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Oscillation of *n-th* Order Neutral Delay Differential Equations

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Oscillation of *n-th* Order Neutral Delay Differential Equations

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Declaration

I certify that this thesis submitted for the degree of Master is the result of my own research, except where otherwise acknowledged, and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution

Signed:.....................................

Ali Ibrahim Ali Zein

Date: 6/6/2005

Dedication

To my parents, my brother, my sister, my son Mohammad, and to the mother of Mohammad.

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Abstract

This thesis is concerned with the oscillation of solutions of a class of *n-th* order nonlinear neutral delay differential equations. The general form of this class of equations contains two delayed arguments.

The thesis presents main concepts and basic definitions of neutral differential equations and oscillation. Also, it presents a practical model for the applications of neutral delay differential equations in distributed networks containing loss less transmission lines.

The thesis contains several recent results in the oscillation theory of that class of *n-th* order nonlinear neutral delay differential equations. It also contains our own results in the subject. And several examples are given to illustrate the main theorems in the thesis.

Our own results involve some improvements and modifications to previous results, besides our new criteria for oscillation of bounded solutions of that class of *n-th* order neutral delay differential equations with oscillating coefficients.

الملخص

تهتم هذه الأطروحة بخاصية التذبذب لحلول فئة معينة من المعادلات التفاضلية الاقترانية المتعادلة غير الخطية من الدرجة النونية.و الشكل العام لهذه الفئة من المعادلات يحتوي على اشتمال الاقتران المجهول على متغيرين متأخرين عن المتغير الذي يمثل الوضع الحالي.

تحتوي هذه الأطروحة على المفاهيم الأساسية للمعادلات التفاضلية الاقترانية المتعادلة و لمفاهيم التذبذب، و كذلك تقدم الأطروحة مثال عملي على تطبيقات المعادلات التفاضلية الاقترانية المتعادلة في الشبكات التي تحتوي على خطوط نقل غير فاقدة للطاقة

و تحتوي الأطروحة على العديد من النتائج الصادرة حديثا في نظرية التذبذب لتلك الفئة من المعادلات التفاضلية الاقترانية المتعادلة غير الخطية من الدرجة النونية التي تتناولها الأطروحة.
و تحتوي الأطروحة كذلك على النتائج الخاصة بنا التي حصلنا عليها في الموضوع. هذا بالإضافة إلى أمثلة متعددة توضح النظريات الرئيسية في الأطروحة.

النتائج الخاصة التي حصلنا عليها في الموضوع كانت عبارة عن تطوير و تحسين لنتائج سابقة بالإضافة لطريقة جديدة للكشف عن التذبذب للحلول المحدودة للفئة من المعادلات المتعادلة من الدرجة النونية التي تتناولها الأطروحة، عندما يكون المعامل الذي تحتوي عليه المعادلة اقتران متذبذب.

Contents

Introduction

Differential equations with deviating arguments (DEWDA) are among the most important equations in applied mathematics. This importance occurs because they provide mathematical models for many real-life systems, in which the rate of change of the system depends not only on its present state but also on one or more past, or future states.

DEWDA, initially introduced in the eighteenth century by Laplace and Condorcet [8]. Bernoulli (1728) while studying the problem of sound vibrating in a tube with finite size, investigated the properties of solutions of the first order of DEWDA, and was the first to work in this area [29]. But the systematic study of such type of differential equations has begun in the twentieth century, in connection with the needs of the applied science and technology [15].

In the late thirties and early forties Minorsky in his study of ship stabilization and automatic steering pointed out very clearly the importance of the consideration of the delay in the feedback mechanism [25]. The great interest in the theory of automatic control and dynamics systems, during these and later years, has certainly contributed significantly to the rapid development of the theory of delay differential equations [8,15,25,].

Myshkis in his book (1950) introduced a general class of equations with delayed arguments [14, 25, 26]. In 1958 G. A. Kamenskii [29] proposed a classification method for a general class of DEWDA, he classified such type of equations into three types, they are: retarded type, neutral type, and advanced type.

Later many specialized books appeared in the subject, such as: El’sgoltz and Norkin (1963), Bellman and Cooke (1963), El’sgoltz (1964)[15], Myskis (1972), Driver (1977), and Hale (1977)[25], (see [24] page 1).

Oscillatory behavior of solutions of DEWDA is one of the most important properties of such type of equations, besides existence of positive solutions, and asymptotic behavior of solutions. This importance comes from the viewpoint of applications. Where these properties provide a qualitative description of solutions of DEWDA.

Since 1950 the oscillation theory of DEWDA has received the attention of several mathematicians as well as other scientists around the world. However, the theory of oscillation of DEWDA has been extensively developed in the last 30 years.

In 1987 Ladde, Lakshmikantham, and Zhang, in their nice book [29], introduced the first systematic treatment of oscillation and non-oscillation theory of DEWDA. In 1991 Gyori, and Ladas introduced one of the most important books in the oscillation theory of DEWDA [24]. The last book is also a good reference for the theory of DEWDA, and it contains several applications. Recently several books appeared that are specialized in the subject of oscillation, such as Bainov and Mishev (1991), and Agarwal (2000). (See references of [5, 31, 37])

In parallel, during the second half of the twentieth century the area of applications of DEWDA has greatly expanded. And now such equations find numerous applications in physics, control theory, power systems engineering, material science, robotics, neural networks, ecology, physiology, immunology, public health, and economics [2, 3, 24, 25, 34, 37].

The simplest type of past dependence in a differential equation is that in which the past dependence is through the state variable and not the derivative of the state variable, the so-called retarded functional differential equations or delay differential equations [25].

When the delayed argument occurs in the derivative of the state variable as well as in the independent variable, the so-called neutral differential equations [25].

Although the oscillatory theory of non-neutral differential equations has been extensively developed during the last three decades, only in the last 10 or 15 years much effort has been devoted to the study of oscillatory behavior of neutral delay differential equations (NDDE). The study of oscillatory behavior of solutions of NDDE, as other types of DEWDA, besides its theoretical interest, is important from the viewpoint of applications. Where NDDE have many applications in natural science, technology, and economics. As examples, NDDE appear in the following problems:

1) Study of vibrating masses attached to an elastic bar [24, 25].
2) Study of distributed networks containing loss less transmission lines [24, 25]. More details are involved in section (1.7) of the research.
3) Problems of economics where the demand depends on current price but supply depends on the price at an earlier time [37].
4) To describe the Flip-Flop circuit, which is the basic element in a digital electronics [34].

However, in the last few years, there has been a growing interest in oscillation of *n-th* order NDDE. Among numerous papers dealing with the subject we refer in particular to [5, 7, 11, 12, 13, 19,20, 28, 31, 32, 35, 39,40].

In fact, the appearance of neutral term in differential equations can cause or destroy oscillation of its solutions. Moreover, in general the theory of neutral differential equations presents complications, which are unfamiliar for non-neutral differential equations. Most of authors obtained sufficient, rather than necessary, conditions for oscillation of higher orders NDDE. However, the conditions assumed differ from authors to authors due to the different techniques they use and different forms of equations they consider. Also, it is interesting to note that the conditions assumed by different authors for similar form of equations are often not comparable [32].

In our research we study the oscillation of a certain class of *n-th* order NDDE, we consider the nonlinear *n-th* order NDDE of the form:

$$[x(t)+p(t)x(\tau(t))]^{(n)}+\delta f(t,x(t),x(\sigma(t)))=h(t) \tag{1}$$

where, $\delta=\pm 1$, $p(t),h(t),\tau(t),\sigma(t)\in C([t_0,\infty),R)$, $t_0\geq 0$, $\tau(t)\leq t$, $\sigma(t)\leq t$, $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\sigma(t)=\infty$, and $f(t,x,y)$ is continuous on $[0,\infty)\times R\times R$.

During the last decade an extensive amount of study has been devoted to obtain sufficient conditions for oscillation of solutions of equation (1), in particular we refer the reader to [11, 12, 18, 19, 20, 31, 39, 40]. For very recent papers we refer to [5, 28, 37]. The conditions assumed by authors differ from authors to authors. This is due to different restrictions assumed by authors on parameters of equation (1), and due to different techniques used by authors.

This research consists of five chapters:

Chapter one: contains the main concepts, definitions, and preliminary material that are essential for the rest of the research. Also it contains a practical example from the electrical engineering on the application of NDDE.

Chapter two: is devoted to the oscillation theory of equation (1), when the function $f(t,x(t),x(\sigma(t)))$ is separable, and depends on t and $x(\sigma(t))$

i.e. when $f(t,x(t),x(\sigma(t))) = q(t)f(x(\sigma(t)))$, where $q(t) \in C([t_0,\infty),[0,\infty))$, and $f \in C(R,R)$.

Chapter three: is devoted to the bounded oscillation of equation (1), when $f(t,x(t),x(\sigma(t))) = q(t)f(x(\sigma(t)))$, where $q(t) \in C([t_0,\infty),[0,\infty))$, and $f \in C(R,R)$

Chapter four: is devoted to the oscillatory behavior of equation (1) when the function $f(t,x(t),x(\sigma(t)))$ depends on t and $x(\sigma(t))$. i.e. when $f(t,x(t),x(\sigma(t))) = f(t,x(\sigma(t)))$.

In Chapter five: we study oscillation of equation (1), with great attention to Zafer's results [39]. Also in this chapter, we introduce our results. Where we improve some results of Zafer [39] for oscillation of equation (1) when $\sigma(t) = t - \sigma$, $\sigma > 0$. Also we establish sufficient conditions for bounded oscillation of equation (1) when the coefficient $p(t)$ is an oscillating function with property $\lim_{t\to\infty} p(t) = 0$.

Prologue to the reader

The main results in this research are variously labeled theorem, lemma, corollary, and remark.

Theorem: contains main results on the oscillation theory.

Lemma: contains helpful results, that are needed or utilizes the proofs of theorems

Corollary: contains consequently results from theorems

Remark: contains notes, or refers to particular cases.

We refer to each one of theorems, lemmas, corollaries, remarks, definitions and examples by triple (A.B.C). Where

A: refers to the chapter number

B: refers to the section number

C: refers to the number of theorem, lemma, corollary, remark, definition, or example. Each category of theorems, lemmas, corollaries, remarks, definitions, or examples has its own sequence of numbering in each section.

Also we refer to most of equations and inequalities by triple (A.B.C). Where (A, and B) as above, and (C) refers to the number equation or inequality. Equations and inequalities have one sequence of numbering in each section.

We refer to equations and inequalities in appendices by (N.M), where

N: refers to the appendix number

M: refers to the number of equation or inequality.

Each proof in the research begins with the word '**Proof:**' and ends with the symbol '□'.

Throughout the research we let $R_+ = [0, \infty)$.

Throughout the research we call $[x(t) + p(t)x(\tau(t))]^{(n)}$ the derivative part of equation (1). And $\delta f(t, x(t), x(\sigma(t)))$ the non-derivative part of equation (1).

Chapter One

Preliminaries

1.0 Introduction

This chapter contains some basic definitions, and results which are essential for the rest of the research. Sections 1.1 and 1.2, introduce the definition of DEWDA, their classification, and definition of NDDE. Section 1.3, gives what is the meaning of solution of NDDE. Sections 1.4 and 1.5, introduce the definition of oscillation and some oscillatory phenomena caused by deviating arguments. Section 1.6, contains basic lemmas related to the subject. Finally in section 1.7 we give a practical example, in details, on the applications of NDDE.

1.1 Differential equations with deviating arguments (DEWDA)

Differential equations with deviating arguments are differential equations in which the unknown function appears with various values of the argument. They are classified into three types; these types are enumerated in the following discussion:

i. Differential equations with retarded argument:

Differential equation with retarded argument is a differential equation with deviating argument in which the highest-order derivative of the unknown function appears for just one value of the argument, and this argument is not less than the remaining arguments of the unknown function and its derivatives appearing in the equation.

ii. Differential equations with advanced argument:

Differential equation with advanced argument is a differential equation with deviating argument in which the highest-order derivative of the unknown function appears for just one value of the argument, and this argument is not larger than the remaining arguments of the unknown function and its derivatives appearing in the equation.

iii. *Differential equations of neutral type*:

Neutral differential equation is a differential equation in which the highest-order derivative of the unknown function is evaluated both with the present state, and at one or more past or future states.

Example 1.1.1:

1) $x'(t) = f(t, x(t), x(t-\tau(t)))$

2) $x''(t) = f(t, x(t), x'(t), x(t-\tau(t)), x'(t-\tau(t)))$

3) $x''(t) = f(t, x(\frac{t}{2}), x'(\frac{t}{2}), x(t), x'(t))$

4) $x''(t) = f(t, x(t), x'(t), x(\tau(t)), x'(\tau(t)))$

5) $x'(t) = f(t, x(t), x(t-\tau), x'(t-\tau))$

6) $x''(t) = f(t, x(t), x'(t), x(t-\tau(t)), x''(t-\tau(t)))$

Equations (1), (2), (3), and (4) are equations with retarded argument if $\tau(t) \geq 0$ in (1) and (2), $t \geq 0$ in (3), and $\tau(t) \leq t$ in (4).

Equations (1), (2), (3), and (4) are equations with advanced argument if $\tau(t) \leq 0$ in (1) and (2), $t \leq 0$ in (3), and $\tau(t) \geq t$ in (4).

Equations (5) and (6) are equations of neutral type.

1.2 Neutral delay differential equations (NDDE)

A neutral delay differential equation is a differential equation in which the highest-order derivative of unknown function appears in the equation both with and without delays (retarded arguments).

Example 1.2.1:

1) $x'(t) = f(t, x(t), x(t-\tau), x'(t-\tau))$, $\tau > 0$, is a first order NDDE.

2) $x''(t) = f(t, x(t), x'(t), x(t-\tau(t)), x'(t-\tau(t)), x''(t-\tau(t)))$, $\tau(t) > 0$, is second order NDDE

Example 1.2.2:

1) $[x(t) + p(t)x(t-\tau)]'' + q(t)x(t-\sigma) = 0$,

Where $p(t), q(t) \in C([0,\infty), R)$, and $\tau, \sigma \in [0,\infty)$. It is a second order NDDE.

2) $[x(t) + p(t)x(\tau(t))]^{(n)} + f(t, x(t), x(\sigma(t))) = h(t)$

Where, $p(t), h(t) \in C([0,\infty), R)$, $\tau(t), \sigma(t) \in C([0,\infty), R_+)$, $\tau(t) < t$, $\sigma(t) < t$.

It is an *n-th* order NDDE.

In general, the behavior of solutions of neutral type equations may be quite different than that of non neutral-equations, and results, which are true for non-neutral equations, may not be true for neutral equations. For example Snow (1965) has shown that even though the characteristic roots of a neutral differential equation may all have negative real parts, it is still possible for some solutions to be unbounded [23, 24].

1.3 Solution of NDDE

Consider *n-th* order NDDE of the form:

$$[x(t)+p(t)x(\tau(t))]^{(n)}+\delta f(t,x(t),x(\sigma(t)))=s(t) \qquad (1.3.1)$$

where, $\delta=\pm 1$, $p(t),s(t)\in C([t_0,\infty),R)$, $\tau(t),\sigma(t)\in C([t_0,\infty),R_+)$ $t_0\geq 0$, $\tau(t)<t$, and $\sigma(t)<t$.

By solution of equation (1.3.1) we mean a real-valued continuous function x on $[t_x,\infty)$ for some $t_x\geq t_0$, such that $x(t)+p(t)x(\tau(t))$ is n-times continuously differentiable and (1.3.1) is satisfied for $t\in[t_x,\infty)$.

In this research we will consider only such solution that satisfies $\sup\{|x(t)|:t\geq T\}>0$, for any $T\geq t_x$. In other words, $|x(t)|\not\equiv 0$ on any infinite interval $[T,\infty)$. Such a solution sometimes is said to be a regular solution.

1.4 Definition of oscillation

There are various definitions for the oscillation of solutions of ordinary differential equations (with or without deviating arguments). In this section we give two different forms of definitions of the oscillation. These forms are most frequently used in literature.

Definition 1.4.1: A non-trivial solution $x(t)$ is said to be oscillatory if it has arbitrarily large zeros for $t\geq t_0$, that is there exists a sequence of zeros $\{t_n\}$ ($x(t_n)=0$), of $x(t)$ such that $\lim_{n\to\infty} t_n=+\infty$, otherwise $x(t)$ is said to be non-oscillatory.

For non-oscillatory solutions there exists a t_1 such that $x(t) \neq 0$, for all $t \geq t_1$.

Definition 1.4.2: A nontrivial solution $x(t)$ is said to be oscillatory if it changes sign on (T, ∞), where T is any number.

As the solution $x(t)$ is continuous, if it is non-oscillatory it must be eventually positive or eventually negative. That is there exists a $T_0 \in R$ such that $x(t)$ is positive for all $t \geq T_0$ or is negative for all $t \geq T_0$.

Example 1.4.1: The equation

$$x'(t) + x(t - \frac{\pi}{2}) = 0 \tag{1.4.1}$$

has oscillatory solutions $x_1(t) = \sin t$, and $x_2(t) = \cos t$.

Example 1.4.2: The equation

$$x'(t) - e^3 x(t-3) = 0 \tag{1.4.2}$$

has a non-oscillatory solution $x(t) = e^t$.

Example 1.4.3: The equation

$$x''(t) + 4x(\frac{\pi}{2} - t) = 0 \tag{1.4.3}$$

has an oscillatory solution $x_1(t) = \sin 2t$, and a non-oscillatory solution $x_2(t) = e^{2t} - e^{\pi - 2t}$.

Example 1.4.4: Consider the equation

$$x'''(t) - \frac{1}{2}x(t - \frac{\pi}{2}) + \frac{1}{2}x(t - \frac{3\pi}{2}) = 0, \quad t \geq 0 \tag{1.4.4}$$

whose solution $x(t) = 1 + \cos t$, it is oscillatory according to Definition1.4.1, and non-oscillatory according to Definition1.4.2. In fact, Definition1.4.1 is more general than Definition1.4.2, and is the most used in literature. Also, it is the one used in this research..

Example 1.4.5: Consider the equation

$$x''(t) - \frac{1}{t}x'(t) + 4t^2 x(t) = 0 \tag{1.4.5}$$

whose solution $x(t) = \sin t^2$, this solution is not periodic but has an oscillatory property.

Example 1.4.6: Consider the NDDE

$$\left[x(t) - \frac{1}{2}x(t-2\pi)\right]' - \frac{1}{2}x(t - \frac{3\pi}{2}) = 0 \tag{1.4.6}$$

It has an oscillatory solution $x(t) = \sin t$.

Example 1.4.7: Consider the NDDE

$$\left[x(t) - e^t x(t-1)\right]''' + \frac{e^{2t}}{e^3} x^3(t-1) = 0 \tag{1.4.7}$$

It has a non-oscillatory solution $x(t) = e^{-t}$, but $x(t) \to 0$ as $t \to \infty$.

1.5 Effects of deviating arguments on oscillation

The oscillation theory of DEWDA presents some new problems, which are not presented in the theory of corresponding ordinary differential equations (ODE). And the known results for oscillation of differential equations may not be true for DEWDA.

In this section we consider some oscillatory and non-oscillatory phenomena caused by deviating arguments, through the discussion of the following examples.

Example 1.5.1: Consider the equation

$$x'(t) + \beta x(t - \frac{\pi}{2\beta}) = 0, \quad \beta \in R \setminus \{0\} \tag{1.5.1}$$

It has oscillatory solutions $x_1(t) = \sin \beta t$, and $x_2(t) = \cos \beta t$.

While the equation

$$x'(t) + \beta x(t) = 0, \quad \beta \in R \setminus \{0\} \tag{1.5.2}$$

has non-oscillatory solution $x(t) = e^{-\beta t}$.

This example shows that first order DEWDA can have oscillatory solution. While, as known, the first order scalar ODE do not possess oscillatory solution.

Example 1.5.2: Consider the equation

$$x''(t) - 9x(t - \frac{\pi}{3}) = 0 \tag{1.5.4}$$

It has oscillatory solution $x_1(t) = \sin 3t$, and $x_2(t) = \cos 3t$. But the equation

$$x''(t) - 9x(t) = 0 \tag{1.5.5}$$

has non-oscillatory solutions $x_1(t) = e^{3t}$, and $x_2(t) = e^{-3t}$

It is obvious that the nature of solution changes completely after the appearance of deviating argument in the equation.

Example 1.5.3: Consider the equation

$$x''(t) + x(2\pi - t) = 0 \tag{1.5.6}$$

It has an oscillatory solution $x_1(t)=\cos t$, and non-oscillatory solution $x_2(t)=e^t-e^{2\pi-t}$.

Here, in second order DEWDA one solution is oscillatory, but the other is non-oscillatory. And this case can never occur in second order linear ODE, where either all solutions are oscillatory or all solutions are non-oscillatory.

1.6 Some basic lemmas

This section contains basic lemmas that are needed later in the research. First two lemmas are due to Kiguradze.

Lemma 1.6.1: ([29, 40]).

Let $y\in C^n([0,\infty),R)$ be of constant sign, let $y^{(n)}(t)$ be of constant sign and not identically equal to zero in any interval $[t_0,\infty)$, $t_0\geq 0$, and $y(t)y^{(n)}(t)\leq 0$. Then:

i. there exists a $t_1\geq t_0$ such that $y^{(k)}(t)$, $k=1,...,n-1$, is of constant sign on $[t_1,\infty)$,

ii. there exists an integer l, $0\leq l\leq n-1$, which is even if n is odd and is odd if n is even (i.e. $n-l$ is odd) , such that:

$$y(t)y^{(k)}(t)>0,\ k=0,1,...,l,\ \ t\geq t_1 \tag{1.6.1}$$

$$(-1)^{n+k-1}y(t)y^{(k)}(t)>0,\ k=l+1,...,n-1,\ \ t\geq t_1 \text{ and} \tag{1.6.2}$$

iii. $|y(t)| \geq \dfrac{(t-t_1)^{n-1}}{(n-1)\ldots(n-l)}\left|y^{(n-1)}(2^{n-l-1}t)\right|,$ $\quad t \geq t_1$

(1.6.3)

Depending on this lemma we have Definition 1.6.1, which is used later.

Definition 1.6.1: The function $y(t)$ satisfying (1.6.1), (1.6.2), and (1.6.3) is said to be a function of degree l.

Lemma 1.6.2: ([29, 40])

Assume that the function y together with its derivatives of order up to $n-1$ is absolutely continuous and of constant sign on the interval (t_0, ∞). Moreover,

$$y^{(n)}(t)y(t) \geq 0$$

Then either

$$y^{(k)}(t)y(t) \geq 0 \quad k = 0,1,\ldots,n-1 \tag{1.6.4}$$

Or one can find a number l, $0 \leq l \leq n-2$, which is even when *n* is even and odd when *n* is odd (i.e. $n-l$ is even), such that:

$$y(t)y^{(k)}(t) > 0,\ k = 0,1,\ldots,l \tag{1.6.5}$$

$$(-1)^{n+k}y(t)y^{(k)}(t) > 0,\ k = l+1,\ldots,n-1 \tag{1.6.6}$$

and inequality (1.6.3) is satisfied.

Lemma 1.6.3: ([5, 29])

Assume that the hypotheses of Lemma 1.6.1 (or Lemma 1.6.2) hold. Assume further that y satisfies the following relation:

$$y^{(n-1)}(t)y^{(n)}(t) \leq 0 \quad \text{for all } t \geq t_1,$$

Then for every λ, $0 < \lambda < 1$, there exists constants $M > 0$, and $M_1 > 0$, such that

$$|y(\lambda t)| \geq M t^{n-1} |y^{(n-1)}(t)| \quad \text{for all large } t \tag{1.6.7}$$

$$|y'(\lambda t)| \geq M_1 t^{n-2} |y^{(n-1)}(t)| \quad \text{for all large } t \tag{1.6.8}$$

Lemma 1.6.4: ([29])

If y is as in lemma 1.6.1, and for some $k = 0,1,\ldots,n-2$,

$$\lim_{t\to\infty} y^{(k)}(t) = c \qquad c \in R \tag{1.6.9}$$

Then

$$\lim_{t\to\infty} y^{(k+1)}(t) = 0 \tag{1.6.10}$$

Lemma 1.6.5: ([32])

Let $n \geq 3$ be an odd integer, $\alpha(t) \in C([0,\infty),[0,\infty))$, $0 < \alpha(t) \leq \alpha_0$, and $y \in C^n([0,\infty),R)$ such that $(-1)^i y^{(i)}(t) > 0$, $0 \leq i \leq n-1$, and $y^{(n)}(t) \leq 0$. Then

$$y(t-\alpha(t)) \geq \frac{(\alpha(t))^{n-1}}{(n-1)!} y^{(n-1)}(t) \qquad \text{for } t \geq \alpha_0. \tag{1.6.11}$$

Proof: By Taylor's expansion we have

$$y(t-\alpha(t)) = y(t) + (-\alpha(t))y'(t) + \frac{(-\alpha(t))^2}{2!} y''(t) + \ldots$$

$$+ \frac{(-\alpha(t))^{n-1}}{(n-1)!} y^{(n-1)}(t) + \frac{(-\alpha(t))^n}{n!} y^{(n)}(t-\mu\alpha(t))$$

where $0 \leq \mu \leq 1$. Thus

$$y(t-\alpha(t)) \geq \frac{(\alpha(t))^{n-1}}{(n-1)!} y^{(n-1)}(t)$$

for $t \geq \alpha_0$, since n is an odd integer. Hence the lemma is proved. □

1.7 Mathematical model by using NDDE for a particular problem

As mentioned previously, NDDE find numerous applications in natural science, and technology. For instance, they are frequently used for the

study of distributed networks containing lossless transmission lines. In this section we discuss, in details, a certain example of lossless transmission lines connected to a nonlinear elements. This example is due to Brayton [6, 25]. For more about theory of transmission lines see [9].

Treatment of transmission lines is more complicated than that of ordinary networks. Whereas the physical dimensions of electric networks are very much smaller than the operating wavelength. But transmission lines are usually a considerable fraction of a wavelength and may even be many wavelengths long. The circuit elements in ordinary electric networks can be considered discrete, and they described by lumped parameters. But transmission line is a distributed-parameter network, and must be described by circuit parameters that are distributed through its length. However this situation is the case in long transmission lines (as in power systems engineering) [9], also the case in small channels under very high frequency (as in high speed computers where the lossless transmission lines are used to interconnect switching circuits) [23, 24].

Consider lossless transmission line connected between lumped elements as shown in fig.1.1

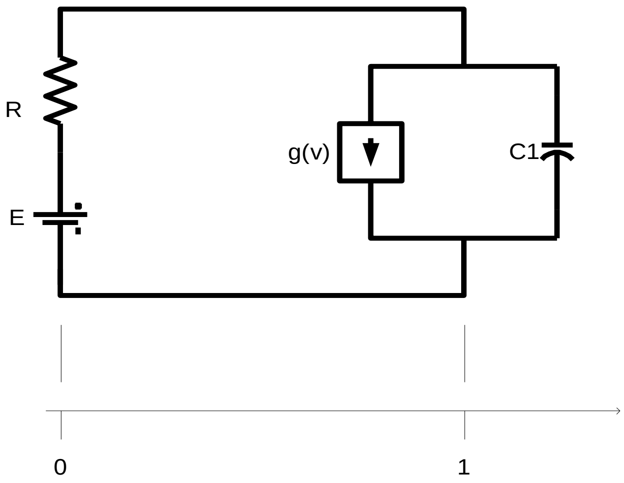

fig.1.1

The length of the line can be normalized to unity without loss of generality. Where $g(v)$ is a nonlinear function of voltage v and gives the current in the indicated box in the direction shown. And the behavior in the line can be described by the following pair of partial differential equations:

$$L\frac{\partial i}{\partial t} = -\frac{\partial v}{\partial x}, \qquad C\frac{\partial v}{\partial t} = -\frac{\partial i}{\partial x}, \qquad 0 < x < 1, \qquad t > 0 \tag{1.7.1}$$

Where $i(x,t)$, $v(x,t)$ are the current in the line and the voltage to ground respectively at the point x and at time t. And L, C are, respectively, the inductance and capacitance of the line per unit length.

Since the elements at the two sides of the transmission line are lumped elements, we can introduce the following boundary conditions:

$$E - v(0,t) - Ri(0,t) = 0, \qquad C_1\frac{dv(1,t)}{dt} = i(1,t) - g(v(1,t)) \tag{1.7.2}$$

It is clear that the problem modeled by system of partial differential equations with certain boundary conditions. Also we can write the system of equations using matrices.

$$\begin{bmatrix} \frac{\partial i}{\partial t} \\ \frac{\partial v}{\partial t} \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial i}{\partial x} \\ \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad 0 < x < 1, \quad t > 0 \tag{1.7.3}$$

The general solution of this system is:

$$i(x,t) = \sqrt{\frac{C}{L}} \left[\phi(x - \frac{t}{\sqrt{LC}}) - \psi(x + \frac{t}{\sqrt{LC}}) \right] \tag{1.7.4}$$

$$v(x,t) = \phi(x - \frac{t}{\sqrt{LC}}) + \psi(x + \frac{t}{\sqrt{LC}}) \tag{1.7.5}$$

Put $s = \frac{1}{\sqrt{LC}}$, and $z = \sqrt{\frac{L}{C}}$, then

$$2\phi(x - st) = v(x,t) + zi(x,t) \tag{1.7.6}$$

$$2\psi(x + st) = v(x,t) - zi(x,t) \tag{1.7.7}$$

This implies:

$$2\phi(-st) = v(1, t + \frac{1}{s}) + zi(1, t + \frac{1}{s}) \tag{1.7.8}$$

$$2\psi(st) = v(1, t - \frac{1}{s}) - zi(1, t - \frac{1}{s}) \tag{1.7.9}$$

Using these expressions in the general solution and using first boundary condition at $t - \frac{1}{s}$ we obtain:

$$i(1,t) - ki(1, t - \frac{2}{s}) = \alpha - \frac{1}{z} v(1,t) - \frac{k}{z} v(1, t - \frac{2}{s}) \tag{1.7.10}$$

Where $k = \frac{z - R}{z + R}$, $\alpha = \frac{2E}{z + R}$

Inserting second boundary condition and let $x(t) = v(1,t)$, we obtain the equation:

$$x'(t) - kx'(t - \frac{2}{s}) = f(x(t), x(t - \frac{2}{s})) \qquad (1.7.11)$$

Where

$$f(x(t), x(t - \frac{2}{s})) = \frac{1}{C_1}\left[\alpha - \frac{1}{z}x(t) - \frac{k}{z}x(t - \frac{2}{s}) - g(x(t)) + g(x(t - \frac{2}{s}))\right] \qquad (1.7.12)$$

Let $\tau = \frac{2}{s} \Rightarrow \tau > 0$, we get:

$$[x(t) - kx(t - \tau)]' = f(x(t), x(t - \tau)) \qquad (1.7.13)$$

It is a first order NDDE, where $x(t)$ represents the voltage as function of time at destination elements, i.e. at the elements that the power transferred to.

Chapter Two

Oscillation of *n-th* Order NDDE When the Non-Derivative Part is Separable

2.0 Introduction

In this chapter we consider the oscillation of *n-th* order NDDE when the non-derivative part is separable, i.e. we consider an equation of the form:

$$[x(t)+p(t)x(\tau(t))]^{(n)}+q(t)f(x(\sigma(t)))=0 \qquad (2.0.1)$$

where, $p(t),q(t),\tau(t),\sigma(t)\in C([t_0,\infty),R)$, $t_0\geq 0$, $\tau(t)$ and $\sigma(t)$ are delayed arguments, and $f\in C(R,R)$.

NDDE of form (2.0.1) is the most familiar form that appears in literature, and so many results are known for the oscillation of this form of equations. Our aim in this chapter is to present some of the oscillation results that recently have been obtained for this form of equations under different restrictions.

In section 2.1 we introduce sufficient conditions for the oscillation of equation (2.0.1) with constant delays i.e. when $\tau(t)=t-\tau$ and $\sigma(t)=t-\sigma$. In Section 2.2 we study some oscillation results when the delayed arguments are commute, i.e. $\tau(\sigma(t))=\sigma(\tau(t))$. Finally, in section 2.3 we present oscillation criteria for the solutions of equation (2.0.1) with variable delays.

2.1 Oscillation of *n-th* order NDDE with constant delays

Consider the equation

$$[x(t)-p(t)x(t-\tau)]^{(n)}+q(t)f(x(t-\sigma))=0 \qquad (2.1.1)$$

where, $n\geq 1$ be an odd integer, $p(t),q(t)\in C([t_0,\infty),R_+)$, $t_0\geq 0$, $\tau,\sigma\in(0,\infty)$, $f\in C(R,R)$ such that $xf(x)>0$ for $x\neq 0$, and f is nondecreasing.

The results of this section are contained in Theorems 2.1.1 and 2.1.2, each theorem presents its own sufficient conditions for the oscillation of solutions of equation (2.1.1).

Theorem 2.1.1: Suppose that $0\leq p(t)\leq 1$. If f satisfies the sub-linearity condition

$$\int_0^{\pm\alpha}\frac{ds}{f(s)}<\infty \quad \text{for any } \alpha>0 \qquad (2.1.2)$$

and

$$\int_{t_0}^{\infty}q(s)ds=\infty \qquad (2.1.3)$$

Then every solution of equation (2.1.1) is oscillatory.

Proof: On the contrary, assume that $x(t)$ is a non-oscillatory solution of equation (2.1.1), without loss of generality, we assume that $x(t)>0$ for $t\geq t_1\geq t_0$. (The proof is similar for $x(t)<0$). Set

$$z(t)=x(t)-p(t)x(t-\tau) \qquad (2.1.4)$$

From (2.1.1), and (2.1.4) we have

$$z^{(n)}(t) = -q(t)f(x(t-\sigma)) \leq 0 \text{ for } t \geq t_1 + \sigma \tag{2.1.5}$$

That is $z^{(n)}(t) \leq 0$, for $t \geq t_1 + \sigma$. It follows that $z^{(j)}(t)\{ j = 0,1,...,n-1\}$ is strictly monotone and of constant sign eventually. Hence $z(t) < 0$ or $z(t) > 0$ from large t, say for $t \geq t_2 \geq t_1 + \sigma$. Let us consider these two cases:

<u>*Case1:*</u> Suppose that $z(t) < 0$, for $t \geq t_2$. Since n is odd, $z(t) < 0$ implies that $z'(t) < 0$ from a certain point on, say for $t \geq t_3 \geq t_2$ (see Lemma 1.6.2). Consequently: $\lim_{t\to\infty} z(t) = \lambda < 0$, and so $\lim_{t\to\infty} \inf x(t) \neq 0$. Let $\varepsilon > 0$ such that $x(t) > \varepsilon$ for $t \geq t_4 \geq t_3$.

Using the facts that $x(t) > \varepsilon$ for $t \geq t_4$, and f is nondecreasing, it follows from (2.1.1) that for large t

$$z^{(n)}(t) + q(t)f(\varepsilon) \leq 0 \tag{2.1.6}$$

Integrating (2.1.6) from t_4 to t and using condition (2.1.3), then we have $z^{(n-1)}(t) \to -\infty$ as $t \to \infty$, which implies that $z(t) \to -\infty$ as $t \to \infty$. Consequently, $x(t)$ is unbounded. But $z(t) < 0$ for $t \geq t_2$, it follows from (2.1.4) that $x(t) < p(t)x(t-\tau) < x(t-\tau)$, so $x(t)$ is bounded, which is a contradiction.

<u>*Case2:*</u> Suppose that $z(t) > 0$, for $t \geq t_2$. From (2.1.4) it follows that $z(t) < x(t)$, for $t \geq t_2$.Using this fact and the fact that f is nondecreasing. Then from (2.1.1) we have:

$$z^{(n)}(t)+q(t)f(z(t-\sigma))\le 0 \tag{2.1.7}$$

If $z'(t)>0$ eventually, $z(t)$ is increasing , then integrating inequality (2.1.7) from t_2 to t and letting $t\to\infty$ we get $z^{(n-1)}(t)\to-\infty$ as $t\to\infty$, which implies that $z(t)\to-\infty$ as $t\to\infty$, but this contradicts the fact that $z(t)>0$. Hence $z'(t)<0$ for $t\ge t_3$.

Since $z(t)>0$, it satisfies conditions of lemma 1.6.1. And since $z'(t)<0$, then the number l in lemma 1.6.1 must be zero ($l=0$). Therefore by Lemma 1.6.1 we have

$$(-1)^i z^{(i)}(t)>0,\ 0\le i\le n, \text{ for } t\ge t_4\ge t_3 \tag{2.1.8}$$

Since n is odd, and (2.1.8) holds, follows that $z(t)$ satisfies the hypotheses of lemma 1.6.5. By result of lemma 1.6.5, we obtain:

$$z(t-\sigma)\ge Bz^{(n-1)}(t)\ ,\quad t\ge T\ge t_4 \tag{2.1.9}$$

where $B=\dfrac{\sigma^{n-1}}{(n-1)!}$. Hence from (2.1.1) and (2.1.9) we get

$$z^{(n)}(t)+q(t)f(Bz^{(n-1)}(t))\le z^{(n)}(t)+q(t)f(z(t-\sigma)) \tag{2.1.10}$$

$$\le z^{(n)}(t)+q(t)f(x(t-\sigma))=0,\qquad t\ge T$$

Dividing (2.1.10) by $f(Bz^{(n-1)}(t))$, and integrating the resulting inequality from T to t we obtain

$$-\frac{1}{B}\int_{Bz^{(n-1)}(t)}^{Bz^{(n-1)}(T)}\frac{ds}{f(s)}+\int_T^t q(s)ds\le 0 \tag{2.1.11}$$

Letting $t\to\infty$ in (2.1.11) and using conditions (2.1.2) and (2.1.3) leads to a contradiction. Hence the theorem is proved. ☐

Example 2.1.1: Consider the equation

$$\left[x(t)-\frac{1}{2}x(t-2\pi)\right]' + \frac{3}{2}\sin^2 t\left(x(t-\frac{\pi}{2})\right)^{\frac{1}{3}} = 0 \qquad (2.1.12)$$

Here $n=1$, $p(t)=\frac{1}{2}$, $q(t)=\frac{3}{2}\sin^2 t$, $f(x)=x^{\frac{1}{3}}$, $\tau(t)=t-2\pi$, and $\sigma(t)=t-\frac{\pi}{2}$.

$f(x)$ satisfies the sub-linear condtion (2.1.2), $q(t)$ satisfies condtion (2.1.3), so we see that all condtions of Theorem 2.1.1 are fulfilled. Therefore every solution of equation (2.1.12) is oscillatory. In fact, $x(t)=\sin^3 t$ is such solution.

Theorem 2.1.2: Suppose that $p(t)=p\in(0,1)$. If f satisfies the generalized linear condition

$$\lim_{x\to 0}\inf\frac{f(x)}{x} = M \in (0,\infty), \qquad (2.1.13)$$

$f(xy)=f(x)f(y)$ for any two continuous functions x and y, and

$$\lim_{t\to\infty}\sup\int_{t-\sigma}^{t} q(s)\,f\left(\frac{(s-t)^{n-1}}{(n-1)!}\right)ds > 1 \qquad (2.1.14)$$

Then every solution of equation (2.1.1) is oscillatory.

Proof: On the contrary, assume that $x(t)$ is a non-oscillatory solution of (2.1.1). Without any loss of generality, assume that $x(t)>0$ for $t\ge t_1\ge t_0$. Set $z(t)$ as in (2.1.4), since $p(t)=p$ is constant ,then (2.1.4) will be

$$z(t)=x(t)-px(t-\tau) \qquad (2.1.15)$$

From (2.1.1), and (2.1.15) we have:

$$z^{(n)}(t) = -q(t)f(x(t-\sigma)) \le 0 \text{ for } t\ge t_1+\sigma \qquad (2.1.16)$$

That is $z^{(n)}(t) \le 0$ for $t \ge t_1 + \sigma$. It follows that $z^{(j)}(t)\{ j = 0,1,\ldots,n-1\}$ is strictly monotone and of constant sign eventually. Hence $z(t) < 0$ or $z(t) > 0$ for large t, say for $t \ge t_2 \ge t_1 + \sigma$. Let us consider these two cases:

<u>*Case1:*</u> Suppose that $z(t) < 0$, for $t \ge t_2$. Since n is odd, $z(t) < 0$ implies that $z'(t) < 0$ from a certain point on, say for $t \ge t_3 \ge t_2$ (see Lemma 1.6.2). Therefore we have:

$$\lim_{t \to \infty} z(t) = \lambda < 0 \tag{2.1.17}$$

From (2.1.15), it follows that $px(t-\tau) = x(t) - z(t)$ and hence

$$\liminf_{t \to \infty} x(t) > -\frac{\lambda}{p} > 0 \tag{2.1.18}$$

Integrating both sides of (2.1.16) from t_3 to t and letting $t \to \infty$ with the use of (2.1.18) we have $z^{(n-1)}(t) \to -\infty$ as $t \to \infty$, which implies that $z(t) \to -\infty$ as $t \to \infty$. Consequently, $x(t)$ is unbounded. Hence there exists a sequence of real numbers $\{s_n\}$ such that $s_n \to \infty$, $x(s_n) \to \infty$ as $n \to \infty$ and $x(s) \le x(s_n)$ for $s \le s_n$. Now

$$z(s_n) = x(s_n) - px(s_n - \tau) > (1-p)x(s_n)$$

and hence $z(s_n) \to \infty$ as $s_n \to \infty$, which is a contradiction to our earlier conclusion. So $z(t) < 0$ for $t \ge t_3$ is impossible.

<u>*Case2:*</u> Suppose that $z(t) > 0$ for $t \ge t_2$. Then either $z'(t) < 0$ or $z'(t) > 0$ eventually. If $z'(t) > 0$, $t \ge t_3 \ge t_2$ the contradiction is as follows. In this case

$\lim_{t\to\infty} z(t) = \mu > 0$. Consequently, from (2.1.15) it follows that

$$\liminf_{t\to\infty} x(t) = \liminf_{t\to\infty} (z(t) + px(t-\tau)) > 0 \tag{2.1.19}$$

Integrating (2.1.16) from t_3 to t, with the use of (2.1.19), then we see that $z^{(n-1)}(t) \to -\infty$ as $t \to \infty$, which implies that $z(t) \to -\infty$ as $t \to \infty$, but this contradicts the fact that $z(t) > 0$ eventually. Next, assume that $z'(t) < 0$, $t \ge t_3$. By use of lemma 1.6.1, and since $z'(t) < 0$, the number l in lemma 1.6.1 must be zero ($l = 0$). Hence, we conclude that

$$(-1)^i z^{(i)}(t) > 0,\ 0 \le i \le n, \text{ for } t \ge T \ge t_3 \tag{2.1.20}$$

Since n is odd and (2.1.20) holds then we can apply Lemma 1.6.5, for $z(s)$, and $t > s$, by result of Lemma 1.6.5, we have:

$$z(s) = z\big(t-(t-s)\big) \ge \frac{(s-t)^{n-1}}{(n-1)!} z^{(n-1)}(t)$$

From the fact that $x(t) \ge z(t)$ we have:

$$x(s) \ge z(s) \ge \frac{(s-t)^{n-1}}{(n-1)!} z^{(n-1)}(t) \tag{2.1.21}$$

Replacing t and s by $t-\sigma$ and $s-\sigma$ respectively in inequality (2.1.21) we get

$$x(s-\sigma) \ge \frac{(s-t)^{n-1}}{(n-1)!} z^{(n-1)}(t-\sigma) \tag{2.1.22}$$

Then, from (2.1.15) it follows that

$$z^{(n)}(s) + q(s) f(x(s-\sigma)) = 0 \tag{2.1.23}$$

With the use of (2.1.22) and (2.1.23) and the fact that f is nondecreasing we obtain

$$z^{(n)}(s)+q(s)f\left(\frac{(s-t)^{n-1}}{(n-1)!}z^{(n-1)}(t-\sigma)\right)\le 0 \qquad (2.1.24)$$

Now using the fact that $f(xy)=f(x)f(y)$ for any two continuous functions x and y. Then from (2.1.24) we have

$$z^{(n)}(s)+q(s)f\left(\frac{(s-t)^{n-1}}{(n-1)!}\right)f\left(z^{(n-1)}(t-\sigma)\right)\le 0 \qquad (2.1.25)$$

Integrating both sides of (2.1.25) with respect to s from $t-\sigma$ to t we get

$$z^{(n-1)}(t)-z^{(n-1)}(t-\sigma)+\int_{t-\sigma}^{t}q(s)f\left(\frac{(s-t)^{n-1}}{(n-1)!}\right)f\left(z^{(n-1)}(t-\sigma)\right)ds\le 0,$$

Consequently,

$$z^{(n-1)}(t)-z^{(n-1)}(t-\sigma)+f\left(z^{(n-1)}(t-\sigma)\right)\int_{t-\sigma}^{t}q(s)f\left(\frac{(s-t)^{n-1}}{(n-1)!}\right)ds\le 0, \qquad (2.1.26)$$

Dividing both sides of (2.1.26) by $z^{(n-1)}(t-\sigma)$ and using the fact that $z^{(n-1)}(t)>0$ {see (2.1.20)}, we obtain

$$\frac{z^{(n-1)}(t)}{z^{(n-1)}(t-\sigma)}-1+\frac{f\left(z^{(n-1)}(t-\sigma)\right)}{z^{(n-1)}(t-\sigma)}\int_{t-\sigma}^{t}q(s)f\left(\frac{(s-t)^{n-1}}{(n-1)!}\right)ds\le 0 \qquad (2.1.27)$$

Further from (2.1.20), we conclude that $z^{(n-2)}(t)$ is negative, increasing, and concave down, so $\lim_{t\to\infty} z^{(n-2)}(t)=c\in R$, and from lemma 1.6.4 we have:

$$\lim_{t\to\infty} z^{(n-1)}(t)=0 \qquad (2.1.28)$$

Taking limit superior of both sides of (2.1.27) and using (2.1.28), we see that

$$\limsup_{t\to\infty}\int_{t-\sigma}^{t}q(s)f\left(\frac{(s-t)^{n-1}}{(n-1)!}\right)ds\le\frac{1}{M},$$

which is a contradiction to the condition (2.1.14). Hence the proof is completed. □

Example 2.1.2: Consider the equation

$$\left[x(t)-\frac{1}{2}x(t-\frac{\pi}{2})\right]'+3x(t-\frac{\pi}{4})=0 \qquad (2.1.29)$$

All conditions of Theorem 2.1.2 are satisfied. Therefore every solution of equation (2.1.29) is oscillatory. Indeed, $x(t)=\sin 2t$ is an oscillatory solution of this equation.

Note that equation (2.1.29) satisfies all conditions of Theorem 2.1.1, except the condition of sub-linearity, hence if we use Theorem 2.1.1 we have no conclusion about the oscillatory of equation (2.1.29).

Remark 2.1.1: Theorem 2.1.1 is due to Das [11], which is an extension of Theorem 4 of Graef [16]. Theorem 2.1.2 is due to Das and Mishra [12].

2.2 Oscillation of *n-th* order NDDE with commute delayed arguments

In this section we study the oscillatory behavior of NDDE when the delayed arguments are commute, i.e. $\tau(\sigma(t))=\sigma(\tau(t))$ for $t\geq t_0\geq 0$.

Consider the NDDE, with commute delayed arguments, of the form:

$$\left[x(t)+p(t)x(\tau(t))\right]^{(n)}+q(t)f(x(\sigma(t)))=0 \qquad (2.2.1)$$

where, $p(t),q(t),\tau(t),\sigma(t)\in C([t_0,\infty),R_+)$, $t_0\geq 0$, $q(t)\not\equiv 0$ on any half line $[t_*,\infty)$, $\tau(t)\leq t$, $\sigma(t)\leq t$, $\lim_{t\to\infty}\tau(t)=\infty$, $\lim_{t\to\infty}\sigma(t)=\infty$, and $f\in C(R,R)$ such that $xf(x)>0$ for $x\neq 0$.

Further the following assumptions are made for their use in this section:

(1) $f(u+v) \le f(u) + f(v),$ if $u,v > 0$

(2) $f(u+v) \ge f(u) + f(v),$ if $u,v < 0$

(3) $f(ku) \le kf(u),$ if $k \ge 0$ and $u > 0$, for each $k \in K$,

where $K = \{k : p(t) = k$ for some $t \in [t_0,\infty)\}$.

(4) $f(ku) \ge kf(u),$ if $k \ge 0$ and $u < 0$, for each $k \in K$

(5) $f(u)$ is bounded away from zero if u is bounded away from zero

(6) $\int_{t_0}^{\infty} q(s)ds = \infty$

(7) $\tau(t) \in C^1([t_0,\infty),R)$ and $\tau'(t) \ge b$, where b is positive constant.

(8) There exists a positive constant M such that $p(\sigma(t))q(t) \le Mq(\tau(t))$

The main results in this section are contained in the following theorems:

Theorem 2.2.1: Assume that conditions (1)-(8) hold. Then

i. If n is even, every solution of equation (2.2.1) is oscillatory

ii. If n is odd, any solution of equation (2.2.1) is either oscillatory or tends to zero as $t \to \infty$.

Proof: Suppose that equation (2.2.1) has a non-oscillatory solution $x(t)$. Without loss of generality, assume that $x(t)$ is eventually positive (The proof is similar when $x(t)$ is eventually negative). That is $x(t) > 0$, $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$, and $x(\tau(\sigma(t))) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$.

Set

$$z(t) = x(t) + p(t)x(\tau(t)). \tag{2.2.2}$$

Since $p(t)$ is nonnegative then $z(t) > 0$ for $t \ge t_1$. Using the fact that $\tau(\sigma(t)) = \sigma(\tau(t))$ for $t \ge t_1$, then from (1), and (3) we have:

$$f(z(\sigma(t))) = f\big(x(\sigma(t)) + p(\sigma(t))x(\tau(\sigma(t)))\big)$$

$$\le f(x(\sigma(t))) + p(\sigma(t)) f(x(\sigma(\tau(t)))), \tag{2.2.3}$$

Using (2.2.1) and (2.2.2) we obtain

$$z^{(n)}(t) = -q(t) f(x(\sigma(t))) \tag{2.2.4}$$

From (2.2.3) and (2.2.4) we have:

$$z^{(n)}(t) + q(t) f(z(\sigma(t))) \le z^{(n)}(t) + q(t)\big(f(x(\sigma(t))) + p(\sigma(t)) f(x(\sigma(\tau(t))))\big).$$

Hence

$$z^{(n)}(t) + q(t) f(z(\sigma(t))) \le q(t) p(\sigma(t)) f(x(\sigma(\tau(t)))) \tag{2.2.5}$$

Since $x(\sigma(t)) > 0$ for $t \ge t_1$, $z^{(n)}(t) \le 0$ and so $z^{(i)}(t)$ is monotonic for $i = 0,1,\ldots,n-1$. Therefore, $z^{(n-1)}(t) > 0$ or $z^{(n-1)}(t) < 0$ eventually. If $z^{(n-1)}(t) \le 0$ then from the facts that $z^{(n)}(t) \le 0$ and $q(t) \not\equiv 0$, lead that $z(t) < 0$ eventually, so a contradiction. Hence there exists $t_2 \ge t_1$ such that $z^{(n-1)}(t) > 0$ for $t \ge t_2$.

From (2.2.1), and the fact that $\tau'(t) \ge b > 0$, we have:

$$z^{(n)}(\tau(t))\tau'(t) + q(\tau(t)) f(x(\sigma(\tau(t))))\tau'(t) = 0 \tag{2.2.6}$$

Let $t_3 \ge t_2$ such that $z^{(n-1)}(\tau(t)) > 0$ for $t \ge t_3$. Then integrate (2.2.6) from t_3 to ∞, we get:

$$\int_{t_3}^{\infty} q(\tau(s)) f(x(\sigma(\tau(s))))\tau'(s)ds = z^{(n-1)}(\tau(t_3)) - L.$$

where $L = \lim_{t\to\infty} z^{(n-1)}(t)$. Since $z^{(n-1)}(t) > 0$ eventually, we show that:

$$\int_{t_3}^{\infty} q(\tau(s)) f(x(\sigma(\tau(s))))\tau'(s)ds < \infty \tag{2.2.7}$$

Using (2.2.7), with (7), and (8), follows that

$$\int_{t_3}^{\infty} q(s) p(\sigma(s)) f(x(\sigma(\tau(s))))ds < \infty \tag{2.2.8}$$

Integrating (2.2.5) , and using (2.2.8) , we show that:

$$\int_{t_3}^{\infty} q(s) f(z(\sigma(s)))ds < \infty \tag{2.2.9}$$

Since (5) and (6) hold, then (2.2.9) implies that $\liminf_{t\to\infty} z(t) = 0$. But $z(t)$ is positive and monotonic, so $z(t) \to 0$ as $t \to \infty$. So $z(t)$ is decreasing, implies that $z'(t) \le 0$ eventually. For $n > 1$, $z'(t) \to 0$ as $t \to \infty$ since $z'(t)$ is monotonic and $z(t) > 0$ ($z(t)$ is concave up) (you can see lemma 1.6.4). Hence $z'(t)$ is increasing, implies that $z''(t) > 0$. For $n > 2$, $z''(t)$ is eventually positive and satisfies $z''(t) \to 0$ as $t \to \infty$ since $z'(t)$ is monotonic and negative. Continuing in this manner we have:

$$z^{(i)}(t) z^{(i+1)}(t) \le 0, \text{ for } i = 0,1,...,n-1 \tag{2.2.10}$$

with strict inequality holding for $i < n-1$. If n is even, using (2.2.10) and the fact that $z^{(n)}(t) \le 0$ we reach to $z(t) < 0$, and this contradicts $z(t) > 0$. If n is odd, then $x(t) < z(t) \to 0$ as $t \to \infty$, and this completes the proof. □

Example 2.2.1: Consider the NDDE

$$\left[x(t) + (2 + \cos t)x(t - 2\pi)\right]'' + (3 + \cos t)x(t - 4\pi) = 0 \tag{2.2.11}$$

Here $n=2$, $p(t)=2+\cos t$, $q(t)=3+\cos t$, $f(x)=x$, $\tau(t)=t-2\pi$, and $\sigma(t)=t-4\pi$. The delayed arguments $\tau(t)$ and $\sigma(t)$ aere commute i.e. $\tau(\sigma(t))=\sigma(\tau(t))=t-6\pi$, function $f(x)$ satisfies condtions (1)-(5), and $q(t)$ satisfies the divergent integral in condtion (6). Also condtions (7) and (8) are satisfied by (2.2.11). Thus all condtions of Theorem 2.2.1 (i) are satisfied. Therefore we can conclude that every solution of equation (2.2.11) is oscillatory. In fact, $x_1(t)=\dfrac{\cos t}{3+\cos t}$, and $x_2(t)=\dfrac{\sin t}{3+\cos t}$ are oscillatory solutions of (2.2.11).

Example 2.2.2: The NDDE

$$\left[x(t)+pe^{-\pi}x(t-\pi)\right]''' +2\sqrt{2}(p-1)e^{-\frac{7\pi}{4}}x(t-\frac{7\pi}{4})=0, \qquad (2.2.12)$$

where $p>1$, satisfies the conditions of Theorem 2.2.1 (ii). So every solution of (2.2.12) is either oscillatory or tends to zero as $t\to\infty$. In fact, (2.2.12) has an oscillatory solution $x(t)=e^{-t}\sin t$.

Example 2.2.3: Consider the NDDE

$$\left[x(t)+px(t-\ln 2)\right]^{(n)}+\left[\frac{1+2p}{e^2}\right]x(t-2)=0 \qquad (2.2.13)$$

where n is odd, and $p\geq 0$. All conditions of Theorem 2.2.1 (ii) are satisfied. So every solution of (2.2.13) is either oscillatory or tends to zero as $t\to\infty$. Indeed, (2.2.13) has the non-oscillatory solution $x(t)=e^{-t}\to 0$ as $t\to\infty$.

Theorem 2.2.2: Suppose that $n\geq 2$ is even, and conditions (1), (2), (7) hold. Moreover, if the following conditions are satisfied:

(9) $f(x)$ is nondecreasing on R

(10) $p(t) \le p$, $p > 0$ is a constant

(11) There exists a continuous and nonnegative function $Q(t)$ such that

$$q(t) \ge Q(t) + Q\left(\tau^{-1}(t)\right)$$

where $\tau^{-1}(t)$ is the inverse function of $\tau(t)$. If

$$\int^{\infty} Q(t)dt = \infty \qquad (2.2.14)$$

Then every solution of equation (2.2.1) oscillates.

Proof: Without loss of generality ,Suppose that (2.2.1) has an eventually positive solution $x(t)$, say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$, for $t \ge t_1$ for some $t_1 \ge t_0$.

Set $z(t) = x(t) + p(t)x(\tau(t))$. Since $p(t)$ is nonnegative then $z(t) > 0$ for $t \ge t_1$.

By (2.2.1) we have:

$$z^{(n)}(t) = -q(t) f(x(\sigma(t)) \le 0, \quad t \ge t_1 \qquad (2.2.15)$$

By lemma 1.6.1, there exists $t_2 \ge t_1$ such that $z'(t) > 0$, $z^{(n-1)}(t) > 0$, for $t \ge t_2$. Using (1) we have:

$$f(x(t)) + f(x(\tau(t))) \ge f\left(x(t) + x(\tau(t))\right)$$

$$\ge f\left(\frac{x(t) + p(t)x(\tau(t))}{\max\{1, p\}}\right) = f\left(\frac{z(t)}{\max\{1, p\}}\right) \qquad (2.2.16)$$

Using (11) we have

$$q(t) f(x(\sigma(t))) \ge \left[Q(t) + Q\left(\tau^{-1}(t)\right)\right] f(x(\sigma(t))) \qquad (2.2.17)$$

Integrating (2.2.17) from t_2 to t, gives that:

$$\int_{t_2}^{t} q(s) f(x(\sigma(s)))ds \geq \int_{t_2}^{t} Q(s) f(x(\sigma(s)))ds + \int_{t_2}^{t} Q\left(\tau^{-1}(s)\right) f(x(\sigma(s)))ds \qquad (2.2.18)$$

Replace s by $\tau(s)$ in the second integral of right side of (2.2.18), we get:

$$\int_{t_2}^{t} q(s) f(x(\sigma(s)))ds \geq \int_{t_2}^{t} Q(s) f(x(\sigma(s)))ds + \int_{\tau^{-1}(t_2)}^{\tau^{-1}(t)} Q(s) f(x(\sigma(\tau(s))))\tau'(s)ds \qquad (2.2.19)$$

From (2.2.19), and using condition (7) with (2.2.16) we obtain:

$$\int_{t_2}^{t} q(s) f(x(\sigma(s)))ds \geq \min\{1,b\} \int_{t_3}^{\delta(t)} Q(s)\big(f(x(\sigma(s))) + f(x(\tau(\sigma(s))))\big)ds$$

$$\geq \min\{1,b\} \int_{t_3}^{\delta(t)} Q(s) f\left(\frac{z(\sigma(s))}{\max\{1,p\}}\right)ds \qquad (2.2.20)$$

where

$$\delta(t) = \min\{t, \tau^{-1}(t)\}, \quad t_3 = \max\{t_2, \tau^{-1}(t_2)\} \qquad (2.2.21)$$

Choose $t_4 \geq t_3$, such that $z(\sigma(t)) \geq z(t_3)$ for $t \geq t_4$. Then using (2.2.20), and the fact that f is nondecreasing, we obtain:

$$\int_{t_2}^{t} q(s) f(x(\sigma(s)))ds \geq \min\{1,b\}\, f\left(\frac{z(t_3)}{\max\{1,p\}}\right) \int_{t_3}^{\delta(t)} Q(s)ds, \; t \geq t_4 \qquad (2.2.22)$$

Taking $t \to \infty$ in (2.2.22), and using (2.2.14), we have:

$$\int_{t_2}^{t} q(s) f(x(\sigma(s)))ds = \infty \qquad (2.2.23)$$

But, Integrating (2.2.15) from t_2 to t, and using $z^{(n-1)}(t) > 0$, we have:

$$\int_{t_2}^{t} q(s) f(x(\sigma(s)))ds = -z^{(n-1)}(t) + z^{(n-1)}(t_2) \leq z^{(n-1)}(t_2) \qquad (2.2.24)$$

Taking $t \to \infty$ in (2.2.24). The result contradicts (2.2.23).Thus the proof is completed. □

Example 2.2.4: The NDDE

$$\left[x(t) + 51e^{-\pi} x(t-\pi)\right]^{(IV)} + 200e^{-3\pi} x(t-3\pi) = 0,\ t > 0 \qquad (2.2.25)$$

satisfies all conditions of Theorem 2.2.2, by setting $Q(t) = 100e^{-3\pi}$. So every solution of (2.2.25) is oscillatory. In fact, $x(t) = e^{-t}\sin t$ is such a solution.

Theorem 2.2.3: Suppose that $n \geq 2$ is even, and conditions (1), (2), (7), and (9) hold. Moreover, if the following conditions are satisfied:

(12) There exists a function $g(u) \in C([0,\infty),[0,\infty))$ such that

$$g(u) > 0 \text{ for } u > 0$$

$$f(uv) \leq g(u) f(v) \text{ for } u > 0,\ v > 0$$

$$f(uv) \geq g(u) f(v) \text{ for } u > 0,\ v < 0$$

(13)There exists a continuous and nonnegative function $Q(t)$ such that

$$q(t) \geq Q(t) + Q\left(\tau^{-1}(t)\right) g\left(p(\tau^{-1}(t))\right)$$

If $$\int^{\infty} Q(t)dt = \infty \qquad (2.2.26)$$

Then every solution of equation (2.2.1) is oscillatory.

Proof: Without loss of generality, Suppose that (2.2.1) has an eventually positive solution $x(t)$, say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$, for $t \geq t_1$ for some $t_1 \geq t_0$.

Set $z(t)=x(t)+p(t)x(\tau(t))$. Since $p(t)$ is nonnegative then $z(t)>0$ for $t\geq t_1$.

From (2.2.1) we have (2.2.15).

By lemma 1.6.1, there exists $t_2\geq t_1$ such that $z'(t)>0$, $z^{(n-1)}(t)>0$, for $t\geq t_2$. Using (1), and (12), we have

$$f(x(t))+g(p(t))f(x(\tau(t)))\geq f(x(t))+f\big(p(t)x(\tau(t))\big)$$

$$\geq f\big(x(t)+p(t)x(\tau(t))\big)=f(z(t)),\quad t\geq t_2 \qquad (2.2.27)$$

Using (13) we have

$$q(t)f(x(\sigma(t)))\geq\left[Q(t)+Q\big(\tau^{-1}(t)\big)g\big(p(\tau^{-1}(t))\big)\right]f(x(\sigma(t))) \qquad (2.2.28)$$

Integrating (2.2.28) from t_2 to t, using (2.2.27), with (7), and proceeding the same as in proof of Theorem 2.2.2. Then we have:

$$\int_{t_2}^{t}q(s)f(x(\sigma(s)))ds\geq\int_{t_2}^{t}Q(s)f(x(\sigma(s)))ds+\int_{t_2}^{t}Q\big(\tau^{-1}(s)\big)g\big(p(\tau^{-1}(s))\big)f(x(\sigma(s)))ds$$

$$=\int_{t_2}^{t}Q(s)f(x(\sigma(s)))ds+\int_{\tau^{-1}(t_2)}^{\tau^{-1}(t)}Q(s)g\big(p(s)\big)f(x(\sigma(\tau(s))))\tau'(s)ds$$

$$\geq\min\{1,b\}\int_{t_3}^{\delta(t)}Q(s)\big(f(x(\sigma(s)))+g(p(s))f(x(\tau(\sigma(s))))\big)ds$$

$$\geq\min\{1,b\}\int_{t_3}^{\delta(t)}Q(s)f\big(z(\sigma(s))\big)ds \qquad (2.2.29)$$

where $\delta(t)$ and t_3 are defined by (2.2.21).

Choose $t_4\geq t_3$, such that $z(\sigma(t))\geq z(t_3)$ for $t\geq t_4$. In view of f be nondecreasing from (2.2.29) we have:

$$\int_{t_2}^{t}q(s)f(x(\sigma(s)))ds\geq\min\{1,b\}\ f\big(z(t_3)\big)\int_{t_3}^{\delta(t)}Q(s)ds,\ t\ \geq t_4 \qquad (2.2.30)$$

Taking $t \to \infty$ in (2.2.30), we obtain:

$$\int_{t_2}^{t} q(s) f(x(\sigma(s)))ds = \infty$$

On the other hand, Integrating (2.2.15) from t_2 to t, we have

$$\int_{t_2}^{t} q(s) f(x(\sigma(s)))ds = -z^{(n-1)}(t) + z^{(n-1)}(t_2) \le z^{(n-1)}(t_2)$$

This is a contradiction. Thus the proof is completed. ☐

Example 2.2.5: The NDDE

$$\left[x(t) + 2e^{t-\frac{3\pi}{2}} x(t - \frac{3\pi}{2})\right]'' + 2\left(e^{t-\frac{7\pi}{2}} + e^{\frac{-7\pi}{2}}\right) x(t - \frac{7\pi}{2}) = 0, t > 0 \qquad (2.2.31)$$

satisfies all conditions of Theorem 2.2.3, by setting $Q(t) = \frac{1}{2} e^{\frac{-7\pi}{2}}$, and $g(u) = 2u$. So every solution of (2.2.31) is oscillatory. In fact, $x(t) = e^{-t} \sin t$ is such a solution.

Note that in Example 2.2.4, equation 2.2.25 also satisfies the conditions of Theorem 2.2.3, but equation (2.2.31) does not satisfy conditions of Theorem 2.2.2, since $p(t) = 2e^{t-\frac{3\pi}{2}}$ is not bounded. And note that conditions of Theorem 2.2.1 are not satisfied for equation (2.2.31) since the condition (8) failed. If you return to Example 2.2.1, you can see that equation (2.2.11) satisfies the condition of Theorem 2.2.3 by using $Q(t) = 1$, and $g(u) = u$. From advantages of Theorem 2.1.1, that it considers odd and even orders. From drawbacks of Theorem 2.2.3, that it needs extra work to find function $g(u)$.

Remark 2.2.1: Theorem 2.2.1 is due to Graef and Spikes [19], Theorems 2.2.2 and 2.2.3 are special cases of Theorems 3 and 4 of Li [31]. Where Li in [31] consider equation 2.2.1 when the derivative part has several delays. Also, his results are true for neutral equations rather than NDDE. So the results of Theorems 2.2.2, and 2.2.3 are still true if conditions $\tau(t) \le t$, and $\sigma(t) \le t$ are omitted.

2.3 Oscillation of *n-th* order NDDE with variable delays.

This section establishes sufficient conditions for oscillation of the solutions of NDDE

$$[x(t) + p(t)x(\tau(t))]^{(n)} + q(t)f(x(\sigma(t))) = 0, \quad n \ge 2 \tag{2.3.1}$$

The main results are contained in Theorems 2.3.1, 2.3.2, and 2.3.3.

Throughout theorems 2.3.1, and 2.3.2, the following conditions are assumed to be hold:

(1) $p(t), q(t) \in C([t_0,\infty), R)$, $t_0 \ge 0$, such that $0 \le p(t) < 1$, and $q(t) > 0$.

(2) $\tau(t) \in C([t_0,\infty), R)$, $\tau(t) \le t$, and $\lim\limits_{t\to\infty} \tau(t) = \infty$

(3) $\sigma(t) \in C^1([t_0,\infty), R)$, $\sigma'(t) > 0$, $\sigma(t) \le t$, and $\lim\limits_{t\to\infty} \sigma(t) = \infty$

(4) $f \in C(R,R)$, $xf(x) > 0$ for $x \ne 0$, $f(x)\operatorname{sgn} x \ge \beta|x|^{\alpha}$, $\alpha \ge 1$, $\beta > 0$.

The following lemma will be required in this section

Lemma 2.3.1: Let $z(t)$ be a positive function of degree l (As in Definition 1.6.1), $l \ge 2$. Then

$$z'(t) \ge \int_{t_1}^{t} z^{(l)}(s)\frac{(t-s)^{l-2}}{(l-2)!}\,ds \tag{2.3.2}$$

Proof: We did the proof in Appendix A. □

For notational purposes we assume that:

$$a_{n-1}(t) = \beta(1 - p(\sigma(t)))^{\alpha} q(t) \tag{2.3.3}$$

and for $n \geq 4$ and all $l \in \{1,2,...,n-3\}$,

$$a_l(t) = \int_t^{\infty} \beta(1 - p(\sigma(s)))^{\alpha} q(s) \frac{(s-t)^{n-l-2}}{(n-l-2)!} ds \tag{2.3.4}$$

Note: All inequalities presented below are assumed to hold eventually.

Theorem 2.3.1: Assume that $\alpha > 1$ and for all $l \in \{1,2,...,n-1\}$ such that $n + l$ is odd

$$\int^{\infty} \left(\sigma^{l}(t) a_l(t) - M_l \frac{\sigma'(t)}{\sigma(t)} \right) dt = \infty \text{ for some } M_l > 0 \qquad (2.3.5)_l$$

Further assume that for n odd $p(t) \leq p < 1$, and for n even

$$\int_{t_1}^{\infty} a_1(s)(s - t_1) ds = \infty \tag{2.3.6}$$

Then

i. If n is even, every solution of equation (2.3.1) is oscillatory.

ii. If n is odd, any solution of equation (2.3.1) is either oscillatory or tends to zero as $t \to \infty$.

Proof: Assume that $x(t)$ is a non-oscillatory solution of equation (2.3.1). Without loss of generality, we may assume that $x(t) > 0$. (The proof is similar for $x(t) < 0$).

Set

$$z(t) = x(t) + p(t)x(\tau(t)) \tag{2.3.7}$$

Thus $z(t) > 0$ eventually. From (2.3.1), and (2.3.7) we have:

$$z^{(n)}(t) = -q(t) f(x(\sigma(t))) < 0 \tag{2.3.8}$$

That is $z^{(n)}(t) < 0$, and consequently $z^{(j)}(t)\{\ j = 0,1,\ldots,n-1\}$ are monotonic and of constant signs eventually. By Lemma 1.6.1 there exists an integer $l \in \{0,1,\ldots,n-1\}$, such that $n+l$ is odd, and:

$$z^{(i)}(t) > 0, \quad 0 \le i \le l \tag{2.3.9}$$

$$(-1)^{n+i-1} z^{(i)}(t) > 0, \quad l+1 \le i \le n-1 \tag{2.3.10}$$

Now we consider the following two cases:

<u>*Case 1*</u>**:** Let $l \ge 1$. Follows from (2.3.9) that $z'(t) > 0$. So $z(t)$ is increasing. Using this fact and from (2.3.7) we have:

$$x(t) = z(t) - p(t)x(\tau(t)) \ge z(t) - p(t)z(\tau(t)) \ge (1-p(t))z(t),$$

$$x(t) \ge (1-p(t))z(t), \tag{2.3.11}$$

Since $x(\sigma(t)) > 0$ and from condition (4), and (2.3.11) we have:

$$f(x(\sigma(t)) \ge \beta(x(\sigma(t)))^{\alpha} \ge \beta(1-p(\sigma(t)))^{\alpha} z^{\alpha}(\sigma(t)), \tag{2.3.12}$$

Using (2.3.8), and (2.3.12) we have:

$$z^{(n)}(t) + \beta(1-p(\sigma(t)))^{\alpha} q(t) z^{\alpha}(\sigma(t)) \le 0, \tag{2.3.13}$$

If $l \le n-3$ then by integrating (2.3.13) from t to ∞ $(n-l-1)$ times, in view of (2.3.9) and (2.3.10), we get (2.3.14), (We show this process in Appendix B),

$$z^{(l+1)}(t) \le -\int_t^{\infty} \beta q(s) z^{\alpha}(\sigma(s))(1-p(\sigma(s)))^{\alpha} \frac{(s-t)^{n-l-2}}{(n-l-2)!} ds \tag{2.3.14}$$

Using notation in (2.3.4), with the fact that $z(t)$ is increasing we obtain:

$$z^{(l+1)}(t)+a_l(t)z^{\alpha}(\sigma(t))\le 0 \tag{2.3.15}$$

Note, from (2.3.13), and notation (2.3.3), the inequality (2.3.15) holds for $l=n-1$.

Define

$$w_l(t)=\sigma^l(t)\frac{z^{(l)}(t)}{z^{\alpha}(\sigma(t))} \tag{2.3.16}$$

From (2.3.9) $z^{(l)}(t)>0$, then $w_l(t)>0$ and further

$$w_l'(t)=l\sigma^{l-1}(t)\sigma'(t)\frac{z^{(l)}(t)}{z^{\alpha}(\sigma(t))}+\sigma^l(t)\frac{z^{(l+1)}(t)}{z^{\alpha}(\sigma(t))}$$

$$-\sigma^l(t)\frac{\alpha z^{(l)}(t)z^{\alpha-1}(\sigma(t))}{z^{2\alpha}(\sigma(t))}z'(\sigma(t))\sigma'(t) \tag{2.3.17}$$

Using Lemma 2.3.1 for $n>2$, we have:

$$z'(t)\ge\int_{t_0}^{t}\frac{(t-s)^{l-2}}{(l-2)!}z^{(l)}(s)ds \tag{2.3.18}$$

The inequality (2.3.10) implies that $z^{(l+1)}(t)<0$, so $z^{(l)}(t)$ is decreasing. Therefore, for $t\ge s$ implies that $z^{(l)}(t)\le z^{(l)}(s)$. Using this fact and inequality (2.3.18), we obtain:

$$z'(t)\ge\int_{t_0}^{t}\frac{(t-s)^{l-2}}{(l-2)!}z^{(l)}(s)ds\ge z^{(l)}(t)\frac{(t-t_0)^{l-1}}{(l-1)!} \tag{2.3.19}$$

Hence (2.3.19) implies that for any $\lambda_l>1$,

$$z'(t)\ge\frac{1}{\lambda_l(l-1)!}t^{l-1}z^{(l)}(t),\quad \text{for large time} \tag{2.3.20}$$

Also (2.3.20) is satisfied for $n=2$. In this case $l=1$ and $\lambda_l=1$. Then from (2.3.20), and the fact that $z^{(l)}(t)$ is decreasing, we have:

$$z'(\sigma(t)) \geq \frac{1}{\lambda_l(l-1)!}\sigma^{l-1}(t)z^{(l)}(\sigma(t)) \geq \frac{1}{\lambda_l(l-1)!}\sigma^{l-1}(t)z^{(l)}(t) \qquad (2.3.21)$$

Use inequalities (2.3.15), and (2.3.21), with equation (2.3.17) we get:

$$w_l'(t) \leq l\sigma^{l-1}(t)\sigma'(t)\frac{z^{(l)}(t)}{z^{\alpha}(\sigma(t))} - \sigma^{l}(t)a_l(t)$$

$$-\frac{\alpha\sigma^{2l-1}(t)\sigma'(t)z^{\alpha-1}(\sigma(t))}{\lambda_l(l-1)!}\left[\frac{z^{(l)}(t)}{z^{\alpha}(\sigma(t))}\right]^2 \qquad (2.3.22)$$

Now Let us consider the following two sub-cases:

Case 1.1: If $z(t)$ is bounded eventually. Since $z(t) > 0$, $z'(t) > 0$, implies that $z''(t) < 0$ (otherwise $z(t)$ is unbounded), and this case is possible only if $l = 1$. Since $z(t) > 0$, $z'(t) > 0$, $z''(t) < 0$, follows that there exists:

$$\lim_{t\to\infty} z(t) = c > 0 \qquad (2.3.23)$$

Since $l = 1$ in this case, (2.3.15) will be:

$$z''(t) + a_1(t)z^{\alpha}(\sigma(t)) \leq 0 \qquad (2.3.24)$$

Integrating (2.3.24) from t to ∞, and using (2.3.23), we have:

$$L - z'(t) \leq -\int_t^{\infty} a_1(s)z^{\alpha}(\sigma(s))ds \qquad (2.3.25)$$

Where $L = \lim_{t\to\infty} z'(t)$. Since $z'(t) > 0$ eventually, (2.3.25) implies that:

$$-z'(t) \leq -\int_t^{\infty} a_1(s)z^{\alpha}(\sigma(s))ds \qquad (2.3.26)$$

Integrate (2.3.26) from t_1 to ∞, ($t_1 > t$), and taking into account (2.3.23) and monotonicity of $z^\alpha(\sigma(t))$, (we show this in Appendix C), The result is:

$$z(t) \le c - z^\alpha(\sigma(t))\int_{t_1}^{\infty} a_1(s)(s-t_1)ds \qquad (2.3.27)$$

But being $z(t)$ is bounded and $l = 1$, this is possible only if n is even. Therefore, combining condition (2.3.6) with inequality (2.3.27) lead to a contradiction to positivity of $z(t)$. Thus, this sub-case is impossible, and $z(t)$ must be unbounded.

Case 1.2: If $z(t)$ is unbounded. Then for every $K > 0$, and all sufficiently large t:

$$z^{\alpha-1}(\sigma(t)) \ge K \qquad (2.3.28)$$

Take $K = \dfrac{\lambda_l l^2 (l-1)!}{4\alpha M_l}$.

Combining (2.3.22) and (2.3.28) leads to

$$w_l'(t) \le -\sigma^l(t)a_l(t) + \frac{\lambda_l l^2(l-1)!\sigma'(t)}{4\alpha K\sigma(t)}$$

$$-\frac{\alpha K\sigma^{2l-1}(t)\sigma'(t)}{\lambda_l(l-1)!}\left[\frac{z^{(l)}(t)}{z^\alpha(\sigma(t))} - \frac{\lambda_l l(l-1)!}{2\alpha K\sigma^l(t)}\right]^2$$

$$\le -\sigma^l(t)a_l(t) + \frac{\lambda_l l^2(l-1)!\sigma'(t)}{4\alpha K\sigma(t)} \qquad (2.3.29)$$

Hence,

$$w_l'(t) \le -\sigma^l(t)a_l(t) + M_l\frac{\sigma'(t)}{\sigma(t)} \qquad (2.3.30)$$

Integrating (2.3.30) from t_1 to t we get

$$w_l(t) \le w_l(t_1) - \int_{t_1}^{t} \left[\sigma^l(s) a_l(s) - M_l \frac{\sigma'(s)}{\sigma(s)} \right] ds \tag{2.3.31}$$

Letting $t \to \infty$ in (2.3.31), and with use of condition $(2.3.5)_l$ we get $w_l(t) \to -\infty$. This contradicts to positivity of $w_l(t)$ and we conclude that this sub-case is also impossible. Hence, Case1 is impossible.

<u>*Case 2:*</u> Let $l = 0$. This case is possible only when n is odd. Therefore, for n even the proof of our theorem is complete, (so part (i) of the theorem is proved). To complete the proof we shall show that $\lim_{t\to\infty} x(t) = 0$.

Since $z(t) > x(t) > 0$, it is sufficient to verify that $\lim_{t\to\infty} z(t) = 0$.

Since $l = 0$, from (2.3.9), and (2.3.10), we have

$$(-1)^i z^{(i)}(t) > 0, \; 0 \le i \le n \tag{2.3.32}$$

Thus $z'(t) < 0$, and $z''(t) > 0$.This implies that $\lim_{t\to\infty} z(t)$ exists and is nonnegative and finite. Assume that $\lim_{t\to\infty} z(t) = c_1 > 0$. Then $z(t) > c_1$, eventually. Choose $0 < \varepsilon < c_1 \dfrac{1-p}{p}$. Obviously, $z(\tau(t)) < c_1 + \varepsilon$ for all large t.

Then from (2.3.7) we have:

$$x(t) > z(t) - p(t) z(\tau(t)) > c_1 - p(c_1 + \varepsilon) > \frac{c_1 + \varepsilon}{c_1 + \varepsilon} [c_1 - p(c_1 + \varepsilon)]$$

$$x(t) > (c_1 + \varepsilon) c_2 > c_2 z(t) \tag{2.3.33}$$

where $0 < c_2 = \dfrac{c_1 - p(c_1 + \varepsilon)}{c_1 + \varepsilon}$. Using inequality (2.3.33) with (2.3.8) and (4) we have

$$z^{(n)}(t)+c_2{}^{\alpha}\beta q(t)z^{\alpha}(\sigma(t))\le 0 \qquad (2.3.34)$$

Integrating an equality $z^{(n)}(t)=z^{(n)}(t)$ from t to ∞ $(n-1)$ times, from t_1 to ∞ once and using (2.3.32) , we get: (See details in Appendix D)

$$z(t_1)\ge -\int_{t_1}^{\infty}\frac{(u-t_1)^{n-1}}{(n-1)!}z^{(n)}(u)du \qquad (2.3.35)$$

Substituting (2.3.34) into (2.3.35) and using $z(\sigma(t))\ge c_1$ we obtain

$$z(t_1)\ge c_2{}^{\alpha}c_1{}^{\alpha}\beta\int_{t_1}^{\infty}\frac{(u-t_1)^{n-1}}{(n-1)!}q(u)du \qquad (2.3.36)$$

which implies

$$\int_{t_1}^{\infty}u^{n-1}q(u)du<\infty \qquad (2.3.37)$$

But in view of $(2.3.5)_{n-1}$ we have

$$\infty=\int_{t_1}^{\infty}\sigma^{n-1}(u)\beta\big(1-p(\sigma(u))\big)^{\alpha}q(u)du\le\int_{t_1}^{\infty}\beta u^{n-1}q(u)du \qquad (2.3.38)$$

which contradicts (2.3.37). Consequently, $\lim_{t\to\infty} z(t)=0$. And the proof is completed. □

Corollary 2.3.1: Assume that n=2, $\alpha>1$ for some $M_1>0$,

$$\int^{\infty}\left(\sigma(t)\beta\big(1-p(\sigma(t))\big)^{\alpha}q(t)-M_1\frac{\sigma'(t)}{\sigma(t)}\right)dt=\infty,$$

and

$$\int_{t_1}^{\infty}\beta\big(1-p(\sigma(s))\big)^{\alpha}q(s)(s-t_1)ds=\infty$$

Then equation (2.3.1) {of second order}is oscillatory.

Corollary 2.3.2: Assume that n=3, $\alpha > 1$, $p(t) \le p < 1$ and for some $M_2 > 0$,

$$\int^{\infty}\left(\sigma^2(t)\beta(1 - p(\sigma(t)))^{\alpha} q(t) - M_2 \frac{\sigma'(t)}{\sigma(t)}\right)dt = \infty,$$

Then any solution of equation (2.3.1) {of third order} is either oscillatory or tends to zero as $t \to \infty$.

Example 2.3.1: Consider the equation

$$\left(x(t) + \frac{1}{t}x(t-1)\right)'' + (t^2+1)(x(\sqrt{t}))^3 = 0, \quad t \ge 1 \tag{2.3.39}$$

By corollary 2.3.1, equation (2.3.39) is oscillatory .

Example 2.3.2: Consider the equation

$$\left(x(t) + e^{-t}x(t - \ln 2)\right)^{(3)} + 5e^t(x(\sqrt{t}))^3 = 0, \quad t \ge 1 \tag{2.3.40}$$

By corollary 2.3.2, every non-oscillatory solution of equation (2.3.40) tends to zero as $t \to \infty$.

Theorem 2.3.2: Assume that $\alpha = 1$, for all $l \in \{1,2,...,n-1\}$ such that $n + l$ is odd

$$\int^{\infty}\left(\sigma^l(t)a_l(t) - \frac{\lambda_l l^2(l-1)!\sigma'(t)}{4\sigma(t)}\right)dt = \infty \quad \text{for some} \quad \lambda_l > 1 \tag{2.3.41$_l$}$$

and for n odd $p(t) \le p < 1$. Then

i. If n is even, every solution of equation (2.3.1) is oscillatory.
ii. If n is odd, any solution of equation (2.3.1) is either oscillatory or tends to zero as $t \to \infty$.

Proof: We proceed the same as in proof of Theorem 2.3.1. Follow the same steps until (2.3.22). Substitute $\alpha = 1$, in (2.3.22), we have:

$$w_l'(t) \le l\sigma^{l-1}(t)\sigma'(t)\frac{z^{(l)}(t)}{z(\sigma(t))} - \sigma^l(t)a_l(t) \; - \frac{\sigma^{2l-1}(t)\sigma'(t)}{\lambda_l(l-1)!}\left[\frac{z^{(l)}(t)}{z(\sigma(t))}\right]^2$$

$$= -\sigma^l(t)a_l(t) + \frac{\lambda_l l^2(l-1)!\sigma'(t)}{4\sigma(t)} \; - \frac{\sigma^{2l-1}(t)\sigma'(t)}{\lambda_l(l-1)!}\left[\frac{z^{(l)}(t)}{z(\sigma(t))} - \frac{\lambda_l l(l-1)!}{2\sigma^l(t)}\right]^2$$

$$\le -\sigma^l(t)a_l(t) + \frac{\lambda_l l^2(l-1)!\sigma'(t)}{4\sigma(t)} \tag{2.3.42}$$

Integrating from t_1 to t, we get:

$$w_l(t) \le w_l(t_1) - \int_{t_1}^{t}\left[\sigma^l(s)a_l(s) - \frac{l^2\lambda_l(l-1)!\sigma'(s)}{4\sigma(s)}\right]ds \tag{2.3.43}$$

Letting $t \to \infty$ we get $w_l(t) \to -\infty$. This contradicts to positivity of $w_l(t)$ and we conclude that Case 1 is impossible. The rest of the proof is the same as that in the proof of Theorem 2.3.1. □

Remark 2.3.1: We obtain the constant λ_l in condition $(2.3.41)_l$ from (2.3.20). Note that the constant M_l in condition $(2.3.5)_l$ is arbitrary.

In next theorem, Theorem 2.3.3, we assume that the following conditions hold:

(1) $p(t), q(t) \in C([t_0, \infty), R_+)$, $t_0 \ge 0$, $0 \le p(t) \le 1$, $f \in C(R, R)$, $xf(x) > 0$ for $x \ne 0$ and $f(x)$ is nondecreasing on R.

(2) $\tau(t), \sigma(t) \in C([t_0, \infty), R)$, $\tau(t) \le t$, $\sigma(t) \le t$, $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$

Theorem 2.3.3: Assume that $n \ge 2$ is an even integer. If

$$\int_{t_0}^{\infty} q(s) f([1-p(\sigma(s))]c) ds = \infty \tag{2.3.44}$$

$$\int_{t_0}^{\infty} q(s) f(-[1-p(\sigma(s))]c) ds = -\infty \tag{2.3.45}$$

hold for every $c > 0$, then every solution of equation (2.3.1) is oscillatory.

Proof: Without loss of generality, Suppose that (2.3.1) has an eventually positive solution $x(t)$, say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$, for $t \geq t_1$ for some $t_1 \geq t_0$.(The proof is similar for $x(t)$ being eventually negative).

Set

$$z(t) = x(t) + p(t)x(\tau(t)) \tag{2.3.46}$$

Since $p(t)$ is nonnegative then $z(t) > 0$ for $t \geq t_1$.

From (2.3.1), and (2.3.46) we have:

$$z^{(n)}(t) = -q(t) f(x(\sigma(t))) \leq 0, \text{ for } t \geq t_1 \tag{2.3.47}$$

By lemma 1.6.1, there exists $t_2 \geq t_1$ such that $z(t) > 0$, $z'(t) > 0$, $z^{(n-1)}(t) > 0$, and $z^{(n)}(t) < 0$ for $t \geq t_2$. Thus $z(t)$ is increasing.

Choose $t_3 \geq t_2$ such that $\tau(t) \geq t_2$ for $t \geq t_3$. Then from (2.3.46) and the fact that $z(t)$ is increasing, we have:

$$\begin{aligned} x(t) &= z(t) - p(t)x(\tau(t)) \\ &\geq z(t) - p(t)z(\tau(t)) \\ &\geq (1-p(t))z(t), \quad t \geq t_3 \end{aligned} \tag{2.3.48}$$

Let $t_4 \geq t_3$ such that $\sigma(t) \geq t_3$ for $t \geq t_4$. Since $f(x)$ is nondecreasing, combining (2.3.47) with (2.3.48) we have:

$$z^{(n)}(t)+q(t)f\left([1-p(\sigma(t))]z(\sigma(t))\right)\le 0\,,\ t\ge t_4 \qquad (2.3.49)$$

Since $z(t)$ is increasing, $z(\sigma(t))\ge z(t_3)$ for $t\ge t_4$. Therefore we have:

$$z^{(n)}(t)+q(t)f\left([1-p(\sigma(t))]z(t_3)\right)\le 0\,,\qquad t\ge t_4 \qquad (2.3.50)$$

Integrating (2.3.50) from t_4 to t, we obtain

$$\int_{t_4}^{t} q(s)f\left([1-p(\sigma(s))]z(t_3)\right)ds \le -z^{(n-1)}(t)+z^{(n-1)}(t_4)\,,\ \ t\ge t_4 \qquad (2.3.51)$$

But $z^{(n-1)}(t)>0$ for $t\ge t_4$. Then (2.3.51) implies that:

$$\int_{t_4}^{t} q(s)f\left([1-p(\sigma(s))]z(t_3)\right)ds \le z^{(n-1)}(t_4)\,,\ \ t\ge t_4 \qquad (2.3.52)$$

This contradicts (2.3.44), and the proof is completed. □

Example 2.3.2: Consider the NDDE

$$\left[x(t)+\frac{1}{p}x(t-\pi)\right]^{(VI)}+\left(1-\frac{1}{p}\right)x(t-2\pi)=0\,, \qquad (2.3.53)$$

where $p>1$. It satisfies all conditions of Theorem 2.3.3. Thus every solution of equation (2.3.53) is oscillatory. For example, $x(t)=-\cos t$ is such a solution.

Remark 2.3.2: Theorems 2.3.1 and 2.3.2 are due to Lackova [28], they are extensions of Dzurina results [13]. Theorem 2.3.3 is a special case of Theorem 1 of Li [31]. Where Li in [31]establish sufficient conditions for oscillation of equation (2.3.1) when the derivative part contains several delays. Also the results of Theorem 2.3.3 still hold if the condition $\sigma(t)\le t$ is omitted.

Chapter Three

Bounded Oscillation of *n-th* Order NDDE When the Non-Derivative Part is Separable

3.0 Introduction

In this chapter we interested in the bounded oscillation of *n-th* order NDDE when the non-derivative part is separable, i.e. we consider an equation of the form:

$$[x(t)+p(t)x(\tau(t))]^{(n)}+\delta q(t)f(x(\sigma(t)))=h(t) \tag{3.0.1}$$

where, $\delta=\pm 1$, $p(t),q(t),\tau(t),\sigma(t),h(t)\in C([t_0,\infty),R)$, $t_0\geq 0$, $\tau(t)$ and $\sigma(t)$ are delayed arguments, and $f\in C(R,R)$.

Many results are known for the oscillation of bounded solutions of equation (3.0.1). It seems that most of these results consider cases when $p(t)=p\in R$ and $p(t)>0$ (or $p(t)<0$). And few results are known when the coefficient $p(t)$ is oscillatory. In this chapter we present some recent results for bounded oscillation of equation (3.0.1) for several cases of the coefficient $p(t)$.

In section 3.1 we introduce some results of oscillation of bounded solutions of equation (3.0.1), when the equation has positive or negative coefficients. In section 3.2 we study the oscillation of bounded solutions when the coefficient is nonnegative and delayed arguments are commute. Finally, In section 3.3 we present oscillation of bounded solutions when the equation has oscillating coefficients.

3.1 Bounded oscillation of *n-th* order NDDE with positive or negative coefficients

Consider the following NDDE:

$$[x(t) - p(t)x(\tau(t))]^{(n)} + \delta q(t) f(x(\sigma(t))) = h(t) \tag{3.1.1}$$

$$[x(t) + p(t)x(\tau(t))]^{(n)} + \delta q(t) f(x(\sigma(t))) = h(t) \tag{3.1.2}$$

Where $\delta = \pm 1$, and the following conditions are made for their use:

(1) $p(t), q(t) \in C([t_0, \infty), R_+)$, $t_0 \geq 0$

(2) $q(t)$ is not identically zero on any half-line of the form $[t_*, \infty)$ for any $t_* \geq t_0$,

(3) $\tau(t), \sigma(t) \in C([t_0, \infty), R_+)$, $\tau(t) < t$, $\sigma(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$, and $\tau(t)$ is monotone.

(4) $f \in C(R, R)$ such that $xf(x) > 0$ for $x \neq 0$.

(5) $h(t), r(t) \in C([t_0, \infty), R)$, and $r(t)$ is an oscillating function such that:

$\lim_{t\to\infty} r(t) = 0$ and $r^{(n)}(t) = h(t)$.

(6) $\int^{\infty} s^{n-1} q(s) ds = \infty$

In this section we present necessary and sufficient conditions under which solutions of equations (3.1.1) and (3.1.2) are either oscillatory or else satisfy $\lim_{t\to\infty} x(t) = 0$.

The following Lemma will be needed in this section

Lemma 3.1.1:(see lemma 1 in [37], and lemma 3 in [40])

Let $\tau(t)$ be a continuous monotone function with $\lim_{t\to\infty} \tau(t) = \infty$. Set

$$z(t) = x(t) + p(t)x(\tau(t)) \tag{3.1.3}$$

If $x(t)$ is eventually positive, $\lim_{t\to\infty} \inf x(t) = 0$ and $\lim_{t\to\infty} z(t) = L \in R$ exists. Then $L = 0$ provided that for some real numbers p_1, p_2, p_3, and p_4 the function $p(t)$ is in one of the following ranges:

i. $p_1 \le p(t) \le 0$

ii. $0 \le p(t) \le p_2 < 1$

iii. $1 < p_3 \le p(t) \le p_4$

Proof: By (3.1.3) we have

$$z(\tau^{-1}(t)) - z(t) = x(\tau^{-1}(t)) + p(\tau^{-1}(t))x(t) - x(t) - p(t)x(\tau(t))$$

Therefore,

$$\lim_{t\to\infty} \{x(\tau^{-1}(t)) + p(\tau^{-1}(t))x(t) - x(t) - p(t)x(\tau(t))\} = 0 \tag{3.1.4}$$

Let $\{t_n\}$ be a sequence of real numbers such that

$$\lim_{n\to\infty} t_n = \infty \quad \text{and} \quad \lim_{n\to\infty} x(t_n) = 0 \tag{3.1.5}$$

From (3.1.4) and (3.1.5) we obtain

$$\lim_{n\to\infty} \left\{x(\tau^{-1}(t_n)) - p(t_n)x(\tau(t_n))\right\} = 0 \tag{3.1.6}$$

Since $x(t)$ is eventually positive, $x(\tau^{-1}(t_n)) > 0$.

If (i) holds, then $-p(t_n)x(\tau(t_n)) \ge 0$, and it follows from (3.1.6) that:

$$\lim_{n\to\infty} x(\tau^{-1}(t_n)) = 0 \tag{3.1.7}$$

And so from (3.1.3), and (3.1.7)

$$L = \lim_{n\to\infty} z(\tau^{-1}(t_n)) = \lim_{n\to\infty} \{x(\tau^{-1}(t_n)) + p(\tau^{-1}(t_n))x(t_n)\} = 0$$

In the case of (ii) by replacing t by $\tau(t)$ in (3.1.4) and using (3.1.5), we have:

$$\lim_{n\to\infty} \{[p(t_n)-1]x(\tau(t_n)) - p(\tau(t_n))x(\tau(\tau(t_n)))\} = 0 \tag{3.1.8}$$

As $[p(t_n)-1]x(\tau(t_n)) < 0$, and $-p(\tau(t_n))x(\tau(\tau(t_n))) \le 0$, it follows that:

$$\lim_{n\to\infty} x(\tau(t_n)) = 0 \tag{3.1.9}$$

Then

$$L = \lim_{n\to\infty} z(t_n) = \lim_{n\to\infty} \{x(t_n) + p(t_n)x(\tau(t_n))\} = 0$$

Finally , if (iii) holds, then by replacing t by $\tau^{-1}(t)$ in (3.1.4) we obtain

$$\lim_{n\to\infty} \{x(\tau^{-1}(\tau^{-1}(t_n))) + [p(\tau^{-1}(\tau^{-1}(t_n))) - 1]x(\tau^{-1}(t_n))\} = 0$$

implies that

$$\lim_{n\to\infty} x(\tau^{-1}(t_n)) = 0 \tag{3.1.10}$$

Then

$$L = \lim_{n\to\infty} z(\tau^{-1}(t_n)) = \{x(\tau^{-1}(t_n)) + p(\tau^{-1}(t_n))x(t_n)\} = 0$$

Thus the proof is complete. □

Theorem 3.1.1: Let (1)–(6) be satisfied**,** and $\inf_{t\ge t_0}[t-\tau(t)] > 0$ .If there exist positive numbers p_1 and p such that $p(t)$ satisfies $1 < p \le p(t) \le p_1 < \infty$.

Then

i. every bounded solution $x(t)$ of equation (3.1.1) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$, when $(-1)^n\delta = 1$

ii. every bounded solution $x(t)$ of equation (3.1.1) is oscillatory, when $(-1)^n \delta = -1$

Proof: Let $x(t)$ be a non-oscillatory bounded solution of equation (3.1.1). Without loss of generality, we may assume that $x(t)$ is eventually positive (The proof is similar when $x(t)$ is eventually negative). Set

$$z(t) = x(t) - p(t)x(\tau(t)) - r(t) \tag{3.1.11}$$

From (3.1.1) and (3.1.11) we have

$$z^{(n)}(t) = -\delta q(t) f(x(\sigma(t))) \tag{3.1.12}$$

Therefore $z^{(n)}(t)$ is of constant sign eventually. Hence, $z(t)$ must be monotonic and of constant sign eventually, that is $z(t) > 0$ or $z(t) < 0$ eventually. If $z(t) > 0$. By (3.1.11) $x(t) - p(t)x(\tau(t)) > r(t)$. Therefore $x(t) - p(t)x(\tau(t)) > 0$ (otherwise contradicts $r(t)$ being oscillatory). Hence $x(t) > p(t)x(\tau(t))$, But $p(t) \geq p$. Follows that

$$x(t) > p(t)x(\tau(t)) > px(\tau(t)) = px(t-(t-\tau(t))) \tag{3.1.13}$$

Assume that $\beta(t) = t - \tau(t)$, then

Let $\beta_0 = \inf_{t \geq t_0} \beta(t) = \inf_{t \geq t_0} [t - \tau(t)] > 0$, then from (3.1.13) we have:

$$x(t) > px(t-\beta_0) \tag{3.1.14}$$

$$x(t-\beta_0) > px(t-2\beta_0)$$

Using (3.1.14), we have:

$$x(t) > p^2 x(t-2\beta_0)$$

So for every positive integer n, we obtain:

$$x(t) > p^n x(t - n\beta_0), \quad \text{or}$$

$$x(t + n\beta_0) > p^n x(t) \tag{3.1.15}$$

By using the fact that $\beta_0 = \inf_{t \ge t_0}[t - \tau(t)] > 0$, inequality (3.1.15) implies that $x(t) \to \infty$ as $t \to \infty$, and this contradicts that $x(t)$ being bounded. Hence $z(t) < 0$. Moreover, $z(t)$ is bounded; since $x(t)$, and $p(t)$ are bounded, and $\lim_{t \to \infty} r(t) = 0$.

From (3.1.12), we have

$$\delta z^{(n)}(t) = -q(t) f(x(\sigma(t)) \le 0 \tag{3.1.16}$$

Thus $\delta z^{(n-1)}(t)$ is decreasing function for $t \ge t_1$, for some $t_1 \ge t_0$, and so we can have:

$$\delta z^{(n-1)}(t) > 0 \quad \text{for} \quad t \ge t_1 \tag{3.1.17}$$

$$\delta z^{(n-1)}(t) < 0 \quad \text{for} \quad t \ge T_1 \ge t_1 \tag{3.1.18}$$

suppose that (3.1.18) holds. Then

$$\delta z^{(n-1)}(t) \le \delta z^{(n-1)}(T_1) < 0 \quad \text{for} \quad t \ge T_1 \tag{3.1.19}$$

Integrating (3.1.19) from T_1 to t, we get

$$\delta z^{(n-2)}(t) \to -\infty \quad \text{as} \quad t \to \infty \tag{3.1.20}$$

Therefore, (3.1.16), (3.1.18), and (3.1.20) lead to $\delta z(t) \to -\infty$ as $t \to \infty$. But this contradicts to $z(t)$ being bounded, and so (3.1.17) hold.

If $\underline{\delta = 1}$, then (3.1.16), and (3.1.17) implies that $z^{(n)}(t) \le 0$, and $z^{(n-1)}(t) > 0$. By Lemma 1.6.2, there exists a t_1 and a number $l \in \{0,1\}$ with $n-l$ is even, such that for all $t \ge t_1$

$$z^{(i)}(t) < 0, \qquad i = 0,1,...,l, \tag{3.1.21}$$

$$(-1)^{i-l} z^{(i)}(t) < 0, \qquad i = l,...,n-1. \tag{3.1.22}$$

Note that $l \in \{0,1\}$. If n is even $l = 0$; (otherwise $l \ge 2$, so $z(t) < 0$, $z'(t) < 0$, and $z''(t) < 0$, this implies that $z(t)$ is unbounded). If n is odd $l = 1$; (otherwise $l \ge 3$, so $z(t) < 0$, $z'(t) < 0$, and $z''(t) < 0$, this implies that $z(t)$ is unbounded).

If $\underline{\delta = -1}$, then $z^{(n)}(t) \ge 0$, by Lemma 1.6.1, there exists a t_1 and a number $l \in \{0,1\}$(otherwise $z(t)$ is unbounded) ,with $n-l$ is odd, such that for all $t \ge t_1$ (3.1.21), and (3.1.22) are satisfied.

So, in general, there exists t_1 and number $l \in \{0,1\}$ with

$$(-1)^{n-l}\delta = 1 \tag{3.1.23}$$

such that (3.1.21), and (3.1.22) are satisfied for $t \ge t_1$.

Now integrate equation (3.1.16) from t to ∞ and see that:

$$L - \delta z^{(n-1)}(t) + \int_t^\infty q(s) f(x(\sigma(s))ds = 0 \tag{3.1.24}$$

where $L = \lim_{t\to\infty} \delta z^{(n-1)}(\infty)$. From (3.1.17) $L > 0$, so

$$-\delta z^{(n-1)}(t) + \int_t^\infty q(s) f(x(\sigma(s))ds \le 0 \tag{3.1.25}$$

Now by (3.1.21), and (3.1.22) (see Lemma 1.6.4), we have

$$\lim_{t\to\infty} z^{(i)}(t) = 0 \quad \text{for } \; i = 1,2,\ldots,n-2 \tag{3.1.26}$$

Now, if we integrate (3.1.25) $(n-2)$ times from t to ∞,we get

$$(-1)^{n-1}\delta z'(t) + \frac{1}{(n-2)!}\int_t^\infty q(s)(s-t)^{n-2} f(x(\sigma(s))ds \le 0 \tag{3.1.27}$$

Let $L_1 = \lim_{t\to\infty} z(t)$ and integrating (3.1.27) from T to ∞, we obtain

$$\frac{1}{(n-1)!}\int_T^\infty q(s)(s-T)^{n-1} f(x(\sigma(s))ds \le (-1)^n \delta\left[L_1 - z(T)\right] \tag{3.1.28}$$

Note that we evaluate the integrals (3.1.27), and (3.1.28) using integration by parts in view of (3.1.26). (This is similar to work in Appendix B).

In view of (6) and the fact that $z(t)$ is bounded, from (3.1.28) we conclude that $\lim_{t\to\infty} \inf f(x(t)) = 0$, or

$$\lim_{t\to\infty} \inf x(t) = 0 \tag{3.1.29}$$

Let $(-1)^n \delta = 1$. We shall now proceed to show that $\lim_{t\to\infty} x(t) = 0$

From (3.1.23) $l = 0$. So (3.1.21), and (3.1.22) implies that $z'(t) > 0$, and $z''(t) < 0$. Hence, $z(t)$ approaches to a finite limit as t tends infinity. Hence, by Lemma 3.1.1 $\lim_{t\to\infty} z(t) = L_1 = 0$. Since $z(t) < 0$ and $z(t) \to 0$ as $t \to \infty$, given $\varepsilon > 0$ there exists a T_* such that

$$z(t) > -\varepsilon, \quad \text{for all } t \ge T_*.$$

So, $\quad x(t) - p(t)x(\tau(t)) > -\varepsilon + r(t)$

Since $\lim_{t\to\infty} r(t) = 0$, there exists $\lambda \in (0,1)$ such that

$\lambda\big(x(t) - p(t)x(\tau(t))\big) > -\varepsilon$, so

$x(t) - p(t)x(\tau(t)) > -\dfrac{\varepsilon}{\lambda}$, Let $\dfrac{\varepsilon}{\lambda} = \mu$. Then

$p(t)x(\tau(t)) < \mu + x(t)$, use $\beta(t) = t - \tau(t)$, then

$p(t)x(t - \beta_0) < \mu + x(t)$, use $p(t) \geq p$, so

$px(t) < \mu + x(t + \beta_0)$

$p^2 x(t) < \mu + p\mu + x(t + 2\beta_0)$,

$\vdots$

$$p^n x(t) < \mu + p\mu + ... + p^{n-1}\mu + x(t + n\beta_0) \qquad (3.1.30)$$

Letting M be a bound of $x(t)$ and simplifying (3.1.30), we obtain that

$$x(t) < \mu \frac{p^{-n} - 1}{1 - p} + Mp^{-n} = \frac{\varepsilon}{\lambda}\frac{p^{-n} - 1}{1 - p} + Mp^{-n} \qquad (3.1.31)$$

Because p^{-n} goes to zero as n tends to infinity, and ε is arbitrary, from (3.1.31) we have $x(t) \to 0$ as $t \to \infty$ as desired.

Suppose that $(-1)^n \delta = -1$. From (3.1.23) $l = 1$, Because $z(t)$ is bounded and $l = 1$, $\lim_{t \to \infty} z(t)$ exists. In view of (3.1.29), it follows from Lemma 3.1.1 that $z(t) \to 0$ as $t \to \infty$. But this contradicts the fact that $z(t)$ is negative and decreasing, and hence proves that $x(t)$ is oscillatory.□

Example 3.1.1: Consider the equation

$$\left[x(t) - (5 - e^{-t})x(t - 2\pi)\right]'' + 2(3 + \cos 2t)\frac{x(t - \pi)}{1 + x^2(t - \pi)} = 2e^{-t}\sin t \qquad (3.1.32)$$

Here $n = 2$, $\delta = +1$ $p(t) = 5 - e^{-t}$, $q(t) = 2(3 + \cos 2t)$, $f(x) = \frac{x}{1+x^2}$, $\tau(t) = t - 2\pi$, $\sigma(t) = t - \pi$, and $h(t) = 2e^{-t}\sin t$. See that $4 \le p(t) < 5$, $\inf_{t \ge t_0}[t - \tau(t)] = 2\pi > 0$. And it is easy to see that all conditions of Theorem 3.1.1 (i) are fulfilled, so every bounded solution of equation (3.1.32) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$. Indeed, $x(t) = \cos t$ is an oscillatory solution of the equation.

Example 3.1.2: Consider the equation

$$\left[x(t) - \frac{3}{2}x(t - \ln 8)\right]''' - \frac{2e^{\frac{2t}{3}}}{27t}x^3(t - \ln t) = 0 \tag{3.1.33}$$

it satisfies all conditions of Theorem 3.1.1 (i), so every bounded solution of equation (3.1.33) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$. For example, $x(t) = e^{-\frac{t}{3}}$ is a solution of the equation, which tends to zero as $t \to \infty$.

Example 3.1.3: Consider the equation

$$\left[x(t) - ex(t - \frac{\pi}{2})\right]' + (1+e)(3 + \cos 4t)\frac{x(t - \frac{\pi}{4})}{1 + x^2(t - \frac{\pi}{4})} = 0 \tag{3.1.34}$$

it satisfies all conditions of Theorem 3.1.1 (ii), so every bounded solution of equation (3.1.34) is oscillatory. Indeed, $x(t) = \sin 2t$ is such a solution of the equation.

Theorem 3.1.2: Let (1)-(6) be satisfied. And assume that $0 \le p(t) \le p_2 < 1$ or $1 < p_3 \le p(t) \le p_4$ for some real numbers p_2, p_3 and p_4.

Then

i. every bounded solution $x(t)$ of equation (3.1.2) is oscillatory, when $(-1)^n \delta = 1$

ii. every bounded solution $x(t)$ of equation (3.1.2) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$ when $(-1)^n \delta = -1$

Proof: Let $x(t)$ be a non-oscillatory bounded solution of equation (3.1.2). Without any loss of generality we may assume that $x(t)$ is eventually positive. (The proof is similar when $x(t)$ is eventually negative). Set

$$z(t) = x(t) + p(t)x(\tau(t)) - r(t) \tag{3.1.35}$$

From (3.1.2) and (3.1.35) we have:

$$z^{(n)}(t) = -\delta q(t) f(x(\sigma(t))) \tag{3.1.36}$$

Therefore $z^{(n)}(t)$ is of constant sign eventually. Hence, $z(t)$ must be monotonic and of constant sign eventually. If $z(t) < 0$. By (3.1.35) $x(t) + p(t)x(\tau(t)) < r(t)$. Implies that, $r(t) > 0$, and this contradicts $r(t)$ being oscillatory. Therefore $z(t) > 0$. Moreover, $z(t)$ is bounded; since $x(t)$, and $p(t)$ are bounded, and $\lim_{t\to\infty} r(t) = 0$.

If we proceed in a similar way to that in proof of Theorem 3.1.1, (with use of Lemmas 1.6.1, and 1.6.2). Then we conclude that there exists a t_1 and a number $l \in \{0,1\}$ with

$$(-1)^{n-l} \delta = -1 \tag{3.1.37}$$

such that for all $t \geq t_1$.

$$z^{(i)}(t) > 0, \qquad i = 0,1,\ldots,l, \tag{3.1.38}$$

$$(-1)^{i-l} z^{(i)}(t) > 0, \quad i = l,\ldots,n-1. \tag{3.1.39}$$

Now by (3.1.38), and (3.1.39) (see Lemma 1.6.4), we have:

$$\lim_{t\to\infty} z^{(i)}(t) = 0 \quad \text{for } i = 1,2,\ldots,n-2 \tag{3.1.40}$$

Integrate equation (3.1.36) from t to ∞ and see that:

$$L - \delta z^{(n-1)}(t) + \int_t^\infty q(s) f(x(\sigma(s))ds = 0 \tag{3.1.41}$$

where $L = \lim_{t\to\infty} \delta z^{(n-1)}(\infty)$. From (3.1.39) $L > 0$, so

$$-\delta z^{(n-1)}(t) + \int_t^\infty q(s) f(x(\sigma(s))ds \leq 0 \tag{3.1.42}$$

Then, in view of (3.1.40), by integrating equation (3.1.42) $(n-2)$ times from t to ∞,we get

$$(-1)^{n-1}\delta z'(t) + \frac{1}{(n-2)!}\int_t^\infty q(s)(s-t)^{n-2} f(x(\sigma(s))ds \leq 0 \tag{3.1.43}$$

Let $L_1 = \lim_{t\to\infty} z(t)$ and integrating (3.1.43) from T to ∞, we obtain

$$\frac{1}{(n-1)!}\int_T^\infty q(s)(s-T)^{n-1} f(x(\sigma(s))ds \leq (-1)^n \delta\left[L_1 - z(T)\right] \tag{3.1.44}$$

Note that: we get integrals (3.1.43), and (3.1.44) using integration by parts. (This is similar to the work in appendix B).

In view of (6) and the fact that $z(t)$ is bounded, follows from(3.1.44) that

$\lim_{t\to\infty} \inf f(x(t)) = 0$, or

$$\lim_{t\to\infty}\inf x(t)=0 \qquad (3.1.45)$$

Let $(-1)^n\delta=-1$. We shall now proceed to show that $\lim_{t\to\infty} x(t)=0$

From (3.1.37) $l=0$. So (3.1.38), and (3.1.39) implies that $z'(t)<0$, and $z''(t)>0$. Hence, $z(t)$ approaches to a finite limit as t tends infinity. By Lemma 3.1.1, $\lim_{t\to\infty} z(t)=L_1=0$. From (3.1.35), $x(t)+p(t)x(\tau(t))=z(t)+r(t)$ and since $\lim_{t\to\infty} r(t)=\lim_{t\to\infty} z(t)=0$, then $\lim_{t\to\infty} x(t)=0$.

Let $(-1)^n\delta=1$, From (3.1.37) $l=1$. So (3.1.38), and (3.1.39) implies that $z'(t)>0$, and $z''(t)<0$. Since $z(t)$ is bounded , $\lim_{t\to\infty} z(t)$ exists. it follows from Lemma 3.1.1 that $z(t)\to 0$ as $t\to\infty$. But this contradicts the fact that $z(t)$ is positive and increasing, and hence proves that $x(t)$ is Oscillatory. □

Example 3.1.4: Consider the equation

$$\left[x(t)+(6+e^{-t})x(t-\pi)\right]''+5(1+\sin^2 t)\frac{x(t-\pi)}{1+x^2(t-\pi)}=2e^{-t}\cos t \qquad (3.1.46)$$

it satisfies all conditions of Theorem 3.1.2 (i), so every bounded solution of equation (3.1.46) is oscillatory. Indeed, $x(t)=\sin t$ is an oscillatory solution of the equation.

Example 3.1.5: Consider the equation

$$\left[x(t)+\frac{2}{3}x(t-5\ln 3)\right]^{III}+\frac{3e^{\frac{4t}{5}}}{125t}x^5(t-\ln t)=0 \qquad (3.1.47)$$

it satisfies all conditions of Theorem 3.1.2 (ii), so every bounded solution of equation (3.1.47) is either oscillatory or satisfies $\lim_{t\to\infty} x(t)=0$.For example, $x(t)=e^{-\frac{t}{5}}$ is a solution of the equation, which tends to zero as $t\to\infty$.

Example 3.1.6: Consider the equation

$$\left[x(t)+(5+2\sin t)x(\frac{t}{2}-\frac{\pi}{4})\right]^{III}+5(1+\sin^2 t)\frac{x(\frac{t}{2})}{1+x^2(\frac{t}{2})}=0 \qquad (3.1.48)$$

it satisfies all conditions of Theorem 3.1.2 (ii), so every bounded solution of equation (3.1.48) is oscillatory or satisfies $\lim_{t\to\infty} x(t)=0$. In fact, $x(t)=\sin 2t$ is such a solution of the equation.

In the next theorem, Theorem 3.1.3, we present the conditions under which equations (3.1.1), and (3.1.2) have non-oscillatory solutions.

The following conditions are made to be used:

(7) $\int^{\infty} s^{n-1}|h(s)|ds<\infty$

(8) $\int^{\infty} s^{n-1}q(s)ds<\infty$

Theorem 3.1.3: Let (1), (3), (7) and (8) be satisfied, and assume that the function f satisfies the Lipschitz condition with a constant L on an interval $[a,b]$, where a and b are positive real numbers which depend on the range of $p(t)$ and may chosen as follows:

i. $\frac{a}{b}<\frac{1-p}{1-p_1}$, when $1<p\le p(t)\le p_1$ in equation (3.1.1)

ii. $\frac{a}{b}<1-p_2$, when $0\le p(t)\le p_2<1$, in equation (3.1.2)

iii. $\frac{a}{b} < \frac{p_3 - 1}{p_4 - 1}$, when $1 < p_3 \le p(t) \le p_4$, in equation (3.1.2)

where p, p_1, p_2, p_3, and p_4 are real numbers.

Then equations (3.1.1), and (3.1.2) have bounded positive solutions.

Proof: see [37]. □

Example 3.1.7: Consider the equation

$$\left[x(t) + (4 + e^{-2t})x(\frac{t}{2})\right]'' - 16e^{-2t}(2 + 2e^{-2t} + e^{-4t})\frac{2x(\frac{t}{2})}{1 + x^2(\frac{t}{2})} = -12e^{-2t} \qquad (3.1.49)$$

$p(t) = 4 + e^{-2t}$, take $p_3 = 4$ and $p_4 = 5$ then (iii) will be satisfied when $4a < 3b$. Choose $a = 1$ and $b = 2$. Then Theorem 3.1.3 implies that there is a positive solution $x(t) \in [1,2]$. Note that $x(t) = 1 + e^{-4t}$ is a solution of the equation.

Finally, In view of Theorem 3.1.3, we obtain the following necessary and sufficient conditions for oscillation of equations (3.1.1), and (3.1.2).

Theorem 3.1.4: Let (1)-(5), and (7) be satisfied. If f satisfies the Lipschitz condition on an interval $[a,b]$, where a and b are as in Theorem 3.1.3. Then the conclusions of Theorems 3.1.1 and 3.1.2 hold if and only if (6) is satisfied.

Remark 3.1.1: All theorems in this section are due to Zafer and Dahiya [40], and Yilmaz and Zafer [37]. We collect several theorems from [3], and [37] in two main theorems (Theorems 3.1.1, and 3.1.2).

3.2 Bounded Oscillation of *n-th* order NDDE with nonnegative coefficient and commuting delayed arguments

In this section we assume that the delayed arguments are commute, i.e. $\tau(\sigma(t)) = \sigma(\tau(t))$.

Consider the following equation:

$$[x(t) + p(t)x(\tau(t))]^{(n)} - q(t)f(x(\sigma(t))) = 0 \tag{3.2.1}$$

where, $p(t), q(t), \tau(t), \sigma(t) \in C([t_0, \infty), R_+)$, $t_0 \geq 0$, $q(t) \not\equiv 0$ on any half line $[t_*, \infty)$ for any $t_* \geq t_0$, $\tau(t) \leq t$, $\sigma(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \infty$, $\lim_{t\to\infty} \sigma(t) = \infty$, and $f \in C(R, R)$ such that $xf(x) > 0$ for $x \neq 0$.

Further, assume that the following assumptions are to be hold:

(1) $f(u+v) \leq f(u) + f(v),$ if $u, v > 0$,

(2) $f(u+v) \geq f(u) + f(v),$ if $u, v < 0$,

(3) $f(ku) \leq kf(u),$ if $k \geq 0$ and $u > 0$, and

$f(ku) \geq kf(u),$ if $k \geq 0$ and $u < 0$, for each $k \in K$

where $K = \{k : p(t) = k$ for some $t \in [t_0, \infty)\}$.

(4) $f(u)$ is bounded away from zero if u is bounded away from zero

(5) $p(t)$ is bounded

(6) $\int_{t_0}^{\infty} q(s)ds = \infty$

(7) $\tau(t) \in C^1([t_0, \infty), R_+)$ and $\tau'(t) \geq b$, where b is positive constant.

(8) There exists a positive constant M such that $p(\sigma(t))q(t) \leq Mq(\tau(t))$

Theorem 3.2.1: Suppose that conditions (1)-(8) hold. Then:

i. If n is even, any bounded solution of (3.2.1) is either oscillatory or tends to zero as $t \to \infty$

ii. If n is odd, every bounded solution of (3.2.1) is oscillatory.

Proof: Suppose that $x(t)$ is a bounded and non-oscillatory solution of equation (3.2.1). Without loss of generality, we may assume that $x(t)$ is eventually positive, (The proof is similar when $x(t)$ is eventually negative), say $x(t) > 0$, $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$, and $x(\tau(\sigma(t))) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Set

$$z(t) = x(t) + p(t)x(\tau(t)) \tag{3.2.2}$$

Since $p(t)$ is nonnegative then $z(t) > 0$ for $t \geq t_1$. And since $x(t)$, and $p(t)$ are bounded, then $z(t)$ is also bounded From (3.2.1) and (3.2.2) we have

$$z^{(n)}(t) = q(t) f(x(\sigma(t))) \tag{3.2.3}$$

That is $z^{(n)}(t) \geq 0$, implies that $z^{(i)}(t)$ is monotonic and of constant sign eventually for $i = 0,1,...,n-1$.

Using the fact that $\tau(\sigma(t)) = \sigma(\tau(t))$ for $t \geq t_1$, and from (1), and (3) we have:

$$\begin{aligned} f(z(\sigma(t))) &= f\big(x(\sigma(t)) + p(\sigma(t))x(\tau(\sigma(t)))\big) \\ &\leq f(x(\sigma(t))) + p(\sigma(t)) f(x(\sigma(\tau(t)))), \end{aligned} \tag{3.2.4}$$

Hence

$$q(t) f(z(\sigma(t))) \leq q(t) f(x(\sigma(t))) + q(t) p(\sigma(t)) f(x(\sigma(\tau(t)))) \tag{3.2.5}$$

Now, using (3.2.3), and (3.2.5) with (8) we obtain the following:

$$q(t)f(z(\sigma(t))) \le z^{(n)}(t) + Mq(\tau(t))f(x(\sigma(\tau(t)))) \qquad (3.2.6)$$

If $n \ge 2$, then $z^{(n-1)}(t) < 0$ eventually, say for $t \ge t_2 \ge t_1$; since $z^{(n-1)}(t) > 0$, $z^{(n)}(t) \ge 0$ and $q(t) \not\equiv 0$ imply that $z(t) \to \infty$ as $t \to \infty$, and this contradicts $z(t)$ being bounded.

Replace t by $\tau(t)$ in (3.2.3), and multiply by $\tau'(t)$, we have:

$$z^{(n)}(\tau(t))\tau'(t) - q(\tau(t))f(x(\sigma(\tau(t))))\tau'(t) = 0 \qquad (3.2.7)$$

Let $t_3 \ge t_2$ such that $z^{(n-1)}(\tau(t)) < 0$ for $t \ge t_3$. Then integrate (3.2.7) from t_3 to ∞, we get:

$$\int_{t_3}^{\infty} q(\tau(s))f(x(\sigma(\tau(s))))\tau'(s)ds = L - z^{(n-1)}(t_3).$$

where $L = \lim_{t\to\infty} z^{(n-1)}(t)$. Since $z^{(n-1)}(t) < 0$ eventually, we show that:

$$\int_{t_3}^{\infty} q(\tau(s))f(x(\sigma(\tau(s))))\tau'(s)ds < \infty \qquad (3.2.8)$$

Using (3.2.8), with (7), and (8), follows that

$$M\int_{t_3}^{\infty} q(\tau(s))f(x(\sigma(\tau(s))))ds < \infty \qquad (3.2.9)$$

Integrate (3.2.6) , and use (3.2.9) we show that

$$\int_{t_3}^{\infty} q(s)f(z(\sigma(s)))ds < \infty \qquad (3.2.10)$$

Since (4) and (6) hold, then (3.2.10) implies that $\liminf_{t\to\infty} z(t) = 0$. And since $z(t)$ is monotonic, $z(t) \to 0$ as $t \to \infty$. So $z(t)$ is decreasing, implies that $z'(t) \le 0$ eventually. For $n > 1$, $z'(t) \to 0$ as $t \to \infty$ since $z'(t)$ is monotonic and $z(t) > 0$ ($z(t)$ concave up) (you can see lemma 1.6.4). Hence $z'(t)$ is increasing, implies that $z''(t) > 0$. For $n > 2$, $z''(t)$ is eventually positive and satisfies $z''(t) \to 0$ as $t \to \infty$ since $z'(t)$ is monotonic and negative. Continuing in this manner we have:

$$z^{(i)}(t)z^{(i+1)}(t) \le 0, \text{ for } i = 0,1,\dots,n-1 \tag{3.2.11}$$

If n is even, then $x(t) < z(t) \to 0$ as $t \to \infty$ and (i) is proved. If $n \ge 3$ is odd, then (3.2.11) and the fact that $z^{(n)}(t) \ge 0$ contradicts $z(t) > 0$. If $n = 1$, i.e. $z^{(n)}(t) = z'(t) \ge 0$ so there exists $\varepsilon > 0$ and $T \ge t_1$ such that $z(\sigma(t)) \ge \varepsilon$ for $t \ge T$. From (7), and equation (3.2.1) we have

$$z'(\tau(t))\tau'(t) \ge bq(\tau(t))f(x(\sigma(\tau(t)))) \tag{3.2.12}$$

Integrating (3.2.12) and using that fact that $z(t)$ is bounded, we have:

$$\int_T^\infty q(\tau(s))f(x(\sigma(\tau(s))))ds < \infty \tag{3.2.13}$$

Then, by integration of (3.2.6) we obtain

$$z(t) + M\int_T^t q(\tau(s))f(x(\sigma(\tau(s))))ds \ge z(T) + \varepsilon\int_T^t q(s)ds \tag{3.2.14}$$

Let $t \to \infty$ in (3.2.14) . The left side of (3.2.14) in view of (3.2.13) and boundedness of $z(t)$ must be finite. But from (6) the right side of (3.2.14) tends to infinity. which is a contradiction. Thus the proof is complete □

Example 3.2.1: The NDDE

$$[x(t) + 3x(t-\pi)]''' - 2x(t - \frac{3}{2}\pi) = 0 \tag{3.2.15}$$

satisfies the conditions of Theorem 3.2.1 (ii). So every bounded solution of (3.2.15) is oscillatory. Indeed, (3.2.15) has oscillatory solutions $x_1(t) = \sin t$, $x_2(t) = \cos t$

Example 3.2.2: The NDDE

$$\left[x(t) + 2x(t - \frac{\pi}{2})\right]'' - 4x(t - \pi) = 0 \qquad (3.2.16)$$

satisfies the conditions of Theorem 3.2.1 (i). So every bounded solution of (3.2.16) is either oscillatory or tends to zero as $t \to \infty$. In fact, (3.2.16) has oscillatory solutions $x_1(t) = \sin 2t$, $x_2(t) = \cos 2t$

Example 3.2.3: The NDDE

$$[x(t) + px(t - \ln 2)]^{(n)} - \left[\frac{1+2p}{3}\right]x(t - \ln 3) = 0 \qquad (3.2.17)$$

where n is even, $p > 0$. All conditions of Theorem 3.2.1 (i) are satisfied. So every solution of (3.2.17) is either oscillatory or tends to zero as $t \to \infty$. In fact, (3.2.17) has the non-oscillatory solution $x(t) = e^{-t} \to 0$ as $t \to \infty$.

The following example shows that the condition of boundedness cannot be dropped.

Example 3.2.4: The NDDE

$$[x(t) + 2x(t-1)]^{(n)} - (e+2)e^{\frac{2}{3}(t-1)} x^{\frac{1}{3}}(t-1) = 0 \qquad (3.2.18)$$

has unbounded solution $x(t) = e^t$, which is non-oscillatory.

Remark 3.2.1: Theorem 3.2.1 is due to Graef and Spikes [19].

3.3 Bounded oscillation of *n-th* order NDDE with oscillating coefficients

In this section we shall study the oscillatory behavior of bounded solutions of NDDE of the form

$$[x(t)+p(t)x(\tau(t))]^{(n)}+q(t)f(x(\sigma(t)))=h(t) \qquad (3.3.1)$$

where $n\geq 2$, and the following conditions are assumed to hold:

(1) $p(t),q(t),\tau(t),h(t)\in C([t_0,\infty),R)$, $t_0\geq 0$

(2) $p(t)$ and $h(t)$ are oscillating functions

(3) $q(t)$ is nonnegative

(4) $\sigma(t)\in C^1([t_0,\infty),R)$, $\sigma'(t)>0$, $\sigma(t)\leq t$, $\tau(t)\leq t$, and $\lim\limits_{t\to\infty}\tau(t)=\lim\limits_{t\to\infty}\sigma(t)=\infty$

(5) $f\in C(R,R)$, is nondecreasing function, and $xf(x)>0$ for $x\neq 0$.

(6) There exists an oscillating function $r(t)\in C^n([t_0,\infty),R)$, such that:

$$r^{(n)}(t)=h(t)$$

Theorem 3.3.1: Assume that n is odd, conditions (1)-(6) hold, and so do the following conditions:

(7) $\lim\limits_{t\to\infty}p(t)=\lim\limits_{t\to\infty}r(t)=0$

(8) $\int\limits_{t_0}^{\infty}s^{n-1}q(s)ds=\infty$

Then every bounded solution of equation (3.3.1) is either oscillatory or tends to zero as $t\to\infty$.

Proof: Suppose that $x(t)$ is a bounded non-oscillatory solution of equation (3.3.1). Further, assume that $x(t)$ does not tend to zero as $t\to\infty$

Without loss of generality, let $x(t)$ be eventually positive (the proof is similar when $x(t)$ is eventually negative). Say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$.

Set

$$z(t) = x(t) + p(t)x(\tau(t)) - r(t) \tag{3.3.2}$$

By (3.3.1) and (3.3.2) we have

$$z^{(n)}(t) = -q(t) f(x(\sigma(t))) < 0 \quad \text{for } t \ge t_1 \tag{3.3.3}$$

Thus $z^{(n)}(t) < 0$. It follows that $z^{(i)}(t)$ ($i = 0,1,\ldots,n-1$) is monotonic and of constant sign eventually. Since $p(t)$ and $r(t)$ are oscillating functions, there exists a $t_2 \ge t_1$ such that $z(t) > 0$ for $t \ge t_2$. From (3.3.2), by using (7), and the fact that $x(t)$ is bounded there exists a $t_3 \ge t_2$, such that $z(t)$ is also bounded for $t \ge t_3$. Applying Lemma 1.6.1, there exists $t_4 \ge t_3$, such that, for $t \ge t_4$,

$$z^{(i)}(t) > 0 \qquad \text{for } i = 0,1,\ldots,l \tag{3.3.4}$$

$$(-1)^{i-l} z^{(i)}(t) > 0 \quad \text{for } i = l, l+1,\ldots,n-1 \tag{3.3.5}$$

Because n is odd and $z(t)$ is bounded, then $l = 0$ (otherwise $z(t)$ is not bounded). Therefore, from (3.3.4), and (3.3.5) for $t \ge t_4$ we have:

$$(-1)^i z^{(i)}(t) > 0 \text{ for } i = 0,1,\ldots,n-1 \tag{3.3.6}$$

Thus $z'(t) < 0$ for $t \ge t_4$, so $z(t)$ is decreasing. Hence, we may write $\lim_{t\to\infty} z(t) = L$ $(0 \le L < \infty)$. Let $L > 0$, then there exists a constant $c > 0$ and $t_5 \ge t_4$, such that $z(t) > c > 0$, for $t \ge t_5$.

Since $x(t)$ is bounded by (7) $\lim_{t\to\infty} p(t)x(\tau(t)) = 0$. Using this fact with facts that $\lim_{t\to\infty} r(t) = 0$ and $z(t) > c > 0$, imply that there exists a constant $c_1 > 0$, such that $x(t) = z(t) - p(t)x(\tau(t)) + r(t) > c_1 > 0$ for $t \ge t_6$ for some $t_6 \ge t_5$. Choose $t_7 \ge t_6$, such that $x(\sigma(t)) > c_1 > 0$ for $t \ge t_7$. From (3.3.3), and the fact that $f(x)$ is nondecreasing function, we have

$$z^{(n)}(t) \le -q(t) f(c_1) < 0 \quad \text{for } t \ge t_7 \tag{3.3.7}$$

Multiplying (3.3.7) by t^{n-1} and integrating it from t_7 to t, we have:

$$\int_{t_7}^{t} s^{n-1} z^{(n)}(s)ds \le -f(c_1)\int_{t_7}^{t} s^{n-1} q(s)ds \tag{3.3.8}$$

Let

$$I(s) = \int s^{n-1} z^{(n)}(s)ds \tag{3.3.9}$$

by using integration by parts repeatedly, (we show this in Appendix E) we have:

$$I(t) = t^{n-1} z^{(n-1)}(t) - (n-1)t^{n-2} z^{(n-2)}(t) + (n-1)(n-2)t^{n-3} z^{(n-3)}(t) -$$

$$\ldots - (n-1)(n-2)(n-3)...3.2tz'(t) + (n-1)(n-2)(n-3)...3.2z(t)$$

Therefore, from (3.3.8) we obtain:

$$I(t) - I(t_7) \le -f(c_1)\int_{t_7}^{t} s^{n-1} q(s)ds \tag{3.3.10}$$

In view of (3.3.6), $I(t) > 0$ for $t \ge t_4$. Follows from (3.3.10) that:

$$-I(t_7) \le -f(c_1)\int_{t_7}^{t} s^{n-1} q(s)ds$$

By (8) we obtain

$$-I(t_7) \le -f(c_1)\int_{t_7}^{\infty} s^{n-1}q(s)ds = -\infty$$

This is a contradiction. Hence, $L > 0$ is impossible. Therefore, $L = 0$ is the only possible case. That is, $\lim_{t\to\infty} z(t) = 0$. Since $x(t)$ is bounded, by (7), and from (3.3.2) we obtain

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) - \lim_{t\to\infty} p(t)x(\tau(t)) + \lim_{t\to\infty} r(t) = 0$$

So , $\lim_{t\to\infty} x(t) = 0$. Thus the proof is complete. □

Example 3.3.1: Consider the NDDE

$$\left[x(t) + e^{-\frac{t}{2}}e^{-\frac{\pi}{2}}\sin(\frac{t}{2})x(\frac{t}{2}-\frac{\pi}{2})\right]^{III} + e^{-\pi}x(t-\pi) = e^{-t}\cos t \, , \qquad (3.3.11)$$

Here $n = 3$, $p(t) = e^{-\frac{t+\pi}{2}}\sin(\frac{t}{2})$, $q(t) = e^{-\pi}$, $f(x) = x$, $\tau(t) = \frac{t-\pi}{2}$, $\sigma(t) = t - \pi$, and $h(t) = e^{-t}\cos t$. It is easy to see that all conditions of Theorem 3.3.1. are satisfied. So every solution of (3.3.11) is either oscillatory or tends to zero as $t \to \infty$.In fact, $x(t) = e^{-t}\sin t$ is oscillatory solution of (3.3.11).

Theorem 3.3.2: Assume that n is even, conditions (1)-(7) hold, and so do the following conditions:

(9) There is a function $\varphi(t)$, such that $\varphi(t) \in C^1[t_0,\infty)$,

$$\lim_{t\to\infty} \sup \int_{t_0}^{t} \varphi(s)q(s)ds = \infty$$

and

$$\lim_{t\to\infty} \sup \int_{t_0}^{t} \left[\frac{(\varphi'(s))^2}{\varphi(s)\sigma'(s)\sigma^{n-2}(s)} \right] ds < \infty$$

for the function $\varphi(t)$.

Then every bounded solution of equation (3.3.1) is oscillatory.

Proof: Suppose that $x(t)$ is a bounded non-oscillatory solution of equation (3.3.1). Without loss of generality, let $x(t)$ be eventually positive (the proof is similar when $x(t)$ is eventually negative). That is, let $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$.

Set $z(t)$ as in (3.3.2). Then, by (3.3.1) and (3.3.2) we have (3.3.3)

Thus $z^{(n)}(t) < 0$, follows that $z^{(i)}(t)$ $(i = 0,1,...,n-1)$ is monotonic and of constant sign eventually. Since $p(t)$ and $r(t)$ are oscillating functions, there exists a $t_2 \geq t_1$ such that $z(t) > 0$ for $t \geq t_2$. From (3.3.2), by using (7), and the fact that $x(t)$ is bounded there exists a $t_3 \geq t_2$, such that $z(t)$ is also bounded for $t \geq t_3$.

Applying Lemma 1.6.1, there exists $t_4 \geq t_3$, such that, for $t \geq t_4$ (3.3.4), and (3.3.5) are satisfied.

Because n is even and $z(t)$ is bounded $l = 1$ (otherwise $z(t)$ is not bounded). Therefore from (3.3.4) and (3.3.5) we have:

$$(-1)^{i+1} z^{(i)}(t) > 0 \text{ for } i = 1,2,...,n-1, \quad \text{for } t \geq t_4 \qquad (3.3.12)$$

Thus $z'(t) > 0$ for $t \geq t_4$, so $z(t)$ is increasing. Since $x(t)$ is bounded, by (7) $\lim_{t\to\infty} p(t)x(\tau(t)) = 0$. Then, by (3.3.2), there exists a $t_5 \geq t_4$ and an integer μ ($\mu > 1$), such that

$$x(t) = z(t) - p(t)x(\tau(t)) + r(t) > \frac{1}{\mu} z(t) > 0, \quad \text{for} \quad t \geq t_5$$

Choose $t_6 \geq t_5$, such that

$$x(\sigma(t)) > \frac{1}{\mu} z(\sigma(t)) > 0, \quad \text{for} \quad t \geq t_6 \tag{3.3.13}$$

From (3.3.3), (3.3.13), and from the fact that f is nondecreasing function, we have

$$z^{(n)}(t) \leq -q(t) f\left(\frac{1}{\mu} z(\sigma(t))\right) = -q(t) \frac{f\left(\frac{1}{\mu} z(\sigma(t))\right)}{z(\sigma(t))} z(\sigma(t)), \quad \text{for} \quad t \geq t_6 \tag{3.3.14}$$

Since $z(t)$ is bounded and increasing, $\lim_{t\to\infty} z(t) = L \quad (0 < L < \infty)$. By the continuity of f, we have

$$\lim_{t\to\infty} \frac{f\left(\frac{1}{\mu} z(\sigma(t))\right)}{z(\sigma(t))} = \frac{f\left(\frac{L}{\mu}\right)}{L} > 0$$

Hence, there exists $t_7 \geq t_6$, such that:

$$\lim_{t\to\infty} \frac{f\left(\frac{1}{\mu} z(\sigma(t))\right)}{z(\sigma(t))} \geq \frac{f\left(\frac{L}{\mu}\right)}{2L} = \alpha > 0 \quad \text{for} \quad t \geq t_7 \tag{3.3.15}$$

By (3.3.14), and (3.3.15), we have:

$$z^{(n)}(t) \leq -\alpha q(t) z(\sigma(t)), \quad \text{for} \quad t \geq t_7 \tag{3.3.16}$$

Assume that

$$w(t)=\frac{z^{(n-1)}(t)}{z(\frac{1}{\mu}\sigma(t))} \tag{3.3.17}$$

From (3.3.12), there exists $t_8 \ge t_7$, such that $w(t)>0$ for $t \ge t_8$. Since $z(t)$ is increasing, there exists $t_9 \ge t_8$, such that $z(\sigma(t)) > z(\frac{1}{\mu}\sigma(t)) > 0$ for $t \ge t_9$.

From (3.3.17), we obtain:

$$w'(t)=\frac{z(\frac{1}{\mu}\sigma(t))z^{(n)}(t)-z'(\frac{1}{\mu}\sigma(t))z^{(n-1)}(t)\frac{\sigma'(t)}{\mu}}{z^2(\frac{1}{\mu}\sigma(t))}$$

$$=\frac{z^{(n)}(t)}{z(\frac{1}{\mu}\sigma(t))}-\frac{1}{\mu}w(t)\frac{z'(\frac{1}{\mu}\sigma(t))}{z(\frac{1}{\mu}\sigma(t))}\sigma'(t) \tag{3.3.18}$$

From (3.3.12) we have $z'(t)>0$, and $z^{(n-1)}(t)>0$ for $t \ge t_9$. Since $\sigma(t) \le t$ and $\sigma'(t)>0$. By Lemma 1.6.3, there exists $M_1 > 0$ and a $t_{10} \ge t_9$ for $\lambda = \frac{1}{\mu}$ (λ as in Lemma 1.6.3),such that:

$$z'(\frac{1}{\mu}\sigma(t)) \ge M_1\sigma^{n-2}(t)z^{(n-1)}(\sigma(t)) \ge M_1\sigma^{n-2}(t)z^{(n-1)}(t) \ge M_1\sigma^{n-2}(t)\sigma'(t)z^{(n-1)}(t)$$

for $t \ge t_{10}$.

Hence

$$z'(\frac{1}{\mu}\sigma(t)) \ge M_1\sigma^{n-2}(t)\sigma'(t)z^{(n-1)}(t) \quad \text{for } t \ge t_{10} \tag{3.3.19}$$

Therefore, combining (3.3.16), (3.3.18) and (3.3.19) we have:

$$w'(t) \le -\alpha q(t)-\frac{M_1}{\mu}w^2(t)\sigma^{n-2}(t)\sigma'(t) \quad \text{for } t \ge t_{10} \tag{3.3.20}$$

From (3.3.20), we have

$$\alpha q(t) \le -w'(t) - \frac{M_1}{\mu} w^2(t)\sigma^{n-2}(t)\sigma'(t) \quad \text{for} \quad t \ge t_{10} \tag{3.3.21}$$

Multiplying (3.3.21) by $\varphi(t)$ and integrating it from t_{10} to t, we obtain

$$\alpha \int_{t_{10}}^{t} \varphi(s)q(s)ds \le -\int_{t_{10}}^{t} \varphi(s)w'(s)ds - \frac{M_1}{\mu}\int_{t_{10}}^{t} \varphi(s)w^2(s)\sigma^{n-2}(s)\sigma'(s)ds$$

$$= -\varphi(t)w(t) + \varphi(t_{10})w(t_{10}) + \int_{t_{10}}^{t} \varphi'(s)w(s)ds - \frac{M_1}{\mu}\int_{t_{10}}^{t} \varphi(s)w^2(s)\sigma^{n-2}(s)\sigma'(s)ds$$

$$\le \varphi(t_{10})w(t_{10}) - \frac{M_1}{\mu}\int_{t_{10}}^{t} \varphi(s)\sigma^{n-2}(s)\sigma'(s)\left[w(s) - \frac{\mu\phi'(s)}{2M_1\varphi(s)\sigma^{n-2}(s)\sigma'(s)}\right]^2 ds$$

$$+ \int_{t_{10}}^{t} \frac{\mu[\varphi'(s)]^2}{4M_1\varphi(s)\sigma^{n-2}(s)\sigma'(s)} ds$$

$$\le \varphi(t_{10})w(t_{10}) + \int_{t_{10}}^{t} \frac{\mu[\varphi'(s)]^2}{4M_1\varphi(s)\sigma^{n-2}(s)\sigma'(s)} ds\,.$$

Therefore, by (9)

$$\infty = \alpha \lim_{t\to\infty} \sup \int_{t_{10}}^{t} \varphi(s)q(s)ds$$

$$\le \varphi(t_{10})w(t_{10}) + \frac{\mu}{4M_1}\lim_{t\to\infty} \sup \int_{t_{10}}^{t} \frac{[\varphi'(s)]^2}{\varphi(s)\sigma^{n-2}(s)\sigma'(s)} ds < \infty$$

This is a contradiction. Hence the proof is complete. □

Example 3.3.2: The NDDE

$$\left[x(t) + e^{-\frac{t}{2}}e^{-\frac{\pi}{2}}\sin(\frac{t}{2})x(\frac{t}{2} - \frac{\pi}{2})\right]'' + \sqrt{2}e^{-\frac{5\pi}{4}}x(t - \frac{5\pi}{4}) = -e^{-t}\sin t \tag{3.3.22}$$

satisfies all conditions of Theorem 3.3.2. By choosing $\varphi(t)=1$. So every solution of (3.3.22) is oscillatory. In fact, $x(t)=e^{-t}\sin t$ is such a solution.

Remark 3.3.1: Theorems 3.3.1 and 3.3.2 are special cases of the results of Bolat and Akin [5]. Where, the results of [5] are also applicable when the non-derivative part has several delays.

Chapter Four

Oscillation of *n-th* Order NDDE When the Non-Derivative Part is Dependent on the Independent Variable and the Unknown Function with Delayed Argument

4.0 Introduction

In this chapter we consider the oscillation of *n-th* order NDDE when the non-derivative part is a function of independent variable, and unknown function with delayed argument, i.e. we consider an equation of the form

$$[x(t)+p(t)x(\tau(t))]^{(n)}+\delta f(t,x(\sigma(t)))=0 \tag{4.0.1}$$

Where, $\delta=\pm 1$, $p(t),\tau(t),\sigma(t)\in C([t_0,\infty),R)$, $t_0\geq 0$, $\tau(t)$ and $\sigma(t)$ are delayed arguments, and $f:[t_0,\infty)\times R\to R$ is continuous

Many results are known for the case when the non-derivative part is separable, as in chapters 2, and 3. But when the non-derivative part is in more general form as in equation (4.0.1), it seems that few authors only deal with the subject. And so this case still needs a lot of work. However, it is clear that the cases of chapters 2, and 3, are special cases of equation (4.0.1).

In section 4.1 we study the oscillation of equation (4.0.1) when the coefficient $p(t)$ is non-negative. In section 4.2 we discuss the oscillation of equation (4.0.1) when the coefficient $p(t)$ is non-positive.

4.1 Oscillation of *n-th* order NDDE with non-negative coefficient

Consider the NDDE

$$[x(t)+p(t)x(\tau(t))]^{(n)}+\delta f(t,x(\sigma(t)))=0 \qquad (4.1.1)$$

where $\delta=\pm 1$, and the following conditions are assumed to be hold:

(1) $p(t)\in C([t_0,\infty),R)$, $t_0\geq 0$

(2) $\tau(t),\sigma(t)\in C([t_0,\infty),R_+)$, $\tau(t)\leq t$, $\sigma(t)\leq t$, and $\lim\limits_{t\to\infty}\tau(t)=\lim\limits_{t\to\infty}\sigma(t)=\infty$

(3) $f:[t_0,\infty)\times R\to R$ is continuous, $uf(t,u)\geq 0$ for $u\neq 0$ and $t\geq t_0$, and

$f(t,u)\not\equiv 0$ on $[t_*,\infty)\times R\setminus\{0\}$ for every $t_*\geq t_0$

(4) If $u(t)>0$ ($u(t)<0$) is a continuous with $\lim\limits_{t\to\infty}\inf|u(t)|>0$, then

$$\int^{\infty} f(s,u(s))ds=\infty(-\infty)$$

For notational purposes, we let

$$z(t)=x(t)+p(t)x(\tau(t)) \qquad (4.1.2)$$

The main results of this section are introduced in Theorem 4.1.1. Firstly, we introduce the following Lemmas, which are useful in the proof of Theorem 4.1.1, and in proving of results of the next section.

Lemma 4.1.1: (see [18], [20])

Suppose that $\delta=+1$. If (1)-(4) hold and $x(t)$ is an eventually positive (negative) solution of equation (4.1.1). Then:

i. $z^{(n-1)}(t)$ is eventually decreasing (increasing) and satisfies $z^{(n-1)}(t)\to L<\infty(>-\infty)$ as $t\to\infty$.

ii. If $L>-\infty(<\infty)$, then $\lim\limits_{t\to\infty}\inf|x(t)|=0$.

iii. If $z(t) \to 0$ as $t \to \infty$, then $z^{(i)}(t)$ is monotonic and

$z^{(i)}(t) \to 0$ as $t \to \infty$, and $z^{(i)}(t) z^{(i+1)}(t) < 0$ for $i = 0,1,\ldots,n-1$ (4.1.3)

iv. Let $z(t) \to 0$ as $t \to \infty$. If n is even, then $z(t) < 0$ ($z(t) > 0$) for $x(t) > 0$ ($x(t) < 0$).If n is odd, then $z(t) > 0$ ($z(t) < 0$)for $x(t) > 0$ ($x(t) < 0$).

Proof: Suppose that equation (4.1.1) has an eventually positive solution $x(t)$. (The proof is similar when $x(t)$ is eventually negative). That is $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. By equation (4.1.1):

$$z^{(n)}(t) = -f(t, x(\sigma(t))) \leq 0 \ , \quad t \geq t_1 \tag{4.1.4}$$

Using (3), then (4.1.4) implies that $z^{(n-1)}(t)$ is decreasing and tends to $L < \infty$ as $t \to \infty$. Thus (i) holds.

If $L > -\infty$, then integrating equation (4.1.1) from t_1 to t, and then letting $t \to \infty$, we have

$$\int_{t_1}^{\infty} f(s, x(\sigma(s)))ds = z^{(n-1)}(t_1) - L < \infty \tag{4.1.5}$$

Therefore, condition (4) implies that $\lim_{t\to\infty} \inf x(t) = 0$, so (ii) holds.

Now, to prove (iii), suppose that $z(t) \to 0$ as $t \to \infty$. From part (i) $z^{(n-1)}(t)$ is decreasing and $z^{(n-1)}(t) \to L < \infty$ as $t \to \infty$. Let us consider the following cases:

<u>Case1</u>: If $z^{(n-1)}(t) \to L < 0$ as $t \to \infty$, then there exists $L_1 < 0$ and $t_2 \ge t_1$ such that $z^{(n-1)}(t) \le L_1$ for $t \ge t_2$. This with $z^{(n)}(t) \le 0$, leads to $z(t) \to -\infty$ as $t \to \infty$. Which is a contradiction to the assumption $z(t) \to 0$ as $t \to \infty$.

<u>Case2</u>: If $z^{(n-1)}(t) \to L > 0$ as $t \to \infty$, then $z^{(n-1)}(t) \ge L$ for $t \ge t_1$, which also, in view of $z^{(n)}(t) \le 0$,contradicts $z(t) \to 0$ as $t \to \infty$.

So, from Cases1, and 2, we conclude that $z^{(n-1)}(t) \to 0$ as $t \to \infty$. Since $z^{(n-1)}(t)$ is decreasing, $z^{(n-1)}(t) > 0$ for $t \ge t_1$. Hence, if $n \ge 2$, then $z^{(n-2)}(t)$ is increasing and so $z^{(n-2)}(t) \to L_2 > -\infty$ as $t \to \infty$. **If** $\underline{L_2 < 0}$**,** then $z^{(n-2)}(t) \le L_2$ for $t \ge t_1$contradicting $z(t) \to 0$ as $t \to \infty$. **If** $\underline{L_2 > 0}$**,** then there exist $L_3 > 0$ and $t_3 \ge t_1$ such that $z^{(n-2)}(t) \ge L_3$for $t \ge t_3$, again this contradicts $z(t) \to 0$ as $t \to \infty$. Therefore, $z^{(n-2)}(t) \to 0$ as $t \to \infty$. Since $z^{(n-2)}(t)$ is increasing, we have $z^{(n-2)}(t) < 0$ for $t \ge t_1$. Continuing in this manner we obtain (4.1.3).

To prove part (iv), if n is even, then from (4.1.3), and from the fact that $z^{(n)}(t) < 0$, it follows that $z(t) < 0$. If n is odd, then from (4.1.3), and from the fact that $z^{(n)}(t) < 0$, it follows that $z(t) > 0$. □

Lemma 4.1.2: (see [18], [20])

Suppose that $\delta = -1$. If (1)-(4) hold and $x(t)$ is an eventually positive (negative) solution of equation (4.1.1). Then:

i. $z^{(n-1)}(t)$ is eventually increasing (decreasing) and satisfies $z^{(n-1)}(t) \to L > -\infty (< \infty)$ as $t \to \infty$.

ii. If $L < \infty (> -\infty)$, then $\lim_{t\to\infty} \inf |x(t)| = 0$.

iii. If $z(t) \to 0$ as $t \to \infty$, then $z^{(i)}(t)$ is monotonic for $i = 0,1,...,n-1$, and (4.1.3) holds.

iv. Let $z(t) \to 0$ as $t \to \infty$. If n is even, then $z(t) > 0$ ($z(t) < 0$) for $x(t) > 0$ ($x(t) < 0$).If n is odd, then $z(t) < 0$ ($z(t) > 0$)for $x(t) > 0$ ($x(t) < 0$).

Proof: Suppose that equation (4.1.1) has an eventually positive solution $x(t)$.(The proof is similar when $x(t)$ is eventually negative). Then there exists $t_1 \geq t_0$ such that $x(t) > 0, x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. By equation (4.1.1), $z^{(n)}(t) = f(t, x(\sigma(t))) \geq 0$, so $z^{(n-1)}(t)$ is increasing and tends to $L > -\infty$ as $t \to \infty$. Thus (i) holds.

If $L < \infty$, then integrating equation (4.1.1) from t_1 to t, and then letting $t \to \infty$, we have

$$\int_{t_1}^{\infty} f(s, x(\sigma(s)))ds = L - z^{(n-1)}(t_1) < \infty \tag{4.1.6}$$

condition (4), with (4.1.6) imply that $\liminf_{t \to \infty} x(t) = 0$, so (ii) holds.

Now, to prove (iii), suppose that $z(t) \to 0$ as $t \to \infty$. Part (i) implies that $z^{(n-1)}(t)$ is increasing and $z^{(n-1)}(t) \to L > -\infty$ as $t \to \infty$. We have two cases:

<u>*Case1*:</u> If $z^{(n-1)}(t) \to L > 0$ as $t \to \infty$, then there exists $L_1 > 0$ and $t_2 \geq t_1$ such that $z^{(n-1)}(t) \geq L_1$ for $t \geq t_2$. This contradicts the fact that $z(t) \to 0$ as $t \to \infty$.

<u>*Case2*:</u> If $z^{(n-1)}(t) \to L < 0$ as $t \to \infty$, then $z^{(n-1)}(t) \leq L$ for $t \geq t_1$, which also contradicts $z(t) \to 0$ as $t \to \infty$.

So, from Cases1 and 2, we conclude that $z^{(n-1)}(t) \to 0$ as $t \to \infty$. Since $z^{(n-1)}(t)$ is increasing, $z^{(n-1)}(t) < 0$ for $t \geq t_1$. Hence, if $n \geq 2$, then $z^{(n-2)}(t)$ is decreasing and so $z^{(n-2)}(t) \to L_2 < \infty$ as $t \to \infty$. **If** $\underline{L_2 > 0}$, then $z^{(n-2)}(t) \geq L_2$ for $t \geq t_1$ contradicting $z(t) \to 0$ as $t \to \infty$. **If** $\underline{L_2 < 0}$, then there exist $L_3 > 0$ and $t_3 \geq t_1$ such that $z^{(n-2)}(t) \leq L_3$ for $t \geq t_3$, again this contradicts $z(t) \to 0$ as $t \to \infty$. Therefore, $z^{(n-2)}(t) \to 0$ as $t \to \infty$. Since $z^{(n-2)}(t)$ is decreasing, we have $z^{(n-2)}(t) > 0$ for $t \geq t_1$. Continuing in this manner we obtain (4.1.3).

Then to prove (iv), if n is even, then from (4.1.3), and from the fact that $z^{(n)}(t) > 0$, it follows that $z(t) > 0$. If n is odd, then from (4.1.3), and from the fact that $z^{(n)}(t) > 0$, it follows that $z(t) < 0$. □

Theorem 4.1.1: Suppose that conditions (1)-(4) hold. If there exists a constant $p_1 > 0$ such that

$$0 \leq p(t) \leq p_1 < 1 \tag{4.1.7}$$

Then

i. If $\delta = +1$ and n is even, then equation (4.1.1) is oscillatory, while if n is odd, then any solution $x(t)$ of equation (4.1.1) is either oscillatory or $x(t) \to 0$ as $t \to \infty$.

ii. If $\delta = -1$ and n is even, then either $x(t)$ is oscillatory, $|x(t)| \to \infty$, or $x(t) \to 0$ as $t \to \infty$, while if n is odd, then either $x(t)$ is oscillatory or $|x(t)| \to \infty$ as $t \to \infty$.

Proof: Suppose that equation (4.1.1) has an eventually positive solution $x(t)$. (The proof is similar when $x(t)$ is eventually negative). Say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$.

***Proof of part* (i):** by Lemma 4.1.1(i) we have that $z^{(n-1)}(t)$ is eventually decreasing and converges to $L \geq -\infty$ as $t \to \infty$. If $L < 0$, then this implies that $z(t)$ is eventually negative, which contradicts $x(t) > 0$ for $t \geq t_1$. Hence, $L \geq 0$, follows from Lemma 4.1.1(ii), that $\lim_{t\to\infty} \inf x(t) = 0$.

Now, since $z^{(n)}(t) \leq 0$, then $z^{(i)}(t)$ is monotonic for $i = 0,1,...,n-1$. Thus $z(t)$ is monotonic, so $z(t) \to M$ as $t \to \infty$. If $M < 0$ implies that $x(t) < 0$, so $M \geq 0$. Suppose that $M > 0$, then we have two possible cases:

<u>*Case1:*</u> if $z(t)$ is increasing. From (4.1.2), and since $p(t)$ is non-negative, we have $x(t) \leq z(t)$. Using this fact with (4.1.7) we have:

$$z(t) = x(t) + p(t)x(\tau(t)) \leq x(t) + p(t)z(\tau(t)) \leq x(t) + p_1 z(t), \quad \text{so}$$

$$z(t)[1 - p_1] \leq x(t)$$

Since $0 \leq p_1 < 1$, we get a contradiction to $\lim_{t\to\infty} \inf x(t) = 0$.

<u>*Case2:*</u> If $z(t)$ is decreasing, let $\varepsilon = 1 - p_1 > 0$. Then

$$z(t) = x(t) + p(t)x(\tau(t)) \leq x(t) + p(t)z(\tau(t)) \leq x(t) + p_1 z(\tau(t)), \quad \text{so}$$

$$z(t) \leq x(t) + p_1 z(\tau(t)) \tag{4.1.8}$$

Dividing (4.1.8) by $z(\tau(t))$, we get

$$\frac{z(t)}{z(\tau(t))} \leq \frac{x(t)}{z(\tau(t))} + p_1$$

Since $M > 0$, and $z(t)$ is decreasing, we have:

$$\frac{z(t)}{z(\tau(t))} \le \frac{x(t)}{M} + p_1 \tag{4.1.9}$$

Since $\lim_{t\to\infty} \tau(t) = \infty$, and $p_1 + \frac{\varepsilon}{2} < 1$, there exists $t_2 \ge t_1$ such that

$$\frac{z(t)}{z(\tau(t))} \ge p_1 + \frac{\varepsilon}{2} \qquad \text{for } t \ge t_2 \tag{4.1.10}$$

From (4.1.9), and (4.1.10), we have:

$$x(t) \ge \frac{\varepsilon M}{2} \qquad \text{for } t \ge t_2 \tag{4.1.11}$$

Which again contradicts $\lim_{t\to\infty} \inf x(t) = 0$. Hence, we have $z(t) \to 0$ as $t \to \infty$, so by Lemma 4.1.1(iii), the (4.1.3) holds. To complete the proof, note that Lemma 4.1.1(iv) implies that $z(t) < 0$ for n even and $z(t) > 0$ for n odd. But $z(t) < 0$ contradicts $x(t) > 0$. While, in the case of $z(t) > 0$, follows $x(t) \le z(t) \to 0$ as $t \to \infty$.

***Proof of part* (ii):** from Lemma 4.1.2(i) $z^{(n-1)}(t)$ is eventually increasing and satisfies $z^{(n-1)}(t) \to L > -\infty$ as $t \to \infty$. **If** $\underline{L < 0}$, then this leads to $z(t) < 0$ eventually, which contradicts $x(t) > 0$. **Thus** $\underline{L \ge 0}$. See the following cases:

<u>*Case1:*</u> If $L = \infty$, then this leads to $z(t) \to \infty$ as $t \to \infty$. From (4.1.2), and (4.1.7) we have:

$$z(t) = x(t) + p(t)x(\tau(t)) \le x(t) + p_1 z(\tau(t)) \le x(t) + p_1 z(t) \tag{4.1.12}$$

so $$z(t)[1 - p_1] \le x(t)$$

Therefore, $x(t) \to \infty$ as $t \to \infty$

Case2: If $0 \le L < \infty$, then Lemma 4.1.2(ii) implies that $\lim_{t\to\infty} \inf x(t) = 0$. Since $z(t)$ is monotonic and positive, $z(t) \to K \ge 0$ as $t \to \infty$. **If** $z(t)$ **is increasing**, then $K > 0$ and (4.1.12) holds, then it follows from (4.1.12) that $z(t)[1 - p_1] \le x(t)$, contradicting to $\lim_{t\to\infty} \inf x(t) = 0$. **If** $z(t)$ **is decreasing** and $K > 0$, then K is finite, and since $\lim_{t\to\infty} \tau(t) = \infty$, we have $\frac{z(t)}{z(\tau(t))} \to 1$ as $t \to \infty$. Let $\varepsilon = 1 - p_1 > 0$. Then there exists $t_2 \ge t_1$ such that

$$\frac{z(t)}{z(\tau(t))} > 1 - \frac{\varepsilon}{2} \quad \text{for } t \ge t_2 \tag{4.1.13}$$

Then, from (4.1.7), and (4.1.13) we have:

$$K < z(t) \le x(t) + p_1 x(\tau(t)) \le x(t) + p_1 z(\tau(t))$$

$$< x(t) + \frac{p_1 z(t)}{(1 - \frac{\varepsilon}{2})} = x(t) + \frac{2 p_1 z(t)}{(1 + p_1)}$$

Then we have,

$$x(t) > \left[1 - \frac{2p_1}{(1+p_1)}\right] z(t) = \left[\frac{(1-p_1)}{(1+p_1)}\right] z(t) \tag{4.1.14}$$

Since $\varepsilon = 1 - p_1$, and $1 + p_1 < 2$, then from (4.1.14) we obtain:

$$x(t) > \left[\frac{(1-p_1)}{(1+p_1)}\right] z(t) > \frac{\varepsilon K}{2} \quad \text{, and so}$$

$$x(t) > \frac{\varepsilon K}{2} \tag{4.1.15}$$

And this contradicts $\lim_{t\to\infty} \inf x(t) = 0$. Thus for $0 \le L < \infty$, $z(t) \to 0$ as $t \to \infty$, then $x(t) \le z(t) \to 0$ as $t \to \infty$. Therefore, from Cases 1, and 2, we have that either $x(t) \to \infty$ or $x(t) \to 0$ for n even or n odd.

To complete the proof, note that if $x(t) \to 0$ as $t \to \infty$, then $z(t) \to 0$ as $t \to \infty$, and for n odd, Lemma 4.1.2(iv) implies that $z(t) < 0$, which is impossible in view of (4.1.7). Thus the proof is complete. □

Example 4.1.1: Consider the NDDE

$$\left[x(t) + \frac{1}{4}x(t - \ln 2)\right]^{(n)} + \frac{3}{2e}x(t-2)\exp\left\{\frac{1}{t-2}\ln\left[\frac{|x(t-2)| + e^{-(t-2)}}{2}\right]\right\} = 0, \quad t > 2$$

(4.1.16)

Here $p(t) = \frac{1}{4}$, $f(t,u) = \frac{3}{2e}u\exp\left\{\frac{1}{t-2}\ln\left[\frac{|u| + e^{-(t-2)}}{2}\right]\right\}$, $\tau(t) = t - \ln 2$,

$\sigma(t) = t - 2$, $f(t,u)$ satisfies the divergent integral in condtion (4). And all conditions of Theorem 4.1.1(i) are satisfied. If n is odd, then $x(t) = e^{-t}$ is a solution of (4.1.16) such that $x(t) \to 0$ as $t \to \infty$.

Example 4.1.2: The NDDE

$$\left[x(t) + \frac{e^{\frac{1}{2}}}{2}x(t - \frac{1}{2})\right]^{(n)} - \frac{3e}{2}x(t-2)\exp\left\{\frac{1}{t-2}\ln\left[\frac{|x(t-2)| + e^{t-2}}{2}\right]\right\} = 0, t > 2 \quad (4.1.17)$$

satisfies all conditions of Theorem 4.1.1(ii). If n is either even or odd, then $x(t) = e^{t}$ is a solution of (4.1.17) such that $x(t) \to \infty$ as $t \to \infty$. Also, note that $x(t) = -e^{t}$ is a solution and satisfies that $|x(t)| \to \infty$ as $t \to \infty$.

Example 4.1.3: Consider the NDDE

$$[x(t) + px(t - 2\pi)]^{(2k)} + (-1)^{k+1}(p+1)x(t - 2\pi) = 0 \quad (4.1.18)$$

Where $0 < p < 1$. If k is odd, then (4.1.18) satisfies conditions of Theorem 4.1.1(i) , and so it may have oscillatory solutions. Indeed,

$x(t) = \sin t$ is such a solution. If k is even, then (4.1.18) satisfies conditions of Theorem 4.1.1(ii), and so it may have oscillatory solutions. In fact, $x(t) = \sin t$ is such a solution.

Example 4.1.4: Consider the NDDE

$$[x(t) + px(t-2\pi)]^{(2k-1)} + (-1)^{k+1}(p+1)x(t-\frac{\pi}{2}) = 0 \qquad (4.1.19)$$

Where $0 < p < 1$. If k is odd, then (4.1.19) satisfies conditions of Theorem 4.1.1(i) , and so it may have oscillatory solutions. Indeed, $x(t) = \sin t$ is such a solution. If k is even, then (4.1.19) satisfies conditions of Theorem 4.1.1(ii), and so it may have oscillatory solution. In fact, $x(t) = \sin t$ is such a solution.

Now, let us consider the following equation, which is a special case of equation (4.1.1)

$$[x(t) + p(t)x(\tau(t))]^{(n)} + \delta q(t) f(x(\sigma(t))) = 0 \qquad (4.1.20)$$

where $\delta = \pm 1$, and the following conditions are assumed to be hold:

(H1) $p(t), q(t) \in C([t_0, \infty), R_+)$, $t_0 \geq 0$, , and $q(t) \not\equiv 0$ on any half line $[t_*, \infty)$, for every $t_* \geq t_0$

(H2) $\tau(t), \sigma(t) \in C([t_0, \infty), R_+)$, $\tau(t) \leq t$, $\sigma(t) \leq t$, and $\lim\limits_{t \to \infty} \tau(t) = \lim\limits_{t \to \infty} \sigma(t) = \infty$

(H3) $f \in C(R, R)$, such that $uf(u) > 0$ for $u \neq 0$

(H4) $f(u)$ is bounded away from zero if u is bounded away from zero, and

$$\int^{\infty} q(s)ds = \infty$$

Remark 4.1.1: If conditions (H1)-(H4) hold, and (4.1.7) is satisfied, then the results of Theorem 4.1.1 are true for equation (4.1.20). To prove this fact, we can proceed the same as in proof of Theorem 4.1.1. Note that conditions (H1)-(H4) are special case of conditions (1)-(4). Hence, we can apply Theorem 4.1.1 directly for equation (4.1.20), which is a special case of equation (4.1.1).

Remark 4.1.2: Theorem 4.1.1 is due to Graef and Spikes, it is theorem 2 at [20], which is also generalizes Theorems 4 and 20 in [18]. Where Theorems 4, and 20 of [18] consider equation of form like equation (4.1.20). However, Remark 4.1.1 collect together the results of Theorems 4, and 20 of [18], for equation (4.1.20).

4.2 Oscillation of *n-th* order NDDE with non-positive coefficient

Consider the NDDE

$$[x(t) - p(t)x(\tau(t))]^{(n)} + \delta f(t, x(\sigma(t))) = 0 \qquad (4.2.1)$$

where $\delta = \pm 1$, and the following conditions are assumed to be hold:

(1) $p(t) \in C([t_0, \infty), R_+)$, $t_0 \geq 0$, such that $0 \leq p(t) \leq p_2 < 1$, for some real number p_2.

(2) $\tau(t), \sigma(t) \in C([t_0, \infty), R_+)$, $\tau(t) \leq t$, $\sigma(t) \leq t$, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$

(3) $f : [t_0, \infty) \times R \to R$ is continuous, $uf(t,u) \geq 0$ for $u \neq 0$ and $t \geq t_0$, and $f(t,u) \not\equiv 0$ on $[t_*, \infty) \times R \setminus \{0\}$ for every $t_* \geq t_0$

(4) If $u(t) > 0$ ($u(t) < 0$) is a continuous with $\lim_{t\to\infty} \inf |u(t)| > 0$, then

$$\int^{\infty} f(s, u(s))ds = \infty(-\infty)$$

(5) $\tau(t)$ is increasing.

For notational purposes, we let

$$z(t) = x(t) - p(t)x(\tau(t)) \tag{4.2.2}$$

Also, let

$$\tau_1(t) = \tau^{-1}(t), \text{ and } \quad \tau_k(t) = \tau_{k-1}(\tau^{-1}(t)) \text{ for } k = 2,3,... \tag{4.2.3}$$

The main results of this section are introduced in Theorem 4.2.1, but at beginning, we introduce Lemmas 4.2.1, and 4.2.2, to utilize the proof of the Theorem 4.2.1. Also it is important to see the following remark.

Remark 4.2.1: The results of Lemmas 4.1.1, and 4.1.2 are also true in this section since conditions (2)-(4) in this section are the same as conditions (2)-(4) in section 4.1.1, and condition (1) in this section is a special case of condition (1) in section 4.1.1, so it is obvious that Lemmas 4.1.1, and 4.1.2 are also can be used in this section.

Lemma 4.2.1: (see [20])

Suppose that $\delta = +1$, If (1)-(4) hold and $x(t)$ is an eventually positive (negative) solution of (4.2.1), then $z(t) \to 0$ as $t \to \infty$.

Proof: Suppose that $x(t)$ is an eventually positive solution of equation (4.2.1). Then there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. From Lemma 4.1.1, parts (i), and (ii) we have that

$z^{(n-1)}(t)$ is eventually decreasing and satisfies $z^{(n-1)}(t) \to L < \infty$ as $t \to \infty$, and if $L > -\infty$, then $\liminf_{t\to\infty} x(t) = 0$. Let us consider the following cases:

<u>*Case1:*</u> If $L = -\infty$, by successive integration of equation (4.2.1), we get that $z(t) \to -\infty$ as $t \to \infty$. So exists $t_2 \geq t_1$, such that $z(t) < 0$ for $t \geq t_2$. Then $z(t) = x(t) - p(t)x(\tau(t)) < 0$ for $t \geq t_2$, implies that $x(t) < p(t)x(\tau(t))$, and since $p(t) < 1$, we have,

$$x(t) < p(t)x(\tau(t)) < x(\tau(t)) \Rightarrow x(t) < x(\tau(t)) \tag{4.2.4}$$

From (4.2.4), it follows that $x(t)$ is bounded, and this contradicts the fact that $z(t) \to -\infty$ as $t \to \infty$.

<u>*Case2:*</u> If $-\infty < L < 0$, then there exists $L_1 < 0$, such that $z(t) \leq L_1$ eventually. Since $p(t) \leq p_2$, so $p(t)x(\tau(t)) \leq p_2 x(\tau(t))$, then

$$L_1 \geq z(t) = x(t) - p(t)x(\tau(t)) \geq x(t) - p_2 x(\tau(t)) > -p_2 x(\tau(t))$$

implies that $L_1 > -p_2 x(\tau(t))$, and this contradicts $\liminf_{t\to\infty} x(t) = 0$.

<u>*Case 3:*</u> If $L > 0$, then, we eventually have $x(t) \geq z(t) \geq L_2$ for some $L_2 > 0$, and this contradicts $\liminf_{t\to\infty} x(t) = 0$.

So, from Cases 1,2, and 3, we conclude that $L = 0$, i.e. $z^{(n-1)}(t) \to 0$ as $t \to \infty$. Since $z^{(n-1)}(t)$ is decreasing, we have $z^{(n-1)}(t) > 0$ eventually. So $z^{(n-2)}(t)$ is increasing. If $z^{(n-2)}(t)$ is eventually positive, then, this gives that $z(t)$ has a positive lower bound, and since $x(t) \geq z(t)$, this contradicts $\liminf_{t\to\infty} x(t) = 0$. So eventually $z^{(n-2)}(t) < 0$ If $z^{(n-2)}(t) \to L_3 < 0$ as $t \to \infty$, then $z(t) \leq L_4$ for some $L_4 < 0$, and this contradicts $\liminf_{t\to\infty} x(t) = 0$.

Therefore, $z^{(n-2)}(t)$ is increasing and tends to zero as $t \to \infty$. Continuing this process we get $z(t) \to 0$ as $t \to \infty$. The proof when $x(t)$ is an eventually negative is similar, so it is omitted. □

Lemma 4.2.2: (see [20])

Suppose that $\delta = -1$, If (1)-(4) hold and $x(t)$ is an eventually positive (negative) solution of (4.2.1), then either $|x(t)| \to \infty$ as $t \to \infty$, or $z^{(i)}(t)$ is monotonic and (4.1.3) holds.

Proof: Suppose that $x(t)$ is an eventually positive solution of equation (4.2.1). Then there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. From Lemma 4.1.2, parts (i), and (ii) we have that $z^{(n-1)}(t)$ is eventually increasing and satisfies $z^{(n-1)}(t) \to L \leq \infty$ as $t \to \infty$, and if $L < \infty$, then $\liminf_{t \to \infty} x(t) = 0$. Let us consider the following cases:

<u>*Case1:*</u> If $L = \infty$, follows that $z(t) \to \infty$, as $t \to \infty$. Hence, $z(t) \leq x(t) \to \infty$ as $t \to \infty$.

<u>*Case2:*</u> If $L < 0$, then eventually $z(t) \leq L_1$ for some $L_1 < 0$. But $z(t) > -p(t)x(\tau(t)) \geq -p_2 x(\tau(t))$, so this contradicts $\liminf_{t \to \infty} x(t) = 0$.

<u>*Case 3:*</u> If $L > 0$, then eventually $x(t) \geq z(t) \geq L_2$ for some $L_2 > 0$, and this contradicts $\liminf_{t \to \infty} x(t) = 0$.

If $L < \infty$, then we conclude that $L = 0$, i.e. $z^{(n-1)}(t) \to 0$ as $t \to \infty$. Since $z^{(n-1)}(t)$ is increasing, then $z^{(n-1)}(t) < 0$ eventually. So $z^{(n-2)}(t)$ is decreasing. If $z^{(n-2)}(t)$ is eventually negative then, this give that $z(t)$ has a negative upper bound, and this contradicts $\liminf_{t\to\infty} x(t)=0$. So eventually $z^{(n-2)}(t)>0$. If $z^{(n-2)}(t)\to L_3>0$ as $t\to\infty$, then $z(t)\geq L_4$ for some $L_4>0$, and this again contradicts $\liminf_{t\to\infty} x(t)=0$. Therefore, $z^{(n-2)}(t)$ is decreasing and tends to zero as $t\to\infty$. Continuing this process we obtain (4.1.3). The proof when $x(t)$ is an eventually negative is similar. □

Theorem 4.2.1: Suppose that conditions (1)-(5) hold. Then:

i. If $\delta=+1$, then any solution $x(t)$ of equation (4.2.1) is either oscillatory or satisfies $x(t)\to 0$ as $t\to\infty$.

ii. If $\delta=-1$, then either $x(t)$ is oscillatory, $x(t)\to 0$ as $t\to\infty$, or $|x(t)|\to\infty$ as $t\to\infty$.

Proof: Suppose that equation (4.2.1) has an eventually positive solution $x(t)$. (The proof is similar when $x(t)$ is eventually negative). Say $x(t)>0$, $x(\tau(t))>0$, and $x(\sigma(t))>0$ for $t\geq t_1$ for some $t_1\geq t_0$.

***Proof of part* (i):** by Lemma 4.2.1, $z(t)\to 0$ as $t\to\infty$. Then by Lemma 4.1.1(iii) it follows that (4.1.3) holds. We have the following cases:

<u>*Case1*:</u> If n is even, then from Lemma 4.1.1(iv) we have $z(t)<0$. This with (4.2.2) imply that:

$$x(t)<p(t)x(\tau(t))<p_2x(\tau(t)) \quad \text{for } t\geq t_1. \tag{4.2.5}$$

Substitute $\tau^{-1}(t)$ instead of t and use notation (4.2.3), we get $x(\tau_1(t))<p_2x(t)$, then use (4.2.5), we have $x(\tau_1(t))<(p_2)^2x(\tau(t))$. Follow this process, and using notation (4.2.3), we have $x(\tau_k(t)) < (p_2)^{k+1} x(\tau(t))$ for any positive integer k. This implies $x(t) \to 0$ as $t \to \infty$, since $0 \le p_2 < 1$.

<u>*Case2:*</u> If n is odd, then from Lemma 4.1.1(iv) we have $z(t) > 0$. Using the fact that $z(t) \to 0$ as $t \to \infty$, we conclude that $0 < z(t) < K_1$ for some constant $K_1 > 0$, from which with (4.2.2) we have that $0 < x(t) = p(t)x(\tau(t)) + z(t) < p_2 x(\tau(t)) + K_1$. **If** $x(t)$ is unbounded, then there exists an increasing sequence $\{s_k\}$ with $s_1 > t_1$, $s_k \to \infty$ and $x(s_k) \to \infty$ as $k \to \infty$, and $x(s_k) = \max\{x(t) : s_1 \le t \le s_k\}$. For each k,

$$x(s_k) < p_2 x(\tau(s_k)) + K_1 \le p_2 x(s_k) + K_1, \qquad \text{so we have}$$

$$(1 - p_2)x(s_k) \le K_1 \tag{4.2.6}$$

But $0 \le p_2 < 1$, so (4.2.6) leads to a contradiction. Therefore, $x(t)$ is bounded and **if** $x(t) \not\to 0$ as $t \to \infty$, there exists $K_2 > 0$ such that $\limsup_{t\to\infty} x(t) = K_2$. So, there exists an increasing sequence $\{r_k\}$ with $r_1 > t_1$, $r_k \to \infty$ and $x(r_k) \to K_2$ as $k \to \infty$. From (1), and (4.2.2), it follows that

$$p_2 x(\tau(r_k)) \ge x(r_k) - z(r_k) \tag{4.2.7}$$

Since $K_2 > 0$, there exists $\varepsilon > 0$ such that $(1 + p_2)\varepsilon < (1 - p_2)K_2$ which implies that

$$0 < p_2(K_2 + \varepsilon) < K_2 - \varepsilon. \tag{4.2.8}$$

But for k sufficiently large, $x(\tau(r_k)) < K_2 + \varepsilon$, so by (4.2.7), and (4.2.8), we have

$$K_2 - \varepsilon > p_2 x(\tau(r_k)) \ge x(r_k) - z(r_k)$$

As $k \to \infty$, $z(r_k) \to 0$ so we obtain a contradiction to $x(r_k) \to K_2$ as $k \to \infty$. So $x(t)$ is either oscillatory or $x(t) \to 0$ as $t \to \infty$.

***Proof of part* (ii):** By Lemma 4.2.2, we have that either $x(t) \to \infty$ as $t \to \infty$, or (4.1.3) holds. So our goal is to show that if (4.1.3) holds, then $x(t) \to 0$ as $t \to \infty$. We can consider the following cases, when (4.1.3) holds:

<u>*Case1*:</u> If n is odd, by Lemma 4.1.2(iv) $z(t) < 0$, this with (4.2.2) imply that $x(t) < p(t)x(\tau(t)) < p_2 x(\tau(t))$, it then follows (as above in part (i)) that $x(\tau_k(t)) < (p_2)^{k+1} x(\tau(t))$ for each integer $k \geq 1$. Since $0 \leq p_2 < 1$, we have that $x(t) \to 0$ as $t \to \infty$.

<u>*Case2*:</u> If n is even, by Lemma 4.1.2(iv) $z(t) > 0$, and by (4.1.3) we see that $z'(t) < 0$. So $z(t)$ is decreasing. Therefore, exists constant $M > 0$, such that $0 < z(t) < M$, so we get $0 < x(t) < p_2 x(\tau(t)) + M$. Now, **If** $x(t)$ is unbounded, then there exists an increasing sequence $\{T_j\}$ with $T_1 > t_1$, $T_j \to \infty$ and $x(T_j) \to \infty$ as $j \to \infty$, and $x(T_j) = \max\{x(t) : T_1 \leq t \leq T_j\}$. For each j,

$$x(T_j) < p_2 x(\tau(T_j)) + M \leq p_2 x(T_j) + M, \qquad \text{so we have}$$

$$(1 - p_2) x(T_j) \leq M \tag{4.2.9}$$

But $0 \leq p_2 < 1$, so (4.2.9) leads to a contradiction. Therefore, $x(t)$ is bounded and if $x(t) \not\to 0$ as $t \to \infty$, there exists constant $N > 0$ such that

$\lim_{t\to\infty} \sup x(t) = N$. So, there exists an increasing sequence $\{S_j\}$ with $S_1 > t_1$, $S_j \to \infty$ and $x(S_j) \to N$ as $j \to \infty$. From (1), and (4.2.2), it follows that

$$p_2 x(\tau(S_j)) \geq x(S_j) - z(S_j) \tag{4.2.10}$$

Since $N > 0$, there exists $\varepsilon > 0$ such that $(1 + p_2)\varepsilon < (1 - p_2)N$ which implies that

$$0 < p_2(N + \varepsilon) < N - \varepsilon \tag{4.2.11}$$

But for j sufficiently large, $x(\tau(S_j)) < N + \varepsilon$, so by (4.2.10), and (4.2.11), we have

$$N - \varepsilon > p_2 x(\tau(S_j)) \geq x(S_j) - z(S_j)$$

As $j \to \infty$, $z(S_j) \to 0$ so we obtain a contradiction to $x(S_j) \to N$ as $j \to \infty$.

□

Example 4.2.1: The NDDE

$$\left[x(t) - \frac{1}{6}x(t - \ln 3)\right]^{(n)} + \frac{1}{2e}x(t-2)\exp\left\{\frac{1}{t-2}\ln\left[\frac{|x(t-2)| + e^{-(t-2)}}{2}\right]\right\} = 0, \quad t > 2 \tag{4.2.12}$$

satisfies all conditions of Theorem 4.2.1(i). If n is odd, then $x(t) = e^{-t}$ is a solution of (4.2.12) such that $x(t) \to 0$ as $t \to \infty$.

Example 4.2.2: The NDDE

$$\left[x(t) - \frac{1}{2}x(t-1)\right]^{(n)} + \frac{2^{n-1}}{e^2}(e^2 - 2)x(t-2)\exp\left\{\frac{1}{t-2}\ln\left[\frac{|x(t-2)| + e^{-2(t-2)}}{2}\right]\right\} = 0, t > 2 \tag{4.2.13}$$

satisfies all conditions of Theorem 4.2.1(i). If n is even, then $x(t)=e^{-2t}$ is a solution of (4.2.13) such that $x(t)\to 0$ as $t\to\infty$.

Example 4.2.3: The NDDE

$$\left[x(t)-\frac{e^{\frac{1}{2}}}{2}x(t-\frac{1}{2})\right]^{(n)}-\frac{e^{2}}{2}x(t-3)\exp\left\{\frac{1}{t-3}\ln\left[\frac{|x(t-3)|+e^{t-3}}{2}\right]\right\}=0\,,\quad t>2$$

(4.2.14)

satisfies all conditions of Theorem 4.2.1(ii). If n is either even or odd, then $x(t)=e^{t}$ is a solution of (4.2.14) such that $x(t)\to\infty$ as $t\to\infty$. Also, note that $x(t)=-e^{t}$ is a solution and satisfies that $|x(t)|\to\infty$ as $t\to\infty$.

Example 4.2.4: Consider the NDDE

$$[x(t)-px(t-2\pi)]^{(2k)}+(-1)^{k+1}(1-p)x(t-2\pi)=0 \qquad (4.2.15)$$

where $0<p<1$. If k is odd, then (4.2.15) satisfies conditions of Theorem 4.2.1(i) , and so it may have oscillatory solutions. Indeed, $x(t)=\sin t$ is such a solution. If k is even, then (4.2.15) satisfies conditions of Theorem 4.2.1(ii), and so it may have oscillatory solutions. indeed, $x(t)=\sin t$ is such a solution.

Example 4.2.5: Consider the NDDE

$$[x(t)-px(t-2\pi)]^{(2k-1)}+(-1)^{k+1}(1-p)x(t-\frac{\pi}{2})=0 \qquad (4.2.16)$$

where $0<p<1$. If k is odd, then (4.2.16) satisfies conditions of Theorem 4.2.1(i) , and so it may have oscillatory solutions. Indeed, $x(t)=\sin t$ is such a solution. If k is even, then (4.2.16) satisfies conditions of Theorem

4.2.1(ii), and so it may have oscillatory solutions. In fact, $x(t) = \sin t$ is such a solution.

Remark 4.2.2: Theorem 4.2.1 is due to Graef and Spikes, it is theorem 3 at [20], which also generalizes, and modifies Theorem 7 in [18]. Where theorem 7 in [18], is interested in equations of forms like (4.1.20), with $p(t)$ the same as in (1) ,which is special case of equation (4.2.1).

Chapter Five

Oscillation of *n-th* Order NDDE When the Non-Derivative Part is Dependent on the Independent Variable and the Unknown Function with and without Delay

5.0 Introduction

In this chapter we consider the oscillation of *n-th* order NDDE when the non-derivative part is a function of the independent variable, and the unknown function with and without delayed argument, i.e. we consider an equation of the form

$$[x(t)+p(t)x(\tau(t))]^{(n)}+f(t,x(t),x(\sigma(t)))=0 \qquad (5.0.1)$$

Where, $p(t),\tau(t),\sigma(t)\in C([t_0,\infty),R)$, $t_0\geq 0$, $\tau(t)$ and $\sigma(t)$ are delayed arguments, and $f:[t_0,\infty)\times R\times R\to R$ is continuous.

This chapter contains results of Zafer [39], and our results for the oscillation of equation (5.0.1). In section 5.1, we introduce the results of Zafer, where he established sufficient conditions for oscillation of equation (5.0.1) when the order is even, and also when the order is odd for unbounded solutions.

While our results are introduced in sections 5.2, and 5.3. In section 5.2, we establish sufficient conditions for oscillation of equation (5.0.1) when $\sigma(t)=t-\sigma$, $\sigma>0$. In section 5.3 we study bounded oscillation of equation (5.0.1) with oscillating coefficients. In our work, we depend on the ideas of Zafer [39]. Also we benefit from the following papers, Parhi

[32], Dahiya and Zafer [10], Bolat and Akin [5], Shen [35], and B. Li [30].

5.1 Oscillation of *n-th* order NDDE

Consider the NDDE

$$[x(t)+p(t)x(\tau(t))]^{(n)}+f(t,x(t),x(\sigma(t)))=0 \qquad (5.1.1)$$

where $n \geq 2$, and the following conditions are always assumed to hold:

(1) $p(t) \in C([0,\infty),R)$, such that $0 \leq p(t) < 1$

(2) $\tau(t),\sigma(t) \in C([0,\infty),R_+)$, $\tau(t) < t$, $\sigma(t) \leq t$, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$

(3) $f:[0,\infty)\times R\times R \to R$ is continuous, and $yf(t,x,y) > 0$ for $xy > 0$.

This section contains two theorems, each one presents certain sufficient conditions for oscillation of solutions of equation (5.1.1).

Theorem 5.1.1: Assume that $\phi(t)$ is a non-negative continuous function on $[0,\infty)$, and that $w(t) > 0$ for $t > 0$ is continuous and nondecreasing on $[0,\infty)$ with:

$$|f(t,x,y)| \geq \phi(t)w\left(\frac{|y|}{[1-p(\sigma(t))][\sigma(t)]^{n-1}}\right), \qquad (5.1.2)$$

$$\int_0^{\pm\alpha}\frac{dx}{w(x)} < \infty, \quad \text{for every } \alpha > 0, \qquad (5.1.3)$$

and

$$\int^{\infty}\phi(t)dt = \infty \qquad (5.1.4)$$

Then:

i. If n is even, every solution $x(t)$ of equation (5.1.1) is oscillatory.

ii. If n is odd, every unbounded solution $x(t)$ of equation (5.1.1) is oscillatory.

Proof: Assume that equation (5.1.1) has a non-oscillatory solution $x(t)$. Without loss of generality, let $x(t)$ be eventually positive (the proof is similar when $x(t)$ is eventually negative). Set

$$z(t) = x(t) + p(t)x(\tau(t)) \tag{5.1.5}$$

In view of (1), and (2), $x(t)$ and $x(\tau(t))$ become eventually positive, and so $z(t)$ is also eventually positive. From equation (5.1.1), we have

$$z^{(n)}(t) = -f(t.x(t), x(\sigma(t))) < 0 \tag{5.1.6}$$

implies that $z(t)z^{(n)}(t) < 0$ eventually . Now, by applying Lemma 1.6.1 there exists $t_1 \geq 0$, and an integer l, $0 \leq l \leq n-1$ with $n-l$ odd, such that for $t \geq t_1$

$$z(t)z^{(k)}(t) > 0,\ k = 0,1,...,l, \tag{5.1.7}$$

$$(-1)^{n+k-1} z(t)z^{(k)}(t) > 0,\ k = l+1,...,n-1,\ \text{and} \tag{5.1.8}$$

$$|z(t)| \geq \frac{(t-t_1)^{n-1}}{(n-1)...(n-l)} \left| z^{(n-1)}(2^{n-l-1}t) \right|, \tag{5.1.9}$$

If n is even, the integer l associated with $z(t)$ is greater than or equal to 1. But if n is odd then $l \in \{0,2,...,n-1\}$, and since the solution $x(t)$ is unbounded for odd orders, then $z(t)$ is unbounded, and hence $l \geq 2$.

Therefore, either n is odd or even, then $l \geq 1$. Hence $z(t)$ is increasing for $t \geq t_1$.

From (5.1.5) it is clear that $x(t) \leq z(t)$, so $x(\tau(t)) \leq z(\tau(t))$. Using this fact, and the fact that $z(t)$ is increasing, it follows that

$$z(t) = x(t) + p(t)x(\tau(t)) \leq x(t) + p(t)z(\tau(t)) \leq x(t) + p(t)z(t) \quad \text{for} \quad t \geq t_1,$$

Hence,

$$x(t) \geq [1 - p(t)]z(t) \quad \text{for} \quad t \geq t_1 \tag{5.1.10}$$

From (5.1.9), and the fact that $z(t)$ is increasing, we have

$$z(t) > z(2^{l-n+1}t) \geq \frac{2^{(l-n+1)(n-1)}}{(n-1)...(n-l)}(t-t_2)^{n-1}z^{(n-1)}(t), \quad \text{for} \quad t \geq t_2 = 2^{n-l-1}t_1$$

Therefore, by choosing $t_3 > t_2$, arbitrarily large, we have

$$z(t) \geq ct^{n-1}z^{(n-1)}(t), \quad \text{for} \quad t \geq t_3 \tag{5.1.11}$$

where $c > 0$ is an appropriate constant dependent upon l and n.

Let $t_4 \geq t_3$ be such that $\sigma(t) \geq t_3$, for all $t \geq t_4$. From (5.1.10), and (5.1.11), we get

$$\frac{x(\sigma(t))}{[1 - p(\sigma(t))][\sigma(t)]^{n-1}} \geq cz^{(n-1)}(\sigma(t)), \quad \text{for} \quad t \geq t_4 \tag{5.1.12}$$

using the fact that $z^{(n-1)}(t)$ is decreasing, we have

$$\frac{x(\sigma(t))}{[1 - p(\sigma(t))][\sigma(t)]^{n-1}} \geq cz^{(n-1)}(t) \quad \text{for} \quad t \geq t_4 \tag{5.1.13}$$

using (5.1.2), and (5.1.13), then it follows from (5.1.6), that

$$z^{(n)}(t) + \phi(t)w\left(cz^{(n-1)}(t)\right) \leq 0 \tag{5.1.14}$$

Setting $y(t) = cz^{(n-1)}(t)$, we have

$$y'(t) + c\phi(t)w(y(t)) \le 0 \tag{5.1.15}$$

dividing (5.1.15) by $w(y(t))$, and integrating from t_4 to t, we obtain

$$\int_{y(t_4)}^{y(t)} \frac{ds}{w(s)} + c\int_{t_4}^{t} \phi(r)dr \le 0, \text{ then}$$

$$c\int_{t_4}^{t} \phi(r)dr \le \int_{y(t\ )}^{y(t_4)} \frac{ds}{w(s)} \tag{5.1.16}$$

Since $y'(t) < 0$, so $y(t)$ is decreasing. And since $y(t) > 0$, it follows that $\lim_{t\to\infty} y(t) = L \ge 0$. If $L \ne 0$, then by (5.1.16) we must have

$$\int_{t_4}^{\infty} \phi(t)dt < \infty \tag{5.1.17}$$

which contradicts (5.1.4). In the case when $L = 0$, letting $t \to \infty$ in (5.1.16) and using (5.1.3), we again obtain (5.1.17). Thus the proof is complete. □

Example 5.1.1: Consider the NDDE

$$[x(t) + p(t)x(\tau(t))]^{(n)} + p(t)|x(\sigma(t))|^{\beta} \operatorname{sgn}(x(\sigma(t))) = 0 \tag{5.1.18}$$

where $0 < \beta < 1$, and n is even. Use $w(x) = |x|^{\beta}$, Then, if

$$\int^{\infty} [1 - p(\sigma(t))]^{\beta} [\sigma(t)]^{\beta(n-1)} p(t)dt = \infty$$

by Theorem 5.1.1, every solution of (5.1.18) is oscillatory.

In the next theorem, Theorem 5.1.2, we have different sufficient conditions for the oscillation of equation (5.1.1). Firstly, we introduce the following lemma that is useful in the proof of Theorem 5.1.2

Lemma 5.1.1: (see theorems 2.1.1, and 2.1.3 in [29])

If $q(t), \sigma(t) \in C(R_+, R_+)$, $\sigma(t) < t$, and $\lim_{t\to\infty} \sigma(t) = \infty$. If either

$$\lim_{t\to\infty} \inf \int_{\sigma(t)}^{t} q(s)ds > \frac{1}{e} \tag{5.1.19}$$

or $\sigma(t)$ is nondecreasing and

$$\lim_{t\to\infty} \sup \int_{\sigma(t)}^{t} q(s)ds > 1 \tag{5.1.20}$$

Then the differential inequality

$$u'(t) + q(t)u(\sigma(t)) \le 0 \tag{5.1.21}$$

has no eventually positive solutions.

Theorem 5.1.2: Assume that there exists a continuous nonnegative function $\psi(t)$ defined on $[0,\infty)$, which is not identically zero on any half-line of the form $[t_*,\infty)$, such that $yf(t,x,y) \ge \psi(t)y^2$ for all $t \ge 0$.

If either

$$\lim_{t\to\infty} \inf \int_{\sigma(t)}^{t} [\sigma(s)]^{n-1}[1 - p(\sigma(s))]\psi(s)ds > \frac{(n-1)2^{(n-1)(n-2)}}{e} \tag{5.1.22}$$

or $\sigma(t)$ is nondecreasing and

$$\lim_{t\to\infty} \sup \int_{\sigma(t)}^{t} [\sigma(s)]^{n-1}[1 - p(\sigma(s))]\psi(s)ds > (n-1)2^{(n-1)(n-2)} \tag{5.1.23}$$

Then

i. If n is even, every solution $x(t)$ of equation (5.1.1) is oscillatory.

ii. If n is odd, every unbounded solution $x(t)$ of equation (5.1.1) is oscillatory.

Proof: Proceed the same as in the proof of Theorem 5.1.1, until step in which we get (5.1.12). Use the fact that $yf(t,x,y) \geq \psi(t)y^2$, and since $x(\sigma(t)) > 0$ eventually, we have

$$f(t,x,x(\sigma(t))) \geq \psi(t)x(\sigma(t)) \tag{5.1.24}$$

From (5.1.12) we have:

$$x(\sigma(t)) \geq cz^{(n-1)}(\sigma(t))[1-p(\sigma(t))][\sigma(t)]^{n-1}, \quad \text{for } \ t \geq t_4 \tag{5.1.25}$$

where $c = \dfrac{2^{(l-n+1)(n-1)}}{(n-1)...(n-l)}$, $l \in \{1,3,...,n-1\}$ if n is even, and $l \in \{2,4,....n-1\}$

if n is odd.

From (5.1.24), and (5.1.25), we obtain

$$f(t,x,x(\sigma(t))) \geq c\psi(t)\ z^{(n-1)}(\sigma(t))[1-p(\sigma(t))][\sigma(t)]^{n-1}, \text{ for } t \geq t_4 \tag{5.1.26}$$

Notice that c takes on its smallest value when $l = 1$, and using (5.1.6), and (5.1.26), we have

$$z^{(n)}(t) + \frac{2^{(2-n)(n-1)}}{(n-1)}[\sigma(t)]^{n-1}[1-p(\sigma(t))]\psi(t)z^{(n-1)}(\sigma(t)) \leq 0 \tag{5.1.27}$$

Let $u(t) = z^{(n-1)}(t)$ in (5.1.27), we have

$$u'(t) + \frac{2^{(2-n)(n-1)}}{(n-1)}[\sigma(t)]^{n-1}[1-p(\sigma(t))]\psi(t)u(\sigma(t)) \leq 0 \tag{5.1.28}$$

Since (5.1.22) or (5.1.23) is satisfied, by Lemma 5.1.1 inequality (5.1.28) has no eventually positive solution. But from (5.1.8), $u(t) = z^{(n-1)}(t)$ is eventually positive, so a contradiction. This completes the proof. □

Corollary 5.1.1: If n is odd, then we can deduce Theorem 5.1.2, with assumptions weaker than (5.1.22), and (5.1.23). Since in the case of odd order we assume that the solution $x(t)$ is unbounded, then we can take c to be smallest at $l=2$ since $l \in \{2,4,...,n-1\}$. Therefore take smallest c as $c=\dfrac{2^{(3-n)(n-1)}}{(n-1)(n-2)}$, and then assumptions (5.1.22), and (5.1.23) will be

$$\lim_{t\to\infty}\inf \int_{\sigma(t)}^{t} [\sigma(s)]^{n-1}[1-p(\sigma(s))]\psi(s)ds > \frac{(n-1)(n-2)2^{(n-1)(n-3)}}{e} \tag{5.1.29}$$

or $\sigma(t)$ is nondecreasing and

$$\lim_{t\to\infty}\sup \int_{\sigma(t)}^{t} [\sigma(s)]^{n-1}[1-p(\sigma(s))]\psi(s)ds > (n-1)(n-2)2^{(n-1)(n-3)} \tag{5.1.30}$$

Corollary 5.1.2: If we consider the linear NDDE

$$[x(t)+p(t)x(\tau(t))]^{(n)}+q(t)x(\sigma(t))=0 \tag{5.1.31}$$

where p,τ,σ as stated above, and $q(t)$ is a continuous nonnegative function defined on $[0,\infty)$, and not identically zero on any half-line of the form $[t_*,\infty)$,. Then the results of Theorem 5.1.2, are true if we directly put $\psi(t)=q(t)$ in conditions (5.1.22), and (5.1.23).

Example 5.1.2: The NDDE

$$\left[x(t)+\frac{1}{2}x(t-2\pi)\right]''+2\left(e^{\frac{\pi}{2}}+\frac{1}{2}e^{\frac{-3\pi}{2}}\right)x(t-\frac{\pi}{2})=0 \tag{5.1.32}$$

satisfies the conditions of Theorem 5.1.2. Therefore, every solution of (5.1.32) is oscillatory. Indeed, $x(t)=e^{t}\sin t$ is such solution.

5.2 Oscillation of *n-th* order NDDE when the non-derivative part has constant delay

In this section we interested in the oscillation of *n-th* order NDDE, when the non-derivative part of the equation contains constant delay. i.e. we will consider the following equation

$$\left[x(t)+p(t)x(\tau(t))\right]^{(n)}+f(t,x(t),x(t-\sigma))=0 \qquad (5.2.1)$$

where $n \geq 2$, and the following conditions are always assumed to hold:

(1) $p(t) \in C([0,\infty),R)$, such that $0 \leq p(t) < 1$

(2) $\tau(t) \in C([0,\infty),R_+)$, $\tau(t) < t$, $\lim_{t\to\infty} \tau(t) = \infty$, and $\sigma > 0$

(3) $f:[0,\infty)\times R\times R \to R$ is continuous, and $yf(t,x,y) > 0$ for $xy > 0$.

In this section we will establish sufficient conditions for the oscillation of equation (5.2.1) by using the ideas used in Theorem 5.1.2. i.e. we proceed with equation (5.2.1) to get first order delay differential equation, but instead of using the criteria in Lemma 5.1.1, we use an improved criteria, and this improves our results.

We need the following Lemmas for our work:

Lemma 5.2.1: (see Theorem 1 in [30])

If $\sigma > 0$, $q(t) \in C(R_+,R_+)$, $\int_t^{t+\sigma} q(s)ds > 0$ for $t \geq t_0$ for some $t_0 > 0$, and

$$\int_{t_0}^{\infty} q(t)\ln\left(e\int_t^{t+\sigma} q(s)ds\right)dt = \infty \qquad (5.2.2)$$

Then (5.2.3) is oscillatory, where

$$u'(t)+q(t)u(t-\sigma)=0 \qquad (5.2.3)$$

Lemma 5.2.2: (see [24] page 67)

If $\sigma > 0$, $q(t) \in C(R_+, R_+)$. Then the differential inequality

$$u'(t) + q(t)u(t-\sigma) \le 0 \tag{5.2.4}$$

has an eventually positive solution if and only if equation (5.2.3) has an eventually positive solution.

Theorem 5.2.1: Assume that there exists a continuous nonnegative function $\psi(t)$ defined on R_+, which is not identically zero on any half-line of the form $[t_*, \infty)$, such that $yf(t,x,y) \ge \psi(t)y^2$ for all $t \ge 0$. If

$$\int_t^{t+\sigma} [s-\sigma]^{n-1}[1-p(s-\sigma)]\psi(s)ds > 0 \quad \text{for } t \ge t_0 \text{ for some } t_0 > 0, \tag{5.2.5}$$

and

$$\int_{t_0}^{\infty} [t-\sigma]^{n-1}[1-p(t-\sigma)]\psi(t)\ln\left(e\int_t^{t+\sigma} c[s-\sigma]^{n-1}[1-p(s-\sigma)]\psi(s)ds \right)dt = \infty \tag{5.2.6}$$

where $c = \dfrac{2^{(2-n)(n-1)}}{(n-1)}$

Then:

i. If n is even, every solution $x(t)$ of equation (5.2.1) is oscillatory.

ii. If n is odd, every unbounded solution $x(t)$ of equation (5.2.1) is oscillatory.

Proof: Proceed the same as in the proof of Theorem 5.1.2, instead of $\sigma(t)$ use $t-\sigma$, until step in which we get (5.1.28). Using $\sigma(t) = t-\sigma$, then (5.1.28) will be

$$u'(t)+\frac{2^{(2-n)(n-1)}}{(n-1)}[t-\sigma]^{n-1}[1-p(t-\sigma)]\psi(t)u(t-\sigma)\leq 0 \qquad (5.2.7)$$

From conditions (5.2.5) and (5.2.6), and by using Lemmas 5.2.1, and 5.2.2, it follows that (5.2.7) has no eventually positive solution. But from (5.1.8), $u(t)=z^{(n-1)}(t)$ is eventually positive, so a contradiction. This completes the proof. □

Remark 5.2.1: depending on the same technique i.e. by using the results of oscillation of first order delay differential equations we can establish many results for oscillation of equations (5.1.1) and (5.2.1). And the results of Theorems 5.1.2, and 5.2.1 are just some applications of this observation. Also, it is interesting to note that the problem of establishing conditions for the oscillation of first order delay differential equations has been the subject of many investigations. You can see, [14, 30, 26, 36, 38]. However, in Theorem 5.2.1, we get conditions that are improved than that of Theorem 5.1.2 in the case of $\sigma(t)=t-\sigma,\ \sigma>0$. {see example 5.2.1}. One can get different, and may be more improved, results if he follows the known results of oscillation of first order delay differential equation. In this situation, we would like to refer to paper [36], where in this paper the authors presented a survey for the most known results of oscillation of first order delay differential equations.

Example 5.2.1: Consider the NDDE

$$\left[x(t)+\frac{1}{2}x(t-1)\right]''+\frac{2e^{0.5\sin t-1}}{t-1}x(t-1)=0,\ t>1 \qquad (5.2.8)$$

since equation (5.2.8) is linear we use $\psi(t)=\dfrac{2e^{0.5\sin t-1}}{t-1}$.

If we check the condition (5.1.22) of Theorem 5.1.2, then

$$\lim_{t\to\infty}\ \inf\frac{1}{2}\int_{t-1}^{t}[s-1]\frac{2e^{0.5\sin s-1}}{s-1}ds=\lim_{t\to\infty}\ \inf\int_{t-1}^{t}e^{0.5\sin s-1}ds<\frac{1}{e}$$

Therefore, condition (5.1.22) is not satisfied. Also you can see that condition (5.1.23) is not satisfied. So by using Theorem 5.1.2, we can not have a conclusion about the oscillation of equation (5.2.8).

Now let us check the conditions of Theorem 5.2.1,

$\dfrac{1}{2}\displaystyle\int_{t}^{t+1}[s-1]\frac{2e^{0.5\sin s-1}}{s-1}ds=\int_{t}^{t+1}e^{0.5\sin s-1}ds>0$, for $t>1$, hence (5.2.5) holds. Then check for condition (5.2.6)

$$\frac{1}{2}\int_{1}^{\infty}[t-1]\frac{2e^{0.5\sin t-1}}{t-1}\ln\left(\frac{e}{2}\int_{t}^{t+1}[s-1]\frac{2e^{0.5\sin s-1}}{s-1}ds\right)dt=\int_{1}^{\infty}e^{0.5\sin t-1}\ln\left(e\int_{t}^{t+1}e^{0.5\sin s-1}ds\right)dt$$

using Jensen’s Inequality (see appendix F), then

$$\int_{1}^{\infty}e^{0.5\sin t-1}\ln\left(e\int_{t}^{t+1}e^{0.5\sin s-1}ds\right)dt\geq\int_{1}^{\infty}e^{0.5\sin t-1}\left(\int_{t}^{t+1}0.5\sin s\,ds\right)dt \tag{5.2.9}$$

$$\Longrightarrow\int_{1}^{\infty}e^{0.5\sin t-1}\left(\int_{t}^{t+1}0.5\sin s\,ds\right)dt=\frac{-1}{2e}\int_{1}^{\infty}e^{0.5\sin t}\left(\cos(t+1)-\cos t\right)dt$$

$$=\frac{-1}{2e}\int_{1}^{\infty}e^{0.5\sin t}\left(\cos(t+\frac{1}{2}+\frac{1}{2})-\cos(t+\frac{1}{2}-\frac{1}{2})\right)dt$$

Use the following identities

$$\cos(t+\frac{1}{2}+\frac{1}{2})=\cos(t+\frac{1}{2})\cos\frac{1}{2}-\sin(t+\frac{1}{2})\sin\frac{1}{2}$$

$$\cos(t+\frac{1}{2}-\frac{1}{2})=\cos(t+\frac{1}{2})\cos\frac{1}{2}+\sin(t+\frac{1}{2})\sin\frac{1}{2}$$

Then we have:

$$\int_1^{\infty} e^{0.5\sin t-1}\left(\int_t^{t+1} 0.5\sin s ds\right)dt \;= \frac{\sin\frac{1}{2}}{e}\int_1^{\infty} e^{0.5\sin t}\sin(t+\frac{1}{2})dt$$

$$= \frac{\sin\frac{1}{2}}{e}\int_1^{\infty} e^{0.5\sin t}\left(\sin t\cos\frac{1}{2}+\cos t\sin\frac{1}{2}\right)dt \qquad (5.2.10)$$

Note that $\int_1^{t} e^{0.5\sin t}\cos t dt$ is bounded, and $\int_0^{2\pi} e^{0.5\sin t}\sin t dt > 0$, so we conclude that $\frac{1}{2}\int_1^{\infty}[t-1]\frac{2e^{0.5\sin t-1}}{t-1}\ln\left(\frac{e}{2}\int_t^{t+\sigma}[s-1]\frac{2e^{0.5\sin s-1}}{s-1}ds\right)dt = \infty$, hence condition (5.2.6) is satisfied. Therefore by Theorem 5.2.1 every solution of equation (5.2.8) is oscillatory.

5.3 Bounded oscillation of *n-th* order NDDE with oscillating coefficients

Consider the NDDE

$$[x(t)+p(t)x(\tau(t))]^{(n)} + f(t,x(t),x(\sigma(t))) = 0 \qquad (5.3.1)$$

where $n \geq 2$, and the following conditions are always assumed to hold:

(1) $p(t) \in C([0,\infty),R)$ is an oscillatory function, and $\lim_{t\to\infty} p(t) = 0$.

(2) $\tau(t),\sigma(t) \in C([0,\infty),R_+)$, $\tau(t) \leq t$, $\sigma(t) \leq t$, and $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\sigma(t) = \infty$

(3) $f : [0,\infty)\times R\times R \to R$ is continuous, and $yf(t,x,y) > 0$ for $xy > 0$.

The main results of this section are contained in Theorems 5.3.1, and 5.3.2. In the first theorem we present results for bounded oscillation of even order, while the second theorem deals with case of odd order.

Theorem 5.3.1: Assume that $\phi(t)$ is a non-negative continuous function on R_+, and that $w(t) > 0$ for $t > 0$ is continuous and nondecreasing on R_+ with:

$$|f(t,x,y)| \geq \phi(t) w\left(\frac{|y|}{[\sigma(t)]^{n-1}}\right), \tag{5.3.2}$$

and

$$\int_0^{\pm\alpha} \frac{dx}{w(x)} < \infty, \quad \text{for every } \alpha > 0. \tag{5.3.3}$$

If n is even and

$$\int^{\infty} \phi(t) dt = \infty \tag{5.3.4}$$

Then every bounded solution $x(t)$ of equation (5.3.1) is oscillatory.

Proof: Assume that equation (5.3.1) has a bounded non-oscillatory solution $x(t)$. Without loss of generality, let $x(t)$ be eventually positive (the proof is similar when $x(t)$ is eventually negative). That is, let $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_0 \geq 0$.

Set

$$z(t) = x(t) + p(t)x(\tau(t)) \tag{5.3.5}$$

From (5.3.1) and (5.3.5) we have

$$z^{(n)}(t) = -f(t, x(t), x(\sigma(t))) < 0 \tag{5.3.6}$$

That is $z^{(n)}(t) < 0$. It follows that $z^{(i)}(t)$ ($i = 0,1,\ldots,n-1$) is strictly monotonic and of constant sign eventually. Since $p(t)$ is an oscillatory function, there exists a $t_1 \geq t_0$ such that as $t \geq t_1$, $z(t) > 0$ eventually. Since $x(t)$ is bounded, and by using the fact that $\lim_{t\to\infty} p(t) = 0$, then it follows from (5.3.5) that there is a $t_2 \geq t_1$, such that $z(t)$ is also bounded for $t \geq t_2$.

Now, by applying Lemma 1.6.1, there exists $t_3 \geq t_2$, and an integer l, $0 \leq l \leq n-1$ with $n-l$ odd, such that for $t \geq t_3$

$$z(t)z^{(k)}(t) > 0, \; k = 0,1,\ldots,l, \tag{5.3.7}$$

$$(-1)^{n+k-1} z(t)z^{(k)}(t) > 0, \; k = l+1,\ldots,n-1, \text{ and} \tag{5.3.8}$$

$$|z(t)| \geq \frac{(t-t_3)^{n-1}}{(n-1)\ldots(n-l)} \left| z^{(n-1)}(2^{n-l-1}t) \right|, \tag{5.3.9}$$

Since n is even, and $z(t)$ is bounded it follows that $l = 1$ (otherwise $z(t)$ is not bounded). And from (5.3.7) we have $z'(t) > 0$, so $z(t)$ is increasing.

Since $x(t)$ is bounded, by (1) it follows that $\lim_{t\to\infty} p(t)x(\tau(t)) = 0$. Then using this fact and by (5.3.5) there exists a $t_4 \geq t_3$ such that

$$x(t) \geq \lambda z(t) \quad \text{for} \quad t \geq t_4 \tag{5.3.10}$$

where λ is some number in $(0,1)$.

From (5.3.9), and the fact that $z(t)$ is increasing, we have

$$z(t) \geq z(2^{2-n}t) \geq \frac{2^{(2-n)(n-1)}}{(n-1)} (t-t_5)^{n-1} z^{(n-1)}(t), \quad \text{for} \quad t \geq t_5 = 2^{n-2}t_3$$

Therefore, by choosing $t_6 > t_5$, arbitrarily large, we have

$$z(t) \geq ct^{n-1} z^{(n-1)}(t), \quad \text{for} \quad t \geq t_6 \tag{5.3.11}$$

where $c > 0$ is an appropriate constant dependent upon n.

Let $t_7 \geq t_6$ such that $\sigma(t) \geq t_6$ for all $t \geq t_7$, then it follows from (5.3.11) that

$$z(\sigma(t)) \geq c[\sigma(t)]^{n-1} z^{(n-1)}(\sigma(t)) \; \text{for} \quad t \geq t_7 \tag{5.3.12}$$

Let $t_8 = \max\{t_4, t_7\}$ then by (5.3.10), and (5.3.12) we have

$$x(\sigma(t)) \geq \lambda z(\sigma(t)) \geq \lambda c[\sigma(t)]^{n-1} z^{(n-1)}(\sigma(t)) \quad \text{for} \quad t \geq t_8$$

Hence, we have:

$$\frac{x(\sigma(t))}{[\sigma(t)]^{n-1}} \geq \lambda c z^{(n-1)}(\sigma(t)), \quad \text{for} \quad t \geq t_8 \tag{5.3.13}$$

using the fact that $z^{(n-1)}(t)$ is decreasing, it follows from (5.3.13) that

$$\frac{x(\sigma(t))}{[\sigma(t)]^{n-1}} \geq \lambda c z^{(n-1)}(t), \quad \text{for} \quad t \geq t_8 \tag{5.3.14}$$

using (5.3.2), and (5.3.14), then it follows from (5.3.6), that

$$z^{(n)}(t) + \phi(t) w\left(\lambda c z^{(n-1)}(t)\right) \leq 0 \tag{5.3.15}$$

Setting $y(t) = \lambda c z^{(n-1)}(t)$, we have

$$y'(t) + \lambda c \phi(t) w\left(y(t)\right) \leq 0 \tag{5.3.16}$$

dividing (5.3.16) by $w\left(y(t)\right)$, and integrating from t_8 to t, we obtain

$$\int_{y(t_8)}^{y(t)} \frac{ds}{w(s)} + \lambda c \int_{t_8}^{t} \phi(r) dr \leq 0, \quad \text{then}$$

$$\lambda c \int_{t_8}^{t} \phi(r) dr \leq \int_{y(t\ )}^{y(t_8)} \frac{ds}{w(s)} \tag{5.3.17}$$

Since $y'(t) < 0$, so $y(t)$ is decreasing. And since $y(t) > 0$, it follows that $\lim_{t \to \infty} y(t) = L \geq 0$. If $L \neq 0$, then by (5.3.17) we must have

$$\int_{t_8}^{\infty} \phi(t) dt < \infty \tag{5.3.18}$$

which contradicts (5.3.4). In the case when $L=0$, letting $t \to \infty$ in (5.3.17) and using (5.3.3), we again obtain (5.3.18). Thus the proof is complete. □

Example 5.3.1: Consider the NDDE

$$[x(t)+p(t)x(\tau(t))]^{(n)}+q(t)|x(\sigma(t))|^{\beta}G\left(\frac{x(\sigma(t))}{[\sigma(t)]^{n-1}}\right)\operatorname{sgn}(x(\sigma(t)))=0 \qquad (5.3.19)$$

where, n is even, p and σ as stated above, $0<\beta<1$, $q(t)$ is a continuous nonnegative function defined on $[0,\infty)$, and not identically zero on any half-line of the form $[t_*,\infty)$, and $G(x)$ is a continuous function which is positive and nondecreasing for all $x \in R$.

Use $w(x)=|x|^{\beta}G(x)$, then if

$$\int^{\infty} [\sigma(t)]^{\beta(n-1)}q(t)dt=\infty$$

by Theorem 5.3.1 every bounded solution of equation (5.3.19) is oscillatory.

In the next theorem, Theorem 5.3.2, besides conditions (1)-(3) we further assume that:

(4) $0<t-\sigma(t)\le\sigma_0$, where σ_0 is positive constant

Theorem 5.3.2: Assume that $\phi(t)$ is a non-negative continuous function on R_+, and that $w(t)>0$ for $t>0$ is continuous and nondecreasing on R_+ with:

$$|f(t,x,y)| \ge \phi(t)w\left(\frac{|y|}{[t-\sigma(t)]^{n-1}}\right), \qquad (5.3.20)$$

and

$$\int_0^{\pm\alpha} \frac{dx}{w(x)} < \infty, \quad \text{for every } \alpha > 0. \tag{5.3.21}$$

If n is odd and

$$\int^{\infty} \phi(t)dt = \infty \tag{5.3.22}$$

Then every bounded solution $x(t)$ of equation (5.3.1) is oscillatory.

Proof: Assume that equation (5.3.1) has a bounded non-oscillatory solution $x(t)$. Without loss of generality, let $x(t)$ be eventually positive (the proof is similar when $x(t)$ is eventually negative). That is, let $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_0 \geq 0$.

Set $z(t)$ as in (5.3.5). Then from (5.3.1) and (5.3.5) we have (5.3.6)

From (5.3.6) $z^{(n)}(t) < 0$ eventually. It follows that $z^{(i)}(t)$ ($i = 0,1,\dots,n-1$) is strictly monotonic and of constant sign eventually. Since $p(t)$ is an oscillatory function, there exists a $t_1 \geq t_0$ such that as $t \geq t_1$, $z(t) > 0$ eventually. Since $x(t)$ is bounded, and by using the fact that $\lim_{t\to\infty} p(t) = 0$, then it follows from (5.3.5) that there is a $t_2 \geq t_1$, such that $z(t)$ is also bounded for $t \geq t_2$.

Now, applying Lemma 1.6.1, there exists $t_3 \geq t_2$, and an integer l, $0 \leq l \leq n-1$ with $n-l$ odd, such that for $t \geq t_3$ the (5.3.7), and (5.3.8) are satisfied. Since n is odd and $z(t)$ is bounded then $l = 0$. Hence from relations (5.3.7), and (5.3.8) we have

$$(-1)^k z^{(k)}(t) > 0\,,\ k = 0,1,\ldots,n-1 \tag{5.3.23}$$

Since n is odd, from (4) and (5.3.23) follows that we can apply Lemma 1.6.5. Using Lemma 1.6.5, we have

$$z(\sigma(t)) = z\big(t-(t-\sigma(t))\big) \geq \frac{[t-\sigma(t)]^{n-1}}{(n-1)!} z^{(n-1)}(t) \quad \text{for } t \geq t_3 + \sigma_0$$

Hence,

$$z(\sigma(t)) \geq \frac{[t-\sigma(t)]^{n-1}}{(n-1)!} z^{(n-1)}(t) \quad \text{for } t \geq t_3 + \sigma_0 \tag{5.3.24}$$

Since $x(t)$ is bounded, by (1) it follows that $\lim_{t\to\infty} p(t)x(\tau(t)) = 0$. Then using this fact and by (5.3.5) there exists a $t_4 \geq t_3$ such that

$$x(t) \geq \lambda z(t) \quad \text{for} \quad t \geq t_4 \tag{5.3.25}$$

where λ is some number in $(0,1)$.

Let $t_5 \geq t_4$ such that $\sigma(t) \geq t_4$ for all $t \geq t_5$, then it follows from (5.3.25) that

$$x(\sigma(t)) \geq \lambda z(\sigma(t)) \quad \text{for} \quad t \geq t_5 \tag{5.3.26}$$

Let $t_6 = \max\{t_5, t_3 + \sigma_0\}$. Then, by (5.3.25) and (5.3.26) we have

$$x(\sigma(t)) \geq \lambda z(\sigma(t)) \geq \lambda \frac{[t-\sigma(t)]^{n-1}}{(n-1)!} z^{(n-1)}(t) \quad \text{for} \quad t \geq t_6$$

then we get

$$\frac{x(\sigma(t))}{[t-\sigma(t)]^{n-1}} \geq \frac{\lambda z^{(n-1)}(t)}{(n-1)!} \quad \text{for} \quad t \geq t_6 \tag{5.3.27}$$

using (5.3.20), and (5.3.27), then it follows from (5.3.6), that

$$z^{(n)}(t) + \phi(t) w\big(\lambda c z^{(n-1)}(t)\big) \leq 0 \tag{5.3.28}$$

where $c = \dfrac{1}{(n-1)!}$

Setting $y(t) = \lambda c z^{(n-1)}(t)$, we have

$$y'(t) + \lambda c \phi(t) w\big(y(t)\big) \le 0 \tag{5.3.29}$$

Dividing (5.3.29) by $w\big(y(t)\big)$, and integrating from t_6 to t, we obtain

$$\int_{y(t_6)}^{y(t)} \frac{ds}{w(s)} + \lambda c \int_{t_6}^{t} \phi(r)dr \le 0, \text{ then}$$

$$\lambda c \int_{t_6}^{t} \phi(r)dr \le \int_{y(t\ )}^{y(t_6)} \frac{ds}{w(s)} \tag{5.3.30}$$

Since $y'(t) < 0$, so $y(t)$ is decreasing. And since $y(t) > 0$, it follows that $\lim_{t\to\infty} y(t) = L \ge 0$. If $L \ne 0$, then by (5.3.30) we must have

$$\int_{t_6}^{\infty} \phi(t)dt < \infty \tag{5.3.31}$$

which contradicts (5.3.22). In the case when $L = 0$, letting $t \to \infty$ in (5.3.30) and using (5.3.21), we again obtain (5.3.31). Thus the proof is complete. □

Example 5.3.2: Consider the NDDE

$$\big[x(t) + p(t)x(\tau(t))\big]^{(n)} + q(t)\left|x(\sigma(t))\right|^{\beta} G\left(\frac{x(\sigma(t))}{\big[t-\sigma(t)\big]^{n-1}}\right)\mathrm{sgn}(x(\sigma(t))) = 0 \tag{5.3.32}$$

where, n is odd , p and σ as stated above, $0 < \beta < 1$, $q(t)$ is a continuous nonnegative function defined on $[0,\infty)$, and not identically zero on any half-line of the form $[t_*,\infty)$, and $G(x)$ is a continuous function which is positive and nondecreasing for all $x \in R$.

Use $w(x) = |x|^{\beta} G(x)$, then if

$$\int^{\infty} [t - \sigma(t)]^{\beta(n-1)} q(t)dt = \infty$$

by Theorem 5.3.2 every bounded solution of equation (5.3.32) is oscillatory.

The results of this section can be directly generalized to the non-homogeneous form with certain conditions on the forcing term. The following remark shows such conditions for oscillation of non-homogenous NDDE.

Remark 5.3.1: consider the following non-homogenous NDDE

$$[x(t) + p(t)x(\tau(t))]^{(n)} + f(t, x(t), x(\sigma(t))) = h(t) \qquad (5.3.33)$$

where $h(t) \in C([0,\infty), R)$. If conditions (1)-(3) are satisfied, condition (4) holds for odd orders, and there exists an oscillatory function $r(t) \in C^n([0,\infty), R)$ such that $r^{(n)}(t) = h(t)$, and $\lim_{t \to \infty} r(t) = 0$. Then the results of Theorems 5.3.1, and 5.3.2 still remain true for equation (5.3.33).

Proof: To prove this remark we proceed the same as in proofs of Theorems 5.3.1, and 5.3.2. But here we set $z(t) = x(t) + p(t)x(\tau(t)) - r(t)$.

All steps of proof of this remark is exactly the same as that in proofs of Theorems 5.3.1, and 5.3.2. So the proof here is omitted. □

Appendix A

In this appendix we prove Lemma 2.3.1.

Our aim is to prove that if $z(t)$ is positive function of degree l, $l \geq 2$.

Then

$$z'(t) \geq \int_{t_1}^{t} z^{(l)}(s) \frac{(t-s)^{l-2}}{(l-2)!} ds \tag{A.1}$$

Proof: Since $z(t)$ is positive function of degree l, by Lemma 1.6.1:

$$z^{(i)}(t) > 0, \quad 0 \leq i \leq l \tag{A.2}$$

In view of (A.2), we prove (A.1) by using the identity:

$$z^{(l)}(t) = z^{(l)}(t) \tag{A.3}$$

Integrate (A.3) from t_1 to t we have:

$$z^{(l-1)}(t) - z^{(l-1)}(t_1) = \int_{t_1}^{t} z^{(l)}(s) ds \tag{A.4}$$

From (A.1) $z^{(l-1)}(t_1) > 0$, so it follows from (A.4) that:

$$z^{(l-1)}(t) \geq \int_{t_1}^{t} z^{(l)}(s) ds \tag{\textbf{A.5}}$$

Integrate (A.5) from t_1 to t, and use $z^{(l-2)}(t_1) > 0$, we have:

$$z^{(l-2)}(t) \geq \int_{t_1}^{t} \left(\int_{t_1}^{w} z^{(l)}(s) ds \right) dw \tag{A.6}$$

Integrate the right side of (A.6) by parts. (The formula of integration by parts is $\int u dv = uv - \int v du$).

Let $u = \int_{t_1}^{w} z^{(l)}(s) ds \Rightarrow du = z^{(l)}(w) dw$. Let $dv = dw \Rightarrow v = w$, so

$$\int_{t_1}^{t}\left(\int_{t_1}^{w} z^{(l)}(s)ds\right)dw = \left[w\int_{t_1}^{w} z^{(l)}(s)ds\right]_{t_1}^{t} - \int_{t_1}^{t} wz^{(l)}(w)dw$$

$$= t\int_{t_1}^{t} z^{(l)}(s)ds - \int_{t_1}^{t} sz^{(l)}(s)ds = \int_{t_1}^{t}(t-s)z^{(l)}(s)ds \qquad \text{(A.7)}$$

Hence from (A.6) and (A.7) we get:

$$z^{(l-2)}(t) \geq \int_{t_1}^{t}(t-s)z^{(l)}(s)ds \qquad \textbf{(A.8)}$$

Again integrate (A.8) from t_1 to t, and use $z^{(l-3)}(t_1) > 0$, we have:

$$z^{(l-3)}(t) \geq \int_{t_1}^{t}\left(\int_{t_1}^{w}(w-s)z^{(l)}(s)ds\right)dw \qquad \text{(A.9)}$$

Integrate the right side of (A.9) using formula of integration by parts:

Let $u = \int_{t_1}^{w}(w-s)z^{(l)}(s)ds = w\int_{t_1}^{w} z^{(l)}(s)ds - \int_{t_1}^{w} sz^{(l)}(s)ds \Rightarrow du = \left[\int_{t_1}^{w} z^{(l)}(s)ds\right]dw$

Let $dv = dw \Rightarrow v = w$, then

$$\int_{t_1}^{t}\left(\int_{t_1}^{w}(w-s)z^{(l)}(s)ds\right)dw = \left[w\int_{t_1}^{w}(w-s)z^{(l)}(s)ds\right]_{t_1}^{t} - \int_{t_1}^{t} w\left(\int_{t_1}^{w} z^{(l)}(s)ds\right)dw$$

$$= t\int_{t_1}^{t}(t-s)z^{(l)}(s)ds - \int_{t_1}^{t} w\left(\int_{t_1}^{w} z^{(l)}(s)ds\right)dw \qquad \text{(A.10)}$$

Again use integration of parts for second integration in the right side of (A.10):

Let $u = \int_{t_1}^{w} z^{(l)}(s)ds \Rightarrow du = z^{(l)}(w)dw$. Let $dv = wdw \Rightarrow v = \frac{w^2}{2}$, then

$$\int_{t_1}^{t} w\left(\int_{t_1}^{w} z^{(l)}(s)ds\right)dw = \left[\frac{w^2}{2}\int_{t_1}^{w} z^{(l)}(s)ds\right]_{t_1}^{t} - \int_{t_1}^{t}\frac{w^2}{2}z^{(l)}(w)dw$$

$$= \int_{t_1}^{t} \frac{t^2}{2} z^{(l)}(s)ds \; - \int_{t_1}^{t} \frac{s^2}{2} z^{(l)}(s)ds \tag{A.11}$$

Now, from (A.9), (A.10), and (A.11) we have:

$$z^{(l-3)}(t) \geq \; t\int_{t_1}^{t} (t-s) z^{(l)}(s)ds - \int_{t_1}^{t} \frac{t^2}{2} z^{(l)}(s)ds + \int_{t_1}^{t} \frac{s^2}{2} z^{(l)}(s)ds$$

$$\geq \int_{t_1}^{t} (\frac{t^2}{2} - ts + \frac{s^2}{2}) z^{(l)}(s)ds = \int_{t_1}^{t} \frac{(t-s)^2}{2} z^{(l)}(s)ds \text{, so}$$

$$z^{(l-3)}(t) \geq \int_{t_1}^{t} \frac{(t-s)^2}{2} z^{(l)}(s)ds \tag{\textbf{A.12}}$$

Again integrate (A.12) from t_1 to t, and use $z^{(l-4)}(t_1) > 0$, we have:

$$z^{(l-4)}(t) \geq \int_{t_1}^{t} \left(\int_{t_1}^{w} \frac{(w-s)^2}{2} z^{(l)}(s)ds \right) dw \tag{A.13}$$

Integrate the right side of (A.13) using formula of integration by parts:

Let $u = \int_{t_1}^{w} \frac{(w-s)^2}{2} z^{(l)}(s)ds = w^2 \int_{t_1}^{w} z^{(l)}(s)ds - w\int_{t_1}^{w} s z^{(l)}(s)ds + \int \frac{s^2}{2} z^{(l)}(s)ds$

$$\Rightarrow \; du = \left[\int_{t_1}^{w} (w-s) z^{(l)}(s)ds \right] dw$$

Let $dv = dw \;\Rightarrow\; v = w$, then

$$\int_{t_1}^{t} \left(\int_{t_1}^{w} \frac{(w-s)^2}{2} z^{(l)}(s)ds \right) dw = \left[w \int_{t_1}^{w} \frac{(w-s)^2}{2} z^{(l)}(s)ds \right]_{t_1}^{t} - \int_{t_1}^{t} w \left(\int_{t_1}^{w} (w-s) z^{(l)}(s)ds \right) dw$$

$$= t\int_{t_1}^{t} (\frac{t^2}{2} - ts + \frac{s^2}{2}) z^{(l)}(s)ds - \int_{t_1}^{t} w \left(\int_{t_1}^{w} (w-s) z^{(l)}(s)ds \right) dw \tag{A.14}$$

Then, use integration by parts for second integration in the right side of(A.14):

Let $u = \int_{t_1}^{w} (w-s) z^{(l)}(s) ds = w\int_{t_1}^{w} z^{(l)}(s) ds - \int_{t_1}^{w} s z^{(l)}(s) ds \;\Rightarrow du = \left[\int_{t_1}^{w} z^{(l)}(s) ds \right] dw$

Let $dv = wdw \;\Rightarrow\; v = \dfrac{w^2}{2}$, then

$$\int_{t_1}^{t} w \left(\int_{t_1}^{w} (w-s) z^{(l)}(s) ds \right) dw = \left[\frac{w^2}{2} \int_{t_1}^{w} (w-s) z^{(l)}(s) ds \right]_{t_1}^{t} - \int_{t_1}^{t} \frac{w^2}{2} \left(\int_{t_1}^{w} z^{(l)}(s) ds \right) dw$$

$$= \frac{t^2}{2} \int_{t_1}^{t} (t-s) z^{(l)}(s) ds - \int_{t_1}^{t} \frac{w^2}{2} \left(\int_{t_1}^{w} z^{(l)}(s) ds \right) dw \qquad \text{(A.15)}$$

Also, integrate by parts the second integral in the right side of (A.15), we have:

$$\int_{t_1}^{t} \frac{w^2}{2} \left(\int_{t_1}^{w} z^{(l)}(s) ds \right) dw = \int_{t_1}^{t} \frac{t^3}{6} z^{(l)}(s) ds - \int_{t_1}^{t} \frac{s^3}{6} z^{(l)}(s) ds \qquad \text{(A.16)}$$

From (A.13), (A.14), (A.15), and (A.16) we have:

$$z^{(l-4)}(t) \geq t \int_{t_1}^{t} \left(\frac{t^2}{2} - ts + \frac{s^2}{2} \right) z^{(l)}(s) ds - \frac{t^2}{2} \int_{t_1}^{t} (t-s) z^{(l)}(s) ds + \int_{t_1}^{t} \frac{t^3}{6} z^{(l)}(s) ds - \int_{t_1}^{t} \frac{s^3}{6} z^{(l)}(s) ds$$

$$\geq \int_{t_1}^{t} \left(\frac{t^3 - 3t^2 s + 3ts^2 - s^3}{6} \right) z^{(l)}(s) ds = \int_{t_1}^{t} \frac{(t-s)^3}{3!} z^{(l)}(s) ds \text{, hence}$$

$$z^{(l-4)}(t) \geq \int_{t_1}^{t} \frac{(t-s)^3}{3!} z^{(l)}(s) ds \qquad \textbf{(A.17)}$$

By following the above process, in view of (A.2), we can show that

$$z^{(l-5)}(t) \geq \int_{t_1}^{t} \frac{(t-s)^4}{4!} z^{(l)}(s) ds \qquad \textbf{(A.18)}$$

From **(A.5), (A.8), (A.12), (A.17),** and **(A.18)** in general we have:

$$z^{(l-k)}(t) \geq \int_{t_1}^{t} \frac{(t-s)^{k-1}}{(k-1)!} z^{(l)}(s) ds, \tag{A.19}$$

If $l-k=1$, then (A.19) implies that

$$z'(t) \geq \int_{t_1}^{t} \frac{(t-s)^{l-2}}{(l-2)!} z^{(l)}(s) ds \tag{A.20}$$

Thus the proof is completed. □

Appendix B

In this appendix we shall show the inequality (2.3.14), by using inequality (2.3.13). We do this by integrating (2.3.13) from t to ∞ $(n-l-1)$ times, in view of (2.3.9), (2.3.10), and $l \le n-3$. Let us rewrite these inequalities. {(2.3.13) is (B.1), (2.3.14) is (B.2), (2.3.9) is (B.3), and (2.3.10) is (B.4).}

$$z^{(n)}(t) + \beta\left(1 - p(\sigma(t))\right)^{\alpha} q(t) z^{\alpha}(\sigma(t)) \le 0\,, \qquad \text{(B.1)}$$

$$z^{(l+1)}(t) \le -\int_t^{\infty} \beta q(s) z^{\alpha}(\sigma(s))\left(1 - p(\sigma(s))\right)^{\alpha} \frac{(s-t)^{n-l-2}}{(n-l-2)!} ds \qquad \text{(B.2)}$$

$$z^{(i)}(t) > 0\,, \quad 0 \le i \le l \qquad \text{(B.3)}$$

$$(-1)^{n+i-1} z^{(i)}(t) > 0\,, \quad l+1 \le i \le n-1 \qquad \text{(B.4)}$$

For simplicity suppose that $\psi(t) = \beta\left(1 - p(\sigma(t))\right)^{\alpha} q(t) z^{\alpha}(\sigma(t))$

Proof: Integrate (B.1) from t to ∞

$$z^{(n-1)}(\infty) - z^{(n-1)}(t) \le -\int_t^{\infty} \psi(s) ds \qquad \text{(B.5)}$$

From (B.4) $z^{(n-1)}(t) > 0$, using this fact with (B.5) we have:

$$-z^{(n-1)}(t) \le -\int_t^{\infty} \psi(s) ds \qquad \text{(B.6)}$$

Integrate (B.6) from t to ∞,

$$-z^{(n-2)}(\infty) + z^{(n-2)}(t) \le -\int_t^{\infty}\left(\int_w^{\infty} \psi(s) ds\right) dw \qquad \text{(B.7)}$$

Use integration by parts for the right side of (B.7)

Let $u = \int_w^{\infty} \psi(s) ds \;\Rightarrow\; du = -\psi(w) dw$. Let $dv = dw \;\Rightarrow\; v = w$,

$$\int_{t}^{\infty}\left(\int_{w}^{\infty}\psi(s)ds\right)dw=\left[w\int_{w}^{\infty}\psi(s)ds\right]_{t}^{\infty}+\int_{t}^{\infty}w\psi(w)dw=\int_{t}^{\infty}(s-t)\psi(s)ds \tag{B.8}$$

If $l \le n-3$, from (B.4) $z^{(n-2)}(t)<0$, using this with (B.7), and (B.8) we have:

$$z^{(n-2)}(t)\le -\int_{t}^{\infty}(s-t)\psi(s)ds \tag{B.9}$$

Again integrate (B.9) from t to ∞, we have

$$z^{(n-3)}(\infty)-z^{(n-3)}(t)\le -\int_{t}^{\infty}\left(\int_{w}^{\infty}(s-w)\psi(s)ds\right)dw \tag{B.10}$$

If $l<n-4$, From (B.4) $z^{(n-3)}(t)>0$, therefore from (B.10) we obtain

$$-z^{(n-3)}(t)\le -\int_{t}^{\infty}\left(\int_{w}^{\infty}(s-w)\psi(s)ds\right)dw \tag{B.11}$$

Use integration by parts for the right side of (B.11). However, this is similar to our work in Appendix A, (see A.10-A.11), so we put the result directly:

$$-z^{(n-3)}(t)\le -\int_{t}^{\infty}\frac{(s-t)^2}{2}\psi(s)ds \tag{B.12}$$

Again integrate (B.12) t to ∞, with use of $z^{(n-4)}(t)<0$ (If $l\le n-5$), we have

$$z^{(n-4)}(t)\le -\int_{t}^{\infty}\frac{(s-t)^3}{3!}\psi(s)ds \tag{B.13}$$

In general If $l\le n-k-1$, we have

$$(-1)^k\, z^{(n-k)}(t)\le -\int_{t}^{\infty}\frac{(s-t)^{k-1}}{(k-1)!}\psi(s)ds \tag{B.14}$$

If $n-k=l+1$, i.e. we integrate (B.1) $(n-l-1)$, then we have:

$$(-1)^{n-l-1}\, z^{(l+1)}(t) \le -\int_t^\infty \frac{(s-t)^{n-l-2}}{(n-l-2)!}\psi(s)ds \tag{B.15}$$

But $n-l-1$ is even, hence:

$$z^{(l+1)}(t) \le -\int_t^\infty \frac{(s-t)^{n-l-2}}{(n-l-2)!}\psi(s)ds \tag{B.16}$$

Substitute $\psi(s)=\beta(1-p(\sigma(s)))^\alpha q(s)z^\alpha(\sigma(s))$, we have:

$$z^{(l+1)}(t) \le -\int_t^\infty \beta q(s)z^\alpha(\sigma(s))(1-p(\sigma(s)))^\alpha \frac{(s-t)^{n-l-2}}{(n-l-2)!}ds \tag{B.17}$$

Thus the proof is completed. □

Appendix C

In this appendix we will show inequality (2.3.27) from inequality (2.3.26). Where $\lim_{t\to\infty} z(t) = c > 0$. The inequality (2.3.26) is (C.1):

$$-z'(t) \le -\int_t^\infty a_1(s) z^\alpha(\sigma(s))ds \tag{C.1}$$

Proof: Integrate (C.1) from t_1 to ∞,

$$-z(\infty) + z(t_1) \le -\int_{t_1}^\infty \left(\int_w^\infty a_1(s) z^\alpha(\sigma(s))ds \right) dw \tag{C.2}$$

Use formula of integration by parts for the right side of (C.2)

Let $u = \int_w^\infty a_1(s) z^\alpha(\sigma(s))ds \Rightarrow du = -a_1(w) z^\alpha(\sigma(w))dw$.

Let $dv = dw \Rightarrow v = w$,

$$\int_{t_1}^\infty \left(\int_w^\infty a_1(s) z^\alpha(\sigma(s))ds \right) dw = \left[w \int_w^\infty a_1(s) z^\alpha(\sigma(s))ds \right]_{t_1}^\infty + \int_{t_1}^\infty w a_1(w) z^\alpha(\sigma(w))dw$$

$$= -t_1 \int_{t_1}^\infty a_1(s) z^\alpha(\sigma(s))ds + \int_{t_1}^\infty s a_1(s) z^\alpha(\sigma(s))ds$$

$$= \int_{t_1}^\infty (s - t_1) a_1(s) z^\alpha(\sigma(s))ds \tag{C.3}$$

From (C.2), (C.3), with use of $\lim_{t\to\infty} z(t) = c$, we obtain:

$$z(t_1) \le c - \int_{t_1}^\infty (s - t_1) a_1(s) z^\alpha(\sigma(s))ds \tag{C.4}$$

From (C.4) we have

$$z(t_1) \le c - \int_{t_1}^\infty (s - t_1) a_1(s) z^\alpha(\sigma(s))ds \;\; z(t_1) \le c - z^\alpha(\sigma(t_1)) \int_{t_1}^\infty a_1(s)(s - t_1)ds \tag{C.5}$$

But $z(t)$ is increasing and $t_1 > t$, then (C.5) implies that:

$$z(t) \leq c - z^{\alpha}(\sigma(t))\int_{t_1}^{\infty} a_1(s)(s - t_1)ds \tag{C.6}$$

Inequality (2.3.27) is (C.6). Thus the proof is completed. □

Appendix D

In this appendix we will show inequality (2.3.35). {(2.3.35) is (D.1)}

$$z(t_1) \geq -\int_{t_1}^{\infty} \frac{(s-t_1)^{n-1}}{(n-1)!} z^{(n)}(s)ds \tag{D.1}$$

Where $\quad (-1)^i z^{(i)}(t) > 0,\ 0 \leq i \leq n \quad (n$ is odd) (D.2)

For convenience, we use the notation $\quad y(\infty) = \lim_{t\to\infty} y(t)$

Proof: Consider the equality

$$z^{(n)}(t) = z^{(n)}(t) \tag{D.3}$$

Integrate (D.3) from t to ∞,

$$z^{(n-1)}(\infty) - z^{(n-1)}(t\ ) = \int_{t}^{\infty} z^{(n)}(s)ds \tag{D.4}$$

From (D.2) $z^{(n-1)}(t) > 0$, then (D.4) implies that:

$$z^{(n-1)}(t\ ) \geq -\int_{t}^{\infty} z^{(n)}(s)ds \tag{D.5}$$

Integrate (D.5) from t to ∞,

$$z^{(n-2)}(\infty) - z^{(n-2)}(t\ ) \geq -\int_{t}^{\infty}\left(\int_{w}^{\infty} z^{(n)}(s)ds\right)dw \tag{D.6}$$

From (D.2) $z^{(n-2)}(t) < 0$, and by using integration by parts for the right side of (D.6), we have:

$$-z^{(n-2)}(t\ ) \quad \geq -\int_{t}^{\infty}(s-t)z^{(n)}(s)ds \tag{D.7}$$

Integrate (D.7) from t to ∞,

$$-z^{(n-3)}(\infty) + z^{(n-3)}(t\ ) \geq -\int_{t}^{\infty}\left(\int_{w}^{\infty}(s-t)z^{(n)}(s)ds\right)dw \tag{D.8}$$

Again, by using of integration by parts for the integral of right side of (D.8), with use of $z^{(n-3)}(t) > 0$, (We did similar thing in Appendix B), then we have:

$$z^{(n-3)}(t\ ) \ \geq -\int_t^{\infty} \frac{(s-t)^2}{2} z^{(n)}(s)ds \tag{D.9}$$

If we complete in the same manner we obtain, in general, the following:

$$(-1)^{k-1}\ z^{(n-k)}(t) \geq -\int_t^{\infty} \frac{(s-t)^{k-1}}{(k-1)!} z^{(n)}(s)ds \tag{D.10}$$

If $n-k=1$, i.e. we integrate (D.3) $(n-1)$ times, then we have:

$$-z'(t) \geq -\int_t^{\infty} \frac{(s-t)^{n-2}}{(n-2)!} ds \tag{D.11}$$

If $n=k$, then we have

$$z(t) \geq -\int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} ds \tag{D.12}$$

However, if we integrate (D.11) from t to ∞, we obtain (D.12). Hence, by integration of (D.11) from t_1 to ∞, we obtain:

$$z(t_1) \geq -\int_t^{\infty} \frac{(s-t_1)^{n-1}}{(n-1)!} ds \tag{D.13}$$

Thus the proof is completed. □

Appendix E

In this appendix we evaluate the integral (3.3.9), which is the following:

$$I(t)=\int t^{n-1}z^{(n)}(t)dt \qquad \text{(E.1)}$$

Since t^{n-1} can be differentiated repeatedly to become zero, we use integration by parts repeatedly. And for simplicity we use the tabular integration method, which is, in fact, a method of organization the repeatedly integration by parts. (You can see books of calculus).

$$\begin{array}{lcl}
\underline{t^{n-1}\text{ and its derivatives}} & & \underline{z^{(n)}(t)\text{ and its integrals}} \\
t^{n-1} & (+) & z^{(n)}(t) \\
(n-1)t^{n-2} & (-) & z^{(n-1)}(t) \\
(n-1)(n-2)t^{n-3} & (+) & z^{(n-2)}(t) \\
(n-1)(n-2)(2-3)t^{n-4} & & z^{(n-3)}(t) \\
\vdots & \vdots & \vdots \\
(n-1)(n-2)(2-3)...3.2t & (-) & \vdots \\
(n-1)(n-2)(2-3)...3.2 & (+) & z'(t) \\
 & & z(t)
\end{array}$$

We add the products of the functions connected by the arrows, with the middle sign changed, to obtain:

$$I(t)=t^{n-1}z^{(n-1)}(t)-(n-1)t^{n-2}z^{(n-2)}(t)+(n-1)(n-2)t^{n-3}z^{(n-3)}(t)-$$

$$\ldots-(n-1)(n-2)(n-3)...3.2tz'(t)+(n-1)(n-2)(n-3)...3.2z(t) \qquad \text{(E.2)}$$

Appendix F

Jensen's Inequality

There are several forms for Jensen's inequality. In this appendix we choose an easy form to help us in our work in section 5.2.

Theorem F.1 [4]: Let $\phi:(-\infty,\infty)\to(-\infty,\infty)$ be continuous and convex downward. If f and g are continuous on $[a,b]$ with $g(t)\geq 0$, and $\int_a^b g(t)dt>0$. Then

$$\phi\left[\int_a^b f(t)g(t)dt \Big/ \int_a^b g(t)dt\right] \leq \int_a^b \phi\big(f(t)\big)g(t)dt \Big/ \int_a^b g(t)dt \qquad \text{(F.1)}$$

Example F.1: take $\phi(t)=-\ln t$. It is continuous and convex downward on $(0,\infty)$. Take $f(t)=e^{05\sin t}$, $g(t)=1$, and interval $[t,t+1]$. Apply Theorem (F.1) we have:

$$\ln\left(e\int_t^{t+1} e^{0.5\sin s-1}ds\right)=\ln\left(\int_t^{t+1} e^{0.5\sin s}ds\right)\geq \int_t^{t+1}\ln\left(e^{0.5\sin s}\right)=\int_t^{t+1}0.5\sin s ds$$

$$\Rightarrow \qquad \ln\left(e\int_t^{t+1} e^{0.5\sin s-1}ds\right)\geq \int_t^{t+1}0.5\sin s ds \qquad \text{(F.2)}$$

Therefore from (F.2) we can obtain:

$$\int_1^{\infty} e^{0.5\sin t-1}\ln\left(e\int_t^{t+1} e^{0.5\sin s-1}ds\right)dt \geq \int_1^{\infty} e^{0.5\sin t-1}\left(\int_t^{t+1}0.5\sin s ds\right)dt \qquad \text{(F.3)}$$

(F.3) is the relation (5.3.9).

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