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Assessing Non-Atomicity in Groups of Divisibility

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Assessing Non-Atomicity in Groups of Divisibility

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Abstract

An integral domain D is *atomic* if every non-zero non-unit is a product of irreducibles. More generally, D is *quasi-atomic* if every non-zero non-unit divides some product of atoms. Arbitrary integral domains, however, cannot be assumed to be quasi-atomic in general; factorization in a non-atomic D can be subtle. We outline a novel method of qualifying the quasi-atomicity of D by studying ascending filtrations of localizations of D and the associated groups of divisibility. This approach yields structure theorems, cochain complexes, and cohomological results. We take care to present examples of integral domains exhibiting the spectrum of factorization behavior and we relate the results of our new method to factorization in D.

Chapter 1

Introduction

Factorization in an integral domain D loosely concerns decomposing elements or ideals into products. We say a non-zero non-unit element $d \in D$ is *atomic* if d is a product of irreducibles, and we say that D is an *atomic domain* if every non-zero non-unit is atomic. On the other hand, if D contains no irreducibles, then we say D is an *antimatter domain* (first described in [7]). An arbitrary integral domain may be atomic, antimatter, or neither; interest in non-atomic domains has grown recently, especially since [7] and [19]. The spectrum of factorization behaviors between the extremes of atomic and antimatter domains is explored with some depth in [24]. We present background information on integral domains in Appendix A. Our study presents some novel constructions that are sensitive to factorization in integral domains. Our methods exploit the *quasi-atomic* elements of an integral domain; first described in [4], quasi-atomic elements are non-zero non-unit divisors of atomic elements.

The study of non-atomic domains is not new. In [34], Zaks provides the first discussion on half factorial domains without assuming atomicity. In [22], Krull studies integral domains and avoids the assumption of atomicity. In [25], Mott considers localizations of arbitrary integral domains, and in [1], Anderson and Zafrullah characterize weakly factorial domains (which are not necessarily atomic).

Many results in commutative algebra follow from studying objects by proxy. For example, rather than study a ring R directly, we study R by proxy using some algebraic object $A(R)$ related to R such that properties of R are detectable in $A(R)$. More generally, with *a priori* knowledge of some category $\mathbf{D}$, we can study a category $\mathbf{C}$ by studying functors of the form $F : \mathbf{C} \to \mathbf{D}$ or $G : \mathbf{D} \to \mathbf{C}$

and the natural transformations of those functors. Examples of this technique include construction of pre-sheaves [31], group representations [15], and projective resolutions [17]. An exemplary use of this method in commutative algebra is to study an integral domain D by studying its group of divisibility $G(D)$. For any integral domain D there exists a group of divisibility $G(D)$ with operation induced by multiplication in D and a naturally defined map $\eta : D \setminus \{0\} \to G(D)$. The map η can be extended to a semi-valuation $\nu : \mathbb{F} \setminus \{0\} \to G(D)$. Valuations, first examined in detail in [22], have several properties we use throughout this study. The domain D and its quotient field $\mathbb{F}$ is commutative so $G(D)$ is abelian. Moreover, the group operation on $G(D)$ is compatible with a partial order $\leqslant$ induced by the divisibility relation in D. The group of divisibility $G(D)$ is therefore a partially ordered abelian group that encodes multiplicative relationships from D as the group operation on $G(D)$ and encodes divisibility relationships from D as a partial order on $G(D)$ that is compatible with the group operation. We assume all groups in the sequel are abelian, and we refer to partially ordered gropus as po-groups. Irreducible elements from D (through η) are minimal and positive in the partial order on $G(D)$, which we denote $\leqslant$. Hence, we refer to the minimal positive elements of an arbitrary po-group G as atoms. Quasi-atomic elements in D are the non-unit proper divisors of atomic elements; in $G(D)$, these are elements that are strictly bounded above by atomic elements. Thus for an arbitrary po-group G, we say an element $g \in G^+$ is quasi-atomic if there exists an atomic $a \in G^+$ such that $g \leqslant a$. We denote the subgroup generated by the atoms of G as $A(G)$ and the subgroup generated by the quasi-atomic elements as $Q(G)$. Just as we refer to an integral domain with no irreducibles as antimatter, we define an *antimatter po-group* as a po-group with no atoms. We summarize background information on po-groups in general and groups of divisibility in particular in Appendix B.

Neatly summarized in [26] and expanded thoroughly in [27], groups of divisibility are helpful in solving ring-theoretic problems using the following general method: formulate a ring-theoretic problem in the context of the group of divisibility, solve the problem using group-theoretic techniques, then interpret the result back in the ring-theoretic context. The crown jewel of this approach characterizes Bézout domains as precisely those domains that have lattice-ordered groups of divisibility.

Theorem 1.0.1 (Krull, Kaplansky, Jaffard, Ohm)**.** *Let G be an a po-group. Then G is lattice-ordered if and only if there exists a Bézout domain D such that $G = G(D)$.*

We omit the proof, which constructs an integral domain using a group algebra over a field.

Another example of studying objects by proxy is the study of *filtrations*, as in [3]: if an object M admits chains $M \supset \cdots \supset M_n \supset M_{n-1} \supset \cdots$ or $M \subseteq \cdots \subset M_n \subset M_{n+1} \subseteq \cdots$, we can glean information about M by studying the sequence $\{M_n\}_{n\in\mathbb{Z}}$. Previous studies had a factorization flavor, such as [22], [32], and [25], all of which studied filtrations or sequences of morphisms between algebraic objects. In [22], Krull demonstrated a one-to-one correspondence between the prime ideals of arbitrary valuation rings and the convex subgroups of the associated groups of divisibility. In [32], Sheldon extended Krull's work by demonstrating a one-to-one correspondence between the prime ideals of a Bézout domain, D, and the prime filters of the positive cone of the group of divisibility $G(D)$. In [25], Mott was the first to establish that prime filters of a positive cone correspond to subgroups that are both convex and directed, which generalizes the correspondences developed by Krull and Sheldon: there exists a one-to-one correspondence between convex and directed subgroups within $G(D)$ and the saturated multiplicatively closed sets within D. Mott's generalization seems quite natural considering that a subgroup of a po-group $H \subseteq G$ is convex if and only if G/H is partially ordered as showed in [11].

The connection between saturated multiplicatively closed sets in D and convex directed subgroups within $G(D)$ suggests that localizations are vital in understanding factorization. To find filtrations of an integral domain D that are sensitive to factorization behavior, we therefore turn our attention to filtrations induced by localization at saturated multiplicatively closed sets. Localization can have a dramatic effect on the factorization structure of the resulting integral domain. Indeed, localizing at elements that are not atomic can often destroy factorization information, whereas localizing at sets generated by irreducibles can create new irreducibles as demonstrated in Example 3.1.1.

We study sequences of saturated multiplicatively closed sets $S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq D$ from which we construct an ascending filtration of localizations $D \subseteq D_{S_0} \subseteq D_{S_1} \subseteq D_{S_2} \subseteq \cdots \subseteq \mathbb{F}$. We select the sets S_i carefully such that this filtration is sensitive to factorization information. We study this filtration by qualifying the resulting groups of divisibility using Mott's correspondence. We develop some homological tools to investigate chains of convex subgroups in arbitrary groups of divisibility, allowing us to qualify the depth of non-atomicity in a po-group or domain. We use Mott's correspondence to construct a functor between domain localizations and po-group projections of groups of divisibility. In this way, factorization questions about filtrations of localizations of D reduce to questions about the sequences of the quotient po-groups of $G(D)$. These sequences provide structure theorems, induce a menagerie of cochain complexes of order-preserving homomorphisms, and yield homological information related to factorization. We summarize background information on (co)chain complexes and (co)homology groups in Appendix C.

Examples of rings that are neither atomic nor antimatter are abundant. We begin our discussion in Chapter 2 where we present rings exemplifying a variety of different behaviors related to factorization and relevant to our findings in the sequel. In Chapter 3, we construct the primary objects of study, the quasi-atomic sequences, and we elaborate upon the properties of those sequences. In Chapter 4, we present a general method for extracting cochain complexes from arbitrary sequences of D-module epimorphisms, and we use this method to compute cohomological information from the quasi-atomic sequences. In Chapter 5, we present several structure theorems for po-groups in general and groups of divisibility in particular. Our approach depends upon commutative rings, partially ordered groups, and homological algebra; we present the requisite background material and previously known results in the appendices. Background on integral domains is presented in Appendix A, background material on po-groups and groups of divisibility is presented in Appendix B, and background material on cochain complexes and cohomological algebra is presented in Appendix C.

In the sequel, we denote the positive integers as $\mathbb{N} = \{1, 2, \ldots\}$, the non-negative integers as $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and the non-negative rationals as $\mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x \geqslant 0\}$.

Chapter 2

Motivating Examples

In this chapter, we establish examples illustrating the range of factorization behaviors in arbitrary integral domains. We adopt the relaxed notions of atomicity first described in [4].

Definition 2.0.1. Let D be an integral domain and $d \in D$ an arbitrary non-zero non-unit.

(i) If d is a product of irreducibles we say d is *atomic*.

(ii) If there exists an atomic d' such that dd' is atomic in D, we say d is *almost atomic*.

(iii) If there exists any non-zero non-unit d' such that dd' is atomic in D, we say d is *quasi-atomic*.

If every non-zero non-unit of D is *atomic* (*almost atomic*, *quasi-atomic*), we say D is an atomic (almost atomic, quasi-atomic, respectively) domain. If D has no irreducible elements, then we say D is an *antimatter* domain.

We now present examples of almost atomic domains that are not atomic, quasi-atomic domains that are not almost atomic, and antimatter domains. We also present an example of a domain with mixed behaviors (some non-zero non-unit elements are quasi-atomic and some elements are not). Perhaps surprisingly, the more mysterious of these examples, the antimatter domains, are easiest to produce. Every field is antimatter. Example 2.0.2 is an antimatter domain that is not a field.

Example 2.0.2. Let $\mathbb{F}$ be any field and X an indeterminate over $\mathbb{F}$. From the algebraic closure of the quotient field $\overline{\mathbb{F}(X)}$ let $\mathcal{X}$ denote the subset $\mathcal{X} = \{X^\alpha \mid \alpha \in \mathbb{Q}^+\}$. Let $R' = \mathbb{F}[\mathcal{X}]$ and let $\mathfrak{m}$ be the maximal ideal in R' generated by $\mathcal{X}$. Let $R = R'_{\mathfrak{m}}$. Then R is antimatter. Indeed, every non-zero non-unit is associate to some monomial, say uX^{α}, which is not reducible since $uX^{\alpha} = uX^{\alpha/2}X^{\alpha/2}$.

Examples of quasi-atomic domains that are not almost atomic can be challenging to produce. In Example 2.0.3, we present an example of a quasi-atomic domain that is not almost atomic. This example was first discussed in the context of atomicity in [24].

Example 2.0.3. The ring $R = \mathbb{Z}[X] + X^2\mathbb{R}[X]$, first presented in [24], is quasi-atomic and not almost atomic. Certainly R is a subring of $\mathbb{R}[X]$, which is a UFD. Any non-zero $f \in \mathbb{Z}[X]$ with $\deg(f) \leqslant 1$ is therefore atomic or a unit; atomic elements are quasi-atomic. Inductively, assume that any $g \in R$ with $\deg(g) < n$ is quasi-atomic and let $f \in R$ have $\deg(f) = n$. Either $f = f_1 f_2$ for non-units $f_1, f_2 \in R$ or not. If not, f is irreducible therefore f is quasi-atomic. If $f = f_1 f_2$, then $\deg(f) = \deg(f_1) + \deg(f_2)$. Thus, $\deg(f_i) \leqslant \deg(f)$. If this inequality is strict for both f_1 and f_2, then both are quasi-atomic and therefore f is quasi-atomic.

The only remaining case is to check $f \in R$ such that every factorization $f = f_1 f_2$ has $\deg(f_1) = 0$ or $\deg(f_2) = 0$. If $\operatorname{ord}(f) \leqslant 1$, we can always write $f = m\hat{f}$ for some maximal $m \in \mathbb{Z}$ and some $\hat{f} \in R$. Following the maximality of m, if there exists some $n \in \mathbb{Z}$ such that $n \mid \hat{f}$ then $n \in U(\mathbb{Z})$. Here m is an atomic element of R. It is thus sufficient to show $\hat{f}$ is quasi-atomic. Of course, either $\hat{f}$ factors as $\hat{f} = \hat{f}_1 \hat{f}_2$ for some non-zero non-units $\hat{f}_1, \hat{f}_2 \in R$ or not. If not, then $\hat{f}$ is irreducible and therefore is atomic. If $\hat{f} = \hat{f}_1 \hat{f}_2$, then $\deg(\hat{f}_i) > 0$ for both $i = 1, 2$ (for otherwise $\hat{f}_i \in \mathbb{Z}$, and is therefore a unit, contradicting our choice of both $\hat{f}_i$ as non-zero non-units). Then $f = m\hat{f} = m\hat{f}_1\hat{f}_2$. This is a contradiction: we are in our final remaining case where $f \in R$ and every factorization $f = f_1 f_2$ has $\deg(f_1) = 0$ or $\deg(f_2) = 0$. Hence, $\hat{f}$ is irreducible and thus f is atomic. Therefore, f is quasi-atomic.

This establishes that R is quasi-atomic. We still must establish that R is not almost atomic by demonstrating the existence of an element that is not almost atomic. To this end, let $r \in \mathbb{R} \setminus \mathbb{Q}$ and consider $f = rx^2 \in R$. Certainly rx^2 is not irreducible, for $rx^2 = 2 \cdot \left(\frac{r}{2}x^2\right)$ where 2 and $\frac{r}{2}x^2$ are non-units. If f is almost atomic, there are some atoms $g_1, g_2, \ldots, g_N \in R$ and $h_1, h_2, \ldots, h_M \in R$ such that $rx^2 g_1 g_2 \cdots g_N = h_1 \cdots h_M$.

Substituting $x \mapsto 0$ yields that both sides must be zero here, and so some irreducible h_i satisfies $h_i(0) = 0$. Hence, $\operatorname{ord}(h_i) = 1$, for otherwise h_i is not irreducible. For this h_i, we can write $h_i = aX + r_2X^2 + r_3X^3 + \cdots + r_QX^Q$; if a is a non-unit, then $h_i = a(X + \frac{r_2}{a}X^2 + \cdots + \frac{r_Q}{a}X^Q)$, contradicting the irreducibility of h_i. Hence, we can write $h_i = \pm X + r_2X^2 + \cdots + r_QX^Q$.

In the quotient field of R we have $\frac{h_i}{X} = \pm 1 + r_2X + r_3X^2 + \cdots + r_QX^{Q-1}$, and we have the relationship $rXg_1g_2\cdots g_N = h_1\cdots h_{i-1}\frac{h_i}{X}h_{i+1}\cdots h_M$. But now the substitution $x \mapsto 0$ reveals both sides evaluate to 0, so some h_j evaluates to zero. Applying the same argument as before, we obtain a new index j with the same property; without loss of generality we can assume $i < j$ and obtain

$$rg_1g_2\cdots g_N = h_1\cdots h_{i-1}\cdot\frac{h_i}{X}h_{i+1}\cdots h_{j-1}\frac{h_j}{X}h_{j+1}\cdots h_M.$$

Note that each $h_k \in R$ for $k \neq i, j$, so each of their constant terms is an integer. Also, substituting $x \mapsto 0$ provides that $\frac{h_i}{X}$ and $\frac{h_j}{X}$ evaluates to ± 1 and each $g_k \in R$ has an integer constant term also. Hence, since $r \in \mathbb{R} \setminus \mathbb{Q}$, we have that the left-hand side $rg_1(0)\cdots g_N(0) \in \mathbb{R} \setminus \mathbb{Q}$. But $h_1(0)h_2(0)\cdots h_M(0) \in \mathbb{Z}$. This is a contradiction; no such g_k, h_k can exist, and hence rX^2 is not almost atomic. △

Example 2.0.4 presents a monoid that, on the surface, seems to have no business being almost atomic, but every generator is almost atomic. We can investigate a generalized polynomial ring based on this monoid.

Example 2.0.4. In this example, we construct an almost atomic (additive) monoid generated over $\mathbb{N}$ that is not atomic. Let $p \in \mathbb{Z}$ be an odd prime and let $M \subseteq \mathbb{Q}^+$ be the additive sub-monoid $M = \langle 1, 2/p, 2/p^2, 2/p^3, \ldots \rangle$. The monoid element $1 \in M$ is irreducible. To see this, decompose 1 into a sum with coefficients $\alpha_i \in \mathbb{N}_0$.

$$1 = \alpha_0 \cdot 1 + \alpha_1\frac{2}{p} + \alpha_2\frac{2}{p^2} + \cdots + \alpha_n\frac{2}{p^n}$$

where $\alpha_n \neq 0$ (for otherwise we would not write this element) and, without loss of generality, each α_i satisfies $0 \leqslant \alpha_i < p$ for otherwise we can write $\alpha_i = p + \alpha_i'$ and we can write $\alpha_i\frac{2}{p^i} = (p + \alpha_i')\frac{2}{p^i}$ or $\frac{2}{p^{i-1}} + \alpha_i'\frac{2}{p^i}$, and collect elements with common powers p together. Moreover, $M \subseteq \mathbb{Q}$ so M inherits the usual ordering on $\mathbb{Q}$. Hence, $\alpha_0 = 0$, for otherwise the right-hand side is too large. Now we have

$$1 = \alpha_1\frac{2}{p} + \alpha_2\frac{2}{p^2} + \cdots + \alpha_n\frac{2}{p^n}$$

where each $0 \leqslant \alpha_i < p$ for each $1 \leqslant i \leqslant n$ and $\alpha_n \neq 0$. Hence, we have

$$p^n = 2\alpha_1 p^{n-1} + 2\alpha_2 p^{n-2} + \cdots + 2\alpha_n.$$

Reducing both sides *modulo* p reveals $0 \equiv 2\alpha_n (\text{mod } p)$, which contradicts our choice of α_n as $0 < \alpha_n < p$. Hence, we cannot write 1 this way and therefore 1 is irreducible. Any element of the form $2/p^n$ is not irreducible since $2/p^n = 2p^m/p^{n+m} = \frac{2}{p^{n+m}} + \frac{2}{p^{n+m}} + \cdots + \frac{2}{p^{n+m}}$ for any $m \in \mathbb{N}$.

Hence, in the ring $R = \mathbb{F}_2[X^m : m \in M]_{\mathfrak{m}}$ where $\mathfrak{m}$ is the maximal ideal generated by all monomials X^m, $X \in R$ is a uniquely irreducible monomial and each X^{2/p^n} is not irreducible. It seems as if R ought to be quasi-atomic, but R is almost atomic. Indeed, we always have the factorization

$$\begin{aligned}
X^2 + X^{2+2/p^n} =& X^2(1 + X^{2/p^n}) \\
=& X^{p^n \frac{2}{p^n}} \left(1 + X^{2/p^n}\right) \\
=& X^{2/p^n} X^{(p^n-1)\frac{2}{p^n}} \left(1 + X^{2/p^n}\right) \\
=& X^{2/p^n} \left(X^{(p^n-1)\frac{2}{p^n}} + X^{(p^n-1)\frac{2}{p^n} + \frac{2}{p^n}}\right) \\
=& X^{2/p^n} \left(X^2 + X^{(p^n-1)\frac{2}{p^n}}\right) \\
=& X^{2/p^n} \left(X + X^{\frac{p^n-1}{2}\frac{2}{p^n}}\right)^2
\end{aligned}$$

Note $p^n - 1$ is even so $\frac{p^n-1}{2} \in \mathbb{Z}$ and $X^2 + X^{2+2/p^n} = X^2\left(1 + X^{2/p^n}\right)$ with $\left(1 + X^{2/p^n}\right) \in U(R)$.

$$\begin{aligned}
uX^2 =& X^{2/p^n} \left(X + X^{\frac{p^n-1}{2}\frac{2}{p^n}}\right)^2 \\
X^{2/p^n} =& u\left(\frac{X}{X + X^{2a/p^n}}\right)^2
\end{aligned}$$

in the quotient field of R where $a = \frac{p^n-1}{2}$. We claim that $X + X^{2a/p^n}$ is also irreducible in R, and hence X^{2/p^n} is a ratio of atomic elements, i.e. almost atomic. To see that $X + X^{2a/p^n}$ is irreducible, write $X + X^{2a/p^n} = fg$ for some $f, g \in R$. Since $f, g \in R$, we can write $f = \sum_i X^{m_i}$ and $g = \sum_j X^{n_j}$ for some $m_i, n_j \in M$ (over $\mathbb{F}_2$, coefficients can all be taken to be 1). Since $fg = X + X^{2a/p^n}$, we have some monoid elements from $\{m_i\}$ and $\{n_j\}$ that sum to $1 \in M$ (to account for the term X). Hence, either $m_i = 1$ for some i or $n_j = 1$ for some j. If $m_i = 1$ for some i, then there exists some

$n_j = 0$, in which case g is a unit. If $n_j = 1$ for some j, then there exists some $m_i = 0$, in which case f is a unit. Hence, $X + X^{2a/p^n}$ is irreducible. Since $X + X^{2a/p^n}$ is irreducible and X is irreducible, $X^{2/p^n} = u\frac{X^2}{(X+X^{2a/p^n})^2}$ is a ratio of atomic elements that is not atomic. △

A proof that all non-zero non-units of R from Example 2.0.4 are almost atomic may be available by writing elements of R in a canonical form that is amenable to factorization. However, this is a rather heavy-handed way to provide an example of an almost atomic ring. We have, instead, Example 2.0.5.

Example 2.0.5. Let $R = \mathbb{Z}[X] + X^2\mathbb{Q}[X]$. If $f \in \mathbb{Z}[X]$ is irreducible in $\mathbb{Z}[X]$ then f is irreducible in R. To see this, assume $f \in \mathbb{Z}[X]$ is irreducible and assume $f = gh$ for some $g, h \in R$. As elements of $\mathbb{Q}[X]$, which is a UFD, we see that

In the quotient field of R, any non-zero non-unit $f \in R$ may be written $\frac{\widehat{f}}{m}$ where m is the least common multiple of the denominators of the coefficients of f and $\widehat{f} \in \mathbb{Z}[X]$. Moreover, for any prime integer $p \in \mathbb{Z}$, p is irreducible in R. To see this, assume $f, g \in R$ such that $p = fg$. Note that $\deg(f) = \deg(g) = 0$. Hence, $f, g \in \mathbb{Z}$ where p is prime. Hence, $p \mid f$ or $p \mid g$.

Here is an example of a ring with mixed quasi-atomic and antimatter behavior.

Example 2.0.6. Let X, Y be indeterminates over $\mathbb{F}_2$ and $R' = \mathbb{F}_2[X, Y]$. The quotient field of R' is $\mathbb{F}_2(X, Y)$ and has algebraic closure $\overline{\mathbb{F}_2(X, Y)}$. Let $\mathcal{Y} = \{Y^\alpha \mid \alpha \in \mathbb{Q}^+\} \subseteq \overline{\mathbb{F}_2(X, Y)}$. Set $R'' = R'[\mathcal{Y}] = \mathbb{F}_2[X, Y, \mathcal{Y}]$. In R'', the ideal $(X, Y, \mathcal{Y}) = \mathfrak{m}$ is maximal. Set $R = R''_{\mathfrak{m}}$. Then R is Bézout (each finitely generated ideal is principal), and so $G(R)$ is lattice-ordered (by the Jaffard-Kaplansky-Krull-Ohm Theorem). Note that elements of the form X^n are atomic, whereas elements of the form Y^α seem to be associated to antimatter behavior. We later make rigorous the notion of an antimatter element to describe elements of the form Y^α. △

An overring of an antimatter domain may be antimatter or it may not be. Using examples from [7], we present antimatter domains with atomic overrings in Example 2.0.7 , and in Example 2.0.8 we present an antimatter domain whose overrings are all antimatter.

Example 2.0.7. Let X, Y be indeterminate over $\mathbb{F}_2$. Let $T = \overline{\mathbb{F}_2(X)}$ be the algebraic closure of the quotient field of $\mathbb{F}_2[X]$ and define $\mathcal{X} = \{X^r \mid r \in \mathbb{Q}^+\} \subseteq T$. Set $R_0 := \mathbb{F}_2[\mathcal{X}, Y]$ and denote the quotient field for R_0 as $\mathbb{K}$. Define $\mathcal{Y} = \left\{ \frac{Y}{X^n} \mid n \in \mathbb{N} \right\} \subseteq \mathbb{K}$. Define

$$R_1 = R_0[\mathcal{Y}] = \mathbb{F}_2 \left[X^r, Y, \frac{Y}{X^n} \mid r \in \mathbb{Q}^+, n \in \mathbb{N} \right]$$

and let $\mathfrak{m} \subseteq R_1$ be the ideal generated by all monomials of the form $X^r Y^n$ where $r \in \mathbb{Q}^+$ and $n \in \mathbb{N}_0$ such that if $n = 0$ then $r > 0$. Since $R_1/\mathfrak{m} \cong \mathbb{F}_2$, $\mathfrak{m}$ is maximal. Define $R := (R_1)_{\mathfrak{m}}$.

Note that in R, every non-zero non-unit element is associate to some X^r with $0 \neq r \in \mathbb{Q}^+$ or is associate to some $\frac{Y^n}{X^r}$ with $n \in \mathbb{N}$ and $r \in \mathbb{Q}$. We claim R is a valuation domain that is antimatter with prime spectrum $(0) \subseteq \mathfrak{p} \subseteq \mathfrak{n}$ where $\mathfrak{n}$ is generated by $\{X^r \mid r \in \mathbb{Q}^+\}$. To demonstrate that R is a valuation domain, we show that for any pair of non-zero non-units, one divides the other. Let $f, g\mathfrak{n}R$; we can write $f = uX^r$ for some $u \in U(R)$, $0 \neq r \in \mathbb{Q}^+$ or $f = u\frac{Y^n}{X^r}$ for some $u \in U(R)$, $n \in \mathbb{N}$, $r \in \mathbb{Q}$, and we can write $g = vX^s$ for some $v \in U(R)$ and $0 \neq s \in \mathbb{Q}^+$, or $g = v\frac{Y^m}{X^s}$ for some $v \in U(R)$, $m \in \mathbb{N}$, $s \in \mathbb{Q}$.

If $f = uX^r$ and $g = vX^s$ for some $r, s \in \mathbb{Q}^+$, then $r \leqslant s$ or $s \leqslant r$ since $\mathbb{Q}$ is totally ordered. In the former case, $f \mid g$, and in the latter case, $g \mid f$. If $f = uX^r$ for some $r \in \mathbb{Q}^+$ and $g = v\frac{Y^m}{X^s}$ for some $m \in \mathbb{N}$ and $s \in \mathbb{Q}$, we always have that $Y^m = X^r\frac{Y^m}{X^r}$ so $f \mid g$. This is the same case as when $f = u\frac{Y^n}{X^r}$ for some $n \in \mathbb{N}$ and $r \in \mathbb{Q}$ and $g = vX^s$ for some $s \in \mathbb{Q}^+$. Lastly, if $f = u\frac{Y^n}{X^r}$ and $g = v\frac{Y^m}{X^s}$, we have either $n \leqslant m$ or $m \leqslant n$ since $\mathbb{N}$ is totally ordered. If $n \leqslant m$, we can write $\frac{Y^m}{X^s} = \frac{Y^{m-n}}{X^s}X^r\frac{Y^n}{X^r}$ so $f \mid g$. The case where $m \leqslant n$ proceeds similarly. Thus, R is a valuation domain whose prime ideals are linearly ordered.

All monomials are divisible by any uX^r for $0 \neq r \in \mathbb{Q}^+$, and so the maximal ideal in R is $\mathfrak{n} = (\{X^r \mid 0 \neq r \in \mathbb{Q}^+\})$. However, $(Y) \neq \mathfrak{n}$ is also a prime ideal. To see this, presume that $fg \in (Y)$, say $fg = h \cdot Y^n$ for some $h \in R$ and $n \in \mathbb{N}$. As before, we can write $f = uX^r$ or $f = u\frac{Y^n}{X^r}$, and we can write g as vX^s or $v\frac{Y^m}{X^s}$. We may also write $h = wX^t$ for some $w \in U(R)$ and $0 \neq t \in \mathbb{Q}^+$ or $h = w\frac{Y^\ell}{X^t}$ for some $w \in U(R)$, $\ell \in \mathbb{N}$, and $t \in \mathbb{Q}$. The degrees of each side (with respect to Y) must match, so $\deg_Y(f) + \deg_Y(g) = \deg_Y(h) + \deg_Y(Y^n)$. In particular, we have that $\deg_Y(f) \geqslant 1$ (so $f = u\frac{Y^n}{X^r} \in (Y)$) or $\deg_Y(g) \geqslant 1$ (so $g = v\frac{Y^m}{X^s} \in (Y)$). Hence, (Y) is a prime ideal.

These are all the non-zero prime ideals in R. To see this, let $(0) \subset I \subset \mathfrak{p} \subseteq \mathfrak{m} \subseteq R$ be any ideal and let $0 \neq f \in I$. We cannot write $f = uX^r$ for some $r \in \mathbb{Q}^+$, for otherwise $f \in \mathfrak{m} \setminus \mathfrak{p}$, contradicting our choice of f as a generator of I. So we write $f = u\frac{Y^n}{X^r}$ for some $n \in \mathbb{N}$, $r \in \mathbb{Q}$, $u \in U(R)$. If $n > 1$, then this element factors non-trivially, but f was chosen arbitrarily, so I cannot be prime. If $n = 1$, then $(f) = (\frac{Y}{X^r}) = (Y)$. Now if $\mathfrak{p} \subset I \subset \mathfrak{m} \subseteq R$, any non-zero $f \in I$ must take the form X^r, in which case I can only be prime if $I = \mathfrak{m}$, violating our choice of I. Hence, we have a 2-dimensional valuation domain with prime spectrum $(0) \subseteq (Y) \subseteq \Big(\{X^r\}_{0 \neq r \in \mathbb{Q}^+}\Big) \subseteq R$.

The value group for this domain is $\mathbb{Z} \oplus \mathbb{Q}$ ordered lexicographically, which we see by the canonical map sending $X^r \mapsto (0, r)$ and sending $\frac{Y^n}{X^r} \mapsto (n, r)$. Note that elements like $(1, 0)$, while positive, are not minimally positive because any $(0, r) \leqslant (1, 0)$ with $r \in \mathbb{Q}^+$. In fact, R has no irreducibles. Moreover, if we localize R at $\mathfrak{p} = (Y)$, we obtain an atomic ring with a unique irreducible, Y. △

Example 2.0.8. Similar to the previous example, construct $\mathbb{F}_2[X^r, Y^s, \frac{Y^s}{X^k} \mid r, s \in \mathbb{Q}^+, k \in \mathbb{N}]_{\mathfrak{m}}$. This time, R is an antimatter valuation domain with value group $\mathbb{Q} \oplus \mathbb{Q}$ ordered lexicographically; every overring is antimatter. Every element is associate to some X^α or some $X^\alpha Y^\beta$. △

We may be interested in whether overrings preserve atomic (or almost atomic or quasi-atomic) domains. Certainly, an atomic domain can have an antimatter overring; we merely need to consider the quotient field for a realization. However, fields are vacuously atomic, and so this seems to be an unsatisfying angle. We later see that every integral domain has a minimal antimatter overring, which may not be a proper overring. This seems to suggest that, as we climb overrings toward the quotient field, we tend toward antimatter behavior.

The domain in Example 2.0.9 was first presented in [7, Example 2.7]; we revisit this example in some detail later. This integral domain exhibits behavior that is mixed between atomic and antimatter in a few different ways.

Example 2.0.9. Let $\mathbb{F}$ be any field and X, Y, indeterminates over $\mathbb{F}$. Let $R' = \mathbb{F}[X, Y]$ have algebraic closure of its quotient field $\overline{\mathbb{F}(X, Y)}$. Set $\mathcal{W} = \left\{Y^\alpha, \frac{Y^\alpha}{X^j} : j \in \mathbb{N}, \alpha \in \mathbb{Q}^+\right\}$ and define the ring $R'' = R'[\mathcal{W}] = \mathbb{F}[X, Y, \mathcal{W}]$. In R'', the ideal $(X, Y, \mathcal{W}) = \mathfrak{m}$ is maximal. Set $R = R''_{\mathfrak{m}}$.

An arbitrary non-zero non-unit of R is associate to some X^n for $n \in \mathbb{N}$ or is associate to some $\frac{Y^\alpha}{X^n}$ for $0 \neq \alpha \in \mathbb{Q}^+$, $n \in \mathbb{Z}$. The ring R is a 2-dimensional valuation domain with value group $\mathbb{Q} \oplus \mathbb{Z}$ ordered lexicographically with the natural semi-valuation defined by mapping $X^n \mapsto (0, n)$ and mapping $\frac{Y^\alpha}{X^n} \mapsto (\alpha, n)$. Note that $(0, 1)$ is minimal positive: if $(0, 0) \leqslant (\alpha, n) \leqslant (0, 1)$ for some $\alpha \in \mathbb{Q}^+$ and $n \in \mathbb{Z}$ then since $(0, 0) \leqslant (\alpha, n)$, we have that $0 < \alpha$ or $0 = \alpha$ and $0 \leqslant n$. Since $(\alpha, n) \leqslant (0, 1)$, we have $\alpha < 0$ or $\alpha = 0$ and $n \leqslant 1$. Hence, $\alpha = 0$ and $0 \leqslant n \leqslant 1$.

The element X, which has minimally positive value $(0, 1)$, is irreducible (and uniquely so, among monomials). Thus, the atomic elements are precisely the elements of the form X^n. Moreover, for any $n \in \mathbb{N}_0, \alpha \in \mathbb{Q}^+$, we have that $X^n \mid Y^\alpha$, so every non-unit element is atomic (of the form X^n) or has a countably infinite set of atomic divisors (of the form X^nY^α). In this sense, R exhibits behavior mixed between antimatter and atomic. △

We provide some explanatory framework for these varied factorization properties and behaviors in the sequel.

Chapter 3

Quasi-Atomic Sequences

In this dissertation, our primary objects of study are sequences of localizations of integral domains and their associated groups of divisibility, constructed so as to discern factorization information. In Section 3.1, we construct these sequences, and in Section 3.2, we discuss their properties.

3.1 Constructions

We study localizations of integral domains because factorization in a ring is very sensitive to localization, as demonstrated in Example 3.1.1. Recall we say a valuation domain is *discrete* if its value group is discrete.

Example 3.1.1. Let D be a discrete valuation domain with Krull dimension 2 and prime spectrum $(0) \subset \mathfrak{p} \subset \mathfrak{m}$. The maximal ideal, $\mathfrak{m}$, is principal, say $\mathfrak{m} = (x)$. Any element of $\mathfrak{m} \setminus \mathfrak{p}$ can be uniquely factored into a power of x (up to associates). Since D is a valuation domain with an irreducible element, all irreducibles are associate to x, and so all atomic elements are associate to some x^n. However, any $0 \neq y \in \mathfrak{p}$ cannot be written as a pure power of x, so any such y is not atomic. However, there are only two possible overrings of D (up to isomorphism), the quotient field of D and a 1-dimensional discrete valuation domain with prime spectrum $(0) \subseteq \mathfrak{p}_{\mathfrak{p}} \subseteq D_{\mathfrak{p}}$.

The set of non-unit, non-atomic elements (i.e. $\mathfrak{p} \setminus \{0\}$) is certainly multiplicatively closed, but its saturation is $D \setminus \{0\}$. The saturation of $\mathfrak{p} \setminus \{0\}$ in D is $D \setminus \{0\}$, because the set complement of the saturation is a union of prime ideals. In D the only options are $\mathfrak{p}$, $\mathfrak{m}$, or R. The corresponding localization, $D_{D \setminus \{0\}}$ is precisely the quotient field of D; every element in the localization is a unit,

$$
\begin{array}{ccccccc}
D & \xrightarrow{\epsilon_0} & D_{S_1} & \xrightarrow{\epsilon_1} & D_{S_2} & \xrightarrow{\epsilon_2} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
G(D) & & G(D_{S_1}) & & G(D_{S_2}) & &
\end{array}
$$

Figure 3.1: The quasi-atomic localization sequence and the groups of divisibility in each degree.

so there are no irreducibles in the localization. △

That is to say, the set of non-atomic elements is generally not multiplicatively closed, which suggests we should localize at atomic elements. However, localizing at a set is equivalent to localizing at the saturation of that set. Of course, if t is in the saturation of a set, $S \subseteq R$, then there exists some $s \in S$ such that $t \mid s$. In particular, if S consists of atomic elements, then the saturation of S consists of the quasi-atomic elements. To select our localization sets as generated by atoms is equivalent to localize at all quasi-atoms.

We sequentially localize at the set of all quasi-atomic elements, providing a chain of localizations $D = D_{S_0} \subseteq D_{S_1} \subseteq D_{S_2} \subseteq D_{S_3} \subseteq \cdots$ where we select $S_0 = U(D)$ and each S_{n+1} to be the saturation of the set of all atoms in D_{S_n}. In the sequel, we refer to this sequence as the *quasi-atomic localization sequence*. More information on localization is presented in Appendix A. Every integral domain has an associated quotient field $\mathbb{F}$, unit group $U(D)$, and po-group of divisibility $G(D) = \mathbb{F}^\times/U(D)$, which we define and describe in some detail in Appendix B. We expand the quasi-atomic localization sequence by considering the groups of divisibility in each degree as in Figure 3.1.

The first natural question to arise is whether we can "fill in" the bottom edges of each square to make the diagram in Figure 3.1 commute. We answer this in the affirmative in Theorem 3.1.4 establishing there exists a functor between integral domains and po-groups of the group of divisibility. To describe the the group of divisibility as a functor we first must describe the category of integral domains and the category of po-groups.

We take the morphisms in the category of integral domains to be ring monomorphisms. We take the morphisms in the category of po-groups as the *o-homomorphisms*, described in Appendix B. An o-homomorphism is a group homomorphism $\phi : G_1 \to G_2$ between po-groups that preserves the partial order on G_1: if $x \leqslant y$ in G_1 then $\phi(x) \leqslant \phi(y)$ in G_2. In Appendix B, we describe certain distinguished o-homomorphisms called o-epimorphisms in detail, since they play a central role in the category of po-groups due to Mott's correspondence connecting o-epimorphisms and localizations of integral domains. An o-homomorphism $\phi : G_1 \to G_2$ is an *o-epimorphism* if ϕ is a surjective o-homomorphism such that $\phi(G_1^+) = G_2^+$.

Definition 3.1.2. (a) Let **Dom** be the category whose objects are all integral domains and whose morphisms are ring homomorphisms such that $1 \mapsto 1$. Let **Dom*** be the subcategory with the same objects as **Dom** but restricting the morphisms to include only ring monomorphisms.

(b) Let **Pog** be the category whose objects are all abelian po-groups and whose morphisms are po-group o-homomorphisms. Let **Pog*** have the same objects as **Pog** but restricting the morphisms to include only po-group o-epimorphisms.

For a category **C**, we denote the collection of all objects of **C** as Ob(**C**) and the morphisms of **C** as Mor(**C**).

Definition 3.1.3. Define $\mathfrak{G}$ on objects, by $\mathfrak{G}(D) = G(D)$. Define $\mathfrak{G}$ on a morphism $\epsilon : D_1 \to D_2$ by defining $\mathfrak{G}(\epsilon)$ as the o-homomorphism $\phi_\epsilon : G(D_1) \to G(D_2)$ defined by $\frac{a}{b}U(D_1) \to \frac{\epsilon(a)}{\epsilon(b)}U(D_2)$.

Theorem 3.1.4. *In Definition 3.1.3, $\mathfrak{G}$ is a covariant functor $\mathfrak{G}$: **Dom*** $\to$ **Pog**.*

Proof. Let $\epsilon : D_1 \to D_2$ be an arbitrary arrow from **Dom***. We construct $\mathfrak{G}(\epsilon) : G(D_1) \to G(D_2)$ explicitly. First observe that $\epsilon(U(D_1)) \subseteq U(D_2)$. To see this, if $u \in U(D_1)$ then $uu' = 1$ for some $u' \in U(D_1)$; since ϵ is a homomorphism of rings, we conclude $\epsilon(u)\epsilon(u') = 1$. An arbitrary element from $G(D_1)$ is a co-set of the form $\frac{a}{b}U(D_1)$ where $a, b \in D_1 \setminus \{0\}$, so $\epsilon(a), \epsilon(b) \in D_2 \setminus \{0\}$. Furthermore, if $\frac{a}{b}U(D_1) = \frac{a'}{b'}U(D_1)$ then there exists some $u \in U(D_1)$ so that $ab' = ua'b$. Thus $\epsilon(a)\epsilon(b') = \epsilon(u)\epsilon(a')\epsilon(b)$ where $\epsilon(u) \in U(D_2)$, so $\frac{\epsilon(a)}{\epsilon(b)}U(D_2) = \frac{\epsilon(a')}{\epsilon(b')}U(D_2)$. Hence the naturally defined map $\phi : G(D_1) \to G(D_2)$ induced by ϵ given by $\frac{a}{b}U(D_1) \to \frac{\epsilon(a)}{\epsilon(b)}U(D_2)$ is well-defined.

Moreover, this map is a morphism in **Pog**. We verify if $x, y \in G(D)$ then $\phi(xy) = \phi(x)\phi(y)$. We set $x = \frac{a}{b}U(D)$, $y = \frac{a'}{b'}U(D)$. Then $xy = \frac{aa'}{bb'}U(D)$. Hence, $\phi(xy) = \frac{\epsilon(aa')}{\epsilon(bb')}U(D')$. Since ϵ is a ring homomorphism $\epsilon(aa') = \epsilon(a)\epsilon(a')$, $\epsilon(bb') = \epsilon(b)\epsilon(b')$, and $\phi(xy) = \frac{\epsilon(a)}{\epsilon(b)}\frac{\epsilon(a')}{\epsilon(b')}U(D') = \phi(x)\phi(y)$. To establish ϕ is a morphism in **Pog** all that remains is to establish that ϕ is order-preserving. If $\frac{a}{b}U(D_1) \leqslant \frac{a'}{b'}U(D_1)$ then $U(D_1) \leqslant \frac{a'b}{ab'}U(D_1)$, so there exists some $x \in D_1$ such that $\frac{a'b}{ab'} = x$. Hence $a'b = xab'$ and therefore $\epsilon(a')\epsilon(b) = \epsilon(x)\epsilon(a)\epsilon(b')$. Thus, $\frac{\epsilon(a')\epsilon(b)}{\epsilon(a)\epsilon(b')} = \epsilon(x) \in D_2$. In particular, $\frac{\epsilon(a')\epsilon(b)}{\epsilon(a)\epsilon(b')}U(D_2)$ is non-negative so $\frac{\epsilon(a)}{\epsilon(b)}U(D_2) \leqslant \frac{\epsilon(a')}{\epsilon(b')}U(D_2)$.

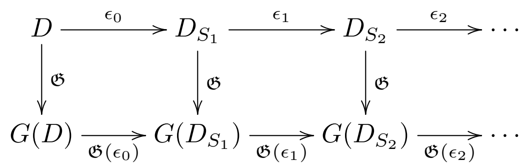


Figure 3.2: The diagram of quasi-atomic localization sequence and its image under $\mathfrak{G}$.

All that remains is to verify that $\mathfrak{G}$ respects composition of morphisms and respects identity morphisms. If $\epsilon : D_1 \to D_2$ is the identity ring monomorphism, then $D_1 = D_2$ and ϕ maps $\frac{a}{b}U(D_1) \mapsto \frac{\epsilon(a)}{\epsilon(b)}U(D_2) = \frac{a}{b}U(D_1)$. Thus, $\mathfrak{G}(\epsilon)$ is the identity morphism. Lastly, let $\epsilon_{1,2} : D_1 \to D_2$ and $\epsilon_{2,3} : D_2 \to D_3$ be a pair of composable morphisms from $\mathbf{Dom}^*$. Then $\epsilon_{2,3} \circ \epsilon_{1,2}$ is precisely the map $\epsilon_{1,3} : D_1 \to D_3$ is a ring monomorphism. Hence, $\mathfrak{G}(\epsilon_{1,3}) : G(D_1) \to G(D_3)$ is the map $\frac{a}{b}U(D_1) \mapsto \frac{\epsilon_{2,3}\circ\epsilon_{1,2}(a)}{\epsilon_{2,3}\circ\epsilon_{1,2}(b)}U(D_3)$. This certainly factors into the composition of arrows $\mathfrak{G}(\epsilon_{2,3}) \circ \mathfrak{G}(\epsilon_{1,2})$. $\mathfrak{G}$ is covariant by construction. □

All that remains is to establish that these squares are commutative. Indeed, let $x \in D_{S_n}$. The map downward map $D_{S_n} \to G(D_{S_n})$ is induced by the natural semi-valuation $x \mapsto xU(D_{S_n})$. The map $\mathfrak{G}(\epsilon_n) : G(D_{S_n}) \to G(D_{S_{n+1}})$ is the canonical projection $xU(D_{S_n}) \mapsto xU(D_{S_{n+1}})$. Going round the other direction, $x \mapsto \frac{x}{1}$ by ϵ_n and then $\frac{x}{1} \mapsto \frac{x}{1}U(D_{S_{n+1}}) = xU(D_{S_{n+1}})$. Thus, we answer our first naturally arising question in the affirmative and construct the commutative diagram in Figure 3.2. After establishing Theorem 3.1.4, we direct the reader to Sections 3.2 and B.2 for extended remarks on the morphisms in **Pog**; these morphisms require some delicate treatment.

Another natural question concerning this sequence is whether we can exploit any properties or structures of the po-groups $G(D_{S_i})$ to expand upon the diagram of Figure 3.2 further. Again, the answer is yes, thanks to Theorem B.21 (first established in [25]), which allows us to write each $G(D_{S_i}) = G(D)/H_i$ for an o-ideal H_i. Choosing $H_0 = \{e_G\}$, the o-ideals may be written in a linearly ordered ascending chain $H_0 \subseteq H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots \subseteq G(D)$ since the sequence $\{S_i\}$ is a linearly ordered ascending chain. Furthermore, the maps $\mathfrak{G}(\epsilon_i) : \frac{G(D)}{H_i} \to \frac{G(D)}{H_{i+1}}$ map $\frac{a}{b}H_i \mapsto \frac{a}{b}H_{i+1}$ so they are the canonical group epimorphisms. Hence, we denote $\mathfrak{G}(\epsilon_i) = \pi_i$. Following Theorem B.21, each $H_i = \langle \nu(S_i) \rangle$ where the map $\nu : D \to G(D)$ is defined by $x \mapsto xU(D)$.

We find the following notation helpful: for a po-group G, we denote the subgroup generated by the atomic elements as $A(G)$ and we denote the subgroup generated by the quasi-atomic elements as $Q(G)$.

Theorem 3.1.5. *Let D be an integral domain and $G(D)$ the group of divisibility. Then any $x \in D$ is quasi-atomic if and only if $xU(D) \in Q(G(D))^+$.*

Proof. Let $x, y \in D$ such that xy is atomic. Since $x \in D$, $xU(D) \in G(D)^+$, and since xy is atomic, $xyU(D) \in A(G(D))$. Hence, $U(D) \leqslant xU(D) \leqslant xyU(D)$, so by definition $xU(D)$ is quasi-atomic in $G(D)$ (similarly, $yU(D)$ is also quasi-atomic). Now let $\frac{a}{b}U(D) \in Q(G(D))^+$. Since this is positive, we have some $x \in D$ such that $\frac{a}{b} = x \in D$. Since $\frac{a}{b}U(D) \in Q(G(D))$ there exists some element $a \in A(G(D))$ such that $U(D) \leqslant xU(D) \leqslant a$. Since $a \in A(G(D))$, we may write $a = (a_1U(D))\,(a_2U(D))^{-1}$ for some $a_1U(D), a_2U(D) \in A(G(D))^+$. We obtain the chain of order relations $xU(D) \leqslant a_2xU(D) \leqslant a_1U(D)$ so $x \mid a_1$ in D. We conclude x is quasi-atomic in D. □

We later elaborate further upon the connection between these subgroups, especially with respect to saturated subsets of integral domains (c.f. Theorem 3.1.6, Lemma B.5, and Theorem B.21).

Mott's correspondence demonstrates that $\langle \nu(S_{n+1}) \rangle$ is an o-ideal of $G(D_{S_n})$. Since each S_{n+1} is the saturation of the atoms of D_{S_n}, S_{n+1} is the collection of all quasi-atoms (and units). Thus $H_n = Q(G(D_{S_n}))$. Due to this, we can more generally define the quasi-atomic quotient sequence for directed po-groups G, even without an underlying integral domain D.

$$G = G_0 \to G_1 = \frac{G_0}{Q(G_0)} \to G_2 = \frac{G_1}{Q(G_1)} \to \cdots$$

However, although Mott's correspondence ensures that each $H = Q(G(D))$ is an o-ideal, it is not necessarily clear that $Q(G)$ is an o-ideal for an arbitrary po-group G.

Theorem 3.1.6. *For a directed po-group G, $Q(G)$ is the smallest o-ideal in G containing $A(G)$.*

Proof. The subgroup $Q(G)$ is generated by the elements $g \in G^+$ that can be bounded above by an atomic element, so its generators are by construction all positive. Hence, $Q(G)$ is directed. To establish convexity, let $q \in Q(G)^+$ and $x \in G^+$ such that $e \leqslant x \leqslant q$. Since $q \in Q(G)^+$, there exists an atomic $a \in A(G)$ such that $q \leqslant a$. By transitivity, $e \leqslant x \leqslant a$ so $x \in Q(G)$. Now let $\mathcal{O}$ be the set of all o-ideals in G containing $A(G)$. Certainly $Q(G) \in \mathcal{O}$, hence we obtain the obvious containment $\bigcap_{H \in \mathcal{O}} H \subseteq Q(G)$. To complete the argument it is sufficient to show $Q(G)^+ \subseteq H$ for each $H \in \mathcal{O}$. If $x \in Q(G)^+$ then $x \leqslant a$ for some $a \in A(G)^+ \subseteq H$. By the convexity of H, we have that $x \in H$. □

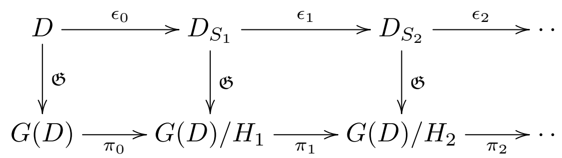


Figure 3.3: The commutative diagram of quasi-atomic localization sequence and its associated quasi-atomic quotient sequence. The arrows ϵ_n denote set inclusion, the arrows π_n denote po-group o-epimorphisms, and the arrows $\mathfrak{G}$ are the maps induced by the natural transformation id $\to \mathfrak{G}$.

Definition 3.1.7. Let G be a po-group. Define $G_0 = G$ and define $G_{n+1} = G_n/Q(G_n)$ for each $n \geqslant 0$ and let $\pi_n : G_n \to G_{n+1}$ be the associated o-epimorphism. We call this the *quasi-atomic quotient sequence* for G.

In this section, we began with an integral domain and we constructed the quasi-atomic localization sequence by iteratively localizing at the set of all quasi-atomic elements. We used the group of divisibility to establish a functor between the category of integral domains $\mathbf{Dom}^*$ and the category of po-groups **Pog**, and we began the study of the image of the quasi-atomic localization sequence under this functor. The morphisms in the bottom row of Figure 3.2 can be rewritten as in Figure 3.3.

3.2 Properties

In this section, we establish some properties of groups of divisibility and the quasi-atomic sequences. In particular, the morphisms in **Pog** require some special care. As we shall see, not every epic morphism in **Pog** needs to be a surjective function, but the o-epimorphisms we have defined, following the definition by Fuchs in [11], play a keystone role in the category. We can exploit properties of $\mathfrak{G}$ to explore the difference between epic morphisms in **Pog** and our o-epimorphisms.

Definition 3.2.1. Let $\mathbf{C}$ be a category and f a morphism in $\mathbf{C}$.

(a) If for any $g, h \in \mathrm{Mor}(\mathbf{C})$, $gf = hf$ implies $g = h$, we say f is *right-cancellable.*

(b) If for any $g, h \in \mathrm{Mor}(\mathbf{C})$, $fg = fh$ implies $g = h$, we say f is *left-cancellable.*

If f is right-cancellable, we say f is a *epimorphism* (or *epic*). If f is left-cancellable, we say f is a *monomorphism* (or *monic*). If f is right- and left-cancellable, we say f is a *bimorphism.* If f has a left and right inverse, we say f is an *isomorphism.*

As defined in [12], every o-epimorphism $\pi : G \to G/H$ is a group epimorphism and hence is right-cancellable with all group epimorphisms. In particular, f cancels with all o-homomorphisms, so f is epic as an o-homomorphism. However, it may be that f is epic in **Pog** but is not epic as a group homomorphism.

Example 3.2.2. We examine the product order on $\mathbb{Z} \oplus \mathbb{Z}$; declare $(a, b) \leqslant (c, d)$ if and only if $a \leqslant c$ and $b \leqslant d$. Let $f : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ be defined by mapping $(a, b) \mapsto 6a + 15b$. Then f is clearly not surjective as a function. We claim that f is epic in the category of po-groups. Indeed, f is order-preserving: if $(a, b) \leqslant (c, d)$ then $a \leqslant c$ and $b \leqslant d$ so $6a + 15b \leqslant 6c + 15d$. Hence, it is sufficient to show that f cancels on the right with respect to all po-group homomorphisms.

Let $g, h : \mathbb{Z} \to G$ be po-group homomorphisms. If $gf = hf$ then $g \circ f(a, b) = h \circ f(a, b)$ so $g(6a + 10b) = h(6a + 15b)$. Now since $g, h : \mathbb{Z} \to G$ have $\mathbb{Z}$ as their domains, g and h are determined by $g(1)$ and $h(1)$. For any $x \geqslant 0$, we have that $g(x) = g(\underbrace{1 + 1 + \cdots + 1}_{x \text{ times}}) = xg(1)$ so $(6a + 10b)g(1) = (6a + 15b)h(1)$ (and for $x < 0$ we have $-(6a + 10b)g(1) = -(6a + 15b)h(1)$). We conclude $(6a + 15b)(g(1) - h(1)) = 0$ in G or has order $6a + 10b$ in G. However, partially ordered groups are torsion-free. To see this, assume $x^n = 1_G$ for some $x \in G$ and $n \geqslant 1$. Then $x^n \leqslant x^{n+1}$ so $1_G \leqslant x$ and, moreover, $1_G \leqslant x \leqslant x^2 \leqslant \cdots \leqslant x^{n-1} \leqslant 1_G$. Since $\leqslant$ is antisymmetric, we have that $x = 1_G$. Hence, we conclude $g(1) = h(1)$ so $g = h$.

Note that f is not epic as an arrow in the category of abelian groups much less the category of all groups **Grp**. For a concrete realization of this, set $G = \mathbb{Z}/3\mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$ defined by $g(1) = \overline{1}$ and $h : \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$ by $h(1) = \overline{2}$. Certainly, $g \neq h$. However $hf = gf = 0$. △

Any f we hope to be epic in **Pog** only needs to cancel on the right with respect to o-homomorphisms, but to be epic in the category of all groups, f needs to cancel on the right with respect to all group homomorphisms. In this regard, the hurdle is lower in subcategories to be regarded as epic (or mono, or iso). Hence, the term *o-epimorphism* (ostensibly first used by Fuchs in [12]) may be misleading: *a priori* not every epic morphism in **Pog** is an o-epimorphism. Example 3.2.2 demonstrates that not every epic morphism in **Pog** is surjective.

Of course, Mott's Correspondence (Theorem B.21) connects o-ideals in $G(D)$ to saturated multiplicatively closed sets of D. Each o-ideal is the kernel for a naturally induced o-epimorphism, but not every o-epimorphism has an o-ideal kernel (all we require is convexity, not that the kernel is direct). On the other hand, every localization of D at a saturated multiplicatively closed set is an overring of D, but not every overring of D is a localization. We use Lemma 3.2.3 several times to establish certain properties of the quasi-atomic sequences.

Lemma 3.2.3. *Let D_1 and D_2 be integral domains. If $G(D_1) = G(D_2)$ then $D_1 = D_2$.*

Proof. If $G(D_1) = G(D_2)$ then the identities of these groups coincide, so $U(D_1) = U(D_2)$. Since these groups coincide, their generating sets coincide. These are directed groups, so they are generated by their positive elements. The set of positive elements is the set of associate classes of domain elements. Hence, the associate classes coincide and the unit groups coincide, so $D_1 = D_2$. □

Example 3.2.4. Let D be any integral domain with quotient field $\mathbb{F}$ and overring D'. Since D' is an overring, the quotient field of D' is also $\mathbb{F}$ so $G(D') = \mathbb{F}^\times/U(D')$. Of course, $G(D) = \mathbb{F}^\times/U(D)$, and so the canonical ring monomorphism $\epsilon : D \to D'$ induces a canonical group epimorphism $G(D) \twoheadrightarrow G(D')$ with kernel $U(D')/U(D) \subseteq \mathbb{F}^\times/U(D)$. Thus this canonical group epimorphism is of the form $\pi : G \twoheadrightarrow G/H$ for a subgroup H. It has been established that G/H is a po-group under the inherited quotient order from G if and only if H is convex in the partial order from G. Thus, we have that $U(D')/U(D)$ is a convex subgroup of $G(D)$. Moreover, if $U(D')/U(D)$ is an o-ideal, then we have $G(D') = \mathbb{F}^\times/U(D') = \frac{\mathbb{F}^\times/U(D)}{U(D')/U(D)}$ so by Mott's Correspondence, $G(D') = G(D_S)$. Applying Lemma 3.2.3, we conclude $D' = D_S$ for some saturated multiplicatively closed set $S \subseteq D$. On the other hand, $U(D')/U(D)$ is not an o-ideal then $D' \neq D_S$ for any saturated multiplicatively closet set S. △

The quasi-atomic sequences themselves reveal information about factorization in D. To this end, we make use of the following definitions.

Definition 3.2.5. Let D be a domain with quasi-atomic localization sequence $\left\{\epsilon_i : D_{S_i} \to D_{S_{i+1}}\right\}_i$. Let G be a po-group with quasi-atomic quotient sequence $\{\pi_i : G(D)/H_i \twoheadrightarrow G(D)/H_{i+1}\}_i$.

(a) Define the *quasi-atomic length* of D as $\operatorname{len}_q(D) := \inf\{m \in \mathbb{N}_0 : S_m = S_{m+1} = S_{m+2} = \cdots\}$.

(b) Define the *quasi-atomic length* of G as $\operatorname{len}_q(G) := \inf\{m \in \mathbb{N}_0 : H_m = H_{m+1} = H_{m+2} = \cdots\}$.

Lemma 3.2.6. *If D is antimatter then $\operatorname{len}_q(D) = 0$. If D is quasi-atomic then $\operatorname{len}_q(D) = 1$. If $\operatorname{len}_q(D) = N \geqslant 1$ and $D_{S_N} = \mathbb{F}$ then $D_{S_{N-1}}$ is quasi-atomic.*

Proof. Note that if D is antimatter, then $S_1 = U(D)$, so $D_{S_1} \cong D$. Thus, the localization sequence is $D = D = D = \cdots$ with quotient sequence $G(D) = G(D) = G(D) = \cdots$. On the other hand, if D is a quasi-atomic domain, then $S_1 = D\backslash\{0\}$, so $D_{S_1} = \mathbb{F}$. In this special case, the localization sequence is then $D \subseteq \mathbb{F} = \mathbb{F} = \mathbb{F} = \cdots$ and the quotient sequence is $G(D) \twoheadrightarrow \{0\} \twoheadrightarrow \{0\} \twoheadrightarrow \{0\} \twoheadrightarrow \cdots$. Lastly, If $\text{len}_q(D) = N \geqslant 1$ and $D_{S_N} = \mathbb{F}$, then $\mathbb{F} = D_{S_N} = (D_{S_{N-1}})_{S_N}$ so S_N, which consists of all quasi-atoms, satisfies $S_N = D_{S_{N-1}} \setminus \{0\}$. □

In this sense, antimatter domains have quasi-atomic sequences that stabilize, and quasi-atomic domains have trivial quasi-atomic sequences. Clearly the quasi-atomic sequences only capture non-quasi-atomic information.

Theorem 3.2.7. *Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a family of po-groups, $G = \oplus_\lambda G_\lambda$ in the product order, let $\{n_\lambda \mid \lambda \in \Lambda\}$ be a net of natural numbers such that $\text{len}_q(G_\lambda) = n_\lambda$ for each λ and such that the supremum $N = \sup\{n_\lambda \mid \lambda \in \Lambda\}$ is finite. Then $\text{len}_q(G) = N$. Moreover, (i) if each G_λ is quasi-atomic in the limit, then G is quasi-atomic in the limit and (ii) if each G_λ is antimatter in the limit, then G is antimatter in the limit.*

Proof. We index the following quotient groups:

$$\begin{aligned} G_0 &:= \oplus_\lambda G_\lambda \\ G_n &:= G_{n-1}/Q(G_{n-1}) \text{ for any } n \geqslant 1 \\ G_\lambda^{(0)} &:= G_\lambda \\ G_\lambda^{(n)} &:= G_\lambda^{(n-1)}/Q(G_\lambda^{(n-1)}) \text{ for any } n \geqslant 1 \end{aligned}$$

By Corollary B.9, $G_n \simeq \oplus_\lambda G_\lambda^{(n)}$ for any $n \geqslant 0$. Since each G_λ is n_λ-atomic, each $G_\lambda^{(n_\lambda)}$ is trivial. Hence, we obtain the sequence $G_0 \to G_1 \to G_2 \to \cdots$, which is o-isomorphic to the sequence, in direct sum notation

$$\oplus_\lambda G_\lambda^{(0)} \to \oplus_\lambda G_\lambda^{(1)} \to \oplus_\lambda G_\lambda^{(2)} \to \ldots$$

Since each $n_\lambda \leqslant N$, we have that this sequence terminates by the N^{th} step and no earlier. This establishes the first statement of the corollary. The second statement is proved similarly with all sequences stabilizing to a non-trivial group by the N^{th} step, rather than terminating. □

We can extend Definition 3.2.5 to elements rather than domains or po-groups. An arbitrary non-zero element $d \in D$ may be a unit in $D_{S_0} = D$ or not. If d is a non-unit, d may remain a non-unit in D_{S_1} or not. If d remains a non-unit in D_{S_1}, then d may remain a non-unit in D_{S_2} or not, and so on.

Definition 3.2.8. Let D be an integral domain, G a po-group, $x \in D$ and $g \in G$.

(a) Define $\text{len}_q(x) = \inf \{m \in \mathbb{N}_0 : x \in U(D_{S_m})\}$.

(b) Define $\text{len}_q(g) = \inf \{m \in \mathbb{N}_0 : g = 1 \in G/H_m\}$.

Theorem 3.2.9. *$len_q(D) = len_q(G(D))$.*

Proof. If D has quasi-atomic length n, then $D_{S_n} = D_{S_{n+k}}$ for each $k \in \mathbb{N}_0$. Their quotient fields and unit groups thus coincide so $G(D_{S_n}) = G(D_{S_{n+k}})$. On the other hand, if $G(D_{S_n}) = G(D_{S_{n+k}})$ then $D_{S_n} = D_{S_{n+k}}$ by Lemma 3.2.3. $\square$

Definition 3.2.10. Let D be an integral domain with the usual quasi-atomic localization sequence $\{\epsilon_n : D_{S_n} \to D_{S_{n+1}}\}$. Let $S^* = \varinjlim S_n$ and $D^* = \varinjlim D_{S_n}$.

(a) We say D^* is the *antimatter limit* and that $S^* \subseteq D$ is the *quasi-atomic limit* of D.

(b) If $D^* = \mathbb{F}$, we say D is *quasi-atomic in the limit*, otherwise, we say D is *antimatter in the limit.*

Definition 3.2.11. Let G be a po-group with quasi-atomic quotient sequence $\{\pi_n : G_n \to G_{n+1}\}$ where $H_n = \text{Ker}(\pi_n)$. Let $H^* = \varinjlim H_n$ and $G^* = \varinjlim G_n$.

(a) We say G^* is the *antimatter limit* and that $H^* \subseteq G$ is the *quasi-atomic limit* of G.

(b) If G^* is trivial then we say G^* is *quasi-atomic in the limit*, otherwise we say G is *antimatter in the limit.*

Lemma 3.2.12. *D^* is an antimatter overring of D, and D is quasi-atomic in the limit if and only if $len_q(x) < \infty$ for any non-zero non-unit $x \in D$. G^* is an antimatter quotient po-group of G, and G is quasi-atomic in the limit if and only if $len_q(x) < \infty$ for any $x \in G$.*

Proof. Assume $x \in D^*$ is an irreducible. The direct limit is the union, so $D^* = \varinjlim D_{S_i} = \cup_i D_{S_i}$. Hence, $x \in D_{S_i}$ for some i (and $D_{S_i} \subseteq D_{S_j}$ for every $i \leqslant j$). Moreover, if x is irreducible in any D_{S_j}, then $x \in U(D_{S_{j+1}})$, contradicting our choice of x as irreducible in D^*. Thus x is not irreducible in any D_{S_j} with $j \geqslant i$. Thus, for each $j \geqslant i$, we have some factorization $x = y_j z_j$ with $y_j, z_j \in D_{S_j} \setminus U(D_{S_j})$.

To see that G^* is antimatter, assume by contradiction that some $x \in (G^*)^+$ is an atom. We can write $x = g + H^*$ for some $g \in G$, and since x is non-negative we have some $h \in H^*$ such that $g + h \geqslant 0$. But $H^* = \varinjlim H_i = \cup_i H_i$, so there must exist some $n \in \mathbb{N}_0$ such that $h \in H_n$. Hence, $x = g + H_n = g + H_{n+1} = \cdots = g + H^*$. But if x is an atom in G/H_n then $x \in H_{n+1}$ and hence x is trivial in G^*, contradicting our choice of x as an atom.

Lastly, assume $G^* = \{0\}$. Since $G^* = G/(H^*)$ and $G/H^* = \{0\}$, we have $G = H^*$, i.e. $G = \cup_i H_i$. Any $g \in G$ has some corresponding minimal n such that $g \in H_n$, and hence $\text{len}_q(g) = n$. On the other hand, if each $g \in G$ has a corresponding $n \in \mathbb{N}_0$ such that $\text{len}_q(g) = n$, then each $G \subseteq \cup_i H_i = H^*$. □

This allows us to provide a rigorous definition of an *antimatter* element of a domain.

Definition 3.2.13. Let D be an integral domain and let G be a group. Any $x \in D$ (or $g \in G$) with infinite quasi-atomic length is called an *antimatter element*.

Corollary 3.2.14. *Let D be an integral domain and G a directed po-group. D is antimatter in the limit if and only if D has an antimatter element. G is antimatter in the limit if and only if G has an antimatter element.*

Proof. This follows directly from Lemma 3.2.12. We always have $D^* \subseteq \mathbb{F}$. If $D^* = \mathbb{F}$ then each element has finite quasi-atomic length and hence D^* has no antimatter elements. If $D^* \subset \mathbb{F}$ then every non-zero non-unit $x \in D^*$ comes from some $x \in D$ with infinite quasi-atomic length. Similarly, if $G^* \neq \{0\}$ then there exists some $g \in G$ such that for each n, $g + H_n \neq H_n$. □

Example 3.2.15. Let D be an integral domain with group of divisibility $G(D) = \mathbb{Q} \oplus \mathbb{Z} \oplus \mathbb{Z}$ ordered lexicographically by declaring $(a, b, c) \leqslant (\alpha, \beta, \gamma)$ if and only if any of the following conditions hold: (i) $a < \alpha$, (ii) $a = \alpha$ and $b < \beta$, or (iii) $a = \alpha$ and $b = \beta$ and $c \leqslant \gamma$. Then D has quasi-atomic quotient sequence

$$\mathbb{Q} \oplus \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Q} \oplus \mathbb{Z} \longrightarrow \mathbb{Q}$$

Indeed, in $\mathbb{Q} \oplus \mathbb{Z} \oplus \mathbb{Z}$, we have the (unique) minimal positive element $(0, 0, 1)$, which generates the quasi-atomic subgroup $0 \oplus 0 \oplus \mathbb{Z}$. In $\frac{\mathbb{Q} \oplus \mathbb{Z} \oplus \mathbb{Z}}{0 \oplus 0 \oplus \mathbb{Z}} \simeq \mathbb{Q} \oplus \mathbb{Z}$, we have the (unique) minimal positive element $(0, 1)$, which generates the quasi-atomic subgroup $0 \oplus \mathbb{Z}$. Lastly, $\mathbb{Q}$ has no minimal positive elements. Hence, $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Q}$ has quasi-atomic length 2, but $G(D)^* = \mathbb{Q} \neq \{1_G\}$.

We can realize this example constructively. Let $\mathbb{F}$ be any field and define $R' = \mathbb{F}[X, Y, Z]$. In the algebraic closure of the quotient field, $\overline{\mathbb{F}(X, Y, Z)}$ let $\mathcal{W} = \left\{Z^\alpha, \frac{Y}{X^j}, \frac{Z^\alpha}{Y^j} : \alpha \in \mathbb{Q}^+, j \in \mathbb{N}\right\}$. Set $D' = R'[\mathcal{W}] = \mathbb{F}[X, Y, Z, \mathcal{W}]$. In D' the ideal generated by monomials $\mathfrak{m} = (X, Y, Z, \mathcal{W})$ is maximal. Set $D = D'_{\mathfrak{m}}$.

Every element of D is associate to some monomial, so every element of D is of the form $uX^nY^mZ^\alpha$ for some $u \in U(D)$, $n \in \mathbb{N}_0$, $m \in \mathbb{N}_0$, $\alpha \in \mathbb{Q}^+$. However, there is a unique irreducible, X. Localizing at quasi-atoms yields a new unique irreducible, Y. Localizing again yields an antimatter domain in which every element is associate to some Z^α. In particular, each element of the form $X^nY^mZ^\alpha$ with $\alpha \neq 0$ is an antimatter element. △

Certainly this example also illustrates that quasi-atomic length does not determine whether antimatter elements exist, only whether the quasi-atomic sequences stabilize. However, information about the quasi-atomic length of elements does inform us on the quasi-atomic length of a domain (or po-group).

Lemma 3.2.16. *Let D be an integral domain such that each $0 \neq x \in D$ has $len_q(x) \in \mathbb{N}_0$. Let G be a directed po-group such that each $0 \neq x \in G^+$ has $len_q(x) \in \mathbb{N}_0$. Then $len_q(D) = \sup_{x\in D}\{len_q(x)\}$ and $len_q(G) = \sup_{x\in G^+}\{len_q(x)\}$.*

Proof. Let $x \in D$ such that $\text{len}_q(x) = N$. Since x is a non-unit in the $(N-1)^{th}$ stage and becomes a unit in the N^{th} stage of localization, $S_N \neq S_{N-1}$ so $\text{len}_q(D) \geqslant N$. On the other hand, if $\text{len}_q(D) = N$, then $S_N = S_{N+k}$ for each $k \geqslant 0$. In particular, any element of finite quasi-atomic length has at most N stages of localization before they become a unit. Thus, for each $x \in D$, $\text{len}_q(x) \leqslant N$, and so $\sup_{x\in D}\{\text{len}_q(x)\} \leqslant N$. The proof is similar, *mutatis mutandis*, for G. □

Note that we require the assumption that all elements in D or G^+ to have finite quasi-atomic length. Indeed, the po-group $G = \mathbb{Z} \oplus \mathbb{Q}$ ordered lexicographically has $\text{len}_q(G) = 1$ but any element of the form $x = (0, \frac{a}{b})$ with $\frac{a}{b} \neq 0$ has $\text{len}_q(x) = +\infty$.

We now see a few natural routes of inquiry. For one thing, we may ask whether an antimatter overring for D contains D^*, or we may ask whether the intersection of all antimatter rings containing D will also contain D^*. In fact, it seems this is reasonable, due to the universal properties of localization and direct limits, which we used to construct D^*. Unfortunately, we have the following counter-example to this idea.

Example 3.2.17. From $\overline{\mathbb{F}_2(X)}$, we select the set $\mathcal{X} = \{X^\alpha \mid \alpha \geqslant 1, \alpha \in \mathbb{Q}^+\}$, and $\mathcal{Y} = \{X^\alpha \mid 0 \neq \alpha \in \mathbb{Q}^+\}$. Define $R_1 = \mathbb{F}_2[\mathcal{X}]_{\mathfrak{m}}$ where $\mathfrak{m}$ is the ideal generated by $\mathcal{X}$ and $R_2 = \mathbb{F}_2[\mathcal{Y}]_{\mathfrak{n}}$ where $\mathfrak{n}$ is the ideal generated by $\mathcal{Y}$. Certainly $R_1 \subseteq R_2$ and R_2 is antimatter. Both have the same quotient field, in which every non-zero element is of the form X^r for some $r \in \mathbb{Q}$ so R_2 is an overring of R_1. However, R_1^* coincides with the quotient field, whereas $R_2^* = R_2$ since R_2 is antimatter.

In fact, it seems that attaining antimatter rings through the process of localization and the process of integral extensions may yield two related but wholly different approaches to constructing an antimatter ring containing some D.

Thinking about universality, since $\varinjlim D_{S_i} = D_{\varinjlim S_i}$, we may wonder if the quasi-atomic limit $G(D)^*$ coincide with the quasi-atomic limit $\mathfrak{G}(D^*) = G(D^*)$? This is, in fact, true. To establish this, we look toward direct limits; the quasi-atomic sequences are direct systems indexed by $\mathbb{N}_0$. Denote $\varinjlim D_{S_i}$ as D^* and denote $\varinjlim G$ as G^*. In Example A.27, we demonstrate that for an ascending chain of saturated multiplicatively closed sets, $S_0 \subseteq S_1 \subseteq \cdots$, $\varinjlim D_{S_i} = D_{\varinjlim S_i}$. We have a similar result for quotient po-groups and ascending chains of o-ideals, $H_0 \subseteq H_1 \subseteq \cdots$ in Lemma 3.2.18.

Lemma 3.2.18. *Let $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ be an ascending chain of o-ideals of a po-group G. Then $G/\varinjlim H_i \simeq \varinjlim G/H_i$.*

Proof. Note that $\varinjlim G/H_i$ comes equipped with o-epimorphisms $\phi_n : G/H_n \to \varinjlim G/H_i$. In particular, for $n = 0$, we have an o-epimorphism $\phi : G \to \varinjlim G/H_i$. Hence, $\varinjlim G/H_i \simeq G/\operatorname{Ker}(\phi)$. Denote $H' = \varinjlim H_i$. It is sufficient to show that $H' \simeq \operatorname{Ker}(\phi)$.

If $x \in H'$ then $x \in H_n$ for some n. We decompose $\phi : G \to \varinjlim G/H_i$ into the chain

$$G \xrightarrow{\pi_n} G/H_n \xrightarrow{\phi_n} G/\operatorname{Ker}(\phi).$$

Since $x \in H_n$, $\pi_n(x)$ is trivial so $\phi(x) = \phi_n \circ \pi_n(x)$ is trivial. On the other hand, let $x \in \operatorname{Ker}(\phi)$. For each n, since $H_n \subseteq H'$, we have the well-defined o-epimorphism $\psi_n : G/H_n \to G/H'$. Since $\varinjlim G/H_i$ is universal, the maps of the form $\psi_n : G/H_n \to G/H'$ can be written $\psi_n = \overline{\psi} \circ \phi_n$ where $\overline{\psi} : G/\operatorname{Ker}(\phi) \to G/H'$ and the maps $\phi_n : G/H_n \to G/\operatorname{Ker}(\phi)$ are associated with the direct limit $\varinjlim G/H_i \simeq G/\operatorname{Ker}(\phi)$. If $x \in \operatorname{Ker}(\phi)$, then $\phi_n(x + H_n)$ is trivial in $G/\operatorname{Ker}(\phi)$, and so $x \in \operatorname{Ker}(\psi_n) = H'$. □

This yields Theorem 3.2.19.

Theorem 3.2.19. $G(D^*) = G(D)^*$

Proof. Since D^* is the direct limit of D_{S_i}, $G(D^*)$ is the group of divisibility of this direct limit. Elaborated upon in A.27, we have that $D^* = D_{\varinjlim S_i}$. This is a localization at a saturated set. By Mott's correspondence $G(D^*) = G(D)/H$ for $H = \langle \nu(\varinjlim S_i) \rangle$, and according to Lemma 3.2.18, $G(D)^* = \varinjlim G(D)/H_i \simeq G(D)/\varinjlim H_i$.

It is sufficient to show that $\varinjlim H_i = \langle \nu(\varinjlim S_i) \rangle$. If $g \in (\varinjlim H_i)^+ = \cup_i H_i^+$, then $g \in H_n^+$ for some n. Hence, $g + H_{n-1}$ is quasi-atomic in $G(D)/H_{n-1} = G(D_{S_{n-1}})$. We may write $g = aU(D)$ for some $a \in D$ since g is non-negative; here, a is a unit in D_{S_n} since $g + H_{n-1}$ is quasi-atomic in $G(D)/H_{n-1}$. Hence, $a \in S_n \subseteq \varinjlim S_i$, so $g \in H = \langle \nu(\varinjlim S_i) \rangle$.

On the other hand, if $g \in \langle \nu(\varinjlim S_i) \rangle^+$, then g is a finite sum of non-negative elements, say $g = g_1 + g_2 + \cdots + g_n$ where each $g_j = a_j U(D)$ for some $a_j \in \varinjlim S_i = \cup_i S_i$. In particular each $a_j \in S_{n_j}$ for some $n_j \in \mathbb{N}_0$; hence, $g_j = a_j U(D) \in G(D)$ is in the kernel of the natural o-epimorphism $G(D) \to G(D)/H_{n_j}$. Thus, g is a sum of elements from the subgroups $\{H_n\}$, so $g \in \varinjlim H_i = \cup_i H_i$. □

Example 3.2.20. Let R be any n-dimensional discrete valuation domain. Then R is quasi-atomic in the limit and has quasi-atomic length n. On the other hand, let R be any n-dimensional valuation domain with value group $\mathbb{Q} \oplus \left(\oplus_{i=1}^{n-1} \mathbb{Z}\right)$ ordered lexicographically: declare $(a_i) \leqslant (b_i)$ if and only if, for some j with $1 \leqslant j < n$, if $j \neq n$ then $a_i = b_i$ for each $1 \leqslant i < j$ and $a_i < b_i$ for $i = j$ and, if $j = n$ then $a_i = b_i$ for each $1 \leqslant i < n$ and $a_n \leqslant b_n$. Then R also has quasi-atomic length n, but is antimatter in the limit.

Theorem 3.2.21. *Let D be an integral domain with at least one irreducible such that every proper overring of D is quasi-atomic. Then $len_q(D) \leqslant 2$ and D is quasi-atomic in the limit.*

Proof. We have the quasi-atomic localization sequence

$$D_{S_0} \subseteq D_{S_1} \subseteq D_{S_2} \subseteq \cdots$$

where $S_0 = U(D)$. If D is quasi-atomic, then the localization sequence is stable at degree 1 and $D^* = \mathbb{F}$, yielding the rather trivial sequence $D_{S_0} \subseteq \mathbb{F}$. Since D is not quasi-atomic and has at least one irreducible, D_{S_1} is a proper overring. Moreover, since every proper overring of D is quasi-atomic, then D_{S_1} is quasi-atomic, and so $D_{S_2} = \mathbb{F}$. □

The Krull dimension of a ring imposes constraints on factorization behavior in that ring, and consequently the Krull dimension of the ring is related to the stability of the quasi-atomic sequences. Recall Example 3.2.20, which provides an n-dimensional valuation domain with quasi-atomic length n. However, the relationship between Krull dimension and quasi-atomic length is not a direct one: recall the ring in Example 2.0.9 $R = \mathbb{F}[X, Y^{\alpha}, \frac{Y^{\alpha}}{X^{n}} : \alpha \in \mathbb{Q}^{+}, n \in \mathbb{N}]_{\mathfrak{m}}$. R has infinite Krull dimension, but $G(R) \simeq \mathbb{Q} \oplus \mathbb{Z}$, so $\text{len}_q(G(R)) = 1$. Valuation domains provide more control over the relationship between Krull dimension and quasi-atomic length.

Theorem 3.2.22. *Let V be a valuation domain. Then $\text{len}_q(V) \leqslant \dim(V)$.*

Proof. If $\dim(V) = +\infty$ then $\text{len}_q(V) < \dim(V)$ rather vacuously. If $\dim(V)$ is finite, then for any saturated multiplicatively closed set $S \subseteq V$ such that $U(V) \neq S$ and $0 \notin S$, $\dim(V_S) \leqslant \dim(V) - 1$. Hence the localization sequence $V_{S_0} \subseteq V_{S_1} \subseteq \cdots$ must stabilize before the N^{th} step. □

Of course, for integral domains that are not valuation domains, the relationship between Krull dimension and quasi-atomic length is not so neat. For an example, consider any infinite dimensional UFD, such as $R = \mathbb{F}[X_1, X_2, X_3, \ldots]$. Since R is a UFD, R is quasi-atomic. In particular, $\text{len}_q(R) = 1$, but $\dim(R) = +\infty$. However, much can be said from the context of valuation domains.

Corollary 3.2.23. *Let V be a valuation domain. If $\dim(V) = 0$ then $\text{len}_q(V) = 0$ and $V^* = V$. If $\dim(V) = 1$ and V has an irreducible element, then V is atomic, $\text{len}_q(V) = 1$, and V^* coincides with the quotient field of V. If $\dim(V) \geqslant 2$, then V is not quasi-atomic.*

Proof. If $\dim(V) = 0$ then V is a field, its own quotient field, and hence $V^* = V$ and $\text{len}_q(V) = 0$. On the other hand, if any ring R is quasi-atomic then $\text{len}_q(R) \leqslant 1$ (with inequality in the case of fields, which are antimatter and vacuously atomic).

All that remains is to consider the 1-dimensional case. If $\dim(V) = 1$ then V has prime spectrum $(0) \subseteq \mathfrak{m}$. If V has an irreducible, then this irreducible is unique since V is a valuation domain. Moreover, for any non-zero non-unit $y \in V$, we have that $\mathfrak{m} = \sqrt{(y)}$, and so $x \in \sqrt{(y)}$. In particular, there exists some $n \geqslant 0$ such that $x^n \in (y)$, so there exists some $r \in V$ such that $x^n = ry$; we select n to be minimal with respect to this property. We claim that r is a unit. Indeed, if r is a non-zero non-unit, then we have that $\mathfrak{m} = \sqrt{(r)}$, and so $x \in \sqrt{(r)}$. Thus, there exists some $m \geqslant 0$ such that $x^m \in (r)$, so there exists some $s \in V$ such that $x^m = sr$. In the quotient field of V, we have that $r = \frac{x^m}{s}$, so $x^n = \frac{x^m}{s} y$. Since V is an integral domain, elements cancel; we conclude that $m = 0$, for otherwise we contradict the minimality of n. Hence, we have that $sr = 1$ and so r is a unit and y is therefore atomic. □

Chapter 4

Cohomology Groups

We introduce a general method of constructing a cochain complex of D-modules from a sequence of D-module epimorphisms. We show that applying this process to sequences of po-group o-epimorphisms results in a cochain complex of po-group o-epimorphisms, extracting cochain complexes of po-group o-epimorphisms from the quasi-atomic quotient sequence.

4.1 Cochain complexes and D-module epimorphisms

Let $G_0, G_1, G_2, \ldots$ be a sequence of D-modules together with D-module epimorphisms. For each $n \geqslant 0$, define $Q_n := \mathrm{Ker}(\pi_n)$. Consider an arbitrary submodule $A_n \subseteq Q_n$. Since each π_n is a D-module epimorphism, we have that the sets $\pi_n^{-1}(A_{n+1})$ and $\pi_n^{-1}(Q_{n+1})$ are D-submodules of G_n. Since $A_{n+1} \subseteq Q_{n+1}$, we have that $\pi_n^{-1}(A_{n+1}) \subseteq \pi_n^{-1}(Q_{n+1})$. Define $\widehat{Q}_n = \pi_n^{-1}(Q_{n+1})$ and $\widehat{A}_n = \pi_n^{-1}(A_{n+1})$. We have the chain of D-submodules $A_n \subseteq Q_n \subseteq \widehat{A}_n \subseteq \widehat{Q}_n \subseteq G_n$; see Figure 4.1.

Now for each $n \in \mathbb{N}_0$, define $d_n^{\widehat{Q}}$ to be the map $d_n^{\widehat{Q}} : \widehat{Q}_n \to \widehat{Q}_{n+1}$ obtained by restricting the domain and codomain of π_n. The resulting sequence $\cdots \xrightarrow{d_{n-2}^{\widehat{Q}}} \widehat{Q}_{n-1} \xrightarrow{d_{n-1}^{\widehat{Q}}} \widehat{Q}_n \xrightarrow{d_n^{\widehat{Q}}} \widehat{Q}_{n+1} \xrightarrow{d_{n+1}^{\widehat{Q}}} \cdots$ is thus a cochain complex; inspecting Figure 4.1, note that composing any two adjacent maps gives the zero map. For any $x \in \widehat{Q}_{n-1}$, $d_n^{\widehat{Q}} \circ d_{n-1}^{\widehat{Q}}(x) = d_n^{\widehat{Q}}\left(d_{n-1}^{\widehat{Q}}(x)\right)$, but $d_{n-1}^{\widehat{Q}}(x) \in Q_n = \mathrm{Ker}(\pi_n) = \mathrm{Ker}(d_n^{\widehat{Q}})$, so $d_n^{\widehat{Q}} \circ d_{n-1}^{\widehat{Q}}(x)$ is trivial. We denote this cochain complex $\widehat{Q}_\bullet$. Observe that $\widehat{Q}_\bullet$ is exact, since $\mathrm{Ker}(d_n^{\widehat{Q}}) = \mathrm{Im}(d_{n-1}^{\widehat{Q}})$.

Similarly, we define $d_n^{\widehat{A}} : \widehat{A}_n \to \widehat{A}_{n+1}$, $d_n^Q : Q_n \to Q_{n+1}$, and $d_n^A : A_n \to A_{n+1}$ by restricting π_n appropriately. For each of these sequences of maps, inspecting Figure 4.1 reveals again that

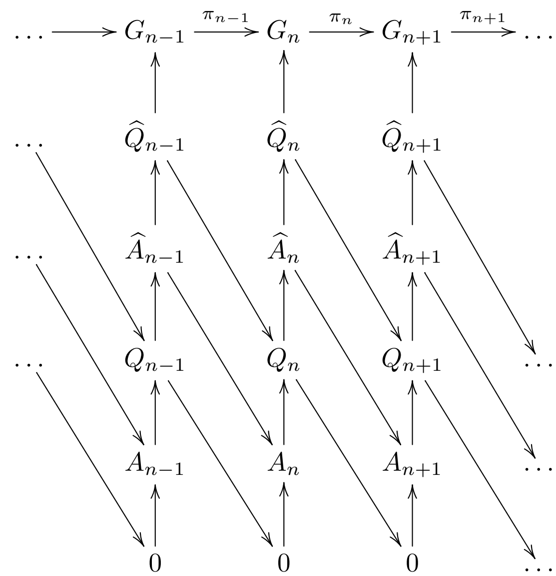


Figure 4.1: A detailed commutative diagram of an arbitrary sequence of D-module epimorphisms with D-submodules $A_n \subseteq Q_n = \mathrm{Ker}(\pi_n)$ and their respective inverse images $\widehat{A}_n$, $\widehat{Q}_n$.

composing any two adjacent maps results in the zero map. Hence, we obtain the cochain complexes $\widehat{A}_\bullet$, $Q_\bullet$, and $A_\bullet$.

$$\widehat{Q}_\bullet = \cdots \longrightarrow \widehat{Q}_{n-1} \longrightarrow \widehat{Q}_n \longrightarrow \widehat{Q}_{n+1} \longrightarrow \cdots \tag{4.1.1}$$

$$\widehat{A}_\bullet = \cdots \longrightarrow \widehat{A}_{n-1} \longrightarrow \widehat{A}_n \longrightarrow \widehat{A}_{n+1} \longrightarrow \cdots \tag{4.1.2}$$

$$Q_\bullet = \cdots \longrightarrow Q_{n-1} \longrightarrow Q_n \longrightarrow Q_{n+1} \longrightarrow \cdots \tag{4.1.3}$$

$$A_\bullet = \cdots \longrightarrow A_{n-1} \longrightarrow A_n \longrightarrow A_{n+1} \longrightarrow \cdots \tag{4.1.4}$$

We use the differentials in these cochain complexes to induce differentials $\frac{\widehat{Q}_n}{A_n} \to \frac{\widehat{Q}_{n+1}}{A_{n+1}}$, $\frac{\widehat{Q}_n}{Q_n} \to \frac{\widehat{Q}_{n+1}}{Q_{n+1}}$, and so on to form cochain complexes. This construction follows the following general format: for each $n \in \mathbb{N}_0$, we always have have the canonical D-module epimorphism of the form $\phi_n : \widehat{Q}_n \twoheadrightarrow \widehat{Q}_n/A_n$. Moreover, the composition $\phi_{n+1} \circ d_n^{\widehat{Q}} : \widehat{Q}_n \to \frac{\widehat{Q}_{n+1}}{A_{n+1}}$ is a D-module epimorphism. Since $A_n \subseteq \mathrm{Ker}(\phi_{n+1} \circ d_n^{\widehat{Q}})$, the composition $\phi_{n+1} \circ d_n^{\widehat{Q}}$ factors through ϕ_n, so there exists a D-module epimorphism $\overline{d}_n : \frac{\widehat{Q}_n}{A_n} \to \frac{\widehat{Q}_{n+1}}{A_{n+1}}$ such that $\phi_{n+1} \circ d_n^{\widehat{Q}} = \overline{d}_n \circ \phi_n$. We repeat the process to construct differentials for the complexes $\frac{\widehat{Q}_\bullet}{A_\bullet}$, $\frac{\widehat{Q}_\bullet}{Q_\bullet}$, $\frac{\widehat{Q}_\bullet}{\widehat{A}_\bullet}$, $\frac{\widehat{A}_\bullet}{A_\bullet}$, $\frac{\widehat{A}_\bullet}{Q_\bullet}$, $\frac{Q_\bullet}{A_\bullet}$. This yields the following nontrivial quotient cochain complexes of D-modules.

$$\widehat{Q}_\bullet/A_\bullet = \cdots \longrightarrow \widehat{Q}_{n-1}/A_{n-1} \longrightarrow \widehat{Q}_n/A_n \longrightarrow \widehat{Q}_{n+1}/A_{n+1} \longrightarrow \cdots \tag{4.1.5}$$

$$\widehat{Q}_\bullet/Q_\bullet = \cdots \longrightarrow \widehat{Q}_{n-1}/Q_{n-1} \longrightarrow \widehat{Q}_n/Q_n \longrightarrow \widehat{Q}_{n+1}/Q_{n+1} \longrightarrow \cdots \tag{4.1.6}$$

$$\widehat{Q}_\bullet/\widehat{A}_\bullet = \cdots \longrightarrow \widehat{Q}_{n-1}/\widehat{A}_{n-1} \longrightarrow \widehat{Q}_n/\widehat{A}_n \longrightarrow \widehat{Q}_{n+1}/\widehat{A}_{n+1} \longrightarrow \cdots \tag{4.1.7}$$

$$\widehat{A}_\bullet/A_\bullet = \cdots \longrightarrow \widehat{A}_{n-1}/A_{n-1} \longrightarrow \widehat{A}_n/A_n \longrightarrow \widehat{A}_{n+1}/A_{n+1} \longrightarrow \cdots \tag{4.1.8}$$

$$\widehat{A}_\bullet/Q_\bullet = \cdots \longrightarrow \widehat{A}_{n-1}/Q_{n-1} \longrightarrow \widehat{A}_n/Q_n \longrightarrow \widehat{A}_{n+1}/Q_{n+1} \longrightarrow \cdots \tag{4.1.9}$$

$$Q_\bullet/A_\bullet = \cdots \longrightarrow Q_{n-1}/A_{n-1} \longrightarrow Q_n/A_n \longrightarrow Q_{n+1}/A_{n+1} \longrightarrow \cdots \tag{4.1.10}$$

Repeating the above process again yields quotients of quotients $\frac{\widehat{Q}_\bullet/A_\bullet}{\widehat{A}_\bullet/A_\bullet}$, $\frac{\widehat{Q}_\bullet/A_\bullet}{Q_\bullet/A_\bullet}$, $\frac{\widehat{A}_\bullet/A_\bullet}{Q_\bullet/A_\bullet}$, $\frac{\widehat{Q}_\bullet/Q_\bullet}{\widehat{A}_\bullet/Q_\bullet}$, but these are chain isomorphic in each degree to the cochain complexes $\widehat{Q}_\bullet/\widehat{A}_\bullet$, $\widehat{Q}_\bullet/Q_\bullet$, $\widehat{A}_\bullet/Q_\bullet$, and $\widehat{Q}_\bullet/\widehat{A}_\bullet$, respectively.

Observe that each $\widehat{Q}_n/Q_n = \pi_n(\widehat{Q}_n) = Q_{n+1}$ and each $\widehat{A}_n/Q_n = \delta_n(\widehat{A}_n) = A_{n+1}$. Hence, the complex $\widehat{A}_\bullet/Q_\bullet$ in the n^{th} degree is precisely the complex $A_\bullet$ in the $(n+1)^{th}$ degree, which is to say $\widehat{A}_\bullet/Q_\bullet$ is $A_\bullet$ shifted to the left by one degree. Similarly, $\widehat{Q}_\bullet/Q_\bullet$ is the complex $Q_\bullet$ shifted to the left by one, and $\widehat{Q}_\bullet/\widehat{A}_\bullet$ is the complex $Q_\bullet/A_\bullet$ shifted to the left by one.

Lemma 4.1.1. *The sequences $\widehat{Q}_\bullet$ and $\widehat{Q}_\bullet/A_\bullet$ are exact.*

Proof. Since $\widehat{Q}_n = \pi^{-1}(\mathrm{Ker}(\pi_{n+1})$, we have that $\mathrm{Ker}(d_{n+1}^{\widehat{Q}}) = \mathrm{Im}(d_n^{\widehat{Q}})$. Since $A_n \subseteq Q_n = \mathrm{Ker}(\pi_n)$, we have that the quotient complex is also exact. ∎

Lemma 4.1.2. *The complexes $A_\bullet$, $Q_\bullet$, $Q_\bullet/A_\bullet$, $\widehat{A}_\bullet/Q_\bullet$, $\widehat{A}_\bullet/A_\bullet$, $\widehat{Q}_\bullet/Q_\bullet$, and $\widehat{Q}_\bullet/\widehat{A}_\bullet$ are trivial.*

Proof. Each map is induced by π_n with kernel Q_n. The complexes $A_\bullet$, $Q_\bullet$, and $Q_\bullet/A_\bullet$ are trivial. Furthermore, since $\pi_n(\widehat{A}_n) = A_{n+1} \subseteq Q_{n+1}$, we have that $\widehat{A}_\bullet/Q_\bullet$ and $\widehat{A}_\bullet/A_\bullet$ are trivial. Similarly, $\widehat{Q}_\bullet/Q_\bullet$ is also trivial. Since $\pi_n(\widehat{Q}_n) = Qn + 1 \subseteq \widehat{A}_{n+1}$, we have that $\widehat{Q}_\bullet/\widehat{A}_\bullet$ is trivial. ∎

Lemma 4.1.3. *If the complex $A_\bullet \neq Q_\bullet$ then $\widehat{A}_\bullet$ is not exact and is non-trivial.*

Proof. If $\widehat{A}_\bullet$ is exact then $A_n = \mathrm{Ker}(\pi_n) = Q_n$ and $A_\bullet = Q_\bullet$. ∎

4.2 Quasi-Atomic Cohomology Groups

Consider the special case that each G_n is a po-group, each π_n is an o-epimorphism with $\mathrm{Ker}(\pi_n) = Q(G_n)$ and each $A_n = A(G_n)$. If we only wish to consider cochain complexes of po-groups, then we must only consider the complexes above whose kernels are o-ideals. Quotient complexes with kernels from $A_\bullet$ or $\widehat{A}_\bullet$ are not partially ordered unless $A_\bullet = Q_\bullet$. Hence, although the quotient cochain complexes $\widehat{Q}_\bullet/A_\bullet$, $\widehat{Q}_\bullet/\widehat{A}_\bullet$, $\widehat{A}_\bullet/A_\bullet$, and $Q_\bullet/A_\bullet$ are cochain complexes of abelian groups, they lack convexity in their kernels in general, so they are sequences of pre-ordered groups not po-groups.

This leaves us with $A_\bullet$, $Q_\bullet$, $\widehat{A}_\bullet$, $\widehat{Q}_\bullet$, $\widehat{A}_\bullet/Q_\bullet$, and $\widehat{Q}_\bullet/Q_\bullet$. Hence, we have the quotient cochain complexes of po-groups $\widehat{Q}_\bullet/Q_\bullet$, which is the shift of $Q_\bullet$, and $\widehat{A}_\bullet/Q_\bullet$, which is the shift of $A_\bullet$. The cohomology groups relevant to po-groups, then, are the trivial cochain complexes $A_\bullet$, $Q_\bullet$ (together with their shifts), the exact cochain complex $\widehat{Q}_\bullet$, and the non-trivial non-exact $\widehat{A}_\bullet$.

Definition 4.2.1. For a cochain complex $X_\bullet$ of po-group o-epimorphisms obtained from a quasi-atomic quotient sequence $G = G_0 \twoheadrightarrow G_1 \twoheadrightarrow \cdots$, define the n^{th} cohomology group for $X_\bullet$ as $H^i(G, X_\bullet) = \mathrm{Ker}(d^X_{n+1})/\mathrm{Im}(d^X_n)$.

From the cochain complexes of po-groups obtained from the quasi-atomic quotient sequence, we obtain the cohomology groups in Table 4.1; in general, these are not po-groups since each $H^i(G, \widehat{A}_\bullet) = Q_i/A_i$ and each A_i is not convex in Q_i in general. We have the following two short exact sequences of cochain complexes of po-group o-homomorphisms

Complex	$i = 0$	$i \geqslant 1$
$\widehat{Q}_\bullet$	$H^0(G, \widehat{Q}_\bullet) = Q_0$	$H^i(G, \widehat{Q}_\bullet) = Q_i/Q_i \cong 0$
$\widehat{A}_\bullet$	$H^0(G, \widehat{A}_\bullet) = Q_0$	$H^i(G, \widehat{A}_\bullet) = Q_i/A_i$
$Q_\bullet$	$H^0(G, Q_\bullet) = Q_0$	$H^i(G, Q_\bullet) = Q_i/(1) \cong Q_i$
$A_\bullet$	$H^0(G, A_\bullet) = A_0$	$H^i(G, A_\bullet) = A_i/(1) \cong A_i$
$\widehat{Q}_\bullet/Q_\bullet$	$H^0(G, \widehat{Q}_\bullet/Q_\bullet) = Q_1$	$H^i(G, \widehat{Q}_\bullet/Q_\bullet) = Q_{i+1}/(1) \cong Q_{i+1}$
$\widehat{A}_\bullet/Q_\bullet$	$H^0(G, \widehat{A}_\bullet/Q_\bullet) = A_1$	$H^i(G, \widehat{A}_\bullet/Q_\bullet) = A_{i+1}/(1) \cong A_{i+1}$

Table 4.1: Cohomology groups from cochain complexes 4.1.3, 4.1.4, 4.1.6, 4.1.9, 4.1.2, and 4.1.1.

$$0 \longrightarrow Q_\bullet \longrightarrow \widehat{A}_\bullet \longrightarrow \widehat{A}_\bullet/Q_\bullet \longrightarrow 0 \tag{4.2.1}$$

$$0 \longrightarrow Q_\bullet \longrightarrow \widehat{Q}_\bullet \longrightarrow \widehat{Q}_\bullet/Q_\bullet \longrightarrow 0 \tag{4.2.2}$$

Following Theorem C.6, short exact sequences 4.2.1 and 4.2.2 yield long exact sequences in cohomology 4.2.3 and 4.2.4, which are abelian groups in general but may not be partially ordered.

$$\cdots \longrightarrow H^i(G, Q_\bullet) \longrightarrow H^i(G, \widehat{A}_\bullet) \longrightarrow H^i(G, \widehat{A}_\bullet/Q_\bullet) \longrightarrow H^{i+1}(G, Q_\bullet) \longrightarrow \cdots \tag{4.2.3}$$

$$\cdots \longrightarrow H^i(G, Q_\bullet) \longrightarrow H^i(G, \widehat{Q}_\bullet) \longrightarrow H^i(G, \widehat{Q}_\bullet/Q_\bullet) \longrightarrow H^{i+1}(G, Q_\bullet) \longrightarrow \cdots \tag{4.2.4}$$

Following Table 4.1, and for each $i \geqslant 1$ defining $\epsilon_i : A_i \hookrightarrow Q_i$ as the canonical inclusion and $\phi_i : Q_i \twoheadrightarrow Q_i/A_i$ as the canonical surjection, the long exact sequence 4.2.3 is written as the long exact sequence

$$Q_0 \xrightarrow{=} Q_0 \xrightarrow{0} A_1 \xrightarrow{\epsilon_1} Q_1 \xrightarrow{\phi_1} Q_1/A_1 \xrightarrow{0} A_2 \xrightarrow{\epsilon_2} \cdots \tag{4.2.5}$$

This long exact sequence splits into a countable set of short exact sequences of abelian groups, the first of the form $0 \to Q_0 \to Q_0 \to 0$, and the rest of the form $0 \to A_i \xrightarrow{\epsilon_i} Q_i \xrightarrow{\phi_i} Q_i/A_i \to 0$ for $i \in \mathbb{N}$. Similarly, following Table 4.1, the long exact sequence 4.2.4 is written as

$$Q_0 \xrightarrow{=} Q_0 \xrightarrow{0} Q_1 \xrightarrow{=} Q_1 \xrightarrow{0} 0 \longrightarrow Q_2 \xrightarrow{=} Q_2 \xrightarrow{0} 0 \longrightarrow \cdots \tag{4.2.6}$$

This long exact sequence splits into a countable set of short exact sequences of abelian groups of the form $0 \to Q_i \to Q_i \to 0$ for each $i \in \mathbb{N}_0$. In summary, the cochain complexes obtained from the quasi-atomic quotient sequence leads naturally to the long exact sequences in cohomology, which leads to the countable short exact sequences of abelian groups from 4.2.5

$$0 \to A_n \to Q_n \to Q_n/A_n \to 0$$

and exploring these short exact sequences reduces to exploring the atomic subgroup A_n and its interrelation with the quasi-atomic subgroup Q_n.

4.3 Interrelations Between Cohomology and Factorization

The sequences $0 \to A_n \to Q_n \to Q_n/A_n \to 0$ are suggestively descriptive of the gap between atomic po-groups and quasi-atomic po-groups; this gap seems to be related to peculiar factorization behavior. For the section, we violate our previous notational convention and write po-groups multiplicatively with identity 1_G, so as to exploit the multiplicative notation inherited from D and its quotient field $\mathbb{F}$. Consider, for example, torsion in $\frac{Q_n}{A_n} = H^n(G, \widehat{A}_\bullet) \subseteq \frac{G_n}{A_n} \simeq \frac{G_{n-1}/Q_{n-1}}{\widehat{A}_n/Q_{n-1}}$. Let $x' \in G(D)^+$ such that $x = \pi_{n-1} \circ \pi_{n-2} \circ \cdots \circ \pi_1 \circ \pi_0(x') \in Q_n$ and such that xA_n and is k-torsion. Since $x' \in G(D)^+$, we can write $x' = yU(D)$ for some $y \in D$. We see that $xA_n = yA_n$.

Since x is k-torsion, $(yU(D))^k A_n = A_n$. Thus, there exists some $\alpha, \beta \in A_n$ such that $(yU(D))^k \alpha = \beta$. Since $a, b \in A_n \subseteq G_n \simeq \frac{G_{n-1}}{Q_{n-1}}$, we can write $a = \frac{\alpha_1}{\beta_1} Q_{n-1}$, $b = \frac{\gamma_1}{\delta_1} Q_{n-1}$ for some $\alpha_1, \beta_1, \gamma_1, \delta_1 \in D$ such that $\frac{\alpha_1}{\beta_1} Q_{n-1} \in A_n$ and $\frac{\gamma_1}{\delta_1} \in A_n$. Hence $(yU(D))^k \frac{\alpha_1}{\beta_1} Q_{n-1} = \frac{\gamma_1}{\delta_1} Q_{n-1}$. Thus, there exists some $a', b' \in Q_{n-1} \subseteq G_{n-2}/Q_{n-2}$ so that $(\frac{g_1}{g_2} U(D))^k \frac{\alpha_1}{\beta_1} a' Q_{n-2} = \frac{\gamma_1}{\delta_1} b' Q_{n-2}$. We write $a' = \frac{\alpha_2}{\beta_2} Q_{n-2}$ and $b' = \frac{\gamma_2}{\delta_2} Q_{n-2}$. Proceeding like this iteratively, we terminate in a finite number of steps at $G_0 = G(D)$. That is to say, we have collected $\alpha_1, \beta_1, \gamma_1, \delta_1$ from A_n, and we have collected each other $\alpha_i, \beta_i, \gamma_i, \delta_i$ from Q_{n-i}. We obtain that

$$y^k \frac{\alpha_1 \alpha_2 \cdots \alpha_n}{\beta_1 \beta_2 \cdots \beta_n} U(D) = \frac{\gamma_1 \gamma_2 \cdots \gamma_n}{\delta_1 \delta_2 \cdots \delta_n} U(D)$$

where $y, \alpha_i, \beta_i, \gamma_i, \delta_i \in D$, where $\frac{\alpha_1}{\beta_1} Q_{n-1}, \frac{\gamma_1}{\delta_1} Q_{n-1} \in A_n$, and each $\frac{\alpha_i}{\beta_i} Q_{n-i}, \frac{\gamma_i}{\delta_i} Q_{n-i} \in Q_{n-i+1}$ for each $i \geqslant 2$. In particular, we have some unit u such that

$$y^k = u \frac{\beta_1 \cdots \beta_n \cdot \gamma_1 \cdots \gamma_n}{\alpha_1 \cdots \alpha_n \cdot \gamma_1 \cdots \gamma_n}$$

That is to say, y is the k^{th} root of the right hand side; in particular, if y k-torsion in Q_n/A_n, then y is the k^{th} root of some ratio of elements, each with quasi-atomic length at most n.

Constructing examples of this phenomenon requires some care. We use the results of [29], [9], and [8] to construct this example, where we iteratively adjoin indeterminates to a ring. This allos us to strictly control the irreducible elements in each iteration.

Example 4.3.1. Similar to the ring from Example 2.0.6, but with a different choice of base field, let $R = (\mathbb{Q}[X, Y, \mathcal{Y}])_{\mathfrak{m}}$ where $\mathcal{Y} = \{Y^\alpha\}_{\alpha \in \mathbb{Q}^+}$. There is nothing special about $\mathbb{Q}$ in this example; any field $\mathbb{F}$ such that $\sqrt{-1} \notin \mathbb{F}$ will do the trick. Consider the quotient ring R/I where $I = (X^2 + Y^2)$.

Set $R_0 = R/I$. I is a prime ideal, so R/I is an integral domain. To see this, we demonstrate I is a prime ideal. Assume $(X^2 + Y^2) \mid fg$ in R. We write $f = \sum_{n=1}^{N} c_n X^{\alpha_n} Y^{\beta_n/\gamma_n}$ where $N \in \mathbb{N}$, each $\alpha_n, \beta_n, \gamma_n$ are non-negative integers, each $\gamma_n \neq 0$, and each $c_n \in \mathbb{Q}$. Similarly, we can write

$g = \sum_{m=1}^{M} c'_m X^{\alpha'_m} Y^{\beta'_m/\gamma'_m}$ where $M \in \mathbb{N}$, each $\alpha'_m, \beta'_m, \gamma'_m$ are non-negative integers, each $\gamma'_m \neq 0$, and each $c'_m \in \mathbb{Q}$. Recall that $\mathbb{Q}[X, Y]$ is a UFD and that for a non-negative integer, L, the substitution map $\phi_L : R_0 \to R_0$ defined by mapping $y \mapsto y^L$ and fixing all other elements is a ring endomorphism. Thus, there exists a sufficiently large choice of L such that $\phi_L(fg) = \phi_L(f)\phi_L(g)$ has all integer exponents, say $\phi_L(fg) = \sum_{k=1}^{K} c''_k X^{\alpha''_k} Y^{\delta_k}$, and so may be treated as an element of $\mathbb{Q}[X, Y]$. Certainly, then, if $(X^2 + Y^2) \mid fg$ in R, then $\phi_L(X^2 + Y^2) = (X^2 + Y^{2L}) \mid \phi_L(fg)$ as elements of $\mathbb{Q}[X, Y]$. Clearly, then, since X^2+Y^{2L} is irreducible in $\mathbb{Q}[X, Y]$, which is a UFD, it must be prime; since $(X^2 + Y^{2L}) \mid \phi_L(f)\phi_L(g)$ as elements of $\mathbb{Q}[X, Y]$, we have that $(X^2 + Y^{2L}) \mid \phi_L(f)$ or $(X^2 + Y^{2L}) \mid \phi_L(g)$. Thus, $(X^2 + Y^2) \mid f$ or $(X^2 + Y^2) \mid g$ in R. Hence $(X^2 + Y^2)$ is prime.

We now construct a ring from R_0 in which X is the only irreducible monomial. Let $A_0 \subseteq R_0$ be the set of all irreducibles in R/I that are not associate to X (we denote associates with the symbol $\sim$). For each associate class of irreducibles in A_0, select a representative irreducible α and an indeterminate over R_0, say Z_α. In the quotient field of $R_0[Z_\alpha]$, select the element $\frac{\alpha}{Z_\alpha}$. Collect these together into the set $\mathcal{Z}_0 = \left\{Z_\alpha, \frac{\alpha}{Z_\alpha} \mid \alpha \in \mathrm{Irr}(R_0), \alpha \not\sim X\right\}$ and define $R_1 = R_0[\mathcal{Z}_0]$. Following [9, Lemma 2.5], we have that $U(R_2) = U(R_1)$ so these new divisibility relationship are, indeed, nontrivial. Further, also from [9, Lemma 2.6], since X is non-associate to any $\alpha \in A_0$ by construction, X remains irreducible in the domain R_1. We iteratively construct the sets

$$\mathcal{Z}_i = \left\{Z_\alpha, \frac{\alpha}{Z_\alpha} \mid \alpha \in \mathrm{Irr}(R_i), \alpha \not\sim X\right\}$$

and define $R_{i+1} = R_i[\mathcal{Z}_i]$. This provides the direct system $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$.

The direct limit of this system $\varinjlim R_i$ is an integral domain in which X is uniquely irreducible, so almost atomic elements must be associate to some value of Moreover, $X^2 + Y^2 = 0$ remains a valid expression in the direct limit. Thus, each Y^α is not almost atomic for any $0 < \alpha < 2$, for otherwise Y^α can be written as an integer power of X. However, since $Y^2 = -X^2$, Y^2 is atomic so each Y^α for $0 < \alpha < 2$ is quasi-atomic but not almost atomic. For example, elements such as $Y^{1/3}$ are not almost atomic, but $(Y^{1/3})^6$ is atomic. In $Q(G(D))/A(G(D))$, $Y^{1/3}A(G(D))$ has finite order. △

Chapter 5

Structure Within Partially Ordered Abelian Groups

In this section, we exploit split exactness in short exact sequences of o-homomorphisms to develop structure theorems. The basic idea behind this section is to apply ordinary group-theoretic structure theorems to the po-group setting, with an eye toward groups of divisibility. Recall the maps in the category of po-groups are handled differently than in the category of groups, to study the splitting of exact sequences of po-groups we must pay careful attention to the maps.

5.1 Factoring a po-group

Perhaps we wish to write every po-group as the group of divisibility of an integral domain. This is not always possible, as every group of divisibility is directed, and it is easy to construct po-groups that are not directed:

Example 5.1.1. Let $G = \mathbb{Q}$ with partial ordering $a \leqslant b \Leftrightarrow b - a \in \mathbb{N}$. The positive elements of G are precisely $\mathbb{N}$. Hence $A(G) = Q(G) = \mathbb{Z}$. Note that no direct sum decomposition is possible, since $\mathbb{Z}$ is not a direct summand of $\mathbb{Q}$. However, this po-group under the given order is not generated by its positive elements and hence is not a group of divisibility for any domain. $\triangle$

In [26], Mott and Hill independently proved the following example of a directed po-group that is not a group of divisibility, first put forth by Jaffard in [20].

Example 5.1.2. Let $G = \mathbb{Z} \oplus \mathbb{Z}$ under the product order and let J be the subgroup of G such that $(a, b) \in J$ if and only if $a + b \in 2\mathbb{Z}$. Let J inherit the product order. Then J is not a group of divisibility. To see this, we use the property of semi-valuations that if $x, y \in D$ then $\nu(x + y)$ is an upper bound on all lower bounds of $\nu(x)$ and $\nu(y)$.

Note that $(3, 1)$ and $(2, 2)$ are both elements of J that are mutually incomparable under the inherited order. However, if $J = G(D)$ for some D, then there must exist some $x, y \in D$ such that $\nu(x) = (3, 1)$ and $\nu(y) = (2, 2)$. Of course, $\nu(x + y)$ must be an upper bound for all mutual lower bounds of $\nu(x)$ and $\nu(y)$ in the group of divisibility. Hence, if $\nu(x+y) = (a, b)$, then $a \geqslant 2$ and $b \geqslant 1$. Moreover, if $a = 2$ then $b \geqslant 1$, and if $b = 1$ then $a \geqslant 3$. In each and every case, $\nu(x + y) \geqslant \nu(x)$ or $\nu(x + y) \geqslant \nu(y)$, and hence $(x + y)D \subseteq xD$ or $(x + y)D \subseteq yD$. If $(x + y)D \subseteq (x)D$, then $x + y \in (x)D$ so $x + y = rx$ for some $r \in D$. Thus, $y = (r - 1)x \in (x)D$ and therefore $(y)D \subseteq (x)D$. If $(x + y)D \subseteq (y)D$ then we similarly conclude $(x)D \subseteq (y)D$. In either case, $\nu(x)$ and $\nu(y)$ are comparable, contradicting our choice of x and y. △

When we are provided the luxury that our po-group is a group of divisibility, we may wish to decompose $G(D)$ into a product (sum, coproduct, etc.) of other po-groups or other groups of divisibility. Unfortunately, groups of divisibility may not be written as a sum of an atomic part and an antimatter part. For an example, consider the ring constructed in Example 4.3.1. We cannot write the set of non-atomic elements of $G(R)$ as a summand of $G(R)$ since the quasi-atoms "run into" the atoms.

The hypothesis that every group of divisibility splits into a direct sum of po-groups, then, is demonstrated to be false by Example 4.3.1. Now the question reduces to assessing when the group of divisibility splits. Recall a short exact sequence of groups splits when the associated surjection has a right inverse. We expand our inquiry to ask when groups of divisibility (or perhaps more generally partially ordered abelian groups) split. Of course, to determine structure of groups it is enough to determine splitting homomorphisms; in po-groups, we also require a splitting order.

Definition 5.1.3. Let G be a po-group and $H \subseteq G$ be an o-ideal, and let $\pi : G \to G/H$ be the natural o-epimorphism. If there exists a right-inverse $\pi^{-1} : \frac{G}{H} \to G$, then $H \oplus \frac{G}{H}$ admits the *induced splitting order* defined by $(h_1, g_1+H) \leqslant (h_2, g_2+H)$ if and only if $h_1+\pi^{-1}(g_1+H) \leqslant h_2+\pi^{-1}(g_2+H)$ where π is the natural o-epimorphism $\pi : G \to G/H$.

The existence of right-inverses for an o-epimorphism plays a central role in this splitting order, clearly.

Lemma 5.1.4. *Let G be a po-group and $H \subseteq G$ be an o-ideal, and let $\pi : G \to G/H$ be the natural o-epimorphism. If there exists a right-inverse $\pi^{-1} : \frac{G}{H} \to G$, the induced splitting order on $H \oplus \frac{G}{H}$ is a partial order.*

Proof. For reflexivity, it is clear that $h+\pi^{-1}(g+H) = h+\pi^{-1}(g+H)$ so $(h, g+H) \leqslant (h, g+H)$. For transitivity, assume $(h_1, g_1+H) \leqslant (h_2, g_2+H) \leqslant (h_3, g_3+H)$. Since $(h_1, g_1+H) \leqslant (h_2, g_2+H)$, we have that $h_1+\pi^{-1}(g_1+H) \leqslant h_2+\pi^{-1}(g_2+H)$ in G. Since $(h_2, g_2+H) \leqslant (h_3, g_3+H)$, we have that $h_2+\pi^{-1}(g_2+H) \leqslant h_3+\pi^{-1}(g_3+H)$. But $\leqslant$ is transitive in G so $h_1+\pi^{-1}(g_1+H) \leqslant h_3+\pi^{-1}(g_3+H)$ in G and so $(h_1, g_1 + H) \leqslant (h_3, g_3 + H)$ in $H \oplus \frac{G}{H}$.

We verify the induced splitting order is antisymmetric. To see this, assume that we have $(h_2, g_1+H) \leqslant (h_2, g_2+H)$ in $H \oplus \frac{G}{H}$ and vice versa. Then $h_1+\pi^{-1}(g_1+H) \leqslant_G h_2+\pi^{-1}(g_2+H)$ and vice versa. By antisymmetry in $\leqslant_G$, we have that $h_1+\pi^{-1}(g_1+H) = h_2+\pi^{-1}(g_2+H)$. In particular, since π^{-1} is an o-homomorphism, $h_2 - h_1 = \pi^{-1}(g_1 - g_2 + H) \in H$. Hence, $\pi^{-1}(g_1 - g_2 + H) \in H$. But then $\pi \circ \pi^{-1}(g_1 - g_2 + H) = H$. Since $\pi^{-1} : \frac{G}{H} \to G$ is the right inverse of $\pi : G \to G/H$, we have that $\pi \circ \pi^{-1} = \mathrm{id}_{G/H}$, so we have that $g_1 - g_2 + H = H$ and therefore $g_1 + H = g_2 + H$. Now $\pi^{-1}(g_1 + H) = \pi^{-1}(g_2 + H)$ so we obtain the following.

$$
\begin{aligned}
h_1 + \pi^{-1}(g_1 + H) =& h_2 + \pi^{-1}(g_2 + H) \\
h_1 + \pi^{-1}(g_1 + H) =& h_2 + \pi^{-1}(g_1 + H) \\
h_1 =& h_2
\end{aligned}
$$

Hence, $(h_1, g_1 + H) = (h_2, g_2 + H)$, demonstrating antisymmetry. □

Theorem 5.1.5. *Let H be a convex subgroup of a po-group G. Consider the natural short exact sequence of po-group o-homomorphisms:*

$$0 \longrightarrow H \xrightarrow{\iota} G \xrightarrow{\pi} G/H \longrightarrow 0$$

If there exists a right-inverse o-homomorphism $\pi \circ \pi^{-1} = id_{G/H}$ then $H \oplus \frac{G}{H}$ is a po-group under the partial order defined by $(h_1, g_1+H) \leqslant (h_2, g_2+H)$ if and only if $h_1+\pi^{-1}(g_1+H) \leqslant h_2+\pi^{-1}(g_2+H)$ in G. Furthermore, under this partial order, there exists a po-group o-isomorphism $\phi : G \to H \oplus \frac{G}{H}$

commutative in the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H & \xrightarrow{\iota} & G & \xrightarrow{\pi} & G/H & \longrightarrow & 0 \\
 & & \updownarrow {\scriptstyle =} & & \downarrow {\scriptstyle \phi} & & \updownarrow {\scriptstyle =} & & \\
0 & \longrightarrow & H & \xrightarrow{\iota'} & H \oplus \frac{G}{H} & \xrightarrow{\pi'} & G/H & \longrightarrow & 0
\end{array}
$$

Proof. If π has a right inverse group homomorphism then G splits as an abelian group. In particular, we have a pair of group isomorphisms, $\phi : G \to H \oplus \frac{G}{H}$ given by $g \mapsto (g - \pi^{-1} \circ \pi(g), \pi(g))$ and $\psi : H \oplus \frac{G}{H} \to G$ given by $(h, g + H) \mapsto h + \pi^{-1}(g + H)$.

These are simply the well-defined group isomorphisms we obtain from usual abelian group theory, before considering that our underlying groups are partially ordered. We verify that $H \oplus \frac{G}{H}$ is a po-group under the induced splitting order (i.e. the group operation is compatible). For any $(h_3, g_3 + H) \in H \oplus \frac{G}{H}$, if $(h_1, g_1 + H) \leqslant (h_2, g_2 + H)$ then we obtain the following.

$$
\begin{aligned}
h_1 + \pi^{-1}(g_1 + H) &\leqslant h_2 + \pi^{-1}(g_2 + H) \\
(h_1 + \pi^{-1}(g_1 + H)) + (h_3 + \pi^{-1}(g_3 + H)) &\leqslant h_2 + \pi^{-1}(g_2 + H) + (h_3 + \pi^{-1}(g_3 + H)) \\
(h_1 + h_3) + \pi^{-1}(g_1 + g_3 + H) &\leqslant (h_2 + h_3) + \pi^{-1}(g_2 + g_3 + H) \\
(h_1 + h_3, g_1 + g_3 + H) &\leqslant_{H \oplus \frac{G}{H}} (h_2 + h_3, g_2 + g_3 + H) \\
(h_1, g_1 + H) + (h_3, g_3 + H) &\leqslant (h_2, g_2 + H) + (h_3, g_3 + H)
\end{aligned}
$$

Hence, $H \oplus \frac{G}{H}$ is a po-group under the induced order. All that remains is to verify that ϕ is an o-isomorphism, which is to say that ϕ^{-1} exists and ϕ is an o-epimorphism. We already have that ϕ is an isomorphism so ϕ^{-1} exists; it is sufficient to verify ϕ is an o-epimorphism.

In the induced splitting order, $(g_1 - \pi^{-1} \circ \pi(g_1), \pi(g_1)) \leqslant (g_2 - \pi^{-1} \circ \pi(g_2), \pi(g_2))$ if and only if $g_1 - \pi^{-1} \circ \pi(g_1) + \pi^{-1} \circ \pi(g_1) \leqslant g_2 - \pi^{-1} \circ \pi(g_2) + \pi^{-1} \circ \pi(g_2)$, which reduces to $g_1 \leqslant g_2$. Thus, ϕ is order-preserving; all that remains is to verify that $\phi(G^+) = (H \oplus \frac{G}{H})^+$ to establish that ϕ is an o-epimorphism (and therefore an o-isomorphism). For any positive $x \in (H \oplus \frac{G}{H})^+$ under the induced splitting order, we have that $x = (h, g + H)$ for some $g \in G$ and $h \in H$ such that $0 \leqslant h + \pi^{-1}(g + H)$. Also, if $x = \phi(g)$ for some g, then $x = (g - \pi^{-1} \circ \pi(g), g + H)$. Since ϕ is a group isomorphism, the map is surjective so $x = (h, g + H) = (g_0 - \pi^{-1} \circ \pi(g_0), \pi(g_0))$ for some $g_0 \in G$. By the partial order induced on $H \oplus \frac{G}{H}$ by G, $0 \leqslant (g_0 - \pi^{-1} \circ \pi(g_0), \pi(g_0))$ if and only if $0 \leqslant g_0 - \pi^{-1} \circ \pi(g_0) + \pi^{-1} \circ \pi(g_0) = g_0$. Hence, x is the image of the positive element g_0, so $\phi(G^+) = (H \oplus \frac{G}{H})^+$. Thus $\phi : G \to H \oplus \frac{G}{H}$ is an o-isomorphism, where this direct sum is under the induced splitting order. □

Lemma 5.1.6. *If G splits as in Theorem 5.1.5, the product order on $H \oplus \frac{G}{H}$ is finer than the induced splitting order.*

Proof. If $(h_1, g_1 + H) \leqslant (h_2, g_2 + H)$ in the product order, then $g_1 + H \leqslant g_2 + H$ and $h_1 \leqslant h_2$. Hence we have that $h_1 + \pi^{-1}(g_1 + H) \leqslant h_2 + \pi^{-1}(g_2 + H)$. □

Assuming G splits as described in Theorem 5.1.5, it is natural to ask when the induced partial order on $H \oplus \frac{G}{H}$ coincides with the product or the (co)lexicographic order. Močkoř answered the colexicographic component of this question in [27], and here we extend that answer to the product component:

Theorem 5.1.7. *Let $H \subseteq G$ be a convex subgroup, let $\pi : G \to G/H$ the canonical o-epimorphism, and let π^{-1} be an o-homomorphism that is the right inverse of π. Then $G \simeq H \oplus \frac{G}{H}$ under the induced splitting partial order. (i) If $G^+ = \{g \in G \mid \pi(g) > H \text{ or } g \in H^+\}$, then the induced splitting partial order coincides with the colexicographic partial order. (ii) If $G^+ = \left\{g \in G \mid g - \pi^{-1} \circ \pi(g) \in H^+\right\}$ then the induced splitting partial order coincides with the product partial order.*

Proof. We first recapitulate the proof by Močkoř of the colexicographic component, part (i). To show the colexicographic order is equivalent to the splitting order, we first show the colexicographic order is finer than the induced splitting order, and then we show the induced splitting order is finer than the colexicographic order.

Let $(h_1, g_1 + H) \leqslant (h_2, g_2 + H)$ in $H \oplus \frac{G}{H}$ under the colexicographic order. We have two cases: first, $g_1 + H < g_2 + H$ or second, $g_1 + H = g_2 + H$ and $h_1 \leqslant h_2$. In the first case, $g_1 + H < g_2 + H$ so $H < g_2 - g_1 + H$. By our assumption on G, $g_2 - g_1 \in G^+$. Also, $h_2 - h_1 + \pi^{-1}(g_2 - g_1 + H) \in g_2 - g_1 + H$ so $(h_1, g_1 + H) \leqslant (h_2, g_2 + H)$. In the second case, if $g_1 + H = g_2 + H$ and $h_1 \leqslant h_2$, then $\pi^{-1}(g_2 - g_1 + H) = \pi^{-1}(H) = 0$, and $h_2 - h_1 + \pi^{-1}(g_2 - g_1 + H) = h_2 - h_1 \geqslant 0$. Hence, $(h_1, g_1 + H) \leqslant (h_2, g_2 + H)$ under the splitting order. This demonstrates that, under assumption (i), the colexicographic order is finer than the induced splitting order.

Now we show the induced splitting order is finer than colexicographic under assumption (i). Let $(h_1, g_1 + H), (h_2, g_2 + H) \in H \oplus \frac{G}{H}$ in the induced order such that $(h_1, g_1 + H) \leqslant (h_2, g_2 + H)$.

Then $h_1 + \pi^{-1}(g_1 + H) \leqslant_G h_2 + \pi^{-1}(g_2 + H)$ by definition of our induced order. Since π^{-1} is an o-homomorphism, $\pi^{-1}(g_2+H)-\pi^{-1}(g_1+H) = \pi^1(g_2-g_1+H)$ and $\pi^{-1}(g_2-g_1+H)+(h_2-h_1) \geqslant 0$. Denote $g := \pi^{-1}(g_2-g_1+H)+(h_2-h_1)$. Note $g \geqslant 0$. By assumption G, since $g \in G^+$, we have that $g+H > H$ or $g \in H^+$. If $g+H > H$, then applying π to both sides of $\pi^{-1}(g_2 - g_1 + H) + (h_2 - h_1)$ yields that $g_2 - g_1 + H > H$, i.e. $g_2 + H > g_1 + H$. Hence, $(h_1, g_1 + H) \leqslant (h_2, g_2 + H)$ under the colexicographic order. On the other hand, if $g \in H^+$, then $\pi^{-1}(g_2 - g_1 + H) + (h_2 - h_1) \geqslant 0$ in H. Applying π to both sides reveals that $g_2 - g_1 + H = H$, so $g_1 + H = g_2 + H$. Thus, we have that $g = h_2 - h_1 + \pi^{-1}(g_2 - g_1 + H) = h_2 - h_1 \geqslant 0$, so $h_1 \leqslant h_2$, and so $(h_1, g_1 + H) \leqslant (h_2, g_2 + H)$ in the lexicographic order. This establishes that the induced order is finer than the lexicographic order under assumption (i), and hence this establishes that under assumption (i) the colexicographic and induced splitting orders coincide.

We now demonstrate that, under assumption (ii), that the induced product order and induced splitting order coincide. Assume $G^+ = \left\{g \in G \mid g - \pi^{-1} \circ \pi(g) \in H^+\right\}$. We show that if $G^+ = \left\{g \in G \mid g - \pi^{-1} \circ \pi(g) \in H^+\right\}$ then the product order is finer than the induced splitting order. Say that $(h_1, g_1+H) \leqslant (h_2, g_2+H)$ in $H \oplus \frac{G}{H}$ under the product order so that $g_1+H \leqslant g_2+H$ and $h_1 \leqslant h_2$. We have $g_2 - g_1 + H \geqslant H$ so $\pi^{-1}(g_2 - g_1 + H) \geqslant 0$. Of course, since $h_2 - h_1 \geqslant 0$ we have $(h_2 - h_1) + \pi^{-1}(g_2 - g_1 + H) \geqslant 0$. Thus, $(h_1, g_1 + H) \leqslant (h_1, g_2 + H)$ in the induced splitting order on G.

All that remains is to show the induced splitting order is finer than the product order. Let $(h_1, g_1 + H) \leqslant (h_2, g_2 + H)$ in the induced splitting order so $h_1 + \pi^{-1}(g_1 + H) \leqslant h_2 + \pi^{-1}(g_2 + H)$. In particular, we have the non-negativity of $x := h_2 - h_1 + \pi^{-1}(g_2 - g_1 + H) \geqslant 0$. Since π is order preserving, we have $\pi(x) > H$, and by our assumption on G, we have $x - \pi^{-1} \circ \pi(x) \geqslant 0$. Of course, $\pi(x) = \pi(h_2 - h_1 + \pi^{-1}(g_2 - g_1 + H)) = g_2 - g_1 + H$, and $x - \pi^{-1} \circ \pi(x) = h_2 - h_1$, and so $(h_1, g_1 + H) \leqslant (h_2, g_2 + H)$ under the product order. □

Theorem 5.1.7 provides the required conditions to determine when the induced splitting order coincides with the product order or the colexicographic order. We can immediately obtain structure theorems governing the antimatter limit of groups. For a po-group G with quasi-atomic quotient sequence $\{\pi_n : G_n \to G_{n+1}\}$, denote $\varinjlim H_i = H^*$, $\varinjlim G/H_i \simeq G/H^* = G^*$ and recall the canonical o-epimorphism $\pi^* : G \to G^*$ has $\mathrm{Ker}(\pi^*) \simeq H^*$.

Corollary 5.1.8. *If π^* has a right-inverse, then $G \simeq H^* \oplus G^*$ under the splitting partial order.*

(i) If $G^+ = \{g \in G \mid \pi^*(g) > H^* \text{ or } g \in (H^*)^+\}$, then the induced splitting partial order coincides with the colexicographic partial order. (ii) If $G^+ = \left\{g \in G \mid g - (\pi^*)^{-1} \circ \pi^*(g) \in (H^*)^+\right\}$ then the induced splitting partial order coincides with the product partial order.

Proof. This corollary follows directly from Theorem 5.1.5 and Theorem 5.1.7. □

Hence, when the o-epimorphism $\pi^* : G \to G^*$ has a right inverse, G splits into an antimatter part and a quasi-atomic part in the sense that elements uniquely factor into a product of a quasi-atomic element $h \in H^*$ with an antimatter element $g \in G^*$.

Example 5.1.9. Recall the ring from Example 2.0.6 denoted $R = (\mathbb{F}_2[X, Y, \mathcal{Y}])_{\mathfrak{m}}$ with the set $\mathcal{Y} = \{Y^\alpha \mid \alpha \in \mathbb{Q}^+\} \subseteq \overline{\mathbb{F}_2(X,Y)}$. The group of divisibility is $G(R) = \mathbb{Z} \oplus \mathbb{Q}$ in the product order with $G(R)^* = \mathbb{Q}$. Certainly $G(R)$ splits. Each non-zero non-unit can be written as uX^nY^α where $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{Q}^+$ are not both zero; in the group of divisibility, this element can be written $(n, 0) + (0, \alpha)$ uniquely. Y^α is the antimatter factor and X^n is the quasi-atomic factor.

Iteratively applying Theorem 5.1.7 to the degrees of the quasi-atomic quotient sequence splits the quasi-atomic component of the decomposition from Corollary 5.1.8, yielding a more complete decomposition theorem for quasi-atomic quotient sequences.

Theorem 5.1.10. *Let G be a po-group with quasi-atomic quotient sequence such that each non-zero differential, $\pi_n : G_n \to G_{n+1} = G_n/Q(G_n)$, has a right inverse o-homomorphism. Then each $G_n \simeq Q(G_n) \oplus G_{n+1}$ under the induced splitting order. For a po-group G with $\mathrm{len}_q(G) = N \in \mathbb{N}_0$,*

(i) if G is quasi-atomic in the limit, then $G \simeq \oplus_{n=0}^N Q(G_n)$ and

(ii) if G is antimatter in the limit, then $G \simeq \left(\oplus_{n=0}^N Q(G_n)\right) \oplus G^*$.

Moreover, if for each $n \leqslant N$, $G_n^+ = \{g \in G_n \mid \pi_n(g) > Q(G_n) \text{ or } g \in Q(G_n)^+\}$, then the induced splitting partial order on $H_n \oplus \frac{G_n}{H_n}$ coincides with the colexicographic partial order. Likewise, for each n, if we have $G_n^+ = \left\{g \in G_n \mid g - \pi_n^{-1} \circ \pi_n(g) \in Q(G_n)^+\right\}$, then the induced splitting partial order coincides with the product partial order.

Proof. We proceed by induction on N. For $N = 0$, then G is antimatter, and hence is antimatter in the limit, and hence $G \simeq \{0\} \oplus G^*$. This may be dissatisfying as a base case due to the degeneracy of antimatter groups; using Theorems 5.1.5 and 5.1.7, we have the base case $N = 1$.

Now assume that if $\text{len}_q(G) < N$ and each map in the quasi-atomic quotient sequence has a right-inverse o-epimorphism, then the theorem is true. Let $\text{len}_q(G) = N$ such that each map in the quasi-atomic quotient sequence has a right-inverse. In the sequence $G = G_0 \to G_1 \to \cdots \to G_N$, we have that G_1 has $\text{len}_q(G) = N-1$. Moreover, the inverse maps from G provide that the quasi-atomic quotient sequence $G_1 \to G_2 \to \cdots \to G_N$ has right-inverses. From Theorems 5.1.5 and 5.1.7, we may write $G_0 \simeq Q(G_0) \oplus G_1$, and by our hypothesis on G_1, we may write each $G_n \simeq Q(G_n) \oplus G_{n+1}$ for $0 \leqslant n$.

If G is quasi-atomic in the limit, then G_1 is quasi-atomic in the limit and $G_1 \simeq \oplus_{n=1}^{N} Q(G_n)$. Hence, $G_0 \simeq \oplus_{n=0}^{N} Q(G_n)$. On the other hand, if G is antimatter in the limit, then G_1 is antimatter in the limit, $G_1^* = G^*$, and $G_1 \simeq (\oplus_{n=1}^{N} Q(G_n)) \oplus (G_1)^*$. Hence, $G \simeq (\oplus_{n=0}^{N} Q(G_n)) \oplus G^*$.

We must lastly verify when the partial orders coincide with the colexicographic or product ordering. We only check the colexicographic case since the product case proceeds similarly, *mutatis mutandis*. If for each n, $G_n^+ = \{g \in G_n \mid \pi_n(g) > Q(G_n) \text{ or } g \in Q(G_n)^+\}$, then the induced splitting order on each $H_n \oplus \frac{G_n}{H_n}$ for $n \geqslant 1$ coincides with the colexicographic order due to our inductive assumption on G_1. Thus, we need only verify that the induced splitting order on $H_0 \oplus \frac{G_0}{H_0}$ coincides with the colexicographic order. Of course, since $G_0^+ = \{g \in G_n \mid \pi_n(g) > Q(G_n) \text{ or } g \in Q(G_n)^+\}$, Theorem 5.1.5 implies that the order on $H_0 \oplus \frac{G_0}{H_0}$ is colexicographic. $\square$

We intuitively think Theorem 5.1.10 as stating: if the G is quasi-atomic in the limit and has finite quasi-atomic length and splits at every degree, then every element of G may be uniquely written as a sum of terms, one from each quasi-atomic part of the sequence. This provides a weak version of a universal factorization property. Similarly, if G is antimatter in the limit and has finite quasi-atomic length and splits at every degree, we also have a sort of universal property: every element of G may be uniquely written as a sum of terms, one from each quasi-atomic part of the sequence and one term from antimatter limit G^*.

5.2 Structure Theorems Exemplified

In Example 5.2.1 we present integral domains whose groups of divisibility constructed to satisfy these structure theorems. In particular, we present a domain with finite quasi-atomic length and quasi-atomic in the limit, a domain with finite quasi-atomic length and antimatter in the limit, a domain with infinite quasi-atomic length and quasi-atomic in the limit, and a domain with infinite quasi-atomic length and antimatter in the limit.

Example 5.2.1. (i) In this example, we present a domain that has finite quasi-atomic length and is quasi-atomic in the limit. Let $\mathbb{F}$ be any field with indeterminates X, Y. Let $R' = \mathbb{F}[X,Y]$ have quotient field $\mathbb{K}$. Set $\mathcal{W} = \left\{ \frac{Y}{X^j} : j \in \mathbb{N} \right\} \subseteq \mathbb{K}$ and define $R'' = R'[\mathcal{W}] = \mathbb{F}[X, Y, \mathcal{W}]$. In R'', the ideal $(X, Y, \mathcal{W}) = \mathfrak{m}$ is maximal. Set $R = R''_{\mathfrak{m}}$. Then R is a 2-dimensional discrete valuation domain with value group $\mathbb{Z} \oplus \mathbb{Z}$ ordered lexicographically. The quasi-atomic localization sequence for R proceeds as follows: $R_0 = R_{S_0} = R$ has precisely one irreducible X, $R_1 = R_{S_1}$ sees X become a unit and obtain a new ring in which Y is the only irreducible (with group of divisibility o-isomorphic to $\mathbb{Z}$), R_2 sees Y become a unit and yields the quotient field. Hence, $\text{length}_q(R) = 2$. For the map $\pi_0 : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$, we have the right-inverse $\pi_0^{-1} : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ defined by $x \mapsto (0, x)$. These are both order-preserving. △

(ii) In this example, first presented in [7, Example 2.7], we present two domains that with finite quasi-atomic length and yet antimatter in the limit. First, let $\mathbb{F}$ be any field with indeterminate X. Set $\mathcal{X} = \{X^\alpha \mid \alpha \in \mathbb{Q}^+\} \subseteq \overline{\mathbb{F}(X)}$ and define $R = \mathbb{F}[\mathcal{X}]_{\mathfrak{m}}$ where $\mathfrak{m} = (\mathcal{X})$ is the maximal ideal generated by all monomials. Then $G(R) = \mathbb{Q}$, which is antimatter and therefore antimatter in the limit with $\text{len}_q(G(R)) = 0$.

If this is too short to be satisfying, consider a different ring R; let $\mathbb{F}$ be any field with indeterminates X, Y, let $R' = \mathbb{F}[X,Y]$ have algebraic closure of its quotient field $\overline{\mathbb{F}(X,Y)}$. Set $\mathcal{W} = \left\{ \frac{Y^\alpha}{X^j} : j \in \mathbb{N}, \alpha \in \mathbb{Q}^+ \right\}$ and define $R'' = R'[\mathcal{W}] = \mathbb{F}[X, Y, \mathcal{W}]$. In R'', the ideal $(X, Y, \mathcal{W}) = \mathfrak{m}$ is maximal. Set $R = R''_{\mathfrak{m}}$. Then R is a 2-dimensional valuation domain with value group $\mathbb{Q} \oplus \mathbb{Z}$ ordered lexicographically with unique atom $(0,1)$ (corresponding to $X \in R$. The quasi-atomic localization sequence for R proceeds as follows: $R_0 = R_{S_0} = R$ has precisely one irreducible X and by localizing at X, we obtain $R_1 = R_{S_1}$, an antimatter ring where every element is associate to some Y^α. Now $R_{S_k} = R_{S_1}$ for every $k \geqslant 1$, so $R^* = R_{S_1}$ and $\text{length}_q(R) = 1$. The map $\pi_0 : \mathbb{Q} \oplus \mathbb{Z} \to \mathbb{Q}$ with kernel $0 \oplus \mathbb{Z}$ has the order-preserving right-inverse $\pi_0^{-1} : \mathbb{Q} \to \mathbb{Q} \oplus \mathbb{Z}$ defined by $x \mapsto (x, 0)$. △

(iii) In this example, we present a domain that has infinite quasi-atomic length and is quasi-atomic in the limit. Let $\mathbb{F}$ be any field with indeterminates $\{X_n\}_{n \in \mathbb{N}}$. Let $R' = \mathbb{F}[\{X_n\}]$ have quotient field $\mathbb{K}$. Set $\mathcal{Y} = \left\{ \frac{X_{n+1}}{X_n^j} : j \in \mathbb{N} \right\} \subseteq \mathbb{K}$ and define $R'' = R'[\mathcal{Y}] = \mathbb{F}[\{X_n\}_n, \mathcal{Y}]$. In R'', the ideal $(\{X_n\}_n, \mathcal{Y}) = \mathfrak{m}$ is maximal. Set $R = R''_{\mathfrak{m}}$. Then R is a discrete valuation domain with

$\dim_{\text{Krull}}(R) = +\infty$, and we write the value group as $\oplus_{n\in\mathbb{N}}\mathbb{Z}$ ordered lexicographically (not to be confused with our previous approaches with the colexicographic order). The quasi-atomic localization sequence for R proceeds as follows. $R_0 = R_{S_0} = R$ has precisely one irreducible X_1 (up to units). By localizing at X_1, we obtain $R_1 = R_{S_1}$, a valuation domain with unique irreducible X_2 (up to units). In the group of divisibility, this localization corresponds to the o-epimorphism $\oplus_{i\geqslant 1}\mathbb{Z} \to \oplus_{i\geqslant 2}\mathbb{Z}$ with kernel $\mathbb{Z}\oplus 0\oplus 0\oplus\cdots$. For each $n \geqslant 2$, by localizing at X_n, we obtain $R_n = R_{S_n}$, a valuation domain with unique irreducible X_{n+1} (up to units). This corresponds to the o-epimorphism $\oplus_{i\geqslant n}\mathbb{Z} \to \oplus_{i>n}\mathbb{Z}$ with kernel $\mathbb{Z}\oplus 0\oplus 0\oplus\cdots$.

Note $R_{S_n} \neq R_{S_{n+1}}$ for any $n \in \mathbb{N}_0$, and so $\text{len}_q(R) = +\infty$. However, an arbitrary non-zero non-unit element $x \in R$ is associate to some monomial $f = u\prod_{i=1}^{m} X_{n_i}^{k_i}$, and for any $N > \max_i\{n_i\}$, we see f is a unit. Hence, $\varinjlim S_i = R\setminus\{0\}$ so $R^* = \mathbb{F}$. Thus, R is quasi-atomic in the limit. Each o-epimorphism of the form $\pi_n : \oplus_{i\geqslant n}\mathbb{Z} \to \oplus_{i>n}\mathbb{Z}$ has right-inverse $\pi_n^{-1} : \oplus_{i>n}\mathbb{Z} \to \oplus_{i\geqslant n}\mathbb{Z}$ defined by $(x_{n+1}, x_{n+2}, \ldots) \mapsto (0, x_{n+1}, x_{n+2}, \ldots)$. △

(iv) Lastly, we present a domain that has infinite quasi-atomic length and is antimatter in the limit. To this end, let $\mathbb{F}$ be any field with indeterminates Y and $\{X_n\}_{n\in\mathbb{N}}$. Let $R' = \mathbb{F}[Y, \{X_n\}]$ have algebraic closure of its quotient field $\overline{\mathbb{F}(Y, \{X_n\})}$. Define

$$\mathcal{W} = \left\{Y^\alpha, \frac{Y^\alpha}{X_n^j}, \frac{X_{n+1}}{X_n^j} : n \in \mathbb{N}_0, j \in \mathbb{N}, \alpha \in \mathbb{Q}, \alpha > 0\right\} \subseteq \overline{\mathbb{F}(Y, \{X_n\})}.$$

Define $R'' = R'[\mathcal{W}] = \mathbb{F}[Y, \{X_n\}_n, \mathcal{W}]$. In R'', the ideal $(Y, \{X_n\}_n, \mathcal{W}) = \mathfrak{m}$ is maximal. Set $R = R''_{\mathfrak{m}}$. Then R is a valuation domain with $\dim_{\text{Krull}}(R) = +\infty$. Writing the value group for this ring is a little unusual: write $G_1 = \mathbb{Q}$ and write $G_2 = \oplus_{n\in\mathbb{N}}\mathbb{Z}$ under the colexicographic order. The value group is then $G_1 \oplus G_2$ in the lexicographic order, which we will write as $\mathbb{Q}\oplus(\oplus_{n\in\mathbb{N}}\mathbb{Z})$ with this order in mind.

The quasi-atomic localization sequence for R proceeds similarly to the previous example, and, just as in that example, $R_{S_n} \neq R_{S_{n+1}}$ for any $n \in \mathbb{N}_0$. This is associated to an o-epimorphism of the form $\mathbb{Q}\oplus(\oplus_{i\geqslant n}\mathbb{Z}) \to \mathbb{Q}\oplus(\oplus_{i>n}\mathbb{Z})$ with kernel $0\oplus(\mathbb{Z}\oplus 0\oplus 0\oplus\cdots)$, which has right-inverse $\pi_n^{-1} : \mathbb{Q}\oplus(\oplus_{i>n}\mathbb{Z}) \to \mathbb{Q}\oplus(\oplus_{i\geqslant n}\mathbb{Z})$ defined by $(q, (x_{n+1}, x_{n+2}, \ldots)) \mapsto (q, (0, x_{n+1}, x_{n+2}, \ldots))$. Note that, just as in the previous example, $\text{len}_q(R) = +\infty$. However, Y^α is an antimatter element. △

One might regard Example 5.2.1 as somewhat trivial; after all, we construct integral domains with groups of divisibility that have already split, and then we apply our splitting theorem. However, these examples provide concrete and simple models of integral domains that satisfy the splitting properties described and inform our intuition about factorization in rings whose groups of divisibility split.

Examples 5.2.1(iii) and (iv) may be considered worse than (i) and (ii) because the quasi-atomic quotient sequence is o-isomorphic in each degree to $G(R)$. For Example 5.2.1(iii), we see by writing $\mathbb{Z} \simeq \mathbb{Z}/0$, the quasi-atomic quotient sequence

$$\oplus_{n\in\mathbb{N}}\mathbb{Z} \to \left(\frac{\mathbb{Z}}{\mathbb{Z}} \oplus \frac{\mathbb{Z}}{0} \oplus \frac{\mathbb{Z}}{0} \oplus \cdots\right) \to \left(\frac{\mathbb{Z}}{\mathbb{Z}} \oplus \frac{\mathbb{Z}}{\mathbb{Z}} \oplus \frac{\mathbb{Z}}{0} \oplus \cdots\right) \to \cdots$$

This sequence is o-isomorphic in each degree to the stable sequence $\oplus_{n\in\mathbb{N}}\mathbb{Z} \to \oplus_{n\in\mathbb{N}}\mathbb{Z} \to \cdots$, which is precisely $G(R)$. However, the o-epimorphisms in the quasi-atomic quotient sequence are not o-isomorphisms. Hence, although this is a sequence of po-group o-epimorphisms that are not o-isomorphisms (and therefore $\mathrm{len}_q(G) = +\infty$), this may be a dissatisfying example of a quasi-atomic quotient sequence that neither stabilizes or terminates.

In Example 5.2.3, we present an example of a domain B that is quasi-atomic in the limit, has infinite quasi-atomic length, and yet does not have a quasi-atomic quotient sequence that is o-isomorphic in each degree to $G(B)$. To construct this example, we require a Lemma from [11].

Lemma 5.2.2. *Let Λ be a well-ordered index set and let $\{G_\lambda\}_{\lambda\in\Lambda}$ be a net of directed po-groups such that each G_λ is linearly ordered. Then $G = \oplus_{\lambda\in\Lambda}G_\lambda$ ordered lexicographically is a lattice-ordered group.*

Proof. Since each G_λ is directed and linearly ordered, the direct sum is certainly directed and partially ordered; it suffices to check that two arbitrary elements $a, b \in G^+$ have a join, or a least upper bound. Write $a = (a_\lambda)$ and $b = (b_\lambda)$. Define $(c_\lambda) \in G$ so that, for each λ, $c_\lambda = a_\lambda \vee b_\lambda$. Since each G_λ is totally ordered, we have that $c_\lambda = a_\lambda$ or $c_\lambda = b_\lambda$. Let (d_λ) be any other upper bound on a and b.

Since $a \leqslant (d_\lambda)$, we have some critical index λ_0 such that $a_\lambda = d_\lambda$ for each $\lambda < \lambda_0$ and such that $a_{\lambda_0} < d_{\lambda_0}$. Similarly, we have a critical index λ_1 such that $b_\lambda = d_\lambda$ for each $\lambda < \lambda_1$ and such that $b_{\lambda_1} < d_{\lambda_1}$. We can assume without loss of generality that $\lambda_0 \leqslant \lambda_1$. Certainly $a_\lambda = b_\lambda = c_\lambda = d_\lambda$ for $\lambda < \lambda_0$. Again note we have $c_{\lambda_0} = a_{\lambda_0} \vee b_{\lambda_0}$ so $c_{\lambda_0} = a_{\lambda_0}$ or $c_{\lambda_0} = b_{\lambda_0}$. But $a_{\lambda_0} < d_{\lambda_0}$ and

$b_{\lambda_0} < d_{\lambda_0}$ so $c_{\lambda_0} < d_{\lambda_0}$, and hence $(c_\lambda) < (d_\lambda)$. Hence, (c_λ) is a least upper bound on (a_λ) and (b_λ). □

Note Lemma 5.2.2 can be relaxed slightly when Λ is finite: for the greatest index λ' we only need $G_{\lambda'}$ to be lattice ordered, not necessarily linearly ordered.

Example 5.2.3. Let $\{V_n\}_{n\in\mathbb{N}}$ be any sequence of integral domains such that each V_n is a discrete n-dimensional valuation domain with group of divisibility $G(V_n) = \oplus_{i=1}^{n}\mathbb{Z}$ ordered lexicographically. Each $G(V_n)$ is totally ordered, so by Lemma 5.2.2, the po-group $G = \oplus_{n\in\mathbb{N}}G(V_n)$ under the lexicographic order is lattice ordered. By Theorem 1.0.1, there exists a Bézout domain B such that $G = G(B)$. Since $G(B)$ is a direct sum of quasi-atomic groups of divisibility ordered lexicographically, the quasi-atomic quotient sequence is

$$G(B) = \oplus_{n\geqslant 1}G(V_n) \to \oplus_{n\geqslant 2}G(V_n) \to \oplus_{n\geqslant 3}G(V_n) \to \cdots$$

where each map is a non-trivial o-epimorphism and this sequence never stabilizes. △

Example 5.2.3 is, in some senses, as good an example as we can hope, for we can construct a very large family of groups of divisibility with quasi-atomic sequences that do not stabilize.

5.3 Complementing Quasi-Atoms

We complete this chapter by using some group theoretic ideas to relax the notion of the "antimatter part" of a ring. Recalling for a domain D, we defined $D^* = \varinjlim D_{S_n}$ as the antimatter limit of D.

Definition 5.3.1. Let G be a po-group with o-ideal $H \subseteq G$. We define the *o-ideal complement* of H to be any subgroup $H' \subseteq G$ such that $H \cap H' = \{0\}$ and such that, for any $g \in G$, there exists some $n \in \mathbb{N}$ such that $ng \in H \oplus H'$. We define a *maximal o-ideal complement* of H to be any o-ideal complement maximal with respect to inclusion of subgroups.

Lemma 5.3.2. *Let G be a po-group with o-ideal H. Then a maximal quasi-atomic complement of H exists.*

Proof. Let $\mathfrak{J}$ denote the set of all subgroups of G with a trivial intersection with H; that is to say, set $\mathfrak{J} = \{J \subseteq G \mid J \cap H = \{0\}\}$. Since $\{0\} \subseteq \mathfrak{J}$, we have that $\mathfrak{J}$ is nonempty. Any chain in $\mathfrak{J}$, say

$\{J_\lambda\}_{\lambda\in\Lambda}$, certainly has an upper bound, namely $\cup_\lambda J_\lambda$. Applying Zorn's Lemma yields a maximal element of $\mathfrak{J}$, which we denote H'. We claim that H' is an o-ideal complement, i.e. for all $g \in G$, there is a strictly positive $n \in \mathbb{N}$ such that $ng \in H \oplus H'$.

Notice that if $g \in H' \subseteq H \oplus H'$, then our claim is established. If not, then the maximality of H' implies $\langle g, H'\rangle$ has nontrivial intersection with H. In particular, for some $n > 0$, $h \in H'$, and $\alpha \in H$, we have that $ng + h = \alpha$. Clearly then $ng \in H \oplus H'$. This establishes that H' is an o-ideal complement of H. □

A maximal o-ideal complement of H, which we have called H', is not unique, even with respect to order considerations. If we consider $\mathbb{Z} \oplus \mathbb{Z}$ under the lexicographic ordering, $H = Q(G)$ is uniquely determined (it is the subgroup $\mathbb{Z} \oplus 0$) but we have many choices for H'. The subgroups generated by $(0, 1)$ and $(1, 1)$ are two distinct choices, for example.

Notice that partially ordered abelian groups with nontrivial elements are necessarily torsion free. For an element x of finite order, we have $e \leqslant x \leqslant x^2 \leqslant x^3 \leqslant \cdots \leqslant x^n = e$. Antisymmetry in the partial order, $\leqslant$, insists that $x = e$. This leads us to the following theorem.

Lemma 5.3.3. *Let G be a directed po-group with o-ideal $H \subseteq G$. If there exists some o-ideal complement of H that is divisible, say $H' \subseteq G$, then $G = H \oplus H'$ as sets.*

Proof. Assume that $H' \subseteq G$ is an o-ideal complement of H that is divisible. Then $H' \cap H = \{0\}$ and for any $g \in G$ there exists some $n \in \mathbb{N}$ such that $ng \in H \oplus H'$, say $ng = h + h'$ for $h \in H, h' \in H'$. Moreover, since H' is divisible, we can write $h' = nh''$ for some $h'' \in H'$ and hence $n(g-h'') = h \in H$. Applying the natural o-epimorphism $\pi : G \to G/H$ yields $n(g - h'') + H = H$. However, G/H is a po-group since H is an o-ideal, so G/H is torsion-free and we conclude $g - h'' + H = H$. In particular, $g \in H \oplus H'$. Hence, $G \subseteq H \oplus H'$ (as abelian groups). But H and H' are subgroups so $G = H \oplus H'$. □

Corollary 5.3.4. *If $H \subseteq G$ is an o-ideal and H' is an o-ideal complement such that $G = H \oplus H'$, the order on G coincides with the induced splitting order on $H \oplus H'$ from Definition 5.1.3.*

Observe that if any po-group G is divisible, it is necessarily antimatter. Example 5.3.5 shows this theorem is immediately applicable to domains with pathological factorization behaviors since the quasi-atomic complement is divisible when certain domain elements admit n^{th} roots for any $n \in \mathbb{N}$.

Example 5.3.5. Consider the ring $R = \mathbb{F}[\mathcal{X}]_{\mathfrak{m}}$ where $\mathcal{X} = \{X^\alpha \mid \alpha \in \mathbb{Q}^+\}$ and $\mathfrak{m} = (\mathcal{X})$ from Example 5.2.1(ii). This ring has no irreducible elements, and so has no quasi-atomic elements. Thus we can trivially write $G = H' \oplus \{1_G\}$. Furthermore, H' is divisible. To demonstrate that H' is divisible, notice that multiplication in the ring is equivalent to the group operation on G. We may write any $g \in G$ as $g = X^\alpha U(R)$ and $ng = X^{n\alpha}U(R)$. Hence, for any $g \in G$, we have $g = X^\alpha U(R) = (X^{\alpha/n})^n U(R) = ng_0$ where $g_0 = X^{\alpha/n}U(R)$. △

Rather than considering the o-ideal complement of $Q(G)$, we could consider the o-ideal complement of $H = H^* = \varinjlim Q(G_i)$ in a variant of Definition 5.3.1. Indeed, we say that H' is an o-ideal complement of H^* if $H' \cap H^* = \{0\}$ and for any $g \in G$, there exists some $n \geqslant 1$ such that $ng \in H^* \oplus H'$. Since H^* consists of all group elements with finite quasi-atomic length, H' must be generated by some antimatter elements. Moreover, if $G = G(D)$ for an integral domain D and H' is an o-ideal complement of H^*, then every non-zero non-unit element $x \in D$ is the root of some product $x^n = yz$ for some $y, z \in D$ with $\text{len}_q(y) < \infty$ and such that $zU(D) \in H'$ is antimatter or a unit.

As in previous chapters, many notions in this section are rather vacuous in the case of quasi-atomic domains (including almost atomic and atomic domains). For example, if $\text{len}_q(D) < \infty$ and D is quasi-atomic in the limit, then $H^* = G(D)$ and so an o-ideal complement of H^* can only be trivial $H = \{1_G\}$, providing the vacuous factorization $G(D) = G(D) \oplus \{1_G\}$. This includes cases of quasi-atomic domains such as UFDs.

Appendices

Appendix A Integral Domains

Integral domains are unital commutative rings lacking non-zero zero divisors. In Section A.1 we review the cancellativity property of integral domains and the pre-order on an integral domain induced by divisibility. In Section A.2 we discuss polynomial extensions, in Section A.3 we describe quotient rings and their associated canonical ring epimorphisms, in Section A.4 we describe integral domain localizations, in Section A.5 we describe direct limits.

A.1 Basic Facts and Notation for Integral Domains

The rings $\mathbb{Z}$, $\mathbb{Z}[X]$, and $\mathbb{Q}$ are all three examples of integral domains. $\mathbb{Q}$ is also a field (all fields are integral domains). If D is an integral domain and X is indeterminate over D then $D[X]$ is an integral domain. Similarly, if $S \subseteq D$ is a multiplicatively closed set with $0 \notin S$, then D_S is an integral domain. If D is an integral domain and $I \subseteq D$ then D/I is an integral domain if and only if I is a prime ideal. We refer to integral domains as simply *domains*.

Domains are *cancellative*: for any non-zero x, y, z, if $xy = xz$ then $y = z$. Invertible elements of D are known as *units*, and the set of units in D form a multiplicative group, which we denote $U(D)$; for example, if D is an integral domain with indeterminate X, then $U(D) = U(D[X])$. Every integral domain has a *field of fractions* or *quotient field* $\mathbb{F}$. This field may be formally constructed by defining an equivalence relation on $D \times (D \setminus \{0\})$ by setting $(a, b) \sim (c, d)$ if and only if $ad - bc = 0$; we denote the equivalence class of (a, b) as $\frac{a}{b}$. For example, the field of fractions for $\mathbb{Z}$ is precisely $\mathbb{Q}$. Further, the map $\epsilon : D \to \mathbb{F}$ defined by $\epsilon(x) = \frac{x}{1}$ is a ring monomorphism. Since $D \cong \epsilon(D) \subseteq \mathbb{F}$, we can assume without loss of generality that $D \subseteq \mathbb{F}$.

We say a non-zero non-unit $x \in D \setminus \{0\}$ is an *irreducible element* when any factorization, say $x = yz$, is trivial in the sense that $y \in U(D)$ or $z \in U(D)$. We sometimes refer to irreducible elements merely as *irreducibles* or *atoms*. If x is an element such that $x \mid yz$ implies $x \mid y$ or $x \mid z$, then we say x is *prime*. Any prime element of any integral domain is irreducible but not conversely. We say that $x, y \in D$ are *associated by units* or simply *associates* when $x = uy$ for some $u \in U(D)$. We denote the set of irreducibles in D as $\operatorname{Irr}(D)$. If an element $x \in D$ can be written as a finite product of atoms, we say x is *atomic* or an *atomic element*. If every non-zero non-unit element of D is atomic, we refer to D as an *atomic domain*. On the other hand, an arbitrary integral domain need not contain any atoms (and hence no atomic elements). For example, as we saw above, $\mathbb{Q}$ is an integral domain, but it is also a field. Each non-zero $x \in \mathbb{Q}$ is a unit, so $\mathbb{Q}$ has no irreducibles. Following [7], whenever D contains no irreducible elements we say D is an *antimatter domain*.

If every non-zero non-unit element of D can be written uniquely as a product of irreducibles (up to a permutation of the order of those irreducibles and up to associates), then we say that D is a *unique factorization domain* (or a UFD). If D is a UFD then $D[X]$ is also a UFD. Hence, since $\mathbb{Z}$ is a UFD, we can inductively obtain that any $\mathbb{Z}[X_1, \ldots, X_N]$ is a UFD. In a UFD, every irreducible element is prime. Note any field $\mathbb{F}$ is not only antimatter, it is also a UFD: since $\mathbb{F}$ contains no non-zero non-units, $\mathbb{F}$ vacuously satisfies the condition that every non-zero non-unit factors into irreducibles.

Following [4], we use two relaxations of the notion of atomicity that are critical to our treatment of integral domains. Recall that every integral domain D has a quotient field $\mathbb{F}$. If D' is any integral domain such that $D \subseteq D' \subseteq \mathbb{F}$, then the quotient field of D' is $\mathbb{F}$ and we say D' is an *overring*. A non-zero non-unit $x \in D$ is said to be *almost atomic* if there exists some atomic $a \in D$ such that ax is atomic. We also say that x is *quasi-atomic* when there exists any $y \in D$ such that yx is atomic. Note that if $x \in D$ is atomic then x is almost atomic. If x is almost atomic then x is quasi-atomic. Examples of almost atomic domains and quasi-atomic domains appear in Chapter 2.

Integral domains admit an additional structure induced by divisibility; this additional structure is a relation that is compatible with multiplication. We say reflexive and transitive relations are *pre-orders* as in [11], although some authors refer to these as *quasi-orders*, as in [33], [10], and [16]. We denote a pre-order with the symbol $\preceq$. We say antisymmetric pre-orders are *partial orders* as in [11], and to distinguish that these pre-orders are antisymmetric we denote a parital order with the symbol $\leqslant$. Define the *divisibility pre-order* for any $x, y \in \mathbb{F} \setminus \{0\}$ by declaring $x \preceq y$ if and only if $\frac{y}{x} \in D$. The divisibility pre-order on an integral domain D is compatible with multiplication in the sense that for any $x, y, z \in D$ if $x \preceq y$ then $xz \preceq yz$. However, the divisibility pre-order is not antisymmetric: for any non-zero non-unit $x \in D$ and for any unit $u \in D$, $x \preceq ux$ and $ux \preceq x$ but $ux \neq x$.

Example A.1. Consider the divisibility pre-order $\preceq$ on the multiplicative subgroup of $\mathbb{Q}$ induced by divisibility in $\mathbb{Z}$: say $\frac{a}{b} \preceq \frac{c}{d}$ if and only if $\frac{c/d}{a/b} \in \mathbb{Z}$. Now $\frac{2}{15} \preceq -\frac{4}{3}$ since $-\frac{4/3}{2/15} = -10 \in \mathbb{Z}$ but $-\frac{2}{15} \not\preceq \frac{3}{2}$ since $-\frac{3/2}{2/15} = -\frac{45}{4} \notin \mathbb{Z}$.

By sacrificing antisymmetry up to units, then the order $\preceq$ on $\mathbb{F}^{\times}$ can be completed to a partial order by considering only associate classes with the canonical map $\eta : \mathbb{F}^\times \mapsto \mathbb{F}^\times/U(D)$ as in [11].

Example A.2. Although $\preceq$ on $\mathbb{Q}$ induced by divisibility in $\mathbb{Z}$ induced a pre-order $\preceq$, we have that $\frac{a}{b} \preceq -\frac{a}{b}$ and $-\frac{a}{b} \preceq \frac{a}{b}$ for any $a, b \in \mathbb{Z}$ with $b \neq 0$. Yet $\frac{a}{b} \neq -\frac{a}{b}$. However, we have that $\frac{a}{b}U(\mathbb{Z}) = \left\{\pm\frac{a}{b}\right\} = -\frac{a}{b}U(\mathbb{Z})$. Hence the relation on $\mathbb{Q}^\times/U(\mathbb{Z})$ induced by $\preceq$ is a partial order. We denote this induced order as $\leqslant$. △

A partial order $\leqslant$ such that any two elements are comparable (in the sense that $x \leqslant y$ or $y \leqslant x$) is known as a *total order* (some authors refer to these as *linear orders*, *simple orders*, *full orders*, or simply *orders* [11]). If a total ordering $\leqslant$ on a set X satisfies the additional property that $\emptyset \neq X' \subseteq X$ implies X' has a least element, then we say $\leqslant$ is a *well-ordering*. In any partially ordered set X we define the *meet* or *greatest lower bound* for two elements $x, y \in X$ as any element $z \in X$ such that $z \leqslant x$ and $z \leqslant y$ and if $z' \leqslant x$ and $z' \leqslant y$ then $z' \leqslant z$. The meet of an arbitrary x and y is not guaranteed to exist. If the meet of x and y exists, we denote it as $z = x \wedge y$. We dually define the *join* as the least upper bound, denoted $x \vee y$.

A partial order on a set X such that the meet (join, respectively) for arbitrary pairs always exists is called a *meet semi-lattice order* (*join semi-lattice order*, respectively) and we say X is a meet semi-lattice (*join semi-lattice*, respectively). A partial order that is both a meet semi-lattice order and a join semi-lattice order is called a *lattice order*. If an order on X is a lattice order, we say X is a lattice.

A.2 Polynomial Extensions of Integral Domains

We follow [18] to define polynomial rings and [23] to generalize these slightly. We denote commutative rings with identity as R and integral domains D.

Define addition in $R[X] := \oplus_{i\in\mathbb{N}_0} R$ by $(a_0, a_1, \ldots) + (b_0, b_1, \ldots) = (a_0 + b_0, a_1 + b_1, \ldots)$ and with multiplication defined by $(a_0, a_1, \ldots)(b_0, b_1, \ldots) = (c_0, c_1, \ldots)$ where each $c_n = \sum_{i=0}^{n} a_{n-i}b_i$. These are well-defined because each sequence (a_n) and (b_n) have a finite number of non-zero coordinates. Note X here is merely a formal symbol we call an *indeterminate*. On the other hand, if R is a subring of some algebraic object $R \subseteq T$ and $t \in T$, we denote the set of formal sums $\left\{\sum_{n=0}^{N} r_n t^n \mid N \in \mathbb{N}_0, r_n \in R\right\}$ as $R[t]$. Note that $R[t]$ is not necessarily freely generated as an R-module, depending upon our selection of t; for example, if $R = \mathbb{Z}$ and $t = \sqrt{-1} \in \mathbb{C}$, we have that $t^2 \in R$. To see the connection between these constructions beyond mere notation, we have the following Lemma.

Lemma A.3. *Let $R \subseteq T$ be any extension of commutative rings and let $t \in T$ be transcendental over R. Then $R[X] \cong R[t]$.*

Proof. The isomorphism $\phi : R[X] \to R[t]$ is defined by mapping $(a_0, a_1, \ldots) \mapsto \sum_{n \geqslant 0} a_n t^n$. □

Lemma A.3 allows us to think of polynomial rings as constructed by adjoining transcendental elements. We define the *degree* of a polynomial $f \in R[X]$, denoted $\deg(f)$, as the largest choice of index, n, such that $a_n \neq 0$. Note that if $R = D$ is an integral domain then given two polynomials, $f, g \in D[X]$, we have that $\deg(fg) = \deg(f) + \deg(g)$. We similarly define the *order* or *co-degree* of $f \in R[X]$, denoted $\operatorname{ord}(f)$, as the smallest choice of index n such that $a_n \neq 0$.

Lemma A.4. *Let $D \subseteq T$ be a containment of integral domains. For any non-zero $t \in T$, $D[t]$ is an integral domain.*

Proof. $D[t]$ is a subring of T, which is an integral domain, and so contains no non-zero zero divisors. □

We refer to elements of $R[X]$ as *polynomials*. For any polynomial, $f = (a_0, a_1, \ldots)$, we abuse notation by writing $f = f(X) = a_0 + a_1 X + \cdots$. Since f is in the direct sum, there exists some $a_n \neq 0$ such that $a_{n+k} = 0$ for each $k \in \mathbb{N}$. So we can write $f(X) = a_0 + a_1 X + \cdots + a_n X^n$. Note here X is merely a formal symbol we call an *indeterminate* but, again, we can think of X as a transcendental element over R. We define a polynomial ring over a finite set of indeterminates $R[X_1, X_2, \ldots, X_n]$ inductively as $R[X_1, \ldots, X_{n-1}][X_n]$, which is isomorphic to $R[t_1, \ldots, t_{n-1}][t_n]$ for some set of R-algebraically independent elements $\{t_1, t_2, \ldots, t_n\}$. The polynomials of the form $\prod_{i=1}^{N} X_i^{n_i} \in R[X_1, \ldots, X_N]$ where each $n_i \in \mathbb{N}$ are known as *monomials*.

We define the degree of a polynomial in multiple variables in the case that R is an integral domain, i.e. $R = D$. For any $f \in D[X_1, X_2, \ldots, X_n]$, selecting $X_i \in \{X_1, \ldots, X_n\}$, we have that

$$f = a_0 + a_1 \left(\prod_{j=1}^{N_1} X_j^{k_{1,j}} \right) + \cdots + a_m \left(\prod_{j=1}^{N_m} X_j^{k_{m,j}} \right)$$

where each $k_{i,j} \in \mathbb{N}$. We can select some X_i and determine the degree of f with respect to X_i by writing the above as

$$
\begin{aligned}
f =& a_0 + a_1 \left(\prod_{j=1, j\neq i}^{N_1} X_j^{k_{1,j}} \right) X_i + \cdots + a_m \left(\prod_{j=1, j\neq i}^{N_m} X_j^{k_{m,j}} \right) X_i^{k_{m,i}} \\
=& \alpha_0 + \alpha_1 X_i + \cdots + \alpha_m X_i^{k_{m,i}}
\end{aligned}
$$

where each $\alpha_i \in D[X_1, \dots X_{i-1}, X_{i+1}, \dots, X_n]$. Now we may define $\deg_{X_i}(f) = k_{m,i}$. On the other hand, for a monomial $f \in D[X_1, \dots, X_n]$, say $f = a \prod_{j=1}^{N} X_j^{k_j}$, we may define the *total degree* as $\deg_{tot}(f) = \sum_j k_j$. Since these degrees are elements of $\mathbb{N}$, they are linearly orderable and any finite subset has a maximum. We may define the total degree of a polynomial as the greatest total degree among the monomials of f. For example, for $f \in \mathbb{Z}[X, Y]$ given by $f(X, Y) = 2X^2 + XY^3$, we have $\deg_Y(f) = 3$ and $\deg_X(f) = 2$, and $\deg_{tot}(f) = 4$.

We retain the property that $\deg_{X_i}(fg) = \deg_{X_i}(f) + \deg_{X_i}(g)$ when we specify an indeterminate in this way. In fact, defining $D' = D[X_1, X_2, \dots, X_N]$, we see that D' is a free D-module; if M is the set of all distinct monomials in D', then the set $\{1\} \cup M$ is an D-module basis for D'.

In many examples in the sequel, we adjoin non-transcendental elements to R (or D). If D is a subring of some integral domain T and $\mathcal{X} \subseteq T$ is a set of non-zero elements, we define

$$
D[\mathcal{X}] = \left\{ \sum_{n=0}^{N} r_n \prod_i x_{n,i}^{m_{n,i}} \mid r_n \in D, x_{n,i} \in \mathcal{X}, m_{n,i} \in \mathbb{N}_0 \right\}
$$

where we interpret $x^0 = 1 \in D$ for any $x \in \mathcal{X}$. Note $\mathcal{X}$ may be an infinite set, but each element $f \in D[\mathcal{X}]$ consists only of at most a finite number of monomials, each of which is built from at most a finite subset of $\mathcal{X}$. If D has quotient field $\mathbb{F}$ and $T = \mathbb{F}$ then elements of $\mathcal{X}$ are rational functions, so we abuse terminology and refer to $D[\mathcal{X}]$. These definitions formalize the idea of polynomial rings of rational functions, but since we did not specify T, this definition allows for a rather general treatment of polynomial-like constructions. We define the set of *monomials* as the set of all finite products of the form $\prod_i x_i^{m_i}$ such that each $x_i \in \mathcal{X}$.

Example A.5. Let D be a domain and X, Y indeterminates. Define $R' = D[X, Y] = (D[X])[Y]$. Since D is an integral domain, $D[X]$ is an integral domain and likewise $D[X, Y]$ is an integral domain. Then R' has a quotient field, $R' \subseteq \mathbb{F}$. We have elements of the form $\frac{Y}{X^n} \in \mathbb{F}$ for each $n \in \mathbb{N}$; let $\mathcal{X} = \left\{ \frac{Y}{X^n} \mid n \in \mathbb{N} \right\}$. Let $R = R'[\mathcal{X}]$. Then Y is irreducible in R' but Y is not irreducible in R since

$Y = X^n \frac{Y}{X^n}$ for any choice of $n \geqslant 1$. △

By selecting T as the algebraic closure of the quotient field for R we obtain a polynomial ring of "algebraic" functions (to be precise, functions of algebraic elements).

Example A.6. Let D be as in the previous example and let $\mathbb{F}$ be the quotient field for $D[X,Y]$. Select $T = \overline{\mathbb{F}}$. Then we can select $\mathcal{X} = \left\{X^{2^n}, \frac{Y}{X^{2^n}} \mid n \in \mathbb{Z}\right\}$. Note these exponents can be positive or negative, so the ring $D[X,Y][\mathcal{X}]$ has elements like X^4 and $X^{1/4}$ and note that Y is divisible by any allowable power of X: $Y = X^4(Y/X^4) = X^{1/8}(Y/X^{1/8})$, and so on. △

Polynomial rings over indeterminates are ubiquitous, so we establish some properties.

Theorem A.7. *Let D be an integral domain and X indeterminate over D. Then $U(D[X]) = U(D)$.*

Proof. If $u \in U(D)$, then there exists some $v \in U(D)$ such that $uv = 1$. This relationship still holds in $D[X]$. On the other hand, if $f \in U(D[X])$, then there exists some $g \in D[X]$ such that $fg = 1$. If $fg = 1$ then certainly $\deg(1) = 0$, and so we have that $\deg(fg) = n + m = 0$, so $n = m = 0$ and thus $f, g \in D$. That is to say, units in $D[X]$ are elements of D. □

Lemma A.8. *Let $f \in D[X]$ be associate to a monomial over D and let g be a proper divisor of f. Then g is also a monomial over D.*

Proof. We induct on the degree of f. For $\deg(f) = 1$, we have that $f = uX$. If g is a proper divisor with $\deg(g) = 1$, say $g = g_0 + g_1 X$ for some $g_0, g_1 \in D$, then there exists some $0 \neq h \in D[X]$ such that $g_0 h + g_1 h X = uX$. Thus, $g_0 h = 0$ and $u = g_1 h$. Since $h \neq 0$ we see that $g_0 = 0$ and g_1 is a unit, say $g_1 = v \in U(D)$, so g takes the form $g = vX$. Hence, g is a monomial.

On the other hand, assume that for monomials of sufficiently low degree, say $\deg(f) < n$, if g is a proper divisor of f then g is also a monomial. Now consider $f = uX^n$ and let g be a proper divisor. Set $\ell = \operatorname{ord}(g)$. Note $\ell \leqslant \deg(f) = n$. Hence, $f = gh$ may be written $\frac{f}{X^\ell} = \frac{g}{X^\ell} h$. where $\frac{f}{X^\ell}, \frac{g}{X^\ell} \in D[X]$. Now $\frac{f}{X^\ell}$ is a monomial of strictly lower degree than f so our inductive hypothesis implies $\frac{g}{X^\ell}$ is a monomial, so g is a monomial. □

Theorem A.9. *Let D be an integral domain and $\mathcal{X}$ a set of indeterminates over D. The set of monomials in $D[\mathcal{X}]$ is a saturated multiplicatively closed set.*

Proof. That monomials are multiplicatively closed is clear; it is sufficient to check saturation. Let $f \in D[\mathcal{X}]$ be associate to a monomial, say $f = u \prod_{i=1}^{N} X_i^{n_i}$ where each $X_i \in \mathcal{X}$. We induct on N

by considering $f \in D[X_1, \ldots, X_N]$. For $N = 1$, we simply cite Lemma A.8: the proper divisors of a monomial in $D[X]$ are all monomials.

Now assume that if $f \in D[X_1, \ldots, X_N]$ is a monomial then any proper divisor of f is also a monomial. Let $f \in D[X_1, \ldots, X_{N+1}]$ be a monomial. We can regard $f \in D[X_1, X_2, \ldots, X_N][X_{N+1}]$. Since f is a monomial (over D) in $D[X_1, \ldots, X_{N+1}]$, f is also a monomial (over $D[X_1, X_2, \ldots, X_N]$) in $D[X_1, X_2, \ldots, X_N][X_{N+1}]$. By Lemma A.8, we see that any proper divisor of g must be a monomial (over $D[X_1, X_2, \ldots, X_N]$) in $D[X_1, X_2, \ldots, X_N][X_{N+1}]$. We write $g = X_{N+1}^{n_{N+1}}\hat{g}$ where $\hat{g} \in D[X_1, X_2, \ldots, X_N]$, $n_{N+1} \in \mathbb{N}$.

Now, $\hat{g}$ is either a unit or a proper divisor of a monomial (and is hence a monomial). Inductively, we determine $\hat{g}$ is a monomial in $D[X_1, \ldots, X_N]$ and so g is a monomial in $D[X_1, \ldots, X_{N+1}]$.

$\square$

Overrings of polynomial extensions, as subsets of the quotient field, consist of rational functions. We look more closely at this idea.

Example A.10. Let $D = \mathbb{Z}[X, Y]$ have quotient field $\mathbb{F}$. Then $\frac{X}{Y} \in \mathbb{F}$. Define $\mathcal{D}$ as all overrings D' such that $\frac{X}{Y} \in D'$. Then $\cap_{D' \in \mathcal{D}} D'$ is an integral domain we denote $\mathbb{Z}[X, Y, \frac{X}{Y}]$. $\triangle$

Also note that polynomial extensions $D[\mathcal{X}]$ could be alternatively defined with a universal mapping property by considering all D-algebras freely generated over D by sets with the same cardinality as $\mathcal{X}$:

Theorem A.11. *Let D be an integral domain, $\mathcal{X}$ a set of indeterminates over D, $\epsilon : \mathcal{X} \to D[\mathcal{X}]$ the canonical inclusion map. For any commutative D-algebra A and set function $f : \mathcal{X} \to A$, there exists a unique D-algebra homomorphism $\phi : D[\mathcal{X}] \to A$ such that $f = \phi \circ \epsilon$.*

A.3 Quotient Rings and Integral Domains

Since we only consider commutative rings, right- and left-ideals coincide so all ideals are two-sided. We say a subset $I \subseteq R$ is an *ideal* of R when I is additively and subtractively closed and for any $x, r \in R$, if $x \in I$ then $rx \in I$. We say an ideal I is *prime* when, for any $a, b \in R$, if $ab \in I$ then $a \in I$ or $b \in I$. We define $\operatorname{Spec}(R)$ to be the set of all prime ideals of R and, given any fixed ideal $I \subseteq R$. Following the standard notation from algebraic geometry, define $V(I)$ to be the set of all prime ideals $\mathfrak{p}$ satisfying $I \subseteq \mathfrak{p}$. Since all ideals are two-sided, for any ring R with ideal $I \subseteq R$ we have a natural ring epimorphism of the form $R \twoheadrightarrow R/I$ defined by $r \mapsto r + I$.

Theorem A.12. *Let D be an integral domain and $I \subseteq D$ an ideal. There exists a one-to-one correspondence $V(I) \cong Spec(D/I)$.*

Lemma A.13. *If R is commutative with identity and ideal $I \subseteq R$, then R/I is a domain if and only if I is a prime ideal, and R/I is a field if and only if I is a maximal ideal.*

Proof. Assume I is prime. If $a+I, b+I \in R/I$ are chosen such that $(a+I)(b+I) = I$, then $ab \in I$, and since I is prime, $a \in I$ or $b \in I$. Hence, $a + I = I$ or $b + I = I$. On the other hand, if R/I has no zero divisors, then $(a + I)(b + I) = I$ implies $a + I = I$ or $b + I$. Hence, if $ab \in I$, then $ab + I = (a + I)(b + I) = I$, and so we conclude $a \in I$ or $b \in I$, so I is prime.

Assume R/I is a field. Then R has no ideals properly containing I according to Theorem A.12 so I is maximal. Now assume I is maximal. Then any nontrivial element $a + I \in R/I$ corresponds to some $a \notin I$. By the maximality of I, the ideal $(a, I) = R$, so $1 \in (a, I)$. Thus, there exists some $r \in R$ and $i \in I$ such that $1 = ra+i$. For this r, we have that $1+I = ra+I = (r+I)(a+I)$, so $a + I$ is invertible. □

Example A.14. The map $\pi : \mathbb{Z}[X] \twoheadrightarrow \mathbb{Z}$ defined by mapping $f(X) \mapsto f(0)$ is a ring epimorphism. The kernel of π is (X), and so we have $\mathbb{Z}[X]/(X) \cong \mathbb{Z}$ by the natural map $f(X) + (X) \mapsto f(0)$. Hence, if $\mathfrak{p}$ is a prime in $\mathbb{Z}[X]$ such that $(X) \subseteq \mathfrak{p}$, then there exists some $\mathfrak{q} \subseteq \mathbb{Z}$ such that $\mathfrak{p} = (\mathfrak{q}, X)$. Hence, the prime ideals of $\mathbb{Z}[X]$ that contain X are of the form (q, X) where q is a prime of $\mathbb{Z}$. △

Example A.15. Let $D = \mathbb{Z}$. Since $\mathbb{Z}$ is a PID, it is also a UFD, so all irreducible elements are prime. Furthermore, all prime ideals are of the form $\mathfrak{p} = (p)$ for some prime element, $p \in \mathbb{Z}$, or $p = 0 \in \mathbb{Z}$. Thus, for each prime $p \in \mathbb{Z}$, the ring $\mathbb{Z}/(p)\mathbb{Z}$ is an integral domain. Furthermore, each $(0) \neq (p)$ is also maximal, and thus $\mathbb{Z}/(p)\mathbb{Z}$ is a field. △

Example A.16. Let $D = \mathbb{C}[X, Y]$. Then $Y^2 = X^3 + X + 1$ defines a non-singular elliptic curve, thus the ideal $I = (X^3 + X + 1 - Y^2)$ is a prime ideal. We conclude D/I is an integral domain. △

Ring epimorphisms and polynomial extensions lead to some interesting constructions. Sometimes these are isomorphic to rings of rational functions described in Section A.2. For example, a ring with rational functions like $D[X, Y, \frac{X}{Y}]$ may be obtained through the natural ring isomorphism $D[X, Y, \frac{X}{Y}] \cong \frac{D[X,Y,Z]}{(YZ-X)}$ induced by the ring epimorphism $D[X, Y, Z] \twoheadrightarrow \frac{D[X,Y,Z]}{(YZ-X)}$. However, this example betrays us: a ring epimorphism with prime ideal kernels is not always equivalent to a ring of rational functions.

Example A.17. For an algebraic α over $\mathbb{Q}$ with minimal polynomial $f \in \mathbb{Q}[x]$, we have that $\mathbb{Q}[\alpha] \cong \mathbb{Q}[x]/(f(x))$. For example $\mathbb{Q}[i] \cong \mathbb{Q}[X]/(X^2+1)$. This is not isomorphic to any subring of the quotient field of $\mathbb{Q}[X]$. To see why, note that $U(\mathbb{Q}[X]) = U(\mathbb{Q}) = \mathbb{Q} \setminus 0$. The unit group of $\mathbb{Q}$ is all of $\mathbb{Q} \setminus 0$, and the only elements of finite order are ± 1. On the other hand, $i \in U(\mathbb{Q}[i])$ and $i^4 = 1$. No element of $U(\mathbb{Q}[X])$ has fourth order. Hence, $\mathbb{Q}[i]$ is not isomorphic to any subring of the quotient field of any polynomial ring $\mathbb{Q}[\mathcal{X}]$ Moreover, this is not isomorphic to any subring of $\mathbb{Q}(X)$. To see this, note that if $(\frac{f}{g})^4 = 1$ for any $f, g \in \mathbb{Q}[X]$, then $f^4 = g^4$ in $\mathbb{Q}[X]$, which is a UFD. $\triangle$

For a domain D and indeterminates $\mathcal{X}$, $\operatorname{Spec}(D[\mathcal{X}]/(\mathcal{X})) \cong V((\mathcal{X}))$ ando $D[\mathcal{X}]/(\mathcal{X}) \cong D$. In particular, $(\mathcal{X})$ is always prime since D is an integral domain. Of course, by Lemma A.9, proper divisors of monomials are monomials so perhaps the primality of $(\mathcal{X})$ is not surprising. But furthermore, $(\mathcal{X})$ is maximal in $D[\mathcal{X}]$ if and only if (0) is maximal in D. This proves Theorem A.18.

Theorem A.18. *The ideal $(\mathcal{X}) \subseteq D[\mathcal{X}]$ generated by all monomials is maximal if and only if D is a field.*

Our primary concern is with integral domains, which leads us to the following universal mapping property for quotient domains.

Theorem A.19. *Let D be an integral domain, $I \subseteq D$ be an ideal, and $\pi : D \twoheadrightarrow D/I$ the canonical epimorphism. Let $\phi : D \to D'$ be a ring homomorphism with $I \subseteq \operatorname{Ker}(\phi)$. Then there exists a unique ring homomorphism $\overline{\phi} : D/I \to D'$ such that $\phi = \overline{\phi} \circ \pi$.*

A.4 Localizations of Integral Domains

We say a set is *multiplicatively closed* when $0 \notin S$, $1_D \in S$ and when $s, t \in S$ implies $st \in S$. Since we investigate integral domains, which lack zero divisors, we assume $0 \notin S$ for any multiplicatively closed set S. We say a multiplicatively closed set in D, say S, is *saturated* when, for any $s \in S$ and $t \in D$, if $t \mid s$ then $t \in S$. Since $1_D \in S$ any saturated S has $U(D) \subseteq S$. Indeed, if $u \in U(D)$, since $u \mid 1_D$ and S is saturated, we have that $u \in S$. Recall Theorem A.9, in $D[\mathcal{X}]$ the set of monomials is saturated and multiplicatively closed. Hence, the set of monomials, say M, is not saturated, but it is "nearly" saturated in the sense that M contains all of its non-unit divisors.

For any integral domain, D, and any multiplicatively closed set, S, the *localization* of D at S, denoted D_S, is a subset of the field of fractions $\mathbb{F}$ wherein all denominators are from S. That is to say, $D_S = \left\{ \frac{r}{s} \mid r \in D, s \in S \right\}$. The natural monomorphism $\epsilon : D \to \mathbb{F}$ induces a ring monomorphism $\epsilon : D \to D_S$ defined by mapping $x \mapsto \frac{x}{1}$. Note the field of fractions for D_S is precisely the field of fractions for D, namely $\mathbb{F}$ and we have the containment $D \subseteq D_S \subseteq \mathbb{F}$. We could alternatively characterize D_S as the intersection of all subrings of $\mathbb{F}$ containing D in which each element of S is a unit. Following unfortunate historical notation in the mathematics community, we follow [18]: when S is the set complement of a prime ideal, $S^C = \mathfrak{p}$, we denote $D_S = D_{\mathfrak{p}}$.

For saturated sets S_1, $S_2 \subseteq D$, $S_1 \cap S_2$ is also saturated. For any $x \in S_1 \cap S_2$, if $y \mid x$ then $y \in S_1$ and $y \in S_2$ since both are saturated. Hence, $y \in S_1 \cap S_2$. Given any subset of $S \subseteq D$ we may saturate that set by intersecting all saturated subsets of D containing S, which we say is the *saturation* of S. Note that if S is multiplicatively closed and its saturation is $\widehat{S}$ then $D_S = D_{\widehat{S}}$. Indeed, for any $s \in S$ and $s' \mid s$ with $s' \notin S$, we can write $s = s't$ for some t. We see that $\frac{td}{s} \in D_S$. But $\frac{td}{s} = \frac{td}{s't} = \frac{d}{s'}$. Thus, $D_{\widehat{S}} \subseteq D_S$. But also since $S \subseteq \widehat{S}$, we have $D_S \subseteq D_{\widehat{S}}$, so we conclude $D_S = D_{\widehat{S}}$.

One might suspect localizing at non-atomic elements could create integral domains with interesting factorization behaviors. In fact, problems arise from localizing at (subsets of) the non-atomic elements of D. In Example A.21, we see that the set of non-atomic elements is not multiplicatively closed in general.

Example A.20. Let $\mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x \geqslant 0\}$ be the additive sub-semigroup of $\mathbb{Q}$. Let $M \subseteq \mathbb{Q}^+$ be any additive sub-semigroup of $\mathbb{Q}^+$ such that M is monoid-isomorphic to a numerical semigroup (a cofinitely generated sub-semigroup of $\mathbb{N}$). We extend our ring-theoretic concept of factorization to monoids. We say $m \in M$ is an *atom* or an *irreducible* when $m = m_1 + m_2$ for some $m_1, m_2 \in M$ implies $m_1 = 0$ or $m_2 = 0$, we say m is *atomic* if m is a finite sum of atoms, and we say M is *atomic* if each non-zero $m \in M$ is atomic.

Let $\mathbb{F}$ be any field and X an indeterminate over $\mathbb{F}$. Define $\mathcal{X} = \{X^m \mid m \in M\}$. Let $R' = \mathbb{F}[\mathcal{X}]$. Let $\mathfrak{m} \subseteq R'$ be the ideal generated by $\mathcal{X}$. Then $\mathfrak{m}$ is maximal and every non-zero non-unit in the ring $R = R'_{\mathfrak{m}}$ is a sum of elements that are associate to some monomial in $\mathcal{X}$. Moreover, the multiplicative behavior in R is governed by the additive behavior in M. In [13], it is shown that since M is monoid isomorphic to a numerical semigroup, if 0 is not a limit point in M then M is atomic, which forces R to be atomic. $\triangle$.

Example A.21. Let X be an indeterminate over $\mathbb{F}_2$. Let $D' = \mathbb{F}_2[X, X^{2/3}, X^{2/9}, \ldots]$; D' is an integral domain since $\mathbb{F}_2$ is a field. Let $\mathfrak{m} \subseteq D'$ be the maximal ideal generated by the set of all non-unit monomials. Define $D = D'_{\mathfrak{m}}$. Note X is the only irreducible monomial. In particular $X^{2/3^n}$ is a non-unit and satisfies the relationships $X^{2/3^n} = (X^{2/3^{n+1}})^3$, and so is not irreducible. Moreover, $X^{2/3^n}$ is not atomic. The monomials are saturated and multiplicatively closed; if $X^{2/3^n}$ can be written as a product of atoms, then those atoms are necessarily associates of monomials, and X is uniquely irreducible among all monomials. It is impossible to write $X^{2/3^n} = X^m$ with an $m, n \in \mathbb{N}$. Thus, $X^{2/3^n}$ is not atomic and yet $(X^{2/3^n})^{3^n} = X^2$ is atomic. △

Example A.21 suggests we need to compute the multiplicative closure of non-atomic elements if we want to invert non-atomic elements. Even if the set of non-atomic elements of D is multiplicatively closed, Example 3.1.1 shows we may lose too much information when we invert non-atomic elements. This example is particularly instructive, because localizing at the set of all non-atomic elements yields the quotient field, in which all elements are units, forcing all factorizations to be trivial.

Examples A.21 and 3.1.1 suggest localization at non-atomic elements cannot be expected to reveal helpful factorization information. On the other hand, we could localize at the atoms of D. Indeed, localizing at a (saturated) multiplicatively closed set generated by irreducibles can lead to new units and new irreducibles. All localizations create new units, but Example A.22 demonstrates how a single localization can also create new irreducibles. Example A.23 demonstrates how a cochain of localizations can create new irreducibles at every degree.

Example A.22. Let D be a UFD with independent indeterminates X and Y and consider the ring $D_0 = D[X, Y, \frac{X}{Y}, \frac{X}{Y^2}, \ldots]$. The monomial Y is irreducible, but X is not irreducible. Let S be the saturated multiplicatively closed subset of D_0 generated by Y, namely $S = \{uY^n \mid u \in U(D_0), n \geqslant 0\}$. Arbitrary elements of $(D_0)_S$ take the form $\frac{r}{s}$ where $r \in D_0$ and $s \in S$. $Y \in (D_0)_S$ is a unit and $\frac{X}{1_{D_0}}$ is thus now irreducible in $(D_0)_S$. △

Example A.23. Example A.22 extends easily. Let $\mathbb{F}$ be any field with indeterminates $\{X_n \mid n \in \mathbb{N}\}$. Construct the polynomial extension $D' = \mathbb{F}[X_1, X_2, \ldots, \frac{X_2}{X_1^n}, \frac{X_3}{X_2^n}, \ldots \mid n \in \mathbb{N}]$. Define $\mathfrak{m} \subseteq D'$ to be the maximal ideal generated by all non-unit monomials, and set $D = D'_{\mathfrak{m}}$. Note X_1 is uniquely irreducible among monomials. We thus pick S to be the saturation of the atomic elements $\{1, X_1, X_1^2, X_1^3, \ldots\}$. In D_S, the element X_1 is now a unit and X_2 is now uniquely irreducible among monomials. Indeed, X_2 divides every other monomial and the monomials of the form $\frac{X_2}{X_1^j}$ are now associate since X_1 is a unit. Repeating the process iteratively yields a sequence of integral domains, $D_1 \subseteq D_2 \subseteq D_3 \subseteq \cdots$ in which X_i is a uniquely irreducible monomial in D_i. $\triangle$

We often localize polynomial rings at a saturated multiplicatively closed set generated by monomials. Recall by Lemma A.9 that a set $\mathcal{X}$ of independent indeterminates over D is saturated and multiplicatively closed. Moreover, $f \in D[\mathcal{X}] \setminus (\mathcal{X})$ if and only if $f(0) \neq 0$; hence, an arbitrary $f \in D[\mathcal{X}]$ is a unit in $D[\mathcal{X}]_{(\mathcal{X})}$ if and only if $f(0) \neq 0$. Recall Theorem A.18, wherein we demonstrate the ideal $(\mathcal{X})$ is maximal if and only if D is a field. Hence, setting $\mathfrak{m} \subseteq \mathbb{F}[\mathcal{X}]$ as the ideal generated by all monomials (which is maximal), in $\mathbb{F}[\mathcal{X}]_{\mathfrak{m}}$ every non-zero non-unit is associate to a monomial.

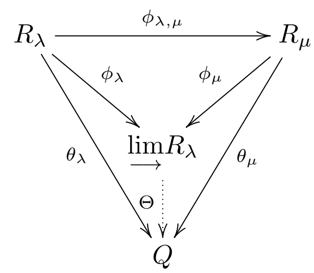
A.5 Direct Limits of Integral Domains

We occasionally investigate direct systems and their direct limits, so we provide some details behind these constructions. We say a set Λ is *directed* if there exists a pre-order on Λ, say $\preceq$, such that if, for any $\lambda, \mu \in \Lambda$, there exists some $\gamma \in \Lambda$ such that $\lambda \preceq \gamma$ and $\mu \preceq \gamma$. Let $\{R_\lambda \mid \lambda \in \Lambda\}$ be a family of algebraic objects (semigroups, monoids, groups, ideals, rings, modules, fields, algebras, etc.) indexed by a directed set, Λ. Let $\{\phi_{\lambda,\mu} : R_\lambda \to R_\mu \mid \lambda \preceq \mu\}$ be a family of homomorphisms such that $\phi_{\lambda,\lambda} = \mathrm{id}_{R_\lambda}$ and for any $\lambda, \mu, \gamma \in \Lambda$ such that $\lambda \preceq \mu \preceq \gamma$, then $\phi_{\lambda,\gamma} = \phi_{\mu,\gamma} \circ \phi_{\lambda,\mu}$. We say the collection $\left\{R_\lambda, \{\phi_{\lambda,\mu}\}_{\mu\in\Lambda}\right\}_{\lambda\in\Lambda}$ is a *direct system* over Λ.

For the direct system $\{R_\lambda\}$ and morphisms $\{\phi_{\lambda,\mu}\}$, we construct the direct limit with the disjoint union $\dot{\cup}_\lambda R_\lambda$ as the underlying set by applying an equivalence relation: for any $r_\lambda, s_\gamma \in R_\gamma$, we declare $r_\lambda \sim s_\gamma$ if and only if there exists some $\mu \in \Lambda$ such that $\phi_{\lambda,\mu}(r_\lambda) = \phi_{\gamma,\mu}(s_\gamma)$. The result, $R/\sim$, is called the *direct limit* of the direct system $\left\{R_\lambda, \{\phi_{\lambda,\mu}\}_{\mu\in\Lambda}\right\}_{\lambda\in\Lambda}$. We denote the direct limit as $\varinjlim R_\lambda = (\amalg_\lambda R_\lambda)/\sim$.

More generally, direct limits can be constructed in categories that admit coproducts by simply replacing the notion of the disjoint union with the construction of the coproduct and by formalizing the above construction in terms of a universal mapping property. The direct limit comes equipped with a sequence of morphisms $\{\phi_\lambda\}_\lambda$ where each ϕ_λ is a morphism of the form $\phi_\lambda : R_\lambda \to \varinjlim R_\lambda$ such that $\phi_\mu \circ \phi_{\lambda,\mu} = \phi_\lambda$ for each $\lambda \preceq \mu$. The direct limit is an object, $\varinjlim R_\lambda$, together with morphisms $\left\{\phi_\lambda : R_\lambda \to \varinjlim R_\lambda\right\}$ such that for any object S together with morphisms $\{\theta_\lambda : R_\lambda \to S\}$ satisfying $\theta_\mu \circ \phi_{\lambda,\mu} = \theta_\lambda$ for each $\lambda \preceq \mu$, there exists a unique morphism

$\Theta : \varinjlim R_\lambda \to S$ such that each $\theta_\lambda = \Theta \circ \phi_\lambda$.



Example A.24. Let D be an integral domain and consider the ascending chain of D-module monomorphisms:

$$M_0 \overset{\epsilon_0}{\hookrightarrow} M_1 \overset{\epsilon_1}{\hookrightarrow} M_2 \overset{\epsilon_2}{\hookrightarrow} \cdots$$

Without loss of generality, we may assume each $M_n \subseteq M_{n+1}$ by considering ϵ_n to be the canonical inclusion map, $\epsilon_n : M_n \to M_{n+1}$ given by mapping $x \mapsto x/1$. Let $\Lambda = \{0, 1, 2, \ldots\}$ under the usual total order. The family of homomorphisms $\{\phi_{n,m}\}$ may be taken to be the composition of inclusion monomorphisms where $\phi_{n,m} = \epsilon_{m-1} \circ \epsilon_{m-2} \circ \cdots \circ \epsilon_n$ and where $\phi_{n,n} = \mathrm{id}_{M_n}$. Thus, we have a direct system.

For an element of the disjoint union $x \in \dot{\cup}_n M_n$ there exists some N such that $x \in M_N$. For any $x_n \in M_n$ and $y_m \in M_m$, we define the equivalence relation $x_n \sim y_m$ if and only if there exists some $k \in \Lambda$ such that $\phi_{n,k}(x_n) = \phi_{m,k}(y_m)$. Certainly $\phi_{n,k}(x_n) = x_n/1$ and $\phi_{m,k}(y_m) = y_m/1$. Thus, we declare $x_n \sim y_m$ if and only if $x_n = y_m$ (thus we can select $n = m$). Under the equality relation we have that $\varinjlim M_n = \dot{\cup}_n M_n$.

Equivalently, to demonstrate $\dot{\cup}_n M_n$ is the direct limit, we can show $\dot{\cup}_n M_n$ satisfies an appropriate universal mapping property. For each M_i, let $\phi_i : M_i \to \dot{\cup}_n M_n$ denote the canonical injection. Let Q be a D-module and consider a family of D-module homomorphisms $\theta_n : M_n \to Q$. For any $m \in \dot{\cup}_n M_n$, there exists some i and some $m' \in M_i$ such that $m = \phi_i(m')$. We map $m \mapsto \theta_i(m')$ to define $\Theta : \dot{\cup}_n M_n \to Q$. Since each ϕ_i is well-defined, Θ is well-defined and moreover $\theta_i = \Theta \circ \phi_i$ by construction.

This result does not use that each M_n is D-module. In fact, we have a similar result if we have a chain of sets connected with monomorphisms. We have a similar result with chains of ideals in a ring connected with ideal containment or a chain of rings (fields, groups, modules, algebras) connected with ring monomorphisms (field, group, module, algebra monomorphisms, respectively).

In these cases we may regard these as ascending chains of objects connected by canonical inclusion morphisms where the direct limit is the union of the chain. △

Example A.25. Let D be an integral domain with an irreducible element and quotient field $\mathbb{F}$. Set $S_0 = U(D)$ and $D_0 := D_{S_0}$, let $S_1 \subseteq D_0$ be the saturation of some nonempty subset of the irreducibles in D_0. For each $s \in S_1$, select an indeterminate over D_0, say $x_0(s)$, and set $\mathcal{X}_0 := \{x_0(s) \mid s \in S_1\}$ and $\mathcal{Y}_0 := \left\{\frac{s}{x_0(s)} \mid s \in S_1\right\}$. Define $D_1 = D_0 [\mathcal{X}_0] [\mathcal{Y}]$. Then $D_0 \subseteq D_1$.

From [9, Lemma 2.5] we have that $U(D_1) = U(D_0)$. From [9, Lemma 2.6], we also have if $\pi \in Irr(D_0)$ and π not associate to any $s \in S_1$ then $\pi \in Irr(D_1)$. Iteratively, for $i > 0$, let $S_{i+1} \subseteq D_i$ be a nonempty subset of irreducibles in D_i. For each $s \in S_{i+1}$, select an indeterminate over D_i, say $x_i(s)$, and set $\mathcal{X}_i := \{x_i(s) \mid s \in S_{i+1}\}$ and $\mathcal{Y}_i := \left\{\frac{x_i(s)}{s} \mid s \in S_{i+1}\right\}$. Define $D_{i+1} = D_i [X_i] [\mathcal{Y}_i]$. This yields the chain of monomorphisms

$$D_0 \subseteq D_1 \subseteq D_2 \subseteq D_3 \subseteq \cdots$$

where each degree "removes the irreducibility" of a specified subset of irreducibles. In this case, any $x \in D$ that associate to some $s \in S$ ends up as a unit in the direct limit. △

Example A.26. Consider now the case of a sequence of integral domain localizations; let D be an integral domain and $\Sigma \subseteq D$ a saturated multiplicatively closed subset of D. Note Σ is partially ordered by the divisibility relation and is directed since, if $s, t \in \Sigma$, then $s \mid st$ and $t \mid st$. We take the index set to be $\Lambda = \Sigma$.

For any $s \in \Sigma$, denote the multiplicatively closed set generated by s as $\widehat{s} = \{1, s, s^2, \dots\}$. For any $s, t \in \Sigma$, if $s \mid t$, then there exists a map $\phi_{s,t} : D_{\widehat{s}} \to D_{\widehat{t}}$. To define this map, notice an arbitrary $\alpha \in D_{\widehat{s}}$ is of the form $\frac{x}{s^n}$ for some n. Since $s \mid t$, we may write $t = \tau s$ and notice that element $\frac{\tau^n x}{t^n} \in D_{\widehat{t}}$. We obtain the ring homomorphism $\phi_{s,t} : D_{\widehat{s}} \to D_{\widehat{t}}$ defined by $\frac{x}{s^n} \mapsto \frac{\tau^n x}{t^n}$ or equivalently, $\frac{x}{s^n} \mapsto \frac{(\frac{t}{s})^n x}{t^n}$. Certainly, if $s \mid t$ and $t \mid v$, then $\phi_{s,t} \circ \phi_{t,v} = \phi_{s,v}$. Thus $\{D_{\widehat{s}}\}_{s \in \Sigma}$ is a direct system indexed by Σ according to the divisibility pre-order.

Note that each $\phi_{s,t}$ is a ring monomorphism and so this example may be regarded as a special case of Example A.24. In fact, $\varinjlim D_{\widehat{s}}$ is naturally isomorphic to D_Σ. Certainly $\varinjlim D_{\widehat{s}} \subseteq D_\Sigma$, because if $x \in \varinjlim D_{\widehat{s}}$ then x is represented by an element in the disjoint union, so $x \in D_{\widehat{s}}$ for some s. Thus, we can write $x = \frac{d}{s}$ for some $d \in D$. But also if $x \in D_\Sigma$ then $x = \frac{d}{s}$ for some $d \in D$ and $s \in \Sigma$ and consequently $x \in D_{\hat{s}}$. Thus, x is in the disjoint union. That is to say, this example may be regarded as an alternative definition of D_Σ as $D_\Sigma = \varinjlim D_{\widehat{s}}$.

Equivalently, we can merely demonstrate D_Σ satisfies the appropriate universal mapping property. For each $s \in S$, we always have the canonical inclusion maps $\phi_s : D_{\widehat{s}} \to D_\Sigma$. If we have a system of ring monomorphisms $f_s : D_{\widehat{s}} \to Q$ for any integral domain Q, we can define $f : D_\Sigma \to Q$ in the following way. For each $\frac{d}{s} \in D_\Sigma$, we have $\frac{d}{s} \in D_{\widehat{s}}$ so $\frac{d}{s} = \phi_s\left(\frac{d}{s}\right)$. This allows us to simply define $f(\frac{d}{s}) = f_s(\frac{d}{s})$. Moreover, $f_i = f \circ \phi_i$ by construction. △

Localization can be characterized in terms of universal properties. If D is an integral domain and S is a multiplicatively closed set, we have the canonical inclusion $\epsilon : D \to D_S$. We may characterize D_S as the integral domain such that if $\phi : D \to T$ is any ring monomorphism and $\phi(S) \subseteq U(T)$, then there exists a unique $\overline{\phi} : D_S \to T$ such that $\phi = \overline{\phi} \circ \epsilon$. The universality of localization and the universality of direct limits are compatible in the sense that direct limits and localization commute.

Example A.27. Consider the localizations $D_{S_0} \subseteq D_{S_1} \subseteq \cdots$. These must sharea common quotient field, $\mathbb{F}$. We claim $D_{\varinjlim S_i} = \varinjlim D_{S_i}$. To see this, let D be an integral domain and $S_0 \subseteq S_1 \subseteq \cdots$ be an ascending chain of saturated multiplicatively closed subsets of D with $0 \notin S_i$ for each i. Then the direct system $\{\{S_i\}, \{\iota_i : S_i \to S_{i+1}\}\}$ (where each ι is a canonical monomorphism) has direct limit $\varinjlim S_i = \cup_i S_i$. Also, $\left\{\{D_{S_i}\}, \left\{\epsilon_i : D_{S_i} \to D_{S_{i+1}}\right\}\right\}$ is a direct system with direct limit $\varinjlim D_{S_i} = \cup_i D_{S_i} \subseteq \mathbb{F}$. If $x \in D_{\varinjlim S_i}$ then $x = \frac{d}{s}$ for some $d \in D$ and $s \in \varinjlim S_i = \cup_i S_i$. Hence, $s \in S_i$ for some particular i, so $\frac{d}{s} \in D_{S_i} \subseteq \cup_i D_{S_i}$. On the other hand, if $x \in \varinjlim D_{S_i} = \cup_i D_{S_i}$ then $x \in D_{S_i}$ for some i. So $x = \frac{d}{s}$ for $s \in S_i \subseteq \cup_i S_i$ so $x \in D_{\varinjlim S_i}$. This establishes our claim.

We may equivalently establish our claim by showing that $\cup_i D_{S_i} = D_{\cup_i S_i}$ with double containment; although this is an equivalent condition, this proof method is not as illustrative of the definition of direct limits. △

Speaking of universality, polynomial extensions and quotient rings are universal, also. Hence, for a direct system $\{\mathcal{X}_\lambda\}$ of indeterminates over D such that if $\lambda \leqslant \mu$ then $\mathcal{X}_\lambda \subseteq \mathcal{X}_\mu$, then we expect that $\varinjlim D[\mathcal{X}_\lambda] = D[\varinjlim \mathcal{X}_\lambda]$. Similarly, for a direct system of prime ideals in D, say $\{\mathfrak{p}_\lambda\}$ so that if $\lambda \leqslant \mu$ then $\mathfrak{p}_\lambda \subseteq \mathfrak{p}_\mu$. In Example A.28, we show $\varinjlim D/\mathfrak{p}_i = D/\varinjlim \mathfrak{p}_i$.

Example A.28. Consider now a sequence of integral domains together with canonical ring epimorphisms, say

$$D_0 \overset{\pi_0}{\twoheadrightarrow} D_1 \overset{\pi_1}{\twoheadrightarrow} D_2 \overset{\pi_2}{\twoheadrightarrow} \cdots$$

wherein each $D_{n+1} \cong D_n/\mathfrak{p}_n$ where $\mathfrak{p}_n \subseteq D_n$ is a prime ideal. The index set here is $\Lambda = \{0, 1, 2, \ldots\}$ under the usual total order. The family of homomorphisms $\{\phi_{n,m}\}$ may be taken to be compositions of the canonical epimorphisms: if $n < m$, define $\phi_{n,m} = \pi_{m-1} \circ \pi_{m-2} \circ \cdots \pi_n$ and define $\phi_{n,n} = \mathrm{id}_{D_n}$. Note that

$$\begin{aligned}\phi_{j,m} \circ \phi_{n,j} =& (\pi_{m-1} \circ \pi_{m-2} \circ \cdots \circ \pi_j) \circ (\pi_{j-1} \circ \pi_{j-2} \circ \cdots \circ \pi_n) \\ =& \pi_{m-1} \circ \pi_{m-2} \circ \cdots \circ \pi_n = \phi_{m,n}\end{aligned}$$

Thus, this is a direct system. Recall that prime ideals of a quotient ring, say R/I, are in one-to-one correspondence with the ideals of R containing I. For each $n \in \Lambda$, the inverse image of each kernel, $\pi_n^{-1}(\mathfrak{p}_{n+1})$, is a prime ideal in D_n. Applying this fact iteratively leads to an ascending chain of prime ideals in D_0

$$\mathfrak{q}_0 \subseteq \mathfrak{q}_1 \subseteq \mathfrak{q}_2 \subseteq \cdots$$

where $\mathfrak{q}_n = \pi_0^{-1} \circ \pi_1^{-1} \circ \cdots \pi_{n-1}^{-1}(\mathfrak{p}_n)$. This, too, is a direct system, with direct limit $\varinjlim \mathfrak{q}_n = \cup_n \mathfrak{q}_n$. Define $\mathfrak{q} := \cup_n \mathfrak{q}_n$. Note that the union of a chain of prime ideals is a prime ideal, and thus $\mathfrak{q}$ is a prime ideal. Thus, $D_0/\varinjlim \mathfrak{q}_n = D_0/\cup_n \mathfrak{q}_n = D_0/\mathfrak{q}$ is an integral domain.

Now, for each n, we have the canonical epimorphism $\psi_n : D_0/\mathfrak{q}_n \twoheadrightarrow D_0/\varinjlim \mathfrak{q}_n$. But we also have, by construction of $\varinjlim D_0/\mathfrak{q}_n$, maps of the form $\theta_n : D_0/\mathfrak{q}_n \twoheadrightarrow \varinjlim D_0/\mathfrak{q}_n$. By the universality mapping property for quotients, each map ψ_n factors as $\psi_n = \overline{\psi}_n \circ \theta_n$. By the universality of $\varinjlim \mathfrak{q}_n$, since $\mathrm{Ker}(\psi_n) \subseteq \mathrm{Ker}(\theta_n)$, θ_n factors as $\theta_n = \overline{\theta}_n \circ \psi_n$. In particular, we conclude that $\overline{\psi}_n \circ \overline{\theta}_n : D_0/\varinjlim \mathfrak{q}_n \to \varinjlim D_0/\mathfrak{q}_n$ is an isomorphism. △

Appendix B Partially Ordered Abelian Groups

In this section, we discuss partially ordered abelian groups (po-groups) and define the group of divisibility of an integral domain. Construction and use of po-groups is somewhat delicate and require more elaboration than usual group theoretic considerations. The properties and facts in this section are not exhaustive but are foundational to our treatment of po-groups; we direct the eager reader to [27] for an in-depth presentation.

In this appendix, we write the group operation multiplicatively unless otherwise stated (in contrast with our notation in Chapters 3 through 5, where we write po-groups additively). In B.1, we review the notion of the inherited quotient order on a quotient po-group, order-preserving group homomorphisms, and o-ideals. In Section B.2, we discuss morphisms between po-groups, and illustrate the connection between these morphisms and convex subgroups. In Section B.3, we construct po-groups using direct sums and we discuss the possible orderings on those constructions. In Section B.5, we define the group of divisibility for an integral domain, and review some of the properties associated with this group. In Theorem B.21, we make explicit the connection between o-ideals and saturated multiplicatively closed subsets of a domain.

B.1 Basics

We assume all groups are abelian.

Definition B.1. Let G be a po-group and $H \subseteq G$ be a subgroup. The group G/H inherits a relation from the partial order G, called the *inherited quotient relation*: say $aH \sim bH$ if and only if there exists some $h \in H$ such that $a \leqslant bh$.

In general, the inherited quotient relation is not a partial order, but a pre-order. We establish exactly when G/H is a po-group in Lemma B.2, but we also establish equivalent conditions in Lemma B.5.

Lemma B.2. *For a po-group G and subgroup $H \subseteq G$, the inherited quotient relation on G/H is always a pre-order. The inherited quotient relation is a partial order if and only if H is convex.*

Proof. The relation $\sim$ is reflexive. Since $\leqslant$ is reflexive and $e \in H$, we have that $g \leqslant ge = g$ for each $g \in G$, providing that $gH \sim gH$. The relation is also transitive: if $aH \sim bH$ and $bH \sim cH$ then we have some $h_1, h_2 \in H$ such that $a \leqslant bh_1$ and $b \leqslant ch_2$. Hence, we have $bh_1 \leqslant ch_2h_1$, so $a \leqslant ch_2h_1$, wherein $h_2h_1 \in H$. Thus, $aH \sim cH$.

Lastly, assume $\sim$ is antisymmetric and let $h_1 \leqslant g \leqslant h_2$. Since $h_1 \leqslant g$, we have $H \sim gH$, and since $g \leqslant h_2$, we have $gH \sim H$. Hence, $gH = H$, so $g \in H$, providing that $\leqslant$ is convex. On the other hand, if $\leqslant$ is convex and $aH, bH \in G/H$ such that $aH \sim bH$ and $bH \sim aH$, then we have some $h_1, h_2 \in H$ such that $a \leqslant bh_1$ and $b \leqslant ah_2$. Thus, we have that $ab^{-1} \leqslant h_1$ and $h_2^{-1} \leqslant ab^{-1}$. Since H is convex, we conclude $ab^{-1} \in H$, i.e. $aH = bH$. ☐

Denote $\sim$ as $\preceq$ when $\sim$ is a pre-order (H is not convex) and denote $\sim$ as $\leqslant$ when $\sim$ is a partial order (H is convex). A po-group homomorphism is a group homomorphism $\phi : G \to G'$ that is *order-preserving* in the sense that if $x \leqslant_G y$ then $\phi(x) \leqslant_{G'} \phi(y)$. Certain order-preserving functions are *order-reflecting* in the sense that $f(x) \leqslant_{G'} f(y)$ implies $x \leqslant_G y$. Other order-preserving functions *order-embedding*: $x \leqslant_G y$ if and only if $f(x) \leqslant_{G'} f(y)$.

Lemma B.3. *Let G be a po-group and $H \subseteq G$ be a subgroup. The canonical group epimorphism $\pi : G \twoheadrightarrow G/H$ is order-preserving. H is convex if and only if π is an o-epimorphism.*

Proof. Let $x \leqslant_G y$. Then $x \leqslant_G y1_G$ and $1_G \in H$, so $xH \preceq yH$. If H is convex, let $y \in (G/H)^+$. Since π is surjective, $y = \pi(x) = xH$ for some $x \in G$, and since y is positive in G/H, there exists some $x' \in xH$ and $h \in H$ such that $x' \geqslant h$. Hence, $x'h^{-1} \geqslant e$, and $x'h^{-1} \in xH$ by construction. Hence, $xH \in \pi(G^+)$, namely $xH = \pi(x'h^{-1})$. On the other hand, if π is an o-epimorphism and $h_1 \leqslant g \leqslant h_2$ in G, then $\pi(h_1) \leqslant \pi(g) \leqslant \pi(h_2)$, where $\pi(h_i) = H = 1_{G/H}$. By antisymmetry, $\pi(g) = 1_{G/H}$ so $g \in \mathrm{Ker}(\pi)$. ☐

Lemma B.3 leans in the direction of isomorphism theorems; we reserve a discussion of these until Section B.2. Following the terminology of [11], we refer to an element $g \in G$ as *non-negative* if $e_G \leqslant_G g$, or *positive* if $e_G <_G g$, and we refer to the subset of all non-negative elements of G as the *positive cone* of G, denoted as G^+. The positive cone is a partially ordered monoid worthy of its own study since the properties of G^+ can determine properties of G. For example, G is directed if and only if G is generated by G^+, i.e. $G = (G^+) \cdot (G^+)^{-1}$. G is linearly ordered under $\leqslant$ if and only if $G = (G^+) \cup (G^+)^{-1}$.

We refer to an irreducible element in an integral domain as an atom; similarly, if an element g in a po-group G is minimal and positive in the partial order $\leqslant_G$, we say g is an *atom*. The subgroup of G generated by all atoms is the *atomic subgroup*, which we denote $A(G)$. Just as we say products of irreducibles in an integral domain are *atomic* ring elements, we say that g is an *atomic* po-group element if g is in the atomic subgroup. Note that a subgroup, $H \subseteq G$ may be directed even if G is not directed; indeed, let G be any subgroup that is not directed with at least one non-negative element, say $e \leqslant g$. Then $\langle g \rangle \subseteq G$ is clearly directed as it is generated by a positive element.

Following Fuchs in [11] and Močkoř in [27], we refer to any subgroup that is both directed and convex as an *o-ideal*. As we shall see, these subgroups share an important connection with integral domains and play a central role in this study.

Example B.4. Not all directed subgroups are convex. For an example of such a subgroup, consider the additive subgroup $2\mathbb{Z} \subseteq \mathbb{Z}$ under the usual total ordering. Also, not all convex subgroups are directed. For an example of such a subgroup, consider again the integers and their group of divisibility, $G(\mathbb{Z}) = \mathbb{Q}^{\times}/U(\mathbb{Z})$ under the partial order induced by divisibility. We claim the subgroup $\langle 2/3 \rangle$ is convex. Indeed, assume $(\frac{2}{3})^n \leqslant x \leqslant (\frac{2}{3})^m$ for some $n, m \in \mathbb{Z}$. In particular, $(\frac{2}{3})^n \leqslant (\frac{2}{3})^m$. Since the ordering here is the divisibility partial order, we have $\frac{(2/3)^m}{(2/3)^n} \in \mathbb{Z}$ so $n = m$. Antisymmetry provides $x = (\frac{2}{3})^n$ and so H is (vacuously) convex. △

B.2 Morphisms of Po-Groups and Isomorphism Theorems

We follow [11] and [27] to define homomorphisms of po-groups as group homomorphisms that are order-preserving (known as *o-homomorphisms*). We denote the category whose objects are po-groups and whose arrows are po-group homomorphisms as **Pog**, the category of po-groups. Following the notation in [11] and [12], we say an order-preserving po-group homomorphism $f : G \to H$ is a *o-epimorphism* when f is an order-preserving group epimorphism such that $f(G^+) = H^+$. As we saw in Example 3.2.2, not every epic morphism in **Pog** is an o-epimorphism (or is even surjective). Despite that o-epimorphisms in the sense of Fuchs are not a complete representation of all epic morphisms in **Pog**, they play a central role in that category.

Lemma B.5. *Let G be a po-group with subgroup $H \subseteq G$. The following are equivalent:*

(i) H is a convex subgroup of G,

(ii) G/H is a po-group under the inherited quotient order $\leqslant_{G/H}$,

(iii) H is the kernel of some o-epimorphism, and

(iv) For any $g_1, g_2 \in G^+$, if $g_1 g_2 \in H$, then $g_1 \in H$ and $g_2 \in H$.

Proof. In Lemmata B.2 and B.3, we establish the equivalency of (i) and (ii), and that (i) implies (iii). We also established the o-epimorphism in (iii) is precisely the canonical group epimorphism $\pi : G \to G/H$. Moreover, if $H = \text{Ker}(\pi)$ for some o-epimorphism $\pi : G \to G'$, a similar argument to Lemma B.3 demonstrates H is convex.

It remains to prove the equivalency of (i) with (iv). Assume H is convex and let $g_1, g_2 \in G^+$ so therefore $e_G \leqslant g_1 \leqslant g_1 g_2$ and $e_G \leqslant g_2 \leqslant g_1 g_2$. Certainly $g_1 g_2 \in G^+$. Convexity in H implies (iv). On the other hand, assume H satisfies (iv) and $h_1 \leqslant g \leqslant h_2$ for some $h_1, h_2 \in H$. Then $h_1^{-1} g$ and $g^{-1} h_2$ are both non-negative elements whose product is in H which satisfies (iv) so we obtain that both $h_1^{-1} g$ and $g^{-1} h_2$ are in H, yielding that $g \in H$, establishing convexity. □

At this point, it is worth noting that isomorphisms in the category of po-groups are distinct from the isomorphisms in the category of posets. Indeed, *order isomorphisms* in the theory of ordered sets are defined as *bijective order-embeddings* between partially ordered sets [6], which are precisely the isomorphisms in the category posets and their monotonic functions **Pos**. The category of po-groups **Pog** is a subcategory of **Pos**, so just as the hurdle was lower in **Pog** compared to the category of all groups in order to be an epimorphisms (monomorphism, isomorphism, respectively), the notion of an order isomorphism from ordered set theory is perhaps impoverished in the po-group setting. At the least, we require morphisms to be o-homomorphisms, not simply order-preserving functions.

To emphasize the difference between order isomorphisms from **Pos** and po-group isomorphisms from **Pog**, we follow the definition of Fuchs (from [11], and [12]): if $f : G \to H$ is a group isomorphism and an o-epimorphism such that $f^{-1} : H \to G$ is also o-epimorphism, then both f and f^{-1} are known as *o-isomorphisms*. If there exists an o-isomorphism $f : G \to H$, we denote this as $G \simeq H$ (to distinguish between group or ring isomorphisms, which we denote with $\cong$). We are generally only concerned with partially ordered groups up to o-isomorphism; if two po-groups are o-isomorphic, we will consider them to be the "same."

We do not have the luxury of all three group isomorphism theorems, as we now see. The first and third group isomorphism theorems from the category of groups have o-isomorphism analogues in the category of abelian po-groups: we simply replace the notion of "normal subgroups" with the notion of "convex subgroups." Since our groups are abelian, normality is automatic for all subgroups.

Theorem B.6 (First o-isomorphism theorem)**.** *Let G and H be abelian po-groups, and $\phi : G \to H$ be an o-homomorphism. Then:*

(i) the kernel of ϕ is a convex subgroup of G,

(ii) the image of ϕ is a subgroup of H, and

(iii) the image of ϕ is o-isomorphic to the quotient group $G/\operatorname{Ker}(\phi)$.

In particular, if ϕ is an o-epimorphism then H is o-isomorphic to $G/\operatorname{Ker}(\phi)$.

Proof. For (i), let $g_1, g_2 \in \operatorname{Ker}(\phi) \subseteq G$ and $x \in G$ such that $g_1 \leqslant x \leqslant g_2$. Then $\phi(g_1) = \phi(g_2) = e_H$, and since ϕ is an o-homomorphism, we then have that $e_H \leqslant \phi(x) \leqslant e_H$. Since H is partially ordered, the order on H is antisymmetric so $\phi(x) = e_H$. For (ii), let $h_1, h_2 \in \phi(G)$. Then there exists some $g_1, g_2 \in G$ such that $h_1 = \phi(g_1)$, $h_2 = \phi(g_2)$. Then $h_1 h_2^{-1} = \phi(g_1 g_2^{-1}) \in \operatorname{Im}(\phi)$. Furthermore, since ϕ is an o-homomorphism, the partial order on $\operatorname{Im}(\phi)$ is compatible with the partial order on H. For (iii), we define the map $\psi : G/\operatorname{Ker}(\phi) \to \operatorname{Im}(\phi)$ by mapping $g\operatorname{Ker}(\phi) \mapsto \phi(g)$. This is certainly a well-defined map: if $g_1 \operatorname{Ker}(\phi) = g_2 \operatorname{Ker}(\phi)$, then $g_1 g_2^{-1} \in \operatorname{Ker}(\phi)$, so $\phi(g_1 g_2^{-1}) = e_H$ and, in particular, $\phi(g_1) = \phi(g_2)$. Furthermore, each $\phi(g) \in \operatorname{Im}(\phi)$ by definition. This map is surjective, as any $h \in \operatorname{Im}(\phi)$, by definition, may be written as $h = \phi(g)$ for some g, and furthermore, $h = \psi(g\operatorname{Ker}(\phi))$. This map is also injective, because $g\operatorname{Ker}(\phi) \in \operatorname{Ker}(\psi)$ implies $\phi(g) = e_H$, i.e. $g \in \operatorname{Ker}(\phi)$. The map ψ is a group homomorphism, because for any $g_1 \operatorname{Ker}(\phi)$, $g_2 \operatorname{Ker}(\phi)$, we have

$$\begin{aligned}\psi\left(\left(g_1 \operatorname{Ker}(\phi)\right)\left(g_2 \operatorname{Ker}(\phi)\right)\right) =& \psi\left(g_1 g_2 \operatorname{Ker}(\phi)\right)\\ =& \phi(g_1 g_2)\\ =& \phi(g_1)\phi(g_2)\\ =& \psi(g_1 \operatorname{Ker}(\phi))\psi(g_2 \operatorname{Ker}(\phi)).\end{aligned}$$

Also ψ is order-preserving. Indeed, say $g_1 \operatorname{Ker}(\phi) \leqslant g_2 \operatorname{Ker}(\phi)$. Then there exists some $x \in \operatorname{Ker}(\phi)$ such that $g_1 \leqslant g_2 x$. Hence ϕ is an o-homomorphism so $\phi(g_1) \leqslant \phi(g_2 x)$ and since $x \in \operatorname{Ker}(\phi)$, we have $\phi(g_2)\phi(x) = \phi(g_2)$. Since $\phi(g_1) \leqslant \phi(g_2)$, $\psi(g_1 \operatorname{Ker}(\phi)) = \phi(g_1)$ and $\psi(g_2 \operatorname{Ker}(\phi)) = \phi(g_2)$ satisfy $\psi(g_1 \operatorname{Ker}(\phi)) \leqslant \psi(g_2 \operatorname{Ker}(\phi))$. Lastly, ψ is an o-epimorphism: by Theorem B.5, since ψ is injective, the kernel is trivial which is, by definition, convex. This establishes ψ as an o-isomorphism. □

Although we have established the po-group analogue to the first group isomorphism theorem, the second group isomorphism theorem does not have an o-isomorphism analogue.

Example B.7. To see that replacing "normal subgroups" with "convex subgroups" does not lead to a po-group analogue of the second isomorphism theorem, let $G = \mathbb{C}$ be the additive group of complex numbers. Order G with the partial order $a + bi \leqslant_G c + di$ if and only if $(c - a) \geqslant_{\mathbb{R}} 0$ and $(d-b) >_{\mathbb{R}} 0$ or $a+bi = c+di$. Define $H_1 = \mathbb{R} \subseteq G$ and define $H_2 = i\mathbb{R} \subseteq G$. Note that $\langle H_1, H_2 \rangle = G$

Note that H_1 inherits the order from G. If $a + bi, c + di \in H_1$ then $0 = b = d$, so $d - b \not> 0$. Hence, $a + bi \leqslant c + di$ in H_1 if and only if $a + bi = c + di$. Thus the partial order from G on H_1 is the equality relation. Also H_2 inherits the order from G. If $a + bi, c + di \in H_2$ then $a = c = 0$, so $c - a \geqslant 0$. Thus, $a + bi \leqslant c + di$ in H_2 if and only if $d - b > 0$, i.e. $d > b$. Hence, we have that $H_1/(H_1 \cap H_2) = H_1/0 = H_1 = \mathbb{R}$ under the equality relation, and we have that $\langle H_1, H_2 \rangle / H_2 = G/H_2$ and $G/H_2 = \mathbb{C}/(i\mathbb{R})$ which is isomorphic as a group to $\mathbb{R}$. However, we claim that the inherited relation on G/H_2 is a total ordering so G/H_2 cannot be o-isomorphic to $H_1/(H_1 \cap H_2)$.

Indeed, let $x, y \in G/H_2$, then $x = (a + bi) + H_2$ and $y = (c + di) + H_2$. Either $a \leqslant c$ or $c \leqslant a$ in $\mathbb{R}$. Without loss of generality, select $a \leqslant c$. Define $\alpha i = (b - d)i$. Then $a + bi \leqslant c + bi$ in G since $a \leqslant c$, so $(a + bi) \leqslant (c + di) + \alpha i$, and so $(a + bi) + H_2 \leqslant (c + di) + H_2$. That is to say, any two po-group elements are comparable so the inherited quotient order on G/H_2 is a total ordering, completing the example. △

Theorem B.8 (Third o-isomorphism theorem)**.** *Let G be a po-group, and $H \subseteq G$ a convex subgroup. Then:*

(i) If H' is a subgroup of G such that $H \subseteq H' \subseteq G$, then H'/H is a subgroup of G/H.

(ii) Every subgroup of G/H is of the form H'/H, for some subgroup $H' \subseteq G$ such that $H \subseteq H' \subseteq G$.

(iii) If H' is a convex subgroup of G such that $H \subseteq H' \subseteq G$, then H'/H is a convex subgroup of G/H.

(iv) Every convex subgroup of G/H is of the form H'/H, for some convex subgroup H' of G such that $H \subseteq H' \subseteq G$.

(v) If H' is a convex subgroup of G such that $H \subseteq H' \subseteq G$, then the quotient group $\frac{G/H}{H'/H}$ is o-isomorphic to G/H'.

Proof. The proofs of (i) and (ii) are identical to the proofs of their analogues in the group-theoretic context, so we omit those proofs. For (iii), let H' be convex in G with $H \subseteq H' \subseteq G$. Consider a chain $aH \leqslant bH \leqslant cH$ in G/H with $aH, cH \in H'/H$. Then $a, c \in H'$. By the definition of the induced quotient partial order on G/H, since $aH \leqslant bH$, there exists some $h_1 \in H$ such that $a \leqslant bh_1$, and since $bH \leqslant cH$, there exists some $h_2 \in H$ such that $b \leqslant ch_2$. Thus, $ah_1^{-1} \leqslant b \leqslant ch_2$ in G. But $H \subseteq H'$ so $h_1^{-1}, h_2 \in H'$, and $a, c \in H'$, so we have bounded b with elements from H', which is convex in G. Thus, $b \in H'$, so $bH \in H'/H$.

For (iv), let $A \subseteq G/H$ be a convex subgroup. By (ii), we can write $A = H'/H$ for some subgroup such that $H \subseteq H' \subseteq G$. We claim that if A is convex in G/H, then H' is convex in G. If A is convex in G/H and if $a \leqslant b \leqslant c$ for some $a, b, c \in G$ such that $a, c \in H'$, then note that the canonical projection $\pi : G \mapsto G/H$ is an o-epimorphism since H is convex. In particular, it is order-preserving, and so $aH \leqslant bH \leqslant cH$. Since $a, c \in H'$, we have that $aH, cH \in A = H'/H$, which is convex, and so we conclude $bH \in A = H'/H$. Hence, $b \in H'$.

For (iv), if H' is a convex subgroup of G such that $H \subseteq H' \subseteq G$, then H'/H is a convex subgroup of G/H by (iii). Thus, the canonical projection, $\pi : G/H \mapsto \frac{G/H}{H'/H}$, is an o-epimorphism. By Theorem B.6 $\mathrm{Im}(\pi) \simeq \frac{G/H}{\mathrm{Ker}(\pi)}$. Of course, $\mathrm{Ker}(\pi)$ is precisely H'/H. ☐

B.3 Ordering Direct Sums of Po-Groups

For any po-groups, G_1, G_2, the direct sum, $G = G_1 \oplus G_2$, is an abelian group under the usual binary operation. It also admits at least three partial orders, the product order, the lexicographic order, and the reverse lexicographic order. The *product order* on $G_1 \oplus G_2$ is defined by saying $(a, b) \leqslant (c, d)$ if and only if $a \leqslant c$ and $b \leqslant d$. This extends naturally to direct sums of sequences of po-groups: if Λ is any index set and $\{G_\lambda\}_{\lambda \in \Lambda}$ is a collection of po-groups indexed by Λ, we may define the product order on $\oplus_{\lambda \in \Lambda} G_\lambda$ by declaring $(g_\lambda)_{\lambda \in \Lambda} \leqslant (h_\lambda)_{\lambda \in \Lambda}$ if and only if $g_\lambda \leqslant h_\lambda$ for every $\lambda \in \Lambda$. Note this definition does not use the property of the direct sum that an element of $\oplus_{\lambda \in \Lambda} G_\lambda$ is a sequence of po-group elements such that all but a finite number of $g_\lambda = e_{G_\lambda}$. Hence, the product order as defined here holds for the direct product $\Pi_\lambda G_\lambda$ as well.

We define the *lexicographic order* on $G_1 \oplus G_2$ or $G_1 \, \Pi \, G_2$ by declaring $(a, b) \leqslant (c, d)$ if and only if $a < c$ or $a = c$ and $b \leqslant d$. By this definition, if G_1 and G_2 are totally ordered then $G_1 \oplus G_2$ (or $G_1 \, \Pi \, G_2$) is totally ordered. The minimal positive elements of $G_1 \oplus G_2$ are precisely the elements of the form $(0, g)$ for some minimal positive $g \in G_2$. For any countable collection of po-groups indexed by some poset, we can iteratively apply the definition of the lexicographic order on $G_1 \oplus G_2$ to obtain a lexicographic order on a direct sum of those po-groups. In particular, the direct sum of any countable collection of totally ordered po-groups is totally ordered. We can reverse the order of this comparison to obtain the colexicographic order: $(a, b) \leqslant (c, d)$ if and only if $b < d$ or $b = d$ and $a \leqslant c$. In this case, the minimal positive elements of $G_1 \oplus G_2$ are from G_1. If G_1, G_2 are totally ordered then both lexicographic orders on $G_1 \oplus G_2$ are total orders.

We more generally define the lexicographic order for uncountable collections in the following way. Let Λ be a partially ordered set with ordering $\leqslant_\Lambda$ and let $\{G_\lambda\}_{\lambda\in\Lambda}$ be a collection of po-groups. Define the lexicographic order on $\oplus_\lambda G_\lambda$ by declaring $(a_\lambda) \leqslant (b_\lambda)$ if and only if (i) $(a_\lambda) = (b_\lambda)$ or (ii) there exists some $\lambda_0 \in \Lambda$ such that $a_{\lambda_0} <_{G_{\lambda_0}} b_{\lambda_0}$ and if $\lambda \leqslant \lambda_0$ then $a_\lambda = b_\lambda$. Just as before, the same definition applies to the product $\Pi_\lambda G_\lambda$ without harm. If Λ is infinite, then the lexicographic ordering on $\oplus_\lambda G_\lambda$ never provides any minimal positive elements (because we need a "last" coordinate).

Just as before, we can proceed backward through the index set to obtain the colexicographic ordering. If we further have that the partial order on Λ is a well-ordering, the minimal positive elements of $\oplus_\lambda G_\lambda$ are precisely the minimal positive elements of $G_{\lambda'}$ where λ' is the minimal among all $\lambda \in \Lambda$.

These generalizations of the (co)lexicographic orders come equipped with a caveat. For an arbitrary collection of totally ordered groups $\{G_\lambda\}$ indexed by a partially-ordered index set Λ, then $\oplus_\lambda G_\lambda$ ordered either lexicographically or colexicographically may not be totally ordered under this alternative definition of the (co)lexicographic ordering. Nor is it enough to assume that Λ is totally ordered to conclude that $\oplus_\lambda G_\lambda$ is totally ordered. However, if Λ is well-ordered, then $\oplus_\lambda G_\lambda$ will be totally-ordered.

We now apply the notion of ordering direct sums to demonstrate that quotient po-groups and direct sums are compatible.

Corollary B.9. *Let Λ be an index set ordered by $\preceq$ and let $\{G_i\}_{i\in\Lambda}$ be a set of po-groups indexed by Λ with corresponding o-ideals $\{H_i\}_{i\in\Lambda}$. Define $G := \oplus_{i\in\Lambda} G_i$ and $H := \oplus_{i\in\Lambda} H_i$ both under the product order (lexicographic order, colexicographic order, respectively). Then there is an o-isomorphism*

$$G/H \simeq \oplus_{i\in\Lambda}(G_i/H_i)$$

where the order on $\oplus_{i\in\Lambda}(G_i/H_i)$ is the product order (lexicographic order, colexicographic order, *respectively).*

Proof. Consider first the case of the product ordered direct sum. Consider first the naturally defined map $\phi : \oplus_{i\in\Lambda} G_i \longrightarrow \oplus_{i\in\Lambda}(G_i/H_i)$. Certainly ϕ is a surjective homomorphism. To see that ϕ is an o-epimorphism, let $\{g_iH_i\}_{i\in\Lambda} \in \oplus_{i\in\Lambda} G_i/H_i$ that is non-negative in the order on $\oplus_{i\in\Lambda} G_i/H_i$. Non-negativity in the product order demands that, for each $i \in \Lambda$, there exists some $h_i \in H_i$ such that $g_ih_i \geqslant_{G_i} e_{G_i}$. Of course, as an element of the direct sum, almost every (all but finitely many) $g_i \in H_i$. Further, $\{g_iH_i\}_i$ is the image of $\{g_ih_i\}_i \in G^+$. Hence, ϕ is an o-epimorphism and we may feel free to apply Theorem B.6 to complete the proof.

Now consider the lexicographic case and the map $\phi : \oplus_{i\in\Lambda} G_i \longrightarrow \oplus_{i\in\Lambda}(G_i/H_i)$. We claim ϕ is an o-epimorphism. Let $\{g_iH_i\}_{i\in\Lambda} \in (\oplus_{i\in\Lambda} G_i/H_i)^+$. Certainly ϕ is a surjective group homomorphism. We find some $\{x_i\}_i \in (\oplus_{i\in\Lambda} G_i)^+$ such that $\phi(\{x_i\}_i) = \{g_iH_i\}_i$. As an element of the direct sum, almost every (all but finitely many) $g_iH_i = H_i$; for these indices, we can select the representative $g_i = e_G$. Positivity in the lexicographic order demands that there exists some $\lambda \in \Lambda$ such that $H_i = g_iH_i$ for each $i \prec \lambda$ and $H_\lambda < g_\lambda H_\lambda$. In particular, if $i \prec \lambda$ then $g_i \in H_i$ and if $i = \lambda$ there exists some $h_\lambda \in H_\lambda$ such that $e_{G_\lambda} < g_\lambda h_\lambda$. Thus, given $\{g_iH_i\}$, define $x_i = g_i$ for each $i \neq \lambda$ and define $x_\lambda = g_\lambda h_\lambda$. Then $\{g_iH_i\} = \phi(\{x_i\})$.

Note $\{x_i\} \in G^+$ under the lexicographic order by construction. Hence, ϕ is an o-epimorphism and we apply Theorem B.6 to complete the proof. *Mutatis mutandis*, this argument applies to the colexicographic order. □

Note the product order and the lexicographic orders are related in the following sense: the underlying sets are equal, and if $(g_\lambda) \leqslant (h_\lambda)$ under the product order then $(g_\lambda) \leqslant (h_\lambda)$ under the (co)lexicographic order. Following the terminology from [30], for two relations on a set, say $\sim$ and $\approx$ on X, for any $a, b \in X$, if $a \sim b$ implies $a \approx b$, then we say $\approx$ is a *coarser* relation than $\sim$ (equivalently, $\sim$ is a *finer* relation than $\approx$). In this sense, the product order is a finer order than the lexicographic order (or the lexicographic order is coarser). The equality partial order is finer than all other partial orders.

B.4 Valuation Domains and Partially Ordered Groups

We introduce valuation domains, which are integral domains that are intricately connected to totally ordered groups, and which play a central role in many of our examples.

Definition B.10. Let $\mathbb{F}$ be any field and D and G a po-group under binary operation $*$ and partial order $\leqslant$. We say $\nu : \mathbb{F} \setminus \{0\} \to G$, such that, for any $a, b \in G$,

(i) $\nu(ab) = \nu(a) * \nu(b)$ and

(ii) $\nu(t) \leqslant \nu(a - b)$ for every $t \in \mathbb{F} \setminus 0$ such that $\nu(t) \leqslant \nu(a)$ and $\nu(t) \leqslant \nu(b)$.

is a *semi-valuation.* If G is lattice-ordered, we say ν is a *demi-valuation.* If G is totally ordered, we say ν is a *valuation.*

Theorem B.11. *Let $\nu : \mathbb{F}\setminus 0 \to G$ be a semi-valuation. If $\nu(1) \geqslant e_G$ then $\{x \in \mathbb{F} \setminus 0 \mid \nu(x) \geqslant 0\} \cup \{0\}$ is a subring of $\mathbb{F}$.*

Proof. Let $T = \{x \in \mathbb{F} \setminus 0 \mid \nu(x) \geqslant 0\} \cup \{0\}$. We can restrict the binary operations from $\mathbb{F}$ to T, and we claim T is closed under both of these operations. We claim T is additively and multiplicatively closed. Let $x, y, z \in T$. Then $\nu(x + y) \geqslant \nu(x)$ and since $\nu(x) \geqslant e_G$, we have $\nu(x + y) \geqslant e_G$, so $x + y \in T$. On the other hand, if $x, y \in T$, then $\nu(xy) = \nu(x) + \nu(y)$. Since $x, y \in T$, $\nu(x) \geqslant e_G$ and $\nu(y) \geqslant e_G$ so $\nu(x) * \nu(y) \geqslant e_G$. Hence, $xy \in T$. Of course, since $\mathbb{F}$ is commutative, T is commutative, and since $\nu(1) \geqslant e_G$, we have $1 \in T$, so T is a commutative ring with identity. □

Definition B.12. Let D be an integral domain with quotient field $\mathbb{F}$. If there exists a valuation $\nu : \mathbb{F} \setminus 0 \to G$ such that $D = \{x \in \mathbb{F} \setminus \{0\} \mid \nu(x) \geqslant 0\}$, we say D is a *valuation domain.*

Lemma B.13. *Let D be an integral domain with quotient field $\mathbb{F}$. The following are equivalent:*

(i) D is a valuation domain

(ii) if $x \in \mathbb{F} \setminus \{0\}$ then $x \in D$ or $x^{-1} \in D$

(iii) if $x, y \in D \setminus 0$ then $x \mid y$ or $y \mid x$.

Proof. We write G additively so that $e_G = 0$ and the operation $*$ is written as addition. For (i) ⇒ (ii), assume D is a valuation domain and let $x \in \mathbb{F}$. Since D is a valuation domain there exists some valuation $\nu : \mathbb{F} \setminus 0 \to G$ where G is a totally ordered group. Since G is totally ordered, for any $x_1, x_2 \in \mathbb{F} \setminus \{0\}$, we have either that $\nu(x_1) \leqslant \nu(x_2)$ or $\nu(x_2) \leqslant \nu(x_1)$. If $\nu(x_1) \leqslant \nu(x_2)$ then $0 \leqslant \nu(x_2) - \nu(x_1) = \nu(x_2 x_1^{-1})$. Hence, $x_2 x_1^{-1} \in D$ or $x_1 x_2^{-1} \in D$. In this case, let $x_1 = x$ and $x_2 = 1_D$. Then either $x \in D$ or $x^{-1} \in D$. For (ii) ⇒ (iii), let $x, y \in D \setminus \{0\}$ and assume that if $z \in \mathbb{F} \setminus \{0\}$ then $z \in D$ or $z^{-1} \in D$. Note $\frac{x}{y}$ or $\frac{y}{x} \in D$, and hence either $y \mid x$ or $x \mid y$.

For (iii) $\Rightarrow$ (i), let D have quotient field $\mathbb{F}$, unit group $U(D)$, and assume any $x, y \in D \setminus 0$ satisfy $x \mid y$ or $y \mid x$. Let $G = G(D) = \mathbb{F}^{\times}/U(D)$ be the group of divisibility (which we will see in greater detail in Section B.5). The partial order on G is a total order: if $\frac{a}{b}U(D), \frac{c}{d}U(D) \in G(D)$, then from (iii) we have $\frac{c/d}{a/b} \in D$ or $\frac{a/b}{c/d} \in D$. The map $\nu : \mathbb{F} \setminus 0 \to G(D)$ given by mapping $\frac{a}{b} \mapsto \frac{a}{b}U(D)$ is a valuation. We have

$$\begin{aligned}\nu\left(\frac{a}{b}\frac{c}{d}\right) =& \frac{a}{b}\frac{c}{d}U(D)\\ =& \left(\frac{a}{b}U(D)\right)\left(\frac{c}{d}U(D)\right)\\ =& \nu\left(\frac{a}{b}\right)\nu\left(\frac{c}{d}\right)\end{aligned}$$

Let $t \in \mathbb{F} \setminus 0$ such that $\nu(t) \leqslant \nu(a)$ and $\nu(t) \leqslant \nu(b)$. Then $\frac{a}{t} \in D$ and $\frac{b}{t} \in D$, and so $\frac{a}{t} - \frac{b}{t} \in D$. In particular, $\frac{a-b}{t} \in D$, so $\nu(t) \leqslant \nu(a - b)$. The order on $G(D)$ is a total order since D is a valuation domain since, for any $\frac{x}{y}U(D)$, $\frac{w}{z}U(D)$ either $\frac{w/z}{x/y} = \frac{wy}{zx} \in D$ or $\frac{zx}{wy} \in D$. Thus ν is a valuation.

Lastly, we show $D = \{x \in \mathbb{F} \mid \nu(x) \geqslant e_G\}$. If $\nu(\frac{a}{b}) \in G(D)^+$ then $U(D) \leqslant \frac{a}{b}U(D)$, and so $\frac{a}{b} \in D$. On the other hand, if $a \in D$ then $\nu(a) = aU(D) \geqslant U(D)$. Hence, D is a valuation domain. □

Theorem B.14. *Let V be a valuation domain. The set Spec(V) is linearly ordered by set inclusion.*

Proof. Set inclusion is always a partial order. It is therefore sufficient to show that any two elements are comparable.

To this end, let $\mathfrak{p}, \mathfrak{q}$ be two distinct primes from V. If $\mathfrak{p} \setminus \mathfrak{q} = \emptyset$ then $\mathfrak{p} \subseteq \mathfrak{q}$ and we are done. If $\mathfrak{p} \setminus \mathfrak{q} \neq \emptyset$, let $p \in \mathfrak{p} \setminus \mathfrak{q}$. Then for any $q \in \mathfrak{q}$, we have that $p \mid q$ or $q \mid p$ from above. If $q \mid p$, then $p \in (q) \subseteq \mathfrak{q}$, which contradicts our choice of p. Hence, $p \mid q$; however, we selected q arbitrarily from $\mathfrak{q}$. That is to say: if $p \in \mathfrak{p} \setminus \mathfrak{q}$, then $p \mid q$ for every $q \in \mathfrak{q}$. Hence, for any $p \in \mathfrak{p} \setminus \mathfrak{q}$, we have that $\mathfrak{q} \subseteq (p)$ so $\mathfrak{q} \subseteq \mathfrak{p}$. □

Example B.15. Let $\mathbb{F}$ be any field, X an indeterminate over $\mathbb{F}$, and let $\mathfrak{m}$ be the maximal ideal in $\mathbb{F}[X]$ generated by X. Every element of the domain $D = \mathbb{F}[X]_{\mathfrak{m}}$ may be written uX^m for some $m \in \mathbb{N}$ and $u \in U(D)$. Hence, any two elements are of the form vX^n and uX^m with $n \leqslant m$ or $m \leqslant n$. Either way, one divides the other so R is a valuation domain. △

Example B.16. Let X, Y be indeterminate over $\mathbb{F}$ and define $T = \mathbb{F}[X, Y]$. In the quotient field of T, we have the set $\mathcal{Z} = \left\{\frac{X}{Y}, \frac{X}{Y^2}, \frac{X}{Y^3}, \ldots\right\}$. Let $T[\mathcal{Z}] = \mathbb{F}[X, Y, \mathcal{Z}]$ have maximal monomial ideal $\mathfrak{m}$ and define $D = T[\mathcal{Z}]_{\mathfrak{m}}$. Each non-zero non-unit element of D is associate to some monomial. Moreover, $Y \mid X$ and Y is irreducible and T is a valuation domain. △

Lemma B.17. *Let V be a valuation domain. If V has any irreducible $\pi \in V$, then every irreducible in V is associate to π and π is prime.*

Proof. Say $\pi, \pi' \in V$ are irreducibles. If $\pi \mid \pi'$, we can write $\pi' = a\pi$ for some $a \in V$. But π' is irreducible, so $a \in U(V)$. On the other hand, if $\pi' \mid \pi$, we can write $\pi = a\pi'$. Again, π is irreducible, so $a \in U(V)$. Hence, all irreducibles are associate to π. It remains to be shown that π is prime.

Let $x, y \in V$ such that $\pi \mid xy$. Since V is a valuation domain we may compare π and x and we see $\pi \mid x$ or $x \mid \pi$. Similarly, we have that $\pi \mid y$ or $y \mid \pi$. If $\pi \mid x$ or $\pi \mid y$, we are done, so we may assume without loss of generality that $x \mid \pi$ and $y \mid \pi$. If $x \mid \pi$, by the irreducibility of π, x is a unit or x is associate to π. Similarly, y is a unit or y is associate to π. Certainly since $\pi \mid xy$, both x and y cannot be units, and so we conclude that x or y is associate to π. □

B.5 The Group of Divisibility for an Integral Domain

Recall that for an integral domain D with quotient field $\mathbb{F}$, a *fractional ideal* is a D-submodule $I \subseteq \mathbb{F}$ and a non-zero element $a \in D \setminus \{0\}$ such that $aI \subseteq D$ is an ideal in D. We say I is *principal* if it is generated by a single element from $\mathbb{F}$. The group of divisibility is formally defined, as in [25], [27], and [21], as the group of non-zero principal fractional ideals, which we denote $G(D)$. This group is, of course, abelian (since $\mathbb{F}$ is commutative), and this group is a po-group under reverse inclusion.

We use an equivalent definition that is somewhat more tractable. Every integral domain D comes equipped with a group of units $U(D)$ and field of fractions, $\mathbb{F} = \left\{\frac{r}{s} \mid r \in D, S \in D \setminus 0\right\}$, whose multiplicative group we denote $\mathbb{F}^\times = \mathbb{F} \setminus 0$. The group $\mathbb{F}^\times/U(D)$ is abelian since $\mathbb{F}$ is commutative and we may extend the divisibility pre-order on D to $\mathbb{F}^\times/U(D)$ by declaring $aU(D) \leqslant bU(D)$ if and only if $\frac{b}{a} \in D$. This relation is a partial order in $\mathbb{F}^\times/U(D)$, since antisymmetry is resolved by only considering elements up to associates. This relation is also compatible with the group operation, and so $\mathbb{F}^\times/U(D)$ is a po-group under this partial order. Furthermore, there exists an o-isomorphism between $\mathbb{F}^\times/U(D)$ and the po-group of non-zero principal fractional ideals (map $\frac{a}{b}U(D)$ to the principal fractional ideal generated by $\frac{a}{b}$). Writing proofs using the field of fractions is often nominally simpler than writing proofs using the notion of non-zero principal fractional ideals, so following [25], we redefine the group of divisibility as $G(D) := \mathbb{F}^{\times}/U(D)$.

Example B.18. Let $D = \mathbb{Z}$. The quotient field is $\mathbb{F} = \mathbb{Q}$, and the unit group is $\{\pm 1\}$. Thus, $G(D) = \mathbb{Q}^{\times}/\{\pm 1\}$. Of course, $\mathbb{Q}^{\times} = \mathbb{Q} \setminus 0$, and so we have $G(D) \simeq \mathbb{Q}^{+} = \{q \in \mathbb{Q} \mid q \geqslant 0\}$. △

Example B.19. Consider an additive sub-monoid of the non-negative rationals $M \subseteq \mathbb{Q}^{+}$ and let x be any indeterminate over $\mathbb{F}_2$. Set $X = \{x^m \mid m \in M\}$ and consider $R = \mathbb{F}_2[X]$. Let $\mathfrak{m} \subseteq R$ be the ideal generated by X and consider $D = R_{\mathfrak{m}}$.

When $M = \mathbb{Q}^{+}$, we can write an arbitrary non-zero non-unit $f \in R$ as $f = ux^q$ for some unit u and $q \in \mathbb{Q}^{+}$. Thus, $G(D) = \left\{\frac{x^q}{x^p}U(D) \mid q, p \in \mathbb{Q}^{+}\right\}$. Now consider the map $\phi : G(D) \to \mathbb{Q}$ defined by mapping, for each $r \in \mathbb{Q}$, $x^r U(D) \mapsto r$. The map ϕ is obviously well-defined by construction, but it is also an invertible group homomorphism. To see this, consider the following.

$$\begin{aligned}\phi\left(\left(x^r U(D)\right)\left(x^s U(D)\right)\right) =& \phi\left(x^{r+s}U(D)\right)\\ =& r + s\\ =& \phi(x^r U(D)) + \phi(x^s U(D))\end{aligned}$$

Under the usual total ordering on $G' = \mathbb{Q}$ ϕ is also order-preserving: if $x^s U(D) \leqslant x^r U(D)$, then $x^{r-s}U(D)$ is non-negative, i.e. $x^{r-s} \in D$, so $r - s \in \mathbb{Q}^{+}$, so $s \leqslant r$. Lastly, $\phi(G(D)^{+}) = \mathbb{Q}^{+}$: if $0 < r$ for some $r \in \mathbb{Q}^{+}$, then $x^r U(D)$ is positive in $G(D)$ and $r = \phi(x^r U(D))$, and if $x^r U(D) \in G(D)^{+}$ then $r > 0$ in $\mathbb{Q}^{+}$. Thus $G(D)$ is o-isomorphic to the additive po-group $\mathbb{Q}$. △

There exists a map from $\nu : D \setminus 0 \longrightarrow G(D)$ defined by $x \mapsto xU(D)$. This may be naturally extended to a semi-valuation on $\mathbb{F}$ by mapping $\frac{x}{y} \mapsto \frac{x}{y}U(D)$ (in [28], Ohm demonstrates ν is a well-defined semi-valuation). This semi-valuation connects the above ring-theoretic factorization definitions to the above po-group-theoretic definitions.

Lemma B.20. *Let D be an integral domain with quotient field $\mathbb{F}$. For any $g \in G(D)$, $g \in G(D)^{+}$ if and only if $g = xU(D)$ for some $x \in D$. For any $g \in G(D)^{+}$, g is minimal in the partial order on $G(D)$ if and only if $g = pU(D)$ for some irreducible $p \in Irr(D)$.*

Proof. An arbitrary $g \in G(D)$ can be written $g = \frac{a}{b}U(D)$. Note that $U(D) \leqslant \frac{a}{b}U(D)$ if and only if $\frac{u(a/b)}{1} \in D$, which is to say $b \mid a$. Hence, any positive element $g \in G(D)$ can be written $g = xU(D)$

for some $x \in D$. If g is minimal positive and $1_G \leqslant h \leqslant g$ then $1_G = h$ or $h = g$. Hence, if $xU(D)$ is minimal positive, and $x_1 \mid x$ in D, then $x_1U(D) \leqslant xU(D)$. Thus, either $x_1U(D) = U(D)$, in which case x_1 is a unit, or $x_1U(D) = xU(D)$, in which case $x_1 = ux$ for some unit u. Thus if $xU(D)$ is minimal positive in $G(D)$, then x is irreducible in D.

On the other hand, if $xU(D) = (yU(D))\,(zU(D))$ in $G(D)^+$, then $x, y, z \in D$ and there exists some unit, $u \in U(D)$, such that $x = uyz$. Hence, if x is irreducible, then y or z is a unit, in which case $yU(D) = U(D)$ or $zU(D) = U(D)$. □

Recall we introduced o-ideals by observing they play a central role in the study of po-groups and their connection with integral domains. We now explore a theorem from [25] to draw connections between saturated multiplicatively closed sets in a domain and the o-ideals (convex and directed subgroups) of the associated group of divisibility. In order to prove this theorem, we exploit Lemma B.5. In that lemma, we connected convexity with property (iv), which is inspired by the saturation of a multiplicatively closed subset in a domain.

Theorem B.21 (Mott's correspondence theorem)**.** *Let D be an integral domain with quotient field $\mathbb{F}$, unit group $U(D)$, and group of divisibility $G(D) = \mathbb{F}^\times/U(D)$. Let $\nu : \mathbb{F}^\times \to G(D)$ be the natural semi-valuation defined by $x \mapsto xU(D)$. Let $\mathcal{S}$ be the set of all saturated multiplicatively closed subsets of D and let $\mathcal{O}$ be the set of all o-ideals of $G(D)$. Then the map $\Theta : \mathcal{S} \to \mathcal{O}$ given by $\Theta(S) = \langle \nu(S) \rangle$ is a one-to-one correspondence. Further, $G(D)/\langle \nu(S) \rangle = G(D_S)$.*

Proof. The map Θ is well-defined: for any $S \in \mathcal{S}$, $\langle \nu(S) \rangle$ is convex (following from the saturation of S) and directed (following from the fact that the generators are from $S \subseteq D$). We also have surjectivity: let $H \subseteq G(D)$ be any o-ideal. The subgroup H is directed, i.e. generated by its positive elements, $H = \langle H^+ \rangle$, and for any $h \in H^+$, we may write $h = x_hU(D)$ for some $x_h \in D$. The set $S = \{x_h \mid h = x_hU(D) \in H^+\}$ is certainly multiplicatively closed since H^+ is multiplicatively closed, and since H is convex, then by Lemma B.5, we see that S is a saturated multiplicatively closed subset of D. Furthermore, $H = \Theta(S)$ by construction.

We also have injectivity: let $H_1 \neq H_2$. Since $H_1 = \langle H_1^+ \rangle$ and $H_2 = \langle H_2^+ \rangle$, and since $H_1 \neq H_2$ we have that $H_1^+ \neq H_2^+$. Without loss of generality, we may select some $xU(D) \in H_2^+ \setminus H_1^+$ for some $x \in D$. Thus, if $H_1 = \Theta(S_1)$ and $H_2 = \Theta(S_2)$, then $x \in S_2 \setminus S_1$, and so $S_1 \neq S_2$. Hence, Θ is a one-to-one correspondence.

All that remains is to show that if $S \in \mathcal{S}$ then $G(D_S) = G(D)/\Theta(S)$. Let $x \in G(D_S)$. Then

$x = \frac{x_1}{x_2}U(D_S)$ for some non-zero $x_1, x_2 \in D_S$. Since $x_1, x_2 \in D_S$, we may write $x_1 = \frac{a}{s}$ and $x_2 = \frac{b}{t}$ for some $a, b \in D$, $s, t \in S$. Then $\frac{x_1}{x_2}U(D_S) = \frac{a}{b}\frac{t}{s}U(D_S)$. Since $s, t \in S \subseteq U(D_S)$, we have that $x = \frac{a}{b}U(D_S) \in G(D)/U(D_S)$. On the other hand, if $xU(D_S) \in G(D)/U(D_S)$ for some $x \in G(D)$, then $x = \frac{a}{b}U(D)$ for some $a, b \in D \setminus \{0\}$. Thus, we may write $xU(D_S) = \frac{a}{b}U(D_S) \in G(D_S)$. □

Theorem B.21 provides a critical tool used in our treatment of integral domains and their groups of divisiblity. The slogan is now "localizations of D correspond to o-epimorphisms of $G(D)$ and vice versa."

Appendix C Facts and Notation for Cochain Complexes

We follow [2], [14], and [5] to define chain complexes of abelian groups and the dual notion of a cochain complex, and we define trivial and exact complexes. We define the sequence of cohomology groups associated to a cochain complex. We consider the edge cases when the cochain complexes are trivial and exact, we interpret cohomology groups as a group-theoretic measurement of "how exact" a given cochain complex is.

C.1 Cochain Complexes

(Co)chain complexes are appealing because they allow for a richer algebraic environment, leading to algebraic properties for chain complexes such as localization, quotient complexes, and tensor products, as well as leading to homological-topological properties like homotopy. In this section, we develop the notion of (co)chain complexes, their chain maps, and quotient (co)chain complexes.

Definition C.1. A *chain complex* $G_\bullet$ of abelian groups is a diagram in the category of abelian groups of the form

$$G_\bullet = \cdots \overset{\partial_{n+2}}{\to} G_{n+1} \overset{\partial_{n+1}}{\to} G_n \overset{\partial_n}{\to} G_{n-1} \overset{\partial_{n-1}}{\to} \cdots$$

such that, for each n, $\partial_n \circ \partial_{n+1}$ is the trivial group homomorphism sending all group elements to the identity. Equivalently, we require $\mathrm{Im}(\partial_n) \subseteq \mathrm{Ker}(\partial_{n-1})$.

We refer to the maps ∂_n as *differentials.* We can, without loss of generality, regard any finite or infinite sequence of maps satisfying $\partial_n \circ \partial_{n+1} = 0$ as a bi-infinite sequence by padding the diagram on either side with infinite sequences of trivial groups. As usual, we can obtain the dual notion by simply reversing the direction of all arrows in the diagram (or, equivalently, reversing the index notation) to obtain a *cochain complex*:

$$G^\bullet = \cdots \overset{\partial_{n-2}}{\to} G_{n-1} \overset{\partial_{n-1}}{\to} G_n \overset{\partial_n}{\to} G_{n+1} \overset{\partial_{n+1}}{\to} \cdots$$

where, again, we require that $\partial_{n-1} \circ \partial_n$ is trivial, i.e. $\mathrm{Im}(\partial_{n-1}) \subseteq \mathrm{Ker}(\partial_n)$. Relaxations of the above concepts are available to categories other than the category of groups.

Example C.2. Let H be any additive abelian group and $G = H \oplus H \oplus H$. Consider the group homomorphisms $\pi_1 : G \to G$ defined by mapping $(x, y, z) \mapsto (x, 0, 0)$ and $\pi_3 : G \to G$ defined by mapping $(x, y, z) \mapsto (0, y, 0)$. Note that $\pi_1 \circ \pi_2$ and $\pi_2 \circ \pi_1$ are both the trivial group homomorphisms sending every element to the identity. Thus, defining $G_n = G$ for each n and defining

$$\partial_n = \begin{cases} \pi_1 \text{ if } & n \equiv 0 (\text{mod } 2) \\ \pi_2 \text{ if } & n \equiv 1 (\text{mod } 2) \end{cases}$$

we obtain a chain complex and a cochain complex.

$$G^\bullet = \cdots \overset{\partial_{n+2}}{\to} G_{n+1} \overset{\partial_{n+1}}{\to} G_n \overset{\partial_n}{\to} G_{n-1} \overset{\partial_{n-1}}{\to} \cdots$$

$$G^\bullet = \cdots \overset{\partial_{n-2}}{\to} G_{n-1} \overset{\partial_{n-1}}{\to} G_n \overset{\partial_n}{\to} G_{n+1} \overset{\partial_{n+1}}{\to} \cdots$$

The key here, again, is that composing any two adjacent arrows yields the trivial map. △

In Example C.2, we see proper containment $\text{Im}(\partial_n) \subsetneq \text{Ker}(\partial_{n-1})$. On the other hand, if we use $G = H \oplus H$ in a similar example to Example C.2, we obtain equality, $\text{Im}(\partial_n) = \text{Ker}(\partial_{n-1})$ or $\text{Im}(\partial_{n-1}) = \text{Ker}(\partial_n)$. This leads us to consider two edge cases.

Definition C.3. Given a (co)chain complex, if each $\partial_n = 0$ then we say the complex is a *trivial complex.* On the other hand, if the differentials in a chain complex satisfy $\text{Im}(\partial_n) = \text{Ker}(\partial_{n-1})$ for every n (or, dually, if the differentials in a cochain complex satisfy $\text{Im}(\partial_{n-1}) = \text{Ker}(\partial_n)$), then we say the complex is an *exact complex.* A complex $A_\bullet$ (or $A^\bullet$) is the *zero complex* if $A_i = 0$ for each i. Certainly if $A_\bullet$ (or $A^\bullet$) is the zero complex, then it is both trivial and exact.

Given a pair of cochain complexes of abelian groups, say $A^\bullet$ with differentials $\{\partial_i\}$, and $B^\bullet$ with differentials $\{d_i\}$, such that for each i we have the subgroup containment $A_i \subseteq B_i$ and such that the differentials are restrictions, $\partial_i = d_i|_{A_i}$, then we may define the quotient cochain complex

$$\frac{B^\bullet}{A^\bullet} = \cdots \xrightarrow{\eta_{i-1}} \frac{B_i}{A_i} \xrightarrow{\eta_i} \frac{B_{i+1}}{A_{i+1}} \xrightarrow{\eta_{i+1}} \cdots$$

where each $\eta_i : B_i/A_i \to B_{i+1}/A_{i+1}$ is defined by the natural map $b + A_i \mapsto d_i(b) + A_{i+1}$. Of course, we obtain all required maps because in each degree we have the well-defined natural quotient map

$\pi_i : B_i \twoheadrightarrow B_i/A_i$, leading to the following exact sequence.

$$\begin{array}{lccccccc}
A^\bullet & = \cdots & \xrightarrow{\partial_{i-1}} & A_i & \xrightarrow{\partial_i} & A_{i+1} & \xrightarrow{\partial_{i+1}} & \cdots \\
 & & & \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi_{i+1}} & & \\
B^\bullet & = \cdots & \xrightarrow{d_{i-1}} & B_i & \xrightarrow{d_i} & B_{i+1} & \xrightarrow{d_{i+1}} & \cdots \\
 & & & \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi_{i+1}} & & \\
B^\bullet/A^\bullet & = \cdots & \xrightarrow{\eta_{i-1}} & B_i/A_i & \xrightarrow{\eta_i} & B_{i+1}/A_{i+1} & \xrightarrow{\eta_{i+1}} & \cdots
\end{array}$$

This idea leads us to investigate (co)chain maps. Given a pair of cochain complexes, say $A^\bullet$ with differentials $\{\partial_i\}$, and $B^\bullet$ with differentials $\{d_i\}$, a *cochain map*, say $f^\bullet : A^\bullet \to B^\bullet$, is a sequence of abelian group homomorphisms, $\{f_n : A_n \to B_n\}_{n \in \mathbb{Z}}$ which forms the following commutative diagram.

$$\begin{array}{lccccccc}
A^\bullet & = \cdots & \xrightarrow{\partial_{i-1}} & A_i & \xrightarrow{\partial_i} & A_{i+1} & \xrightarrow{\partial_{i+1}} & \cdots \\
\downarrow{\scriptstyle f^\bullet} & & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i+1}} & & \\
B^\bullet & = \cdots & \xrightarrow{d_{i-1}} & B_i & \xrightarrow{d_i} & B_{i+1} & \xrightarrow{d_{i+1}} & \cdots
\end{array}$$

If some $f^\bullet : A^\bullet \to B^\bullet$ is a cochain map such that each f_i is an isomorphism, we say $A^\bullet$ and $B^\bullet$ are isomorphic as cochain complexes.

C.2 Cohomology Groups and Long Exact Sequences

From any cochain complex, say $A^\bullet$, we may compute a sequence of *cohomology groups*:

Definition C.4. Let $A^\bullet = \cdots \xrightarrow{\partial_{i-1}} A_i \xrightarrow{\partial_i} A_{i+1} \xrightarrow{\partial_{i+1}} \cdots$ be a cochain complex of abelian groups. We define the i^{th} cohomology group to be $H^i(A^\bullet) := \mathrm{Ker}(\partial_i)/\mathrm{Im}(\partial_{i-1})$.

These are well-defined since $A^\bullet$ is a cochain complex, so $\mathrm{Im}(\partial_i) \subseteq \mathrm{Ker}(\partial_{i+1})$ for each i. Note that an exact cochain complex has $\mathrm{Im}(\partial_i) = \mathrm{Ker}(\partial_{i+1})$. Thus, $A^\bullet$ is exact if and only if $H^i(A^\bullet) = 0$ for each i. If $A^\bullet$ is trivial then $\mathrm{Im}(\partial_i) = 0$ and so $H^i(A^\bullet) \cong \mathrm{Ker}(\partial_{i+1})$.

(Co)homology groups often detect information embedded in short exact sequences of (co)chain complexes. For any cochain complexes $A^\bullet$, $B^\bullet$, and $C^\bullet$, and for cochain maps $f^\bullet : A^\bullet \to B^\bullet$ and $g^\bullet : B^\bullet \to C^\bullet$, then the sequence of cochain maps

$$0^\bullet \longrightarrow A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \longrightarrow 0^\bullet$$

corresponds to the following commutative diagram.

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle d_{i-2}} & & \downarrow{\scriptstyle \partial_{i-2}} & & \downarrow{\scriptstyle \eta_{i-2}} & & \\
0 & \longrightarrow & A_{i-1} & \longrightarrow & B_{i-1} & \longrightarrow & C_{i-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_{i-1}} & & \downarrow{\scriptstyle \partial_{i-1}} & & \downarrow{\scriptstyle \eta_{i-1}} & & \\
0 & \longrightarrow & A_{i} & \longrightarrow & B_{i} & \longrightarrow & C_{i} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_{i}} & & \downarrow{\scriptstyle \partial_{i}} & & \downarrow{\scriptstyle \eta_{i}} & & \\
0 & \longrightarrow & A_{i+1} & \longrightarrow & B_{i+1} & \longrightarrow & C_{i+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_{i+1}} & & \downarrow{\scriptstyle \partial_{i+1}} & & \downarrow{\scriptstyle \eta_{i+1}} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Such a sequence of cochain maps is *short exact* if each row $0 \to A_i \to B_i \to C_i \to 0$ is exact.

Lemma C.5 (Snake Lemma)**.** *Let the following diagram be commutative with each row exact:*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A_{i} & \xrightarrow{f_i} & B_{i} & \xrightarrow{g_i} & C_{i} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_{i}} & & \downarrow{\scriptstyle \partial_{i}} & & \downarrow{\scriptstyle \eta_{i}} & & \\
0 & \longrightarrow & A_{i+1} & \xrightarrow{f_{i+1}} & B_{i+1} & \xrightarrow{g_{i+1}} & C_{i+1} & \longrightarrow & 0.
\end{array}
$$

Then there exists a long exact sequence of the form

$$0 \to \mathrm{Ker}(d_i) \to \mathrm{Ker}(\partial_i) \to \mathrm{Ker}(\eta_i) \xrightarrow{\partial} \mathrm{Coker}(d_i) \to \mathrm{Coker}(\delta) \to \mathrm{Coker}(\eta_i) \to 0.$$

Proof. The crux of the proof lies in constructing the group homomorphism $\delta : \mathrm{Ker}(\eta_i) \to \mathrm{Coker}(d_i)$ from the group homomorphisms g_i, ∂_i, and f_{i+1}. For each $x \in \mathrm{Ker}(\eta_i)$, we can naturally define $\delta_i(x)$ in the following steps: (i) since the group homomorphism g_i is a surjection, we can write $x = g_i(b)$ for some $b \in B_i$, (ii) then $\partial_i(b) \in \mathrm{Ker}(g_{i+1}) = \mathrm{Im}(f_{i+1})$, so we have some $a \in A_{i+1}$ such that the image under the group homomorphism $f_{i+1}(a) = \partial_i(b)$, and (iii) setting $\delta_i(x) = a$.

This map is well-defined: assume $b_1, b_2 \in B_i$ are such that $g_i(b_1) = g_i(b_2) = x$. Then $b_1 - b_2 \in \mathrm{Ker}(g_i) = \mathrm{Im}(f_i)$. Thus, $b_1 - b_2 = f_i(c)$ for some $c \in A_i$. By the commutativity of the lefthand square, $\partial_i(b_1 - b_2) = \partial_i \circ f_i(c) = f_{i+1} \circ d_i(c)$. Hence, $\partial_i(b_1) = \partial_i(b_2) + f_{i+1} \circ d_i(c)$. In particular, $\partial_i(b_1) + \mathrm{Im}(f_{i+1}) = \partial_i(b_2) + \mathrm{Im}(f_{i+1})$.

Notice though that since $g_i(b_1) = x \in \text{Ker}(\eta_i)$ and since the righthand square is commutative, $0 = \eta_i \circ g_i(b_1) = g_{i+1} \circ \partial_i(b_1)$. Thus, $\partial_i(b_1) \in \text{Ker}(g_{i+1}) = \text{Im}(f_{i+1})$, so $\partial_i(b_1) = f_{i+1}(a_1)$ for some $a_1 \in A_{i+1}$. But we just proved that $\partial_i(b_1 - b_2) = \partial_i \circ f_i(a) = f_{i+1} \circ d_i(a)$, so we have

$$\partial_i(b_1 - b_2) = f_{i+1} \circ d_i(a_1 - a_2) = f_{i+1} \circ d_i(a)$$

Since the bottom row is exact, f_{i+1} is an injection, and so we conclude $a_1 - a_2 = d_i(a)$. In particular, $a_1 + \text{Im}(d_i) = a_2 + \text{Im}(d_i)$, establishing well-definedness in δ_i.

We can use f_i to induce a map $\widehat{f_i}$ of the form $\widehat{f_i} : \text{Ker}(d_i) \to \text{Ker}(\partial_i)$ by restricting f_i to $\text{Ker}(d_i)$, i.e. mapping $x \mapsto f_i(x)$. If $a \in \text{Ker}(d_i)$, then $d_i(a) = 0$ so $f_{i+1} \circ d_i(a) = 0$. By the commutativity of the lefthand square, we have that $\partial_i \circ f_i(a) = f_{i+1} \circ d_i(a)$. Thus, $\partial_i \circ f_i(a) = 0$, so $f_i(a) \in \text{Ker}(\partial_i)$.

Similarly, we can use g_i to induce a map $\widehat{g}_i$ of the form $\widehat{g}_i : \text{Ker}(\partial_i) \to \text{Ker}(\eta_i)$ by restricting g_i to $\text{Ker}(\partial_i)$, i.e. mapping $x \mapsto g_i(x)$. If $b \in \text{Ker}(\partial_i)$, by the commutativity of the righthand square, we have that $\eta_i \circ g_i(b) = g_{i+1} \circ \partial_i(b) = g_{i+1}(0) = 0$. Thus, $g_i(b) \in \text{Ker}(\eta_i)$.

We use f_{i+1} to induce a map $\widehat{f}_{i+1} : \text{Coker}(d_i) \to \text{Coker}(\partial_i)$: for any $a + \text{Im}(d_i) \in \text{Coker}(d_i)$, consider $f_{i+1}(a) + \text{Im}(\partial_i)$. Certainly $f_{i+1}(a) \in B_{i+1}$ so $f_{i+1}(a) + \text{Im}(\partial_i) \in \text{Coker}(\partial_i)$. Furthermore, if $a_1 + \text{Im}(d_i) = a_2 + \text{Im}(d_i)$, then $a_1 - a_2 \in \text{Im}(d_i)$ so $a_1 - a_2 = d_i(\alpha)$ for some $\alpha \in A_i$. Thus, $f_{i+1}(a_1) - f_{i+1}(a_2) = f_{i+1} \circ d_i(\alpha)$ and by the commutativity of the left square, $f_{i+1}(a_1) - f_{i+1}(a_2) = \partial_i \circ f_i(\alpha) \in \text{Im}(\partial_i)$. Thus, $f_{i+1}(a_1) + \text{Im}(\partial_i) = f_{i+1}(a_2) + \text{Im}(\partial_i)$, establishing this map is well-defined.

We induce a map $\widehat{g}_{i+1} : \text{Coker}(\partial_i) \to \text{Coker}(\eta_i)$ by defining $b + \text{Im}(\partial_i) \mapsto g_{i+1}(b) + \text{Im}(\eta_i)$. Certainly $g_{i+1}(b) \in C_{i+1}$ so $g_{i+1}(b) + \text{Im}(\eta_i) \in \text{Coker}\,\eta_i$. Furthermore, if $b_1 + \text{Im}(\partial_i) = b_2 + \text{Im}(\partial_i)$, then $b_1 - b_2 = \partial_i(\beta)$ for some $\beta \in B_i$. Thus, $g_{i+1}(b_1 - b_2) = g_{i+1} \circ \partial_i(\beta) = \eta_i \circ g_i(\beta)$. Thus, $g_{i+1}(b_1) - g_{i+1}(b_2) \in \text{Im}(\eta_i)$, establishing well-definedness.

That the resulting sequence is exact at $\text{Ker}(d_i)$, $\text{Ker}(\partial_i)$, $\text{Coker}(\partial_i)$, and $\text{Coker}(\eta_i)$ follows from the fact that we induced the maps on the top and bottom rows, which are assumed to be exact. All that remains is to establish exactness at $\text{Ker}(\eta_i)$ and $\text{Coker}(d_i)$. Consider first

$$\text{Ker}(\partial_i) \xrightarrow{\widehat{g}_i} \text{Ker}(\eta_i) \xrightarrow{\delta_i} \text{Coker}(d_i)$$

The image of $\widehat{g}_i$ is, by construction, $g_i(\text{Ker}(\partial_i))$; we claim this is exactly $\text{Ker}(\delta_i)$. Recall that

$\partial : \text{Ker}(\eta_i) \mapsto \text{Coker}(d_i)$ is defined by $\partial(x) = a + \text{Im}(d_i)$ for $a \in A_{i+1}$ such that $f_{i+1}(a) = \partial_i(b)$ where $g_i(b) = x$.

If $x \in g_i(\text{Ker}(\partial_i))$ then for some $b \in \text{Ker}(\partial_i) \subseteq B_i$ such that $x = g_i(b)$. We then have that $\partial(x) = a + \text{Im}(d_i)$ for some $a \in A_{i+1}$ such that $f_{i+1}(a) = \partial_i(b)$. For this choice, $\partial_i(b) = 0$, and so we have that $f_{i+1}(a) = 0$, i.e. $a \in \text{Ker}(f_{i+1})$. Of course, the bottom row is exact so $a = 0$. Thus, $g_i(\text{Ker}(\partial_i)) \subseteq \ker(\delta_i)$.

On the other hand, if $x \in \text{Ker}(\delta_i) \subseteq \text{Ker}(\eta_i)$, then since g_i is surjective, there exists some $b \in B_i$ such that $g_i(b) = x$. Furthermore, $\eta_i \circ g_i(b) = \eta_i(x) = 0$. But commutativity of the righthand square provides that $g_{i+1} \circ \partial_i(b) = \eta_i \circ g_i(b) = 0$. Thus, $\partial_i(b) \in \text{Ker}(g_{i+1}) = \text{Im}(f_{i+1})$. So we have some $a \in A_{i+1}$ such that $\partial_i(b) = f_{i+1}(a)$ and $\partial(x) = a + \text{Im}(d_i)$. Since $x \in \text{Ker}(\partial)$, we have that $\partial(x) = \text{Im}(d_i)$, so we have that $a \in \text{Im}(d_i)$, say $a = d_i(\alpha)$ for some $\alpha \in A_i$. Thus, $\partial_i(b) = f_{i+1}(a) = f_{i+1} \circ d_i(\alpha) = \partial_i \circ f_i(\alpha)$. Thus, $b - f_i(\alpha) \in \text{Ker}(\partial_i)$. But we selected $b \in B_i$ so that $g_i(b) = x$, and $g_i(b - f_i(\alpha)) \in g_i(\text{Ker}(\partial_i))$. Of course, $g_i \circ f_i = 0$, so we have that $g_i(b) \in g_i(\text{Ker}(\partial_i))$. This establishes exactness at $\text{Ker}(\eta_i)$.

To establish exactness at $\text{Coker}(d_i)$, note that

$$\text{Im}(\partial) = \{a + \text{Im}(d_i) \in A_{i+1}/\text{Im}(d_i) \mid \exists x \in \text{Ker}(\eta_i), b \in B_i \text{ such that } g_i(b) = x \text{ and } f_{i+1}(a) = \partial_i(b)\}$$

If $a + \text{Im}(d_i) \in \text{Im}(\partial)$, then consider $\widehat{f}_{i+1}(a + \text{Im}(d_i))$. Indeed, $\widehat{f}_{i+1}(a + \text{Im}(d_i)) = f_{i+1}(a) + \text{Im}(\partial_i)$, but since $f_{i+1}(a) = \partial_i(b)$, we have that $f_{+1}(a) + \text{Im}(\partial_i) = \text{Im}(\partial_i)$. Thus, $a + \text{Im}(d_i) \in \text{Ker}(\widehat{f}_{i+1})$.

On the other hand, if $a + \text{Im}(d_i) \in \text{Ker}(\widehat{f}_{i+1})$, then $f_{i+1}(a) \in \text{Im}(\partial_i)$, so $f_{i+1}(a) = \partial_i(b)$ for some $b \in B$. Then $g_i(b) \in \text{Ker}(\eta_i)$ since the righthand square commutes and $g_{i+1} \circ f_{i+1}(a) = 0$. In particular, we may write $a + \text{Im}(d_i) = \partial(g_i(b))$, i.e. $a + \text{Im}(d_i) \in \text{Im}(\partial)$. □

Corollary C.6 (Long Exact Sequences)**.** *Given a short exact sequence of cochain maps*

$$0^\bullet \to A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{f^\bullet} C^\bullet \to 0^\bullet$$

There is a long exact sequence in cohomology:

$$\cdots \xrightarrow{\widehat{g}_{n-1}} H^{n-1}(C^\bullet) \xrightarrow{\delta_{n-1}} H^n(A^\bullet) \xrightarrow{\widehat{f}_n} H^n(B^\bullet) \xrightarrow{\widehat{g}_n} H^n(C^\bullet) \xrightarrow{\delta_n} H^{n+1}(A^\bullet) \xrightarrow{\widehat{f}_{n+1}} \cdots$$

Proof. We construct the maps ∂_n similarly to before. We have the diagram

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle d_{n-2}} & & \downarrow{\scriptstyle \partial_{n-2}} & & \downarrow{\scriptstyle \eta_{n-2}} & & \\
0 & \longrightarrow & A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_{n-1}} & & \downarrow{\scriptstyle \partial_{n-1}} & & \downarrow{\scriptstyle \eta_{n-1}} & & \\
0 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \eta_n} & & \\
0 & \longrightarrow & A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \eta_{n+1}} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

We construct the map $\widehat{f_n} : H^n(A^\bullet) \to H^n(B^\bullet)$. First, write $H^n(A^\bullet) = \text{Ker}(d_n)/\text{Im}(d_{n-1})$ and $H^n(B^\bullet) = \text{Ker}(\partial_n)/\text{Im}(\partial_{n-1})$. The map $f_n : A_n \to B_n$ can be used to induce a map $\widehat{f_n}$ by mapping $x + \text{Im}(d_{n-1}) \mapsto f_n(x) + \text{Im}(\partial_{n-1})$. Similarly, we obtain $\widehat{g}_n : \text{Ker}(\partial_n)/\text{Im}(\partial_{n-1}) \to \text{Ker}(\eta_n)/\text{Im}(\eta_{n-1})$ defined by mapping $x + \text{Im}(\partial_{n-1}) \mapsto g_n(x) + \text{Im}(\eta_{n-1})$ which is a well-defined group homomorphism as described in the Snake Lemma.

We define $\partial_n : \text{Ker}(\eta_n)/\text{Im}(\eta_{n-1}) \to \text{Ker}(d_{n+1})/\text{Im}(d_n)$ in the following way: for each $x + \text{Im}(\eta_{n-1}) \in \text{Ker}(\eta_n)/\text{Im}(\eta_{n-1})$, since $g_n : B_n \to C_n$ is surjective, we may write $x = g_n(b)$ for some $b \in B_n$. Since $x \in \text{Ker}(\eta_n)$, we have that $0 = \eta_n(x) = \eta_n \circ g_n(b) = g_{n+1} \circ \partial_n(b)$. Thus, $\partial_n(b) \in \text{Ker}(g_{n+1}) = \text{Im}(f_{n+1})$, so we have some $a \in A_{n+1}$ such that $\partial_n(b) = f_{n+1}(a)$.

We claim $a \in \text{Ker}(d_{n+1})$; indeed, we have $\partial_n(b) = f_{n+1}(a)$ so $\partial_{n+1} \circ f_{n+1}(a) = \partial_{n+1} \circ \partial_n(b)$. But $B^\bullet$ is a cochain complex so $\partial_{n+1} \circ \partial_n(b) = 0$. In particular, $\partial_{n+1} \circ f_{n+1}(a) = 0$, but commutativity of the lefthand square implies $f_{n+2} \circ d_{n+1}(a) = 0$, and so $d_{n+1}(a) \in \text{Ker}(f_{n+2})$. Recall each f_i is an injection, so $\text{Ker}(f_{n+2}) = 0$, so $d_{n+1}(a) = 0$. Thus, the map ∂_n indeed has its image in $\text{Ker}(d_{n+1})/\text{Im}(d_n)$. That ∂_n is a well-defined group homomorphism is established similarly to our construction of ∂ in the proof of the snake lemma, and exactness of each sequence is obvious. □

Let $0^\bullet \to A^\bullet \to B^\bullet \to C^\bullet \to 0^\bullet$ be a short exact sequence. If $X^\bullet$ is one of these cochain complexes and is exact, then each $H^n(X^\bullet) = 0$, simplifying the long exact sequence in cohomology.

For example, if $A^\bullet$ is exact, then $H^n(A_\bullet) = 0$, so we have the following long exact sequence.

$$\cdots \xrightarrow{\widehat{g}_{n-1}} H^{n-1}(C^\bullet) \xrightarrow{\delta_{n-1}} 0 \xrightarrow{\widehat{f}_n} H^n(B^\bullet) \xrightarrow{\widehat{g}_n} H^n(C^\bullet) \xrightarrow{\delta_n} 0 \xrightarrow{\widehat{f}_{n+1}} \cdots$$

That is to say, we obtain that $0 \to H^n(B_\bullet) \to H^n(C_\bullet) \to 0$ is an exact sequence; we conclude that $H^n(B_\bullet) \cong H^n(C_\bullet)$. Similarly, if $B_\bullet$ is exact, then the long exact sequence in cohomology provides that $H^n(C_\bullet) \cong H^{n+1}(A_\bullet)$, providing a shift in degree. If $C_\bullet$ is exact, then the long exact sequence in cohomology provides that $H^n(A_\bullet) \cong H^n(B_\bullet)$.

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