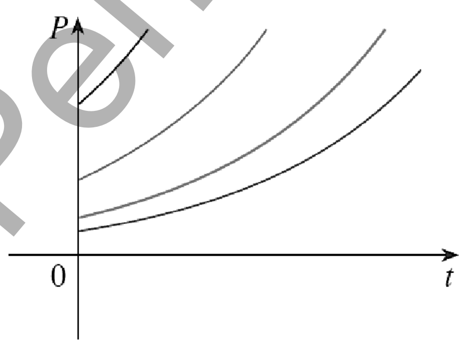


Figure 21

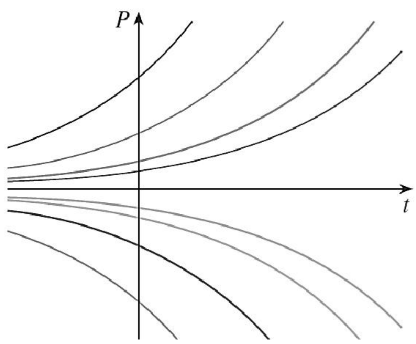


Figure 21

Separable Equations

Definition A ***separable equation*** is a first-order differential equation in which the expression for dy/dx can be factored as a function of x times a function of y. In other words, it can be written in the form:

$$\frac{dy}{dx} = f(x)\cdot\varphi(y) \qquad (2)$$

The name *separable* comes from the fact that the expression on the right side can be "separated" into a function of x and a function of y. Equivalently, if $\varphi(y)\neq 0$, we could solve this equation rewriting it in the differential form:

$$dy = f(x)\cdot\varphi(y)dx, \quad \frac{dy}{\varphi(y)} = \frac{dx}{f(x)}.$$

Then we integrate both sides of the equation:

$$\int\frac{dy}{\varphi(y)} = \int\frac{dx}{f(x)} \qquad (3)$$

Equation 3 implicitly defines y as a function of x. In some cases we may be able to solve for y in terms of x.

Note 1 If $M(x)\cdot N(y)dx + P(x)\cdot Q(y)dy = 0$, then

$$\int\frac{M(x)}{P(x)}dx + \int\frac{Q(y)}{N(y)}dy = C, \quad P(x)\neq 0, \; N(y)\neq 0.$$

Definition A function $\varphi(x,y)$ is called a *homogeneous function of degree n* with respect to the variables x and y, if for any $t\in R$ the identity $\varphi(tx,ty) = t^n\cdot\varphi(x,y)$ holds.

Definition The differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0 \qquad (4)$$

is called a ***homogeneous differential first order equation***, if $M(x,y)$, $N(x,y)$ are homogeneous functions of the same order.

If functions $M(x,y), N(x,y)$ are the uniform functions of one and the same measurement, then the equation (4) is possible to lead to the form

$$\frac{dy}{dx} = \varphi\left(\frac{y}{x}\right) \qquad (5)$$

The substitution $y = x\cdot u(x)$ can be used to converting the equations (4) or (5) to the separable equation.

Example 1 Integrate the differential equation of the model for population growth

$$\frac{dP}{dt} = kP.$$

Solution We write the equation in terms of differentials and integrate both sides:

$$\frac{dP}{P} = kdt \;\Rightarrow\; \int\frac{dP}{P} = \int kdt \;\Rightarrow\; \ln|P| = kt + C,$$

where C is an arbitrary constant. (We could have used a constant C_1 on the left side and another constant C_2 on the right side. But then we could combine these constants by writing $C = C_2 - C_1$). Solving for P, we get

$$P(t) = e^{kt+C} = e^{kt}e^{C} = Ae^{kt} \ , A = const\,.$$

Example 2 Solve the differential equation

$$(y^2 + xy^2)\cdot y' + x^2 - yx^2 = 0\,.$$

Solution We write the equation in terms of differentials and integrate both sides:

$$y^2(1+x)dy = x^2(y-1)dx\,.$$

If $x \neq 0\,, y \neq 0\,, x \neq -1\,, y \neq 1$, then

$$\frac{y^2}{y-1}dy = \frac{x^2}{x+1}dx \Rightarrow \int\left(y+1+\frac{1}{y-1}\right)dy = \int\left(x-1+\frac{1}{x+1}\right)dx \Rightarrow$$

$$\frac{y^2}{2} + y + \ln|y-1| = \frac{x^2}{2} - x + \ln|x+1| + C\,,$$

where C is an arbitrary constant.

Example 3 Find the solution of the initial-value problem

$$\left(x^2 - 3y^2\right)dx + 2xy\,dy = 0,\ y(2) = 1\,.$$

Solution Let us write down the equation in the form:

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(3\frac{y}{x} - \frac{x}{y}\right).$$

Since $\varphi(tx,ty) = 3\frac{ty}{tx} - \frac{tx}{ty} = 3\frac{y}{x} - \frac{x}{y} = \varphi(x,y)$, then the replacement of the unknown function of $y = x\cdot u(x)$ can be introduced and $y' = u(x) + x\cdot u'(x)$.

Substituting these expressions into the initial equation, we will obtain the separable equation:

$$x\cdot u'(x) + u = \frac{1}{2}\left(3u - \frac{1}{u}\right) \Rightarrow xu' = \frac{1}{2}\left(u - \frac{1}{u}\right) \Leftrightarrow xu' = \frac{u^2-1}{2u} \Leftrightarrow x\,du = \frac{u^2-1}{2u}dx \Leftrightarrow$$

$$\Leftrightarrow \frac{2u\,du}{u^2-1} = \frac{dx}{x} \Rightarrow \int\frac{2u\,du}{u^2-1} = \int\frac{dx}{x} \ \Rightarrow\ \ln|u^2-1| = \ln|x| + \ln|C| \ \Leftrightarrow\ u^2 - 1 = Cx;$$

$$\left(\frac{y}{x}\right)^2 - 1 = Cx\,.$$

Since $y(2) = 1$, we have $1 - 4 = 8C$; $C = -3/8$. Therefore the solution to the initial-value problem is

$$\frac{y^2}{x^2} = 1 - \frac{3}{8}x, \quad y = \pm x\sqrt{1 - \frac{3}{8}x}\,.$$

Exercise Set 3.1

In exercises 1 to 12 solve the differential equation.

1. $xdx+\frac{2dy}{y+5}=0$ 2. $xyy'=1-x^2$ 3. $\left(2x+e^x\right)dx-\frac{dy}{y}=0$

4. $xy'=y^2+1$ 5. $e^{x+3y}dy=xdx$ 6. $tgx\cdot\sin^2 ydx+\cos^2 x\cdot ctgydy=0$

7. $y'=\frac{y^2}{x^2}-2$ 8. $y'=-\frac{x+y}{y}$ 9. $ydx+\left(\sqrt{xy}-\sqrt{x}\right)dy=0$

10. $xy'=y\ln\frac{x}{y}$ 11. $(x+2y)dx-xdy=0$ 12. $xy'=\sqrt{y^2-x^2}$

In exercises 13 to 18 find the solution of the differential equation that satisfies the given initial condition.

13. $\left(xy^2+x\right)dx+\left(y-x^2y\right)dy=0, y(0)=1$ 14. $y'\cdot\sin x=y\ln y,\ y\left(\frac{\pi}{2}\right)=e$

15. $\sin y\cdot\cos x\,dy=\cos y\cdot\sin x\,dx,\ y(0)=\frac{\pi}{2}$ 16. $xy'=y+\sqrt{x^2+y^2},\ y(1)=0$

17. $(x+xy)dy+(y-xy)dx=0,\ y(1)=1$ 18. $xy'=y(1+\ln y-\ln x), y(1)=e^2$

19. A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?

20. Suppose you have just poured a cup of freshly brewed coffee with temperature $95^\circ C$ in a room where the temperature is $20^\circ C$. Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. Write a differential equation that expresses Newton's Law of Cooling for this particular situation. Solve the differential equation to find an expression for the temperature of the coffee at time t.

21. One model for the spread of a rumor is that the rate of spread is proportional to the product of the fraction of the population who have heard the rumor and the fraction who have not heard the rumor. Write a differential equation that is satisfied by y and solve the differential equation. A small town has 1000 inhabitants. At 8 AM, 80 people have heard a rumor. By noon half the town has heard it. At what time will 90% of the population have heard the rumor?

Individual Tasks 3.1

1-3. Solve the differential equation.

4-6. Find the solution of the differential equation that satisfies the given initial condition.

1. $xy'-y=y^3$	**II.** 1. $\left(x^2+x\right)y'=2y+1$

2. $x\sqrt{4+y^2}dx - y\sqrt{1+x^2}dy = 0$	2. $4\left(yx^2 + y\right)dy + \sqrt{5+y^2}dx = 0$
3. $3y' = \dfrac{y^2}{x^2} + 9\dfrac{y}{x} + 9$	3. $xy'\cos\dfrac{y}{x} = y\cos\dfrac{y}{x} - x$
4. $y'\cos^2 x = y\ln y,\ \ y\left(\pi/4\right) = e$	4. $xdy - (y+1)dx = 0,\ \ y(2) = 5$
5. $xy' = x\sin\dfrac{y}{x} + y,\ \ y\left(2\right) = \pi$	5. $\left(xy' - y\right)arctg\dfrac{y}{x} = x,\ \ y(1) = 0$
6. $\left(x^2 + y^2\right)dx = 2xydy,\ \ y(4) = 0$	6. $(x^2 - 3y^2)dx + 2xydy = 0,\ y(2) =$

3.2 Linear Equations

Definition A ***first-order linear differential equation*** is the one that can be put into the form:

$$y' + p(x)y = q(x) \quad \left(or \quad A(x)y' + B(x)y + C(x) = 0\right) \qquad (1)$$

where $p(x)$ and $q(x)$ are continuous functions on a given interval.

It turns out that every first-order linear differential equation can be solved in a similar way by multiplying both sides of Equation 1 by a suitable function called an *integrating factor*

$$I(x) = e^{\int p(x)dx} \qquad (2)$$

Thus a formula for the general solution to Equation 1 is provided by the solution

$$y(x) = \frac{1}{I(x)}\left(\int I(x)q(x)dx + C\right) \qquad (3)$$

where $I(x) = e^{\int p(x)dx}$. Instead of memorizing this formula, however, we just remember the form of the integrating factor.

Note 1 It turns out that every first-order linear differential equation can be solved by using the substitution $y = u(x)\cdot v(x)$, where $u(x), v(x)$ are unknown functions. We reduce this equation to the form:

$$u'v + uv' + p(x)uv = q(x), \qquad u'v + u\left(v' + p(x)v\right) = q(x).$$

Since one of the unknown functions can be selected arbitrarily, then $v(x)$ is taken as any particular solution of the equation

$$v' + p(x)v = 0,$$

function $u(x)$ will be determined from the equation

$$u'(x)\cdot v(x) = q(x).$$

Thus, the solution of a linear equation is reduced to the sequential solution of two equations with the divided variables relative to each of the auxiliary functions.

Note 2 A first-order linear differential equation is one that can be put into the form

$$x' + p(y)x = q(y).$$

This linear differential equation can be solved with the help of the substitution

$$x(y)=u(y)\cdot v(y).$$

Note 3 The ***Bernoulli equation*** takes the form

$$y'+p(x)y=q(x)\cdot y^n \text{ or } x'+p(y)x=q(y)x^n,\ n\in R.$$

These equations can be reduced to the appropriate linear equations, but then they are usually solved with the help of the substitution

$$y=u(x)\cdot v(x) \ \ or\ \ x=u(y)\cdot v(y).$$

Example 1 Solve the differential equation $y'+\dfrac{3y}{x}=x^2$.

Solution We use the substitution $y=u\cdot v,\ y'=u'v+uv'$. We obtain the following equation

$$u'v+uv'+\frac{3uv}{x}=x^2 \ \Rightarrow\ u'v+u\left(v'+\frac{3v}{x}\right)=x^2.$$

We solve two equations $v'+\dfrac{3v}{x}=0$ and $u'v=x^2$.

$$\frac{dv}{dx}=-\frac{3v}{x}\Leftrightarrow\frac{dv}{v}=-\frac{3dx}{x}\Rightarrow\int\frac{dv}{v}=-\int\frac{3dx}{x}\Leftrightarrow\ \ln|v|=-3\ln|x|\Rightarrow\ \ v=\frac{1}{x^3}.$$

$$u'\cdot\frac{1}{x^3}=x^2\ \Leftrightarrow\ \frac{du}{dx}=x^5\ \Leftrightarrow\ du=x^5dx\ \Rightarrow\ u=\int x^5dx=\frac{x^6}{6}+C.$$

Multiplying $u(x)$ on $v(x)$, we obtain the general solution of this equation

$$y=\frac{1}{x^3}\left(\frac{x^6}{6}+C\right)\ or\ y=\frac{x^3}{6}+\frac{C}{x^3},\ \ C-const.$$

Example 2 Find the solution of the initial-value problem

$$2ydx+\left(y^2-6x\right)dy=0,\ y(6)=2.$$

Solution It is easy to see that this equation is not linear relative to y. Let us write it down in the form

$$2y\frac{dx}{dy}+y^2-6x=0;\quad \frac{dx}{dy}-\frac{3}{y}x=-\frac{y}{2};\quad x=uv;\quad u'v+uv'-\frac{3}{y}uv=-\frac{y}{2};$$

$$u'v+u\left(v'-\frac{3}{y}v\right)=-\frac{y}{2};\quad \frac{dv}{dy}=\frac{3v}{y};\quad \frac{dv}{v}=\frac{3dy}{y};\quad \ln|v|=3\ln|y|;\quad v=y^3.$$

Then from the equation $u'v=-\dfrac{y}{2}$ we determine the function $u(y)$:

$$u'y^3=-\frac{y}{2};\quad u'=-\frac{1}{2y^2};\ du=-\frac{dy}{2y^2};\quad u=\frac{1}{2y}+C.$$

Let us extract the general solution of the initial equation

$$x=uv;\quad x=\left(\frac{1}{2y}+C\right)y^3;\quad x=Cy^3+\frac{y^2}{2}.$$

Since $y(6)=2$, we have $6=8C+2$, $C=1/2$. Therefore the solution to the initial-value problem is $x=0{,}5\left(y^3+y^2\right)$.

Exercise Set 3.2

In exercises 1 to 9 solve the differential equation.

1. $y'-\dfrac{y}{x}=x$	*2.* $xy'+2y=x^2$	*3.* $xy'+y=xy^2\ln x$
4. $y'+\dfrac{2y}{x}=\dfrac{2\sqrt{y}}{\cos^2 x}$	*5.* $y'-\dfrac{2y}{x+1}=e^x(x+1)^2$	*6.* $y^2dx-(2xy+3)dy=0$
7. $2xydx+\left(y-x^2\right)dy=0$	*8.* $x\,y'-3y=x^4-2x^3+5x$	*9.* $\dfrac{dy}{dx}+\dfrac{y}{x}=-xy^2$

In exercises 10 to 15 find the solution of the differential equation that satisfies the given initial condition.

10. $y'+\dfrac{3}{x}y=\dfrac{2}{x^3},\ y(1)=1$	*11.* $x^2y'+2xy=\ln x,\ y(e)=1$
12. $xy'+y-e^x=0,\ y(a)=b$	*13.* $y'+3y=e^{2x}y^2,\ y(0)=1$
14. $y'+\dfrac{6y}{x}=x^3,\ y(1)=3$	*15.* $y'-ytgx=\dfrac{1}{\cos^2 x},\ y(0)=2$

Individual Tasks 3.2

1-3. Solve the differential equation.

4-6. Find the solution of the differential equation that satisfies the given initial condition.

	II.
1. $y'+\dfrac{4y}{x}=\dfrac{x^2-3x+1}{x^4}$	*1.* $y'+\dfrac{3y}{x}=\dfrac{4x-5}{x^2}$
2. $3xy'-2y=\dfrac{x^3}{y^2}$	*2.* $y'=\dfrac{y}{2y\ln y+y-x}$
3. $\left(1+x^2\right)y'=xy+x^2y^2$	*3.* $x^2y'+2x^3y=y^2\left(1+2x^2\right)$
4. $y'+ytgx=\dfrac{1}{\cos x},\ y(\pi)=5$	*4.* $y'-\dfrac{4y}{x}=3+2x-x^2,\ y(1)=4$
5. $y'-7y=e^{3x}y^2,\ y(0)=2$	*5.* $xdy=\left(e^{-x}-y\right)dx,\ y(1)=1$
6. $y^2dx=\left(x+ye^{-\frac{1}{y}}\right)dy,\ y(0)=-3$	*6.* $y'-\dfrac{y}{x-3}=\dfrac{y^2}{x-3},\ y(1)=-2$

3.3 Higher Order Differential Equations Admitting a Reduction of the Order

Definition A differential equation is called a *differential equation order* n, if it can be represented as follows:

$$F\left(x,y,y',..,y^{(n)}\right)=0 \qquad (1)$$

Definition A function $y=\varphi(x)$ is called a *solution of a differential equation* if the equation is satisfied when $y=\varphi(x)$ and its derivatives are substituted into the equation.

Definition A function $y=\varphi\left(x,c_1,c_2,..,c_n\right)$ is called a *general solution of a differential equation* (1) if the function is satisfied to following conditions:

1. A function $y=\varphi\left(x,c_1,c_2,..,c_n\right)$ is a solution of a differential equation for any fixed values of constants $c_1,c_2,..,c_n$;

2. There are unique values of constants $c_1=c_1^0,\ c_2=c_2^0,\ ..,\ c_n=c_n^0$ for any initial conditions

$$y\left(x_0\right)=y_0,y'\left(x_0\right)=y_0',...,y^{(n-1)}\left(x_0\right)=y_0^{(n-1)} \qquad (2)$$

such that a function $y=\varphi(x,c_1^0,c_2^0,..,c_n^0)$ is a solution of a differential equation and it satisfies initial conditions (2).

Definition A function $y=\varphi(x,c_1^0,c_2^0,..,c_n^0)$ is called a *partial solution* of a differential equation (1) if it can be obtained from a general solution with any fixed values of constants $c_1=c_1^0,\ c_2=c_2^0,\ ..,\ c_n=c_n^0$.

The problem of finding a solution of the differential equation $F\left(x,y,y',..,y^{(n)}\right)=0$ that satisfies the initial condition $y\left(x_0\right)=y_0,y'\left(x_0\right)=y_0',...,y^{(n-1)}\left(x_0\right)=y_0^{(n-1)}$ is called an *initial-value problem*.

The simplest method of solution of a differential equation is *the method of reducing the order of a differential equation*. The essence of the method is that this equation can be reduced to an equation of a lower order by means of a change of variable. Let us consider some types of higher-order equations that allow a decrease in order.

1. $y^{(n)}=f(x)$

The general solution is found by the n -times integration method.

Example 1 Solve the differential equation $y'''=x+\cos x$.

Solution We successively integrate this equation 3 times

$$y''=\int\left(x+\cos x\right)dx=\frac{x^2}{2}+\sin x+c_1;$$

$$y'=\int\left(\frac{x^2}{2}+\sin x+c_1\right)dx=\frac{x^3}{6}-\cos x+c_1x+c_2;$$

$$y=\int\left(\frac{x^3}{6}-\cos x+c_1x+c_2\right)dx=\frac{x^4}{24}-\sin x+c_1\frac{x^2}{2}+c_2x+c_3.$$

2. The equation obviously does not contain a function y*:* $y''=f(x,y')$

With the help of the substitution $y'=p(x),\ y''=p'(x)$ the equation $y''=f(x,y')$ is converted into a first order differential equation. The solution of this equation depends on its type.

We write the general solution in the form of $p=\varphi\left(x,C_1\right)$. We substitute it in for the unknown function $p\left(x\right)=y'\left(x\right)$ and solve the separable equation:

$$\frac{dy}{dx}=\varphi(x,C_1) \Rightarrow dy=\varphi(x,C_1)dx \Rightarrow y=\int\varphi(x,C_1)dx+C_2.$$

Example 2 Find the solution of the initial-value problem

$$xy''=y'\ln\frac{y'}{x},\ y\left(1\right)=e,\ y'\left(1\right)=e^2$$

Solution This differential equation is a second-order equation that does not explicitly contain the variable y. We reduce the order of the equation by substituting $y'=p\left(x\right),\ y''=p'\left(x\right)$.The initial equation is transformed into a homogeneous differential equation of the first order with respect to the unknown function $p\left(x\right)$:

$$xp'=p\ln\frac{p}{x} \Rightarrow p'=\frac{p}{x}\ln\frac{p}{x}.$$

Solve it in a known way

$$\frac{p\left(x\right)}{x}=u\left(x\right),\ p\left(x\right)=x\cdot u(x),\ \ p'=u+xu';\Rightarrow u+xu'=u\ln u$$

$$\frac{du}{u\left(\ln u-1\right)}=\frac{dx}{x};\Rightarrow \ln|\ln u-1|=\ln|x|+\ln|C|;\ \Rightarrow\left|\ln u-1\right|=\left|Cx\right|;\ \Rightarrow$$

$$\ln u-1=\pm Cx;\ \pm C=C_1 \Rightarrow \ln u-1=C_1x;\ \Rightarrow$$

$$u=e^{1+C_1x} \Rightarrow p=xe^{1+C_1x} \Rightarrow y'=xe^{1+C_1x}.$$

We use the initial condition $y'(1)=e^2$ or $p(1)=e^2$:

$$e^2=1\cdot e^{1+C_1};\ 2=1+C_1;\ \ C_1=1.$$

Hence, we obtain the equation

$$y'=xe^{x+1} \Rightarrow y=\int xe^{x+1}dx=xe^{x+1}-e^{x+1}+C_2.$$

From the initial condition $y\left(1\right)=e$ we find the value of the constant C_2:

$$e=e^2-e^2+C_2;\ \Rightarrow\ C_2=e.$$

Thus, a particular solution of the original equation is the function

$$y=(x-1)e^{x+1}+e.$$

3. The equation obviously does not contain a variable x*:* $y''=f(y,y')$

With the help of the substitution $y'=p\left(y\right),\ y''=p'p$ the equation $y''=f(y,y')$ is converted into a first order differential equation. The solution of this equation depends on its type.

Example 3 Find the solution of the initial-value problem

$$yy'+y'^2+yy''=0,\ y(0)=1,\ y'(0)=1.$$

Solution

This equation is a second-order differential equation in which the variable x is obviously not present. We reduce the order of the equation by the substitution $y' = p(y)$, $y'' = p'p$. We obtain the first-order equation:

$$yp + p^2 + ypp' = 0 \Rightarrow p(yp' + p + y) = 0 \Rightarrow \left[\begin{array}{l} p = 0, \\ yp' + p + y = 0. \end{array}\right.$$

We solve a homogeneous differential equation of the first order in a known way:

$$p' = -\frac{p}{y} - 1 \Rightarrow \frac{dp}{dy} = -\frac{p}{y} - 1;$$

$$\frac{p}{y} = u(y), \ p = y \cdot u, \ p' = u + yu';$$

$$u + yu' = -u - 1 \Rightarrow yu' = -2u - 1 \Rightarrow ydu = -(2u+1)dy \Rightarrow$$

$$\frac{du}{2u+1} = -\frac{dy}{y} \Rightarrow \frac{1}{2}\ln|2u+1| = -\ln|y| + \frac{1}{2}\ln|C| \Rightarrow 2u + 1 = \frac{C_1}{y^2}.$$

Exercise Set 3.3

In exercises 1 to 9 solve the differential equation.

1. $y''' = 2x + \cos x$ *2.* $y^{IV} = \frac{y'''}{x}$ *3.* $x^2 y'' + xy' = 1$

4. $yy'' + y'^2 = 1$ *5.* $y''' = x + \sin 3x$ *6.* $x^2 y''' = (y'')^2$

7. $xy'' = y'(\ln y' - \ln x)$ *8.* $y'' tg y = 2(y')^2$ *9.* $xy'' + y' = (y')^2$

In exercises 10 to 15 find the solution of the differential equation that satisfies the given initial condition.

10. $y''' = \frac{x}{(x+2)^5}$, $y(1) = y'(1) = y''(1) = 0$

11. $y'' = e^{2y}$, $y(0) = 0$, $y'(0) = 1$

12. $2yy'' = (y')^2$, $y(-1) = 4$, $y'(-1) = 1$

13. $y'' = \frac{\ln x}{x^2}$, $y(1) = 3$, $y'(1) = 1$

14. $xy''' - y'' = x^2 + 1$, $y(-1) = 0$, $y'(-1) = 1$, $y''(-1) = 0$

15. $(x+1)y'' + xy'^2 = y'$, $y(1) = -2$, $y'(1) = 4$

Individual Tasks 3.3

1-3. Solve the differential equation.

4-6. Find the solution of the differential equation that satisfies the given initial condition.

	II.
1. $y''' = x^2 - \sin x$	*1.* $y'' = x \sin x$
2. $y''' = (y'')^2$	*2.* $2yy'' = 3y'^2 + 4y^2$
3. $xy'' - y' = x^2 e^x$	*3.* $(1 + x^2)y'' - 2xy' = 0$
4. $xy''' = 2$, $y(1) = 0{,}5$	*4.* $y'' = \frac{\ln x}{x^2}$, $y(1) = 3$, $y'(1) = 1$

$y'(1)=y''(1)=0$	5. $xy''' - y'' = x^2+1,\ y(-1)=0$
5. $y''(x^2+1)=2xy'$,	$y'(-1)=1,\ y''(-1)=0$
$y(0)=1,\ y'(0)=3$	6. $y''=e^{3y},\ y(0)=0,\ y'(0)=1$
6. $y^3y''+1=0,\ y(1)=1,\ y'(1)=0$	

3.4 Linear Homogeneous Differential Equations

Definition A differential equation is called a ***second-order linear differential equation***, if it can be represented as follows:

$$P(x)\cdot y'' + Q(x)\cdot y' + R(x)\cdot y = G(x) \qquad (1)$$

where $P(x), Q(x), R(x), G(x)$ are continuous functions.

In this section we study the case where $G(x)=0$ for all values of x. Such equations are called ***homogeneous linear equations***. Thus the form of a second-order linear homogeneous differential equation is

$$P(x)\cdot y'' + Q(x)\cdot y' + R(x)\cdot y = 0 \qquad (2)$$

If $G(x)\neq 0$ for some x, Equation 1 is *nonhomogeneous* and is discussed in Section 3.5.

Theorem 1 If $y_1(x)$ and $y_2(x)$ are both solutions of the linear homogeneous equation (2) and C_1, C_2 are any constants, then the linear combination $y = C_1y_1 + C_2y_2$ is also a solution of Equation 2.

The other fact we need is given by the following theorem which says that the general solution is a linear combination of two *linearly independent* solutions $y_1(x)$ and $y_2(x)$. This means that neither $y_1(x)$ nor $y_2(x)$ is a constant multiple of the other.

Theorem 2 If $y_1(x)$ and $y_2(x)$ are linearly independent solutions of Equation 2, and $P(x)\neq 0$, then the general solution is given by $y = C_1y_1 + C_2y_2$, where C_1 and C_2 are arbitrary constants.

Note 1 Two functions $f_1(x)$ and $f_2(x)$ are linearly independent on the interval $[a,b]$ if and only if $W(f_1,f_2)\neq 0$ for any $x\in[a,b]$, where the determinant

$$W(f_1,f_2)=\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

is called the *Wronskian of the functions $f_1(x)$ and $f_2(x)$*.

In general, it is not easy to discover particular solutions to a second-order linear equation. But it is always possible to do so if the coefficient functions $P(x), Q(x), R(x)$ are *constant functions*, that is, if the differential equation has the form

$$y'' + py' + qy = 0 \qquad (3)$$

where p, q are constants.

After replacing $y=e^{kx}$, we get the equation

$$k^2 + pk + q = 0 \qquad (4)$$

Equation 4 is called the ***auxiliary equation*** (or ***characteristic equation***) of the differential equation (3).

The general solution of the initial equation takes the form:

$$\begin{array}{l} 1.\ y = C_1 e^{k_1 x} + C_2 e^{k_2 x},\ if\ \ k_1 \neq k_2;\ \ k_1, k_2 \in R \\ 2.\ y = e^{k_1 x}(C_1 + C_2 x),\ \ if\ \ k_1 = k_2 \\ 3.\ y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x),\ \ if\ \ k_{1,2} = \alpha \pm i\beta . \end{array}$$

Definition A differential equation is called a ***linear homogeneous differential equations order*** n, if it can be represented as follows:

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_1 y' + a_0 y = 0 \qquad (5)$$

The general solution to the equation (5) is a function

$$y = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n ,$$

where $y_1, y_2, ..., y_n$ are linearly independent solutions of equation (5).

Particular linearly independent solutions of equation (5) can be founded in the form $y = e^{kx}$. To determine k, the following characteristic equation must be formed

$$k^n + a_{n-1}k^{n-1} + a_{n-2}k^{n-2} + \cdots + a_1 k + a_0 = 0 .$$

1. Each real root k of the characteristic equation corresponds to one particular solution of (5) of the form $y = e^{kx}$.

2. Each real root k of order m corresponds to m linearly independent partial solutions of (5): $y_1 = e^{kx}, y_2 = xe^{kx}, \cdots, y_m = x^m e^{kx}$.

3. If $\alpha \pm i\beta$ is a pair of complex roots of a characteristic equation of order m, then it corresponds to $2m$ linearly independent solutions of (5): $e^{\alpha x}\cos\beta x,\ e^{\alpha x}\sin\beta x;\ xe^{\alpha x}\cos\beta x,\ xe^{\alpha x}\sin\beta x;\ ...,\ x^{m-1}e^{\alpha x}\cos\beta x, x^{m-1}e^{\alpha x}\sin\beta x$.

Example 1 Solve the equations

$y'' - 5y' + 6y = 0$; $\quad y'' + 8y' + 16y = 0$; $\quad y'' - 6y' + 13y = 0$.

Solution For each case we compile a characteristic equation, we find its roots, we extract the appropriate linearly independent solutions of a differential equation and their general solution:

a) $k^2 - 5k + 6 = 0 \Rightarrow k_1 = 2,\ k_2 = 3 \Rightarrow y_1 = e^{2x},\ y_2 = e^{3x} \Rightarrow y = C_1 e^{2x} + C_2 e^{3x}$;

b) $k^2 + 8k + 16 = 0 \Rightarrow k_1 = -4,\ k_2 = -4 \Rightarrow y_1 = e^{-4x},\ y_2 = xe^{-4x} \Rightarrow$

$\Rightarrow y = e^{-4x}(C_1 + C_2 x)$;

c) $k^2 - 6k + 13 = 0 \Rightarrow k_{1,2} = 3 \pm 2i \Rightarrow \alpha = 3,\ \beta = 2 \Rightarrow$

$\Rightarrow y_1 = e^{3x}\cos 2x,\ y_2 = e^{3x}\sin 2x \Rightarrow y = e^{3x}\left(C_1\cos 2x + C_2\sin 2x\right)$.

Example 2 Solve the initial-value problem

$$y'' - 5y' + 6y = 0; \quad y(0) = 1,\ \ y'(0) = 0 .$$

Solution Example 1 has determined that the general solution of the differential equation is $y = C_1 e^{2x} + C_2 e^{3x}$.

Differentiating this solution, we get $y' = 2C_1e^{2x} + 3C_2e^{3x}$.

To satisfy the initial conditions we require that

$$y(0) = C_1 + C_2 = 1 \qquad \text{(a)}$$

$$y'(0) = 2C_1 + 3C_2 = 0 \qquad \text{(b)}$$

From (b), we have $C_2 = -\dfrac{2}{3}C_1$ and so (a) gives

$$C_1 - \frac{2}{3}C_1 = 1;\ \frac{1}{3}C_1 = 1;\ C_1 = 3;\ C_2 = -\frac{2}{3}C_1 = -\frac{2}{3}\cdot 3 = -2.$$

Thus the required solution of the initial-value problem is $y = 3e^{2x} - 2e^{3x}$.

Example 3 Solve the equations

a) $y''' - 3y'' - 10y' + 24y = 0$; *b)* $y^{IV} + 3y'' - 4y = 0$; *c)* $y^{IV} + 2y'' + y = 0$

.

Solution For each case we compile characteristic equation, we find its roots, extract the appropriate linearly independent solutions of differential equations and their general solution:

a) $k^3 - 3k^2 - 10k + 24 = 0,\ k_1 = 2,\ k_2 = -3,\ k_3 = 4;\ y = C_1e^{2x} + C_2e^{-3x} + C_3e^{4x}$;

b) $k^4 + 3k^2 - 4 = 0,\quad (k^2 - 1)(k^2 + 4) = 0,\quad k_1 = -1,\ k_2 = 1,\ k_3 = -2i,\ k_4 = 2i;$
$y = C_1e^{-x} + C_2e^{x} + C_3\cos 2x + C_4\sin 2x$;

c) $k^4 + 2k^2 + 1 = 0,\ (k^2 + 1)^2 = 0,\ k_{1,2} = \pm i,\ k_{3,4} = \pm i;\ y_1 = \cos x,\ y_2 = \sin x,$
$y_3 = x\cos x,\quad y_4 = x\sin x;\ y = (C_1 + C_2x)\cos x + (C_3 + C_4x)\sin x$.

Exercise Set 3.4

In exercises 1 to 12 solve the differential equation.

1. $y'' + y' - 2y = 0$ *2.* $y'' - 9y = 0$ *3.* $y'' - 2y' + y = 0$

4. $y'' - 10y' + 25y = 0$ *5.* $y'' + 6y' + 13y = 0$ *6.* $y'' + 36y = 0$

7. $y'' + 2y' - 8y = 0$ *8.* $y'' + 3y' = 0$ *9.* $y'' - 6y' + 34y = 0$

10. $y^{IV} + 4y'' = 0$ *11.* $y^{IV} - 4y''' + 4y'' = 0$ *12.* $y^{IV} + 8y' = 0$

In exercises 13 to 18 find the solution of the differential equation that satisfies the given initial condition.

13. $y'' - 4y' + 3y = 0,\ y(0) = 6,\ y'(0) = 10$ *14.* $y'' + 4y = 0,\ y(0) = 0,\ y'(0) = 2$

15. $y'' + 4y' + 29y = 0,\ y(0) = 0,\ y'(0) = 15$ *16.* $y'' + 3y' = 0,\ y(0) = 0,\ y'(0) = 3$

17. $y''' - 3y'' + 3y' - y = 0,$ $y(0) = 1,\ y'(0) = 2,\ y''(0) = 3$

18. $y''' + y'' - 5y' + 3y = 0,$ $y(0) = 0,\ y'(0) = 1,\ y''(0) = -2$

Individual Tasks 3.4

1-6. Solve the differential equation.

	II.
1. $y'' - y' = 0$	*1.* $y'' - 4y' = 0$
2. $y'' - 4y' + 13y = 0$	*2.* $y'' + 16y = 0$
3. $y'' + 8y' + 16y = 0$	*3.* $y'' - 10y' + 25y = 0$

4. $y'' - 2y' - 15y = 0$	4. $y'' - 12y' + 11y = 0$
5. $y^{IV} + 4y'' = 0$	5. $y^{IV} + 12y''' + 36y'' = 0$
6. $y^{IV} - 6y'' + 9y = 0$	6. $y^{IV} + 27y' = 0$

3.5 Nonhomogeneous Linear Equations

Definition A differential equation is called a ***linear homogeneous differential equations order*** n ***with constant coefficients***, if it can be represented as follows:

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_1 y' + a_0 y = f(x) \tag{1}$$

In this section we learn how to solve second-order nonhomogeneous linear differential equations with constant coefficients, that is, the equations of the form

$$y'' + py' + qy = f(x) \tag{2}$$

where p, q are constants and $f(x)$ is a continuous function. The related homogeneous equation

$$y'' + py' + qy = 0 \tag{3}$$

is called the ***complementary equation*** and plays an important role in the solution of the original nonhomogeneous equation (2).

Theorem 1 The general solution of the nonhomogeneous differential equation (1) can be written as

$$y = \overline{y} + y^* \tag{4}$$

where y^* is a particular solution of Equation 2 and $\overline{y}$ is a general solution of the complementary Equation 3.

Therefore Theorem 1 says that we know the general solution of the nonhomogeneous equation as soon as we know a particular solution y^*. There are two methods for finding a particular solution. *The method of undetermined coefficients* is straightforward but works only for a restricted class of functions $f(x)$. *The method of variation of parameters* can be used for every function $f(x)$ but is usually more difficult to apply in practice.

The Method of Undetermined Coefficients

1. If $f(x) = P_n(x) \cdot e^{\alpha x}$, where $P_n(x)$ is a polynomial of degree n, then try

$$y^* = x^r Q_n(x) \cdot e^{\alpha x},$$

where $Q_n(x)$ is an nth-degree polynomial (whose coefficients are determined by substituting in the differential equation).

2. If $f(x) = e^{ax}\left(P_n(x)\cos bx + Q_m(x)\sin bx\right)$, where $P_n(x)$ is an nth-degree polynomial ($Q_n(x)$ is an m th-degree polynomial), then try

$$y^* = x^r e^{ax}\left(S_N(x)\cos bx + T_N(x)\sin bx\right),$$

where $S_N(x), T_N(x)$ and are N th-degree polynomials ($N = \max\{n, m\}$).

Modification: r number is equal to the multiplicity of the number with respect to the roots of the characteristic equation.

Example 1 Solve the equation $y'' + 3y' - 4y = -4x^2 - 6x + 19$.

Solution The auxiliary equation of $y'' + 3y' - 4y = 0$ is $k^2 + 3k - 4 = 0$ with the roots

$$k_{1,2} = \frac{-3 \pm \sqrt{25}}{2} \Rightarrow k_1 = -4,\ k_2 = 1.$$

So the solution of the complementary equation is $\overline{y} = C_1 e^{-4x} + C_2 e^{x}$.

Since $f(x) = -4x^2 - 6x + 19$ is a polynomial of degree 2, we seek a particular solution of the form of $y^* = Ax^2 + Bx + C$. Then $\left(y^*\right)' = 2Ax + B$ and $\left(y^*\right)'' = 2A$ so, substituting into the given differential equation, we have

$$2A + 3\left(2Ax + B\right) - 4\left(Ax^2 + Bx + C\right) = -4x^2 - 6x + 19,$$

$$-4Ax^2 + \left(6A - 4B\right)x + \left(2A + 3B - 4C\right) = -4x^2 - 6x + 19.$$

Polynomials are equal when their coefficients are equal. Thus

$$\begin{cases} -4A = -4 \\ 6A - 4B = -6 \\ 2A + 3B - 4C = 19 \end{cases} \Leftrightarrow \begin{cases} A = 1 \\ B = 3 \\ C = -2 \end{cases}.$$

A particular solution is therefore $y^* = Ax^2 + Bx + C = x^2 + 3x - 2$.

The general solution is $y = y^* + \overline{y} = C_1 e^{-4x} + C_2 e^{x} + x^2 + 3x - 2$.

Example 2 Solve the equation $y'' - y' - 2y = 4xe^{x}$.

Solution The auxiliary equation of $y'' - y' - 2y = 0$ is $k^2 - k - 2 = 0$ with the roots $k_1 = -1,\ k_2 = 2$. So the solution of the complementary equation is $\overline{y} = C_1 e^{-x} + C_2 e^{2x}$.

For a particular solution we try $y^* = (Ax + B)e^{x}$.

Then $(y^*)' = Ae^{x} + (Ax + B)e^{x} = (Ax + A + B)e^{x}$ and $(y^*)'' = (Ax + 2A + B)e^{x}$ so, substituting into the given differential equation, we have

$$2Ae^{x} + (Ax + B)e^{x} - Ae^{x} - (Ax + B)e^{x} - 2(Ax + B)e^{x} = 4xe^{x},$$

$$A - 2Ax - 2B = 4x.$$

Polynomials are equal when their coefficients are equal. Thus

$$-2A = 4,\ \ A - 2B = 0;\quad A = -2,\ \ B = -1.$$

A particular solution is therefore $y^* = -(2x + 1)e^{x}$.

The general solution is $y = C_1 e^{-x} + C_2 e^{2x} - (2x + 1)e^{x}$.

Example 3 Solve the equation $y'' + y = x\sin x$.

Solution The auxiliary equation of $y'' + y = 0$ is $k^2 + 1 = 0$ with the roots

$$k = \pm i = 0 \pm 1 \cdot i \quad (\alpha = 0,\ \beta = 1).$$

So the solution of the complementary equation is $\overline{y} = C_1 \cos x + C_2 \sin x$.

For a particular solution we try $y^* = x\left((Ax + B)\cos x + (Cx + D)\sin x\right)$.

We find derivatives $(y^*)'$, $(y^*)''$ and substitute them in the assigned equation:

$$y^* = (Ax^2 + Bx)\cos x + (Cx^2 + Dx)\sin x;$$

$$\left(y^*\right)' = \left(2Ax + B\right)\cos x - (Ax^2 + Bx)\sin x + \left(2Cx + D\right)\sin x + (Cx^2 + Dx)\cos x;$$

$$\left(y^*\right)'' = 2A\cos x - 2\left(2Ax + B\right)\sin x - (Ax^2 + Bx)\cos x + 2C\sin x + 2\left(2Cx + D\right)\cos x -$$
$$-(Cx^2 + Dx)\sin x;$$

$$2A\cos x - 2\left(2Ax + B\right)\sin x + 2C\sin x + 2\left(2Cx + D\right)\cos x = x\sin x.$$

These expressions are equal when their coefficients before $\sin x$, $\cos x$, $x\sin x$, $x\cos x$ are equal

$$\begin{cases} 2A + 2D = 0 \\ 4C = 0 \\ -2B + 2C = 0 \\ -4A = 1 \end{cases} \Leftrightarrow \begin{cases} A = -1/4 \\ B = 0 \\ C = 0 \\ D = 1/4 \end{cases}.$$

Therefore the particular solution is $y^* = -\dfrac{x^2}{4}\cos x + \dfrac{x}{4}\sin x$.

The general solution is $y = C_1\cos x + C_2\sin x - \dfrac{x^2}{4}\cos x + \dfrac{x}{4}\sin x$.

The Method of Variation of Parameters

Suppose we have already solved the homogeneous equation $y'' + a_1y' + a_2y = 0$ and written the solution as $\overline{y} = C_1y_1(x) + C_2y_2(x)$, where $C_1, C_2 - const$, $y_1(x)$, $y_2(x)$ are linearly independent solutions. We look for a particular solution of the nonhomogeneous equation of the form

$$y^* = C_1(x)\cdot y_1(x) + C_2(x)\cdot y_2(x) \qquad (5)$$

(This method is called *variation of parameters* because we have varied the parameters C_1, C_2 to make them functions.)

Functions $C_1(x), C_2(x)$ are determined from the system of equations:

$$\begin{cases} C_1'(x)y_1(x) + C_2'(x)y_2(x) = 0, \\ C_1'(x)y_1'(x) + C_2'(x)y_2'(x) = f(x). \end{cases}$$

Example 4 Solve the equation $y'' + 4y = \dfrac{1}{\sin 2x}$.

Solution The auxiliary equation of $y'' + 4y = 0$ is $k^2 + 4 = 0$ with the roots $k = \pm 2i$.

So the solution of the complementary equation is $\overline{y} = C_1\cos 2x + C_2\sin 2x$.

For a particular solution we try

$$y^* = C_1(x)\cos 2x + C_2(x)\sin 2x;$$

$$y_1(x) = \cos 2x, \quad y_1'(x) = -2\sin 2x,$$

$$y_2(x) = \sin 2x, \quad y_2'(x) = 2\cos 2x.$$

Functions $C_1(x), C_2(x)$ are determined from the system of equations:

$$\begin{cases} C_1'(x)\cdot\cos 2x + C_2'(x)\cdot\sin 2x = 0, \\ -2C_1'(x)\cdot\sin 2x + 2C_2'(x)\cdot\cos 2x = \dfrac{1}{\sin 2x}. \end{cases}$$

We solve this system according to Cramars' Rules.

$$\Delta = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2.$$

Then

$$C_1'(x) = \frac{1}{2}\begin{vmatrix} 0 & \sin 2x \\ \dfrac{1}{\sin 2x} & 2\cos 2x \end{vmatrix} = -\frac{1}{2},$$

$$C_2'(x) = \frac{1}{2}\begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \dfrac{1}{\sin 2x} \end{vmatrix} = \frac{1}{2}ctg 2x.$$

Integrating last two equalities, we have:

$$C_1(x) = -\frac{1}{2}x,$$

$$C_2(x) = \frac{1}{4}\ln|\sin 2x|.$$

The general solution is

$$y = \overline{y} + y^* = C_1\cos 2x + C_2\sin 2x - \frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x \ln|\sin 2x|.$$

Exercise Set 3.5

In exercises 1 to 12 solve the differential equation.

1. $y'' - 3y' + 2y = xe^{-x}$ *2.* $y'' - 3y' + 2y = (2x+3)e^x$ *3.* $y'' - 3y' + 2y = e^{2x}$

4. $y'' - 10y' = 10x^2 + 18x$ *5.* $y'' - 8y' + 16y = e^{4x}(1-x)$ *6.* $y'' - 10y' = (3x-4)e^{5x}$

7. $y'' + 9y = 3\sin x$ *8.* $y'' + 9y = 2\sin 3x - 4\cos 3x$ *9.* $y'' + 16y = x\sin 4x$

10. $y^{IV} - y = 3xe^x$ *11.* $y^{IV} - y = \sin x$ *12.* $y^{V} - y^{IV} = 2xe^x$

In exercises 13 to 16 find the solution of the differential equation that satisfies the given initial condition.

13. $y'' + y = 2\cos x$, $y(0) = 1,\ y'(0) = 0$

14. $y'' + 4y = 4(\cos 2x + \sin 2x)$, $y(\pi) = 2\pi,\ y'(\pi) = 2\pi$

15. $y'' + y = 4\sin x - 6\cos x$, $y(0) = 1,\ y'(0) = 18$

16. $y'' + 9y = 2\cos 4x - 3\sin 4x$, $y(0) = 0,\ y'(0) = 12$

In exercises 17 to 22 solve the differential equation.

17. $y'' - 4y' + 5y = \dfrac{e^{2x}}{\cos x}$ *18.* $y'' + 4y' + 4y = e^{-2x}\ln x$ *19.* $y'' + y + ctg^2 x = 0$

20. $y''' + y' = tgx \cdot \frac{1}{\cos x}$ 21. $y'' - y' = e^{2x}\cos e^{x}$ 22. $y'' + 4y = \frac{1}{\sin^2 x}$

Individual Tasks 3.5

1-6. Solve the differential equation.

	II.
$y'' + 8y' = 8x$	$y'' - 5y' = x + 5$
$y'' + 4y' + 3y = 9e^{-3x}$	$y'' - 3y' + 2y = e^{3x}\left(x^2 + x\right)$
$y'' + y = 4\sin x - 6\cos x$	$y'' + 9y = 2\cos 4x - 3\sin 4x$
$y''' + y'' - 2y' = x^2 + x$	$y''' + y'' = 6x$
$y'' - 2y' + y = \frac{e^x}{x}$	$y'' + y = \frac{1}{\sin x}$

3.6 Systems of differential equations

Definition A system of differential equations is called a ***normal system of two differential first order equations***, if it can be represented as follows:

$$\begin{cases} \dfrac{dy}{dx} = f(x, y, z), \\ \dfrac{dz}{dx} = g(x, y, z). \end{cases} \qquad (1)$$

The Method of Exception

The solution of the normal system of two differential first order equations permitted relative to derivatives of two unknown functions $y(x)$ and $z(x)$ (or $x(t), y(t)$), it is reduced to the solution of one differential equation of the second order relative to one of the functions. Let us examine the given method with the help of the following example.

Example 1 Find the general solution of the system of the differential equations

$$\begin{cases} x' = x - y, \\ y' = -4x + y. \end{cases}.$$

Solution We differentiate the first equation of the system $x'' = x' - y'$. Let us replace y' in the last equation with its expression from the second equation of the system $x'' = x' - \left(-4x + y\right)$, $x'' = x' + 4x - y$. Let us replace y in the last equation with its expression from the second equation of the system $x'' = x' + 4x - \left(x - x'\right)$, $x'' - 2x' - 3x = 0$.

The auxiliary equation of $x'' - 2x' - 3x = 0$ is $k^2 - 2k - 3 = 0$ with the roots $k_1 = -1, k_2 = 3$.

So the solution of the complementary equation is $x\left(t\right) = C_1 e^{-t} + C_2 e^{3t}$.

Differentiating this equation with respect to variable t, we will obtain: $x'\left(t\right) = -C_1 e^{-t} + 3C_2 e^{3t}$.

Then we find $y(t)$ from the equation $y = x - x'$:

$$y(t) = C_1e^{-t} + C_2e^{3t} - \left(-C_1e^{-t} + 3C_2e^{3t}\right) = 2C_1e^{-t} - 2C_2e^{3t}.$$

Therefore $\begin{cases} x(t) = C_1e^{-t} + C_2e^{3t} \\ y(t) = 2C_1e^{-t} - 2C_2e^{3t} \end{cases}$ is the general solution of the system of differential equations.

Euler's method to solving uniform systems of differential equations with the constant coefficients

Assume that the system of three equations with three unknown functions is assigned: $\begin{cases} x'(t) = a_{11}x + a_{12}y + a_{13}z, \\ y'(t) = a_{21}x + a_{22}y + a_{23}z, \\ z'(t) = a_{31}x + a_{32}y + a_{33}z. \end{cases}$

We will search for unknown functions in the form $\begin{cases} x(t) = \alpha \cdot e^{kt}, \\ y(t) = \beta \cdot e^{kt}, \\ z(t) = \gamma \cdot e^{kt}. \end{cases}$

Substituting these expressions into the system and converting it, we will obtain the system of linear homogeneous algebraic equations with variables α, β, γ:

$$\begin{cases} (a_{11} - k)\alpha + a_{12}\beta + a_{13}\gamma = 0, \\ a_{21}\alpha + (a_{22} - k)\beta + a_{23}\gamma = 0, \\ a_{31}\alpha + a_{32}\beta + (a_{33} - k)\gamma = 0. \end{cases} \qquad (2)$$

System (2) has non-trivial solutions, if its determinant is equal to zero. We will obtain a cubic equation for determining the number k:

$$\Delta = \begin{vmatrix} a_{11} - k & a_{12} & a_{13} \\ a_{21} & a_{22} - k & a_{23} \\ a_{31} & a_{32} & a_{33} - k \end{vmatrix} = 0 \qquad (3)$$

Equation (3) is called the ***characteristic equation of a reference system***. We solve it, we find values k, for each getting value k we find α, β, γ from system (2), write the linearly independent solutions for each unknown function and compose the general solution of the system.

Example 2 Find the general solution of the system of differential equations

$$\begin{cases} x' = x - y + z, \\ y' = x + y - z, \\ z' = 2x - y. \end{cases}$$

Solution The characteristic equation of this system takes the form

$$\begin{vmatrix} 1-k & -1 & 1 \\ 1 & 1-k & -1 \\ 2 & -1 & -k \end{vmatrix} = 0,$$

$$(1-k)^2(-k) - 1 + 2 - 2(1-k) - (1-k) - k = 0,$$

$$(k-1)(k-2)(k+1) = 0,$$

$$k_1 = 1,\ k_2 = 2,\ k_3 = -1.$$

Let us find the appropriate values α, β, γ for each k from the system of equations

$$\begin{cases} (1-k)\alpha - \beta + \gamma = 0, \\ \alpha + (1-k)\beta - \gamma = 0, \\ 2\alpha - \beta - k\gamma = 0. \end{cases} \qquad (4)$$

If $k = 1$, then

$$\begin{cases} -\beta + \gamma = 0, \\ \alpha - \gamma = 0, \\ 2\alpha - \beta - \gamma = 0, \end{cases} \Rightarrow \begin{cases} \beta = \gamma, \\ \alpha = \gamma, \\ 0 = 0, \end{cases} \Rightarrow \begin{cases} \alpha = 1, \\ \beta = 1, \\ \gamma = 1, \end{cases} \Rightarrow \begin{cases} x_1 = e^t, \\ y_1 = e^t, \\ z_1 = e^t. \end{cases}$$

If $k = 2$, then

$$\begin{cases} -\alpha - \beta + \gamma = 0, \\ \alpha - \beta - \gamma = 0, \\ 2\alpha - \beta - 2\gamma = 0, \end{cases} \Rightarrow \begin{cases} 2\beta = 0, \\ \alpha = \gamma, \\ 0 = 0, \end{cases} \Rightarrow \begin{cases} \alpha = 1, \\ \beta = 0, \\ \gamma = 1, \end{cases} \Rightarrow \begin{cases} x_2 = e^{2t}, \\ y_2 = 0, \\ z_2 = e^{2t}. \end{cases}$$

If $k = -1$, then

$$\begin{cases} 2\alpha - \beta + \gamma = 0, \\ \alpha + 2\beta - \gamma = 0, \\ 2\alpha - \beta + \gamma = 0, \end{cases} \Rightarrow \begin{cases} 2\alpha - \beta + \gamma = 0, \\ 3\alpha + \beta = 0, \end{cases} \Rightarrow \begin{cases} \beta = -3\alpha, \\ \gamma = -5\alpha, \end{cases} \Rightarrow \begin{cases} \alpha = 1, \\ \beta = -3, \\ \gamma = -5, \end{cases} \Rightarrow \begin{cases} x_3 = e^{-t}, \\ y_3 = -3e^{-t}, \\ z_3 = -5e^{-t}. \end{cases}$$

We extract the general solution of the system of differential equations

$$\begin{cases} x(t) = C_1x_1 + C_2x_2 + C_3x_3, \\ y(t) = C_1y_1 + C_2y_2 + C_3y_3, \\ z(t) = C_1z_1 + C_2z_2 + C_3z_3, \end{cases} \Rightarrow \begin{cases} x(t) = C_1e^t + C_2e^{2t} + C_3e^{-t}, \\ y(t) = C_1e^t - 3C_3e^{-t}, \\ z(t) = C_1e^t + C_2e^{2t} - 5C_3e^{-t}. \end{cases}$$

Exercise Set 3.6

In exercises 1 to 14 find the general solution of the system of differential equations.

1. $\begin{cases} x' = 5x + 3y; \\ y' = -3x - y. \end{cases}$

2. $\begin{cases} x' = 2x + y; \\ y' = 3x + 4y. \end{cases}$

3. $\begin{cases} x' = 3x + y; \\ y' = x + 3y. \end{cases}$

4. $\begin{cases} x' = 2x + y; \\ y' = -6x - 3y. \end{cases}$

5. $\begin{cases} y' = -7y + z; \\ z' = -2y - 5z. \end{cases}$

6. $\begin{cases} y' = y + 2z; \\ z' = 4y + 3z. \end{cases}$

7. $\begin{cases} y' = -5y + 2z + e^{x}; \\ z' = y - 6z + e^{-2x}. \end{cases}$

8. $\begin{cases} y_1' = 3y_1 - 2y_2 + x; \\ y_2' = 3y_1 - 4y_2. \end{cases}$

9. $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_1 \end{cases}$

10. $\begin{cases} x'(t) = x - 4y - z; \\ y'(t) = x + y; \\ z'(t) = 3x + z. \end{cases}$

11. $\begin{cases} x'(t) = 2x + y + e^{t}; \\ y'(t) = x + 2y - 3e^{4t}. \end{cases}$

12. $\begin{cases} x'(t) = x - y + 8t; \\ y'(t) = 5x - y. \end{cases}$

13. $\begin{cases} x'(t) = 3x - y + z; \\ y'(t) = x + y + z; \\ z'(t) = 4x - y + 4z. \end{cases}$

14. $\begin{cases} x'(t) = -3x + 4y - 2z; \\ y'(t) = x + z; \\ z'(t) = 6x - 6y + 5z. \end{cases}$

Individual Tasks 3.6

1-3. Find the general solution of the system of differential equations.

	II.
1. $\begin{cases} x' = 2x + y; \\ y' = 3x + 4y. \end{cases}$	1. $\begin{cases} x' = 4x - 8y; \\ y' = -8x + 4y. \end{cases}$
2. $\begin{cases} x'(t) = 3x - 4y + e^{-2t}; \\ y'(t) = x - 2y - 3e^{-2t}. \end{cases}$	2. $\begin{cases} x'(t) = 4x + y - 36t; \\ y'(t) = -2x + y - 2e^{t}. \end{cases}$
3. $\begin{cases} x'(t) = x - 4y - z; \\ y'(t) = x + y; \\ z'(t) = 3x + z. \end{cases}$	3. $\begin{cases} x'(t) = x - 2y - z; \\ y'(t) = -x + y + z; \\ z'(t) = x - z. \end{cases}$

IV MULTIPLE INTEGRALS

In this chapter we extend the idea of a definite integral to double and triple integrals of functions of two or three variables. We will introduce two new coordinate systems in three-dimensional space — cylindrical coordinates and spherical coordinates — that greatly simplify the computation of triple integrals over certain commonly occurring solid regions.

4.1 Double Integrals over Rectangles

We consider a function f of two variables defined on a closed rectangle

$$R = \left\{ (x, y) \in \mathbb{R}^2 \mid a \le x \le b, c \le y \le d \right\}$$

and we first suppose that $f(x,y) \geq 0$. The graph of f is a surface with the equation $z = f(x,y)$.

Let S be the solid that lies above R and under the graph of f, that is,

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq z \leq f(x,y), (x,y) \in \mathbb{R}\}$$

(See Figure 22). Our goal is to find the volume of S.

The first step is to divide the rectangle R into subrectangles. We accomplish this by dividing the interval $[a,b]$ into m subintervals $[x_{i-1}, x_i]$ of the equal width $\Delta x = (b-a)/m$ and dividing $[c,d]$ into n subintervals $[y_{j-1}, y_j]$ of the equal width $\Delta y = (d-c)/n$. By drawing lines parallel to the coordinate axes through the endpoints of these subintervals, as in Figure 23, we form the subrectangles

$$R_{ij} = \{(x,y) \mid x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j\}$$

each with the area $\Delta S = \Delta x \Delta y$.

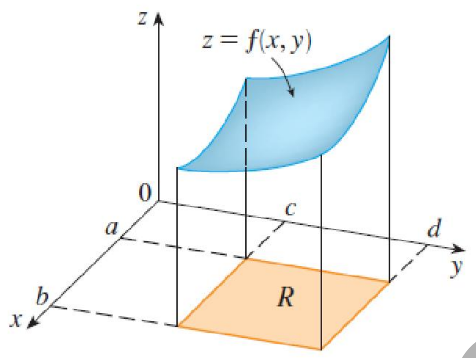


Figure 22

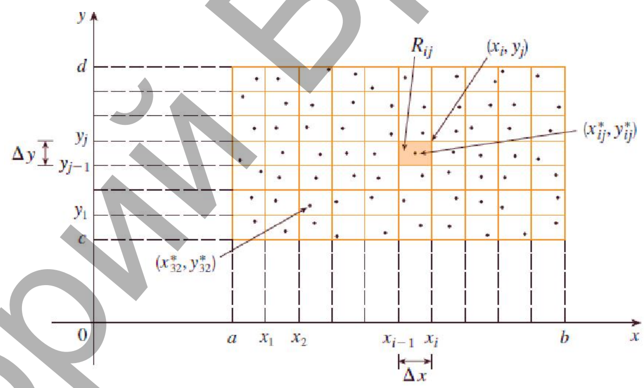


Figure 23

If we choose a ***sample point*** (x^*_{ij}, y^*_{ij}) in each R_{ij}, then we can approximate the part of S that lies above each R_{ij} by a thin rectangular box (or “column”) with base R_{ij} and height $f(x^*_{ij}, y^*_{ij})$ as shown in Figure 24.The volume of this box is the height of the box times the area of the base rectangle: $f(x^*_{ij}, y^*_{ij})\Delta S$.

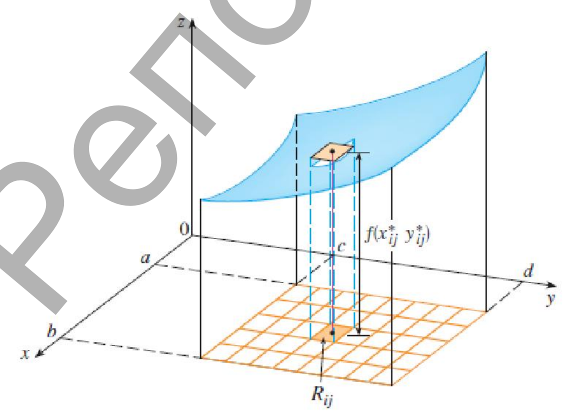


Figure 24

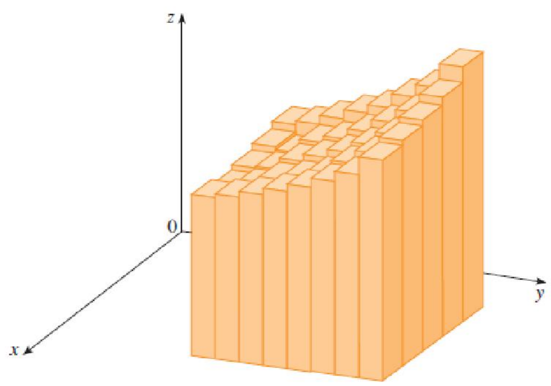


Figure 25

If we follow this procedure for all the rectangles and add the volumes of the corresponding boxes, we get an approximation to the total volume of S:

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta S \tag{1}$$

(See Figure 25). This double sum means that for each subrectangle we evaluate f at the chosen point and multiply by the area of the subrectangle, and then we add the results.

The approximation given in (1) becomes better as m and n become larger and so we would expect that

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta S \tag{2}$$

We use the expression in Equation 2 to define the *volume* of the solid S that lies under the graph of f and above the rectangle R.

Limits of the type that appear in Equation 2 occur frequently, not just in finding volumes but in a variety of other situations as well even when f is not a positive function. So we make the following definition.

Definition The ***double integral*** of f over the rectangle R is

$$\iint_R f(x,y)dS = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta S$$

if this limit exists.

A function f is called *integrable* if the limit in Definition exists. The double integral of f exists provided that f is "not too discontinuous." In particular, if f is bounded, and f is continuous there, except on a finite number of smooth curves, then f is integrable over R.

The sample point (x_{ij}^*, y_{ij}^*) can be chosen to be any point in the subrectangle R_{ij} but if we choose it to be the upper right-hand corner of R_{ij} [namely (x_i, y_j), see Figure 23], then the expression for the double integral looks simpler:

$$\iint_R f(x,y)dS = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\Delta S \tag{3}$$

The sum $\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta S$ is called a *double Riemann sum* and is used as an approximation to the value of the double integral. If f happens to be a *positive* function, then the double Riemann sum represents the sum of volumes of columns, as in Figure 25, and is an approximation to the volume under the graph of f and above the rectangle R.

Properties of Double Integrals

1. $\iint_R (\alpha f_1(x,y) \pm \beta f_2(x,y))dxdy = \alpha\iint_R f_1(x,y)dxdy \pm \beta\iint_R f_2(x,y)dxdy.$

2. If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries

(see Figure 26), then $\iint\limits_R f(x,y)dxdy = \sum\limits_{i=1}^{n} \iint\limits_{R_i} f(x,y)dxdy$.

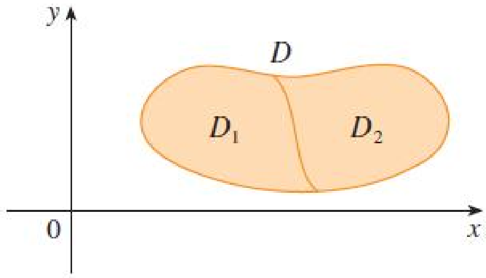


Figure 26

3. If $f(x,y) \ge g(x,y)$ for all (x,y) in R, then $\iint\limits_R f(x,y)dxdy \ge \iint\limits_R g(x,y)dxdy$.

4. If we integrate the constant function $f(x,y) = 1$ over a region D, we get the area of D:

$$S_D = \iint\limits_D 1dS .$$

5. If $f(x,y) \ge 0$, then the volume V of the solid, that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \iint\limits_R f(x,y)dS .$$

6. ***Midpoint rule for double integrals.*** If function $z = f(x,y)$ is continuous in the closed domain R, then there is a point $P_0(x_0, y_0)$ in this region such, that

$$\iint\limits_R f(x,y)dS = f(P_0) \cdot S$$

is the average value of function $z = f(x,y)$ in the region R.

Iterated Integrals

The evaluation of double integrals from first principles is even more difficult, but in this section we see how to express a double integral as *an iterated integral*, which can then be evaluated by calculating two single integrals.

Suppose that f is a function of two variables that is integrable on the rectangle

$$R = \left\{(x,y) \in \mathbb{R}^2 \mid a \le x \le b, c \le y \le d\right\}.$$

We use the notation $\int\limits_c^d f(x,y)dy$ to mean that x is held fixed and $f(x,y)$ is integrated with respect to y from $y = c$ to $y = d$. This procedure is called *partial integration with respect to* y. (Notice its similarity to partial differentiation.)

Now $\int\limits_c^d f(x,y)dy$ is a number that depends on the value of x, so it defines a function of x: $S(x)=\int\limits_c^d f(x,y)dy$.

If we now integrate the function $S(x)$ with respect to x from $x=a$ to $x=b$, we get

$$\int\limits_a^b S(x)dx=\int\limits_a^b\left(\int\limits_c^d f(x,y)dy\right)dx \qquad (1)$$

The integral on the right side of Equation 1 is called an ***iterated integral***. Usually the brackets are omitted. Thus

$$\int\limits_a^b\int\limits_c^d f(x,y)dydx=\int\limits_a^b\left(\int\limits_c^d f(x,y)dy\right)dx \qquad (2)$$

means that we first integrate with respect to y from $y=c$ to $y=d$ and then with respect to x from $x=a$ to $x=b$.

Similarly, the iterated integral

$$\int\limits_c^d\int\limits_a^b f(x,y)dydx=\int\limits_c^d\left(\int\limits_a^b f(x,y)dx\right)dy \qquad (3)$$

means that we first integrate with respect to x (holding y fixed) from $x=a$ to $x=b$ and then we integrate the resulting function of y with respect to y from $y=c$ to $y=d$. Notice that in both Equations 2 and 3 we work *from the inside out.*

Fubini's Theorem If f is continuous on the rectangle

$$R=\left\{(x,y)\in\mathbb{R}^2 \mid a\le x\le b, c\le y\le d\right\},$$

then

$$\iint\limits_R f(x,y)dS=\int\limits_c^d dy\int\limits_a^b f(x,y)dx=\int\limits_a^b dx\int\limits_c^d f(x,y)dy.$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 1 Evaluate the double integral $\iint\limits_R (x-3y^2)dxdy$, where

$$R=\left\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 2,\ 1\le y\le 2\right\}.$$

Solution 1 Fubini's Theorem gives

$$\iint\limits_R (x-3y^2)dxdy=\int\limits_0^2 dx\int\limits_1^2 (x-3y^2)dy=\int\limits_0^2\left[xy-y^3\right]_{y=1}^{y=2}dx=\int\limits_0^2 (x-7)dx=\left[\frac{x^2}{2}-7x\right]_0^2=-12$$

.

Solution 2 Applying Fubini's Theorem again once more, but this time integrating with respect to x first, we have

$$\iint_R f dx dy = \int_1^2 dy \int_0^2 (x-3y^2)dx = \int_1^2 \left[\frac{x^2}{2} - 3xy^2\right]_{x=0}^{x=2} dy = \int_1^2 (2-6y^2)dy = \left[2y-2y^3\right]_1^2 = -12$$

.

If f is continuous on a region D such that $D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$ (see figure 27), then

$$\iint_D f(x,y)dS = \int_a^b dx \int_{g_1(x)}^{g_2(x)} f(x,y)dy \qquad (4)$$

The integral on the right side of (4) is an iterated integral that is similar to the ones we considered in the preceding section, except that in the inner integral we regard x as being constant not only in $f(x,y)$, but also in the limits of integration, $g_1(x)$ and $g_2(x)$.

We also consider plane regions, which can be expressed as

$$D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\},$$

where $h_1(y)$ and $h_2(y)$ are continuous. Two such regions are illustrated in Figure 28.

Using the same methods that were used in establishing (4), we can show that

$$\iint_D f(x,y)dS = \int_c^d dy \int_{h_1(y)}^{h_2(y)} f(x,y)dx \qquad (5)$$

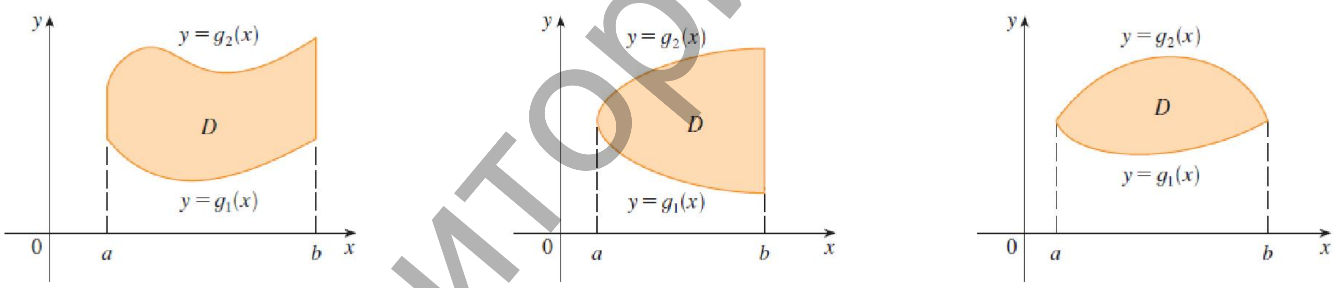


Figure 27

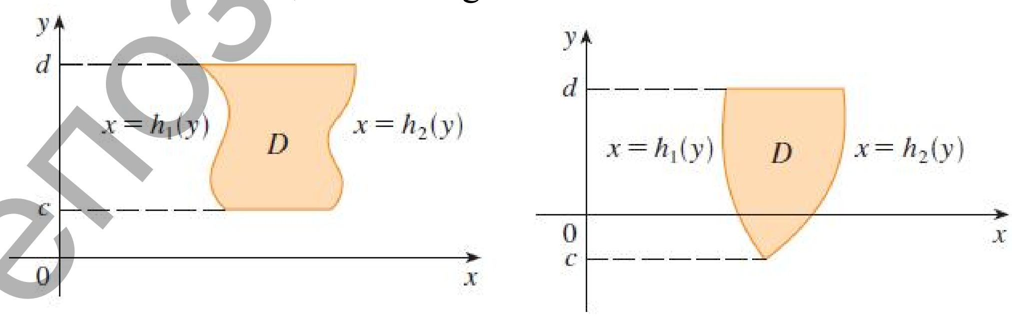


Figure 28

Example 2 Evaluate $\iint_D (x+2y)dxdy$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

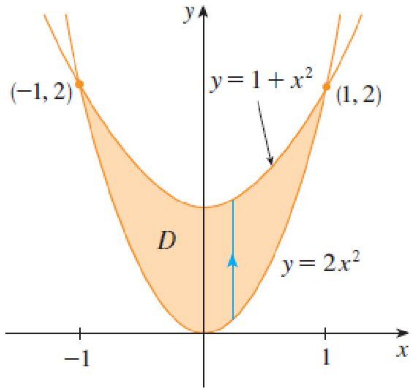


Figure 29

Solution The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = \pm 1$. We note that the region D, sketched in Figure 29, is a type I region but not a type II region and we can write $D = \left\{(x, y) \mid -1 \le x \le 1,\ 2x^2 \le y \le 1 + x^2\right\}$.

Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, Equation 4 gives

$$\iint_D (x + 2y)dxdy = \int_{-1}^{1} dx \int_{2x^2}^{1+x^2} (x + 2y)dy = \int_{-1}^{1} \left[xy + y^2 \right]_{y=2x^2}^{y=1+x^2} dx = \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)dx =$$

$$= \left[-3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x \right]_{-1}^{1} = \frac{32}{15}.$$

Example 3 Evaluate $\iint_D xydxdy$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution The region D is shown in Figure 30. Again it is both of type I and type II, but the description of D as a type I region is more complicated because the lower boundary consists of two parts. Therefore we prefer to express D as a type II region:

$$D = \left\{ (x, y) \mid -2 \le y \le 4,\ \frac{1}{2}y^2 - 3 \le x \le y + 1 \right\}$$

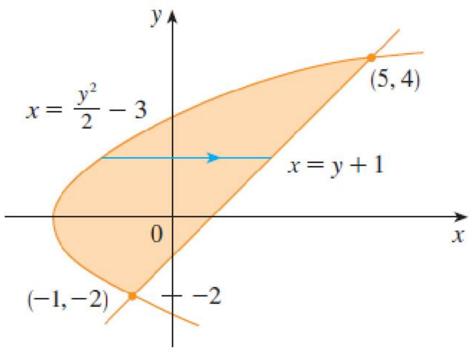


Figure 30

Then (6) gives

$$\iint_D xydxdy=\int_{-2}^{4}dy\int_{\frac{1}{2}y^2-3}^{1+y}xydx=\int_{-2}^{4}\left[\frac{x^2}{2}y\right]_{x=\frac{1}{2}y^2-3}^{x=1+y}dy=\frac{1}{2}\int_{-2}^{4}(-\frac{y^5}{4}+4y^3+2y^2-8y)dy=$$

$$=\frac{1}{2}\left[-\frac{y^6}{6}+y^4+2\frac{y^3}{3}-4y^2\right]_{-2}^{4}=36.$$

Exercise Set 4.1

In Exercise 1 to 6, evaluate the iterated integral.

1. $\int_0^2 dx\int_0^1 (x^2+2y)dy.$ *2.* $\int_{-3}^{8} dy\int_{y^2-4}^{5}(x+2y)dx.$ *3.* $\int_1^2 dx\int_{1/x}^{x}\frac{x^2}{y^2}dy.$

4. $\int_3^4 dx\int_1^2\frac{dy}{(x+y)^2}.$ *5.* $\int_1^2 dx\int_x^{x^2}(2x-y)dy.$ *6.* $\int_0^1 dx\int_0^1\frac{x^2dy}{1+y^2}.$

In Exercise 7 to 12, sketch the region of integration and change the order of integration.

7. $\int_0^1 dx\int_{x^3}^{\sqrt{x}} f(x,y)dy.$ *8.* $\int_0^1 dx\int_{2x}^{3x} f(x,y)dy.$ *9.* $\int_1^e dx\int_0^{\ln x} f(x,y)dy.$

10. $\int_0^1 dy\int_{2-y}^{1+\sqrt{1-y^2}} f(x,y)dx.$ *11.* $\int_{-6}^{2} dx\int_{\frac{x^2}{4}-1}^{2-x} f(x,y)dy.$ *12.* $\int_0^1 dy\int_{-\sqrt{1-y^2}}^{1-y} f(x,y)dx.$

In Exercise 13 to 16, evaluate the double integral $\iint_D f(x,y)\,dx\,dy$, if $f(x,y)=1$ and D is given by:

13. $\{(x,y)|\ x^2=2y,\ 5x-2y-6=0\}$ *14.* $\{(x,y)|\ y=\sqrt{4-x^2},\ y=\sqrt{3x}, x\ge 0\}$

15. $\{(x,y)|\ y=-x,\ y^2=x+2\}$ *16.* $\{(x,y)|\ y=\log_{0,5}x,\ y=1,\ y=-1,\ x\ge 0\}$

In Exercise 17 to 19, evaluate the double integral

17. $\iint_D (x^3+3y)\,dx\,dy$, where $D:\{(x,y)|\ x+y=1,\ y=x^2-1,\ x\ge 0\}$.

18. $\iint_D (x-y)\,dxdy$, where $D:\{(x,y)|\ x+y=2,\ y=0,\ x=y\}$.

19. $\iint_D x^4y\ dxdy$, where $D:\{(x,y)|\ xy=1,\ x=2,\ x=y\}$.

Individual Tasks 4.1

1. Evaluate the iterated integral.

2. Sketch the region of integration and change the order of integration.

3. Evaluate the double integral.

I

1. $\int\limits_0^{\pi} dx \int\limits_0^{1+\cos x} y^2 \sin x\, dy$.

2. $\int\limits_{-1}^{3} dx \int\limits_{2}^{\frac{x+5}{2}} f(x,y)dy + \int\limits_{3}^{6} dx \int\limits_{2}^{6-\frac{2}{3}x} f(x,y)dy$.

3. $\iint\limits_D \frac{x^2}{y^2} dxdy$, where $D:\{(x,y)|\ xy=1,\ x=2,\ x=y\}$.

II

1. $\int\limits_0^{\pi/2} dx \int\limits_{\cos x}^{1} y^4 dy$.

2. $\int\limits_{\frac{1}{2}}^{2} 2y \int\limits_{-\sqrt{y-1}}^{\sqrt{y-1}} f(x,y)dx + \int\limits_{2}^{5} dy \int\limits_{-\sqrt{y-1}}^{3-y} f(x,y)dx$.

3. $\iint\limits_D dxdy$, where $D:\{(x,y)|\ y=2-x,\ y^2=4x+4\}$.

4.2 Double Integrals in Polar Coordinates

Suppose that we want to evaluate a double integral $\iint\limits_D f(x,y)dS$, where D is one of the regions shown in Figure 31. In either case the description of D in terms of rectangular coordinates is rather complicated but D is easily described using polar coordinates.

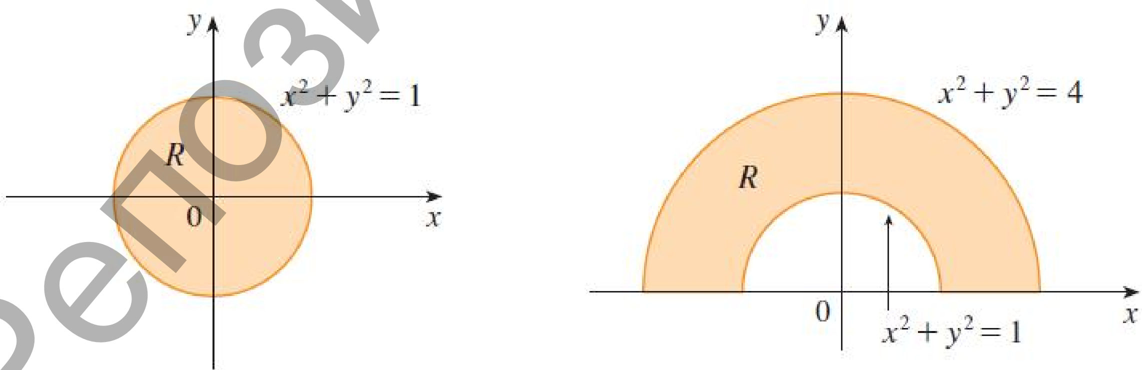


Figure 31

Recall from Figure 32 that the polar coordinates of a point are related to the rectangular coordinates by the equations

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$

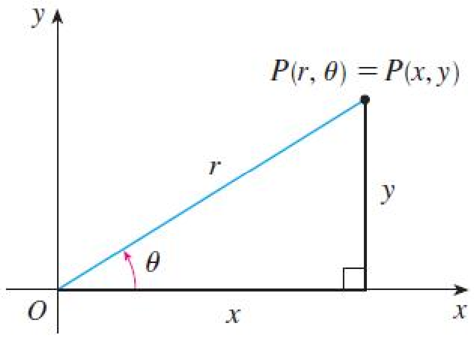


Figure 32

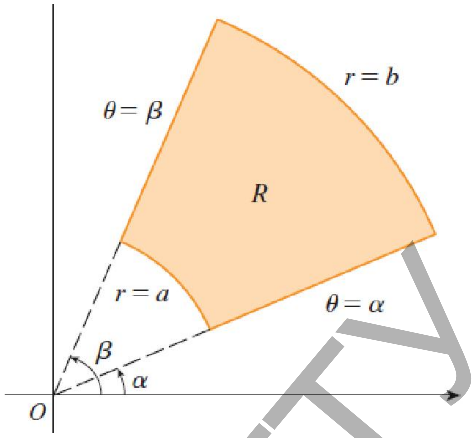


Figure 33

The regions in Figure 31 are special cases of a *polar rectangle* which is shown in Figure 33.

Change to Polar Coordinates in a Double Integral. If f is continuous on a polar rectangle D given by $r_1(\theta)\le r \le r_2(\theta),\ \alpha\le\theta\le\beta$, then

$$\iint\limits_D f(x,y)\,dxdy = \iint\limits_D f(r\cos\theta, r\sin\theta) r\,dr\,d\theta = \int\limits_{\alpha}^{\beta} d\theta \int\limits_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) r\,dr. \quad (1)$$

The formula (1) says that we convert from rectangular to polar coordinates in a double integral by writing $x = r\cos\theta$ and $y = r\sin\theta$, using the appropriate limits of integration for r and θ, and replacing dS by $drd\theta$. Be careful not to forget the additional factor r on the right side of Formula 1.

Example 1 Evaluate

$$\iint\limits_D \sin\pi\left(\frac{x^2}{4}+y^2\right)dxdy\,,$$

where $D:\ \left\{(x,y)\,|\,\frac{x^2}{4}+y^2=1;\ \ \frac{x^2}{16}+\frac{y^2}{4}=1\right\}$.

Solution $x = 2r\cos\theta,\ \ y = r\sin\theta,\qquad I = 2r.$

$$\frac{x^2}{4}+y^2=1 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = 1 \Rightarrow r=1;$$

$$\frac{x^2}{16}+\frac{y^2}{4}=1 \Rightarrow \frac{1}{4}\left(r^2\cos^2\theta+r^2\sin^2\theta\right)=1 \Rightarrow \frac{r^2}{4}=1 \Rightarrow r=2,\quad 0\le\theta\le 2\pi\,.$$

In polar coordinates it is given by $1\le r\le 2,\ 0\le\theta\le 2\pi$. Therefore, by Formula (1):

$$\iint\limits_D \sin\pi\left(\frac{x^2}{4}+y^2\right)dxdy = \iint\limits_D \sin\left(\pi\cdot r^2\right)\cdot 2r\,dr\,d\theta = \int\limits_0^{2\pi} d\theta \int\limits_1^2 \sin\left(\pi r^2\right)\cdot 2r\,dr =$$

$$= 2\int\limits_1^2 \sin\left(r^2\pi\right)d\left(r^2\pi\right) = -2\cos\left(r^2\pi\right)\Big|_1^2 = -2(\cos 4\pi - \cos\pi) = -2(1+1) = -4.$$

Exercise Set 4.2

In Exercise 1 to 6, evaluate the given integral by changing to polar coordinates.

1. $\iint\limits_D \left(1-\dfrac{y^2}{x^2}\right)dxdy$, where $D:\left\{(x,y)|\ x^2+y^2\le\pi^2\right\}$.

2. $\iint\limits_D 6\,dx\,dy$, where $D:\left\{(x,y)|\ x^2+y^2=4x,\ x^2+y^2=6x,\ y=x,\ \ y=0\right\}$.

3. $\iint\limits_D \left(x^2+y^2\right)dx\,dy$, where $D:\left\{(x,y)|\ x^2+y^2\le 4x\right\}$.

4. $\iint\limits_D \dfrac{x\,y}{\sqrt{x^2+y^2}}dx\,dy$, where $D:\left\{(x,y)|\ 1\le x^2+y^2\le 4,\ y=x,\ \ y=0,\ \ x<0,\ \ y<0\right\}$.

5. $\iint\limits_D e^{-x^2-y^2}dxdy$, where $D:\left\{(x,y)|\ x^2+y^2\le R^2\right\}$.

6. $\iint\limits_D \left(x^2+y^2\right)dx\,dy$, where $D:\left\{(x,y)|\ x^2+y^2=4x, x^2+y^2=6x, y\ge x, y\le\sqrt{3}x\right\}$.

Individual Tasks 4.2

1-3. Evaluate the double integral.

I

1. $\int\limits_0^1 dx\int\limits_0^{\sqrt{1-x^2}}\sqrt{\dfrac{1-x^2-y^2}{1+x^2+y^2}}dy$.

2. $\iint\limits_D (12-x-y)dxdy$, where $D:\left\{(x,y)|\ x^2+y^2\le 25\right\}$.

3. $\iint\limits_D \dfrac{x}{\sqrt{x^2+y^2}}dxdy$, where $D:\left\{(x,y)|\ 1\le x^2+y^2\le 16, x\le 0\right\}$.

II

1. $\int\limits_{-2}^0 dx\int\limits_0^{\sqrt{4-x^2}}\sin(x^2+y^2)\,dy$.

2. $\iint\limits_D (6-2x-3y)dxdy$, where $D:\left\{(x,y)|\ x^2+y^2\le 4\right\}$.

3. $\iint\limits_D \dfrac{y\,dxdy}{\sqrt{x^2+y^2}}$, where $D:\left\{(x,y)|\ 1\le x^2+y^2\le 9, y\ge 0\right\}$.

4.3 Applications of Double Integrals

Areas of Figures and Volumes of Bodies

1. If we integrate the constant function $f(x,y)=1$ over a region D, we get the area of D:

$$S_D=\iint_D 1dS=\iint_D dS.$$

2. If the region D is determined in the polar coordinates, we see that the area of the region D bounded by $\alpha\le\theta\le\beta,\quad r_1(\theta)\le r\le r_2(\theta)$, is

$$S=\iint_D r\,dr\,d\theta=\int_{\alpha}^{\beta}d\theta\int_{r_1(\theta)}^{r_2(\theta)}r\,dr.$$

3. If $f(x,y)\ge 0$, then the volume V of the solid that lies above the region D and below the surface $z=f(x,y)$ is

$$V=\iint_D f(x,y)dS.$$

Example 1 Find the volume of the solid that lies under the paraboloid $z=x^2+y^2$, above the xy-plane, and inside the cylinder $x^2+y^2=2x$.

Solution The solid lies above the disk D whose boundary circle has the equation $x^2+y^2=2x$ or, after completing the square, $(x-1)^2+y^2=1$ (See Figure 34 and 35). In polar coordinates we have $x^2+y^2=r^2$ and $x=r\cos\theta$, so the boundary circle becomes $r^2=2r\cos\theta$, or $r=2\cos\theta$.

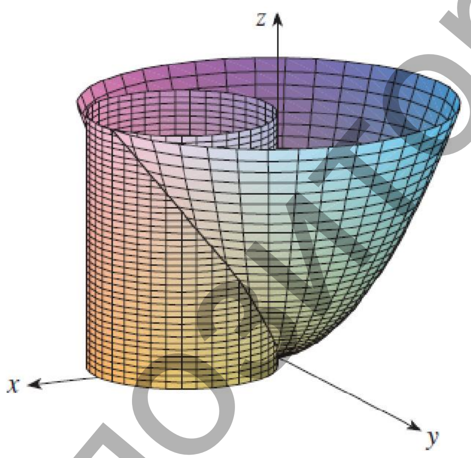


Figure 34

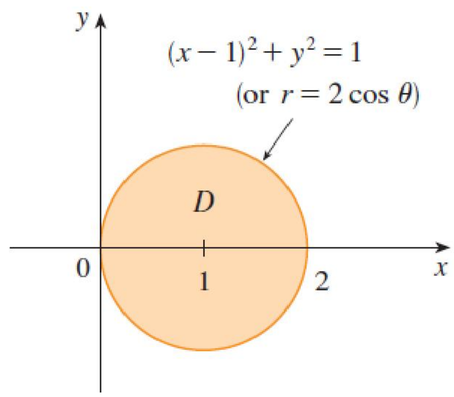


Figure 35

Thus the disk D is given by $D=\{(r,\theta)\,|\,-\pi/2\le\theta\le\pi/2, 0\le r\le 2\cos\theta\}$ and we have

$$V=\iint_D\left(x^2+y^2\right)dS=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}r^2r\,dr=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_0^{2\cos\theta}d\theta=4\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta\,d\theta=$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = 2\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1+2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)\right)d\theta =$$

$$=2\cdot\left[\frac{3}{2}\theta+\sin 2\theta+\frac{1}{8}\sin 4\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{3\pi}{2}.$$

Example 2 Use a double integral to find the area enclosed by one loop of the four leaved rose $r=\cos 2\theta$.

Solution From the sketch of the curve in Figure 36, we see that a loop is given by the region $D=\left\{(r,\theta)\,|\,-\frac{\pi}{4}\le\theta\le\frac{\pi}{4},0\le r\le\cos 2\theta\right\}$.

So the area is

$$S=\iint_D r\,dr\,d\theta=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}d\theta\int_0^{\cos 2\theta}r\,dr=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left[\frac{1}{2}r^2\right]_0^{\cos 2\theta}d\theta=\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos^2 2\theta\,d\theta=\frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1+\cos 4\theta)\,d\theta=$$

$$=\frac{1}{4}\left[\theta+\frac{1}{4}\sin 4\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}=\frac{\pi}{8}.$$

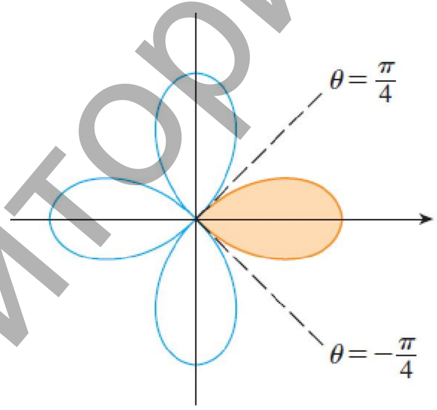


Figure 36

Moments and Centers of Mass

Consider a lamina with a variable density. Suppose the lamina occupies a region D of the xy-plane and its ***density*** (in units of mass per unit area) at a point (x,y) in D is given by $\rho(x,y)$, where $\rho(x,y)$ is a continuous function on D (see Figure 37).

The total mass m of the lamina can be obtained as the limiting value of the approximations:

$$m=\lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}\rho(x_{ij}^*,y_{ij}^*)\Delta S=\iint_D\rho(x,y)dS.$$

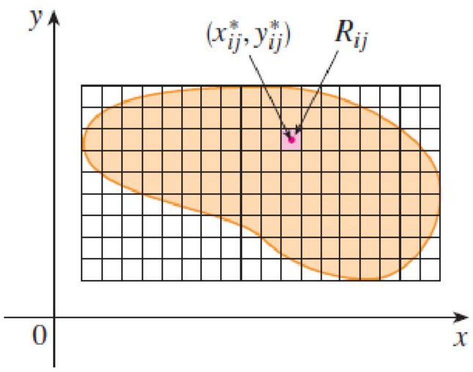


Figure 37

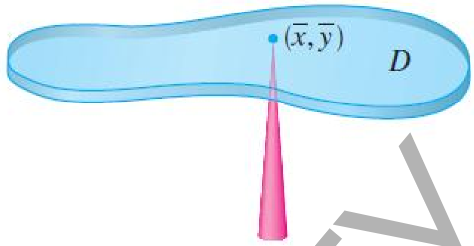


Figure 38

Suppose the lamina occupies a region D and has a density function $\rho(x, y)$. The ***moment*** of the entire lamina ***about the x-axis*** is

$$M_x = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*) \Delta S = \iint_D y\rho(x, y)dS .$$

Similarly, the ***moment about the y-axis*** is

$$M_y = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} x_{ij}^* \rho(x_{ij}^*, y_{ij}^*) \Delta S = \iint_D x\rho(x, y)dS .$$

We define the center of mass $(\overline{x}, \overline{y})$ so that $m\overline{x} = M_y$ and $m\overline{y} = M_x$:

$$\overline{x} = \frac{M_y}{m}, \quad \overline{y} = \frac{M_x}{m}$$

The physical significance is that the lamina behaves as if its entire mass is concentrated at its center of mass. Thus the lamina balances horizontally when supported at its center of mass (see Figure 38).

The ***moment of inertia*** (also called *the second moment*) of a particle of mass m ***about the x-axis*** can be obtained as the limiting value of the approximations:

$$I_x = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \Delta S = \iint_D y^2 \rho(x, y)dS .$$

Similarly, the ***moment of inertia about the y-axis*** is

$$I_y = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \Delta S = \iint_D x^2 \rho(x, y)dS .$$

It is also of interest to consider the ***moment of inertia about the origin***, also called the *polar moment of inertia*:

$$I_0 = \iint_D (x^2 + y^2) \rho(x, y)dS .$$

Exercise Set 4.3

In Exercise 1 to 8, use a double integral to find the area of the region.

1. $y = x^2$, $y = 3x$ *2.* $y^2 = 4 + x$, $x + 3y = 0$ *3.* $x = y^2$, $x = \sqrt{2 - y^2}$

4. $y = 2 - x,\ y^2 = 4x + 4$ *5.* $r = a\sin 3\theta,\ a > 0$ *6.* $r = a\cos 5\theta,\ a > 0$

7. $r = 4(1+\cos\theta)$ *8.* $r\cos\theta = 1,\ \ r = 2$

In Exercise 9 to 14, use polar coordinates to find the volume of the given solid.

9. $x^2 + y^2 = R^2,\ \ x^2 + z^2 = R^2$ *10.* $z = x^2 + y^2,\ \ z = x + y + 10$

11. $x^2 + y^2 = 4x,\ \ 2z = x^2 + y^2,\ \ z = 0$ *12.* $6z = x^2 + y^2,\ \ x^2 + y^2 + z^2 = 27,\ \ z >$

13. $z = 4 - x^2,\ 2x + y = 4,\ x = 0,\ y = 0,\ z = 0$ *14.* $2(x^2 + y^2) - z^2 = 0,\ \ x^2 + y^2 - z^2 = -1$

In Exercise 15 to 17, find the mass and the center of mass of the lamina that occupies the region D and has a given density function $\rho(x, y)$.

15. $D : \{(x, y) |\ x + y = 2,\ x = 2,\ y =$ $\rho(x, y) = 1$

16. $D : \{(x, y) |\ x^2 + y^2 - 2x = 0\}$ $\rho(x, y) = 3{,}5$

17. $D : \{(x, y) |\ y = x^2,\ y = 1\}$ $\rho(x, y) = x^2 y$

Individual Tasks 4.3

1. Use a double integral to find the area of the region.

2. Use polar coordinates to find the volume of the given solid.

3. Find the mass and the center of mass of the lamina that occupies the region D and has a given density function $\rho(x, y)$.

I *1.* $x = y^2 - 2y,\ \ x + y = 0$. *2.* $z = x^2 + y^2,\ \ \ y = x^2,\ y = 1,\ \ \ z = 0$. *3.* $D : \{(x, y) |\ y = \cos x,\ x = 0, x = \pi/4\}$, where $\rho(x, y) = 1$.
II *1.* $y = 4x - x^2,\ \ y = 2x^2 - 5x$. *2.* $x^2 + y^2 = 9,\ \ x^2 + y^2 - z^2 = -9$. *3.* $D : \{(x, y) |\ y = \sin x,\ x = 0, x = \pi/4\}$, where $\rho(x, y) = 1$.

4.4 Triple Integrals

Let f be defined on a rectangular box:

$$B = \{(x, y, z) \in \mathbb{R}^3 \mid a \le x \le b, c \le y \le d, r \le z \le s\}.$$

The first step is to divide B into sub-boxes. We do this by dividing the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of the equal width Δx, dividing $[c, d]$ into m subintervals $[y_{j-1}, y_j]$ of the width Δy, and dividing $[r, s]$ into n subintervals $[z_{k-1}, z_k]$ of the width Δz. The planes through the endpoints of these subintervals parallel to the coordinate planes divide the box B into lmn sub-boxes which are shown in Figure 39. Each sub-box has the volume $\Delta V = \Delta x \Delta y \Delta z$. Then we form the *triple Riemann sum*

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^{*}, y_{ijk}^{*}, z_{ijk}^{*})\Delta V \tag{1}$$

Definition The ***triple integral*** of f over the region B is

$$\iiint_B f(x,y,z)dV = \lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i, y_j, z_k)\Delta V \tag{2}$$

if this limit exists.

Note The triple integral always exists if f is continuous.

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals.

Fubini's Theorem for triple integrals If f is continuous on the rectangular box B, then

$$\iiint_B f(x,y,z)dV = \int_r^s\int_c^d\int_a^b f(x,y,z)dxdydz \tag{3}$$

The iterated integral on the right side of Fubini's Theorem means that first we integrate with respect to x (keeping y and z fixed), then we integrate with respect to y (keeping z fixed), and finally we integrate with respect to z.

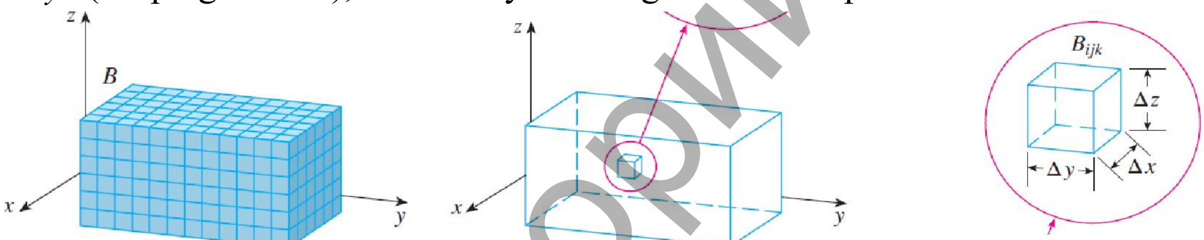


Figure 39

There are five other possible orders in which we can integrate, all of which give the same value. For instance, if we integrate with respect to y, then z, and then x, we have

$$\iiint_B f(x,y,z)dV = \int_a^b\int_r^s\int_c^d f(x,y,z)dydzdx \tag{4}$$

Now we define the *triple integral over a general bounded region E* in three dimensional space (a solid) by much the same procedure that we used for double integrals. We enclose E in a box B. Then we define a function F so that it agrees with f on E, but is 0 for points that are outside E. By definition, $\iiint_B f(x,y,z)dV = \iiint_B F(x,y,z)dV$.

This integral exists if f is continuous and the boundary of E is "reasonably smooth". *The triple integral has essentially the same properties as the double integral.*

We restrict our attention to continuous functions f and to certain simple types of regions. If the solid region E lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x,y,z) \mid (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$$

where D is the projection of E onto the xy-plane as shown in Figure 40, then

$$\iiint\limits_E f(x,y,z)dV = \iint\limits_D \left(\int\limits_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz \right) dS \qquad (5)$$

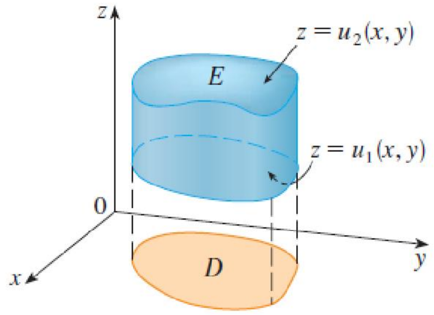


Figure 40

In particular, if the projection D of E onto the xy-plane is given by the following plane region (as in Figure 41)

$$E = \{(x,y,z) \mid a \le x \le b, g_1(x) \le y \le g_2(x), u_1(x,y) \le z \le u_2(x,y)\},$$

then Equation 5 becomes

$$\iiint\limits_E f(x,y,z)dV = \int\limits_a^b \int\limits_{g_1(x)}^{g_2(x)} \int\limits_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdydx \qquad (6)$$

If, on the other hand, D is given by the following plane region (as in Figure 41)

$$E = \{(x,y,z) \mid c \le y \le d, h_1(y) \le x \le h_2(y), u_1(x,y) \le z \le u_2(x,y)\},$$

then Equation 5 becomes

$$\iiint\limits_E f(x,y,z)dV = \int\limits_c^d \int\limits_{h_1(y)}^{h_2(y)} \int\limits_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdxdy \qquad (7)$$

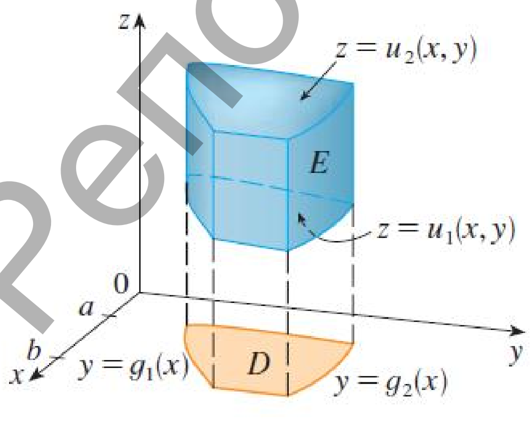


Figure 41

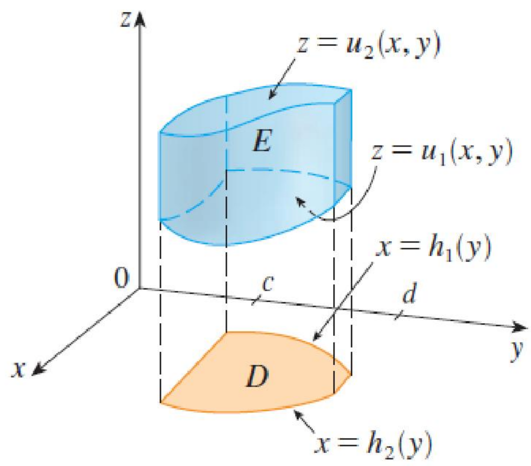


Figure 42

Example 1 Evaluate $\iiint\limits_E z\,dV$, where E is the solid tetrahedron bounded by the four planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.

Solution When we set up a triple integral, it is wise to draw *two* diagrams: one of the solid region E (see Figure 43) and one of its projection D on the xy-plane (see Figure 44). The lower boundary of the tetrahedron is the plane $z=0$ and the upper boundary is the plane $x+y+z=1$ (or $z=1-x-y$), so we use $u_1(x,y)=0$ and $u_2(x,y)=1-x-y$ in Formula 6. Notice that the planes $x+y+z=1$ and $z=0$ intersect in the line $x+y=1$ (or $y=1-x$) in the xy-plane. So the projection of E is the triangular region shown in Figure 44, and we have

$$E=\{(x,y,z)\,|\,0\le x\le 1, 0\le y\le 1-x, 0\le z\le 1-x-y\}.$$

This description of E as a type 1 region enables us to evaluate the integral as follows:

$$\iiint\limits_E z\,dV=\int_0^1\int_0^{1-x}\int_0^{1-x-y} z\,dz=\int_0^1\int_0^{1-x}\left[\frac{z^2}{2}\right]_0^{1-x-y}dy=\frac{1}{2}\int_0^1\int_0^{1-x}(1-x-y)^2\,dy=$$

$$=\frac{1}{2}\int_0^1\left[-\frac{(1-x-y)^3}{3}\right]_0^{1-x}dx=\frac{1}{6}\int_0^1(1-x)^3\,dx=\frac{1}{6}\left[-\frac{(1-x)^4}{4}\right]_0^1=\frac{1}{24}.$$

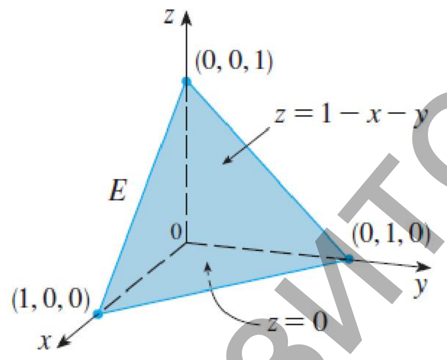


Figure 43

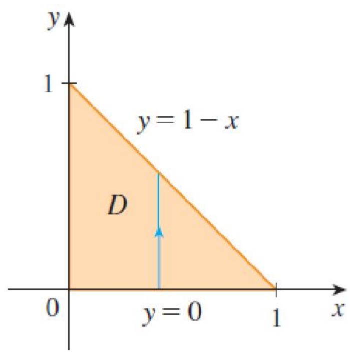


Figure 44

If the solid region E is given by the following form

$$E=\{(x,y,z)\,|\,(y,z)\in D, u_1(y,z)\le x\le u_2(y,z)\},$$

where this time D is the projection of E onto the yz-plane (see Figure 45), when the back surface is $x=u_1(y,z)$, the front surface is $x=u_2(y,z)$, then we have

$$\iiint\limits_E f(x,y,z)\,dV=\iint\limits_D\left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right)dS \qquad (8)$$

Finally, if a region is of the form $E=\{(x,y,z)\,|\,(x,z)\in D, u_1(x,z)\le y\le u_2(x,z)\}$, where D is the projection of E onto the xz-plane, then $y=u_1(x,z)$ is the left surface, and $y=u_2(x,z)$ is the right surface (see Figure 46). For this type of a region we have

$$\iiint\limits_E f(x,y,z)dV = \iint\limits_D \left(\int\limits_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy \right) d \qquad (9)$$

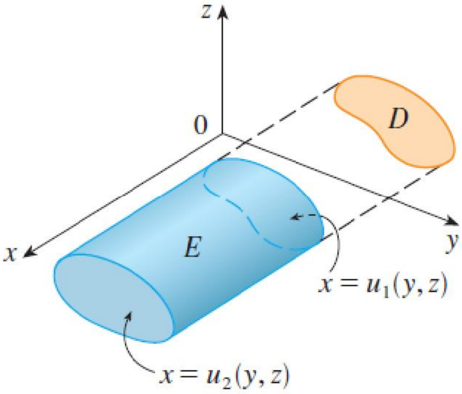


Figure 45

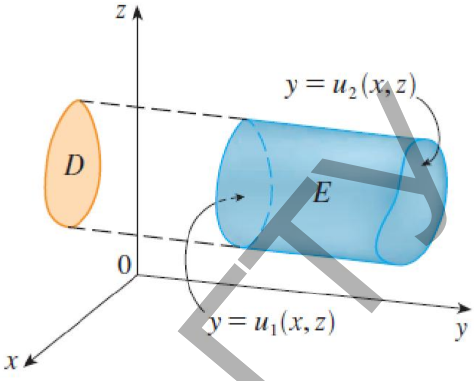


Figure 46

Triple Integrals in Cylindrical Coordinates

In the ***cylindrical coordinate system***, a point P in three-dimensional space is represented by the ordered triple (r,θ,z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P (See Figure 47).

To convert from cylindrical to rectangular coordinates, we use the equations

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z, \quad r \ge 0,\ 0 \le \theta \le 2\pi,\ z \in R.$$

Cylindrical coordinates are useful in problems that involve symmetry about an axis, and the z-axis is chosen to coincide with this axis of symmetry. For instance, the axis of the circular cylinder with Cartesian equation $x^2 + y^2 = c^2$ is the z-axis. In cylindrical coordinates this cylinder has a very simple equation $r = c$ (See Figure 48.) This is the reason for the name "cylindrical" coordinates.

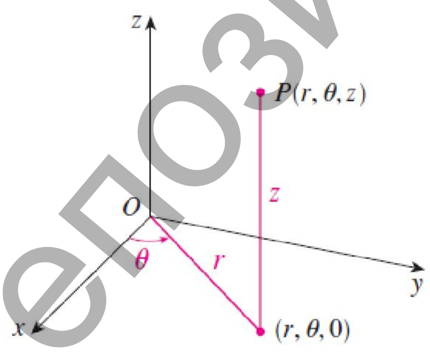


Figure 47

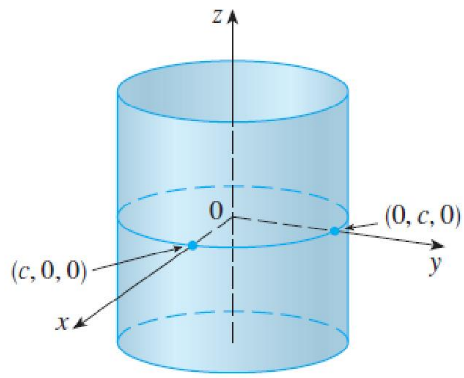


Figure 48

Suppose that E is a region whose projection D on the xy-plane is conveniently described in polar coordinates (see Figure 49). In particular, suppose that f is continuous and $E = \{(x,y,z) \mid (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$, where D is given in polar coordinates by $D = \{(r,\theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$.

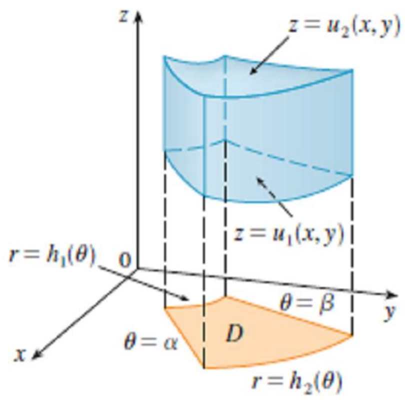


Figure 49

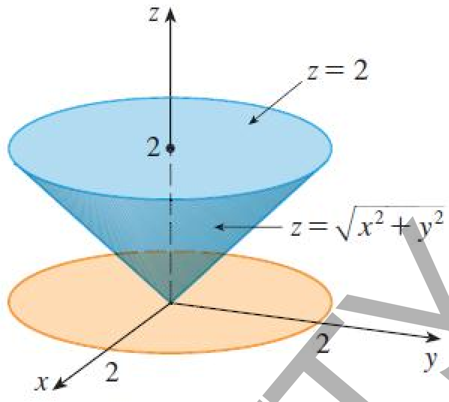


Figure 50

We also know how to evaluate double integrals in polar coordinates. We obtain

$$\iiint\limits_E f(r\cos\theta, r\sin\theta, z) r\,dr\,d\theta\,dz = \int_{\alpha}^{\beta} d\phi \int_{h_1(\theta)}^{h_2(\theta)} r\,dr \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)dz .$$

The last formula is the *formula for triple integration in cylindrical coordinates*. It says that we convert a triple integral from rectangular to cylindrical coordinates by writing $x = r\cos\theta$, $y = r\sin\theta$, leaving z as it is, using the appropriate limits of integration for z, r, and θ, and replacing dV by $r\,dr\,d\theta\,dz$.

Example 2 Evaluate

$$\int_{-2}^{2\pi} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{y^2+x^2}}^{2} (x^2 + y^2)dzdydx .$$

Solution This iterated integral is a triple integral over the solid region

$$E = \left\{ (x, y, z) \,|\, -2 \le x \le 2, -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, \sqrt{x^2+y^2} \le z \le 2 \right\}$$

and the projection of E onto the xy-plane is the disk $x^2 + y^2 \le 4$. The lower surface of E is the cone $z = \sqrt{x^2 + y^2}$ and its upper surface is the plane $z = 2$ (See Figure 50.)

This region has a much simpler description in cylindrical coordinates:

$$E = \{ (r, \theta, z) \,|\, 0 \le \theta \le 2\pi, 0 \le r \le 2, r \le z \le 2 \} .$$

Therefore, we have

$$\int_{-2}^{2\pi} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{y^2+x^2}}^{2} (x^2 + y^2)dzdydx = \int_0^{2\pi}\int_0^2\int_r^2 r^2 r dz dr d\theta = \int_0^{2\pi} d\theta \int_0^2 r^3(2-r)dr =$$

$$= 2\pi \left[\frac{1}{2}r^4 - \frac{1}{5}r^5 \right]_0^2 = \frac{16}{5}\pi .$$

Triple Integrals in Spherical Coordinates

Another useful coordinate system in three dimensions is the *spherical coordinate system*. It simplifies the evaluation of triple integrals over regions bounded by spheres or cones.

The ***spherical coordinates*** (r, θ, ϕ) of a point P in space are shown in Figure 51, where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP.

The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point.

The relationship between rectangular and spherical coordinates can be seen from Figure 52. Triangles OQP and OPP' give $r = \rho \sin\phi$, $z = \rho\cos\phi$.

But $x = r\cos\theta$ and $y = r\sin\theta$, so to convert from spherical to rectangular coordinates, we use the equations

$$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi.$$

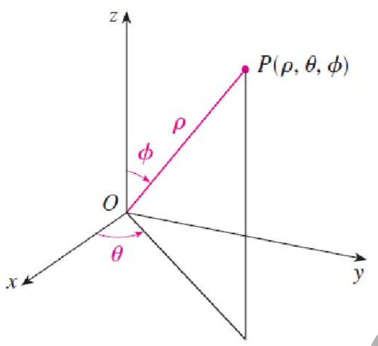


Figure 51

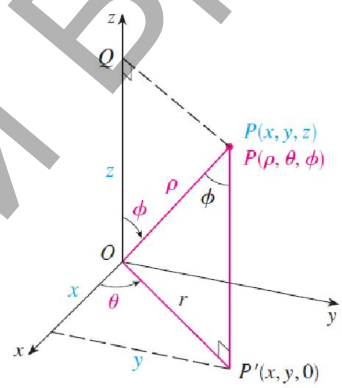


Figure 52

We have obtained the following *formula for triple integration in spherical coordinates*.

$$\iiint\limits_E f(x,y,z)dxdydz = \iiint\limits_E \rho^2 \sin\phi \cdot f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, d\rho\, d\theta\, d\phi.$$

Exercise Set 4.4

In Exercise 1 to 4, define the limits of integration of the triple integral $\iiint\limits_V f(x,y,z)dxdydz$, where:

1. $V: x + y + z = 1,\ x = 0, y = 0, z = 0.$

2. $V: x^2 + y^2 = 4,\ z = 5,\ z = 0.$

3. $V: \dfrac{x^2}{4} + \dfrac{y^2}{1} = \dfrac{z^2}{9},\ z = 3.$

4. $V: z = 1 - x^2 - y^2,\ z = 0.$

In Exercise 5 to 8, calculate:

5. $\displaystyle\int_0^1 dx \int_0^1 dy \int_0^1 \frac{dz}{\sqrt{x+y+z+1}}$

6. $\displaystyle\int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} xyzdz$

7. $\int\limits_0^2 dx \int\limits_0^{\sqrt{4-x^2}} dy \int\limits_0^{\sqrt{4-x^2-y^2}} \frac{dz}{\sqrt{4-x^2-y^2-z^2}}$ 8. $\int\limits_0^2 dx \int\limits_0^{2\sqrt{x}} dy \int\limits_0^{\sqrt{\frac{4x-y^2}{2}}} xdz$

In Exercise 9 to 14, evaluate the triple integral.

9. $\iiint\limits_V x^3y^2zdxdydz,\ V:\{(x,y,z)|0\le x\le 1,\ 0\le y\le x,\ 0\le z\le xy\}.$

10. $\iiint\limits_V x^2y^2dxdydz,\ V:\{(x,y,z)|z=x^2+y^2,\ x^2+y^2=1,\ z=0\}.$

11. $\iiint\limits_V (2x-y+4z)dV,\ V:\{(x,y,z)|\ x+2y+z=2,\ x>0,\ y>0,\ z>0\}.$

12. $\iiint\limits_V (2x-y)dV,\ V:\{(x,y,z)|\ z=x+y+4,\ y^2=4x,\ x=4,\ z=0,\ y>0\}.$

13. $\iiint\limits_V (x-y+z)dV,\ V:\{(x,y,z)|\ y^2=4x,\ z=4-x,\ z=0\}.$

14. $\iiint\limits_V xydxdydz,\ V:\{(x,y,z)|z=x^2+y^2,\ x^2+y^2=4,\ z=0\}.$

In Exercise 15 to 17, write down the equations of given surfaces in cylindrical coordinates:

15. $z=2-x^2-y^2$ 16. $z=\sqrt{x^2+y^2}$ 17. $z=\sqrt{R^2-x^2-y^2}$

In Exercise 18 to 20, determine the type of surfaces given in cylindrical coordinates:

18. $z=5$ 19. $\varphi=\pi/3$ 20. $\rho=2$

21. Change the variables in the integral $\int\limits_0^2 dx \int\limits_0^{\sqrt{4-x^2}} dy \int\limits_{x^2+y^2}^{4} \frac{dz}{\sqrt{x^2+y^2}}$ by cylindrical coordinates.

In Exercise 22 to 24, write down the equations of given surfaces in spherical coordinates:

22. $x^2+y^2+z^2=81$ 23. $z=\sqrt{x^2+y^2}$ 24. $y=\frac{x}{\sqrt{3}}, x\ge 0.$

In Exercise 25 to 27, determine the type of surfaces given in spherical coordinates:

25. $\theta=3\pi/4$ 26. $\varphi=5\pi/6$ 27. $\rho=3$

Individual Tasks 4.4

1-3. Evaluate the triple integral.

I
1. $\int\limits_0^1 dx \int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int\limits_0^1 dz\cdot$

2. $\iiint\limits_V (x-2y+6z)dV$, $V:\{(x,y,z)|\ 2x+3y+z=6,\ x>0,\ y>0,\ z>0\}$.

3. $\iiint\limits_V x^2y^2dxdydz$, $V:\{(x,y,z)|\ z^2=x^2+y^2,\ z=6\}$.

II

1. $\int\limits_0^a dx\int\limits_0^x dy\int\limits_0^{xy} x^3y^3z\,dz$ ·

2. $\iiint\limits_V (x+z)dV$, $V:\{(x,y,z)|\ x+2y+3z=6,\ x>0,\ y>0,\ z>0\}$.

3. $\iiint\limits_V x^2ydxdydz$, $V:\{(x,y,z)|z^2=x^2+y^2,\ x^2+y^2=9,\ z=0\}$.

4.5 Applications of Triple Integrals

1. Let's begin with a special case where $f(x,y,z)=1$ for all points in E. Then a triple integral represents the ***volume*** of E: $\iiint\limits_E dV=V_E$.

2. All the applications of double integrals in Section 4.3 can be immediately extended to triple integrals. For example, if the density function of a solid object that occupies the region E is $\rho(x,y,z)$, in units of mass per unit volume, at any given point (x,y,z), then its ***mass*** is

$$m=\iiint\limits_E \rho(x,y,z)dV$$

and its ***moments*** about the three coordinate planes are

$$M_{xy}=\iiint\limits_E z\rho(x,y,z)dV;\quad M_{yz}=\iiint\limits_E x\rho(x,y,z)dV;\quad M_{xz}=\iiint\limits_E y\rho(x,y,z)dV.$$

The ***center of mass*** is located at the point $(\overline{x},\overline{y},\overline{z})$, where

$$\overline{x}=\frac{M_{yz}}{m},\quad \overline{y}=\frac{M_{xz}}{m},\quad \overline{z}=\frac{M_{xy}}{m}.$$

If the density is constant, the center of mass of the solid is called the ***centroid*** of E. The ***moments of inertia*** about the three coordinate axes are

$$I_x=\iiint\limits_E (z^2+y^2)\rho(x,y,z)dV;\ I_y=\iiint\limits_E (x^2+z^2)\rho(x,y,z)dV;$$

$$I_z=\iiint\limits_E (y^2+x^2)\rho(x,y,z)dV.$$

Example 1 A solid E lies within the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$ (See Figure 53.) The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

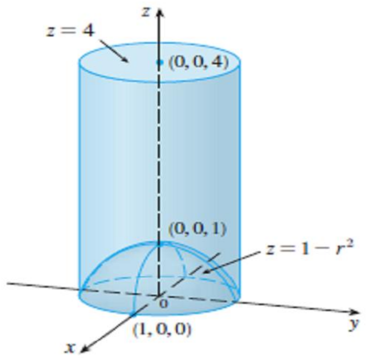


Figure 53

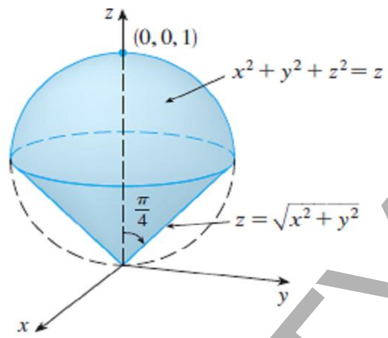


Figure 54

Solution In cylindrical coordinates the cylinder is $r = 1$ and the paraboloid is $z = 1 - r^2$, so we can write $E = \left\{(r,\theta,z)\,|\,0 \le \theta \le 2\pi, 0 \le r \le 1, 1 - r^2 \le z \le 4\right\}$. Since the density at (x,y,z) is proportional to the distance from the z-axis, the density function is

$f(x,y,z) = K\sqrt{x^2+y^2} = Kr$, where K is the proportionality constant.

Therefore, the mass of E is

$$m = \iiint\limits_E \rho(x,y,z)dV = \iiint\limits_E K\sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 (Kr)r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 Kr^2\left[4-(1-r^2)\right]dr\,d\theta =$$

$$= K\int_0^{2\pi} d\theta \int_0^1 \left(3r^2 + r^4\right)dr = 2\pi K\left[r^3 + \frac{r^5}{5}\right]_0^1 = \frac{12\pi K}{5}.$$

Example 2 Use spherical coordinates to find the volume of the solid, that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 \le z$ (See Figure 54.)

Solution Notice that the sphere passes through the origin and has the center $\left(0,0,\frac{1}{2}\right)$. We write the equation of the sphere in spherical coordinates as $\rho^2 = \rho\cos\varphi$ or $\rho = \cos\varphi$.

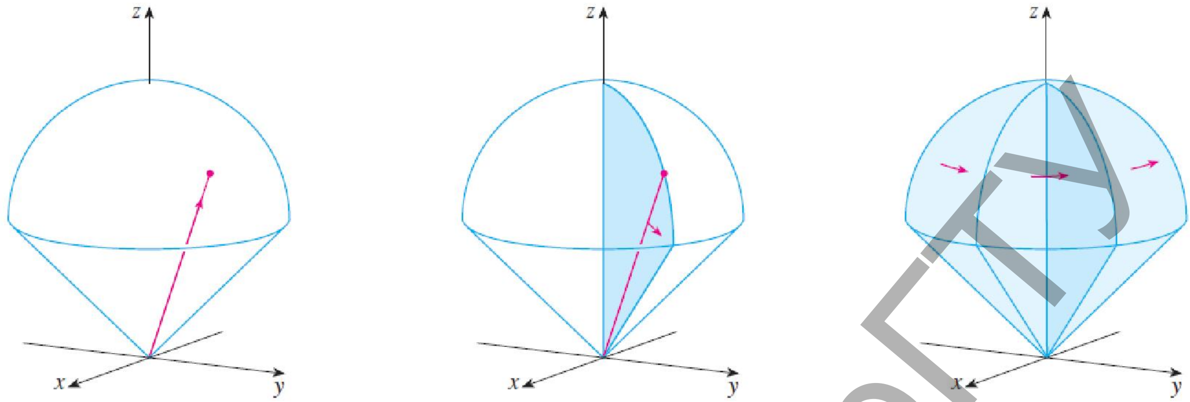
The equation of the cone can be written as this gives $\sin\varphi = \cos\varphi$, or $\varphi = \frac{\pi}{4}$. Therefore the description of the solid E in spherical coordinates is

$$E = \left\{(\rho,\theta,\varphi)\,|\,0 \le \theta \le 2\pi, 0 \le \varphi \le \frac{\pi}{4}, 0 \le \rho \le \cos\varphi\right\}.$$

Figure 55 shows how E is swept out if we integrate first with respect to ρ, then φ, and then θ. The volume of E is

$$V = \iiint\limits_E dV = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\varphi} \rho^2 \sin\varphi\,d\varphi\,d\rho\,d\theta = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\varphi\left[\frac{\rho^3}{3}\right]_0^{\cos\varphi} d\varphi =$$

$$= \frac{2\pi}{3} \int_{0}^{\frac{\pi}{4}} \sin\varphi \cos^3\varphi d\varphi = \frac{2\pi}{3}\left[-\frac{\cos^4\varphi}{4}\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8}.$$



ρ varies from 0 to cos ϕ while ϕ and θ are constant.

ϕ varies from 0 to $\pi/4$ while θ is constant.

θ varies from 0 to 2π.

Figure 55

Exercise Set 4.5

In Exercise 1 to 4, use a triple integral to find the volume of the given solid bounded by given surfaces.

1. $V:\left\{(x,y,z)|\ x^2+y^2=10x,\ x^2+y^2=13x,\ z=\sqrt{x^2+y^2},\ z=0,\ y\ge 0\right\}$

2. $V:\left\{(x,y,z)|\ z=x^2+y^2,\ y=x^2,\ y=1,\ z=0\right\}$

3. $V:\left\{(x,y,z)|\ x^2+z^2=4,\ y=-1,\ y=3\right\}$

4. $V:\left\{(x,y,z)|\ z=x^2+y^2,\ x=y^2,\ x=4,\ z=0\right\}$

In Exercise 5 to 7, find the mass and the center of mass of the solid V with the given density function $\rho(x,y,z)=1$.

5. $V:\left\{(x,y,z)|\ z\ge 8\left(x^2+y^2\right),\ z\le 32\right\}$

6. $V:\left\{(x,y,z)|\ z\ge 9\sqrt{x^2+y^2},\ z\le 36\right\}$

7. $V:\left\{(x,y,z)|\ y\ge 3\sqrt{x^2+z^2},\ y\le 9\right\}$

Individual Tasks 4.5

1. Use a triple integral to find the volume of the given solid.

2. Use a triple integral to find the volume of the given solid bounded by given surfaces.

3. Find the mass and the center of mass of the solid V with the given density function $\rho(x,y,z)=1$.

I

1. $V:\left\{(x,y,z)|\ x^2+y^2+z^2\le 1,\ z^2\ge x^2+y^2\right\}$.

2. $z=x^2+y^2,\quad y=x^2,\ y=1,\quad z=0$.

3. $V:\left\{(x,y,z)\mid z\ge 2\sqrt{x^2+y^2},\ z\le 8\right\}$.
II 1. $V:\left\{(x,y,z)\mid z^2\ge x^2+y^2,\ z\le 6\right\}$. 2. $x^2+y^2=9,\ x^2+y^2-z^2=-9$. 3. $V:\left\{(x,y,z)\mid x\ge 4\sqrt{y^2+z^2},\ x\le 16\right\}$

4.6 Line Integrals

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval $[a,b]$, we integrate over a curve C. Such integrals are called *line integrals,* although "curve integrals" would be better terminology.

We start with a plane curve C given by the parametric equations

$$x = x(t), y = y(t), a \le t \le b \qquad (1)$$

If we divide the parameter interval $[a,b]$ into n subintervals $[t_{i-1},t_i]$ of the equal width and we let $x_i = x(t_i)$ and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with the lengths $\Delta s_1, \Delta s_2, \Delta s_3, ..., \Delta s_n$ (See Figure 56.)

We choose any point $P_i^*(x_i^*, y_i^*)$ in the i-th subarc. (This corresponds to a point t_i^* in $[t_{i-1},t_i]$.) Now if f is any function of two variables whose domain includes the curve C, we evaluate f at the point $P_i^*(x_i^*, y_i^*)$, multiply by the length Δs_i of the subarc, and form the sum $\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$, which is similar to a Riemann sum. Then we take the limit of these sums and make the following definition by analogy with a single integral.

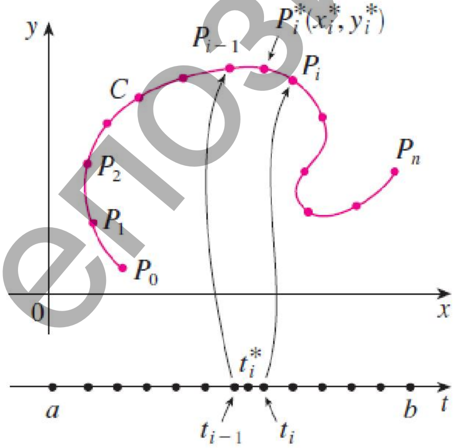


Figure 56

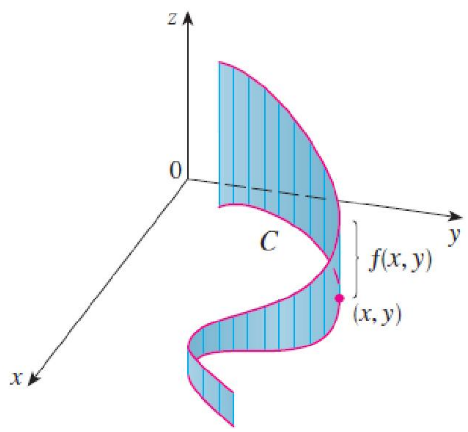


Figure 57

Definition If f is defined on a smooth curve C given by Equations 1, then the ***line integral of*** f ***along C*** is

$$\int_C f(x,y)ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i \tag{2}$$

if this limit exists.

We found that the length of C is $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.

A similar type of argument can be used to show that if f is a continuous function, then the limit in definition always exists and the following formula can be used to evaluate the line integral:

$$\int_C f(x,y)ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \tag{3}$$

The value of the line integral does not depend on the parameterization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

Just as for an ordinary single integral, we can interpret the line integral of a *positive* function as an area. In fact, if $f(x,y) \geq 0$, $\int_C f(x,y)ds$ represents the area of one side of the "fence" or "curtain" in Figure 57, whose base is C and whose height above the point (x,y) is $f(x,y)$.

Example 1 Evaluate $\int_C (2 + x^2 y)ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.

Solution In order to use Formula 3, we first need parametric equation to represent C. Recall that the unit circle can be parameterized by means of the equations $x = \cos t, y = \sin t$ and the upper half of the circle is described by the parameter interval $0 \leq t \leq \pi$ (See Figure 58.)

Therefore Formula 3 gives

$$\int_C (2 + x^2 y)ds = \int_0^\pi (2 + \cos^2 t \sin t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^\pi (2 + \cos^2 t \sin t)\sqrt{\sin^2 t + \cos^2 t}\,dt =$$

$$= \int_0^\pi (2 + \cos^2 t \sin t)dt = \left[2t - \frac{\cos^3 t}{3}\right]_0^\pi = 2\pi + \frac{2}{3}.$$

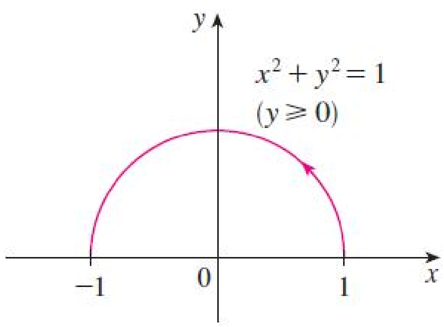


Figure 58

Suppose now that C is a ***piecewise-smooth curve***; that is, C is a union of a finite number of smooth curves $C_1, C_2, C_3, \ldots, C_n$ where, as illustrated in Figure 59, the initial point of C_{i+1} is the terminal point of C_i.

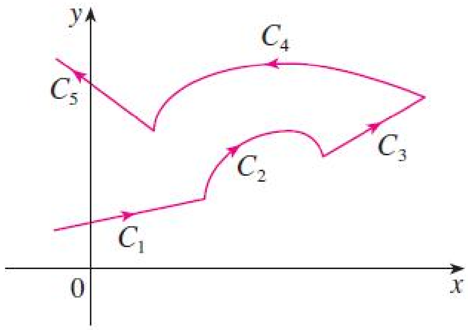


Figure 59

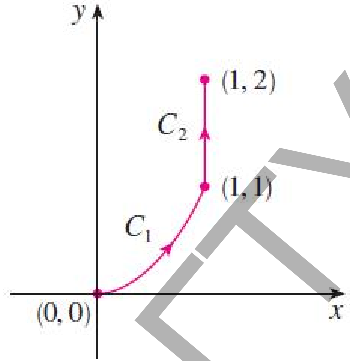


Figure 60

Then we define the integral of f along C as the sum of the integrals of f along each of the smooth pieces of C:

$$\int_C f(x,y)ds = \int_{C_1} f(x,y)ds + \int_{C_2} f(x,y)ds + \ldots + \int_{C_n} f(x,y)ds .$$

Example 2 Evaluate $\int_C 2xds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$.

Solution The curve C is shown in Figure 60. C_1 is the graph of a function of x, so we can choose x as the parameter and the equations for C_1 become $x = x, y = x^2, 0 \le x \le 1$.

Therefore

$$\int_C 2xds = \int_0^1 2x\sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 2x\sqrt{1+4x^2}\,dx = \left[\frac{1}{4}\cdot\frac{2}{3}\left(1+4x^2\right)^{\frac{3}{2}}\right]_0^1 = \frac{5\sqrt{5}-1}{6}.$$

On C_2 we choose y as a parameter, so the equations of C_2 are $x = 1, y = y, 1 \le y \le 2$ and

$$\int_C 2xds = \int_1^2 2\cdot 1\cdot\sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\,dy = \int_1^2 2dy = 2,$$

$$\int_C 2xds = \int_{C_1} 2xds + \int_{C_2} 2xds = \frac{5\sqrt{5}-1}{6} + 2 .$$

Any physical interpretation of a line integral $\int_C f(x,y)ds$ depends on the physical interpretation of the function f. Suppose that $\rho(x,y)$ represents the linear density at a point (x,y) of a thin wire shaped like a curve C. Then the ***mass*** m of C is

$$m = \int_C \rho(x,y)ds\,.$$

The ***center of mass*** of the wire with a density function $\rho(x,y)$ is located at the point $(\overline{x},\overline{y})$, where

$$\overline{x} = \frac{1}{m}\int_C x\rho(x,y)ds\,; \qquad \overline{y} = \frac{1}{m}\int_C y\rho(x,y)ds.$$

Two other line integrals are obtained by replacing Δs_i by either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$ in definition. They are called the ***line integrals of*** f ***along*** C ***with respect to x and y***:

$$\int_C f(x,y)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta x_i\,;$$
$$\int_C f(x,y)dy = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta y_i \qquad (4)$$

When we want to distinguish the original line integral $\int_C f(x,y)ds$ from those in Equation 4, we call it the ***line integral with respect to arc length***.

The following formulas say that line integrals with respect to x and y can also be evaluated by expressing everything in terms of t: $x = x(t), y = y(t), dx = x'(t), dy = y'(t)$.

$$\int_C f(x,y)dx = \int_a^b f(x(t),y(t))yx(t)dt \qquad (5)$$

It frequently happens that line integrals with respect to x and y occur together. When this happens, it is customary to abbreviate by writing

$$\int_C P(x,y)dx + \int_C Q(x,y)dy = \int_C P(x,y)dx + Q(x,y)dy \qquad (6)$$

Example 3 Evaluate $\int_C y^2dx + xdy$, where (a) $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$ and (b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$ (See Figure 61.)

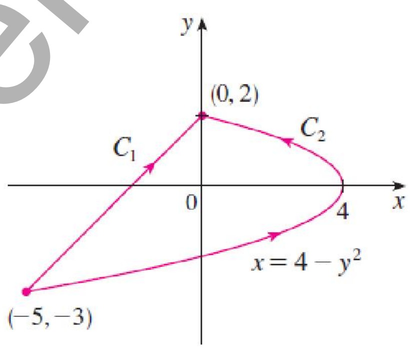


Figure 61

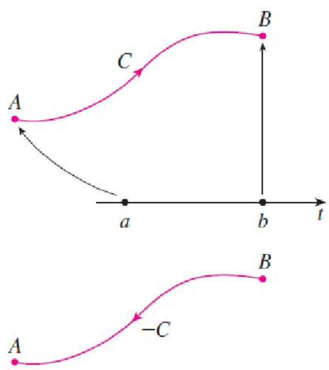


Figure 62

Solution

(a) A parametric representation for the line segment is $x = 5t - 5,\ y = 5t - 3,\ 0 \le t \le 1$. Then $dx = 5dt,\ dy = 5dt$, and Formula 6 gives

$$\int_C y^2 dx + x dy = \int_0^1 (5t-3)^2(5dt) + (5t-5)(5dt) = 5\int_0^1 (25t^2 - 25t + 4)dt =$$

$$= 5\left(\frac{25t^3}{3} - \frac{25t^2}{2} + 4t\right)\Bigg|_0^1 = -\frac{5}{6}.$$

(b) Since the parabola is given as a function of y, let's take as the parameter and write C_2 as $x = 4 - y^2,\ -3 \le y \le 2$. Then $dx = -2ydy$ and by Formulas 6 we have

$$\int_C y^2 dx + x dy = \int_{-3}^{2} y^2(-2y)dy + (4 - y^2)dy = \int_{-3}^{2} (-2y^3 - y^2 + 4)dy =$$

$$= \left(-\frac{y^4}{2} - \frac{y^3}{3} + 4y\right)\Bigg|_{-3}^{2} = 40\frac{5}{6}.$$

In general, a given parameterization $x = x(t),\ y = y(t),\ a \le t \le b$, determines the *orientation* of a curve C, with the positive direction corresponding to increasing values of the parameter t. (See Figure 62, where the initial point A corresponds to the parameter value a and the terminal point B corresponds to b.)

If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation (from initial point B to terminal point A in Figure 62), then we have

$$\int_{-C} f(x,y)dx = -\int_C f(x,y)dx;$$
$$\int_{-C} f(x,y)dy = -\int_C f(x,y)dy \qquad (7)$$

But if we integrate with respect to the arc length, the value of the line integral does *not* change when we reverse the orientation of the curve: $\int_{-C} f(x,y)ds = \int_C f(x,y)ds$.

We now suppose that C is a smooth space curve given by the parametric equations

$$x = x(t), y = y(t), z = z(t), a \le t \le b.$$

If f is a function of three variables that is continuous on some region containing C, then we define the ***line integral of*** f ***along*** C (with respect to the arc length) in a manner similar to that for plane curves: $\int_C f(x,y,z)ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s_i$.

We evaluate it using a formula similar to Formula 3:

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \tag{8}$$

Therefore, as with line integrals in the plane, we evaluate the integrals of the form

$$\int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz \tag{9}$$

by expressing everything (x, y, z, dx, dy, dz) in terms of the parameter t.

Example 4 Evaluate $\int_C y\sin z ds$, where C is the circular helix given by the equations $x = \cos t, y = \sin t, z = t, 0 \le t \le 2\pi$ (See Figure 63.)

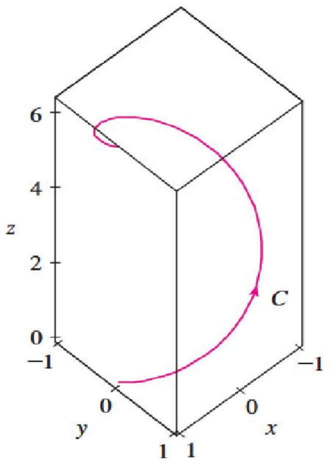


Figure 63

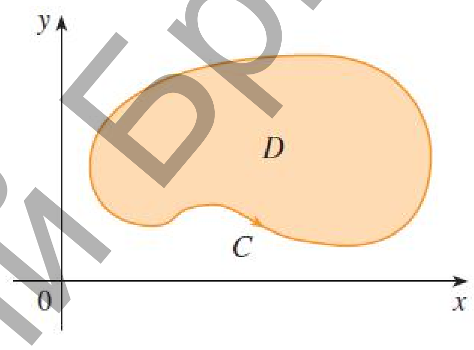


Figure 64

Solution Formula 8 gives

$$\int_C y\sin z ds = \int_0^{2\pi} (\sin t)\sin t\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \int_0^{2\pi} \sin^2 t\sqrt{\sin^2 t + \cos^2 t + 1}dt =$$

$$= \sqrt{2}\int_0^{2\pi} \frac{1}{2}(1-\cos 2t)dt = \frac{\sqrt{2}}{2}\left[t - \frac{1}{2}\sin 2t\right]_0^{2\pi} = \sqrt{2}\pi .$$

Exercise Set 4.6

In Exercise 1 to 4, evaluate the line integral, where L is the given curve.

1. $\int_L x\, dl$, if L is the line segment from $A(0;0)$ to $B(1,2)$.

2. $\int_L \frac{dl}{(x+y)}$, if L is the line segment $y = x + 2$ from $A(2;4)$ to $B(1,3)$.

3. $\int_L \sqrt{2y}\, dl$, if L is given by $\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t), \end{cases}$ $(a > 0)$.

4. $\int_L (x+y)\, dl$, if L is given by $\rho^2 = a^2\cos^2\theta$.

In Exercise 5 to 12, find the mass of the curve C with the density $\rho = \rho(x;y)$:

5. $C: y = 2\sqrt{x}, 0 \le x \le 1, \rho = y$

6. $C: x^2+y^2=6x,\ \rho=x^2$

7. $C: x=3t,\ y=3t^2, z=2t^3$ from $O(0;0;0)$ to $A(3;3;2)$, $\rho=4$

8. $C: x=\cos t,\ y=\sin t, t\in[0;\pi], \rho=y$

9. $C: x=e^{-t}\cos t,\ y=e^{-t}\sin t,\ z=e^{-t},\ t\in[0;+\infty), \rho=y$

10. $C: x=2\cos^3 t,\ y=2\sin^3 t, \rho=|xy|$

11. $C: r=3(1+\cos\theta),\ \rho=2\sqrt{r}$

12. $C: r^2=4\cos 2\theta,\ \rho=2r$

In Exercise 13 to 16, evaluate the line integral, where L is the given curve.

13. $\int\limits_L (x^2-2xy)dx+(y^2-2xy)dy$, if L is $y=x^2$ from $A(-1,1)$ to $B(1,1)$.

14. $\int\limits_L 2xy\,dx - x^2\,dy$, if L is given by OAB: $O(0,0)$, $B(2,0)$, $A(2,1)$.

15. $\int\limits_L x\,dy - y\,dx$, if L is given by $\begin{cases} x=2\cos^3 t, \\ y=2\sin^3 t, \end{cases}$ from $A(2,0)$ to $B(0,2)$.

16. $\int\limits_{L_{AB}} 2xy\,dx + y^2\,dy + z^2\,dz$, if L_{AB} is given by $\begin{cases} x=\cos t, \\ y=\sin t, \\ z=2t. \end{cases}$ from $A(1,0,0)$ to $B(1,0,4\pi)$.

Individual Tasks 4.6

1-3. Evaluate the line integral, where L is the given curve.

I 1. $\int\limits_L \frac{dl}{(x-2y)}$, if L is the line segment from $A(2;1)$ to $B(1,4)$. 2. $\int\limits_L \sqrt{x^2+y^2}\,dl$, if L is given by $\begin{cases} x=a(\cos t+t\sin t); \\ y=a(\sin t - t\sin t). \end{cases}$. 3. $\int\limits_L xydx+zx^2dy+xyzdz$, if L is given by $x=e^t$, $y=e^{-t}$, $z=t^2$, $0\le t\le 1$
II 1. $\int\limits_L \frac{dl}{(2x+y)}$, if L is the line segment from $A(-2;1)$ to $B(1,-3)$. 2. $\int\limits_L \left(3x-2\sqrt[3]{a^2y}\right)dl$, if L is given by $\begin{cases} x=a\cos^3 t, \\ y=a\sin^3 t, \end{cases}$ $(a>0)$. 3. $\int\limits_{L_{AB}} \frac{x^2dy-y^2dx}{\sqrt[3]{x^5}+\sqrt[3]{y^5}}$, if L is given by $\begin{cases} x=2\cos^3 t, \\ y=2\sin^3 t, \end{cases}$ from $A(2,0)$ to $B(0,2)$.

4.7 Green's Theorem

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C (See Figure 64.) We assume that D consists of all points inside C as well as all points on.)

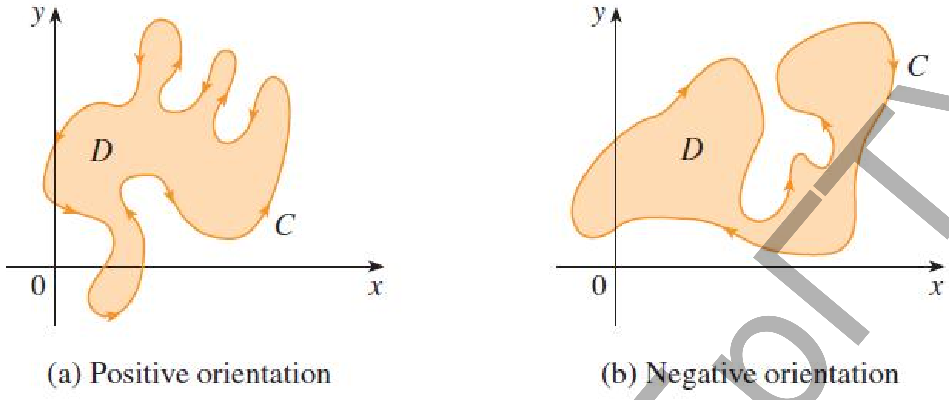


Figure 65

In stating Green's Theorem we use the convention that the *positive orientation* of a simple closed curve C refers to a single *counterclockwise* traversal of C. Thus if C is given by the vector function $\vec{r}(t)$, $a \le t \le b$, then the region D is always on the left as the point $\vec{r}(t)$ traverses C (See Figure 65.)

Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P(x,y)dx + Q(x,y)dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dS \qquad (1)$$

Then Green's Theorem gives the following formulas for the area of D:

$$S = \oint_C xdy = -\oint_C ydx = \frac{1}{2}\oint_C xdy - ydx \qquad (2)$$

Example 1 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution The ellipse has parametric equations $x = a\cos t, y = b\sin t$, where $0 \le t \le 2\pi$. Using the third Formula 2, we have

$$S = \frac{1}{2}\int_C xdy - ydx = \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)dt - (b\sin t)(-a\sin t)dt = \frac{ab}{2}\int_0^{2\pi} dt = \pi ab .$$

Exercise Set 4.7

1. Does the integral $\int_C ydx + xdy$ depend on a form of the curve C?

2. What is the integral $\oint_\gamma x^2dx + y^2dy,\ \gamma:\ x^2 + y^2 = 4$ equal to?

In Exercise 3 to 4, find the area of the domain D bounded by the given curves using the line integral:

3. $D: y = x^2, y^2 = x, 8xy = 1$

4. $D: x = 2\cos t - \cos 2t,\ y = 2\sin t - \sin 2t$

In Exercise 5 to 7, evaluate the line integrals for the given points:

5. $\int\limits_{L_{AB}} (x+3y)dx + (y+3x)dy, A(1;1), B(2;3)$

6. $\int\limits_{L_{AB}} (xy^2 - x^3)dx + (yx^2 - y^3)dy, A(-1;1), B(2;3)$

7. $\int\limits_{L_{AB}} \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy, A(1;0), B(6;8)$

In Exercise 8 to 10, evaluate the line integral, where L is the given curve.

8. $\oint\limits_L y\,dx - x\,dy$, if L is given by $\begin{cases} x = a\cos t, \\ y = b\sin t. \end{cases}$

9. $\oint\limits_L x\,dy$, if L is the triangle bounded by $y = x$, $x = 2$, $y = 0$.

10. $\oint\limits_L (x^2+y^2)dx + (x^2-y^2)dy$, if L is the triangle with vertices $A(0,0), B(1,0), C(0,1)$.

Individual Tasks 4.7

1-2. Evaluate the line integral, where L is the given curve.

I *1.* $\oint\limits_L y^2dx + (x+y)^2dy$, if L is triangle with vertices $A(3,0)$, $B(3,3)$, $C(0,3)$. *2.* $\oint\limits_L y(1-x^2)dx + (1+y^2)dy$, if L is positively oriented circle $x^2+y^2=4$.
II *1.* $\oint\limits_L y^2dx + (x+y)^2dy$, if L is triangle with vertices $A(2,0)$, $B(2,2)$, $C(0,2)$. *2.* $\oint\limits_L (x^2y - x)dx + (y^2x - 2y)dy$, if L is given by $\begin{cases} x = 3\cos t, \\ y = 2\sin t. \end{cases}$

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