**Deanship of Graduate Studies**

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# Necessary and Sufficient Conditions for the Oscillation in a Class of Even Order Neutral Differential Equations

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## Thesis Approval

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2011

## Dedication

To my husband, my kids, my parents, my brothers, and to my sisters.

## Declaration:

I certify that this thesis submitted for the degree of Master is the result of my own research, except where otherwise acknowledged and that this thesis has not been submitted for a higher degree to any other university or institution.

Signed

Kareemah Mohammad Salem Al-Shwaiki

Date: November / 2011

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Kareemah Mohammad Salem Al-Shwaiki

**Abstract**

This thesis is concerned with the oscillation and non oscillation of solutions of a class of even order neutral differential equations. The general form of this class of equations involves two delayed arguments.

The thesis presents main concepts and basic definitions of neutral differential equations and establishes both necessary and sufficient conditions for non oscillatory solutions. Several results in the oscillation theory of that class of even order neutral differential equations are proposed and a number of examples are given to illustrate the main theorems.

This study also investigates deeply and analyzes and compares, in order to understand accurately, the results about necessary and sufficient conditions for the oscillation and non oscillation of solutions of even order neutral differential equations with constant and variable coefficients.

## الملخص

تهتم هذه الأطروحة بخاصية التذبذب وعدم التذبذب لحلول فئة معينه من المعادلات التفاضلية المتعادلة من الدرجة النونية الزوجية ، و الشكل العام لهذه الفئة من المعادلات يحتوي على اشتمال الاقتران المجهول على متغيرين متأخرين عن المتغير الذي يمثل الوضع الحالي.

تقدم هذه الأطروحة على المفاهيم و التعاريف الأساسية للمعادلات التفاضلية المتعادلة، و تؤسس الشروط الضرورية و الكافية لعدم تذبذب الحلول. تحتوي هذه الأطروحة على العديد من النتائج الصادرة في نظرية التذبذب لتلك الفئة من المعادلات التفاضلية المتعادلة من الدرجة الزوجية. تقترح عدد من الأمثلة لتوضيح النظريات الرئيسية.

تحقق هذه الدراسة أيضا بعمق وبالتحليل وبالمقارنة ، من أجل فهم دقيق، في النتائج حول الشروط الضرورية و الكافية للتذبذب وعدمه لحلول المعادلات التفاضلية المتعادلة من الدرجة الزوجية بوجود عوامل ثابتة ومتغيرة.

# Contents

## Introduction

Differential equations with deviating arguments (DEWDA) are among the most important equations in applied mathematics. This importance occurs because they provide mathematical models for many real-life systems, in which the rate of change of the system depends not only on its present state but also could depend on past or future states.

DEWDA initially was introduced in the eighteenth century by Laplace and Condorcet [26] . Bernoulli ( 1728) while studying the problem of sound vibrating in a tube with finite size investigated the properties of solutions of the first order of DEWDA, and was the first to work in this area [1]. However, the systematic study of such type of differential equations has begun in the twentieth century in connection with the needs of applied science and technology [11] .

In the late thirties and early forties Minorsky in his study of ship stabilization and automatic steering pointed out very clearly the importance of the consideration of the delay in the feedback mechanism[8]. The great interest in the theory of automatic control and dynamics systems, during these and later years, has certainly contributed significantly to the rapid development of the theory of delay differential equations [26 , 11 , 8].

Myshkis in his book (1950) introduced a general class of equations with delayed arguments [8]. In 1958 G.A. Kamenskii [5] proposed a classification method for a general class of DEWDA, he classified such type of equations into three types, they are: retarded type, neutral type, and advanced types.

Oscillatory behavior of solutions of DEWDA is one of the most important properties of this type of equations, besides existence of positive solutions, and asymptotic behavior of solutions. This importance comes from the viewpoint of applications where these properties provide a qualitative description of solutions of the DEWDA.

Since 1950 the oscillation theory of DEWDA has received the attentions of several mathematicians as well as other scientists around the world. However, the theory of oscillation of DEWDA has been extensively developed in the last 30 years.

In 1987 Ladde, Lakshmikantham, and Zhang in their well presented book [5], introduced the first systematic treatment of oscillation and non oscillation theory of

DEWDA[4]. In 1991 Gyori, and Ladas introduced one of the most important books in the oscillation theory of DEWDA. The last book is also a extensive reference for the theory of DEWDA, and it contains several applications. Recently several books appeared that are specialized in the subject of oscillation, such as Bainov and Mishev (1991), and Agarwal (2000).

In parallel, during the second half of the twentieth century the area of applications of DEWDA has greatly expanded. Now such equations find numerous applications in physics, control theory, power systems engineering, material science, robotics, neural networks, ecology, physiology, immunology, public health, and economics (see references of [2,3,4,8,12,30]).

The simplest type of past dependence in differential equation is that in which the past dependence is through the state variable and not the derivative of the state variable, in this were DEWDA are the so-called retarded functional differential equations or delay differential equations [8].

When the delayed argument occurs in the derivative of the state variable as well as in the independent variable, the system is called neutral differential equations [8].

Although the oscillatory theory of non-neutral differential equations has been extensively developed during the last three decades, only in the last ten or fifteen years much effort has been devoted to the study of oscillatory behavior of neutral delay differential equations (NDDE). From the viewpoint of applications, the study of oscillatory behavior of solutions of NDDE, the study of other types of DEWDA, and its theoretical interest are all important. Accordingly, NDDE have many applications in natural science, technology, and economics. For more illustration, NDDE appear in the following applications:

1. Study of vibrating masses attached to an elastic bar [4 , 8].
2. Study of distributed networks containing loss-less transmission lines [4 , 8].
3. Problems of economics where the demand depends on current price but supply depends on the price at an earlier time [30].
4. To describe the Flip Flop circuit which is the basic element in a digital electronic [12].

In fact, the appearance of neutral term in differential equations can cause or destroy oscillation of its solutions. Moreover, in general the theory of neutral differential equations presents complications which are unfamiliar for non-neutral differential equations: Most of authors obtained sufficient rather than necessary conditions for oscillation of higher orders NDDE. However, the conditions assumed differ from author to author due to the different techniques they used and different forms of equations they considered. Also, it is interesting to note that the conditions assumed by different researches for similar form of equations are often not comparable, see [17].

In our thesis we study the oscillation of a certain class of even order NDDE of the forms with constant or variable coefficients:

$$\frac{d^n}{dt^n}[x(t)+\lambda x(t-\tau)]+f\left(t,x\big(g(t)\big)\right)=0. \tag{1}$$

$$\frac{d^n}{dt^n}[x(t)+h(t)x(t-\tau)]+f\left(t,x\big(g(t)\big)\right)=0. \tag{2}$$

Throughout this thesis, the following conditions are assumed to hold; $n \geq 2$ is even; $\tau > 0$; $\lambda > 0$; $g \in C[t_0,\infty)$, $lim_{t\to\infty}\, g(t)=\infty$; $f \in C([t_0,\infty)\times\mathbb{R})$; $uf(t,u) \geq 0$ for $(t,u) \in [t_0,\infty)\times\mathbb{R}$, and $f(t,u)$ is nondecreasing in $u \in \mathbb{R}$ for each fixed $t \geq t_0$; $h(t) \in C(\mathbb{R})$ .

The outline of the thesis is as follows:

**Chapter One**: Contains the main concepts, definitions, and preliminary material that are essential for the rest of the thesis.

**Chapter Two**: Is devoted to oscillation and non oscillation theories of equations (3) and (4) for the case of constant coefficients

$$\frac{d^n}{dt^n}[x(t)+\lambda x(t-\tau)]+q(t)x(t-\sigma)=0 \tag{3}$$

$$\frac{d^n}{dt^n}[x(t)+\lambda x(t-\tau)]+q(t)|x(t-\sigma)|^{\gamma}=0 \tag{4}$$

where $\gamma \neq 1$ and $\gamma > 0$.

**Chapter Three and Chapter Four**: highlight on Tanka's results, [20] and [21] respectively. The theory of oscillation for equation (1)(respectively (2)) will be studied since it is of extreme significance over earlier theories. Not only it provides necessary and sufficient conditions for the oscillation, but also compares NDDE with ODE. The detailed proof, the resulting corollaries along with its evidences, and lemmas, will be all presented and proved along with the chapter.

# Chapter One

# Preliminaries

This chapter contains some basic definitions, and results which are essential for the rest of the thesis. Sections 1.1 and 1.2, introduce the definition of DEWDA, their classification, and definition of NDDE. Section 1.3, gives the meaning of solution of NDDE and sections 1.4 and 1.5, introduce the definition of oscillation and some oscillatory phenomena caused by deviating arguments. Section 1.6 contains basic lemmas related to the subject.

## 1.1 Differential equations with deviating arguments (DEWDA)

Differential equations with deviating arguments are differential equations in which the unknown function appears with various values of the argument. They are classified into three types:

i. Differential equations with retarded argument:

Differential equation with retarded argument is a differential equation with deviating argument in which the highest-order derivative of the unknown function appears for just one value of the argument, and this argument is not less than the remaining arguments of the unknown function and its derivatives appearing in the equation.

ii. Differential equations with advanced argument:

Differential equation with advanced argument is a differential equation with deviating argument in which the highest-order derivative of the unknown function appears for just one value of the argument, and this argument is not larger than the remaining arguments of the unknown function and its derivatives appearing in the equation.

iii. Differential equations of neutral type:

Neutral differential equation is a differential equation in which the highest-order derivative of the unknown function is evaluated both with the present state and at one or more past or future states.

**Example 1.1.1:**

1) $x'(t) = f\left(t, x(t), x\big(t-\tau(t)\big)\right)$.
2) $x''(t) = f\left(t, x(t), x'\big(t-\tau(t)\big)\right)$.
3) $x''(t) \ = f(t, x(\frac{t}{2}), x'\left(\frac{t}{2}\right), x(t), x'(t)))$
4) $x''(t) = f(t, x(t), x'(t), x\big(\tau(t)\big), x'\big(\tau(t)\big)$.
5) $x'(t) = f\left(t, x(t), x(t-\tau), x'(t-\tau)\right)$.
6) $x''(t) = f\left(t, x(t), x'(t), x\big(t-\tau(t)\big), x''\big(t-\tau(t)\big)\right)$

Equation (1), (2), (3), and (4) are equations with retarded argument if $\tau(t) > 0$ in (1) and (2), $t > 0$ $in$ (3), $and$ $\tau(t) < t$ $in$ (4).

Equations (1), (2), (3), and (4) are equation with advanced argument if $\tau(t) < 0$ in (1) and (2), $t < 0$ $in$ (3), and $\tau(t) > t$ in (4).

Equations (5) and (6) are equations of neutral type.

## 1.2 Neutral delay differential equations (NDDE)

A neutral delay differential equations is a differential equations in which the highest-order derivative of unknown function appears in the equation both with and without delays (retarded arguments).

**Example 1.2.1:**

1) $x'(t) = f\big(\ t, x(t), x'(t-\tau)\big), \tau > 0$, is a first order NDDE.
2) $x''(t) = \ f\left(t, x(t), x'(t), x\big(t-\tau(t)\big), x'\big(t-\tau(t)\big), x''\big(t-\tau(t)\big)\right)$, $\tau(t) > 0$ is second order NDDE.

In general, the behavior of solutions of neutral type equations may be quite different than that of non neutral- equations, and results, which are true for non- neutral equations, may not be true for neutral equations.

## 1.3 Solution of NDDE

We shall be concerned with the oscillatory behavior of the solutions for even order neutral differential equation of the forms

$$\frac{d^n}{dt^n}[x(t) + \lambda x(t-\tau)] + f\left(t, x\big(g(t)\big)\right) = 0 \qquad (1.3.1)$$

$$\frac{d^n}{dt^n}[x(t) + h(t)x(t-\tau)] + f\left(t, x\big(g(t)\big)\right) = 0. \qquad (1.3.2)$$

The following conditions are assumed to hold: $n \geq 2$ is even ; $\lambda > 0$; $\tau > 0$; $g \in C[t_0, \infty), \lim_{t\to\infty} g(t) = \infty$; $f \in C([t_0, \infty) \times \mathbb{R})$, $uf(t, u) \geq 0$ for $(t, u) \in [t_0, \infty) \times \mathbb{R}$, and $f(t, u)$ is nondecreasing in

$u \in \mathbb{R}$ for each fixed $t \geq t_0$ ; $h(t) \in C(\mathbb{R})$.

By a solution of (1.3.1) or (1.3.2), we mean a function $x(t)$ that is continuous and satisfies (1.3.1) or (1.3.2) on $[t_x, \infty)$ for some$t_x \geq t_0$. Therefore, if $x(t)$ is a solution of (1.3.1) or (1.3.2 ), then $x(t) + \lambda x(t-\tau)$ is n-times continuously on $[t_x, \infty)$ . Note that, $x(t)$ itself is not continuously differentiable.

## 1.4 Definition of oscillation

There are various definitions for the oscillation of solutions of ordinary differential equations (with or without deviating argument). In this section we give two different forms of definitions of the oscillation. These forms are most frequently used in literature.

**Definition 1.4.1:** A non- trivial solution $x(t)$ is said to be oscillatory if it has arbitrarily lagre zeros for t≥ $t_0$ that is, there exists a sequence of zeros $\{t_n\}$, ( $x(t_n) = 0$) of $x(t)$ such thatlim$_{n\to\infty} t_n = +\infty$, otherwise $x(t)$ is said to be non- oscillatory.

For non oscillatory solutions there exists a $t_1$ such that$x(t) \neq 0$, for all $t \geq t_1$.

**Definitions 1.4.2:** A non- trivial solution $x(t)$ is said to be oscillatory if it changes sign on (T,∞), where T is any number.

As the solution $x(t)$ is continuous, if it non oscillatory it must be eventually positive or eventually negative. That is there exists a $T_0 \in$ R such that $x(t)$ is positive for all t≥ $T_0$ or is negative for all t≥ $T_0$.

**Example 1.4.1**: The equation

$$x'(t) + x\left(t-\frac{\pi}{2}\right) = 0 \qquad (1.4.1)$$

Has oscillatory solutions $x_1(t) = \sin t$, and $x_2(t) = \cos t$.

**Example 1.4.2:** The equation

$$x'(t) - e^3 x(t-3) = 0 \qquad (1.4.2)$$

Has non oscillatory solutions $x(t) = e^t$.

**Example 1.4.3:** the equation

$$x''(t) + 4x\left(\frac{\pi}{2} - t\right) = 0 \qquad (1.4.3)$$

Has an oscillatory solution $x_1(t) = \sin 2t$, and a non oscillatory solution $x_2 = e^{2t} - e^{\pi-2t}$.

**Example 1.4.4:** Consider the equation

$$x'''(t) - \frac{1}{2}x\left(t-\frac{\pi}{2}\right) + \frac{1}{2}x\left(t-\frac{3\pi}{2}\right) = 0,\ t \geq 0 \qquad (1.4.4)$$

Whose solution $x(t) = 1 + \cos t$, it is oscillatory according to definition 1.4.1, and non oscillatory according to definition 1.4.2. In fact, definition 1.4.1 is more general than definition 1.4.2 and is the most used in literature, also, it is the one used in this thesis.

**Example 1.4.5:**

$$x''(t) - \frac{1}{t}x'(t) + 4t^2 x(t) = 0 \qquad (1.4.5)$$

Whose solution $x(t) = \sin t^2$ , this solution is not periodic but has an oscillatory property.

**Example 1.4.6:** Consider the NDDE

$$\left[x(t) - \frac{1}{2}x(t-2\pi)\right]' - \frac{1}{2}x\left(t-\frac{3\pi}{2}\right) = 0 \qquad (1.4.6)$$

It has an oscillatory solution x(t)= sin t.

**Example 1.4.7:** Consider the NDDE

$$[\,x(t)-\ e^{t}x\,(t-1]^{'''}+\frac{e^{2t}}{e^{3}}x^{3}(t-1)=0 \qquad (1.4.7)$$

It has a non oscillatory soltuon x(t)= $e^{-t}$, but x(t) → 0 as t→ ∞.

## 1.5 Effects of deviating arguments on oscillation

The oscillation theory of DEWDA presents some new problems, which are not presented in the theory of corresponding ordinary differential equations (ODE). However, results for oscillation of differential equations may not be true for DEWDA.

In this section we consider some oscillatory and non oscillatory phenomena caused by deviating arguments, through the discussion of the following example.

**Example 1.5.1:** Consider the equation

$$x^{'}(t)+\beta x\left(t-\frac{\pi}{2\beta}\right)=0,\quad \beta\in R\setminus\{0\} \qquad (1.5.1)$$

It has oscillatory solution $x_1(t)= \sin\beta t, and\ x_2(t)=\cos\beta t$ . While the equation

$$x^{'}(t)+\ \beta x(t)=0,\quad \beta\in R\setminus\{0\} \qquad (1.5.2)$$

Has non- oscillatory solution$x(t)=e^{-\beta t}$.

This example shows that first order DEWDA can have oscillatory solution. While, as known, the first order scalar ODES that contain the unknown function do not possess oscillatory solution.

**Example 1.5.2:** Consider the equation

$$x^{''}(t)-9x\left(t-\frac{\pi}{3}\right)=0 \qquad (1.5.3)$$

It has oscillatory solution $x_1(t)$ = sin3t, and $x_2$ (t)= cost 3t. But equation

$$x^{''}(t)-\ 9x(t)=0 \qquad (1.5.4)$$

Has non oscillatory solutions $x_1(t) = e^{3t}$, and $x_3(t) = e^{-3t}$

It is obvious that the nature of solution changes completely after the appearance of deviating argument in the equation.

**Example 1.5.3:** Consider the equation

$$x''(t) + x(2\pi - t) = 0 \qquad (1.5.5)$$

It has an oscillatory solution $x_1(t) = \cos t$, and non oscillatory solution $x_2(t) = e^t - e^{2\pi - t}$.

Here, in second order DEWDA one solution is oscillatory, but the other is non oscillatory, and this case can never occur in second order linear ODE, where either all solutions are oscillatory or all solutions are non oscillatory.

## 1.6 Some basic lemmas

This section contains basic lemmas needed later in the thesis.

We begin by classifying all possible non oscillatory solutions of equation (1.3.1) according to their asymptotic behavior as $t \to \infty$.

**Lemma 1.6.1 ([7])**: (Kiguradze ) Let $n \geq 2$ and $\sigma = 1$ *or* $-1$ and let $u \in C^n[T, \infty)$ satisfies

$$\sigma u(t) u^{(n)}(t) < 0 , \quad \text{for } t \geq T .$$

Then there exist an integer $j \in \{0,1,2, \dots \dots , n\}$ and a number $t_0 \geq T$ such that $(-1)^{n-j-1}\sigma = 1$ and

$$\begin{cases} u(t)u^{(i)}(t) > 0, & t \geq t_0 , \quad 0 \leq i \leq j, \\ (-1)^{i-j}\, u(t)u^{(i)}(t) > 0 & t \geq t_0 , \quad j \leq i \leq n \end{cases}$$

**Lemma1.6.2 ([29])**: let $x(t)$ be a non oscillatory solution of (1.3.1) . Then one of the following two cases holds:

(I) There is an integer $j$ with $0 \leq j \leq n$, $(-1)^{n-j-1}\sigma = 1$ and a number $t_0 \geq T$ such that

$$x(t)(Lx)(t) > 0, \qquad t \geq t_0 , \qquad (1.6.1)$$

$$\begin{cases} (Lx)(t)(Lx)^{(i)}(t) > 0 & t \geq t_0 , \quad 0 \leq i \leq j \\ (-1)^{i-j}(Lx)(t)(Lx)^{(i)}(t) > 0, & t \geq t_0 , \quad j \leq i \leq n, \end{cases} \qquad (1.6.2)$$

(II) There is a number $t_0 \geq T$ such that

$$x(t)(Lx)(t) < 0, \qquad t \geq t_0 \qquad (1.6.3)$$

$$(-1)^i (Lx)(t)(Lx)^{(i)}(t) > 0 \qquad t \geq t_0 , \quad 0 \leq i \leq n \qquad (1.6.4)$$

and

$$\lim_{t\to\infty}(Lx)(t) = 0 \ , \quad \lim_{t\to\infty} x(t) = 0 \tag{1.6.5}$$

**Note**: Where $L(t) = [x(t) - h(t)\,x(t-\tau)]$. Furthermore the case (II) can hold only when $(-1)^n\sigma = 1$ and $h(t)$ is eventually positive.

**Definition 1.6.1 ([7]):** Let $\mathcal{N}$ denote the set of all non oscillatory solutions of (1.3.1). For an integer $j$ with $0 \le j \le n$ and $(-1)^{n-j-1}\sigma = 1$, we denote $\mathcal{N}_j$ to be the set of all non oscillatory solutions $x$ of (1.3.1) which satisfy (1.6.1) and (1.6.2). In addition, we denote $\mathcal{N}_0^-$ to be the set of all non oscillatory solutions $x$ of (1.3.1) which satisfy (1.6.3)-(1.6.5).

Lemma (1.6.2) means that every non oscillatory solution $x \in \mathcal{N}$ falls into one and only one of the classes $\mathcal{N}_j$ $(0 \le j \le n, (-1)^{n-j-1}\sigma = 1)$ and $\mathcal{N}_0^-$. More precisely, $\mathcal{N}$ has the following decomposition:

$$\left.\begin{array}{ll} \mathcal{N} = \mathcal{N}_{n-1} \cup \mathcal{N}_{n-3} \cup \dots \cup \mathcal{N}_1 \cup \mathcal{N}_0^- & for\ \sigma = 1 \quad and\ n\ is\ even\,; \\ \mathcal{N} = \mathcal{N}_{n-1} \cup \mathcal{N}_{n-3} \cup \dots \cup \mathcal{N}_2 \cup \mathcal{N}_0 & for\ \sigma = 1 \quad and\ n\ is\ odd\,; \\ \mathcal{N} = \mathcal{N}_{n} \cup \mathcal{N}_{n-2} \cup \dots \cup \mathcal{N}_2 \cup \mathcal{N}_0 & for\ \sigma = -1\ and\ n\ is\ even, \\ \mathcal{N} = \mathcal{N}_{n} \cup \mathcal{N}_{n-2} \cup \dots \cup \mathcal{N}_1 \cup \mathcal{N}_0^- & for\ \sigma = -1\ and\ n\ is\ odd; \end{array}\right\}$$

**Note**: $\mathcal{N}_0^-$ can appear only when $h(t)$ is eventually positive, so if $h(t)$ is either oscillatory or eventually negative, then (1.3.1) cannot possess a non oscillatory solution $x(t)$ satisfies (1.6.3), so that in this case the class $\mathcal{N}_0^-$ should be removed from decomposition from $\mathcal{N}$.

Let $x \in \mathcal{N}_j$ then we see by (1.6.2) that the asymptotic behavior of $(Lx)(t)$ as $t \to \infty$ is as follows:

(i) If $j = 0$, then either

(i-1) $\lim_{t\to\infty}(Lx)(t) = const \neq 0$ or

(i-2) $\lim_{t\to\infty}(Lx)(t) = 0$.

(ii) If $1 \le j \le n-1$, then one of the following three cases holds:

(ii-1) $\lim_{t\to\infty}\frac{(Lx)(t)}{t^j} = const \neq 0;$

(ii-2) $\lim_{t\to\infty}\frac{(Lx)(t)}{t^{j-1}} = const \neq 0;$

(ii-3) $\lim_{t\to\infty}\frac{(Lx)(t)}{t^j} = 0;$ and $\lim_{t\to\infty}\frac{|(Lx)(t)|}{t^{j-1}} = \infty.$

(iii) If $j = n$, then either

(iii-1) $\lim_{t\to\infty}\frac{(Lx)(t)}{t^{n-1}} = const \neq 0$ or

(iii-2) $\lim_{t\to\infty}\frac{|(Lx)(t)|}{t^{n-1}} = \infty.$

**Lemma 1.6.3**([5]):

If $x$ is as in lemma 1.6.1 and for some $k = 0,1, \dots, n-2$ ,

$$\lim_{t\to\infty} x^{(k)}(t) = c, \qquad c \in \mathbb{R} \tag{1.6.6}$$

Then

$$\lim_{t\to\infty} x^{(k+1)}(t) = 0 \tag{1.6.7}$$

# Chapter Two

# Oscillation and Non oscillation for Linear and Nonlinear Equations

## 2.1 Oscillation and non oscillation of linear equation

In this part of the study, equation (A) theorems that present oscillation and non oscillation will be demonstrated. Relying on the coefficient value, this part will be divided into several sections. According to equation (A) outcomes, confirming conditions for oscillation and non oscillation of equation (2.1.1) will be derived taking into consideration the fact that this equation is a special case for equation (A).

We consider the neutral linear functional differential equations of type

$$\frac{d^n}{dt^n}\left[x(t) - h(t)x\big(\tau(t)\big)\right] + \sigma \sum_{i=1}^{N} p_i(t)x\big(g_i(t)\big) = 0 \qquad \text{(A)}$$

Where $n \geq 2, \sigma = 1 \; or -1$, and the following conditions are assumed to hold:

a. $h: [t_0, \infty) \to \mathbb{R}$ is continuous and satisfies $|h(t)| \leq \lambda$ on $[t_0, \infty)$ for some constant $\lambda<1$;

b. $\tau$: $[t_0, \infty) \to \mathbb{R}$ is continuous and increasing, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$;

c. each $p_i: [t_0, \infty) \to (o, \infty)$ is continuous, $1 \leq i \leq N$;

d. each $g_i: [t_0, \infty) \to \mathbb{R}$ is continuous and satisfies $\lim_{t\to\infty} g_i\,(t) = \infty,\; 1 \leq i \leq N$.

From equation (A), we have a special equation

$$\frac{d^n}{dt^n}[x(t) + \lambda x(t-\tau)] + p(t)x(t-\sigma) = 0 \qquad (2.1.1)$$

Where $i = 1, h(t) \equiv -\lambda,\; \lambda > 0, \sigma \in \mathbb{R}, p \in C[t_0, \infty), p(t) > 0$ for $t \geq t_0$.

The main results are contained in theorems 2.1.1 and 2.1.3. Theorems 2.1.1 presents sufficient condition for the non oscillation of equation (2.1.1). Theorems 2.1.3 and 2.1.4 present sufficient condition for the oscillation of equation (2.1.1).

**Case 1**: equation (2.1.1) has a non oscillatory solution if

$$\int_{t_0}^{\infty} t^{n-1} p(t)dt < \infty \qquad (2.1.2)$$

The objective of this case is to obtain criteria for equation (A) to have non oscillatory solutions of two types described in section 1.6:

Type (**I**): $\lim_{t\to\infty} \frac{(Lx)(t)}{t^k} = \text{const} \neq 0$ for some $k \in \{0,1, \ldots n-1\}$;

Type (**II**): $\lim_{t\to\infty} \frac{(Lx)(t)}{t^l} = 0$; and $\lim_{t\to\infty} \frac{|(Lx)(t)|}{t^{l-1}} = \pm\infty$

for some $l \in \{1,2, \ldots n-1\}$.

where $(Lx)(t) = x(t) - h(t)x(\tau(t))$.

- **Solutions of type (I) :** We start with Type (I) solutions, and note that such solutions can be completely characterized in case

$h(t)h(\tau(t)) \geq 0$ is satisfied (which include the case of one-signed).

**Theorem 2.1.1**([7]): Suppose that (2.1.3) holds

$$h(t)h(\tau(t)) \geq 0 \quad \text{for all large } t. \tag{2.1.3}$$

Equation (A) has a non oscillatory solution $x(t)$ satisfying

$$x(t)[x(t) - \mathrm{h}(t)x(\tau(t))] > 0 \tag{2.1.4}$$

and

$$\lim_{t\to\infty} \frac{x(t)-\mathrm{h}(t)x(\tau(t))}{t^k} = \text{const} \neq 0 \tag{2.1.5}$$

for some $k \in \{0,1, \dots, n-1\}$ if and only if

$$\sum_{i=1}^{N} \int_{t_0}^{\infty} t^{n-k-1} [g_i(t)]^k p_i(t) dt < \infty \tag{2.1.6}$$

- **Solutions of type (II)** : We now consider non oscillatory solutions of type II of equation (A), that is, those solutions $x(t)$ which satisfy (2.1.4) and

$$\lim_{t\to\infty} \frac{x(t)-\mathrm{h}(t)x(\tau(t))}{t^l} = 0 \text{ and } \lim_{t\to\infty} \frac{x(t)-\mathrm{h}(t)x(\tau(t))}{t^{l-1}} = \pm\infty \tag{2.1.7}$$

for some $l \in \{1,2, \dots \mathrm{n}-1\}$ such that $(-1)^{n-l-1}\sigma=1$.

If $x(t)$ is one such solution of (A), then integration of (A) gives

$$\sum_{i=1}^{N} \int_{T}^{\infty} t^{n-l-1} |x(g_i(t))| p_i(t) dt < \infty$$

and

$$\sum_{i=1}^{N} \int_{T}^{\infty} t^{n-l} |x(g_i(t))| p_i(t) dt = \infty$$

For some $T > t_0$ sufficiently large. Since (2.1.4) holds, then $(Lx)(t)$ satisfies case (I) from Lemma (1.6.2), so $(Lx)(t)$ is a function of Kiguaradze degree $l$ for some $l \in \{1,2, \dots n-1\}$ such that $(-1)^{n-l-1}\sigma=1$. Thus, there exist positive constants $\alpha, \beta$ and T such that

$$|x(t)| \geq \alpha t^{l-1} \quad \text{and} \quad |x(t)| \leq \beta t^l \quad \text{for} \quad t \geq T,$$

Combining the above inequalities, which follows readily from (2.1.3) and (2.1.7), we see that

$$\sum_{i=1}^{N} \int_{t_0}^{\infty} t^{n-l-1} [g_i(t)]^{l-1} p_i(t) dt < \infty \tag{2.1.8}$$

and

$$\sum_{i=1}^{N} \int_{t_0}^{\infty} t^{n-l} [g_i(t)]^{l} p_i(t) dt = \infty \tag{2.1.9}$$

Thus, (2.1.8) and (2.1.9) are necessary conditions for the existence of a solution $x(t)$ satisfying (2.1.4) and (2.1.7) of equation (A) for which (2.1.3) is satisfied.

The following theorem presents sufficient conditions for the existence of such a solution of (A) in the case where $h(t) \geq 0$ for $t \geq t_0$.

**Theorem 2.1.2**([7])**:** Suppose that $h(t) \geq 0$ for $t \geq t_0$. Let $l \in \{1,2, \ldots n-1\}$ be such that $(-1)^{n-l-1}\sigma=1$. Equation (A) has a non oscillatory solution $x(t)$ satisfying (2.1.4) and (2.1.7) if

$$\sum_{i=1}^{N} \int_{t_0}^{\infty} t^{n-l-1} [g_i(t)]^l p_i(t) dt < \infty \tag{2.1.10}$$

and

$$\sum_{i=1}^{N} \int_{t_0}^{\infty} t^{n-l} [g_i(t)]^{l-1} p_i(t) dt = \infty \tag{2.1.11}$$

**Example 2.1.1**: Consider the special case of (A) with N=1

$$\frac{d^n}{dt^n}\left[x(t) - h(t)x\big(\tau(t)\big)\right] + \sigma p(t)x\big(g(t)\big) = 0 \tag{2.1.12}$$

In addition to (a-d) assume that (2.1.3) is satisfied. Condition (2.1.6), (2.1.10) and (2.1.11) for this equation reduce to

$$\int_{t_0}^{\infty} t^{n-k-1} [g(t)]^k p(t)\, dt < \infty \tag{2.1.13}$$

$$\int_{t_0}^{\infty} t^{n-l-1} [g(t)]^l p(t)\, dt < \infty \tag{2.1.14}$$

and

$$\int_{t_0}^{\infty} t^{n-l} [g(t)]^{l-1} p(t)\, dt = \infty \tag{2.1.15}$$

respectively. Suppose that $g(t)$ satisfies

$$0 < \lim_{t\to\infty} \inf \frac{g(t)}{t} \leq \lim_{t\to\infty} \sup \frac{g(t)}{t} < \infty \tag{2.1.16}$$

(Example of such $g(t)$ are

$$g(t) = t \pm \delta\ , \qquad g(t) = \mu t, \qquad g(t) = t + \sin t$$

Where $\delta$ and $\mu$ are positive constant). Then, the set of (2.1.6) for all $k = 0,1,\ldots,n-1$ reduces to a single condition

$$\int_{t_0}^{\infty} t^{n-1} p(t) dt < \infty \tag{2.1.17}$$

From Theorem 2.1.1 it follows that if (2.1.17) holds , then (2.1.12) has a solution $x(t)$ satisfying (2.1.5) for every $k \in \{0,1,\ldots\ldots,n-1\}$ , and that if

$$\int_{t_0}^{\infty} t^{n-1-\epsilon} p(t) dt = \infty$$

then (2.1.12) cannot have a solution $x(t)$ satisfying (2.1.5) for any $k \in \{0,1,\ldots\ldots,n-1\}$, we note that theorem 2.1.2 is not applicable to equation (2.1.12) subject to (2.1.16), since in this case conditions (2.1.14) and (2.1.15) are not consistent for any $l$, since $g(t)$ is finite as $t \to \infty$.

Next, suppose that $g(t) = t^{\theta}$, where $\theta \in (0,1)$ is a constant. Then, (2.1.14) and (2.1.15) become

$$\int_{t_0}^{\infty} t^{n-l-1+l\theta}\, p(t)\, dt < \infty \tag{2.1.18}$$

and

$$\int_{t_0}^{\infty} t^{n-l+\theta(l-1)}\, p(t)\, dt = \infty \tag{2.1.19}$$

Since $\lim_{t\to\infty} t^{\theta} = \infty$. Which may hold simultaneously; for example, the function $p(t) = t^{\gamma}, \gamma$ being a constant, from inequality (2.1.18) we have $n - l - 1 + l\theta + \gamma < 0 \Rightarrow \gamma < l(1-\theta) - n + 1$, And, from inequality (2.1.19) we have $n - l + \theta(l-1) + \gamma > 0 \Rightarrow \gamma > l - n - \theta(l-1)$, so it satisfies both (2.1.18) and (2.1.19) if $t_0 > 0$ *and* $l - n - \theta(l-1) < \gamma < l(1-\theta) - n + 1$. According to theorem 2.1.2, condition (2.1.18) and (2.1.19) for some

$l \in \{0,1,\ldots\ldots, n-1\}$ with $(-1)^{n-l-l}\sigma=1$ guarantee the existence of a solution $x(t)$ of equation (2.1.12) which has the asymptotic behavior (2.1.7)

**Example 2.1.2:** Consider the equation

$$\frac{d^2}{dt^2}[x(t) - \lambda x(t-\rho)] + (\lambda e^{\rho} - 1)e^{(v-1)t}x(vt) = 0\ ,\ t \ge 0 \tag{2.1.20}$$

where $0 < \lambda < 1$, $\rho > o$ and $v > 0$.

(i) Suppose that $\lambda e^{\rho} > 1$ .Then , (2.1.20) is a special case of (2.1.12) in which $n = 2, \sigma = 1, h(t) = \lambda, \tau(t) = t - \rho, p(t) = (\lambda e^{\rho} - 1)e^{(v-1)t}$ and $g(t) = vt$. From section 1.6 we have $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_0^-$ for (2.1.20).

Note that $\mathcal{N}_0^- \neq \phi$. Since $h(t) = \lambda$ is positive and $(-1)^n\sigma = 1$ is, more specifically, since the proper solution is positive, then we have from lemma(1.6.2) case II in section 1.6 $[x(t) - \lambda x(t-\rho)] < 0$. So $|x(t)| < \lambda|x(t-\rho)|$ and hence $|x(\tau^{-m}(t))| < \lambda^{m}|x(t)|$. Which implies that

$\lim_{t\to\infty} x(t) = 0$. To explain that, the set of all solutions $x(t)$ of equation (A) satisfies $x(t)[x(t) - h(t)x(\tau(t))] < 0$ is $\mathcal{N}_0^-$, and this class $\mathcal{N}_0^-$ is empty if $\sigma = 1$ and $n$ is odd or $\sigma = -1$ and n is even,

note that $\tau^0(t) = t$, $\tau^i(t) = \tau\left(\tau^{i-1}(t)\right)$, $\tau^{-i}(t) = \tau^{-1}\left(\tau^{-(i-1)}(t)\right)$,

since $\tau(t) \le t$. Then, $|x(t)| \le \lambda x(\tau(t))$

$t \le \tau^{-1}(t)$, so we have $|x(\tau^{-1}(t))| \le \lambda|x(t)| \le \lambda^2 x(\tau(t))$

$\tau^{-1}(t) \le \tau^{-2}(t)$, so we have $\left|x\left(\tau^{-2}(t)\right)\right| \le \lambda\left|x\left(\tau^{-1}(t)\right)\right| \le \lambda^2 x(t)$

Continuing in this manner we have

$\left|x\left(\tau^{-m}(t)\right)\right| \leq \lambda^m x(t)$ . Equation (2.1.20) has a solution $x(t) = e^{-t}$ belonging to the class $\mathcal{N}_0^-$.

The possible asymptotic behavior of the members $x(t)$ of $\mathcal{N}_1$ are:

$$\lim_{t\to\infty} \frac{x(t)-\lambda x(t-\rho)}{t} = \text{const} \neq 0 \qquad (2.1.21)$$

$$\lim_{t\to\infty} x(t) - \lambda x(t-\rho) = \text{const} \neq 0 \qquad (2.1.22)$$

$$\lim_{t\to\infty} \frac{x(t)-\lambda x(t-\rho)}{t} = 0 \quad , \ \lim_{t\to\infty} x(t) - \lambda x(t-\rho) = \pm\infty \qquad (2.1.23)$$

If $v < 1$ then, since $g(t) = vt$ satisfies (2.1.16), $p(t) = (\lambda e^{\rho} - 1)e^{(v-1)t}$ and $n = 2$, we have $\int_a^\infty s^{2-1}p(s)ds = \int_a^\infty s(\lambda e^{\rho} - 1)e^{(v-1)s}ds = (\lambda e^{\rho} - 1)\int_a^\infty se^{(v-1)s}\,ds$ use the integration by parts

$$(\lambda e^{\rho} - 1)\int_a^\infty se^{(v-1)s}\,ds = (\lambda e^{\rho} - 1)\ \lim_{A\to\infty}\int_a^A se^{(v-1)s}\,ds$$

$$= (\lambda e^{\rho} - 1)\ \lim_{A\to\infty}\frac{e^{(\upsilon-1)s}}{(\upsilon-1)^2}\left((\upsilon-1)s-1\right)\Big|_a^A$$

$$= (-1)(\lambda e^{\rho} - 1)\ \frac{e^{(\upsilon-1)a}}{(\upsilon-1)^2}\left((\upsilon-1)a-1\right) < \infty$$

Since $\upsilon - 1 < 0$, then (2.1.17) holds, and so (2.1.20) has a solution satisfying (2.1.21)as well as a solution satisfying (2.1.22). However, there is no solution of (2.1.20) which has the asymptotic property (2.1.23), because the condition

(2.1.9) which is necessary for the existence of such a solution is violated for equation (2.1.20).

If $v \geq 1$, then,

$$\int_a^\infty s^{2-1}p(s)\,ds = (\lambda e^{\rho} - 1)\ \lim_{A\to\infty}\int_a^A se^{(v-1)s}\,ds = \infty$$

Since $\upsilon - 1 > 0$, then equation $\int_{t_0}^\infty t^{n-1}p(t)dt = \infty$ *(n = 2)* holds, so that (2.1.20) has neither a solution satisfying (2.1.21) nor a solution satisfying (2.1.22). Since (2.1.8) is not satisfied.

(ii) Suppose that $\lambda e^{\rho} < 1$ .then , (2.1.20) is a special case of (2.1.12) in which

$$n = 2, \sigma = -1, h(t) = \lambda\,, \tau(t) = t - \rho, p(t) = (\lambda e^{\rho} - 1)e^{(v-1)t} \text{ and } g(t) = vt.$$

From section 1.6 we have $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2$ and the possible types of asymptotic behavior of non oscillatory solutions x(t) of (2.1.20) are (2.1.21) and (2.1.22),

$$\lim_{t\to\infty}[x(t) - \lambda x(t-\rho)] = 0 \qquad (2.1.24)$$

and

$$\lim_{t\to\infty} \frac{x(t)-\lambda x(t-\rho)}{t} = \infty \text{ or } -\infty \qquad (2.1.25)$$

Exactly the same statements as in (i) hold for solutions which satisfy (2.1.21) and (2.1.22), depending on whether $v < 1$or $v > 1$. Equation (2.1.20) has a solution $x(t) = e^{-t}$satisfying (2.1.24).

**Case 2**: equation (2.1.1) has oscillatory solution if

$$\int_{t_0}^{\infty} t^{n-1-\varepsilon} p(t)dt = \infty \qquad \text{for some } \varepsilon > 0$$

This result is based on the following theorems due to Joarš and Kusano[7].

- Oscillation criteria : We are interested in the situation in which all

proper solutions of equation (A) are oscillatory. Since this situation is equivalent to the nonexistence of non oscillatory solutions of (A), Jaroš and Kusano obtained conditions under which none of the solution classes appearing in the classification $\mathcal{N}$ in section 1.6 has a member they derivation of the result is based on the following lemma due to Kitamura $[28, p. 487]$ which provide oscillation criteria for functional differential inequalities of the form

$$\{\sigma u^{(n)}(t) + p(t)u(g(t))\}\text{sgn}\, u(t) \leq 0 \qquad (2.1.26)$$

where $n \geq 2,\ \sigma = \pm 1,\ p: [a, \infty) \to (o, \infty)$ is continuous, $g: [a, \infty) \to (o, \infty)$ is continuous, and $\lim_{t\to\infty} g(t) = \infty$.

We introduce the following lemma which is useful in the proof of theorem 2.1.3.

Let $g_*(t) = \min\{g(t), t\}$

**Lemma 2.1.1**([5])**:** Let $\sigma = 1$ and n be even. There is no non oscillatory solution of (2.1.26) if

$$\int_a^{\infty} [g_*(t)]^{n-1} [g(t)]^{-\varepsilon} p(t)dt = \infty \quad \text{for some } \varepsilon \qquad (2.1.27)$$

Let $x(t)$ be a non oscillatory solution of equation (A), let$x(t) - h(t)x(\tau(t)) = (Lx)(t)$ from (A) we have $(Lx)^{(n)}(t) = \sigma \sum_{i=1}^{N} p_i(t) x\left(g_i(t)\right)$, for$g_i(t) \leq t$, that is

$(Lx)^{(n)}(t) \leq \sigma \sum_{i=1}^{N} p_i(t) x\left(g_i(t)\right)$ , for $g_i(t) \leq t$. It follows that $(Lx)^{(i)}(t)$ $\{i = 0,1, \dots, n-1\}$ is strictly monotone and of constant sign eventually. Hence $(Lx)(t) > 0$ or $(Lx)(t) < 0$ from large $t$, that is $x(t) - h(t)x(\tau(t))$ is eventually one-signed, we examine $h(t)$ in two cases:

1) $\boldsymbol{h(t)}$ **is eventually positive** : Since $x(t) - h(t)x(\tau(t))$ is eventually

one-signed. Then, the function $x(t)[x(t) - h(t)x(\tau(t))]$ is either eventually positive or eventually negative.

(1-i) Consider the case where $x(t)[x(t) - h(t)x(\tau(t))] > 0$ for large $t$. Put $v(t) = x(t) - h(t)x(\tau(t))$. Since, in this case, $|v(t)| \leq |x(t)|$ for large $t$, we see from (A) that

$$\{\sigma v^{(n)}(t) + \sum_{i=1}^{N} p_i(t)v(g_i(t))\}\operatorname{sgn} v(t) \leq 0$$

Provided $t$ is large enough. It follows that $v(t)$ is a non oscillatory solution of each of the differential inequalities

$$\{\sigma v^{(n)}(t) + p_i(t)v(g_i(t))\operatorname{sgn} v(t) \leq 0 \quad 1 \leq i \leq N \qquad (2.1.28)_i$$

for all sufficiently large $t$, and that $x(t)$ is a member of $\mathcal{N}_l$ if and only if $v(t)$ is a solution of degree $l$ of $(2.1.28)_i$, for each $1 \leq i \leq N$.

(1-ii) Consider the case where $x(t)[x(t) - h(t)x(\tau(t))] < 0$ for large . Put $w(t) = h(t)x(\tau(t)) - x(t)$. Since $|w(t)| \leq h(t)|x(\tau(t))| \leq \lambda|x(\tau(t))|$.

We find $|w(\tau^{-1}(t))|/\lambda \leq |x(t)|$, which combined with (A), yields

$$\{(-\sigma)w^{(n)}(t) + \lambda^{-1}\sum_{i=1}^{N} p_i(t)w\left(\tau^{-1}(g_i(t))\right)\}\operatorname{sgn} w(t) \leq 0.$$

It follows that

$$\{(-\sigma)w^{(n)}(t) + \lambda^{-1}p_i(t)w\left(\tau^{-1}(g_i(t))\right)\}\operatorname{sgn} w(t) \leq 0, 1 \leq i \leq N \qquad (2.1.29)_i$$

for all sufficiently large $t$, and that $x(t)$ is a member of $\mathcal{N}_0^-$ if and only if $w(t)$ is a solution of degree 0 of $(2.1.29)_i$, for each $1 \leq i \leq N$.

2) **$h(t)$ is eventually negative :** we will be interested in this case and the case where $h(t)$ is oscillatory and such that $h(t)h(\tau(t)) \geq 0$ for all large $t$,in these cases, we must have $\mathcal{N}_0^-$ is empty as was note in section 1.6, then there is no solution of (A) satisfying $x(t)[x(t) - h(t)x(\tau(t))] < 0$ for large $t$, then, the function $v(t) = x(t) - h(t)x(\tau(t))$ satisfies

$$(1 - \lambda)|v(t)| \leq |x(t)| \text{ for large } t \qquad (2.1.30)$$

Provided the Kiguradze degree of $x(t)$ is positive. From (A) and (2.1.30) we see that $v(t)$ satisfies

$$\{\sigma v^{(n)}(t) + (1 - \lambda)p_i(t)v(g_i(t))\operatorname{sgn} v(t) \leq 0, \quad 1 \leq i \leq N \qquad (2.1.31)_i$$

for all sufficiently large $t$, and that $x(t)$ is a member of $\mathcal{N}_l$ if and only if $v(t)$ is a solution of degree $l$ of $(2.1.31)_i$ for each $1 \leq i \leq N$.

Jaros and Kusano derived oscillation criteria for equation (A) to obtain conditions which preclude all the possible solution classes $\mathcal{N}_l, 0 \leq l \leq n$, and $\mathcal{N}_0^-$ appearing in the classification $\mathcal{N}$. That this is indeed possible can be seen from the above observations combined with lemma 2.1.1which apply directly to the functional differential inequalities$(2.1.31)_i$, $1 \leq i \leq N$.

Here, the interesting case where $h(t)$ is eventually negative, so $\mathcal{N}_0^-$ is empty.

The following theorem follows in this manner.

**Theorem 2.1.3**([**5**])**:** Let $\sigma = 1$ and $n$ be even. Suppose that $h(t)$ is eventually negative or that $h(t)$ is oscillatory and satisfies (2.1.3). All proper solutions of (A) are oscillatory if there is $i \in \{0,1,\ldots\ldots,N\}$ such that

$$\int_{t_0}^{\infty}[g_{i*}(t)]^{n-1}[g_i(t)]^{-\varepsilon}\, p_i(t)dt \; = \infty \qquad \text{for some } \varepsilon > 0 \qquad (2.1.32)$$

Where $g_{i*}(t) = \min\{g_i(t), t\}$.

**Proof:** According to classification $\mathcal{N}$, $l \in \{1,3,\ldots, n-1\}$, and $\mathcal{N}_0^-$ are the possible classes of non oscillatory solution of (A) with $\sigma = 1$ and $n$ be even. Our task is, to show that all of these solution classes are empty if the hypotheses of the theorem are satisfied. In this case $\mathcal{N}_0^-$ is necessary empty. Suppose that $\mathcal{N}_l \neq \emptyset$ for some $l \in \{1,3,\ldots, n-1\}$. Then, each of the inequalities in (2.1.31) possesses a non oscillatory solution of degree $l$. However, this impossible, because from Lemma 2.1.1 applied to $(2.1.31)_i$ it follows that (2.1.32) prevent $(2.1.31)_i$ from having a non oscillatory solution of any kind. Thus we must have $\mathcal{N}_l = \emptyset$ for all $l \in \{1,3,\ldots, n-1\}$ .□

**Example 2.1.3 :** Consider the equations

$$\frac{d^n}{dt^n}[x(t) + \lambda x(\alpha t)] + p(t)x(\beta t) = 0 \qquad (2.1.33)$$

Where $0 < \lambda < 1$ , $0 < \alpha < 1$ , $\beta > 0$ and $p: [t_0, \infty) \to (0, \infty)$ is continuous,$t_0 > 0$. This is a special case of (A) in which $\sigma = 1, N = 1, h(t) = -\lambda, \tau(t) = \alpha t, p(t) = p_1(t)$ *and* $g_1(t) = \beta t$ . Noting that $g_*(t) = \min\{1, \beta\}t$, we have all proper solutions of equation (2.1.33) are oscillatory if

$$\int_{t_0}^{\infty} t^{n-1-\varepsilon} p(t)dt = \infty, \qquad \text{for some } \varepsilon > 0$$

**Theorem 2.1.4**([**10**])**:** Consider the equation

$$\frac{d^n}{dt^n}[x(t) + h(t)x(t-\tau)] + p(t)x(t-\sigma) = 0, \; t \geq t_0 \qquad (2.1.34)$$

Where $p, h \in C([t_0, \infty), \mathbb{R})$, and assume that $n$ is even and that the hypotheses $(H1)$ and $(H2)$ are satisfied.

$(H1)$ There exist positive constants $h_1$ and $h_2$ such that $h_1 \leq h(t) \leq h_2$.

$(H2)$ There exists a positive constant $p$ such that

$$p(t) \geq p > 0.$$

Furthermore, assume that $h(t)$ is not eventually negative. Then every solution of equation (2.1.34) oscillates.

The example below illustrates Theorem 2.1.4.

**Example 2.1.4:** The NDDE

$$\frac{d^2}{dt^2}\left[x(t) + \left(\frac{1}{2} + \cos t\right)x(t - 2\pi)\right] + \left(\frac{3}{2} + \cos t\right)x(t - 4\pi) = 0, t \geq 0,$$

Note that there is nonnegative $h_1 = 1/2$ and $h_2 = 3/2$ such that$1/2 \leq h(t) \leq 3/2$, and $p(t) \geq \frac{3}{2} > 0$, satisfies the hypotheses of theorem 2.1.4. Therefore, every solution of this equation oscillates. For example, $x(t) = \frac{\cos t}{3/2 + \cos t}$ is an oscillatory solution.

The following example shows that if we remove the hypotheses $(H2)$ from theorem 2.1.4, the result may not true.

**Example 2.1.5:** The NDDE

$$\frac{d^2}{dt^2}\left[x(t) + \left(\frac{1}{2} + \cos t\right)x(t - 2\pi)\right] + \left(\frac{3}{2} + \cos t\right)x(t - 4\pi) = 0, t \geq 0,$$

Satisfies all the hypotheses of theorem 2.1.4 except $(H2)$. Since $0 < p(t) < 1$, note that $x(t) = t^{1/2}$ is a non oscillatory solution of this equation.

**Remark 2.1.1:** From case 1we have the following result

Equation (2.1.1) with $0 < \lambda < 1$ is oscillatory if

$$\int_{t_0}^{\infty} t^{n-1-\varepsilon} p(t)dt = \infty \qquad \text{for some } \varepsilon > 0 \tag{2.1.35}$$

and equation (2.1.1) is non oscillatory if

$$\int_{t_0}^{\infty} t^{n-1} p(t)dt < \infty \; . \tag{2.1.36}$$

**Remark 2.1.2:** If $p(t) = ct^{-n}$ where $c > 0$ condition (2.1.2) and (2.1.35) fail to be satisfied.

## 2.2 Non Oscillation of nonlinear equation

In this section we discuss the non oscillatory behavior of the equation

$$\frac{d^n}{dt^n}\left[x(t) - h(t)x\big(\tau(t)\big)\right] + \sigma p(t)f\left(x\big(g(t)\big)\right) = 0 \qquad (2.2.1)$$

Where $n \geq 2, \sigma = \pm 1$, and the following conditions are always assumed to hold:

a) $\tau(t) \in C[t_0, \infty)$, $\tau$ is nondecreasing on $[t_0, \infty)$, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$;

b) $h(t) \in C[\tau(t_0), \infty)$, $|h(t)| \leq h < 1$ for $t \geq t_0$, where $h$ is a positive constant, and $h(t)h\big(\tau(t)\big) \geq 0$ for $t \geq t_0$;

c) $p(t) \in C[t_0, \infty)$ and $p(t) > 0$ $for$ $t \geq t_0$;

d) $f(u) \in C\big((-\infty, \infty)\backslash\{0\}\big)$ and $f(u)u > 0$ for $u \neq 0$;

e) $g(t) \in C[t_0, \infty)$ $and$ $\lim_{t\to\infty} g(t) = \infty$.

**Definition 2.2.1:** Equation (2.2.1) is called strictly sublinear if there is a number α such that $0 < \alpha < 1$ and

$$\frac{|f(u_1)|}{|u_1|^\alpha} \geq \frac{|f(u_2)|}{|u_2|^\alpha} \qquad \text{for} \quad |u_1| \leq |u_2|, \quad u_1u_2 > 0 .$$

Equation (2.2.1) is called strictly super linear if there is a number $\beta > 1$ such that

$$\frac{|f(u_1)|}{|u_1|^\beta} \leq \frac{|f(u_2)|}{|u_2|^\beta} \qquad \text{for} \quad |u_1| \leq |u_2|, \quad u_1u_2 > 0 .$$

The equation

$$\frac{d^n}{dt^n}[x(t) + \lambda x(t-\tau)] + p(t)|x(t-\sigma)|^\gamma \, sgn\, x(t-\sigma) = 0$$

Is a special case of equation (2.2.1), is strictly sublinear if $-\infty < \gamma < 1$ and is strictly super linear if $1 < \gamma < \infty$ .

**Note**: We say that a non oscillatory solution $x$ of equation (2.2.1) or the inequality

$$\left\{\sigma x^{(n)}(t) + p(t)f\left(x\big(g(t)\big)\right)\right\} sgn\, x\big(g(t)\big) \leq 0. \qquad (2.2.2)$$

are of class $\mathcal{N}_j$ if $x$ satisfies

$$\begin{cases} (Lx)(t)(Lx)^{(i)}(t) > 0, & 0 \leq i \leq j, \\ (-1)^{i-j}\ (Lx)(t)(Lx)^{(i)}(t) > 0, & j+1 \leq i \leq n, \end{cases}$$

for all sufficiently large t .

**Theorem 2.2.1**([29])**:** Let (a)-(e), (2.2.3) and (2.2.4) be satisfied.

τ is locally Lipschitz continuous on $[t_0\ ,\infty)$ (2.2.3)

h is locally Lipchitz continuous on $[\tau(t_0\ ),\infty)$ (2.2.4)

Assume that equation (2.2.1) is strictly sublinear and $0 \leq j \leq n-1$, $(-1)^{n-j-1}\sigma = 1$ . Assume in addition that $g_*(t) = \min\{g(t), t\}$. satisfies

$$\lim_{t\to\infty} \inf \frac{g_*(t)}{g(t)} > 0 \qquad (2.2.5)$$

Then, a necessary and sufficient condition for (2.2.1) to have a non oscillatory solution of class $\mathcal{N}_j$ is that

$$\int_{t_0}^{\infty} t^{n-j-1}\, p(t)\left|f(c[g(t)]^j)\right| dt < \infty, \qquad \text{for some } c \neq 0. \qquad (2.2.6)$$

**Theorem 2.2.2**([29])**:** Let (a)-(e), (2.2.3) and (2.2.4) be satisfied.

Assume that equation (2.2.1) is strictly super linear and $1 \leq j \leq n-1$ , $(-1)^{n-j-1}\sigma = 1$. Assume in addition that $g_*(t) = \min\{g(t), t\}$. satisfies

$$\lim_{t\to\infty} \inf \frac{g_*(t)}{t} > 0 \qquad (2.2.7)$$

Then, a necessary and sufficient condition for (2.2.1) to have a non oscillatory solution of class $\mathcal{N}_j$ is that

$$\int_{t_0}^{\infty} t^{n-j}\, p(t)\left|f\left(c[g(t)]^{j-1}\right)\right| dt < \infty \qquad \text{for some } c \neq 0 \qquad (2.2.8)$$

**Example 2.2.1:** Consider the equation

$$\frac{d^n}{dt^n}[x(t) - h\sin t\,.\,x(t-2\pi)] + \sigma p(t)|x(t-\tau)|^{\gamma} sgn\, x(t-\tau) = 0 \qquad (2.2.9)$$

where $n \geq 2$ , $\sigma = 1$ or $-1$ , $p \in C[0,\infty)$ , $p(t) > 0$ on $[0,\infty)$ , and $\tau$ , h, $\gamma$ are constants such that $|h| < 1$ , $|\gamma| < \infty$ , $|\tau| < \infty$ . First, notice that
case (II) in lemma 1.6.2 does not occur (that is, the class $\mathcal{N}_0^-$ for equation (2.2.9) is always empty) since the function $h(t) = h\sin t$ takes a nonpositive value on [T, ∞) for all T. Let j be an integer satisfying: $1 \leq j \leq n-1$ and $(-1)^{n-j-1}\sigma = 1$.

**Remark2.2.1:** Theorem 2.2.1 shows that equation (2.2.9) with $-\infty < \gamma < 1$ has a non oscillatory solution of class $\mathcal{N}_j$ if and only if

$$\int_{t_0}^{\infty} t^{n-j-1} p(t) c^{\gamma} |t-\tau|^{\gamma j} sgn(t-\tau)dt < \infty \Longrightarrow \int_{t_0}^{\infty} t^{n-j-1+\gamma j}\, p(t)dt < \infty,$$

While theorem 2.2.2 shows that equation (2.2.9) with $1 < \gamma < \infty$ has a non oscillatory solution of class $\mathcal{N}_j$ if and only if

$$\int_{t_0}^{\infty} t^{n-j} p(t)\left(c^{\gamma}|t-\tau|^{\gamma(j-1)}\right) sgn(t-\tau)dt < \infty \Longrightarrow \int_{t_0}^{\infty} t^{n-j+\gamma(j-1)}\, p(t)dt < \infty.$$

Consider the special case that n is even and $\sigma = 1$ in equation (2.2.9).

We see that if $\gamma < 1$ and the condition

$$\int_{t_0}^{\infty} t^{\gamma(n-1)}\, p(t)dt = \infty \qquad (2.2.10)$$

are satisfied, then all the classes $\mathcal{N}_j$, $j = 1, 3, \ldots, n - 1$, for equation (2.2.9) are empty. Since $\mathcal{N}_0^-$ is also empty, we can conclude the following: Let n be even, $\sigma = 1$ and $\gamma < 1$, then equation (2.2.9) has no non oscillatory solutions if and only if (2.2.10) holds. Similarly, if n be even, $\sigma = 1$ and $\gamma > 1$. Then equation (2.2.9) has no non oscillatory solutions

if and only if $\int_{t_0}^{\infty} t^{(n-1)}\, p(t)dt = \infty$ .

The following results concerning the non oscillatory solution of the

Equations:

$$\left(x(t) + \lambda x(t-\tau)\right)^{(n)} + f\left(t, x\left(g(t)\right)\right) = 0\,,\ \text{for}\ t \geq t_0 \qquad (2.2.11)$$

and

$$\left(x(t) + h(t)x(t-\tau)\right)^{(n)} + \sigma f\left(t, x\left(g(t)\right)\right) = 0 \qquad (2.2.12)$$

Where $\lambda$ is a real number, $\tau > 0\,, n \geq 2$, $\sigma = \pm 1$, $g \in C[t_0, \infty)$, $\lim_{t\to\infty} g(t) = \infty$, $h(t) \in C[t_0 - \tau, \infty)$, $f \in C\left([t_0, \infty) \times (0, \infty)\right)$,

$f\,(t, u) \geq 0$ for $(t, u) \in \left([t_0, \infty) \times (0, \infty)\right)$, and $f\,(t, u)$ is nondecreasing

if $u \in (0, \infty)$ for each fixed $t \in [t_0, \infty)$.

**Theorem 2.2.3** ([27]): Suppose that $|\lambda| \neq 1\,, xf(t, x) > 0\ (x \neq 0)$and $|f(t, x)| \leq |f(t, y)|$ for $|x| < |y|\,, xy > 0$. If

$$\int^{+\infty} s^{n-1}\, |f(s, k)|\; ds < +\infty \qquad (2.2.13)$$

for some $k \neq 0$ , then equation(2.2.11) has a bounded non oscillatory solution.

**Theorem 2.2.4** ([27]): Suppose that $|\lambda| \neq 1\,,\ g(t) \leq t\,, xf(t, x) > 0, (x \neq 0) and\ |f(t, x)| \leq |f(t, y)|\ for\ |x| < |y|\,, xy > 0$. If

$$\int^{+\infty} \left|f\left(t, kR\left(g(t)\right)\right)\right|\; ds < +\infty \qquad (2.2.14)$$

For some $k \neq 0$ , where $R(t) = t^{n-1}$ then (2.2.11) has an unbounded non oscillatory solution.

**Theorem 2.2.5**([18])**:** Let $k$ be an integer with $0 \le k \le n-1$ . Suppose that equation (2.2.15) holds

$$h(t) > -1 \text{ and } h(t) = h(t-\tau), \quad t \ge t_0. \tag{2.2.15}$$

Then equation (2.2.12) has a positive solution $x(t)$ such that

$$x(t) = \left[\frac{c}{1+h(t)} + o(1)\right] t^k \qquad as \quad t \to \infty$$

for some $c > 0$ if and only if

$$\int_{t_0}^{\infty} t^{n-k-1} f(t, c[g(t)]^k) dt < \infty \quad \text{for some} \quad c > 0 .$$

If $\lambda = 1$ in equation (2.2.11) , then the function

$x(t) = \alpha\, t^k$ $(\alpha \in \mathbb{R}, \alpha \ne 0, k \in \mathbb{Z}, 0 \le k \le n-1)$, is a nontrivial solution of the equation $\frac{d^n}{dt^n}[x(t) + \lambda x(t-\tau)] = 0$ ,

and so it is natural to expect that, if $f$ is small enough in some sense, equation (2.2.11) possesses a positive solution $x(t)$ which behaves like

the function $\alpha t^k$ as $t \to \infty$ .

For the case $|\lambda| < 1$, the smallness condition on $f$ is characterized by the integral condition

$$\int_{t_0}^{\infty} t^{n-k-1} f(t, c[g(t)]^k) dt < \infty \text{ for some } c > 0 \tag{2.2.16}$$

In fact, it is known that equation (2.2.11) with $|\lambda| < 1$ has a solution $x(t)$ satisfying $\lim_{t\to\infty} \frac{x(t)}{t^k}$ (exist and is positive finite value) if and only if (2.2.16) holds, see [7], [18].

It has been observed that there is a slight difference between the case $\lambda = -1$ and the case$-1 < \lambda \le 1$. M.Naito discussed the case $\lambda = +1$ and he proved that the same result as the case$|\lambda| < 1$, more precisely, we have the following theorem.

**Theorem 2.2.6**([15]): Let $k$ be an integer with $0 \le k \le n-1$. Then the equation

$$\frac{d^n}{dt^n}[x(t) + x(t-\tau)] + \sigma f\left(t, x\big(g(t)\big)\right) = 0 \tag{2.2.17}$$

has a solution $x(t)$ satisfying $\lim_{t\to\infty}\frac{x(t)}{t^k}$ (exist and is positive finite value) if and only if (2.2.16) holds.

**Remark 2.2.2**: The purpose in [18] is to extend this result to the equation

$$\frac{d^n}{dt^n}[x(t)+h(t)x(t-\tau)]+\sigma f\left(t,x\big(g(t)\big)\right)=0$$

where $h(t)\in C[t_0-\tau,\infty)$ ; $n\geq 2$ ; $\sigma=\pm 1$ for the case equation

(2.2.15), of course (2.2.15) means that $h(t)$ is a $\tau-$periodic function satisfying $h(t)>-1$ , $\geq t_0$ , and hence there are

a constants $\mu$ and $\lambda$ such that $-1<\mu\leq h(t)\leq \lambda<\infty$ for $t\geq t_0$.

**Remark 2.2.3:**

If $\lambda\geq 1$ then the linear equation

$$\frac{d^n}{dt^n}[x(t)+\lambda x(t-\tau)]+p(t)x(t-\sigma)=0$$

has a non oscillatory solution if $\int_{t_0}^{\infty} t^{n-1}\, p(t)dt<\infty$, and the equation

$$\frac{d^n}{dt^n}[x(t)+\lambda x(t-\tau)]+p(t)|x(t-\sigma)|^{\gamma}\, sgn\, x(t-\sigma)=0$$

has a non oscillatory solution if $\int^{\infty} t^{\min\{\gamma,1\}(n-1)}p(t)<\infty$ .

## 2.3 Oscillation of Nonlinear Equation.

We investigate the oscillatory behavior of the equation

$$\left[x(t) + h(t)x\big(\tau(t)\big)\right]^{(n)} + p(t)f\left(x\big(g(t)\big)\right) = 0 \qquad (2.3.1)$$

Where, $h(t), \tau(t) \in C([t_0,\infty), R_+)$, $t_0 \geq 0$, $h(t) \not\equiv 0$ on any half line $[t_0,\infty)$, $\tau(t) \leq t$, $g(t) \leq t$, with commute delayed argument i.e.

$\tau\big(g(t)\big) = g\big(\tau(t)\big), t \geq t_0.$

The following assumptions are made for their use in this section:

(1) $f(u+v) \leq f(u) + f(v)$ if $u, v > 0$

(2) $f(u+v) \geq f(u) + f(v)$ if $u, v < 0$

(3) $f(ku) \leq kf(u)$ if $k \geq 0$ and $u > 0$, for each $k \in K$,

Where $K = \{k: h(t) = k \text{ for some } t \in [t_0,\infty)\}$.

(4) $f(ku) \geq kf(u)$ if $k \geq 0$ and $u < 0$, for each $k \in K$

(5) $f(u)$ is bounded away from zero if $u$ is bounded away from zero .

(6) $\int_{t_0}^{\infty} p(s)\,ds = \infty$ .

(7) $\tau(t) \in C^1([t_0,\infty))$ and $\tau'(t) \geq b$ , where b is positive constant .

(8) There exists a positive constant M such that $h\big(g(t)\big)p(t) \leq Mp\big(\tau(t)\big)$.

The main result is contained in the following theorem:

**Theorem 2.3.1**([10]): Assume that conditions (1)-(8) hold. Then,

If n is even, every solution of equation (2.3.1) is oscillatory.

**Proof:** Suppose that equation (2.3.1) has non oscillatory solution $x(t)$.

Without loss of generality, assume that $x(t)$ is eventually positive (the proof is similar when $x(t)$ is eventually negative). That is $x(t) > 0$,

$x\big(g(t)\big) > 0, x\big(\tau(t)\big) > 0$, and $x\left(\tau\big(g(t)\big)\right) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$.

Set $\quad z(t) = x(t) + h(t)x\big(\tau(t)\big). \qquad (2.3.2)$

Since $h(t)$ is nonnegative then $z(t) > 0$ for $t \geq t_1$ . Using the fact that $\tau(g(t)) = g(\tau(t))$ for $t \geq t_1$ , then from (1), and (3) we have:

$$f\left(z(g\,(t))\right) = f\left(x(g(t)) + h(g(t))x\left(\tau(g(t))\right)\right)$$

$$\leq f\left(x(g(t))\right) + h(g(t))f\left(x\left(g(\tau(t))\right)\right) \qquad (2.3.3)$$

Since $x(g(t)) > 0$, and $x(\tau(g(t)\,)\,) > 0$ from (1) we have (2.3.3)

Now using (2.3.1) and (2.3.2) we obtain

$$z^{(n)}(t) = -p(t)f\left(x(g(t))\right) \qquad (2.3.4)$$

From (2.3.3) and (2.3.4) we have:

$$z^{(n)}(t) + p(t)f\left(z(g(t))\right) \leq z^{(n)}(t) + p(t)\left[f\left(x(g(t))\right) + h(g(t))f\left(x\left(g(\tau(t))\right)\right)\right].$$

Hence

$$z^{(n)}(t) + p(t)f\left(z(g(t))\right) \leq \; p(t)h(g(t))f\left(x\left(g(\tau(t))\right)\right) \qquad (2.3.5)$$

Since $x(g(t)) > 0$ for $t \geq t_1$ , $z^{(n)}(t) \leq 0$ and so, $z^{(i)}(t)$ is monotonic for $i = 0,1, \dots, n -$. Therefore $z^{(n-1)}(t) > 0$ or $z^{(n-1)}(t) < 0$ eventually, if $z^{(n-1)}(t) \leq 0$ then from the facts that $z^{(n)}(t) \leq 0$ and $p(t) \not\equiv 0$ , imply that $z(t) < 0$ eventually , so a contradiction . Hence there exists $t_2 \geq t_1$ such that $z^{(n-1)}(t) > 0$ for $t \geq t_2$ .

From (2.3.1), and the fact that $\tau'(t) \geq b > 0$ , we have:

$$z^{(n)}(\tau(t))\tau'(t) + p(\tau(t))f\left(x\left(g(\tau(t))\right)\right)\tau'(t) = 0 \qquad (2.3.6)$$

Let $t_3 \geq t_2$ such that $z^{(n-1)}(\tau(t)) > 0$ for $t \geq t_3$ , then integrate (2.3.6) from $t_3$ to infinity, we get :

$$\int_{t_3}^{\infty} p(\tau(s))f\left(x\left(g(\tau(s))\right)\right)\tau'(s)ds = z^{(n-1)}(\tau(t_3)) - L\,.$$

where L=$\lim_{t\to\infty} z^{(n-1)}(t)$. Since $z^{(n-1)}(t) > 0$ eventually, we show that:

$$\int_{t_3}^{\infty} p(\tau(s))f\left(x\left(g(\tau(s))\right)\right)\tau'(s)ds < \infty \qquad (2.3.7)$$

Using (2.3.7), with (7), and (8), follows that:

$$\int_{t_3}^{\infty} p(s)h(g(s))f\left(x\left(g(\tau(s))\right)\right)\tau'(s)ds < \infty \qquad (2.3.8)$$

Integrating (2.3.5), and using (2.3.8), we show that:

$$\int_{t_3}^{\infty} p(s)f\left(z(g(s))\right)ds < \infty \qquad (2.3.9)$$

Since (5) and (6) hold, then (2.3.9) implies that $\lim_{t\to\infty} \inf z(t) = 0$.

But $z(t)$ is positive and monotonic, so $z(t) \to 0$ as $t \to \infty$ . So $z(t)$ is decreasing, implies that $z'(t) \leq 0$ eventually. For $n > 1$ , $z'(t) \to 0$

as $t \to \infty$ since $z'(t)$ is monotonic and $z(t) > 0$ ($z(t)$ is concave up ).Hence $z'(t)$ is increasing ,which implies that $z''(t) > 0$ . for $z''(t)$ is monotonic and negative . Continuing in this manner we have:

$$z^{(i)}(t)z^{(i+1)}(t) \leq 0 \text{, for } i = 0,1,\dots,n-1 \qquad (2.3.10)$$

with strict inequality holding for $i < n-1$.If $n$ is <u>even</u> , using (2.3.10) and the fact that $z^{(n)}(t) \leq 0$ ,we get to $z(t) < 0$, and this contradicts $z(t) > 0$, and this complete the proof. □

**Remark 2.3.1**: If $n$ is odd, then $(t) < z(t) \to 0$ as $t \to \infty$ , so that any solution of equation (2.3.1) is either oscillatory or tends to zero as $t \to \infty$.

**Example 2.3.1:** consider the NDDE

$$\frac{d^2}{dt^2}\left(x(t) + (2+\cos t)x(t-2\pi)\right) + (3+\cos t)x(t-4\pi) = 0 \qquad (2.3.11)$$

Here $n = 2$, $h(t) = 2 + cost$ , $p(t) = 3 + cost$ , $f(x) = x$ , $\tau(t) = t - 2\pi$ ,

and $(t) = t - 4\pi$ . The delay arguments $g(t)$ and $\tau(t)$ are commute i.e. $g(\tau(t)) = \tau(g(t)) = t - 6\pi$ , the function $f(x)$ satisfies conditions (1)-(5), and $p(t)$ satisfies the divergent integral in condition (6). Also conditions (7) and (8) are satisfied by (2.3.11). Thus all conditions of theorem 2.3.1 are satisfied. Therefore, we can conclude that every solution of equation (2.3.11) is oscillatory. In fact, $x_1(t) = \frac{\cos t}{3+\cos t}$ , $x_2(t) = \frac{\sin t}{3+\cos t}$ are oscillatory solutions of (2.3.11).

**Example 2.3.2:** consider the NDDE

$$\frac{d^2}{dt^2}\left(x(t) + hx(t-\pi)\right) + 2e^{(3\pi/2)}(he^{-\pi} - 1)x\left(t - \frac{3}{2}\pi\right) = 0, \qquad (2.3.12)$$

Where $h > e^{\pi}$ , equation (2.3.12) has the unbounded oscillatory solution $x(t) = e^{t} \sin t$.

**Remark 2.3.2:** If $\gamma < 1 , 0 < \lambda < 1$ then the equation

$$\frac{d^n}{dt^n}[x(t) + \lambda x(t-\tau)] + p(t)|x(t-\sigma)|^{\gamma}\, sgn\, x(t-\sigma) = 0 \qquad (2.3.13)$$

has no non oscillatory solution if and only if

$\int_{t_0}^{\infty} t^{\gamma(n-1)} p(t)dt = \infty.$

And if $\gamma > 1$, and n is even then it has no non oscillatory solution if and

only if $\int_{t_0}^{\infty} t^{n-1} p(t)dt = \infty.$

In particular equation (2.3.13) with $0 < \lambda < 1$ is oscillatory

If and only if

$$\int^{\infty} t^{\min\{\gamma,1\}(n-1)}\, p(t)dt = \infty \qquad (2.3.14)$$

**Remark 2.3.3:** Equation (2.3.13) is oscillatory if

$$\int^{\infty} t^{\min\{\gamma,1\}(n-1)} \min\{p(t), p(t-\tau)\}dt = \infty \qquad (2.3.15)$$

with $\lambda \geq 1$. (See [19]).

# Chapter Three

## Necessary and Sufficient Conditions for Oscillation of Solution of NDDE Compared with an ODE

In this chapter we will mention main results for the oscillation of

Solution of the equation:

$$\frac{d^n}{dt^n}[x(t)+\lambda x(t-\tau)]+f\Big(t,x\big(g(t)\big)\Big)=0 \qquad (3.1.1)$$

by using new approach in which we provide conditions related to a

certain ODE.

## 3.1 Main result

The following confirm the required results.

**Theorem 3.1.1**([20]) **:** Equation (3.1.1) is oscillatory if and only if

$$x^{(n)}(t)+\frac{1}{1+\lambda}f\Big(t,x\big(g(t)\big)\Big)=0 \qquad (3.1.2)$$

is oscillatory.

Using the known oscillation results for the equations:

$$x^{(n)}(t)+p(t)x(t-\sigma)=0 \qquad (3.1.3)$$

and

$$x^{(n)}(t)+p(t)|x(t-\sigma)|^{\gamma}\operatorname{sgn} x(t-\sigma)=0 \qquad (3.1.4)$$

We can obtain oscillation results for the equations

$$\frac{d^n}{dt^n}[x(t)+\lambda x(t-\tau)]+p(t)x(t-\sigma)=0 \qquad (3.1.5)$$

and

$$\frac{d^n}{dt^n}[x(t)+\lambda x(t-\tau)]+p(t)|x(t-\sigma)|^{\gamma}\, sgn\, x(t-\sigma)=0 \qquad (3.1.6)$$

**Corollary 3.1.1 :**

(i) Equation (3.1.5) is oscillatory if

$$\int^{\infty} t^{n-2} p(t)dt = \infty \tag{3.1.7}$$

(ii) Suppose that (3.1.7) fails, then equation (3.1.5) is oscillatory if

$$\lim_{t\to\infty} \sup\ t\int_t^{\infty} s^{n-2} p(s)ds > (1+\lambda)(n-1)! \tag{3.1.8}$$

or if

$$\lim_{t\to\infty} \inf\ t\int_t^{\infty} s^{n-2} p(s)ds > (1+\lambda)(n-1)!/4 \tag{3.1.9}$$

Equation (3.1.5) has non oscillatory solution if

$$\lim_{t\to\infty} \sup\ t\int_t^{\infty} s^{n-2} p(s)ds < (1+\lambda)(n-2)!/4 \tag{3.1.10}$$

To prove corollary 3.1.1 we need the following oscillation result for equation (3.1.2).

**Lemma 3.1.1**([16]) **:**

(i) Equation (3.1.3) is oscillatory if (3.1.7) holds

(ii) Suppose that (3.1.7) fails, then equation (3.1.3) is oscillatory if

$$\lim_{t\to\infty} \sup\ t\int_t^{\infty} s^{n-2} p(s)ds > (n-1)!$$

or if

$$\lim_{t\to\infty} \inf\ t\int_t^{\infty} s^{n-2} p(s)ds > (n-1)!/4$$

equation (3.1.3) has non oscillatory solution if

$$\lim_{t\to\infty} \sup\ t\int_t^{\infty} s^{n-2} p(s)ds < (n-2)!/4.$$

**Proof of corollary 3.1.1:**

Combining Theorem3.1.1 with Lemma 3.1.1, we obtain corollary 3.1.1.

This completes the proof. □

Now we give an example that illustrates this result:

**Example 3.1.1:** We consider the linear neutral differential equation

$$\frac{d^n}{dt^n}[x(t)+\lambda x(t-\tau)]+ct^{\propto}x(t-\sigma)=0 \qquad (3.1.11)$$

Where $c > 0, \alpha \in \mathbb{R}$, applying corollary 3.1.1, we conclude that:

Equation (3.1.11) is oscillatory if either:

(i) $\alpha = -n \; and \; c > (1+\lambda)(n-1)!/4,$

Since

$$\lim_{t\to\infty}\inf t \int_t^\infty s^{n-2}(cs^{-n})ds = \lim_{t\to\infty}\inf tc\int_t^\infty s^{-2}ds$$

$$=\lim_{t\to\infty}\inf tc \lim_{A\to\infty}\int_t^A s^{-2}ds$$

$$=\lim_{t\to\infty}\inf tc \lim_{A\to\infty}\left(\frac{s^{-1}}{-1}\Big|_t^A\right)$$

$$=\lim_{t\to\infty}\inf tc\left(\lim_{A\to\infty}-\frac{1}{A}-\left(-\frac{1}{t}\right)\right)$$

$$=\lim_{t\to\infty}\inf c > (1+\lambda)(n-1)!/4$$

if $c > (1+\lambda)(n-1)!/4$. Or

(ii) If $\alpha > -n$,

we let $\alpha = -n+\varepsilon$, where $\varepsilon > 1$,

$$\int_t^\infty s^{n-2}(cs^{\alpha})ds = c\int_t^\infty s^{n+(-n+\varepsilon)-2}ds$$

$$= c\int_t^\infty s^{\varepsilon-2}ds$$

$$= c \lim_{A\to\infty}\int_t^A s^{\varepsilon-2}ds = c\lim_{A\to\infty}\frac{s^{\varepsilon-1}}{\varepsilon-1}\Big|_t^A = \infty.$$

Equation (3.1.11) has non oscillatory if either:

(iii) $\alpha = -n$ and $c < (1+\lambda)(n-2)!/4$

$$\lim_{t\to\infty}\sup \mathrm{t}\int_t^\infty s^{n-2}\,p(s)\,ds = \lim_{t\to\infty}\sup \mathrm{t}\int_t^\infty s^{n-2}\,cs^{\propto}\,ds$$

$$= \lim_{t\to\infty}\sup \mathrm{t}\,c\int_t^\infty s^{n-2}\,s^{\propto}\,ds$$

$$= \lim_{t\to\infty}\sup \mathrm{t}\,c\left(\frac{s^{-1}}{-1}\Big|_t^\infty\right)$$

$= \lim_{t\to\infty} \sup t\, c(-1)\left(\lim_{A\to\infty} \frac{1}{A} - \frac{1}{t}\right)$

$= \lim_{t\to\infty} \sup t\, c\left(\frac{1}{t}\right)$

$= \lim_{t\to\infty} \sup\ c$

Hence equation (3.1.11) is non oscillatory if $c < (1+\lambda)(n-2)!/4$

(iv) $\alpha < -n$, so $\alpha + n < 0$ $\lim_{t\to\infty} \sup t \int_t^\infty s^{n-2}\, p(s)\, ds = \lim_{t\to\infty} \sup t \int_t^\infty s^{n-2}\, cs^{\propto}\, ds$

$= \lim_{t\to\infty} \sup t \int_t^\infty s^{n-2}\, cs^{\propto}\, ds$

$= \lim_{t\to\infty} \sup t\ c \int_t^\infty s^{n-2}\, s^{\propto}\, ds$

$= \lim_{t\to\infty} \sup t\ c \int_t^\infty s^{n+\propto-2}\, ds$

$$= \lim_{t\to\infty} \sup t\, c \lim_{A\to\infty} \left.\frac{s^{n+\propto-1}}{n+\propto-1}\right|_t^A$$

since $\alpha + n < 0$

$= \lim_{t\to\infty} \sup t^{n+\propto} \left(\frac{c}{n+\propto-1}\right) = 0$

**Corollary 3.1.2:** equation (3.1.5) is oscillatory if (3.1.12) holds.

$$\int_{t_0}^\infty t^{n-1-\epsilon} p(t)dt = \infty \qquad (3.1.12)$$

equation (3.1.5) has non oscillatory solution if (3.1.13) holds.

$$\int_{t_0}^\infty t^{n-1} p(t)dt < \infty \qquad (3.1.13)$$

To prove corollary 3.1.2 we need the follwing lemmas:

**Lemma 3.1.2**([28] ): Let n be even. If

$$\int^\infty g_*^{(n-1)}(t) g^{-\varepsilon}(t)\, p(t)dt = \infty \qquad \text{for some } \varepsilon > 0,$$

then

$x^{(n)}(t) + \mu p(t) x\big(g(t)\big) = 0$ is oscillatory, and consequently

(3.1.3) is oscillatory.

Where $g_*(t) = \min\{g(t), t\}$ .

**Lemma 3.1.3**([16])**:** Asume that

$$\lim_{t\to\infty} \inf g(t)/t > 0.$$

i. Equation (3.1.3 is strongly oscillatory if and only if either (3.1.7) or

$$\lim_{t\to\infty} \sup t \int_t^{\infty} s^{n-2}\, p(s)ds = \infty.$$

ii. Equation (3.1.3) is strongly non oscillatory if and only if

$$\int_t^{\infty} t^{n-2}\, p(t)dt < \infty$$

and

$$\lim_{t\to\infty} t \int_t^{\infty} s^{n-2}\, p(s)ds = 0 \text{ .}$$

**Lemma 3.1.4**([22])**:** Suppose that $g(t) > t$ for $t \geq t_0$ and

$$\lim_{t\to\infty} \inf g(t)/t < \infty,$$

then equation (3.1.3) is strongly oscillatory if and only if

$$\lim_{t\to\infty} \sup\ t \int_t^{\infty} s^{n-2}\, p(s)ds = \infty,$$

and equation (3.1.3) ) is strongly non oscillatory if and only if

$$\lim_{t\to\infty}\ t \int_t^{\infty} s^{n-2}\, p(s)ds = 0.$$

Note that corollary 3.1.1 implies corollary 3.1.2 .

**Proof of corollary 3.1.2:**Suppose that (3.1.12) holds, from lemma 3.1.2 it follows that the equation

$$x^{(n)}(t) + \mu p(t)x\big(g(t)\big) = 0 \qquad (3.1.14)$$

is oscillatory for all constants $\mu > 0$. Lemma 3.1.3. and lemma 3.1.4 have shown that equation (3.1.14) is oscillatory for all $\mu > 0$ if and only if either (3.1.7) holds or

$$\lim_{t\to\infty} \sup\ t \int_t^{\infty} s^{n-2}\, p(s)ds = \infty,$$

this means that if (3.1.12) holds, then either (3.1.7) or (3.1.8) is satisfied,

and so equation (3.1.5) is oscillatory .

Suppose next that (3.1.13) holds. Then

$$0 \leq \lim_{t\to\infty}\ t \int_t^{\infty} s^{n-2}\, p(s)ds \leq \lim_{t\to\infty}\ \int_t^{\infty} s^{n-1}\, p(s)ds = 0.$$

consequently, if (3.1.13) holds, then (3.1.10) is satisfied, and so (3.1.5) has non oscillatory solution. □

**Corollary 3.1.3:** Equation (3.1.6) is oscillatory if and only if (3.1.15)holds.

$$\int^{\infty} t^{\min\{\gamma,1\}(n-1)} p(t) = \infty \tag{3.1.15}$$

To prove corollary 3.1.3, we need the following result which has been obtained by Kitamura[28, corollary 3.1 ].

Consider the equation

$$x^{(n)}(t) + p(t) f\left(x\big(g(t)\big)\right) = 0 \tag{3.1.16}$$

**Lemma 3.1.5([28] ):** Let $n$ be even, assume that

$$\lim_{t\to\infty} \inf g(t)/t > 0,$$

the condition (2.3.17) is a necessary and sufficient condition for (3.1.16) to be oscillatory

$$\int_{t_0}^{\infty} t^{n-1} p(t) dt = \infty \tag{3.1.17}$$

**Proof of corallary 3.1.3:**

From lemma 3.1.5 equation (3.1.16) is oscillatory ,so with theorem3.1.1 we obtain corollary 3.1.3. □

Let us consider the equation

$$\frac{d^n}{dt^n}\left[x(t) + \bar{\lambda} x(t-\bar{\tau})\right] + \bar{f}\left(t, x\big(\bar{g}(t)\big)\right) = 0 \tag{3.1.18}$$

Where $\bar{\lambda} > 0, \bar{\tau} > 0, \bar{f} \in C([t_0,\infty) \times \mathbb{R}), \bar{g} \in C[t_0,\infty), \lim_{t\to\infty} \bar{g}(t) = \infty,$

$u\bar{f}(t,u) \geq 0 \; for \; (t,u) \in [t_0,\infty) \times \mathbb{R}.$

From theorem 3.1.1, we obtain the following comparison result.

**Corollary 3.1.4:** Suppose that $\bar{\lambda} \leq \lambda, \bar{g}(t) \geq g(t)$ for $t \geq t_0$,and $|\bar{f}(t,u)| \geq |f(t,u)|$ for $(t,u) \in [t_0,\infty) \times \mathbb{R}$. If equation (3.1.1) is oscillatory , then (3.1.18) is oscillatory.

To prove corollary 3.1.4 we need the following result due to H. Onoes [6].

**Lemma 3.1.6**([6].)**:** If the differential inequality

$$x^{(n)}(t) + f\left(t, x\big(g(t)\big)\right) \le 0$$

Has an eventually positive solution, then the differential equation

$$x^{(n)}(t) + f\left(t, x\big(g(t)\big)\right) = 0$$

Has an eventually positive solution.

**Proof of corollary 3.1.4:** Assume that (3.1.18) has non oscillatory solution, then theorem 3.1.1 implies that

$$x^{(n)}(t) + \frac{1}{1+\bar{\lambda}} \bar{f}\left(t, x\big(\bar{g}(t)\big)\right) = 0,$$

has a non oscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t) > 0$ for all $t$. For the case where $x(t) < 0$ for all large $t$, $y(t) \equiv -x(t)$ is an eventually positive solution of

$$y^{(n)}(t) + \frac{1}{1+\bar{\lambda}} \tilde{f}\left(t, y\big(\bar{g}(t)\big)\right) = 0,$$

Where $\tilde{f}(t,u) = -\bar{f}(t,-u)$, and hence the case $x(t) < 0$ can be treated similarly. From Lemma 1.6.1 it follows that $x(t)$ is eventually nondecreasing. In view of the hypothesis of corollary 3.1.4, we see that $x\big(\bar{g}(t)\big) \ge x\big(g(t)\big)$ for all large $t \ge t_0$, and

$$-x^{(n)}(t) = \frac{1}{1+\bar{\lambda}} \bar{f}\left(t, x\big(\bar{g}(t)\big)\right)$$

$$-x^{(n)}(t) \ge \frac{1}{1+\lambda} \bar{f}\left(t, x\big(\bar{g}(t)\big)\right) \ge \frac{1}{1+\lambda} f\left(t, x\big(\bar{g}(t)\big)\right) \ge \frac{1}{1+\lambda} f\left(t, x\big(g(t)\big)\right)$$

We have the differential inequality

$$\frac{1}{1+\lambda} f\left(t, x\big(g(t)\big)\right) + x^{(n)}(t) < 0,$$

that has non oscillatory solution, hence from lemma 3.1.6 the differential equation

$$\frac{1}{1+\lambda} f\left(t, x\big(g(t)\big)\right) + x^{(n)}(t) = 0\ ,$$

has non oscillatory solution, so from theorem 3.1.1 equation 3.1.1 has a non oscillatory solution. This completes the proof.□

## 3.2 Proof of the "if" part of theorem 3.3.1 (Sufficient condition)

We want to prove that if the equation

$$\frac{d^n}{dt^n}[x(t)+\lambda x(t-\tau)]+f\Big(t,x\big(g(t)\big)\Big)=0 \qquad (3.2.1)$$

has a non oscillatory solution, then the equation

$$x^{(n)}(t)+\frac{1}{1+\lambda}f\Big(t,x\big(g(t)\big)\Big)=0 \qquad (3.2.2)$$

has a non oscillatory solution.

The following lemmas are required to complete the proof.

**Lemma 3.2.1**: Let $\lambda \neq 1$ and $l \in \mathbb{N}\cup\{0\}$. Suppose that $u \in C[T-\tau,\infty)$,

$(\Delta u)(t) \in C^1[T,\infty), (\Delta u)(t) \geq 0, (\Delta u)'(t) \geq 0$ for $t \geq T$, and $\lim_{t\to\infty}(\Delta u)'(t)/t^l = 0$. For the case $\lambda > 1$, assume moreover that $\lim_{t\to\infty}\lambda^{-t/\tau}u(t) = 0$. Then

$$u(t)=\frac{1}{1+\lambda}(\Delta u)(t)+o(t^l) \qquad (t\to\infty).$$

Where $(\Delta u)(\mathrm{t}) = u(t)+\lambda u(t-\tau)$.

**Lemma 3.2.2:** Let $\lambda = 1$. Suppose that $u \in C[T-\tau,\infty)$, $u(t) > 0$ for $t \geq T-\tau$. If $(\Delta u)(t)$ is nondecreasing and concave on $[T,\infty)$, then there exists a constant α such that

$$0<\frac{1}{2}(\Delta u)(t)-\alpha \leq u(t) \leq \frac{1}{2}(\Delta u)(t)+\frac{1}{2}(\Delta u)(T+2\tau), \quad t \geq T+2\tau.$$

**Lemma 3.2.3:** Let $\lambda = 1$. suppose that $u \in C[T-\tau,\infty)$, $u(t) > 0$ for $t \geq T-\tau$. *If* $(\Delta u)(t)$ is nondecreasing and convex on $[T,\infty)$, then there exist a constant α such that

$$0<\frac{1}{2}(\Delta u)(t)-\alpha \leq u(t) \leq \frac{1}{2}(\Delta u)(t+\tau)+\frac{1}{2}(\Delta u)(T+2\tau), \qquad t \geq T+2\tau.$$

**Lemma 3.2.4:** let $\lambda = 1$ and $l \in \mathbb{N}$. Suppose that $u \in C[T-\tau, \infty)$, $u(t) > 0$ *for* $t \geq T-\tau$. Assume moreover that $\Delta u \in C^2[T,\infty)$, $(\Delta u)(t) \geq 0, (\Delta u)'(t) \geq 0$ *and either* $(\Delta u)''(t) \leq 0$ or $(\Delta u)''(t) \geq 0$ for $t \geq T$, *and* $\lim_{t\to\infty} (\Delta u)'(t)/t^l = 0.$ *then*

$$u(t) = \frac{1}{2}\,(\Delta u)(t) + o(t^l) \quad (t \to \infty).$$

**Lemma 3.2.5:** Let $u \in C^n[T,\infty)$ satisfy $u(t) \neq 0$ and $u(t)u^{(n)}(t) \leq 0$ for $t \geq T$. Then there exists an integer $k \in \{1,3,\ldots,n-1\}$ such that

$$\begin{cases} u(t)u^{(i)}(t) > 0, & 0 \leq i \leq k-1, \\ (-1)^{i-k}u(t)u^{(i)}(t) \geq 0, & k \leq i \leq n, \end{cases} \qquad (3.2.3)$$

for $t \geq T_1$. In particular, $u'(t) \geq 0$ for $t \geq T_1$.

**Remark 3.2.1:** A function $u(t)$ satisfying (3.2.3) for all large $t$ is called a function of Kiguradze degree $k$. Let $u(t)$ be a function of Kiguradze degree $k \in \{1,3,\ldots,n-1\}$ satisfying $u(t) > 0$ for all large $t$, it can be shown (cf. [2],[3],[19]) that

$$\lim_{t\to\infty} u^{(i)}(t) = 0, \;\; i = k+1, k+2, \ldots, n-1 \qquad (3.2.4)$$

And that one of the following three cases holds:

$$\lim_{t\to\infty} u^{(k)}(t) = const > 0 \;\; and \;\; \lim_{t\to\infty} u^{(k-1)}(t) = \infty; \qquad (3.2.4a)$$

$$\lim_{t\to\infty} u^{(k)}(t) = 0 \;\; and \;\; \lim_{t\to\infty} u^{(k-1)}(t) = \infty; \qquad (3.2.4b)$$

$$\lim_{t\to\infty} u^{(k)}(t) = 0 \;\; and \;\; \lim_{t\to\infty} u^{(k-1)}(t) = const > 0.) \qquad (3.2.4c)$$

If (3.2.4$a$)holds, then we put $l = k$, and if (3.2.4$b$)or (3.2.4$c$)holds, then we put $l = k-1$. Then it is easy to verify that $l \in \{0,1,2,\ldots,n-1\}$,

$$\lim_{t\to\infty} \frac{u\prime(t)}{t^l} = 0 \;\text{ and }\; \lim_{t\to\infty} \frac{u(t)}{t^l} = L \;\; (L > 0 \; or \; L = \infty). \qquad (3.2.5)$$

**Lemma 3.2.6:** Let $\lambda \neq 1$. Suppose that $u \in C[T-\tau,\infty)$, $\Delta u \in C^n[T,\infty)$ and $(\Delta u)(t) > 0$ for $t \geq T$. For the case $\lambda > 1$, assume moreover that $\lim_{t\to\infty} \lambda^{-\frac{t}{\tau}} u(t) = 0$. *If* $(\Delta u)(t)$ is a function of Kiguradze $k$ for some $k \in \{1,3,\dots,n-1\}$ then there exist a constant α and an integer $l \in \{0,1,2,\dots,n-1\}$ such that

$$u(t) \geq \frac{1}{1+\lambda}(\Delta u)(t) - \alpha t^l > 0$$

For all large $t \geq T$.

**Proof of the "if" part of Theorem 3.3.1:** It is sufficient to prove that if equation (3.2.1) has a non oscillatory solution, then equation (3.2.2) has a non oscillatory solution. Let $x(t)$ be a non oscillatory solution of (3.2.1). Without loss of generality, we may assume that $x(t) > 0$ for all large $t$. Then $(\Delta x)(t) > 0$ and $(\Delta x)^{(n)}(t) \leq 0$ for all large t. In view of Lemma 3.2.5, we find that $(\Delta x)(t)$ is a function of Kiguradze degree $k$ for some $k \in \{1,3,\dots,n-1\}$, and hence $\lim_{t\to\infty}(\Delta x)^{(k)}(t) = \text{const}$. Since $0 < x(t) \leq (\Delta x)(t)$ for all large $t$, we have $\lim_{t\to\infty}\lambda^{-t/\tau}x(t) = 0$ if $\lambda > 1$. By Lemmas 3.2.2, 3.2.3 and 3.2.6, there are a constant $\alpha$ and an integer $l \in \{0,1,2,\dots,n-1\}$such that

$$x(t) \geq \frac{1}{1+\lambda}(\Delta x)(t) - \alpha t^l > 0 \text{ for all large } t$$

Put $w(t) = (1+\lambda)^{-1}(\Delta x)(t) - \alpha t^l$. Then $x(t) \geq w(t) > 0$ for all large $t$. From the monotonicity of $f$ it follows that

$$-w^{(n)}(t) = \frac{1}{1+\lambda}(\Delta x)^{(n)}(t) = \frac{1}{1+\lambda} f(t, x(g(t))) \geq \frac{1}{1+\lambda} f(t, w(g(t)))$$

for all large $t$. Lemma 3.1.6 implies that (3.2.2) has a non oscillatory solution.

The proof is complete. □

**Note:** The proof of the lemmas in this section can be found in [ 20 ].

## 3.3 Proof of the "only if" part of the theorem 3.1.1. (Necessary condition)

In this section we give the proof of the "only if" part of Theorem 3.1.1. To this end, we require the following result concerning an "inverse" of the operator$\Delta$.

**Lemma 3.3.1:** Let $T_*$ and $T$ be numbers such that $\max\{t_0, 1\} \leq T_* \leq T - \tau$, and let $k \in \mathbb{N}$ and $M > 0$. Define the set $Y$ as follows:

$$Y = \{y \in C[T_*, \infty): y(t) = 0, t \in [T_*, T], and\ |y(t)| \leq Mt^k, t \geq T\}.$$

Then there exists a mapping $\Phi$ on Y which has the following properties:

(i) $\Phi$ maps $Y$ into $C[T_*, \infty)$;

(ii) $\Phi$ is continuous on Y in the $C[T_*, \infty)$-topology;

(iii) $\Phi$ satisfies $(\Phi y)(t) + \lambda(\Phi y)(t - \tau) = y(t)$ for $t \geq T$ and $y \in Y$;

(iv) If $\lambda = 1$ and $y \in Y$ is nondecreasing on $[T_*, \infty)$, then $(\Phi y)(t) \geq 0$ for $t \geq T_*$;

(v) If $\lambda > 1$, then $\lim_{t\to\infty} \lambda^{-\frac{t}{\tau}}(\Phi y)(t) = 0$ for $y \in Y$.

Here and hereafter, $C[T_*, \infty)$ is regarded as the **Frechet space** of all continuous functions on $[T_*, \infty)$ with the topology of uniform convergence on every compact subinterval of $[T_*, \infty)$.

We divide the proof of Lemma 3.3.1 into the two cases $0 < \lambda \leq 1$ and $\lambda > 1$.

**Proof of Lemma 3.3.1: The case $0 < \lambda \leq 1$.**

For each $y \in Y$, we define the function $\Phi y$ on $[T_*, \infty)$ by

$$(\Phi y)(t) = \begin{cases} \sum_{i=0}^{m} (-\lambda)^i y(t - i\tau), & t \in [T + m\tau, T + (m+1)\tau), \\ & m = 0,1, \dots, \\ 0, & t \in [T_*, T) \end{cases}$$

(i) Let $y \in Y$. Note that$y(T) = 0$. It is obvious that $(\Phi y)(t)$ is continuous on $[T_*, \infty) - \{T + m\tau: m = 0,1,2, \dots\}$. We observe that

$$\lim_{t\to T-0}(\Phi y)(t) = 0 = y(T) = \lim_{t\to T+0}(\Phi y)(t),$$

And that if $m \geq 1$, then

$$\lim_{t\to T+m\tau-0}(\Phi y)(t) = \sum_{i=0}^{m-1}(-\lambda)^i y(T+m\tau-i\tau)$$

$$= \sum_{i=0}^{m-1}(-\lambda)^i y(T+m\tau-i\tau) + (-\lambda)^m y(T)$$

$$= \sum_{i=0}^{m}(-\lambda)^i y(T+m\tau-i\tau)$$

$$= \lim_{t\to T+m\tau+0}(\Phi y)(t).$$

Consequently, $(\Phi y)(t)$ is continuous on $[T_*,\infty)$.

(ii) It suffices to show that if $\{y_j\}_{j=1}^{\infty}$ is a sequence in $C[T_*,\infty)$ converging to $y\in C[T_*,\infty)$ uniformly on every compact subinterval of $[T_*,\infty)$, then $\{\Phi y_j\}$ converges to $\Phi y$ uniformly on every compact subinterval of $[T_*,\infty)$. We claim that $\Phi y_j \to \Phi y$ uniformly on $I_m \equiv [T+m\tau, T+(m+1)\tau], m = 0,1,2,\ldots$ Then we easily conclude that $\{\Phi y_j\}$ converges to $\Phi y$ uniformly on every compact subinterval of $[T_*,\infty)$. Observe that

$$\sup_{t\in I_m}\left|(\Phi y_j)(t)-(\Phi y)(t)\right| \leq \sum_{i=0}^{m}\lambda^i \sup_{t\in I_m}\left|y_j(t-i\tau)-y(t-i\tau)\right|$$

$$\leq \sum_{i=0}^{m}\lambda^i \sup_{t\in I_m}\left|y_j(t)-y(t)\right|$$

For $m = 0,1,2,\ldots.$ then we see that

$$\sup_{t\in I_m}\left|(\Phi y_j)(t)-(\Phi y)(t)\right| \to 0 \ \ (j\to\infty), \ \ m = 0,1,2,\ldots,$$

So that $\{\Phi y_j\}$ converges to $\Phi y$ uniformly on $I_m$ for $m = 0,1,2,\ldots$

(iii) Let $y\in Y$. If $t\in[T, T+\tau)$, then $(\Phi y)(t-\tau) = 0$ and

$$(\Phi y)(t) = y(t) = y(t) - \lambda(\Phi y)(t-\tau).$$

If $t \in [T+m\tau, T+(m+1)\tau], m = 1,2,\ldots$ then

$$(\Phi y)(t) = y(t) + \sum_{i=1}^{m}(-\lambda)^i y(t-i\tau)$$

$$= y(t) - \lambda\sum_{i=1}^{m}(-\lambda)^{i-1} y(t-\tau-(i-1)\tau)$$

$$= y(t) - \lambda \sum_{i=0}^{m-1} (-\lambda)^i y(t - \tau - i\tau)$$

$$= y(t) - \lambda(\Phi y)(t - \tau),$$

Since $t - \tau \epsilon [T + (m-1)\tau, T + m\tau)$.

(iv) Assume that $\lambda = 1$. Let $y \in Y$ be nondecreasing on $[T_*, \infty)$. Notice that $y(t) \geq y(T_*) = 0$ for $t \geq T_*$. It is easy to see that $(\Phi y)(t) = y(t) \geq 0 for\ t \in [T, T + \tau)\ and (\Phi y)(t) = 0$ for $t \in [T_*, T)$. let $t \in [T + m\tau, T + (m+1)\tau], m = 1,2, \ldots$ if $m \geq 1$ is odd, then

$$(\Phi y)(t) = \sum_{j=0}^{(m-1)/2} [y(t - 2j\tau) - y(t - (2j+1)\tau)] \geq 0.$$

If $m \geq 2$ is even, then

$(\Phi y)(t) = \sum_{j=0}^{\left(\frac{m}{2}\right)-1} [y(t - 2j\tau) - y(t - (2j+1)\tau)] + y(t - m\tau) \geq 0.$ Therefore we obtain $(\Phi y)(t) \geq 0$ for $t \geq T_*$. The proof for the case $0 < \lambda \leq 1$ is complete.□

**Proof of Lemma 3.3.1: The case $\lambda > 1$.**

For each $y \in Y$, we assign the function $\Phi y$ on $[T_*, \infty)$ as follows:

$$(\Phi y)(t) = \begin{cases} -\sum_{i=1}^{\infty} (-\lambda)^{-i} y(t + i\tau), & t \in [T - \tau, \infty), \\ (\Phi y)(T - \tau), & t \in [T_*, T - \tau). \end{cases}$$

Let $y \in Y$. Then

$$\left|(-\lambda)^{-i} y(t + i\tau)\right| \leq \lambda^{-i} M (t + i\tau)^k \leq 2^{k-1} M \lambda^{-i} (t^k + i^k \tau^k) \qquad (3.3.1)$$

For $t \geq T - \tau, i = 1,2, \ldots$ Thus we see that the series $\sum_{i=1}^{\infty} (-\lambda)^{-i} y(t + i\tau)$ converges uniformly on every compact subinterval of $[T - \tau, \infty)$, so that $\Phi$ is well-defined, and $(\Phi y)(t)$ is continuous on $[T_*, \infty)$ and satisfies

$$|(\Phi y)(t)| \leq \frac{2^{k-1} M}{\lambda - 1} t^k + L, \qquad t \geq T - \tau$$

For each $y \in Y$, where $L = 2^{k-1} M \tau^k \sum_{i=1}^{\infty} \lambda^{-i} i^k$. This means that (i) and (v) follow.

(ii) Take an arbitrary compact subinterval I of $[T - \tau, \infty)$. Let $\varepsilon > 0$.

There is an integer $q \geq 1$ such that

$$\sum_{i=q+1}^{\infty} \lambda^{-i} M(t+i\tau)^k < \frac{\varepsilon}{3}, \ \ t \in I. \tag{3.3.2}$$

Let $\{y_i\}_{j=1}^{\infty}$ be a sequence in Y converging to y∈Y uniformly on every compact subinterval of $[T_*, \infty)$. There exists an integer $j_0 \geq 1$ such that

$$\sum_{i=1}^{q} \lambda^{-i} \left| y_j(t+i\tau) - y(t+i\tau) \right| < \frac{\varepsilon}{3}, \ \ t \in I, \ \ j \geq j_0.$$

It follows from (3.3.1) and (3.3.2) that

$$\left| (\Phi y_j)(t) - (\Phi y)(t) \right| \leq \sum_{i=1}^{q} \lambda^{-i} \left| y_j(t+i\tau) - y(t+i\tau) \right|$$

$$+ \left| \sum_{i=q+1}^{\infty} (-\lambda)^i y_i(t+i\tau) \right|$$

$$+ \left| \sum_{i=q+1}^{\infty} (-\lambda)^i y(t+i\tau) \right|$$

$$\leq \frac{\varepsilon}{3} + 2.\frac{\varepsilon}{3} = \varepsilon, \ \ t \in I, \ \ j \geq j_0,$$

Which implies that $\Phi y_j$ converges $\Phi y$ uniformly on I. We see that

$\Phi y_j \Rightarrow \Phi y$ uniformly on $[T_*, T-\tau]$, because of $(\Phi y)(t) = (\Phi y)(T-\tau)$ on $[T_*, T-\tau]$ for y∈Y. Consequently, we conclude that Φ is continuous on Y.

(iii) Let y∈Y. Observe that

$$\lambda(\Phi y)(t-\tau) = \sum_{i=1}^{\infty} (-\lambda)^{-(i-1)} y(t+(i-1)\tau)$$

$$= y(t) + \sum_{i=1}^{\infty} (-\lambda)^{-i} y(t+i\tau)$$

$$= y(t) - (\Phi y)(t), \ \ \ t \geq T.$$

The proof for the case λ>1 is complete.□

**Lemma 3.2.2:** Let $w \in C^n[T, \infty)$ be a function of Kiguradze degree $k$ for some $k \in \{1, 3, \ldots, n-1\}$, then $\lim_{t \to \infty} \frac{w(t+\rho)}{w(t)} = 1$ for each $\rho > 0$.

**Proof:** We may assume that $w(t) > 0$ for all large $t$. Recall that $w(t)$ satisfies one of (3.2.4a)- (3.2.4c) If (3.2.4a) holds, then

$$\lim_{t\to\infty}\frac{w(t+\rho)}{w(t)} = \lim_{t\to\infty}\frac{w^{(k)}(t+\rho)}{w^{(k)}(t)} = 1.$$

In exactly the same way, we have $lim_{t\to\infty}\frac{w(t+\rho)}{w(t)} = 1$ for the case (3.2.4c). Assume that (3.2.4b) holds, by the mean value theorem, for each large fixed $t \geq T$, there is a number $\eta(t)$ such that

$$w(t+\rho) - w(t) = \rho w'(\eta(t)) \; and \;\; t < \eta(t) < t+\rho.$$

Thus we obtain

$$\frac{w(t+\rho)}{w(t)} - 1 = \rho\frac{w'(\eta(t))t^{k-1}}{[\eta(t)]^{k-1}w(t)}\left[\frac{\eta(t)}{t}\right]^{k-1}$$

By (3.2.4b) we conclude that $\lim_{t\to\infty}\frac{w'(t)}{t^{k-1}} = 0$ and $\lim_{t\to\infty}\frac{w(t)}{t^{k-1}} = \infty$, so that $\lim_{t\to\infty}\frac{w(t+p)}{w(t)} = 1$.

Now we prove the "only if" part of Theorem 3.1.1.

**Proof of the "only if" part of Theorem 3.1.1:**

We show that if equation (3.2.2) has a non oscillatory solution, then equation (3.2.1) has a non oscillatory solution. Let z(t) be a non oscillatory solution of (3.2.2). Without loss of generality, we may assume that $z(t)$ is eventually positive. Set $w(t) = (1+\lambda)z(t)$. Then $w(t)$ is an eventually positive solution of

$$w^{(n)}(t) + f\left(t, (1+\lambda)^{-1}w(g(t))\right) = 0 \qquad (3.3.3)$$

Lemma 3.2.5 implies that $w(t)$ is a function of Kiguradze degree $k$ for some $k \in \{1,3,\dots,n-1\}$, and one of the cases (3.2.4a)- (3.2.4c) holds. Hence, $\lim_{t\to\infty}\frac{w(t)}{t^k} = \text{const} \geq 0$. From lemma 3.3.6 it follows that

$$w(t+2\tau) \leq \tfrac{3}{2}w(t), \quad t \geq T_1 \qquad (3.3.4)$$

for some $T_1 \geq t_0$.

We can take a sufficiently large number $T \geq T_1$ such that $w^{(i)}(t) > 0$

$(i = 0,1,2,\dots,k-1)$, $w(g(t)) > 0$ for $t \geq T$, and

$$T_* \equiv \min\{T-\tau, \inf\{g(t): t \geq T\}\} \geq \max\{T_1, 1\}.$$

Recall (3.2.4). Integrating (3.3.3),we have

$$w(t) - P(t) = \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} f\left(r, \frac{w(g(r))}{1+\lambda}\right) drds$$

For $t \geq T$, where

$$P(t) = \frac{(t-T)^k}{k!} w^{(k)}(\infty) + \sum_{i=0}^{k-1} \frac{(t-T)^i}{i!} w^{(i)}(T), \quad t \geq T_*,$$

and $w^{(k)}(\infty) = \lim_{t\to\infty} w^{(k)}(t) \geq 0$.

Consider the set $Y$ of functions $y \in C[T_*, \infty)$ which satisfies

$$y(t) = 0 \text{ for } t \in [T_*, T] \text{ and } 0 \leq y(t) \leq w(t) - P(t) \text{ for } t \geq T.$$

Then $Y$ is closed and convex, and there is a constant $M > 0$ such that $|y(t)| \leq Mt^k$ on $[T, \infty)$ for $y \in Y$, by $\lim_{t\to\infty} w(t)/t^k = const \geq 0$. Lemma 3.3.1 implies that there exists a mapping $\Phi$ on $Y$ satisfying (i)-(v) of lemma 3.3.1.

Put $(\Psi y)(t) = (\Phi y)(t) + \frac{P(t)}{4(1+\lambda)}, \ t \geq T_*, \quad y \in Y.$

For each $y \in Y$, we define the mapping $\mathcal{F}: Y \to C[T_*, \infty)$ as follows:

$$(\mathcal{F}y)(t) = \begin{cases} \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \bar{f}(r, (y)(g(r)))drds, & t \geq T, \\ 0, & t \in [T_*, T], \end{cases}$$

Where

$$\bar{f}(t, u) = \begin{cases} f\left(t, (1+\lambda)^{-1} w(g(t))\right), & u \geq (1+\lambda)^{-1} w(g(t)), \\ f(t, u), & 0 \leq u \leq (1+\lambda)^{-1} w(g(t)), \\ 0, & u \leq 0, \end{cases}$$

For $t \geq T$ and $u \in R$. In view of the fact that

$$0 \leq \bar{f}(t, u) \leq f\left(t, (1+\lambda)^{-1} w(g(t))\right), \qquad (t, u) \in [T, \infty) \times R,$$

We see that $\mathcal{F}$ is well defined on Y and maps Y into itself. Since Φ is continuous on Y, by the Lebesgue dominated convergence theorem, we can show that $\mathcal{F}$ is continuous on Y as a routine computation.

Now we claim that $\mathcal{F}$(Y) is relatively compact. We note that $\mathcal{F}$(Y) is uniformly bounded on every compact subinterval of $[T_*, \infty)$, because of $\mathcal{F}(\mathrm{Y}) \subset Y$. By the Ascoli-Arzela theorem, it suffices to verify that $\mathcal{F}$ (Y) is equicontinuous on every compact subinterval of $[T_*, \infty)$. Let $I$ be an arbitrary compact subinterval of $[T_*, \infty)$. If $k = 1$, then

$$0 \leq (\mathcal{F}y)'(t) \leq \int_t^\infty \frac{(s-t)^{n-2}}{(n-2)!} f\left(s, \frac{w(g(s))}{1+\lambda}\right) ds \qquad \text{for } t \geq T \text{ and } y \in Y$$

If $k \geq 3$, then

$$0 \leq (\mathcal{F}y)'(t) \leq \int_T^t \frac{(t-s)^{k-2}}{(k-2)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} f\left(r, \frac{w(g(r))}{1+\lambda}\right) dr ds$$

for $t \geq T$ and $y \in Y$. Thus we see that $\{(\mathcal{F}y)'(t): y \in Y\}$ is uniformly bounded on $I$. The mean value theorem implies that $\mathcal{F}$(Y) is equicontinuous on I. Since $|(\mathcal{F}y)(t_1) - (\mathcal{F}y)(t_2)| = 0$ for $t_1, t_2 \in [T_*, T]$, we conclude that $\mathcal{F}$(Y) is equicontinuous on every compact subinterval of $[T_*, \infty)$.

By applying the Schauder-Tychonoff fixed point theorem to the operator $\mathcal{F}$, there exists a $\tilde{y} \in Y$ such that $\tilde{y} = \mathcal{F}\tilde{y}$.

Put $x(t) = (\Psi\tilde{y})(t)$.Then we obtain

$$(\Delta x)(t) = \tilde{y}(t) + \frac{P(t)+\lambda P(t-\tau)}{4(1+\lambda)}, \quad t \geq T$$

Since

$$(\Delta x)(t) = x(t) + \lambda x(t-\tau) = (\Psi\tilde{y})(t) + \lambda(\Psi\tilde{y})(t-\tau)$$

$$= (\Phi\tilde{y})(t) + \frac{P(t)}{4(1+\lambda)} + \lambda\left[(\Phi\tilde{y})(t-\tau) + \frac{P(t-\tau)}{4(1+\lambda)}\right]$$

$$= [(\Phi\tilde{y})(t) + \lambda(\Phi\tilde{y})(t-\tau)] + \frac{P(t)+\lambda P(t-\tau)}{4(1+\lambda)}$$

By lemma 3.3.1(iii) we obtain

$$(\Delta x)(t) = \tilde{y}(t) + \frac{P(t)+\lambda P(t-\tau)}{4(1+\lambda)} \qquad (3.3.5)$$

And hence $(\Delta x)(t)$ is a function of Kiguradze degree $k$.

Since $P(t) \geq P(t-\tau) \geq P(T) = w(T) > 0 \Longrightarrow P$ is increasing for $t \geq T+\tau$, so that

$$0 < (\Delta x)(t) = \tilde{y}(t) + \frac{P(t)+\lambda P(t-\tau)}{4(1+\lambda)} \leq \tilde{y}(t) + \frac{P(t)+\lambda P(t)}{4(1+\lambda)} = \tilde{y}(t) + \frac{P(t)}{4}$$

But $\tilde{y}(t) \in Y \Longrightarrow \tilde{y}(t) \leq w(t) - P(t)$

$$0 < (\Delta x)(t) \leq w(t) - P(t) + \frac{P(t)}{4} = w(t) - \frac{3}{4}P(t) \qquad (3.3.6)$$

for $t > T+\tau$. We will show that

$$0 < x(t) \leq (1+\lambda)^{-1} w(t) \text{ for all large t.} \qquad (3.3.7)$$

Then the proof of the "only if" part of Theorem 3.1.1 will be complete, since (3.3.5) and (3.3.7) imply that

$$\frac{d^n}{dt^n}[x(t) + \lambda x(t-\tau)] = \tilde{y}^{(n)}(t) = (\mathcal{F}\tilde{y})^{(n)}(t) = -\bar{f}(t, x(g(t)))$$

$$= -f(t, x(g(t)))$$

for all large t, which means $x(t)$ is a non oscillatory solution of (3.2.2).

If $w^{(k)}(\infty) > 0$, then we put $l = k$, and if $w^{(k)}(\infty) = 0$, then we put $l = k-1$. It can be shown that $lim_{t\to\infty} \frac{(\Delta x)'(t)}{t^l} = 0$. Indeed, since

$$\lim_{t\to\infty} (\Delta x)^{(k)}(t) = \lim_{t\to\infty} \tilde{y}^{(k)}(t) + \lim_{t\to\infty} \frac{P^{(k)}(t) + \lambda P^{(k)}(t-\tau)}{4(1+\lambda)}$$

$$= \lim_{t\to\infty} (\mathcal{F}\tilde{y})^{(k)}(t) + \frac{w^{(k)}(\infty)}{4} = \frac{w^{(k)}(\infty)}{4},$$

We see that if $l = k$, then $lim_{t\to\infty} \frac{(\Delta x)'(t)}{t^l} = lim_{t\to\infty} \frac{(\Delta x)^{(k)}(t)}{k!t} = 0$, and that if $l = k-1$, then $lim_{t\to\infty} \frac{(\Delta x)'(t)}{t^l} = lim_{t\to\infty} \frac{(\Delta x)^{(k)}(t)}{(k-1)!} = 0$.

First assume that $\lambda \neq 1$. From Lemma 3.3.6 it follows that $x(t) > 0$ for all large $t \geq T_*$. In view of Lemma 3.2.1 and the fact that

$lim_{t\to\infty}\frac{P(t)}{t^l} = const > 0$, we have

$$x(t) \le \frac{1}{1+\lambda}(\Delta x)(t) + \frac{3}{4(1+\lambda)}P(t)$$

For all large t. Hence, by (3.3.6), we obtain $x(t) \le (1+\lambda)^{-1}w(t)$ for all large t.

Next we assume that $\lambda = 1$ and $l \ne 0$. Since $\tilde{y}(t)(= (\mathcal{F}\tilde{y})(t))$ is nondecreasing in $t \in [T_*, \infty)$, from Lemma 3.3.1 (iv), we see that $(\Phi\tilde{y})(t) \ge 0$ for $t \ge T_*$, so that $x(t) \ge P(t)/[4(1+\lambda)]$ for $t \ge T_*$. Hence, $x(t) > 0$ for $t \ge T$. By using Lemma 3.2.4 and the same argument as in the case λ≠1, we can show that $x(t) \le (1+\lambda)^{-1}w(t)$ for all large t.

Finally we suppose that $\lambda = 1$ and $l = 0$. Then $k = 1$ and $w^{(k)}(\infty) = 0$. Therefore, $P(t) = w(T)$ on $[T_*, \infty)$. As in the case $\lambda = 1$ and $l \ne 0$, we have $x(t) \ge P(t)/[4(1+\lambda)]$ for $t \ge T_*$, which implies that $x(t) > 0$ for $t \ge T_*$. Note that $(\Delta x)'(t) \ge 0$ *and* $(\Delta x)''(t) \le 0$ for $t > T$, since $k = 1$. By Lemma 3.2.2, (3.3.6) and (3.3.4), we conclude that

$$x(t) \le \tfrac{1}{2}(\Delta x)(t) + \tfrac{1}{2}(\Delta x)(T+2\tau)$$

$$\le \tfrac{1}{2}\left[w(t) - \tfrac{3}{4}w(T) + w(T+2\tau) - \tfrac{3}{4}w(T)\right]$$

$$\le \tfrac{1}{2}w(t), \quad t \ge T + 2\tau.$$

The proof is complete. □

## Chapter Four

## Necessary and Sufficient Conditions for the Oscillation of Solution with Positive Variable Coefficients

In this chapter we will mention main results for the oscillation of

Solution of the equation:

$$\frac{d^n}{dt^n}[x(t)+h(t)x(t-\tau)]+f\left(t,x\big(g(t)\big)\right)=0. \qquad (4.1.1)$$

which is a certain kind of generalization of theorem 3.1.1 with the following assumptions (H1) and (H2):

$$(H1) \qquad 0 \le \mu \le h(t) \le \lambda < 1 \quad \text{for } t \in \mathbb{R};$$

$$(H2) \qquad 1 < \lambda \le h(t) \le \mu \qquad \text{for } t \in \mathbb{R}.$$

## 4.1 Main result

Throughout this chapter we use the notation:

$$H_0(t)=1; \qquad H_i(t)=h(t)h(t-\tau)\dots h(t-(i-1)\tau), \quad i=1,2,\dots..$$

We define the function $S(t)$ on $\mathbb{R}$ by

$$S(t)=\begin{cases}\sum_{i=0}^{\infty}(-1)^i H_i(t) & \text{if } (H1)\text{holds},\\ \sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{H_i(t+i\tau)} & \text{if } (H2)\text{holds},\end{cases} \qquad \text{for } t \in \mathbb{R}.$$

Where $S(t)$ converges uniformly on $\mathbb{R}$, and hence S(t) is continuous on $\mathbb{R}$. We will show that

$$0<\frac{1-\lambda}{1-\mu^2}\le S(t)\le\frac{1-\mu}{1-\lambda^2}, \qquad t\in\mathbb{R} \qquad (4.1.2)$$

We note that if $\mu=\lambda=c\neq 1$, then

$$\frac{1-\lambda}{1-\mu^2}=\frac{1-\mu}{1-\lambda^2}=\frac{1}{1+c}, \text{ and } \quad S(t)=\frac{1}{1+c}.$$

**Theorem 4.1.1**([**21**])**:** Suppose that $(H1)$ or $(H2)$holds. Then equation (4.1.1) is oscillatory if and only if

$$x^{(n)}(t) + f\big(t, S\big(g(t)\big)x(g(t))\big) = 0 \qquad (4.1.3)$$

is oscillatory.

Theorem 4.1.1 means that (4.1.1) has a non oscillatory solution if and only if (4.1.3) has a non oscillatory solution.

Theorem 4.1.1 is a generalization of theorem 3.1.1 with $c \neq 1$. Indeed, for the case where $h(t) \equiv c > 0$ and $c \neq 1$, we see that $S(t) = (1+c)^{-1}$ and that (3.1.1) is oscillatory if and only if

$$y^{(n)}(t) + f\left(t, (1+c)^{-1}y\big(g(t)\big)\right) = 0$$

is oscillatory . (Put $x(t) = (1+c)y(t)$).

Now we assume that

$$h(t+\tau) = h(t),\ h(t) \neq 1 \quad and\ h(t) \geq 0 \qquad \text{for } t \in \mathbb{R}. \qquad (4.1.4)$$

since (H1) or (H2) holds, and $S(t) = [1 + h(t)]^{-1}$.

Consequently, from Theorem4.1.1, we have the following result.

**Corollary 4.1.1:** Suppose that (4.1.4)holds. Then (4.1.1) is oscillatory if and only if

$$x^{(n)}(t) + f\left(t, \frac{x(g(t))}{1+h(g(t))}\right) = 0 \qquad (4.1.5)$$

is oscillatory.

The oscillatory behavior of solutions of non-neutral differential equations of the form

$$x^{(n)}(t) + f\left(t, x\big(g(t)\big)\right) = 0 \qquad (4.1.6)$$

Theorem 4.1.1, with the known oscillation results for non-neutral differential equations of the form (4.1.6), can be used to obtain oscillation criteria for the linear neutral differential equation

$$\frac{d^n}{dt^n}[x(t)+h(t)x(t-\tau)]+p(t)x(t-\sigma)=0 \qquad (4.1.7)$$

and for the nonlinear neutral differential equation

$$\frac{d^n}{dt^n}[x(t)+h(t)x(t-\tau)]+p(t)|x(t-\sigma)|^{\gamma-1}x(t-\sigma)=0 \qquad (4.1.8)$$

where $\gamma>0, \gamma\neq 1$ and the following conditions are assumed to hold:

$$\sigma\in R; \quad p\in C[t_0,\infty), \quad p(t)>0 \quad \text{for } t\geq t_0. \qquad (4.1.9)$$

First let us show that S(t) satisfies (4.1.2).

**Lemma 4.1.1**([ **21** ])**:** If (H1) or (H2) holds, then $S(t)$ satisfies (4.1.2).

From theorem 4.1.1, lemma 4.1.1, lemma 3.1.6, we have the following result.

**Corollary 4.1.2:** Suppose that (H1) or (H2) holds. If

$$x^{(n)}(t)+\frac{1-\lambda}{1-\mu^2}f\left(t,x\big(g(t)\big)\right)=0 \qquad (4.1.10)$$

is oscillatory, then (4.1.1) is oscillatory. If

$$x^{(n)}(t)+\frac{1-\mu}{1-\lambda^2}f\left(t,x\big(g(t)\big)\right)=0 \qquad (4.1.11)$$

has a non oscillatory solution, then (4.1.1) has a non oscillatory solution.

**Proof:** Assume that there exists a non oscillatory solution of (4.1.1).

Then Theorem 4.1.1 implies that (4.1.3) has a non oscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t)>0$ for all large $t$, since the case $x(t)<0$ can be treated similarly. Put

$y(t)=(1-\lambda)(1-\mu^2)^{-1}x(t)$. From Lemma 4.1.1we have

$$-y^{(n)}(t) = -\frac{1-\lambda}{1-\mu^2}x^{(n)}(t) = \frac{1-\lambda}{1-\mu^2}f\left(t, S(g(t))x(g(t))\right) \geq \frac{1-\lambda}{1-\mu^2}f\left(t, \frac{1-\lambda}{1-\mu^2}x(g(t))\right)$$
$$= \frac{1-\lambda}{1-\mu^2}f\left(t, y(g(t))\right)$$

For all large $t$. From Lemma 3.1.6 it follows that (4.1.10) has a non oscillatory solution.

Consequently, if (4.1.10) is oscillatory, then (4.1.12) is oscillatory.

Let $y(t)$ be an eventually positive solution of (4.1.11). Thus

$$y^{(n)}(t) + \frac{1-\mu}{1-\lambda^2}f\left(t, y(g(t))\right) = 0$$

$$-x^{(n)}(t)\frac{(1-\mu)}{(1-\lambda^2)} = \frac{1-\mu}{1-\lambda^2}f\left(t, y(g(t))\right) \Rightarrow -x^{(n)}(t) = f\left(t, y(g(t))\right)$$

$$-x^{(n)}(t) = f\left(t, \frac{(1-\mu)}{(1-\lambda^2)}x(g(t))\right)$$

Then Lemma 4.1.1 implies that $x(t) = (1-\lambda^2)(1-\mu)^{-1}y(t)$ is an eventually positive solution of

$$x^{(n)}(t) + f(t, S(g(t))x(g(t))) \leq 0,$$

and hence (4.1.1) has a non oscillatory solution, by Lemma 3.1.6 and Theorem 4.1.1 . This completes the proof.□

The following oscillation result was established by Kitamura [22, Corollaries 5.1 and 3.1].

**Lemma 4.1.2:** assume that (4.1.9) holds. If (3.1.12) holds, then the equation (3.1.3) is oscillatory. If (3.1.13) holds then equation (3.1.3) has a non oscillatory solution.

**Lemma 4.1.3:** let $\gamma > 0$ and $\gamma \neq 1$. Assume that (4.1.9) holds. Then the equation (3.1.4) is oscillatory if and only if equation (3.1.15) holds.

Combining Corollary 4.1.1 with Lemmas 4.1.2 and 4.1.3, we have the following oscillation criteria for (4.1.7) and (4.1.8).

**Corollary 4.1.3:** If (3.1.12) holds, then (4.1.7) is oscillatory. If (3.1.13) holds, then (4.1.7) has a non oscillatory solution.

**Corollary 4.1.4:** Equation (4.1.8) is oscillatory if and only if (3.1.15) holds.

**Remark 4.1.1**: Corollary 4.1.3 with (H1) have been already established by Jaros and Kusano [7, Theorems 3.1 and 4.1]. Corollary 4.1.3 with (H2) extends the results in [9, Theorem 1] and [10, Theorem 7].

**Remark 4.1.2**: Corollary 4.1.4 with (H1) has been obtained by Y. Naito [29] in the case where $h(t)$ is locally Lipschitz continuous.

## 4.2 Proof of the "if" part of theorem 4.1.1.(Sufficient condition)

Want to show that if equation (4.1.1) has a non oscillatory solution, then equation (4.1.3) has a non oscillatory solution.

The following lemmas are required to complete the proof.

**Lemma 4.2.1:** Let H1 and the following condition (4.2.1) hold:

$$\begin{cases} u \in C[T-\tau,\infty), & (\Delta u)(t) \in C^1[T,\infty) \\ (\Delta u)(t) \geq 0, & (\Delta u)'(t) \geq 0 \; for \;\; t \geq T, \\ \lim_{t\to\infty} (\Delta u)'(t)/t^l = 0. & \text{for some } l \in \{0,1,\dots\} \end{cases} \quad \text{and} \qquad (4.2.1)$$

Then

$$u(t) = \mathrm{S(t)}(\Delta u)(t) + o(t^l) \qquad (t \to \infty). \qquad (4.2.2)$$

**Lemma 4.2.2:** Suppose that H2 and (4.2.1) hold. Assume, moreover, that

$\lim_{t\to\infty} \lambda^{-t/\tau} u(t) = 0$. Then (4.2.2) holds.

**Lemma 4.2.3:** Suppose that (H1) or (H2) holds. Let $u \in [T-\tau,\infty)$ satisfy $\Delta u \in C^n[T,\infty)$ and $(\Delta u)(t) > 0$ for $t \geq T$. Assume moreover, that $\lim_{t\to\infty} \lambda^{-\frac{t}{\tau}} u(t) = 0$ if (H2)holds.

if $(\Delta u)(t)$ is a function of Kiguradze degree k for some k $\in \{1,3,\dots,n-1\}$,

then there exist a constant $\alpha > 0$ and an integer $l \in \{0,1,2,\dots,n-1\}$ such that

$$u(t) \geq S(t)[\,(\Delta u)(t) - \alpha t^l] > 0 \text{ for all large } t \geq T. \qquad (4.2.3)$$

**Proof:** Recalling (3.2.5), we have

$$\lim_{t\to\infty} \frac{(\Delta u)'(t)}{t^l} = 0 \text{ and } \lim_{t\to\infty} \frac{(\Delta u)(t)}{t^l} = L \;\; (L > 0 \; or \; L = \infty)$$

For some $l \in \{0,1,2,\dots,n-1\}$. Choose a constant $\alpha > 0$ so small that $\alpha < L$. By lemmas 4.1.1, 4.2.1and 4.2.2, we conclude that (4.2.3) holds.□

**Proof of the "if" part of Theorem 4.1.1:** Let $x(t)$ be a non oscillatory solution of (4.1.1). Without loss of generality, we may assume that $x(t) > 0$ for all large $t$. Then $(\Delta x)(t) > 0$ and $(\Delta x)^{(n)}(t) \leq 0$ for all large $t$. By virtue of Lemma 3.2.5, we see that $(\Delta x)(t)$ is a function of Kiguradze degree $k$ for some k$\in \{1,3,\dots,n-1\}$. From Lemma 4.2.3, there are a constant $\alpha > 0$ and an integer $l \in \{0,1,2,\dots,n-1\}$ such that

$x(t) \geq S(t)[(\Delta x)(t) - \alpha t^l] > 0$ for all large $t$.

Set $w(t) = (\Delta x)(t) - \alpha t^l$. Then $x(t) \geq S(t)w(t) > 0$ for all large $t$. we see that $-w^{(n)}(t) = -(\Delta x)^{(n)}(t) = f(t, x(g(t))) \geq f(t, S(g(t))w(g(t)))$

For all large t. Lemma 3.1.6 implies that (4.1.3) has a non oscillatory solution. The proof is complete. □

## 4.3 Proof of the "only if" part of theorem 4.1.1. (Necessary condition)

In this section we give the proof of the "only if" part of Theorem 4.1.1.

We define the following mapping $\Phi$ which is an "inverse" of the operator $\Delta$.

$$(\Phi v)(t) = \begin{cases} \sum_{i=0}^{\infty}(-1)^i H_i(t) v(t - i\tau) & if\ (H1) holds, \\ \sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{H_i(t+i\tau)} v(t + i\tau) & if\ (H2) holds, \end{cases}$$

For $t \in \mathbb{R}$, where

$$v \in \{v \in C(\mathbb{R}): |v(t)| \le M \max\{t^k, 1\}, t \in \mathbb{R}\} \equiv \boldsymbol{V}, \tag{4.3.1}$$

$M > 0$ and $k \in \{1,2,\dots\}$. The properties of the mapping $\Phi$ are as follows.

**Lemma 4.3.1:** The mapping $\Phi$ is well-defined on $V$ and has the following properties (i)-(iv):

(i) $\Phi$ maps V into C($\mathbb{R}$);

(ii) $\Phi$ is continuous on V in the C($\mathbb{R}$)-topology;

(iii) $\Phi$ satisfies $(\Delta(\Phi \upsilon))(t) = \upsilon(t)$ for $t \in \mathbb{R}$ and $\upsilon \in V$.

(iv) If (H2) holds, then $\lim_{t\to\infty} \lambda^{-\frac{t}{\tau}}(\Phi v)(t) = 0$ for $v \in V$.

**Proof of the "only if" part of Theorem 4.1.1**: We show that if (4.1.3) has a non oscillatory solution, then (4.1.1) has a non oscillatory solution. Let $w(t)$ be a non oscillatory solution of (4.1.3). We may assume that $w(t)$ is eventually positive. Lemma 3.2.5implies that $w(t)$ is a function of Kiguradze degree $k$ for some $k \in \{1,3,\dots,n-1\}$. Hence, (3.2.3) holds and one of the cases (3.2.4a)- (3.2.4c) is satisfied.

We can take a sufficiently large number $T > 1$ such that

$w^{(i)}(t) > 0 (i = 0,1,2,\dots,k-1),\ w(g(t)) > 0\ for\ t \ge T$. Integrating (4.1.3), we have

$$w(t) - P(t) = \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^{\infty} \frac{(r-s)^{n-k-1}}{(n-k-1)!} F(r) dr ds \ge 0 \tag{4.3.2}$$

For $t \ge T$, where $F(t) = f(t, S(g(t)) w(g(t)))$,

$$P(t) = \frac{(t-T)^k}{k!} w^{(k)}(\infty) + \sum_{i=0}^{k-1} \frac{(t-T)^i}{i!} w^{(i)}(T),$$

And $w^{(k)}(\infty) = \lim_{t\to\infty} w^{(k)}(t) \in [0,\infty)$. note that $P(T) = w(T)$ and $P(t) \geq w(T) > 0$ for $t \geq T$. Consider the set Y of functions $y \in C(R)$ which satisfies

$$y(t) = 0 \; for \; t \leq T \text{ and } 0 \leq y(t) \leq w(t) - P(t) \text{ for } t \geq T.$$

Then Y is closed and convex. Set

$$\eta(t) = \begin{cases} \left(\frac{1}{2}\right) P(t), & t \geq T, \\ \left(\frac{1}{2}\right) P(T), & t < T. \end{cases}$$

By $w^{(k)}(\infty) \in [0,\infty)$, there is a constant $M \geq P(T)$ such that $w(t) \leq Mt^k$ and $P(t) \leq Mt^k$ for $t \geq T$. Define the set $V$ by (4.3.1). We easily see $Y \subset V$ and $\eta \in V$.

For each y∈Y, we define the mapping $\mathcal{F}: Y \to C(\mathbb{R})$ as follows:

$$(\mathcal{F}y)(t) = \begin{cases} \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \\ \times \bar{f}\left(r, (\Phi y)(g(r)) + (\Phi\eta)(g(r))\right) dr ds, & t \geq T, \\ 0, \; t \leq T, \end{cases}$$

Where

$$\bar{f}(t,u) = \begin{cases} F(t), & u \geq S(g(t))w(g(t)), \\ f(t,u), & 0 \leq u \leq S(g(t))w(g(t)), \\ 0, & u \leq 0, \end{cases}$$

For $t \geq T$ and $u \in \mathbb{R}$. In view of (4.3.2) and the fact that

$$0 \leq \bar{f}(t,u) \leq F(t), \qquad (t,u) \in [T,\infty) \times \mathbb{R},$$

We see that $\mathcal{F}$is well-defined on Y and maps Y into itself. Since Φ is continuous on Y, by the Lebesgue dominated convergence theorem, we can show that $\mathcal{F}$ is continuous on Y as a routine computation.

Now we claim that $\mathcal{F}$(Y) is relatively compact. We note that $\mathcal{F}$ (Y) is uniformly bounded on every compact subinterval of ℝ, because of $\mathcal{F}(Y) \subset Y$. by the Ascoli-Arzela theorem, it suffices to verify that $\mathcal{F}$ (Y) is equicontinuous on every compact subinterval of ℝ. Let I be an arbitrary compact subinterval of $[T, \infty)$. If $k = 1$, then

$$0 \le (\mathcal{F}y)'(t) \le \int_t^\infty \frac{(s-t)^{n-2}}{(n-2)!} F(s)ds, \quad t \ge T, \quad y \in Y.$$

If $k \ge 3$, then

$$0 \le (\mathcal{F}y)^{'}(t) \le \int_T^t \frac{(t-s)^{k-2}}{(k-2)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} F(r)drds$$

for $t \ge T$ and y∈Y. thus we see that $\{(\mathcal{F}y)'(t): y \in Y\}$ is uniformly bounded on I. the mean value theorem implies that $\mathcal{F}$ (Y) is equicontinuous on I. Since $|(\mathcal{F}y)(t_1) - (\mathcal{F}y)(t_2)| = 0 \; for \; t_1, t_2 \in (-\infty, T]$, we conclude that $\mathcal{F}$ (Y) is equicontinuous on every compact subinterval of ℝ.

By applying the Schauder-Tychonoff fixed point theorem to the operator $\mathcal{F}$, there exists a $\tilde{y} \in Y$ such that $\tilde{y} = \mathcal{F}\tilde{y}$.

Put $x(t) = (\Phi\tilde{y})(t) + (\Phi\eta)(t)$. Then we obtain

$$(\Delta x)(t) = \tilde{y}(t) + \frac{P(t)}{2} = (\mathcal{F}\tilde{y})(t) + \frac{P(t)}{2} > 0, \;\; t \ge T, \tag{4.3.3}$$

By (iii) of Lemma 4.3.1, and hence $(\Delta x)(t)$ is a function of Kiguradze degree $k$.

We will show that

$$0 < x(t) \le S(t)w(t) \text{ for all larg } t. \tag{4.3.4}$$

Then the proof of Theorem 4.1.1 will be complete, since (4.3.3) and (4.4.4)imply that

$$\frac{d^n}{dt^n}[x(t) + h(t)x(t-\tau)] = (\mathcal{F}\tilde{y})^{(n)}(t) = -\bar{f}(t, x(g(t)))$$

$$= -f(t, x(g(t)))$$

For all large $t$, which means that $x(t)$ is a non oscillatory solution of (4.1.1).

If $w^{(k)}(\infty) > 0$, then we put $l = k$, and $if \; w^{(k)}(\infty) = 0$, then we put $l = k - 1$.

By (4.3.3), we find that $\lim_{t\to\infty}(\Delta x)^{(k)}(t) = \left(\frac{1}{2}\right) w^{(k)}(\infty)$, so that $\lim_{t\to\infty} \frac{(\Delta x)^{'}(t)}{t^l} = 0$.

From Lemma 4.2.3 it follows that $x(t) > 0$ for all large $t \geq T$. In view of Lemmas 4.1.1, 4.2.1and 4.2.2, and the fact that $\lim_{t\to\infty} \frac{P(t)}{t^l} = const > 0$, we have

$$\frac{x(t)}{S(t)} \leq \tilde{y}(t) + \frac{P(t)}{2} + \frac{P(t)}{2} \leq w(t) - P(t) + \frac{P(t)}{2} + \frac{P(t)}{2} = w(t)$$

For all large t. This completes the proof. □

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