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GAMMA-CONVERGENCE FOR ONE-DIMENSIONAL NONLOCAL PHASE TRANSITION ENERGIES

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We study the asymptotic behavior as ε goes to 0 of an appropriate scaling of the following nonlocal Allen-Cahn energy,

$$E^s_\varepsilon(u) = \varepsilon^{2s} \iint_{I\times I} \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y + \int_I W(u)\,\mathrm{d}x,$$

where I is an interval in $\mathbb{R}$, and W is a double-well potential. We provide a Γ-convergence result for any $s \in (0,1)$, by extending the case when $s = 1/2$ studied by Alberti, Bouchitté and Seppecher in [2]. We also investigate the convergence as $s \nearrow 1$ of the related *optimal profile problem* to the local counterpart.

1. Introduction

In the present paper, we describe via De Giorgi's Γ-convergence the asymptotic behavior of a nonlocal Allen-Cahn-Ginzburg-Landau-type energy in one-dimension.

Let $I \subset \mathbb{R}$ be an open interval, and let W be a double-well potential with wells at -1 and 1; i.e., a nonnegative function vanishing only at $\{-1,1\}$. For

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any $s \in (0,1)$, and any $\varepsilon > 0$, we consider the following family of functionals,

$$E^s_\varepsilon(u) = \varepsilon^{2s} \iint_{I\times I} \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y + \int_I W(u)\,\mathrm{d}x, \tag{1}$$

naturally defined on the fractional Sobolev spaces $H^s(I)$. As well known, the Γ-convergence is an important tool in Calculus of Variations in order to investigate the asymptotic behavior of variational problems depending on a parameter, and it has become fundamental in dealing with singularly perturbed energies arising in the theory of phase transitions as the one in (1), where the dislocation energy of a double-well potential W is compensated somewhat by a singular perturbation term which avoids the formation of unnecessary interfaces.[1] In this respect, one of the very first examples of Γ-convergence can be found in the by-now classic paper [23] by Modica and Mortola, where the singular perturbation energy is given by a gradient term $\varepsilon^2\|u\|_{H^1}$, in clear accordance with the original theory of phase transitions in fluids by Cahn and Hilliard; that is,

$$E^1_\varepsilon(u) = \varepsilon^2 \int_\Omega |\nabla u(x)|^2\,\mathrm{d}x + \int_\Omega W(u)\,\mathrm{d}x, \quad \text{where } \Omega \subset \mathbb{R}^N.$$

In [23] it has been proved that the Γ-limit in L^1 of $F^1_\varepsilon \equiv E^1_\varepsilon/\varepsilon$ is the following functional defined in $BV(\Omega;\{-1,1\})$ by

$$F^1(u) := \gamma_1 \mathrm{Per}(\{u=1\}); \tag{2}$$

i. e., proportional to the measure of the surface which separates the two phases, of a constant factor γ_1, which is determined by the optimal profile problem below, in view of the equipartition of the energy between the two integral terms in the functional,

$$\gamma_1 := \inf\Big\{ \int_{\mathbb{R}} |v'(x)|^2\,\mathrm{d}x + \int_{\mathbb{R}} W(v)\,\mathrm{d}x : \tag{3}$$
$$v \in H^1_{\mathrm{loc}}(\mathbb{R}),\ \lim_{x\to-\infty} v(x) = -1,\ \lim_{x\to+\infty} v(x) = 1\Big\}.$$

After that, despite a relatively short history, many important extensions and generalizations have been considered. The literature is really too wide to attempt any comprehensive treatment in a single paper; we would like just to mention some of the recent relevant contributions in the nonlocal framework [1–4, 6, 7, 9, 10, 14, 15, 18, 20–22, 26, 27, 30, 31].

For what concern Allen-Cahn energies with nonlocal singular perturbations, the first result can be found in the paper [2], where Alberti, Bouchitté and Seppecher studied the asymptotic behavior of the one-dimentional functional in (1)

[1] We refer to [8] for a detailed presentation of several basic aspects and applications of the Γ-convergence in a very general framework.

in the special case when $s = 1/2$. In such a case, one can prove that the Γ-limit is still local as in (2), but there is no equipartition of the energy and the cost of one transition from -1 to 1 does not come from an optimal profile problem. Precisely, the authors consider the following logarithmic rescaled energies,

$$F_\varepsilon^{1/2}(u) = \varepsilon \iint_{I\times I} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y + \theta_\varepsilon \int_I W(u)\,\mathrm{d}x, \quad \text{for } u \in H^{1/2}(I), \tag{4}$$

where $\theta_\varepsilon \to \infty$ and $\varepsilon \ln(\theta_\varepsilon) \to \gamma_{1/2} \in (0,\infty)$ as $\varepsilon \to 0$. Amongst other results, they proved that $F_\varepsilon^{1/2}$ Γ-converges in $L^1(I)$ to the functional $F^{1/2}$ defined by

$$F^{1/2}(u) = \begin{cases} 8\gamma_{1/2}\mathcal{H}^0(Su) & \text{if } u \in BV(I;\{-1,1\}), \\ +\infty & \text{otherwise}, \end{cases}$$

where the jump set Su is the complement of the set of Lebesgue points of u, and $\mathcal{H}^0$ denotes the counting measure. For the precise assumptions and further clarifications on the scaling above and all the involved quantities, we immediately refer to forthcoming Section 2.

The complete analysis for any $s \in (0,1)$ of the asymptotic behavior of suitably rescaled energies as in (1) in a general smooth domain $\Omega \subset \mathbb{R}^N$, for $N \geq 2$ has been settled in the relevant paper [30] by Savin and Valdinoci, who proved that when $s \in [1/2,1)$ the energy Γ-converges to the classical minimal surface functional, as in the local case (when $s = 1$), while, when $s \in (0,1/2)$ the energy Γ-converges to the nonlocal minimal surface functional. It is worth noticing that most, if not all, the results in [30] still hold for $N = 1$. However, below it will be clarified the relevance of the one-dimensional result in the present paper, in particular in order to establish a bridge between the nonlocal and the local phase transition energies.

Accordingly, in the present paper, we extend the result in [2] by investigating the behavior of the one-dimensional functional (1) when $s \in (1/2,1)$. It is worth noticing that the proof of the Γ-convergence result in the case when $s \in (0,1/2)$, already shown in [30], is immediate since in such a strongly nonlocal framework one has just to reconstruct the nonlocal term in the limit, and for this one can make use of the fact that the constant functions belongs to H^s, and thus they can be plainly chosen to obtain the desired limit estimates; see Section 6. The difficult task is to deal with the case when $s \in [1/2,1)$, because – as well explained in [30] – the nonlocal character of the functional gets localized in the Γ-limit, hence one cannot get rid of the contributions coming from far in order to ≪balance the interaction of these nonlocal contributions with their local counterparts≫.

For any $s \in (1/2, 1)$, we then consider the following functional defined on functions in $H^s_{\rm loc}(I)$,

$$F^s_\varepsilon(u) := \varepsilon^{-1} E^s_\varepsilon(u) = 2(1-s)\varepsilon^{2s-1} \iint_{I\times I} \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{\varepsilon}\int_I W(u)\,\mathrm{d}x, \tag{5}$$

where one can notice the scaling in the energy E^s_ε (for which we refer again to forthcoming Section 2), and a factor $2(1-s)$ in front of the kinetic term, whose presence will be clarified below.

The asymptotic behavior in terms of Γ-convergence of F^s_ε is described by the functional F^s defined by

$$F^s(u) := \gamma_s \mathcal{H}^0(Su), \quad \text{for } u \in BV(I;\{-1,1\}), \tag{6}$$

where the constant γ_s corresponds to the minimal cost in terms of the nonscaled energy F^s_1 of a transition from -1 to 1 on the whole real line, and it is given by the following optimal profile problem,

$$\gamma_s := \inf\Big\{ \iint_{\mathbb{R}\times\mathbb{R}} \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}} W(v)\,\mathrm{d}x : \tag{7}$$
$$v \in H^s_{\rm loc}(\mathbb{R}),\ \lim_{x\to-\infty} v(x) = -1,\ \lim_{x\to+\infty} v(x) = 1 \Big\}.$$

The presence of an optimal profile problem represents the main difference with respect to the case when $s = 1/2$ studied in [2], where on the contrary the limit comes basically from the nonlocal part of the energy and any profile is optimal as far as the transition occurs in a layer of order ε. The optimal profile problem will simplify considerably the proof of the Γ-lim sup inequality; see Section 5.2. On the other hand, we would need to prove that the infimum in (7) is achieved, and for this we will follow the approach in [18], where the Γ-convergence of a very close one-dimensional functional with a nonlocal nonlinear singular perturbation was studied. A key-point in this analysis, as well as in the proof of the Γ-lim inf inequality, will be the investigation of a family of auxiliary optimal profile problems together with the behavior of the energy with respect to monotone rearrangement of functions; see Section 3.

The aforementioned Γ-convergence result is stated in the following

Theorem 1.1. *For any $s \in (1/2,1)$, let $F^s_\varepsilon : H^s(I) \to \mathbb{R}$ and $F^s : BV(I;\{-1,1\}) \to \mathbb{R}$ be defined in (5) and (6), respectively. Then,*

> **Lower Bound Inequality**. *For every $u \in BV(I;\{-1,1\})$ and every sequence $(u_\varepsilon) \subset H^s(I)$ such that $u_\varepsilon \to u$ in $L^1(I)$,*
>
> $$\liminf_{\varepsilon\to 0} F^s_\varepsilon(u_\varepsilon) \geq F^s(u). \tag{LB}$$

Upper Bound Inequality. *For every $u \in BV(I;\{-1,1\})$ there exists a sequence $(\bar{u}_\varepsilon) \subset H^s(I)$ such that $\bar{u}_\varepsilon \to u$ in $L^1(I)$, and*

$$\limsup_{\varepsilon\to 0} F^s_\varepsilon(\bar{u}_\varepsilon) \le F^s(u). \tag{UB}$$

In addition, a natural compactness result is proven, as stated in the following

Theorem 1.2 (Compactness). *Under the same assumptions in* Theorem 1.1, *let $(u_\varepsilon) \subset H^s(I)$ be a sequence such that $F^s_\varepsilon(u_\varepsilon)$ is bounded. Then (u_ε) is precompact in $L^1(I)$ and every cluster point belongs to $BV(I;\{-1,1\})$.*

The strategy of the proof of Theorem 1.2 will closely follow the approach in [2, Theorem 1(i)] with no relevant differences, except in the computations, since a non optimal lower bound for $F^s_\varepsilon(u_\varepsilon)$ will be sufficient to attack the problem by means of Young measures associated to equi-bounded sequences; see Section 4.

Let us come back to the scaling factor $2(1-s)$ in front of the perturbation term in the energy functional F^s_ε. As already mentioned, the family of the nonlocal functionals we are dealing with shares with their local counterpart, i.e., with the Modica-Mortola functionals F^1_ε, the form of the Γ-limit which involves an optimal profile problem. This similarity does not come unexpected: a by-now classical work by Bourgain, Brezis and Mironescu ([5]) bridges the gap between H^s and H^1, showing the convergence of an appropriate scaling of the Gagliardo (semi)norm $[\cdot]_{H^s}$ to $\|u\|_{H^1_0}$, which in one-dimension can be read as follows

$$2(1-s)[u]^2_{H^s} \ \to\ \|u'\|^2_{L^2} \quad \text{as } s \nearrow 1.$$

Consequently, one does expect to deduce the convergence in the limit as $s \to 1^-$ of the *optimal profile for transition* with respect to the nonscaled energy F^s to the classical counterpart with respect to the nonscaled energy F^1. Such a result is not entirely trivial: on one hand the convergence proved in [5] is only pointwise (see also Section 4 in [13]), so it does not really suffice in the study of the convergence of the minimizers; on the other hand, one has to take into account that the condition that the optimal profiles have limits ± 1 at $\pm\infty$ is not closed with respect to the L^1 topology. For this, we will attack such a problem by dealing with the convergence of an auxiliary family of optimal profile problem γ^T_s, defined in (12) below, which guarantees the closure with respect of the L^1 topology. Then, we can deduce the result on the convergence of the nonlocal optimal profile by exploiting the uniformity in s of the estimates of $\gamma_s - \gamma^T_s$ in the limit as $T \to +\infty$. We have the following

Theorem 1.3. *Let γ_s and γ_1 defined by (7) and (3), respectively. Then,*

$$\lim_{s\to 1^-} \gamma_s = \gamma_1. \tag{8}$$

In order to complete the proof of Theorem 1.3 above, in view of the uniformity of our estimates on the auxiliary optimal profile problem, we will be able to apply a stronger convergence result proved by Ponce in the important paper [29], which to some extent does improve the results in [5] and in particular does imply the Γ-convergence of the nonscaled energies F^s to F^1; we refer to Section 7.

Finally, we would like to mention that it could be interesting to deal with even more general nonlocal energies involving spatial inhomogeneity terms, as, for instance, the fractional counterpart of the one treated in the local case in [28]. Consequently, one should take into account a wider class of family of fractional kernels with nonsmooth coefficients, by possibly dealing with the resulting error terms in the same flavour of the papers [11, 12, 16, 17, 24]; that is, by suitably estimating the "nonlocal tail" of the minimizers.

2. Setting and preliminary results

In this section we set the problem we are dealing with, and we add further considerations about the involved quantities and assumptions.

First of all, we recall that, for any $s \in (0,1)$, the fractional Sobolev spaces $H^s(\mathbb{R}^N)$, for $N \geq 1$, is defined through the norm

$$\begin{aligned} \|u\|^2_{H^s(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x + [u]^2_{H^s(\mathbb{R}^N)} \\ &= \int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

For a bounded domain $\Omega \subset \mathbb{R}^N$, the space $H^s(\Omega)$ can be defined similarly, by replacing the domains of integrations with Ω. For further details on the fractional Sobolev spaces, we refer to [13] and the references therein.

As mentioned in the introduction, we will deal with the nonlocal Allen-Cahn energies F^s_ε defined in (5) on functions in $H^s(I)$, where $I \subset \mathbb{R}$ is an interval, and s belongs to $(1/2,1)$. The double-well function $W : \mathbb{R} \to [0,+\infty)$ in the potential term in (5) is a nonnegative continuous function vanishing only in the two phases -1 and 1. Moreover we require that W grows at least linearly at infinity; i.e., that $W(z) \geq C|z|$ at infinity for some $C > 0$.

As customary, the corresponding localization of the energies is defined as follows: for every open set $J \subseteq I$ and every function $u \in H^s(J)$,

$$F_\varepsilon^s(u,J) = 2(1-s)\varepsilon^{2s-1}[u]^2_{H^s(J)} + \frac{1}{\varepsilon}\int_J W(u)\,\mathrm{d}x. \tag{9}$$

Clearly, $F_\varepsilon^s(u,I) \equiv F_\varepsilon^s(u)$ for every $u \in H^s(I)$. In addition, given $J \subseteq I$ and $u \in H^s(J)$, we set $u^{(\varepsilon)}(x) := u(\varepsilon x)$ and $J/\varepsilon := \{x : \varepsilon x \in J\}$. By scaling, it is immediately seen that

$$F_\varepsilon^s(u,J) = F_1^s(u^{(\varepsilon)}, J/\varepsilon).$$

In view of such a scaling property, it is then natural to consider the optimal profile problem defined in (7), whose properties will be investigated in forthcoming Section 3.

We now present an important property of the energy with respect to monotone rearrangements of functions, which will be very useful in the following of the paper. Let $J = (a,b)$. For every $u \in H^s(J)$, consider the non-decreasing rearrangement u^* of u in J given by

$$u^*(a+x) := \sup\{\eta : |\{t \in (a,b) : u(t) < \eta\}| \le x\}, \quad \forall x \in (0, b-a). \tag{10}$$

One can prove that

$$\iint_{J\times J} \frac{|u^*(x)-u^*(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y \le \iint_{J\times J} \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y; \tag{11}$$

see for instance [19, Theorem I.1]. Moreover,

$$\int_J W(u^*)\,\mathrm{d}x = \int_J W(u)\,\mathrm{d}x,$$

which, combined with (11), yields

$$F_\varepsilon^s(u^*,J) \le F_\varepsilon^s(u,J).$$

We conclude this section with a brief observation about the scaling factor in the definition of the energies F_ε^s. Different choices of a scaling factor in ε of the energy E_ε would indeed generate different Γ-limits, which could provide (or not) useful informations on the minimizers. Below, we motivate the choice of the (optimal) scaling in the definition in (5).

By simplicity, we set $I = [-1,1]$, and we define the following partition of I depending on $\delta \in (0,1)$.

$$I^- := [-1,-\delta], \quad I^0 := (-\delta,\delta), \quad I^+ := [\delta,1].$$

We take a function f_δ defined as follows

$$f_\delta(x) = \begin{cases} -1 & \text{se } x \in I^-, \\ x/\delta & \text{se } x \in I^0, \\ 1 & \text{se } x \in I^+. \end{cases}$$

We can estimate the energy $E^s_\varepsilon(f_\delta)$ as follows,

$$\begin{aligned} E^s_\varepsilon(f_\delta) &= \varepsilon^{2s} 2(1-s)[f_\delta]^2_{H^s(I)} + \int_I W(f_\delta)\,\mathrm{d}x \\ &= \varepsilon^{2s} 2(1-s)[f_\delta]^2_{H^s(I)} + C_2\delta \\ &\geq 4(1-s)\iint_{I^-\times I^+} \frac{4}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y + C_2\delta \\ &\sim C_1\varepsilon^{2s}\delta^{1-2s} + C_2\delta \;=\; \eta_\delta(\varepsilon). \end{aligned}$$

By minimizing in δ, and taking $\delta = \varepsilon$, we get

$$\eta_s(\varepsilon) \sim C\varepsilon$$

for some constant C. For any $s \in (1/2, 1)$, such an estimate suggests the optimal scaling $F^s_\varepsilon(u) \equiv \varepsilon^{-1}E_\varepsilon(u)$, as in (5).

Finally, it is worth noticing that in the case when $s = 1/2$, the same computations as above will give $\eta_s(\varepsilon) \sim C\varepsilon\,|\log(\varepsilon)|$, in accordance with the scaling in (4).

3. The optimal profile problem

One can prove that the infimum in (7) is achieved by adapting an argument in [26]; see in particular Remark 2 there. In short, one can take advantage of the decreasing behavior of the energy with respect to monotone rearrangements, and then make use of a plain application of the Direct Method, as firstly seen in [1, Theorem 2.4], where some nonlocal functionals deriving from Ising spin systems are analyzed. However, we will follow and extend the proof in [18, Proposition 3.3], because all the estimates presented below are necessary in order to obtain the Γ-lim inf inequality (LB) in Theorem (1.1), and the bridging result in Theorem 1.3 which will be shown in Section 7. Thereby, we introduce an auxiliary optimal profile problem. For every $T > 0$, let

$$\gamma^T_s := \inf\Big\{ F^s_1(v,\mathbb{R}) : \; v \in H^s_{\mathrm{loc}}(\mathbb{R}),\; v(x) = 1 \;\forall x \geq T,\; v(x) = -1 \;\forall x \leq -T \Big\}. \tag{12}$$

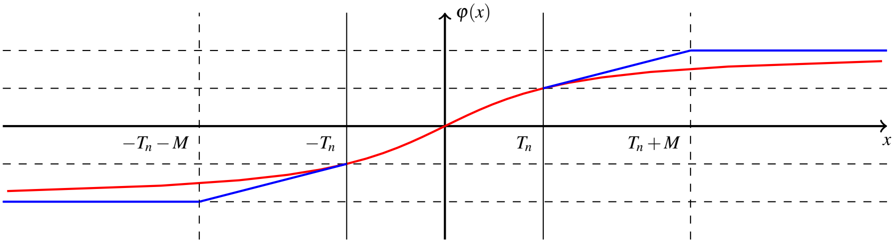

Figure 1: A good competitor for the auxiliary optimal profile problem (12).

By the compactness of the embedding of $H^s([-2T,2T])$ in $L^2([-2T,2T])$, together with the lower semi-continuity of the Gagliardo semi-norm, one can deduce that the minimum in (12) is achieved. Moreover, by truncation and monotone rearrangement, such a minimum is achieved by a function $\varphi^T \in H^s_{\rm loc}(\mathbb{R})$ which is non-decreasing and such that $-1 \le \varphi^T \le 1$. In the proposition below, we prove that the auxiliary problem γ_s^T is a good approximation of the optimal profile problem in (7), and this will also provide the existence result for γ_s.

Proposition 3.1. *The sequence γ_s^T is non-increasing in T, and $\lim_{T\to\infty} \gamma_s^T = \gamma_s$.*

Proof. First of all, by the very definition one can get that γ_s^T is non-increasing in T, and $\gamma_s^T \ge \gamma_s$, which immediately provide the existence of the limit and that

$$\lim_{T\to+\infty} \gamma_s^T \ge \gamma_s.$$

In order to prove the reverse inequality, we will construct a function φ which is a good competitor for γ_s^T. Firstly, for any $\mu > 0$, we can take a function $\psi \in H^s_{\rm loc}(\mathbb{R})$ such that

$$\lim_{x\to-\infty} \psi(x) = -1, \quad \lim_{x\to+\infty} \psi(x) = 1, \quad \text{and} \quad F_1^s(\psi,\mathbb{R}) \le \gamma_s + \mu.$$

Now, we want to modify such a function ψ in a suitable way in order to be admissible for (12), as in Figure 1. To this aim, we consider the function $\Psi \in L^1(\mathbb{R})$ defined by

$$\Psi(x) = \int_{\mathbb{R}} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}y.$$

Notice that $\int_{\mathbb{R}} \Psi(x)\,\mathrm{d}x = [\psi]^2_{H^s(\mathbb{R})}$, and thus $\Psi \in L^1(\mathbb{R})$, so that we can choose a sequence $T_n \to \infty$ such that

$$\Psi(T_n) \to 0 \text{ as } n \to \infty.$$

For any $\delta > 0$, up to subsequences, we can assume that $1 - |\psi(T_n)| < \delta$. We now denote by $J_n := (-T_n, T_n)$ the interval in which the competitor φ will coincide with the function ψ, by extending it affinely to reach the phases. We have,

$$\varphi(x) := \begin{cases} -1 & \text{if } x \in (-\infty, -T_n - M), \\ \frac{\psi(-T_n)+1}{M}(x+T_n) + \psi(-T_n) & \text{if } x \in [-T_n - M, -T_n], \\ \psi(x) & \text{if } x \in J_n, \\ \frac{1-\psi(T_n)}{M}(x-T_n) + \psi(T_n) & \text{se } x \in [T_n, T_n + M], \\ 1 & \text{if } x \in (T_n + M, +\infty). \end{cases}$$

Let us carefully estimate the energy of φ on $\mathbb{R}$. We have

$$\begin{aligned} F_1^s(\varphi, \mathbb{R}) &= F_1^s(\psi, J_n) + F_1^s(\varphi, \mathbb{R} \setminus J_n) + 4(1-s) \iint_{(\mathbb{R}\setminus J_n)\times J_n} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}} \,\mathrm{d}x\,\mathrm{d}y \\ &\leq \gamma_s + \mu + 2(1-s) \iint_{(\mathbb{R}\setminus J_n)\times(\mathbb{R}\setminus J_n)} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}} \,\mathrm{d}x\,\mathrm{d}y \\ &\quad + 4(s-1) \iint_{(\mathbb{R}\setminus J_n)\times J_n} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}} \,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}\setminus J_n} W(\varphi)\,\mathrm{d}x \\ &=: \gamma_s + \mu + 2(1-s)I_1 + 2(1-s)I_2 + I_W. \end{aligned}$$

Now we estimate the contributions given by I_W, I_1, and I_2. We firstly set

$$\omega_\delta := \max_{[-1,-1+\delta]\cup[1-\delta,1]} W. \tag{13}$$

Clearly, $\omega_\delta \to 0$ as $\delta \to 0$, and we have

$$I_W = \int_{\mathbb{R}\setminus J_n} W(\varphi)\,\mathrm{d}x \;\leq\; 2M\omega_\delta. \tag{14}$$

Regarding the integral contribution I_1, we can split it as follows,

$$\begin{aligned} I_1 &= 2\int_{-\infty}^{-T_n}\int_{T_n}^{+\infty} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y + \int_{-\infty}^{-T_n-M}\int_{-\infty}^{-T_n-M} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_{T_n+M}^{+\infty}\int_{T_n+M}^{+\infty} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y + 2\int_{-\infty}^{-T_n-M}\int_{-T_n-M}^{-T_n} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + 2\int_{T_n+M}^{+\infty}\int_{T_n}^{T_n+M} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y + \int_{-T_n-M}^{-T_n}\int_{-T_n-M}^{-T_n} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_{T_n}^{T_n+M}\int_{T_n}^{T_n+M} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &=: 2R_1 + R_2 + R_3 + 2R_4 + 2R_5 + R_6 + R_7, \end{aligned}$$

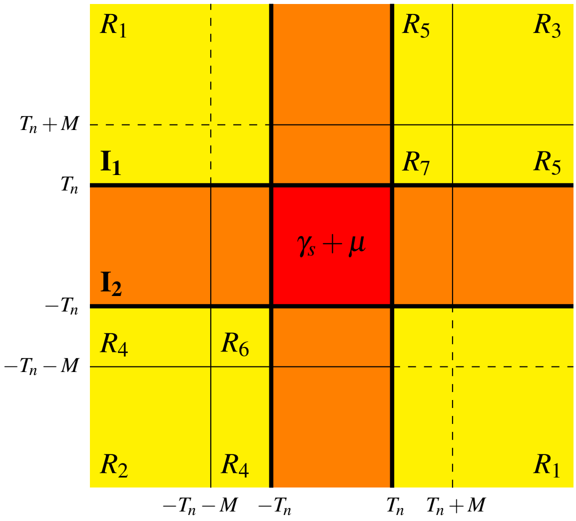

Figure 2: The partition of the plane to compute the energy of the competitor for the auxiliary optimal problem (12).

where the integration domains of the contributions $R_1, \dots, R_8$ can be seen in Figure 2. We immediately notice that $R_2 = R_3 = 0$.

We have

$$R_1 \le 4\int_{-\infty}^{-T_n}\int_{T_n}^{+\infty}(x-y)^{-1-2s}\,\mathrm{d}x\,\mathrm{d}y = \frac{2}{s(2s-1)(2T_n)^{2s-1}},$$

and

$$\begin{aligned} R_4 &= \int_{-\infty}^{-T_n-M}\int_{-T_n-M}^{-T_n}\frac{|1+\psi(-T_n)+(T_n+x)\frac{1+\psi(T_n)}{M}|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{-\infty}^{0}\int_{0}^{M}\frac{|\frac{1+\psi(-T_n)}{M}z|^2}{|z-w|^{1+2s}}\,\mathrm{d}z\mathrm{d}w = \frac{(1+\psi(-T_n))^2}{2s(3-2s)M^{2s-1}}, \end{aligned}$$

where we also used the standard changing variable formula. In a similar way, we can compute R_5 to get

$$R_5 = \frac{(1-\psi(T_n))^2}{2s(3-2s)M^{2s-1}}.$$

For what concerns the contributions R_6 and R_7, we have

$$R_6 = \frac{1}{M^2}\int_{-T_n-M}^{-T_n}\int_{-T_n-M}^{-T_n}\frac{|1+\psi(-T_n)|^2|y-x|^2}{|x-y|^{1+2s}} = \frac{(1+\psi(-T_n))^2}{(1-s)(3-2s)M^{2s-1}},$$

and analogously

$$R_7 = \frac{(1-\psi(T_n))^2}{(1-s)(3-2s)M^{2s-1}}.$$

All in all, we have the following estimate for the integral contribution I_1,

$$\begin{aligned} I_1 &\le \frac{4}{s(2s-1)(2T_n)^{2s-1}} + \frac{(1+\psi(-T_n))^2 + (1-\psi(T_n))^2}{s(3-2s)M^{2s-1}} \\ &\quad + \frac{(1+\psi(-T_n))^2 + (1-\psi(T_n))^2}{(1-s)(3-2s)M^{2s-1}} \\ &\le \frac{4}{s(2s-1)(2T_n)^{2s-1}} + \frac{2\delta^2}{s(3-2s)M^{2s-1}} + \frac{2\delta^2}{(1-s)(3-2s)M^{2s-1}}. \end{aligned} \tag{15}$$

It remains to estimate the contribution I_2; that is,

$$\begin{aligned} I_2 &= 2\int_{-\infty}^{-T_n-M}\int_{-T_n}^{T_n} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y + 2\int_{-T_n-M}^{-T_n}\int_{-T_n}^{T_n} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + 2\int_{T_n}^{T_n+M}\int_{-T_n}^{T_n} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y + 2\int_{T_n+M}^{+\infty}\int_{-T_n}^{T_n} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{16}$$

The first term in the display above can be estimated as follows,

$$\begin{aligned} 2\int_{-\infty}^{-T_n-M}\int_{-T_n}^{T_n} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y &\le 2\int_{-\infty}^{-T_n-M}\int_{-T_n}^{T_n} \frac{4}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &\le \frac{4}{s(2s-1)M^{2s-1}}. \end{aligned}$$

The second term in the right-hand side of (16) can be estimated as follows,

$$\begin{aligned} &2\int_{-T_n-M}^{-T_n}\int_{-T_n}^{T_n} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad = 2\int_{-T_n-M}^{-T_n}\int_{-T_n}^{T_n} \frac{|\psi(y)-\psi(-T_n)-\frac{\psi(-T_n)+1}{M}(x+T_n)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad \le 4\int_{-T_n-M}^{-T_n}\int_{-T_n}^{T_n} \frac{|\psi(y)-\psi(-T_n)|^2}{|y+T_n|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad\quad + \frac{1}{2s}\int_{-T_n-M}^{-T_n}\left(|x+T_n|^{3-2s} - \frac{|x+T_n|}{|T_n-x|^{2s}}\right)\mathrm{d}x \\ &\qquad \le 4M\Psi(-T_n) + \frac{2\delta^2}{s(3-2s)M^{2s-1}}. \end{aligned}$$

Similarly, one can estimate the third and the fourth terms in the right-hand side of (16), to get

$$I_2 \leq 4M(\Psi(-T_n)+\Psi(T_n)) + \frac{4\delta^2}{s(3-2s)M^{2s-1}} + \frac{8}{s(2s-1)M^{2s-1}}. \tag{17}$$

Finally, by combining the estimates in (14), (15) and (17), we have

$$\begin{aligned} \gamma_s^{T_n+M} &\leq F_1(\varphi,\mathbb{R}) \\ &\leq \gamma_s + \mu + \frac{8(1-s)}{s(2s-1)(2T_n)^{2s-1}} + \frac{4(1-s)\delta^2}{s(3-2s)M^{2s-1}} + \frac{4\delta^2}{(3-2s)M^{2s-1}} \\ &\quad + 2M\omega_\delta + 8M(1-s)(\Psi(-T_n)+\Psi(T_n)) + \frac{8(1-s)\delta^2}{s(3-2s)M^{2s-1}} \\ &\quad + \frac{16(1-s)}{s(2s-1)M^{2s-1}}. \end{aligned}$$

By letting $n \to \infty$, it follows

$$\begin{aligned} \lim_{T\to+\infty} \gamma_s^T &= \lim_{n\to+\infty} \gamma_s^{T_n+M} \\ &\leq \gamma_s + \mu + \frac{4(1-s)\delta^2}{s(3-2s)M^{2s-1}} + \frac{4\delta^2}{(3-2s)M^{2s-1}} + 2M\omega_\delta \\ &\quad + \frac{8(1-s)\delta^2}{s(3-2s)M^{2s-1}} + \frac{16(1-s)}{s(2s-1)M^{2s-1}}. \end{aligned}$$

The desired result will thus follow by letting first $\delta \to 0$, then $M \to \infty$, and $\mu \to 0$. □

Remark 3.2. As mentioned in the introduction, it is worth noticing that all the estimates in the proof above are uniform with respect to s when $s \nearrow 1$. This will be crucial in the proof of the convergence of the fractional optimal profile γ_s to γ_1, as we will see in forthcoming Section 7.

We are now in a position to show the following result for the existence of the minimum in (7).

Proposition 3.3. *For any $s \in (0,1)$ the minimum in (7) is achieved by a non-decreasing function $\bar{\varphi} \in H^s_{loc}(\mathbb{R})$ such that $-1 \leq \bar{\varphi} \leq 1$.*

Proof. For any $T > 0$, take a non-decreasing minimizer $\varphi^T : \mathbb{R} \to [-1,1]$ for γ_s^T. By Helly's Theorem, there exists a sequence φ^{T_j} which converges pointwise to a function $\bar{\varphi}$ in $\mathbb{R}$. By construction, such a function $\bar{\varphi}$ is non-decreasing and such that $-1 \leq \bar{\varphi} \leq 1$ in $\mathbb{R}$. By Fatou's Lemma, we have

$$F_1^s(\bar{\varphi}, \mathbb{R}) \leq \liminf_{j \to \infty} \gamma_s^{T_j} = \gamma_s,$$

where we also used the result in Proposition 3.1. Finally, we will show that $\lim_{x \to \pm\infty} \bar{\varphi} = \pm 1$, as proved for instance in [26, Section 4]. Since $\bar{\varphi}$ is non-decreasing in $[-1, 1]$, there exist $\varsigma < 0$ and $\tau > 0$ such that

$$\lim_{x \to -\infty} \bar{\varphi}(x) = \varsigma \quad \text{and} \quad \lim_{x \to +\infty} \bar{\varphi}(x) = \tau.$$

By contradiction, we assume that either $\varsigma \neq -1$ or $\tau \neq 1$. Then, since W is continuous and strictly positive in $(-1, 1)$, we obtain

$$\int_{\mathbb{R}} W(\bar{\varphi})\, dx = +\infty.$$

This is impossible, because, by Fatou's Lemma, we have

$$\int_{\mathbb{R}} W(\bar{\varphi})\, \mathrm{d}x \leq \liminf_{j \to +\infty} \int_{\mathbb{R}} W(\varphi^{T_j})\, \mathrm{d}x \leq \liminf_{j \to +\infty} F_1^s(\varphi^{T_j}) < +\infty. \tag{18}$$

□

Remark 3.4. We would like to notice that the result in Proposition 3.3 plainly implies that γ_s is strictly positive, since it has been proven that the infimum in (7) is achieved by a non-constant function.

4. Compactness

This section is devoted to the proof of the compactness result stated in Theorem 1.2. We will follow the approach of Alberti, Bouchitté and Seppecher in [2], where the case when $s = 1/2$ is studied. The main idea is to make use of Young measures associated to equi-bounded sequences (u_ε) together with the intrinsic properties of the involved energy functional. In order to do this, we would need a lower-bound for $F_\varepsilon^s(u_\varepsilon)$, as stated in the lemma below, which will be useful to estimate the contribution of any jumps of the limit function u.

Lemma 4.1. *Let $(u_\varepsilon) \subset H^s(I)$, $J \subset I$ an interval, and $0 < \delta < 1$. For any $\varepsilon > 0$, consider the sets*

$$A_\varepsilon := \{x \in I : u_\varepsilon \leq -1 + \delta\}, \qquad B_\varepsilon := \{x \in I : u_\varepsilon \geq 1 - \delta\}, \tag{19}$$

and the numbers

$$a_\varepsilon := \frac{|A_\varepsilon \cap J|}{|J|}, \qquad b_\varepsilon := \frac{|B_\varepsilon \cap J|}{|J|}.$$

Then, we have

$$F^s_\varepsilon(u_\varepsilon, J) \geq \frac{8(1-s)\varepsilon^{2s-1}(1-\delta)^2}{s(2s-1)|J|^{2s-1}} \left(1 - \frac{1}{(1-a_\varepsilon)^{2s-1}} - \frac{1}{(1-b_\varepsilon)^{2s-1}}\right) + c_\delta,$$

where the positive constant c_δ does not depend on ε.

Proof. Let $J = (x_0, y_0)$ for some $x_0, y_0 \in \mathbb{R}$, and denote u^*_ε the non-decreasing rearrangement of u_ε in J, as defined in (10). For any $\delta \in (0,1)$, let

$$m_\delta := \min_{-1+\delta \leq t \leq 1-\delta} W(t).$$

We can estimate the energy of u_ε on J as follows,

$$\begin{aligned}
F^s_\varepsilon(u_\varepsilon, J) &\geq F^s_\varepsilon(u^*_\varepsilon, J) \\
&\geq 4\varepsilon^{2s-1}(1-s) \int_{x_0}^{x_0+a_\varepsilon|J|} \int_{y_0-b_\varepsilon|J|}^{y_0} \frac{(u^*_\varepsilon(y) - u^*_\varepsilon(x))^2}{(y-x)^{1+2s}} \, \mathrm{d}y \, \mathrm{d}x \\
&\quad + \frac{1}{\varepsilon} \int_J W(u^*_\varepsilon) \, \mathrm{d}x \\
&\geq 4\varepsilon^{2s-1}(1-s)(1-\delta-1+\delta)^2 \int_{x_0}^{x_0+a_\varepsilon|J|} \int_{y_0-b_\varepsilon|J|}^{y_0} (y-x)^{-1-2s} \, \mathrm{d}y \, \mathrm{d}x \\
&\quad + \frac{|J|}{\varepsilon}(1-a_\varepsilon-b_\varepsilon) m_\delta \\
&= \frac{8\varepsilon^{2s-1}(1-s)(1-\delta)^2}{s(2s-1)|J|^{2s-1}} \left(1 - (1-a_\varepsilon)^{1-2s} - (1-b_\varepsilon)^{1-2s}\right) \\
&\quad + \frac{8\varepsilon^{2s-1}(1-s)(1-\delta)^2}{s(2s-1)|J|^{2s-1}} (1-a_\varepsilon-b_\varepsilon)^{1-2s} \\
&\quad + \frac{|J|}{\varepsilon} (1-a_\varepsilon-b_\varepsilon) \, m_\delta.
\end{aligned}$$

Minimizing with respect to $|J| \, (1-a_\varepsilon-b_\varepsilon)$, it yields

$$F^s_\varepsilon(u_\varepsilon, J) \geq \frac{8\varepsilon^{2s-1}(1-s)(1-\delta)^2}{s(2s-1)|J|^{2s-1}} \left(1 - (1-a_\varepsilon)^{1-2s} - (1-b_\varepsilon)^{1-2s}\right) + c_\delta,$$

where the positive constant

$$c_\delta = \frac{2^{\frac{2s+3}{2s}} (1-\delta)^{\frac{1}{s}} (1-s)^{\frac{1}{2s}}}{(2s-1)^{\frac{2s-1}{2s}} s^{\frac{1}{2s}} m_\delta^{\frac{1}{2s}}}$$

is independent on ε, as desired. □

Proof of Theorem 1.2. Let (u_ε) be a sequence such that $F^s_\varepsilon(u_\varepsilon) < C$; this immediately gives

$$\lim_{\varepsilon\to 0}\int_I W(u_\varepsilon)\,\mathrm{d}x \to 0.$$

Therefore, in view of the behavior of W at infinity, by the Dunford-Pettis theorem, there exist a subsequence, still denoted by u_ε, and a function $u \in L^1(I)$ such that

$$u_\varepsilon \underset{L^1(I)}{\rightharpoonup} u \text{ as } \varepsilon \to 0.$$

It remains to show that $u \in BV(I,\{-1,1\})$ and that the convergence above is strong in $L^1(I)$.

To this aim, we consider the family of measures $\{\nu_x\}$ associated to (u_ε). By Theorem 9 in [32], it follows

$$u = \int_I \left(\int_{\mathbb{R}} \lambda\,\mathrm{d}\nu_x(\lambda)\right)\mathrm{d}x.$$

Therefore, because of the nonnegativeness of the double-well function W, we can apply Theorem 6 in [32] to get

$$\int_I \left(\int_{\mathbb{R}} W(t)\,\mathrm{d}\nu_x(t)\right)\mathrm{d}x \,\leq\, \liminf_{\varepsilon\to 0}\int_I W(u_\varepsilon)\,\mathrm{d}x.$$

From the display above we can deduce that

$$\int_{\mathbb{R}} W(t)\,\mathrm{d}\nu_x(t) = 0.$$

Then, it follows that

$$\nu_x = \theta(x)\delta_{-1} + (1-\theta(x))\delta_1, \quad \text{a. e. } x \in I,$$

where we denoted by δ_{x^*} the Dirac measure concentrated in x^*. This will give

$$u(x) = -\theta(x) + (1-\theta(x)) = 1 - 2\theta(x), \quad \text{a. e. } x \in I$$

for some function $\theta : I \to [-1,1]$.

Take now a point x_0 such that the approximate limit of u in x_0 does exist and is neither -1 nor 1. By applying again [32, Theorem 6], we have

$$\lim_{\varepsilon\to 0} a_\varepsilon = \frac{\int_J \theta(x)\,\mathrm{d}x}{|J|} =: a_0 > 0, \text{ and } \lim_{\varepsilon\to 0} b_\varepsilon = \frac{\int_J (1-\theta(x))\,\mathrm{d}x}{|J|} =: b_0 > 0.$$

Now, we can apply the lower bound on the energy proven in Lemma 4.1; we have

$$\liminf_{\varepsilon\to 0} F^s_\varepsilon(u_\varepsilon, J) \,\geq\, c_\delta > 0.$$

Finally, let Su be the set of points in which the approximate limit of u is neither -1 nor 1. For every $N < \mathcal{H}^0(Su)$ and $x_1, \ldots, x_N \in Su$, let $J_1, \ldots, J_N \subset I$ be disjoint intervals such that $x_i \in J_i$, for $i = 1, \ldots, N$ and the integrals

$$\int_{J_i} \theta(x)\,\mathrm{d}x \text{ and } \int_{J_i} (1-\theta(x))\,\mathrm{d}x$$

are different from zero. As before,

$$F^s_\varepsilon(u_\varepsilon, J_i) \geq c_\delta, \quad i = 1, \ldots, N.$$

Hence, by using the super-additivity of $F^s_\varepsilon(u_\varepsilon, \cdot)$ it follows

$$F^s_\varepsilon(u_\varepsilon, I) \geq N c_\delta.$$

This implies that $\mathcal{H}^0(Su) < +\infty$, and also, by [32, Theorem 9], that the convergence is strong in L^1. □

5. Proof of Theorem 1.1

This section is devoted to the proof of the Γ-convergence result in the case when $s \in (1/2, 1)$ as stated in Theorem 1.1.

5.1. The lower bound inequality

Thanks to the compactness result in Section 4, one can assume that the limit function of any sequence (u_ε) converging in $L^1(I)$ is a function $u \in BV(I, \{-1, 1\})$, and thus whose jump set Su is finite. Consequently, one can find $N := \mathcal{H}^0(Su)$ disjoint subintervals $\{I_i\}_{i=1,\ldots,N}$ such that $Su \cap I_i \neq \emptyset$, for every $i = 1, ..., N$. Now, we take the monotone rearrangement $u^*_{\varepsilon,i}$ of u_ε in I_i. The rearrangement $u^*_{\varepsilon,i}$ is non-decreasing if u is non-decreasing in I_i and non-increasing otherwise. Then, by the super-additivity of $F^s_\varepsilon(u_\varepsilon, \cdot)$ and the monotonicity of the energy with respect to monotone rearrangements, we get

$$\liminf_{\varepsilon \to 0} F^s_\varepsilon(u_\varepsilon, I) \geq \liminf_{\varepsilon \to 0} \sum_{i=1}^N F^s_\varepsilon(u_\varepsilon, I_i) \geq \sum_{i=1}^N \liminf_{\varepsilon \to 0} F^s_\varepsilon(u^*_{\varepsilon,i}, I_i).$$

As a consequence, the lower-bound inequality (LB) will hold thanks to the estimate in the following

Proposition 5.1. *Let J be an open interval, and let $(u_\varepsilon) \subset H^s(J)$ be a sequence of non-decreasing functions. Assume that there exist $\bar{a} < \bar{b} \in J$ such that for any $\delta > 0$ there exists ε_δ such that*

$$u_\varepsilon(\bar{a}) \leq -1 + \delta, \qquad u_\varepsilon(\bar{b}) \geq 1 - \delta$$

for any $\varepsilon \leq \varepsilon_\delta$. Then,

$$\liminf_{\varepsilon\to 0} F^s_\varepsilon(u_\varepsilon) \geq \gamma_s. \tag{20}$$

Proof. We assume that $\liminf_\varepsilon F^s_\varepsilon(u_\varepsilon) < +\infty$. Also, it is not restrictive to assume that $|u_\varepsilon| \leq 1$. We can estimate the energy of the sequence u_ε in $J := (a,b)$ as in the proof of Proposition 3.1. Let

$$U_\varepsilon(x) := \varepsilon^{2s-1} \int_J \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}y.$$

We have

$$\liminf_{\varepsilon\to 0} \int_J U_\varepsilon(x)\,dx < +\infty.$$

Thus, by Fatou's Lemma, we can find $C > 0$, $a < \alpha < \bar{a}$ and $\bar{b} < \beta < b$ such that

$$\liminf_{\varepsilon\to 0} U_\varepsilon(\alpha) < C \quad \text{and} \quad \liminf_{\varepsilon\to 0} U_\varepsilon(\beta) < C. \tag{21}$$

For any $M > 0$, we define the following function v_ε which extends u_ε on the whole $\mathbb{R}$,

$$v_\varepsilon(x) := \begin{cases} -1 & \text{if } x \in (-\infty, \alpha - M\varepsilon), \\ \frac{u_\varepsilon(\alpha)+1}{M\varepsilon}(x-\alpha) + u_\varepsilon(\alpha) & \text{if } x \in [\alpha - M\varepsilon, \alpha], \\ u_\varepsilon(x) & \text{se } x \in (\alpha,\beta), \\ \frac{1-u_\varepsilon(\beta)}{M\varepsilon}(x-\beta) + u_\varepsilon(\beta) & \text{if } x \in [\beta, \beta + M\varepsilon], \\ 1 & \text{if } x \in (\beta + M\varepsilon, +\infty). \end{cases}$$

Since by construction the function v_ε is non-decreasing, it follows $v_\varepsilon(x) \leq -1 + \delta$ for any $x < \bar{a}$, and $v_\varepsilon(x) \geq 1 - \delta$ for any $x > \bar{b}$.

We are ready to estimate the energy of v_ε. Denote by $\tilde{J} := (\alpha, \beta)$; we have

$$\begin{aligned} F^s_\varepsilon(u_\varepsilon, \tilde{J}) &= F^s_\varepsilon(v_\varepsilon, \mathbb{R}) - F^s_\varepsilon(v_\varepsilon, \mathbb{R}\setminus\tilde{J}) \\ &\quad -4(1-s)\varepsilon^{2s-1} \iint_{(\mathbb{R}\setminus\tilde{J})\times\tilde{J}} \frac{|v_\varepsilon(x) - v_\varepsilon(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &\geq \gamma_s - 2(1-s)\varepsilon^{2s-1} \iint_{(\mathbb{R}\setminus\tilde{J})\times(\mathbb{R}\setminus\tilde{J})} \frac{|v_\varepsilon(x) - v_\varepsilon(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad -\frac{1}{\varepsilon}\int_{\mathbb{R}\setminus\tilde{J}} W(v)\,\mathrm{d}x - 4(s-1)\varepsilon^{2s-1} \iint_{(\mathbb{R}\setminus\tilde{J})\times\tilde{J}} \frac{|v_\varepsilon(x) - v_\varepsilon(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &=: \gamma_s - 2(1-s)\tilde{I}_1 - \tilde{I}_W - 2(1-s)\tilde{I}_2. \end{aligned}$$

We can now estimate all the integral contributions in the preceding display by following the estimates of φ in the proof of Proposition 3.1 step by step, by simply replacing $-T_n, T_n$, and M, there with α, β, and $M\varepsilon$, respectively. We have

$$\tilde{I}_1 \leq \frac{4\varepsilon^{2s-1}}{s(1-2s)(\beta-\alpha)^{2s-1}} + \frac{2\delta^2}{s(3-2s)M^{2s-1}} + \frac{2\delta^2}{(1-s)(3-2s)M^{2s-1}},$$

$$\tilde{I}_2 \leq 4M\varepsilon(U_\varepsilon(\alpha)+U_\varepsilon(\beta)) + \frac{4\delta^2}{s(3-2s)M^{2s-1}} + \frac{8}{s(2s-1)M^{2s-1}},$$

and

$$\tilde{I}_W \leq 2M\omega_\delta,$$

where ω_δ is defined in (13).

All in all, for any $\delta > 0$, for any $0 < \varepsilon \leq \varepsilon_\delta$, and for any $M > 0$, we arrive at

$$\begin{aligned} F^s_\varepsilon(u_\varepsilon, \tilde{J}) \geq \; & \gamma_s - \frac{8(1-s)}{s(2s-1)(\beta-\alpha)^{2s-1}}\varepsilon^{2s-1} - \frac{4\delta^2}{(3-2s)M^{2s-1}} - 2M\omega_\delta \\ & -4M\varepsilon(U_\varepsilon(\alpha)+U_\varepsilon(\beta)) + -\frac{16(1-s)\delta^2}{s(3-2s)M^{2s-1}} - \frac{16(1-s)}{s(2s-1)M^{2s-1}}. \end{aligned}$$

We now notice that one can let $\varepsilon \to 0$ in the display above, and then, passing to the limit as δ goes to 0, we finally have

$$\liminf_{\varepsilon\to 0} F^s_\varepsilon(u_\varepsilon, \tilde{J}) \geq \gamma_s - \frac{16(1-s)}{s(2s-1)M^{2s-1}},$$

where we also used the observation in (21). The desired inequality will plainly follow by letting $M \to +\infty$. □

Remark 5.2. The same estimate in (20) does hold in the case when u_ε are non-increasing satisfying the required assumptions with $\bar{a} > \bar{b}$.

Remark 5.3. It is worth remarking that the assumptions in Proposition 5.1 are satisfied by the monotone rearrangement function $u^*_{\varepsilon,i}$. Indeed, up to translations, one can assume $I_i \equiv (-c,c)$ and

$$u|_{I_i}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0. \end{cases}$$

In such a case we have that $E^1_\varepsilon = \{u^*_{\varepsilon,i} < 1-\delta\} \cap (0,c)$ is an interval. Moreover,

$$\|u - u^*_{\varepsilon,i}\|_{L^1} \geq \delta |E^1_\varepsilon|,$$

which yields $|E^1_\varepsilon| \to 0$ as $\varepsilon \to 0$ for any $\delta > 0$. The same happens for the interval $E^{-1}_\varepsilon = \{u^*_{\varepsilon,i} > -1+\delta\} \cap (-c,0)$. This will assure the existence of $\bar{a}$ and $\bar{b}$ as needed in the statement of Proposition 5.1.

5.2. The upper bound inequality

It suffices to deal with the case when the limit function u is such that

$$u(x) = \begin{cases} -1, & \text{if } x \leq x_0, \\ 1, & \text{if } x > x_0, \end{cases}$$

for some $x_0 \in I$.

For any fixed $T > 0$, take the minimizer $\varphi^T \in H^s_{\rm loc}(\mathbb{R})$ for the auxiliary optimal profile problem γ_s^T defined by (12); i.e.,

$$\varphi^T(x) = -1 \quad \forall\, x \leq -T, \quad \varphi^T(x) = 1 \quad \forall\, x \geq T, \quad \text{and} \quad F_1^s(\varphi^T, \mathbb{R}) = \gamma_s^T.$$

We can construct the recovery sequence by taking, for every $\varepsilon > 0$,

$$u_\varepsilon^T(x) := \varphi^T\left(\frac{x - x_0}{\varepsilon}\right), \quad \text{for every } x \in I.$$

We have

$$u_\varepsilon^T \to u \text{ in } L^1(I),$$

and

$$F_\varepsilon^s(u_\varepsilon^T) \;=\; F_1^s(\varphi^T, (I - x_0)/\varepsilon) \;\leq\; F_1^s(\varphi^T, \mathbb{R}) \;=\; \gamma_s^T. \tag{22}$$

Then, Proposition 5.1 yields

$$\lim_{T \to +\infty} \limsup_{\varepsilon \to 0} F_\varepsilon^s(u_\varepsilon^T) \;\leq\; \gamma_s.$$

Finally, by a diagonalization argument we can construct a sequence $\bar{u}_\varepsilon$ converging to u in $L^1(I)$, which satisfies

$$\limsup_{\varepsilon \to 0} F_\varepsilon^s(\bar{u}_\varepsilon) \;\leq\; \gamma_s.$$

Regarding an arbitrary limit function $u \in BV(I, \{-1, 1\})$, the corresponding recovery sequence can be plainly obtained by gluing the sequences constructed above for each single jump of u, and taking into account that the long-range interactions between two different recovery sequences will decay as $\varepsilon \to 0$ since we can always choose $u_\varepsilon^{T_\varepsilon}$ as above such that $T_\varepsilon \varepsilon = O(1)$ as $\varepsilon \to 0$. □

6. The Γ-convergence result in the case when $s \in (0, 1/2)$

For the sake of completeness, we sketch here the proof of the Γ-convergence result of the scaled energies E_ε in the strongly nonlocal case when $s \in (0,1/2)$. As mentioned in the introduction this result is contained in the paper [30] where the authors investigated the problem (with a slightly different kinetic term, in accordance with the corresponding fractional Allen-Cahn equation) in any set $\Omega \in \mathbb{R}^N$, for $N \geq 2$.

Let $I \subset \mathbb{R}$ be an open interval. We firstly notice that the natural scaling is as follows

$$F^s_\varepsilon(u) := \varepsilon^{-2s} E^s_\varepsilon(u) = 2(1-s) \iint_{I\times I} \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{\varepsilon^{2s}} \int_I W(u)\,\mathrm{d}x. \tag{23}$$

For any $s \in (0,1/2)$ we now define

$$X := \Big\{ u \in H^s(I) : u = \chi_E - \chi_{\mathcal{C}E}, \text{ for some } E \subset I \Big\},$$

where we denote by $\mathcal{C}E$ the complement of E, and let

$$F^s(u) := \begin{cases} 2(1-s) \displaystyle\iint_{I\times I} \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y & \text{if } u \in X, \\ +\infty & \text{otherwise.} \end{cases} \tag{24}$$

We have the following

Theorem 6.1. *For any $s \in (0,1/2)$, let $F^s_\varepsilon : H^s(I) \to \mathbb{R}$ and $F^s : H^s \to \mathbb{R}$ be defined in (23) and (24), respectively. Then,*

> **Lower Bound Inequality**. *For every $u \in X$ and every sequence $(u_\varepsilon) \subset H^s(I)$ such that $u_\varepsilon \to u$ in $L^1(I)$,*
>
> $$\liminf_{\varepsilon\to 0} F^s_\varepsilon(u_\varepsilon) \geq F^s(u). \tag{25}$$
>
> **Upper Bound Inequality**. *For every $u \in X$ there exists a sequence $(\tilde{u}_\varepsilon) \subset H^s(I)$ such that $\tilde{u}_\varepsilon \to u$ in $L^1(I)$, and*
>
> $$\limsup_{\varepsilon\to 0} F^s_\varepsilon(\tilde{u}_\varepsilon) \leq F^s(u). \tag{26}$$

Proof. We first prove the inequality in (25). Clearly, it is not restrictive to assume that $\liminf_{\varepsilon\to 0} F^s_\varepsilon(u_\varepsilon)$ is finite. Take a subsequence (u_{ε_k}) such that

$$\lim_{k\to\infty} F_{\varepsilon_k}(u_{\varepsilon_k}) = \liminf_{\varepsilon\to 0} F^s_\varepsilon(u_\varepsilon).$$

Up to subsequences, we can assume that u_{ε_k} converges pointwise to u, a.e. in I. Therefore, by Fatou's Lemma, we have

$$\begin{aligned}\int_I W(u)\,\mathrm{d}x &= \int_I \liminf_{k\to\infty} W(u_{\varepsilon_k})\,\mathrm{d}x \\ &\leq \liminf_{k\to\infty} \int_I W(u_{\varepsilon_k})\,\mathrm{d}x \leq \liminf_{k\to\infty} \varepsilon_k^{2s} F_{\varepsilon_k}(u_{\varepsilon_k}) = 0,\end{aligned}$$

which yields $W(u) = 0$ a.e. in I. Denote by $E := \{u = 1\}$; thus, $u = \chi_E - \chi_{CE}$ and in particular $u \in X$. By pointwise convergence and again by Fatou's Lemma, we plainly obtain the desired lower bound inequality:

$$\begin{aligned}\liminf_{\varepsilon\to 0} F^s_\varepsilon(u_\varepsilon) &= \lim_{k\to\infty} F_{\varepsilon_k}(u_{\varepsilon_k}) \\ &\geq 2(1-s)\iint_{I\times I} \frac{|u_{\varepsilon_k}(x) - u_{\varepsilon_k}(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &\geq 2(1-s)\iint_{I\times I} \frac{|u(x) - u(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y = F^s(u).\end{aligned}$$

The upper bound inequality in (26) is almost immediate. It suffices to choose as a recovery sequence $\tilde{u}_\varepsilon \equiv u$. Indeed, for $u = \chi_E - \chi_{CE}$, we have

$$F^s_\varepsilon(u_\varepsilon) = F^s_\varepsilon(u) = 2(1-s)\iint_{I\times I} \frac{|u(x) - u(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y = F^s(u). \qquad \square$$

7. The link between the fractional and the Modica-Mortola results

This section is devoted to the analysis of the behavior of the fractional optimal profile problems γ_s as $s \to 1^-$, as presented in Theorem 1.3. The key-point of the proof will be passing through the auxiliary optimal profile problems (12) and using the uniformity in s for s close to 1^- in the estimates proven in Section 3, together with a Γ-convergence result by Ponce ([29]). Indeed, on one hand we have proved that the convergence of the auxiliary problems γ_s^T to the optimal profile problem for values of $s \in (1/2, 1)$ is uniform. On the other hand, the validity of such a convergence and such uniformity in the classical Modica-Mortola case, when $s = 1$, is rather straightforward to prove – even though we were not able to find a precise reference to the literature. One can basically repeat the proof of Proposition 3.1: the only sensible difference is in the absence of any nonlocal interaction term, and this only makes computations simpler.

Let $T > 0$. Consider the auxiliary optimal profile problem γ_1^T given by

$$\begin{aligned}\gamma_1^T := \inf\Big\{ \int_{\mathbb{R}} |v'(x)|^2\,\mathrm{d}x + \int_{\mathbb{R}} W(v)\,\mathrm{d}x : \\ v \in H^1_{\mathrm{loc}}(\mathbb{R}),\ v(x) = 1\ \forall x \geq T,\ v(x) = -1\ \forall x \leq -T \Big\}.\end{aligned} \tag{27}$$

We have the following

Proposition 7.1. *Let γ_1^T and γ_1 defined by (27) and (3), respectively. Then,*

$$\lim_{T\to\infty} \gamma_1^T = \gamma_1.$$

Proof of Theorem 1.3. Let $I \subset \mathbb{R}$ be an interval. We notice that the integral functional

$$u \mapsto \int_I W(u)\,\mathrm{d}x$$

is a continuous perturbation. Therefore, we can apply Theorem 8 in [29] in order to deduce the following Γ-convergence result,

$$\Gamma(L^1)\text{-}\lim_{s\to 1^-} F_1^s(u) = \int_I |\nabla u|^2\,\mathrm{d}x + \int_I W(u)\,\mathrm{d}x.$$

Since the constraint on the auxiliary profiles is closed in L^1, it is now sufficient to prove the existence of a pre-compact family of minimizers for the auxiliary problems γ_s^T while keeping T fixed and varying only s in $(1/2,1)$: from the properties of the Γ-convergence and a standard 3ε argument, the result in (8) will follow.

Consider a family u_s^T of non-decreasing minimizers for γ_s^T, which are bounded between -1 and 1, and whose existence is assured in view of the result in Section 3; see in particular Proposition 3.1 there. By Helly's Theorem, we can deduce the existence of a subsequence $u_{s_j}^T$ and a non-decreasing function u_1^T such that

$$\lim_{j\to+\infty} u_{s_j}^T = u_1^T.$$

By Lebesgue's Theorem, such a subsequence converges in L^1 norm: this proves the compactness assumption.

Let now $\mu > 0$. By the uniformity of the aforementioned estimates, there exists $T_0 > 0$ such that

$$\gamma_s - \gamma_s^T < \mu \quad \forall s \in (\bar{s}, 1], \quad \forall T \geq T_0, \tag{28}$$

for some $\bar{s} > \frac{1}{2}$. By Γ-convergence, there exists $s_0 \in (\bar{s}, 1)$, such that for any $1 < s < s_0$ one has

$$\left|\gamma_s^T - \gamma_1^T\right| < \mu. \tag{29}$$

Finally, combining (28) with (29), it yields

$$|\gamma_s - \gamma_1| \leq 3\mu \quad \forall s > s_0.$$

By letting μ to 0, the proof is complete. □

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