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COUPLING BOUNDARY ELEMENT METHOD WITH LEVEL SET METHOD TO SOLVE INVERSE PROBLEM

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Abstract. *The boundary element method and the level set method can be used in order to solve the inverse problem for electric field. In this approach the adjoint equation arises in each iteration step. Results of the numerical calculations show that the boundary element method can be applied successfully to obtain approximate solution of the adjoint equation. The proposed solution algorithm is initialized by using topological sensitivity analysis. Shape derivatives and material derivatives have been incorporated with the level set method to investigate shape optimization problems. The shape derivative measures the sensitivity of boundary perturbations. The coupled algorithm is a relatively new procedure to overcome this problem. Experimental results have demonstrated the efficiency of the proposed approach to achieve the solution of the inverse problem.*

Keywords: inverse problem, boundary element method, level set method

POŁĄCZENIE METODY ELEMENTÓW BRZEGOWYCH I ZBIORÓW POZIOMICOWYCH W ROZWIĄZYWANIU ZAGADNIENIA ODWROTNEGO

Streszczenie*. Metoda elementów brzegowych i metoda zbiorów poziomicowych mogą być wykorzystane to rozwiązania zagadnienia odwrotnego pola elektrycznego. W takim podejściu równanie sprzężone jest rozwiązywane w każdym kroku iteracyjnym. Wyniki obliczeń numerycznych pokazują, że metoda elementów brzegowych może być zastosowana z powodzeniem do uzyskania przybliżonego rozwiązania równania sprzężonego. Proponowany algorytm jest inicjalizowany za pomocą topologicznej analizy wrażliwościowej. Pochodna kształtu i pochodna materialna zostały połączone z metodą zbiorów poziomicowych w celu zbadania problemów optymalizacji kształtu. Pochodna kształtu mierzy wrażliwość perturbacji brzegowych. Zespolony algorytm jest stosunkowo nową procedurą do rozwiązania tego problemu. Wyniki doświadczenia pokazały skuteczność proponowanego podejścia w rozwiązywaniu zagadnienia odwrotnego.*

Słowa kluczowe: zagadnienie odwrotne, metoda elementów skończonych, tomografia impedancyjna

Introduction

The electrical impedance is non-destructive imaging technique [14, 15], which has various applications. For example, it can be used in medical imaging. In our approach the algorithm of the inverse problem bases on the boundary element method (BEM) [4, 13], the gradient technique and the level set method [1, 2, 5–7, 12, 16, 17]. In the gradient technique so-called adjoint equation has to be solved [3, 8–11]. The solution has to be obtained in each iteration step. Numerical techniques give us opportunity to find approximate solutions of differential equations which cannot be solved by means of analytical ones. Among various numerical tools like the finite element method, the finite difference method or BEM we concentrated our attention on the last one. BEM can be effectively employed on condition that partial differential equation can be transformed to integral form. Additionally, the Green's function has to be calculated. In our numerical calculations for simplicity zero order approximation has been chosen. The function which describes electrical conductivity distribution in our system possesses two different nonzero values. Finally, we have successfully solved the inverse problem in two-dimensional system with 16 electrodes. Therefore, the proposed numerical model has been verified.

1. Boundary Element Method

The field studies might be split on two main branches. In that case are defined: the topology of the structure (interface boundary – outside and/or inside), boundary conditions, material coefficients (e.g. conductivity), the internal source or sources etc. The second case concerns with the inverse problem solution. In this case the unknown parameters are searched when the field distribution is known. Unknown shape of the interface could play the role of the unknown parameters and the inverse problem could be called for example Electrical Impedance Tomography (EIT). Normally we only known the field distribution on the most external boundary of the object.

The partial differential equation in its integral form was defined in order to compute the field distribution for the inhomogeneous regions presented in Figure 1:

$$c(\vec{r}_i)\Phi_1(\vec{r}_i) + \int_{\Gamma_1+\Gamma_2+\Gamma_3} \Phi_1(\vec{r})\frac{\partial G_1(\vec{r},\vec{r}_i)}{\partial n} d\Gamma = \int_{\Gamma_1+\Gamma_2+\Gamma_3} G_1(\vec{r},\vec{r}_i)\frac{\partial \Phi_1(\vec{r})}{\partial n} d\Gamma$$

$$c(\vec{r}_i)\Phi_2(\vec{r}_i) + \int_{\Gamma_2} \Phi_2(\vec{r})\frac{\partial G_2(\vec{r},\vec{r}_i)}{\partial n} d\Gamma = \int_{\Gamma_2} G_2(\vec{r},\vec{r}_i)\frac{\partial \Phi_2(\vec{r})}{\partial n} d\Gamma$$

$$c(\vec{r}_i)\Phi_3(\vec{r}_i) + \int_{\Gamma_3} \Phi_3(\vec{r})\frac{\partial G_3(\vec{r},\vec{r}_i)}{\partial n} d\Gamma = \int_{\Gamma_3} G_3(\vec{r},\vec{r}_i)\frac{\partial \Phi_3(\vec{r})}{\partial n} d\Gamma \quad (1)$$

where the value of c coefficient is defined by the location of the point indicated by the position vector $\vec{r}_i$:

$$c(\vec{r}_i) = \begin{cases} 1, & \vec{r}_i \in \Omega \\ \frac{1}{2}, & \vec{r}_i \in \partial\Omega \\ 0, & \vec{r}_i \notin \Omega \end{cases} \quad (2)$$

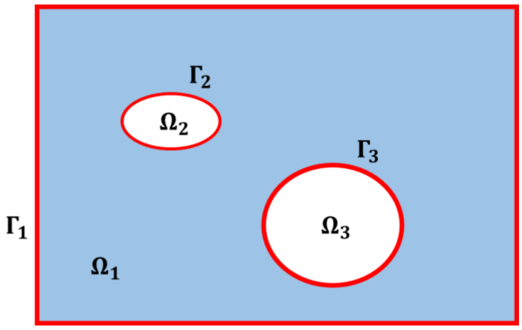

Fig. 1. The example of the structure: region Ω_1 limited by border Γ_1 with two internal objects: Ω_2 limited by border Γ_2 and Ω_3 limited by border Γ_3

The objective is to find interface (inside boundary) using voltage measurements on the periphery of the region. The boundary Γ_1 is known, but the starting shape of interface in request has been chosen randomly (at the beginning of the iteration process). For so called "current electrodes" (nodes of the boundary elements) the Dirichlet boundary conditions are imposed (emulate the voltage source), for the rest (the "voltage electrodes") the homogeneous Neumann boundary condition $\frac{\partial \Phi}{\partial n} = 0$ are imposed.

The state function (electric potential) and its normal derivative on the particular boundary $\Gamma_1, \Gamma_2, \Gamma_3$ of each substructures $\Omega_1, \Omega_2, \Omega_3$ are denoted as:

$$\mathbf{\Phi}_1(\Gamma_1) = \mathbf{\Phi}_1^1,\ \mathbf{\Phi}_1(\Gamma_2) = \mathbf{\Phi}_2^1, \mathbf{\Phi}_1(\Gamma_3) = \mathbf{\Phi}_3^1,$$
$$\mathbf{\Phi}_2(\Gamma_2) = \mathbf{\Phi}_2^2, \mathbf{\Phi}_3(\Gamma_3) = \mathbf{\Phi}_3^3,$$
$$\frac{\partial \mathbf{\Phi}_1(\Gamma_1)}{\partial n} = \left(\frac{\partial \mathbf{\Phi}}{\partial n}\right)_1^1, \frac{\partial \mathbf{\Phi}_1(\Gamma_2)}{\partial n} = \left(\frac{\partial \mathbf{\Phi}}{\partial n}\right)_2^1,$$
$$\frac{\partial \mathbf{\Phi}_1(\Gamma_3)}{\partial n} = \left(\frac{\partial \mathbf{\Phi}}{\partial n}\right)_3^1, \frac{\partial \mathbf{\Phi}_2(\Gamma_2)}{\partial n} = \left(\frac{\partial \mathbf{\Phi}}{\partial n}\right)_2^2,$$
$$\frac{\partial \mathbf{\Phi}_3(\Gamma_3)}{\partial n} = \left(\frac{\partial \mathbf{\Phi}}{\partial n}\right)_3^3 \quad (3)$$

Substitute:

$$A_{ij}^* = \int_{\Gamma_j} \frac{\partial G(\vec{r_i}, \vec{r})}{\partial n} d\Gamma \quad (4)$$

where $A_{ij} = \begin{cases} A_{ij}^*, & i \neq j \\ A_{ij}^* + c(\vec{r_i}), & i = j \end{cases}$

and

$$B_{ij} = \int_{\Gamma_j} G(\vec{r_i}, \vec{r})\, d\Gamma \quad (5)$$

we will rewrite eq. (1) in a matrix form. The same manner was used to mark the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ as the state function and its normal derivative (see equation (3)):

$$\mathbf{A}^1(\Gamma_1) = \mathbf{A}_1^1, \mathbf{A}^1(\Gamma_2) = \mathbf{A}_2^1, \mathbf{A}^1(\Gamma_3) = \mathbf{A}_3^1,$$
$$\mathbf{A}^2(\Gamma_2) = \mathbf{A}_2^2, \mathbf{A}^3(\Gamma_3) = \mathbf{A}_3^3,$$
$$\mathbf{B}^1(\Gamma_1) = \mathbf{B}_1^1, \mathbf{B}^1(\Gamma_2) = \mathbf{B}_2^1,$$
$$\mathbf{B}^1(\Gamma_3) = \mathbf{B}_3^1, \mathbf{B}^2(\Gamma_2) = \mathbf{B}_2^2,$$
$$\mathbf{B}^3(\Gamma_3) = \mathbf{B}_3^3 \quad (6)$$

The voltage on the internal boundary (interface) Γ_2 or Γ_3 fulfill the continuous conditions:

$$\mathbf{\Phi}_1(\Gamma_2) = \mathbf{\Phi}_2(\Gamma_2)$$
$$\mathbf{\Phi}_1(\Gamma_3) = \mathbf{\Phi}_3(\Gamma_3) \quad (7)$$

If the conductivity Ω_1 is equal γ_1, Ω_2 is equal γ_2 and conductivity of Ω_3 is γ_3 (as it was previously noted: the conductivity in objects are constant) then:

$$\gamma_1 \frac{\mathbf{\Phi}_1(\Gamma_2)}{\partial n} = -\gamma_2 \frac{\mathbf{\Phi}_2(\Gamma_2)}{\partial n}$$
$$\gamma_1 \frac{\mathbf{\Phi}_1(\Gamma_3)}{\partial n} = -\gamma_3 \frac{\mathbf{\Phi}_3(\Gamma_3)}{\partial n} \quad (8)$$

where: minus means the opposite direction of the normal unit vector to the border of $\Omega_1 - \Omega_2$ and $\Omega_1 - \Omega_3$.

Rewrite equations for inhomogeneous regions in the matrix form we have got:

$$\begin{bmatrix} \mathbf{A}_2^1 & -\mathbf{B}_2^1 & \mathbf{A}_3^1 & -\mathbf{B}_3^1 \\ \mathbf{A}_2^2 & -\left(-\frac{\gamma_1}{\gamma_2}\right)\mathbf{B}_2^2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_3^3 & -\left(-\frac{\gamma_1}{\gamma_3}\right)\mathbf{B}_3^3 \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_2^1 \\ \left(\frac{\partial \mathbf{\Phi}}{\partial n}\right)_2^1 \\ \mathbf{\Phi}_3^1 \\ \left(\frac{\partial \mathbf{\Phi}}{\partial n}\right)_3^1 \end{bmatrix} = \begin{bmatrix} -\mathbf{A}_1^1\mathbf{\Phi}_1^1 + \mathbf{B}_1^1\left(\frac{\partial \mathbf{\Phi}}{\partial n}\right)_1^1 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad (9)$$

where the following values $\mathbf{\Phi}_2^1, \left(\frac{\partial \mathbf{\Phi}}{\partial n}\right)_2^1, \mathbf{\Phi}_3^1, \left(\frac{\partial \mathbf{\Phi}}{\partial n}\right)_3^1$ are unknown. It is needed to solve the equation (1) for all projection angles.

2. Inverse Problem

To solve the inverse problem with the aid of the level set method there was needed the adjoint equation. Let us consider the following partial differential equation in two-dimensional Cartesian coordinate system:

$$\nabla \cdot [\sigma(\vec{r})\nabla\lambda(\vec{r})] = b(\vec{r}) \quad (10)$$

where $\vec{r}\epsilon\Omega$. We assumed that $\sigma(\vec{r})$ denotes electrical conductivity distribution. The source term $b(\vec{r})$ is defined only on the boundary of the domain Ω. It depends on differences between voltages obtained from measurements and numerical simulations. The source term has to be calculated on each iteration step. The formula for $b(\vec{r})$ is given in [13]. Additionally, we assumed that the adjoint function λ or its normal derivative β is known for all boundary points. The differential problem defined in described manner may be regarded as the boundary value problem for the adjoint equation (10).

Starting point for our research is typical for BEM integral equation [13], where the boundary curve is divided into N elements:

$$c(\vec{r_i})\lambda(\vec{r_i}) + \sum_{j=1}^{N} \int_{\Gamma_j} \lambda(\vec{r})\, h(\vec{r}, \vec{r_i}) d\gamma_j + \int_{\Omega} b(\vec{r}) g(\vec{r}, \vec{r_i}) d\omega =$$
$$= \sum_{j=1}^{N} \int_{\Gamma_j} \beta(\vec{r_i})\, g(\vec{r}, \vec{r_i}) d\gamma_j \quad (11)$$

Equation (11) is valid if the electrical conductivity is constant in the whole domain. In the case of constant boundary elements only three values of the function c are possible. If a given point belongs to boundary of the domain Ω, then the value equals 0.5. The value of function c equals 1, when a given point lies inside of Ω and equals 0 in other cases. The Green's function g may be obtained by solving the fundamental equation and is given by:

$$g(\vec{r}, \vec{r_i}) = \frac{1}{2\pi} ln\left(\frac{A}{|\vec{r} - \vec{r_i}|}\right) \quad (12)$$

In above formula A is a positive constant. Function h represents the derivative of the Green's function in normal direction appointed by unit vector $\vec{n}(\vec{r})$. After calculations we get:

$$h(\vec{r}, \vec{r_i}) = -\frac{1}{2\pi} \frac{\vec{n}(\vec{r}) \cdot (\vec{r} - \vec{r_i})}{|\vec{r} - \vec{r_i}|^2} \quad (13)$$

The vector formula for j-th constant boundary element is given by:

$$\vec{r}_j(\xi) = \vec{r}_{(m)j} + 0{,}5\left(\vec{r}_{(l)j} - \vec{r}_{(f)j}\right)\xi \quad (14)$$

where $\xi\epsilon\langle -1, 1\rangle$. Position vectors $\vec{r}_{(f)j}$, $\vec{r}_{(l)j}$ and $\vec{r}_{(m)j}$ represent the first vertex, the last vertex and the middle point (node) of j-th boundary element, respectively. We use constant boundary elements, therefore:

$$c(\vec{r_i})\lambda(\vec{r_i}) + \sum_{j=1}^{N} \lambda(\vec{r}) \int_{\Gamma_j} h(\vec{r}, \vec{r_i}) d\gamma_j + \int_{\Omega} b(\vec{r}) g(\vec{r}, \vec{r_i}) d\omega =$$
$$= \sum_{j=1}^{N} \beta(\vec{r_i}) \int_{\Gamma_j} g(\vec{r}, \vec{r_i}) d\gamma_j \quad (15)$$

Formulas (12), (13), and (15) can be utilised to solve the adjoint equation (10) because the domain Ω can be decomposed into two subdomains where electrical conductivity is constant for each one.

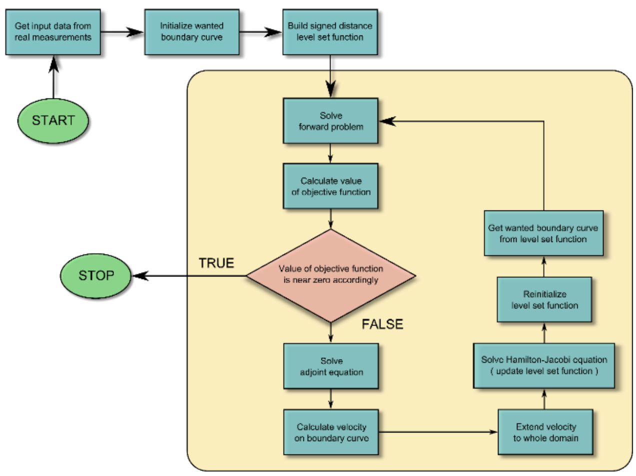

Fig. 2. The scheme of the algorithm to minimize the objective function

The following algorithm is used to find out the shape and the position of the unknown internal obstacles (Fig. 2):

a) make discretization of the outer boundary $\mathbf{\Gamma}_1$ (for example on 32 elements),
b) collect measurements from the electrodes for all projection angles,
c) conduct simulation for randomly placed objects inside,
d) use the Laplace equation and compute electric potential distribution with the aid of BEM,
e) determine difference between the "measured" $\boldsymbol{u}_0$ and computed potential $\mathbf{\Phi}$ and use them for the adjoint equation,
f) update the Hamilton-Jacobi equation,
g) check if it's necessary to reinitialize set level function,
h) get zero level contour of updated φ function and select discretization points,
i) calculate the objective function according to the equation

$$F_c = \frac{1}{2}\sum_{j=1}^{p} F_j = \frac{1}{2}\sum_{j=1}^{p}\left\{\left(\boldsymbol{\Phi}^1{}_j - \boldsymbol{u}_{0j}\right)^T\left(\boldsymbol{\Phi}^1{}_j - \boldsymbol{u}_{0j}\right)\right\} \quad (16)$$

where: $\boldsymbol{\Phi}$ – is computed electric potential distribution; $\boldsymbol{u}_0$ – is measured values for real object, if the objective function is lower than assumed threshold, then "stop", else go to the point (d).

3. Results

Figure 3 shows the solution of the inverse problem. All points in both figures represent vertices of boundary elements. Our algorithm needed about 50 iterations to minimalize the objective function (see Fig. 4). Obtained result is appropriate. Proposed numerical model has been successfully verified. We can make use of BEM in algorithm which solves the inverse problem in EIT.

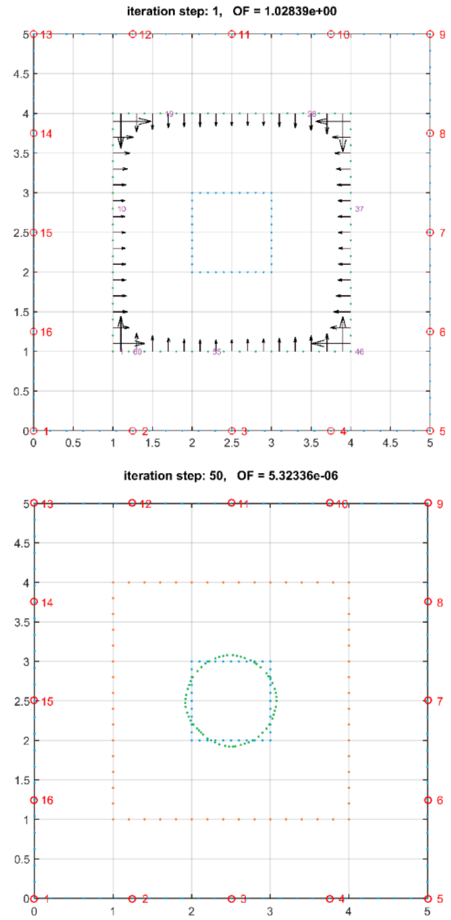

Fig. 3. The solution of the inverse problem (green points). internal square indicated by blue points represents proper position of the interface, orange points represent initial position of the interface, symbol OF denotes the value of the objective function

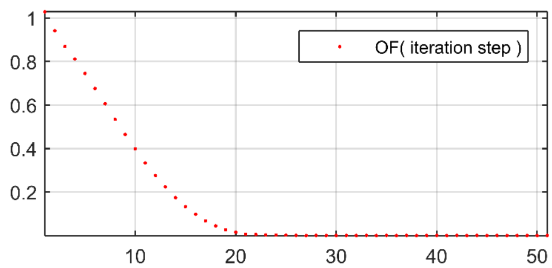

Fig. 4. The objective function versus the number of iteration step

Figure 5 presents the image reconstruction with the one object. Figure 6a shows two obstacles marked by the blue dashed line. Such a test example will be considered as an EIT problem. The phantom object with unknown topology inside was reconstructed. In such case the level set function with four zero level objects will be applied as most appropriate one. For each simulated object the velocity would be calculated under assumption that the conductivities of all objects are known. Unknown structures are marked by the blue line; simulated objects are marked by the red line. For last 300 iterations step the unknown structure was found. The result is presented in Figure 6d. The objective function distribution versus the number of iteration steps is shown in Figure 7.

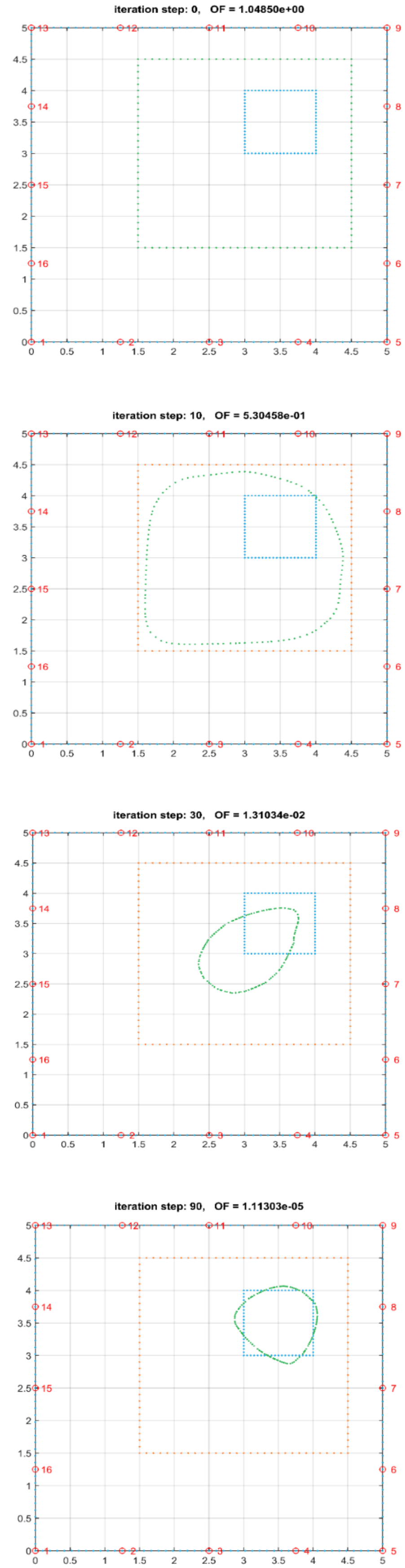

Fig. 5. The image reconstruction of the one object (BEM-LSM)

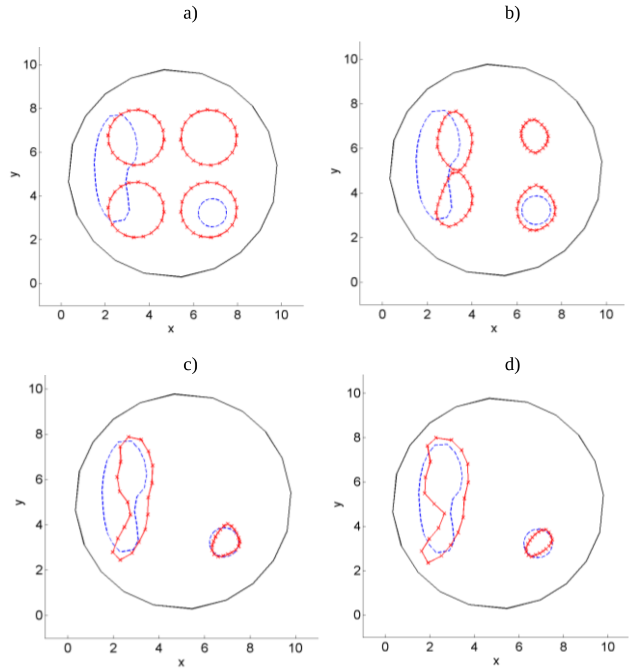

Fig. 6. The black line marks the outside border of the structure with the internal unknown objects (marked by the blue dashed line). The red line marks zero level contours, a) starting point, b) after 98 iterations, one step before merging two objects, c) after 225 iterations with the lowest value of the cost function equal to=0.96; d) after 300 iteration steps when the objective functions again increased

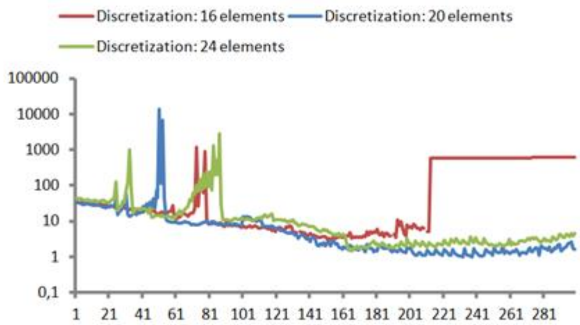

Fig. 7. The objective function distribution versus the number of iteration steps

4. Conclusion

The inverse problem was solved using the combination of the level set function with the boundary element. The level set method and BEM show a way how to compute the interface by updating the Hamilton-Jacobi equation. This is particularly easy when the BEM is applied because the normal derivatives of the state function and adjoint function are the primary values directly achieved after the simulation process. In many cases occurs that the reinitialization is necessary to fix the correct shape of the level set function and also the signed distance function. Experimental results confirmed that presented method is efficient and the only one which is able to change the topology during the iteration process.

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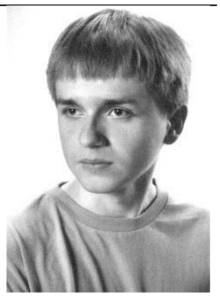

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