LE MATEMATICHE
Vol. LXXV (2020) – Issue I, pp. 167–194
doi: 10.4418/2020.75.1.9

ORLICZ SPACES AND ENDPOINT SOBOLEV-POINCARÉ INEQUALITIES FOR DIFFERENTIAL FORMS IN HEISENBERG GROUPS

ANNALISA BALDI - BRUNO FRANCHI - PIERRE PANSU

In this paper we prove Poincaré and Sobolev inequalities for differential forms in the Rumin's contact complex on Heisenberg groups. In particular, we deal with endpoint values of the exponents, obtaining finally estimates akin to exponential Trudinger inequalities for scalar function. These results complete previous results obtained by the authors away from the exponential case. From the geometric point of view, Poincaré and Sobolev inequalities for differential forms provide a quantitative formulation of the vanishing of the cohomology. They have also applications to regularity issues for partial differential equations.

Received on September 5, 2019

AMS 2010 Subject Classification: 58A10, 35R03, 26D15, 43A80, 53D10, 46E30 46E35
Keywords: Heisenberg groups, differential forms, Sobolev-Poincaré inequalities, Orlicz space, Moser-Trudinger inequality
A. B. and B. F. are supported by the University of Bologna, funds for selected research topics, by MAnET Marie Curie Initial Training Network, by GNAMPA of INdAM (Istituto Nazionale di Alta Matematica "F. Severi"), Italy, and by PRIN of the MIUR, Italy.
P.P. is supported by MAnET Marie Curie Initial Training Network, by Agence Nationale de la Recherche, ANR-10-BLAN 116-01 GGAA and ANR-15-CE40-0018 SRGI. P.P. gratefully acknowledges the hospitality of Isaac Newton Institute, of EPSRC under grant EP/K032208/1, and of Simons Foundation.

1. Introduction

Sobolev and Poincaré inequalities for differential forms on Riemannian and sub-Riemannian manifolds appear naturally in the study of cohomology applications ($L^{q,p}$-cohomology) and geometric group theory (see, e.g. [29] [30], [17] [3]).

At the same time, as for scalar equations, Sobolev and Poincaré inequalities play a crucial role in the study of partial differential equations for differential forms in de Rham complex mainly when regularity issues are concerned. Differential equations for forms have been the object of an increasing interest and of a wide literature in the last few years in connection with A-harmonic equation, calculus of variations, nonlinear elasticity theory, quasiconformal mappings and gas dynamics. Choosing somehow arbitrarily, we mention [27], [21], [36], [20], [34], [12], [19].

A milestone in the theory of Poincaré inequality for differential forms is provided by Corollary 4.2 of [20] that reads as follows: let ω be a ℓ-form in $\mathbb{R}^n$ with distributional coefficients such that $d\omega$ has L^p coefficients with $1<p<n$. Let D be (say) a ball in $\mathbb{R}^n$. Then there exists a closed ℓ-form ω_D such that

$$\|\omega-\omega_D\|_{L^{pn/(n-p)}(D)} \le C\|d\omega\|_{L^p(D)}. \tag{1}$$

A straightforward computation shows that this statement can be formulated as follows:

let α be a closed form in D with L^p coefficients. Then α admits a potential ϕ in $L^{pn/(n-p)}(D)$ (i.e the $L^{(pn/(n-p),p,\ell)}$-cohomology of D vanishes), and

$$\|\phi\|_{L^{pn/(n-p)}(D)} \le C\|\alpha\|_{L^p(D)}.$$

In addition, in [26] the authors prove that ϕ is compactly supported in D when α is (we shall refer to this property as to *Sobolev inequality*) .

In fact, for several applications to regularity issues as well as to the cohomology theory, a weaker form of Poincaré inequality (1) suffices, the so-called *interior* Poincaré inequality or Poincaré inequality with loss of domain (see [30]) that reads as follows:

let D, D' be two balls in $\mathbb{R}^n$, $D \Subset D'$ and let α be a closed form in D' with L^p coefficients. Then α admits a potential ϕ in $L^{pn/(n-p)}(D)$ and

$$\|\phi\|_{L^{pn/(n-p)}(D)} \le C\|\alpha\|_{L^p(D')}.$$

Analogously, we say that the interior Sobolev inequality holds if for any compactly supported closed ℓ-form α in D there exists a potential

We stress that the interior Poincaré inequality, though apparently weaker than the Poincaré inequality without loss of domain, is, under other respects, more general since it is not affected by the geometry of the boundary of D (the word “interior” refers precisely to this feature). Moreover, the interior Sobolev and Poincaré inequalities that we derive from [20], [26] can be extended to the endpoint case $p = 1$ (see [5]) and to the case $p = n$ (this is basically a special instance of the present paper, being a straightforward consequence of [20] and [10]).

The aim of this paper is to establish analogous interior Sobolev-Poincaré inequalities for differential forms in the non-Riemannian setting of Heisenberg groups. Precise definition will be given in Section 2. Here we restrict to say that the n-th Heisenberg group $\mathbb{H}^n$ is a connected, simply connected stratified nilpotent group. Through exponential coordinates, $\mathbb{H}^n$ can be identified with $\mathbb{R}^{2n+1}$ as a differentiable manifold, and the Haar measure of the group is then the Lebesgue measure in $\mathbb{R}^{2n+1}$. In exponential coordinates the identity e of $\mathbb{H}^n$ will coincide with 0. We stress also that $\mathbb{H}^n$ can be endowed with an homogeneous left-invariant metric d that *is not bi-Lipschitz equivalent to the Euclidean distance (even locally)*, since the group is noncommutative (see [33]). In particular, $(\mathbb{H}^n, d)$ is not Riemannian at any scale, and its Hausdorff dimension is $2n+2 =: Q$. Throughout this paper we shall denote by $B(x,r)$ the d-ball of radius $r > 0$ centred at $x \in \mathbb{H}^n$.

We refer to the first layer $\mathfrak{h}_1$ of the stratification of the Lie algebra $\mathfrak{h}$ of $\mathbb{H}^n$ as to the horizontal layer. Since the Lie algebra $\mathfrak{h}$ is stratified nilpotent, $\mathfrak{h}_1$ generates by commutation all the algebra. In addition, the stratification of $\mathfrak{h}$ induces a family of anisotropic dilations δ_λ, $\lambda > 0$ in $\mathbb{H}^n$. Unfortunately, when dealing with differential forms in $\mathbb{H}^n$, the de Rham complex lacks scale invariance under anisotropic dilations. Thus, a substitute for de Rham’s complex, that recovers scale invariance under δ_λ has been defined by M. Rumin, [31]. We refer to Section 5 for a list of the main properties of Rumin’s complex, as well as to [31] and [6] for details of the construction. Here we restrict ourselves to the crucial points we need to state the results proved in this paper:

- For $h = 0, \dots, 2n+1$, the space of Rumin h-forms, E_0^h is the space of smooth sections of a left-invariant subbundle of $\bigwedge^h T^*\mathbb{H}^n$ (that we still denote by E_0^h). Hence it inherits inner products and norms of function spaces. Moreover, let Ξ^h be a basis of E_0^h (see e.g. [2]) then, starting from any vector space $\mathcal{F}$ of real-valued functions, we can define a space of Rumin’s forms $\mathcal{F}(E_0^h)$ in the natural way. Obviously, this notion is invariant under changes of the basis.

- A differential operator $d_c : E_0^h \to E_0^{h+1}$ is defined. It is left-invariant and homogeneous with respect to group dilations. It is a first order homogeneous operator in the horizontal derivatives in degree $\neq n$, whereas *it is a second order homogeneous horizontal operator in degree n.*

- Altogether, $E_0^\bullet$ and the operators d_c form a complex since $d_c \circ d_c = 0$.

- This complex $(E_0^\bullet, d_c)$ is homotopic to de Rham's complex $(\Omega^\bullet, d)$. The homotopy is achieved by differential operators $\Pi_E : E_0^\bullet \to \Omega^\bullet$ and $\Pi_{E_0} : \Omega^\bullet \to E_0^\bullet$ (Π_E has horizontal order ≤ 1 and Π_{E_0} is an algebraic operator).

We can state now the interior Sobolev-Poincaré inequalities for $(E_0^\bullet, d_c)$.

Theorem 1.1 (Poincaré inequality)**.** *Let $B = B(e,1)$ and $B' = B(e,\lambda)$, $\lambda > 1$, be concentric balls of $\mathbb{H}^n$, and take $1 \leq h \leq 2n+1$. We have:*

Ia) if $h \neq n+1$, and $1 \leq p < Q$, for any d_c-exact form $\alpha \in L^p(B', E_0^h)$ there exists a form $\phi \in L^{pQ/(Q-p)}(B, E_0^{h-1})$ such that $d_c\phi = \alpha$ and

$$\|\phi\|_{L^{Qp/(Q-p)}(B,E_0^{h-1})} \leq C\|\alpha\|_{L^p(B',E_0^h)},$$

where $C > 0$ is independent of α.

Ib) denote by $Y = Y(s)$ the Young function (see Section 3 below) $Y(s) := \exp\left(s^{Q/(Q-1)}\right)$. If $h \neq n+1$, for any d_c-exact form $\alpha \in L^Q(B', E_0^h)$ there exists a form $\phi \in L^Y(B, E_0^{h-1})$ such that $d_c\phi = \alpha$ and

$$\|\phi\|_{L^Y(B,E_0^{h-1})} \leq C\|\alpha\|_{L^Q(B',E_0^h)},$$

where $C > 0$ is independent of α. The Orlicz space L^Y is defined in Section 3 below.

IIa) if $h = n+1$, and $1 \leq p < Q/2$, for any d_c-exact form $\alpha \in L^p(B', E_0^{n+1})$ there exists a form $\phi \in L^{pQ/(Q-2p)}(B, E_0^n)$ such that $d_c\phi = \alpha$ and

$$\|\phi\|_{L^{Qp/(Q-2p)}(B,E_0^n)} \leq C\|\alpha\|_{L^p(B',E_0^{n+1})},$$

where $C > 0$ is independent of α.

IIb) denote by $Y = Y(s)$ the Young function $Y(s) := \exp\left(s^{Q/(Q-2)}\right)$. If $h = n+1$, for any d_c-exact form $\alpha \in L^{Q/2}(B', E_0^{n+1})$ there exists a form $\phi \in L^Y(B, E_0^n)$ such that $d_c\phi = \alpha$ and

$$\|\phi\|_{L^Y(B,E_0^n)} \leq C\|\alpha\|_{L^{Q/2}(B',E_0^{n+1})},$$

where $C > 0$ is independent of α.

Theorem 1.2 (Sobolev inequality)**.** *Let $B = B(e,1)$ and $B' = B(e,\lambda)$, $\lambda > 1$, be concentric balls of $\mathbb{H}^n$, and take $1 \leq h \leq 2n+1$. We have:*

Ia) *if $h \neq n+1$, and $1 \le p < Q$, for any compactly supported d_c-exact form $\alpha \in L^p(B, E_0^h)$ there exists a compactly supported form*

$$\phi \in L^{pQ/(Q-p)}(B', E_0^{h-1})$$

such that $d_c\phi = \alpha$ and

$$\|\phi\|_{L^{Qp/(Q-p)}(B',E_0^{h-1})} \le C\|\alpha\|_{L^p(B,E_0^h)},$$

where $C > 0$ is independent of α.

Ib) *denote by $Y = Y(s)$ the Young function $Y(s) := \exp\left(s^{Q/(Q-1)}\right)$ If $h \neq n+1$, for any compactly supported d_c-exact form $\alpha \in L^Q(B, E_0^h)$ there exists a compactly supported form $\phi \in L^Y(B', E_0^{h-1})$ such that $d_c\phi = \alpha$ and*

$$\|\phi\|_{L^Y(B',E_0^{h-1})} \le C\|\alpha\|_{L^Q(B,E_0^h)},$$

where $C > 0$ is independent of α.

IIa) *if $h = n+1$, and $1 \le p < Q/2$, for any compactly supported d_c-exact form $\alpha \in L^p(B, E_0^h) = L^p(B, E_0^{n+1})$ there exists a compactly supported form $\phi \in L^{pQ/(Q-2p)}(B', E_0^n)$ such that $d_c\phi = \alpha$ and*

$$\|\phi\|_{L^{Qp/(Q-2p)}(B',E_0^n)} \le C\|\alpha\|_{L^p(B,E_0^{n+1})},$$

where $C > 0$ is independent of α.

IIb) *denote by $Y = Y(s)$ the Young function $Y(s) := \exp\left(s^{Q/(Q-2)}\right)$. If $h = n+1$, for any compactly supported d_c-exact form $\alpha \in L^{Q/2}(B, E_0^h) = L^{Q/2}(B' E_0^{n+1})$ there exists a compactly supported form $\phi \in L^Y(B', E_0^n)$ such that $d_c\phi = \alpha$ and*

$$\|\phi\|_{L^Y(B',E_0^n)} \le C\|\alpha\|_{L^{Q/2}(B,E_0^{n+1})},$$

where $C > 0$ is independent of α.

Statements Ia) and IIa) of Theorems 1.1 and 1.2 are proved in [4] when $p > 1$ and in [3] when $p = 1$. The proof of statements Ib) and IIb) of Theorems 1.1 and 1.2 will be carried out in the present paper (see Theorem 8.8).

2. Heisenberg groups

We denote by $\mathbb{H}^n$ the n-dimensional Heisenberg group, identified with $\mathbb{R}^{2n+1}$ through exponential coordinates. A point $p \in \mathbb{H}^n$ is denoted by $p = (x,y,t)$, with both $x,y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. If p and $p' \in \mathbb{H}^n$, the group operation is defined by

$$p \cdot p' = (x+x', y+y', t+t' + \frac{1}{2}\sum_{j=1}^{n}(x_j y'_j - y_j x'_j)).$$

The unit element of $\mathbb{H}^n$ is the origin, that will be denote by e. We stress that the $(2n+1)$-dimensional Lebesgue measure in $\mathbb{R}^{2n+1}$ is the Haar measure of $\mathbb{H}^n$. For any $q \in \mathbb{H}^n$, the *(left) translation* $\tau_q : \mathbb{H}^n \to \mathbb{H}^n$ is defined as

$$p \mapsto \tau_q p := q \cdot p.$$

For a general review on Heisenberg groups and their properties, we refer to [8], [35], [18] and to [37]. We limit ourselves to fix some notations.

The Heisenberg group $\mathbb{H}^n$ can be endowed with the homogeneous norm (Korányi norm)

$$\rho(p) = \left(|p'|^4 + p_{2n+1}^2\right)^{1/4},$$

and we define the gauge distance (a true distance, see [35], p. 638, that is left invariant i.e. $d(\tau_q p, \tau_q \bar{p}) = d(p, \bar{p})$ for all $p, \bar{p} \in \mathbb{H}^n$) as

$$d(p, \bar{p}) := \rho(p^{-1} \cdot \bar{p}).$$

Finally, the balls for the metric d are the so-called Korányi balls

$$B(p,r) := \{q \in \mathbb{H}^n;\ d(p,q) < r\}.$$

Notice that Korányi balls are convex smooth sets.

For $r > 0$ small, straightforward computation shows that, if we denote by $B_{\text{Euc}}(e,r)$ the Euclidean ball centred at e of radius r then,

$$B_{\text{Euc}}(e,r^2) \subset B(e,r) \subset B_{\text{Euc}}(e, c_0^2 r).$$

It is well known that the topological dimension of $\mathbb{H}^n$ is $2n+1$, since as a smooth manifold it coincides with $\mathbb{R}^{2n+1}$, whereas the Hausdorff dimension of $(\mathbb{H}^n, d)$ is $Q := 2n+2$ (the so called *homogeneous dimension* of $\mathbb{H}^n$).

We denote by $\mathfrak{h}$ the Lie algebra of the left invariant vector fields of $\mathbb{H}^n$. The standard basis of $\mathfrak{h}$ is given, for $i = 1, \dots, n$, by

$$X_i := \partial_{x_i} - \frac{1}{2} y_i \partial_t, \quad Y_i := \partial_{y_i} + \frac{1}{2} x_i \partial_t, \quad T := \partial_t.$$

The only non-trivial commutation relations are $[X_j, Y_j] = T$, for $j = 1, \dots, n$. The *horizontal subspace* $\mathfrak{h}_1$ is the subspace of $\mathfrak{h}$ spanned by $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$: $\mathfrak{h}_1 := \text{span}\,\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$.
Coherently, from now on, we refer to $X_1, \dots, X_n, Y_1, \dots, Y_n$ (identified with first order differential operators) as the *horizontal derivatives*. Denoting by $\mathfrak{h}_2$ the linear span of T, the 2-step stratification of $\mathfrak{h}$ is expressed by

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2.$$

The stratification of the Lie algebra $\mathfrak{h}$ induces a family of non-isotropic dilations $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$, $\lambda > 0$ as follows: if $p = (x, y, t) \in \mathbb{H}^n$, then

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t).$$

Finally, the vector space $\mathfrak{h}$ can be endowed with an inner product, indicated by $\langle \cdot, \cdot \rangle$, making $X_1, \dots, X_n$, $Y_1, \dots, Y_n$ and T orthonormal.

Throughout this paper, we write also

$$W_i := X_i, \quad W_{i+n} := Y_i \quad \text{and} \quad W_{2n+1} := T, \quad \text{for } i = 1, \dots, n. \tag{2}$$

Following a classical notation ([32]), if $U \subset \mathbb{H}^n$ is an open set, we denote by $\mathcal{D}(U)$ the space of smooth functions in U with compact support, by $\mathcal{D}'(U)$ the space of distributions in U, and by $\mathcal{E}'(U)$ the space of compactly supported distributions in U.

If f is a real function defined in $\mathbb{H}^n$, we denote by ${}^{\mathrm{v}}f$ the function defined by ${}^{\mathrm{v}}f(p) := f(p^{-1})$, and, if $T \in \mathcal{D}'(\mathbb{H}^n)$, then ${}^{\mathrm{v}}T$ is the distribution defined by $\langle {}^{\mathrm{v}}T | \phi \rangle := \langle T | {}^{\mathrm{v}}\phi \rangle$ for any test function ϕ.

Following e.g. [14], p. 15, we can define a group convolution in $\mathbb{H}^n$: if, for instance, $f \in \mathcal{D}(\mathbb{H}^n)$ and $g \in L^1_{\text{loc}}(\mathbb{H}^n)$, we set

$$f * g(p) := \int f(q) g(q^{-1} \cdot p)\, dq \quad \text{for } q \in \mathbb{H}^n. \tag{3}$$

We remind that, if (say) g is a smooth function and P is a left invariant differential operator, then

$$P(f * g) = f * Pg.$$

As in [14], we also adopt the following multi-index notation for higher-order derivatives. If $I = (i_1, \dots, i_{2n+1})$ is a multi–index, we set

$$W^I = W_1^{i_1} \cdots W_{2n}^{i_{2n}}\, T^{i_{2n+1}}. \tag{4}$$

By Poincaré–Birkhoff–Witt theorem, the differential operators W^I form a basis for the algebra of left invariant differential operators in $\mathbb{H}^n$. Furthermore, we set $|I| := i_1 + \cdots + i_{2n} + i_{2n+1}$ the order of the differential operator W^I, and $d(I) := i_1 + \cdots + i_{2n} + 2i_{2n+1}$ its degree of homogeneity with respect to group dilations.

Definition 2.1. Let $\omega \in \mathcal{D}(\mathbb{H}^n)$ be supported in the unit ball $B(e,1)$, and assume $\int \omega(x)\,dx = 1$. If $\varepsilon > 0$, we denote by ω_ε the Friedrichs mollifier $\omega_\varepsilon(x) := \varepsilon^{-Q}\omega(\delta_{1/\varepsilon}x)$.

By [14], if $u \in L^1_{\mathrm{loc}}(\mathbb{H}^n)$, the convolution $u_\varepsilon := u * \omega_\varepsilon$ enjoys the same properties of the usual regularizing convolutions in Euclidean spaces.

3. Orlicz spaces

A Young function A is a function from $[0,\infty)$ into $[0,\infty]$ of the form

$$A(s) = \int_0^s a(\sigma)\,d\sigma \qquad \text{for } s \geq 0,$$

where $a : [0,\infty) \to [0,\infty)$ is a non-decreasing, left-continuous function which is neither identically zero nor identically infinite. If, in particular,

$$0 = a(0) < a(t) < \infty \qquad \text{for } t > 0,$$

as in [1] we shall refer to A as to a N-function.

Let (X,ν) be a non-atomic and σ-finite positive measure space and let A be a Young function. The Orlicz space $L^A(X,\nu)$ is the Banach space of all (equivalence classes of) ν-measurable functions f on X whose Luxemburg norm, defined by

$$\|f\|_{L^A(X,\nu)} := \inf\Big\{\lambda > 0 : \int_X A\big(\lambda^{-1}|f(x)|\big)\,d\nu(x) \leq 1\Big\}$$

is finite. We recover the usual L^p-spaces by taking $A(s) = s^p$ when $1 \leq p < \infty$ and $A(s) \equiv 0$ on $[0,1]$ and $A(s) \equiv \infty$ if $s > 1$ when $p = \infty$ (see e.g. [10], p. 188).

Remark 3.1. We point out that, if $\nu(X) < \infty$, we can replace A by another Young function A_1 provided

$$0 < c_1 \leq \frac{A(s)}{A_1(s)} < c_2 < \infty \qquad \text{for } s \text{ near } \infty.$$

The corresponding Luxemburg norms $\|\cdot\|_{L^A(X,\nu)}$ and $\|\cdot\|_{L^A_1(X,\nu)}$ will be equivalent (see [1], Theorem 8.12).

When $X = D \subset \mathbb{H}^n$ and ν is the Haar measure dx on $\mathbb{H}^n$, we write $\|f\|_{L^A(D)}$ for $\|f\|_{L^A(D,dx)}$.

If A is a Young function, in the sequel we shall denote by $\tilde{A}$ the Young conjugate of A, namely the Young function defined by

$$\tilde{A}(s) := \sup\{rs - A(r)\,;\, r \geq 0\} \qquad \text{for } s \geq 0.$$

Lemma 3.2. *Suppose A is a N-function. If we set* $\tilde{a}(s) := \sup_{a(t) \leq s} t$, *then*

$$\tilde{A}(s) = \int_0^s \tilde{a}(\tau)\, d\tau.$$

In addition, If a is strictly increasing then $\tilde{a} = a^{-1}$.

For a general presentation of Orlicz spaces, we refer e.g. to [1].

Definition 3.3. Let A, B be Young functions. The function B is said to dominate the function A globally (respectively near infinity) if a positive constant c exists such that $A(s) \leq B(cs)$ for $s \geq 0$ (respectively for s greater than some positive number)

Following [10], if $p \in (0, \infty]$ and (A, B) is a couple of Young functions, then we denote by B_p the Young function whose conjugate Young function is given by

$$\tilde{B}_p(s) := \int_0^s r^{p'-1} (\Psi_p^{-1}(r^{p'}))^{p'}\, dr,$$

where $1/p + 1/p' = 1$ and

$$\Psi_p(s) := \int_0^s \frac{B(t)}{t^{1+p'}}\, dt,$$

and (with abuse of notation)

$$A_p(s) := \int_0^s r^{p'-1} (\Phi_p^{-1}(r^{p'}))^{p'}\, dr,$$

where $1/p + 1/p' = 1$ and

$$\Phi_p(s) := \int_0^s \frac{\tilde{A}(t)}{t^{1+p'}}\, dt.$$

We stress explicitly that, by Remark 3.1, we can always choose A, B in such a way A_p and B_p are well defined.

4. Folland-Stein kernels in Heisenberg groups

Following [13], we remind now the notion of *kernel of type* μ.

Definition 4.1. A kernel of type μ is a homogeneous distribution of degree $\mu - Q$ (with respect to group dilations δ_r), that is smooth outside of the origin.

The convolution operator with a kernel of type μ is called an operator of type μ.

Proposition 4.2. *Let $K \in \mathcal{D}'(\mathbb{H}^n)$ be a kernel of type μ.*

i) ${}^{\mathrm{v}}K$ *is again a kernel of type μ;*

ii) WK *and* KW *are associated with kernels of type $\mu - 1$ for any horizontal derivative W;*

iii) If $\mu > 0$, then $K \in L^1_{\mathrm{loc}}(\mathbb{H}^n)$;

Lemma 4.3 (see [13], Lemma 4.12)**.** *There exist kernels $\rho_1, \dots, \rho_{2n}$ of type 1 such that for all $u \in \mathcal{E}'(\mathbb{H}^n)$*

$$u = \sum_j \rho_j * (W_j u).$$

If $f : \mathbb{H}^n \to \mathbb{R}$ is a measurable function, as customary f^* denotes the decreasing rearrangement of f defined by

$$f^*(s) := \sup\{t \geq 0, |\{|f| > t\}| \geq s\}$$

for $s \geq 0$, where $|\{|f| > t\}|$ denotes the Lebesgue measure of the set $\{|f| > t\}$.

Notice that $|f| \leq g$ implies $f^* \leq g^*$. We put also

$$f^{**}(s) := \frac{1}{s} \int_0^s f^*(t)\, dt \geq f^*(s).$$

Notice that, if K is a kernel of type $\alpha < Q$, then $K^*(s) \leq c s^{-(Q-\alpha)/Q}$.

Theorem 4.4 (see [13], Proposition 1.9)**.** *If K is a kernel of type 0 and $1 < p < \infty$, then there exists $C = C(p) > 0$ such that*

$$\|u * K\|_{L^p(\mathbb{H}^n)} \leq C \|u\|_{L^p(\mathbb{H}^n)}.$$

The following theorem together with the subsequent corollaries deals with the continuity of convolution operators between Orlicz spaces in Heisenberg groups and is the counterpart of the analogous statement in the Euclidean setting proved in [10]. We stress that these result still hold in the more general setting of Carnot groups (connected and simply connected, stratified nilpotent Lie groups).

Theorem 4.5. *Suppose $0<\alpha<Q$, and let K be a kernel of type α in $\mathbb{H}^n$. Let (A,B) be a couple of Young functions such that A dominates $B_{Q/\alpha}$ and $A_{Q/\alpha}$ dominates B near infinity.*

If $B_1, B_2 \subset \mathbb{H}^n$ are fixed balls and u is compactly supported in B_2, then

$$\|u*K\|_{L^B(B_1)} \le C\,\|u\|_{L^A(B_2)},$$

where $C=C(A,B,B_1,B_2)$.

If in addition

$$\int_0 \frac{B(t)}{t^{1+Q/(Q-\alpha)}}\,dt<\infty \qquad \text{and} \qquad \int_0 \frac{\tilde{A}(t)}{t^{1+Q/(Q-\alpha)}}\,dt<\infty.$$

then we can drop the assumption on the support of u and we have

$$\|u*K\|_{L^B(\mathbb{H}^n)} \le C\,\|u\|_{L^A(\mathbb{H}^n)},$$

where $C=C(A,B)$.

Proof. The assertion follows from Theorem 4 and Remark 6 in [10], once we prove that the map $u \to u*K$ is of joint-weak type $(1, Q/(Q-\alpha); Q/\alpha, \infty)$ in the sense of [10], formula (3.7), i.e. for any $u \in L^1 + L^{Q/\alpha,\infty}$

$$(u*K)^*(s) \le k_1\Big(s^{-(Q-\alpha)/Q}\int_0^s u^*(r)\,dr + \int_s^\infty u^*(r)r^{-(Q-\alpha)/Q}\,dr\Big) \tag{5}$$

for $s \ge 0$. To prove (5), we notice first that, by Proposition 1.18 of [14], the convolution operator (3) is a convolution operator in the sense of [28]. Thus, by [28], Lemma 1.5

$$\begin{aligned}(u*K)^*(s) &\le (u*K)^{**}(s) \le s u^{**}K^{**}(s) + \int_s^\infty u^*(t)K^*(t)\,dt \\ &\le k_1\Big(s^{-(Q-\alpha)/Q}\int_0^s u^*(t)\,dt + \int_s^\infty u^*(t)t^{-(Q-\alpha)/Q}\,dt\Big).\end{aligned}$$

This proves (5). $\square$

In particular, if we choose $A(t)=t^p$ with $1<p<Q$ and $B(t)=t^q$ with $1/q := 1/p - \alpha/Q$, we obtain the classical Hardy-Littlewood-Sobolev inequality in $\mathbb{H}^n$ (see [14], Proposition 6.2). More precisely we have

Corollary 4.6. *Suppose $0<\alpha<Q$, and let K be a kernel of type α. If $1<p<Q/\alpha$, and $1/q := 1/p-\alpha/Q$, then there exists $C=C(p,\alpha)>0$ such that*

$$\|u*K\|_{L^q(\mathbb{H}^n)} \le C\|u\|_{L^p(\mathbb{H}^n)}$$

for all $u \in L^p(\mathbb{H}^n)$.

Corollary 4.7 (Compare with Trudinger's Theorem: see [1], Theorem 8.27)**.** *Suppose* $0 < \alpha < Q$*, and let* K *be a kernel of type* α*. Let* $B_1, B_2 \subset \mathbb{H}^n$ *be fixed balls, and take*

$$B(s) = \exp(s^{Q/(Q-\alpha)}),$$

then there exists $C = C(p, \alpha) > 0$ *such that*

$$\|u * K\|_{L^B(B_1)} \le C\|u\|_{L^{Q/\alpha}(B_2)}$$

for all $u \in L^Q(\mathbb{H}^n)$*, with* u *compactly supported in* B_2*.*

Proof. Take $A(s) \approx s^{Q/\alpha} \approx \int_0^s t^{(Q-\alpha)/\alpha}\, dt$ at ∞. As pointed out, $\|u\|_{L^{Q/\alpha}(\mathbb{H}^n)} \approx \|u\|_{L^A(\mathbb{H}^n)}$ when u is supported in B_2

By Lemma 3.2

$$\tilde{A}(s) \approx s^{Q/(Q-\alpha)} \qquad \text{at } \infty$$

and thus, provide we take A near zero such that

$$\int_0^1 \frac{\tilde{A}(t)}{t^{1+Q/(Q-\alpha)}}\, dt < \infty$$

we have

$$\Phi_{Q/\alpha}(s) \approx \int_0^s \frac{\tilde{A}(t)}{t^{1+Q/(Q-\alpha)}}\, dt \approx \ln s \qquad \text{at } \infty.$$

Therefore

$$\begin{aligned} A_{Q/\alpha}(s) &\approx \int_0^s r^{\alpha/(Q-\alpha)} \exp\Big(\frac{Q}{Q-\alpha} r^{Q/(Q-\alpha)}\Big)\, dr \\ &\approx \exp\Big(c s^{Q/(Q-\alpha)}\Big) \qquad \text{at } \infty. \end{aligned}$$

This shows that $A_{Q/\alpha}$ dominates $B(s) := \exp\Big(s^{Q/(Q-\alpha)}\Big)$ at infinity. With simular arguments we can prove that $B_{Q/\alpha}$ dominates A at infinity, and the assertion follows.

□

Remark 4.8. In fact, much more precise exponential estimates (the so-called Moser-Trudinger estimates) can be proved: see, e.g., [38], Theorem 14.40, and, in the most recent literature, [11], [25], as well as the careful historical account in [9]. As for the Heisenberg group, we restrict ourselves to quote, within a wide literature, the papers [23], [24] and the references therein.

Corollary 4.9. *Suppose* $0 < \alpha < Q$*. If* K *is a kernel of type* α *and* $\psi \in \mathcal{D}(\mathbb{H}^n)$*,* $\psi \equiv 1$ *in a neighborhood of the origin, then the continuity result Theorem 4.5* *as well as its Corollaries 4.6 and 4.7 still hold if we replace K by* $(1-\psi)K$ *or by* K.

Analogously, if K is a kernel of type 0 and $\psi \in \mathcal{D}(\mathbb{H}^n)$*, then Theorem 4.4 still holds if we replace K by* $(\psi-1)K$.

Proof. We merely need notice that $|(1-\psi)K(x)| \le C_\psi |x|^{\alpha-Q}$, so that $(1-\psi)K$ is of joint-weak type $(1, Q/(Q-\alpha); Q/\alpha, \infty)$ (recall (5)).

Suppose now $\alpha = 0$. Notice that $(\psi-1)K \in L^{1,\infty}(\mathbb{H}^n)$, and therefore also $u \to ((\psi-1)K) * u$ is $L^p - L^p$ continuous by Hausdorff-Young Theorem. This proves that iii) holds true. □

5. Multilinear calculus and Rumin's complex in Heisenberg groups

The dual space of $\mathfrak{h}$ is denoted by $\bigwedge^1 \mathfrak{h}$. The basis of $\bigwedge^1 \mathfrak{h}$, dual to the basis $\{X_1, \dots, Y_n, T\}$, is the family of covectors $\{dx_1, \dots, dx_n, dy_1, \dots, dy_n, \theta\}$ where

$$\theta := dt - \frac{1}{2}\sum_{j=1}^{n}(x_j dy_j - y_j dx_j)$$

is called the *contact form* in $\mathbb{H}^n$. A diffeomorphism ϕ between open subsets of $\mathbb{H}^n$ is called a *contactomorphism* if $\phi^\# \theta$ is pointwise proportional to θ. In other words, contactomorphisms preserve the *contact structure* $\ker(\theta)$.

We denote by $\langle \cdot, \cdot \rangle$ the inner product in $\bigwedge^1 \mathfrak{h}$ that makes $(dx_1, \dots, dy_n, \theta)$ an orthonormal basis.

Coherently with the previous notation (2), we set

$$\omega_i := dx_i, \quad \omega_{i+n} := dy_i \quad \text{and} \quad \omega_{2n+1} := \theta, \quad \text{for } i = 1, \dots, n.$$

We put $\bigwedge_0 \mathfrak{h} := \bigwedge^0 \mathfrak{h} = \mathbb{R}$ and, for $1 \le h \le 2n+1$,

$$\bigwedge^h \mathfrak{h} := \operatorname{span}\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_h} : 1 \le i_1 < \cdots < i_h \le 2n+1\}.$$

In the sequel we shall denote by Θ^h the basis of $\bigwedge^h \mathfrak{h}$ defined by

$$\Theta^h := \{\omega_{i_1} \wedge \cdots \wedge \omega_{i_h} : 1 \le i_1 < \cdots < i_h \le 2n+1\}.$$

To avoid cumbersome notations, if $I := (i_1, \dots, i_h)$, we write

$$\omega_I := \omega_{i_1} \wedge \cdots \wedge \omega_{i_h}.$$

The inner product $\langle \cdot, \cdot \rangle$ on $\bigwedge^1 \mathfrak{h}$ yields naturally a inner product $\langle \cdot, \cdot \rangle$ on $\bigwedge^h \mathfrak{h}$ making Θ^h an orthonormal basis.

The volume $(2n+1)$-form $\theta_1 \wedge \cdots \wedge \theta_{2n+1}$ will be also written as dV.

Throughout this paper, the elements of $\bigwedge^h \mathfrak{h}$ are identified with *left invariant* differential forms of degree h on $\mathbb{H}^n$.

The same construction can be performed starting from the vector subspace $\mathfrak{h}_1 \subset \mathfrak{h}$, obtaining the *horizontal h-covectors*

$$\bigwedge^h \mathfrak{h}_1 := \operatorname{span}\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_h} : 1 \le i_1 < \cdots < i_h \le 2n\}.$$

It is easy to see that

$$\Theta_0^h := \Theta^h \cap \bigwedge^h \mathfrak{h}_1$$

provides an orthonormal basis of $\bigwedge^h \mathfrak{h}_1$.

Keeping in mind that the Lie algebra $\mathfrak{h}$ can be identified with the tangent space to $\mathbb{H}^n$ at $x = e$ (see, e.g. [16], Proposition 1.72), starting from $\bigwedge^h \mathfrak{h}$ we can define by left translation a fiber bundle over $\mathbb{H}^n$ that we can still denote by $\bigwedge^h \mathfrak{h}$. We can think of h-forms as sections of $\bigwedge^h \mathfrak{h}$. We denote by Ω^h the vector space of all smooth h-forms.

The stratification of the Lie algebra $\mathfrak{h}$ induces a family of anisotropic dilations δ_λ, $\lambda > 0$ in $\mathbb{H}^n$. The homogeneous dimension of $\mathbb{H}^n$ with respect to δ_λ, $\lambda > 0$ equals $Q := 2n+2$. Unfortunately, when dealing with differential forms in $\mathbb{H}^n$, the de Rham complex lacks scale invariance under anisotropic dilations. Thus, a substitute for de Rham's complex, that recovers scale invariance under δ_λ has been defined by M. Rumin, [31]. We refer to [31] and [6] for details of the construction. In the present paper, we shall merely need the following list of formal properties.

- For $h = 0, \dots, 2n+1$, the space of Rumin h-forms, E_0^h is the space of smooth sections of a left-invariant subbundle of $\bigwedge^h T^*\mathbb{H}^n$ (that we still denote by E_0^h). Hence it inherits inner products and norms.

- A differential operator $d_c : E_0^h \to E_0^{h+1}$ is defined. It is left-invariant, homogeneous with respect to group dilations. It is a first order homogeneous operator in the horizontal derivatives in degree $\neq n$, whereas *it is a second order homogeneous horizontal operator in degree n*.

- The L^2 (formal) adjoint of d_c is a differential operator d_c^* of the same order as d_c.

- Hypoelliptic "Laplacians" can be formed from d_c and d_c^* (see Definition 6.1 below).

- Altogether, operators d_c form a complex: $d_c \circ d_c = 0$.

- This complex is homotopic to de Rham's complex $(\Omega^\bullet, d)$. The homotopy is achieved by differential operators $\Pi_E : E_0^\bullet \to \Omega^\bullet$ and $\Pi_{E_0} : \Omega^\bullet \to E_0^\bullet$ (Π_E has horizontal order ≤ 1 and Π_{E_0} is an algebraic operator).

In other words, $\Pi_E : E_0^\bullet \to \Omega^\bullet$ and $\Pi_{E_0} : \Omega^\bullet \to E_0^\bullet$ intertwine differentials d_c and d,

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{d_c} & E_0^h & \xrightarrow{d_c} & E_0^{h+1} & \xrightarrow{d_c} & \cdots \\
 & & \Pi_E \Big\downarrow & & \Pi_E \Big\downarrow & & \\
\cdots & \xrightarrow{d} & \Omega^h & \xrightarrow{d} & \Omega^{h+1} & \xrightarrow{d} & \cdots
\end{array}$$

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{d_c} & E_0^h & \xrightarrow{d_c} & E_0^{h+1} & \xrightarrow{d_c} & \cdots \\
 & & \Big\uparrow \Pi_{E_0} & & \Big\uparrow \Pi_{E_0} & & \\
\cdots & \xrightarrow{d} & \Omega^h & \xrightarrow{d} & \Omega^{h+1} & \xrightarrow{d} & \cdots
\end{array}$$

and there exists an algebraic operator $A : \Omega^\bullet \to \Omega^{\bullet -1}$ such that

$$1 - \Pi_{E_0}\Pi_E\Pi_E\Pi_{E_0} = 0$$

on $E_0^\bullet$ and

$$1 - \Pi_E\Pi_{E_0}\Pi_{E_0}\Pi_E = dA + Ad$$

on $\Omega^\bullet$.

6. Rumin's Laplacian and its fundamental solution

Definition 6.1. In $\mathbb{H}^n$, following [31], we define the operators $\Delta_{\mathbb{H},h}$ on E_0^h by setting

$$\Delta_{\mathbb{H},h} = \begin{cases} d_c d_c^* + d_c^* d_c & \text{if} \quad h \neq n, n+1; \\ (d_c d_c^*)^2 + d_c^* d_c & \text{if} \quad h = n; \\ d_c d_c^* + (d_c^* d_c)^2 & \text{if} \quad h = n+1. \end{cases}$$

Notice that $-\Delta_{\mathbb{H},0} = \sum_{j=1}^{2n}(W_j^2)$ is the usual sub-Laplacian of $\mathbb{H}^n$.

For sake of simplicity, once a basis of E_0^h is fixed, the operator $\Delta_{\mathbb{H},h}$ can be identified with a matrix-valued map, still denoted by $\Delta_{\mathbb{H},h}$

$$\Delta_{\mathbb{H},h} = (\Delta_{\mathbb{H},h}^{ij})_{i,j=1,\dots,N_h} : \mathcal{D}'(\mathbb{H}^n, \mathbb{R}^{N_h}) \to \mathcal{D}'(\mathbb{H}^n, \mathbb{R}^{N_h}),$$

where $\mathcal{D}'(\mathbb{H}^n, \mathbb{R}^{N_h})$ is the space of vector-valued distributions on $\mathbb{H}^n$, and N_h is the dimension of E_0^h (see [2]).

This identification makes possible to avoid the notion of currents: we refer to [6] for a more elegant presentation.

Definition 6.2. If a basis of $E_0^\bullet$ is fixed, and $1 \le p \le \infty$, we denote by $L^p(\mathbb{H}^n, E_0^\bullet)$ the space of all sections of $E_0^\bullet$ such that their components with respect to the given basis belong to $L^p(\mathbb{H}^n)$, endowed with its natural norm. Clearly, this definition is independent of the choice of the basis itself.

The notations $L^A(\mathbb{H}^n)$ (A being a Young function), $\mathcal{D}(\mathbb{H}^n, E_0^\bullet)$, $\mathcal{S}(\mathbb{H}^n, E_0^\bullet)$, as well as $W^{m,p}(\mathbb{H}^n, E_0^\bullet)$ have the same meaning.

Combining [31], Section 3, and [7], Theorems 3.1 and 4.1, we obtain the following result.

Theorem 6.3 (see [7], Theorem 3.1)**.** *If $0 \le h \le 2n+1$, then the differential operator $\Delta_{\mathbb{H},h}$ is homogeneous of degree a with respect to group dilations, where $a = 2$ if $h \ne n, n+1$ and $a = 4$ if $h = n, n+1$. It follows that*

i) for $j = 1, \dots, N_h$ there exists

$$K_j = \big(K_{1j}, \dots, K_{N_h j}\big), \quad j = 1, \dots N_h$$

with $K_{ij} \in \mathcal{D}'(\mathbb{H}^n) \cap \mathcal{E}(\mathbb{H}^n \setminus \{0\})$, $i, j = 1, \dots, N$;

ii) if $a < Q$, then the K_{ij}'s are kernels of type a for $i, j = 1, \dots, N_h$

If $a = Q$, then the K_{ij}'s satisfy the logarithmic estimate $|K_{ij}(p)| \le C(1 + |\ln \rho(p)|)$ and hence belong to $L^1_{\mathrm{loc}}(\mathbb{H}^n)$. Moreover, their horizontal derivatives $W_\ell K_{ij}$, $\ell = 1, \dots, 2n$, are kernels of type $Q-1$;

iii) when $\alpha \in \mathcal{D}(\mathbb{H}^n, \mathbb{R}^{N_h})$, if we set

$$\Delta_{\mathbb{H},h}^{-1} \alpha := \Big(\sum_j \alpha_j * K_{1j}, \dots, \sum_j \alpha_j * K_{N_h j}\Big),$$

then $\Delta_h \Delta_{\mathbb{H},h}^{-1} \alpha = \alpha$. Moreover, if $a < Q$, also $\Delta_{\mathbb{H},h}^{-1} \Delta_h \alpha = \alpha$.

iv) if $a = Q$, then for any $\alpha \in \mathcal{D}(\mathbb{H}^n, \mathbb{R}^{N_h})$ there exists $\beta_\alpha := (\beta_1, \dots, \beta_{N_h}) \in \mathbb{R}^{N_h}$, such that

$$\Delta_{\mathbb{H},h}^{-1} \Delta_h \alpha - \alpha = \beta_\alpha.$$

Remark 6.4. If $a < Q$, $\Delta_{\mathbb{H},h}(\Delta_{\mathbb{H},h}^{-1} - {}^{\mathrm{v}}\Delta_{\mathbb{H},h}^{-1}) = 0$ and hence $\Delta_{\mathbb{H},h}^{-1} = {}^{\mathrm{v}}\Delta_{\mathbb{H},h}^{-1}$, by the Liouville-type theorem of [7], Proposition 3.2.

Remark 6.5. From now on, if there are no possible misunderstandings, we identify $\Delta_{\mathbb{H},h}^{-1}$ with its kernel.

Typically, the operator used to invert d_c is $d_c^* \Delta_{\mathbb{H}}^{-1}$. It inverts d_c because d_c commutes with $\Delta_{\mathbb{H}}^{-1}$. Since d_c and d_c^* commute with $\Delta_{\mathbb{H}}$, it is natural that they commute with their inverse. One first shows this for test forms, and then (in a slightly weaker form) for L^1 forms by duality.

Lemma 6.6 (see [4], Lemma 4.11)**.** *If $\alpha \in \mathcal{D}(\mathbb{H}^n, E_0^h)$ and $n \geq 1$, then*

i) $d_c\Delta^{-1}_{\mathbb{H},h}\alpha = \Delta^{-1}_{\mathbb{H},h+1}d_c\alpha, \qquad h = 0,1,\dots,2n, \qquad h \neq n-1, n+1.$

ii) $d_c\Delta^{-1}_{\mathbb{H},n-1}\alpha = d_c d_c^*\Delta^{-1}_{\mathbb{H},n}d_c\alpha \qquad (h = n-1).$

iii) $d_c d_c^* d_c\Delta^{-1}_{\mathbb{H},n+1}\alpha = \Delta^{-1}_{\mathbb{H},n+2}d_c\alpha, \qquad (h = n+1).$

iv) $d_c^*\Delta^{-1}_{\mathbb{H},h}\alpha = \Delta^{-1}_{\mathbb{H},h-1}d_c^*\alpha \qquad h = 1,\dots,2n+1, \qquad h \neq n, n+2.$

v) $d_c^*\Delta^{-1}_{\mathbb{H},n+2}\alpha = d_c^* d_c\Delta^{-1}_{\mathbb{H},n+1}d_c^*\alpha \qquad (h = n+2).$

vi) $d_c^* d_c d_c^*\Delta^{-1}_{\mathbb{H},n}\alpha = \Delta^{-1}_{\mathbb{H},n-1}d_c^*\alpha, \qquad (h = n).$

Lemma 6.7. *Let $h \geq 1$. Let $\omega \in L^1(\mathbb{H}^n, E_0^h)$ be a d_c-closed form. Then $\Delta^{-1}_{\mathbb{H},h}\omega$ is well defined and belongs to $L^1_{\mathrm{loc}}(\mathbb{H}^n, E_0^h)$. Furthermore, $d_c^* d_c\Delta^{-1}_{\mathbb{H},h}\omega = 0$ in distributional sense.*

7. Sobolev spaces in Heisenberg groups

Let $U \subset \mathbb{H}^n$ be an open set and let $1 \leq p \leq \infty$ and $m \in \mathbb{N}$, $W^{m,p}_{\mathrm{Euc}}(U)$ denotes the usual Sobolev space. We want now to introduce intrinsic (horizontal) Sobolev spaces (the so-called Folland-Stein spaces).

Since here we are dealing only with integer order Folland-Stein function spaces, we can give this simpler definition (for a general presentation, see e.g. [13]).

Definition 7.1. If $U \subset \mathbb{H}^n$ is an open set, $1 \leq p \leq \infty$ and $m \in \mathbb{N}$, then the space $W^{m,p}(U)$ is the space of all $u \in L^p(U)$ such that, with the notation of (4),

$$W^I u \in L^p(U) \quad \text{for all multi-indices } I \text{ with } d(I) \leq m,$$

endowed with its natural norm

$$\|u\|_{W^{k,p}(U)} := \sum_{d(I)\leq m} \|W^I u\|_{L^p(U)}.$$

Folland-Stein Sobolev spaces enjoy the following properties akin to those of the usual Euclidean Sobolev spaces (see [13], and, e.g. [15]).

Theorem 7.2. *If $U \subset \mathbb{H}^n$, $1 \leq p < \infty$, and $k \in \mathbb{N}$, then*

i) *$W^{k,p}(U)$ is a Banach space;*

ii) *$W^{k,p}(U) \cap C^\infty(U)$ is dense in $W^{k,p}(U)$;*

iii) if $U = \mathbb{H}^n$, then $\mathcal{D}(\mathbb{H}^n)$ is dense in $W^{k,p}(U)$.

Definition 7.3. If $U \subset \mathbb{H}^n$ is open and if $1 \le p < \infty$, we denote by $\overset{\circ}{W}{}^{k,p}(U)$ the completion of $\mathcal{D}(U)$ in $W^{k,p}(U)$.

Remark 7.4. If $U \subset \mathbb{H}^n$ is bounded, by (iterated) Poincaré inequality (see e.g. [22]), it follows that the norms

$$\|u\|_{W^{k,p}(U)} \quad \text{and} \quad \sum_{d(I)=k} \|W^I u\|_{L^p(U)}$$

are equivalent on $\overset{\circ}{W}{}^{k,p}(U)$ when $1 \le p < \infty$.

If $U \subset \mathbb{H}^n$ is an open set and $1 < p < \infty$, $W^{-k,p}(U)$ is the dual space of $\overset{\circ}{W}{}^{k,p'}(U)$, where $1/p + 1/p' = 1$.

Remark 7.5. It is well known that

$$W^{-k,p}(U) = \{f_0 + \sum_{d(I)=k} W^I f_I \,:\, f_0, f_I \in L^p(U)$$
$$\text{for any multi-index } I \text{ such that } d(I) = k\},$$

and

$$\|u\|_{W^{-k,p}(U)} \approx \inf\{\|f_0\|_{L^p(U)} + \sum_I \|f_I\|_{L^p(U)} \,:\, d(I) = k, f_0 + \sum_{d(I)=k} W^I f_I = u\}.$$

If U is bounded, then we can take $f_0 = 0$.
Finally, we stress that

$$\{f_0 + \sum_{d(I)=k} W^I f_I,\ f_0, f_I \in \mathcal{D}(U) \text{ for any } I \text{ multi-index such that } d(I) = k\}$$

is dense in $W^{-k,p}(U)$.

The spaces $W^{-m,p}(U, E_0^h)$ can be viewed as spaces of currents on $(E_0^\bullet, d_c)$ as in [6], Proposition 3.14. More precisely we have:

Remark 7.6. As in [6], Proposition 3.14, an element of $W^{-m,p}(U, E_0^h)$ can be identified (with respect to our basis) with a N_h-tuple

$$(T_1, \dots, T_{N_h}) \in \left(W^{-m,p}(U, E_0^h)\right)^{N_h}.$$

This is nothing but the intuitive notion of "currents as differential forms with distributional coefficients". The action of $u \in W^{-m,p}(U, E_0^h)$ associated with $(T_1, \dots, T_{N_h})$ on the form $\sum_j \alpha_j \xi_j^h \in \overset{\circ}{W}{}^{m,p'}(U, E_0^h)$ is given by

$$\langle u|\alpha\rangle := \sum_j \langle T_j|\alpha_j\rangle.$$

On the other hand, suppose for the sake of simplicity that U is bounded, then by Remark 7.5 there exist $f_I^j \in L^p(U)$, $j = 1, \dots, N_h$, $i = 1, \dots, 2n+1$ such that

$$\langle u|\alpha\rangle = \sum_j \sum_{d(I)=m} \int_U f_I^j(x) W^I \alpha_j(x)\, dx.$$

Alternatively, one can express duality in spaces of differential forms using the pairing between h-forms and $(2n+1-h)$-forms defined by

$$(\alpha, \beta) \mapsto \int_U \alpha \wedge \beta.$$

Note that this makes sense for Rumin forms and is a nondegenerate pairing. In this manner, the dual of $L^p(U, E_0^h)$ is $L^{p'}(U, E_0^{2n+1-h})$. Hence $W^{-m,p}(U, E_0^h)$ consists of differential forms of degree $2n+1-h$ whose coefficients are distributions belonging to $W^{-m,p}(U)$.

8. Homotopy formulae and Orlicz-Poincaré and Orlicz-Sobolev inequalities

In [20], starting from Cartan's homotopy formula, the authors proved that, if $D \subset \mathbb{R}^N$ is a convex set, $1 < p < \infty$, $1 < h < N$, then there exists a bounded linear map:

$$K_{\mathrm{Euc},h} : L^p(D, \textstyle\bigwedge^h) \to W^{1,p}(D, \textstyle\bigwedge^{h-1})$$

that is a homotopy operator, i.e.

$$\omega = dK_{\mathrm{Euc},h}\omega + K_{\mathrm{Euc},h+1} d\omega \qquad \text{for all } \omega \in C^\infty(D, \textstyle\bigwedge^h)$$

(see Proposition 4.1 and Lemma 4.2 in [20]). More precisely, K_{Euc} has the form

$$K_{\mathrm{Euc},h}\omega(x) = \int_D \psi(y) K_y \omega(x)\, dy,$$

where $\psi \in \mathcal{D}(D)$, $\int_D \psi(y)\, dy = 1$, and

$$\langle K_y\omega(x)|\xi_1 \wedge \cdots \wedge \xi_{h-1})\rangle := \int_0^1 t^{h-1} \langle \omega(tx + (1-t)y)|(x-y) \wedge \xi_1 \wedge \cdots \wedge \xi_{h-1}\rangle.$$

Starting from [20], in [26], Theorem 4.1, the authors define a compact homotopy operator $J_{\mathrm{Euc},h}$ in Lipschitz star-shaped domains in the Euclidean space $\mathbb{R}^N$, providing an explicit representation formula for $J_{\mathrm{Euc},h}$, together with continuity properties among Sobolev spaces. More precisely, if $D \subset \mathbb{R}^N$ is a star-shaped Lipschitz domain and $1 < h < N$, then there exists

$$J_{\mathrm{Euc},h} : L^p(D, \bigwedge^h) \to W_0^{1,p}(D, \bigwedge^{h-1})$$

such that

$$\omega = dJ_{\mathrm{Euc},h}\omega + J_{\mathrm{Euc},h+1}d\omega \qquad \text{for all } \omega \in \mathcal{D}(D, \bigwedge^h).$$

Furthermore, $J_{\mathrm{Euc},h}$ maps smooth compactly supported forms to smooth compactly supported forms.

Following now [4] we want to define preliminarily a kind of "rough" homotopy operator in $(E_0^\bullet, d_c)$. We argue as follows:

Take $D = B(e,1) =: B$ and $N = 2n+1$. If $\omega \in C^\infty(B, E_0^h)$, then we set

$$K = \Pi_{E_0} \circ \Pi_E \circ K_{\mathrm{Euc}} \circ \Pi_E \tag{6}$$

(for the sake of simplicity, from now on we drop the index k - the degree of the form - writing, e.g., K_{Euc} instead of $K_{\mathrm{Euc},hk}$).

Analogously, we can define

$$J = \Pi_{E_0} \circ \Pi_E \circ J_{\mathrm{Euc}} \circ \Pi_E. \tag{7}$$

Then K and J invert Rumin's differential d_c on closed forms of the same degree. More precisely, we have:

Lemma 8.1 (see [4], Lemma 5.7)**.** *If ω is a smooth d_c-exact differential form, then*

$$\omega = d_c K\omega \quad \text{if } 1 \le h \le 2n+1 \qquad \text{and} \qquad \omega = d_c J\omega \quad \text{if } 1 \le h \le 2n+1. \tag{8}$$

In addition, if ω is compactly supported in B, then $J\omega$ is still compactly supported in B.

Lemma 8.2. *Put $B = B(e,1)$. Then:*

i) if $1 < p < \infty$ and $h = 1, \dots, 2n+1$, then $K : W^{1,p}(B, E_0^h) \to L^p(B, E_0^{h-1})$ is bounded;

ii) if $1 < p < \infty$ and $n+1 < h \le 2n+1$, then $K : L^p(B, E_0^h) \to L^p(B, E_0^{h-1})$ is compact;

iii) if $1<p<\infty$ *and* $h=n+1$*, then* $K: L^p(B,E_0^{n+1})\to L^p(B,E_0^n)$ *is bounded.*

Analogous assertions hold for $1\le h\le 2n+1$ *when we replace* K *by* J*. In addition,* $\operatorname{supp} J\omega\subset B$*.*

The operators K and J provide a local homotopy in Rumin's complex, but fail to yield the Sobolev and Poincaré inequalities we are looking for, since, because of the presence of the projection operator Π_E (that on forms of low degree is a first order differential operator) they loose regularity as is stated in Lemma 8.2, ii) above. In order to build "good" local homotopy operators with the desired gain of regularity, we have to combine them with homotopy operators which, though not local, in fact provide the "good" gain of regularity.

Proposition 8.3. *If* $\alpha\in\mathcal{D}(\mathbb{H}^n,E_0^h)$ *for* $p>1$ *and* $h=1,\dots,2n$*, then the following homotopy formulas hold: there exist operators* K_1*,* $\tilde K_1$ *and* K_2*,* $\tilde K_1$ *acting on* $\mathcal{D}(\mathbb{H}^n,E_0^\bullet)$ *such that*

- *if* $h\neq n,n+1$*, then* $\alpha=d_cK_1\alpha+\tilde K_1 d_c\alpha$*, where* K_1 *and* $\tilde K_1$ *are associated with kernels* $k_1,\tilde k_1$ *of type 1;*
- *if* $h=n$*, then* $\alpha=d_cK_1\alpha+\tilde K_2 d_c\alpha$*, where* K_1 *and* $\tilde K_2$ *are associated with kernels* $k_1,\tilde k_2$ *of type 1 and 2, respectively;*
- *if* $h=n+1$*, then* $\alpha=d_cK_2\alpha+\tilde K_1 d_c\alpha$*, where* K_2 *and* $\tilde K_1$ *are associated with kernels* $k_2,\tilde k_1$ *of type 2 and* 1*, respectively.*

The next step consists in proving a *local* Orlicz-Sobolev inequality by means of an approximate homotopy formula, relying Theorems 5.11 and 5.13 of [4].

Suppose first $h\neq n,n+1$. We consider a cut-off function ψ_R supported in a R-neighborhood of the origin, such that $\psi_R\equiv 1$ near the origin. With the notations of Proposition 8.3, we can write

$$k_1=k_1\psi_R+(1-\psi_R)k_1 \qquad\text{and}\qquad \tilde k_1=\tilde k_1\psi_R+(1-\psi_R)\tilde k_1,$$

where

$$k_1=:(k_1)_{\ell,\lambda} \qquad\text{and}\qquad \tilde k_1=:(\tilde k_1)_{\ell,\lambda} \tag{9}$$

are the matrix-valued kernels associated with the operators $\delta_c\Delta_{\mathbb{H},h}$ and $\delta_c\Delta_{\mathbb{H},h+1}$, respectively, as shown in the proof of Proposition 8.3.

Let us denote by $K_{1,R}$, $\tilde K_{1,R}$ the convolution operators associated with $\psi_R k_1$, $\psi_R\tilde k_1$, respectively. Let us fix two balls B_0, B_1 with

$$B\Subset B_0\Subset B_1\Subset B', \tag{10}$$

and a cut-off function $\chi\in\mathcal{D}(B_1)$, $\chi\equiv 1$ on B_0. If $\alpha\in C^\infty(B',E_0^\bullet)$, we set $\alpha_0=\chi\alpha$, continued by zero outside B_1.

We have:

$$\alpha_0 = d_c K_{1,R}\alpha_0 + \tilde{K}_{1,R} d_c \alpha_0 + S_0\alpha_0, \tag{11}$$

where S_0 is

$$S_0\alpha_0 := d_c(\alpha_0 * (1-\psi_R)k_1) + d_c\alpha_0 * (1-\psi_R)\tilde{k}_1.$$

We set

$$T\alpha := K_{1,R}\alpha_0, \qquad \tilde{T} d_c\alpha := \tilde{K}_{1,R} d_c\alpha_0, \qquad S\alpha := S_0\alpha_0. \tag{12}$$

We notice that, provided $R > 0$ is small enough, the definition of T and $\tilde{T}$ does not depend on the continuation of α outside B_0. By (11) we have

$$\alpha = d_c T\alpha + \tilde{T} d_c\alpha + S\alpha \qquad \text{in } B.$$

If $h = n$ we can carry out the same construction, replacing $\tilde{k}_1$ by $\tilde{k}_2$ (keep in mind that $\tilde{k}_2$ is a kernel of type 2, again by Proposition 8.3). Analogously, if $h = n+1$ we can carry out the same construction, replacing k_1 by k_2 (again a kernel of type 2).

Later on, we need the following remark:

Remark 8.4. By construction, if $\operatorname{supp}\alpha \subset B$ then $\operatorname{supp} T\alpha$ is contained in a R-neighborhood of B and then is contained in B_0 provided $R < d(B, \partial B_0)$.

The homotopies T and $\tilde{T}$ provide a the desired "gain of regularity" as stated in following theorem.

Theorem 8.5. *Let $B = B(e,1)$ and $B' = B(e,\lambda)$, $\lambda > 1$, be concentric balls of $\mathbb{H}^n$. If $1 \le h \le 2n+1$, then the operators T, $\tilde{T}$ from $C^\infty(B', E_0^h)$ to $C^\infty(B, E_0^{h-1})$ and S from $C^\infty(B', E_0^h)$ to $C^\infty(B, E_0^h)$ satisfy*

$$d_c T + \tilde{T} d_c + S = I \qquad \text{on } B. \tag{13}$$

Let us denote denote by $A = A(s)$ the Young function defined as follows:

- *if $h \neq n+1$ then $A(s) := \exp\left(s^{Q/(Q-1)}\right)$;*
- *if $h = n+1$ then $A(s) := \exp\left(s^{Q/(Q-2)}\right)$.*

then, if $1 < p < \infty$,

i) $\tilde{T} : W^{-1,p}(B', E_0^{h+1}) \to L^p(B, E_0^h)$ if $h \neq n$, and $\tilde{T} : W^{-2,p}(B, E_0^{n+1}) \to L^p(B, E_0^n)$;

ii) $T : L^p(B', E_0^h) \to W^{1,p}(B, E_0^{h-1})$, $h \neq n+1$, whereas $T : L^p(B', E_0^{n+1}) \to W^{2,p}(B, E_0^n)$, *so that* (13) *still holds in* $L^p(B, E_0^\bullet)$.

In addition,

iii) $T : L^Q(B', E_0^h) \to L^A(B, E_0^{h-1})$ *if* $h \neq n+1$*;*

iv) $T : L^{Q/2}(B', E_0^h) \to L^A(B, E_0^{h-1})$ *if* $h = n+1$*;*

v) the operator S *is a smoothing operator, since* S_0 *maps* $\mathcal{E}'(B, E_0^h)$ *into* $C^\infty(B, E_0^h)$*. In particular, for any* $m, s \in \mathbb{Z}$*,* $m < s$*,* S *is bounded from* $W^{m,p}(B', E_0^h)$ *to* $W^{s,q}(B, E_0^h)$ *for any* $p, q \in (1, \infty)$ *and maps* $W^{m,p}(B', E_0^h)$ *into* $C^\infty(B, E_0^h)$*.*

Proof. Statements i) and ii) are proved in [4], Theorem 5.13. To prove iii) and iv), we notice that T is a convolution operator acting on forms compactly supported in B' and taking value in B. Since T can be viewed as a matrix-valued convolution operator whose entries are truncated kernels, iii) and iv) follow from Corollary 4.9. Finally, statement v) is proved in [4], Theorem 5.14. □

Remark 8.6. As stressed in [4], Remark 7.7, though apparently, in the previous theorem, two different homotopy operators T and $\tilde{T}$ appear. In fact, they coincide when acting on form of the same degree.

The following commutation lemma will be helpful in the sequel.

Lemma 8.7 (see [4], Lemma 5.17)**.** *As above, take* $B := B(e,1)$*,* $B' := B(e,\lambda)$ *with* $\lambda > 1$*, and, as in formula* (10)*, let us fix two balls* B_0*,* B_1 *with*

$$B \Subset B_0 \Subset B_1 \Subset B',$$

and a cut-off function $\chi \in \mathcal{D}(B_1)$*,* $\chi \equiv 1$ *on* B_0*.*

We have:

$$Sd_c\alpha = d_cS\alpha \qquad \text{for all } \alpha \in L^p(\mathbb{H}^n, E_0^h),$$

$1 \leq h \leq 2n+1$.

The following theorem contains the main results of the paper: it yields interior Poincaré inequality and Sobolev inequality for Rumin forms in the limit cases $p = Q$ if $h \neq n+1$ and $p = Q/2$ if $q = n+1$.

Theorem 8.8. *Let* $B = B(e,1)$ *and* $B' = B(e,\lambda)$*,* $\lambda > 1$*, be concentric balls of* $\mathbb{H}^n$*. Take* $1 \leq h \leq 2n+1$*, and denote by* $Y = Y(s)$ *the Young function defined as follows:*

- *if* $h \neq n+1$ *then* $Y(s) := \exp\left(s^{Q/(Q-1)}\right)$*;*
- *if* $h = n+1$ *then* $Y(s) := \exp\left(s^{Q/(Q-2)}\right)$*.*

Then the following Orlicz-Sobolev inequality holds: for any exact form $\alpha \in L^Q(B', E_0^h)$ *if* $h \neq n+1$ *and for any exact form* $\alpha \in L^{Q/2}(B', E_0^{n+1})$ *there exists* $\phi \in L^Y(B, E_0^{h-1})$ *such that* $d_c\phi = \alpha$ *and*

$$\|\phi\|_{L^Y(B,E_0^{h-1})} \leq C\|\alpha\|_{L^Q(B',E_0^h)} \qquad \text{if } h \neq n+1;$$

and

$$\|\phi\|_{L^Y(B,E_0^n)} \leq C\|\alpha\|_{L^{Q/2}(B',E_0^{n+1})}.$$

In addition, if α *is supported in* B*, then the following Orlicz-Sobolev inequality holds: there exists a compactly supported form* $\phi \in L^Y(B', E_0^{h-1})$ *such that* $d_c\phi = \alpha$ *and*

$$\|\phi\|_{L^Y(B',E_0^{h-1})} \leq C\|\alpha\|_{L^Q(B,E_0^h)} \qquad \text{if } h \neq n+1;$$

and

$$\|\phi\|_{L^Y(B',E_0^n)} \leq C\|\alpha\|_{L^{Q/2}(B,E_0^{n+1})}.$$

Proof. Let $\omega \in L^p(B', E_0^h)$ be d_c-closed, where $p = Q$ or $p = Q/2$ according to the degree of ω. By (13) we can write

$$\omega = d_cT\omega + S\omega \qquad \text{in } B. \tag{14}$$

By Theorem 8.5, v), we have $S\omega \in \mathcal{C}^\infty(B, E_0^h)$. Furthermore, $d_cS\omega = 0$ since $d_c\omega = d_c^2T\omega + d_cS\omega$ in B and $d_c\omega = 0$ (by assumption).

Thus we can apply (8) to $S\omega$ and we get $S\omega = d_cKS\omega$, where K is defined in (6). In B, put now

$$\phi := (KS + T)\omega.$$

Trivially $d_c\phi = d_cKS\omega + d_cT\omega = S\omega + d_cT\omega = \omega$, by (14). By Theorems 8.5, if $p = Q$ or $p = Q/2$ according to the degree of ω, we have

$$\begin{aligned} \|\phi\|_{L^Y(B,E_0^{h-1})} &\leq \|KS\omega\|_{L^Y(B,E_0^{h-1})} + \|T\omega\|_{L^Y(B,E_0^{h-1})} \\ &\leq \|KS\omega\|_{L^Y(B,E_0^{h-1})} + C\|\omega\|_{L^p(B',E_0^h)}. \end{aligned} \tag{15}$$

Consider now the term $\|KS\omega\|_{L^Y(B,E_0^{h-1})}$. With the notations of (9), we can write $KS\omega$ as a sum of terms of the form $\kappa * (S\omega)_\ell$, where κ is a kernel of type 1 or 2 according to the degree of $S\omega$.

Keeping in mind Lemma (4.3),

$$\begin{aligned} \kappa * (S\omega)_\ell &= \sum_j \kappa * (\rho_j * W_j(S\omega)_\ell) \\ &= \sum_j (\kappa * \rho_j) * W_j(S\omega)_\ell, \end{aligned}$$

where, by [13], Proposition 1.13, $(\kappa * \rho_j)$ is a kernel of type 2 or 3 according to the degree of $S\omega$.

Thus, combining Corollary 4.7 and Theorem 13, v), we obtain from (15)

$$\|\phi\|_{L^Y(B,E_0^{h-1})} \le C\|\omega\|_{L^p(B',E_0^h)}.$$

This proves the first statement of the theorem.

ii) Interior $\mathbb{H}$-Sobolev$_{p,q}(h)$ inequality: as in formula (10), let us fix two balls B_0, B_1 with

$$B \Subset B_0 \Subset B_1 \Subset B',$$

and a cut-off function $\chi \in \mathcal{D}(B_1)$, $\chi \equiv 1$ on B_0.

Let $\omega \in L^p(B, E_0^h)$ be a compactly supported form such that $d_c\omega = 0$. Since ω vanishes in a neighborhood of ∂B, without loss of generality we can assume that it is continued by zero on B'. In addition, $\omega = \chi\omega$, since $\chi \equiv 1$ on $\operatorname{supp}\omega$.

By (13) we have $\omega = d_cT\omega + S\omega$. On the other hand, since ω vanishes outside B, by its very definition (see (12)) $T\omega$ is supported in B_0 by Remark 8.4, so that also $S\omega$ is supported in B_0.

Again as above $S\omega \in C^\infty(B, E_0^h)$, and $d_cS\omega = 0$. Thus we can apply (8) to $S\omega$ and we get $S\omega = d_cJS\omega$, where J is defined in (7) (that preserves the support). By Lemma 8.1, $JS\omega$ is supported in $B_0 \subset B'$. Thus, if we set $\phi := (JS+T)\omega$, then ϕ is supported in B'. Moreover $d_c\phi = d_cKS\omega + d_cT\omega = S\omega + \omega - S\omega = \omega$.

At this point, we can repeat the estimates (15) and we get eventually

$$\|\phi\|_{L^Y(B',E_0^{h-1})} \le C\|\omega\|_{L^p(B,E_0^h)}.$$

This completes the proof of the theorem. $\square$

Acknowledgments

The authors want to express their warmest thanks to Andrea Cianchi for his help and his invaluable suggestions.

REFERENCES

[1] Robert A. Adams - John J. F. Fournier, *Sobolev spaces*, second ed., Pure and Applied Mathematics (Amsterdam), vol. 140, Elsevier/Academic Press, Amsterdam, 2003.

[2] Annalisa Baldi - Marilena Barnabei - Bruno Franchi, *A recursive basis for primitive forms in symplectic spaces and applications to Heisenberg groups*, Acta Math. Sin. (Engl. Ser.) 32 no. 3 (2016), 265–285.

[3] Annalisa Baldi - Bruno Franchi - Pierre Pansu, *L^1-Poincaré inequalities for differential forms in Heisenberg groups*, Preprint, 2018.

[4] Annalisa Baldi - Bruno Franchi - Pierre Pansu, *Poincaré and Sobolev inequalities for differential forms in Heisenberg groups*, Preprint, 2018.

[5] Annalisa Baldi - Bruno Franchi - Pierre Pansu, *L^1-Poincaré and Sobolev inequalities for differential forms in Euclidean spaces*, Sci. China Math. 62 no. 6 (2019), 1029–1040.

[6] Annalisa Baldi - Bruno Franchi - Nicoletta Tchou - Maria Carla Tesi, *Compensated compactness for differential forms in Carnot groups and applications*, Adv. Math. 223 no. 5 (2010), 1555–1607.

[7] Annalisa Baldi - Bruno Franchi - Maria Carla Tesi, *Hypoellipticity, fundamental solution and Liouville type theorem for matrix-valued differential operators on Carnot groups*, J. Eur. Math. Soc. (JEMS) 11 no. 4 (2009), 777–798.

[8] Andrea Bonfiglioli - Ermanno Lanconelli - Francesco Uguzzoni, *Stratified Lie groups and potential theory for their sub-Laplacians*, Springer Monographs in Mathematics, Springer, Berlin, 2007.

[9] Lu Chen - Guozhen Lu - Caifeng Zhang, *Sharp weighted Trudinger–Moser–Adams inequalities on the whole space and the existence of their extremals*, Calc. Var. Partial Differential Equations 58 no. 4 (2019), 58:132.

[10] Andrea Cianchi, *Strong and weak type inequalities for some classical operators in Orlicz spaces*, J. London Math. Soc. (2) 60 no. 1 (1999), 187–202.

[11] William S. Cohn - Guozhen Lu, *Best constants for Moser-Trudinger inequalities on the Heisenberg group*, Indiana Univ. Math. J. 50 no. 4 (2001), 1567–1591.

[12] Gyula Csató - Bernard Dacorogna - Olivier Kneuss, *The pullback equation for differential forms*, Progress in Nonlinear Differential Equations and their Applications, vol. 83, Birkhäuser/Springer, New York, 2012.

[13] Gerald B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat. 13 no. 2 (1975), 161–207.

[14] Gerald B. Folland - Elias M. Stein, *Hardy spaces on homogeneous groups*, Mathematical Notes, vol. 28, Princeton University Press, Princeton, N.J., 1982.

[15] Bruno Franchi - Raul Serapioni - Francesco Serra Cassano, *Meyers-Serrin type theorems and relaxation of variational integrals depending on vector fields*, Houston J. Math. 22 no. 4 (1996), 859–890.

[16] Sylvestre Gallot - Dominique Hulin - Jacques Lafontaine, *Riemannian geometry*, second ed., Universitext, Springer-Verlag, Berlin, 1990.

[17] Vladimir Goldshtein - Marc Troyanov, *The L_{pq}-cohomology of SOL*, Ann. Fac. Sci. Toulouse Math. (6) 7 no. 4 (1998), 687–698.

[18] Mikhael Gromov, *Carnot-Carathéodory spaces seen from within*, Sub-Riemannian geometry, Progr. Math., vol. 144, Birkhäuser, Basel, 1996, pp. 79–

323.

[19] Tadeusz Iwaniec, *Null Lagrangians, the art of integration by parts*, The interaction of analysis and geometry, Contemp. Math., vol. 424, Amer. Math. Soc., Providence, RI, 2007, pp. 83–102.

[20] Tadeusz Iwaniec - Adam Lutoborski, *Integral estimates for null Lagrangians*, Arch. Rational Mech. Anal. 125 no. 1 (1993), 25–79.

[21] Tadeusz Iwaniec - Carlo Sbordone, *Weak minima of variational integrals*, J. Reine Angew. Math. 454 (1994), 143–161.

[22] David Jerison, *The Poincaré inequality for vector fields satisfying Hörmander's condition*, Duke Math. J. 53 no. 2 (1986), 503–523.

[23] Nguyen Lam - Guozhen Lu, *Sharp Moser-Trudinger inequality on the Heisenberg group at the critical case and applications*, Adv. Math. 231 no. 6 (2012), 3259–3287.

[24] Jungang Li - Guozhen Lu - Maochun Zhu, *Concentration-compactness principle for Trudinger-Moser inequalities on Heisenberg groups and existence of ground state solutions*, Calc. Var. Partial Differential Equations 57 no. 3 (2018), Art. 84, 26.

[25] Guozhen Lu - Maochun Zhu, *A sharp Trudinger-Moser type inequality involving L^n norm in the entire space $\mathbb{R}^n$*, J. Differential Equations 267 no. 5 (2019), 3046–3082.

[26] Dorina Mitrea - Marius Mitrea - Sylvie Monniaux, *The Poisson problem for the exterior derivative operator with Dirichlet boundary condition in nonsmooth domains*, Commun. Pure Appl. Anal. 7 no. 6 (2008), 1295–1333.

[27] Charles B. Morrey, Jr., *Multiple integrals in the calculus of variations*, Classics in Mathematics, Springer-Verlag, Berlin, 2008, Reprint of the 1966 edition [MR0202511].

[28] Richard O'Neil, *Convolution operators and $L(p,q)$ spaces*, Duke Math. J. 30 (1963), 129–142.

[29] Pierre Pansu, *Cohomologie L^p des variétés à courbure négative, cas du degré* 1, Rend. Sem. Mat. Univ. Politec. Torino no. Special Issue (1989), 95–120 (1990), Conference on Partial Differential Equations and Geometry (Torino, 1988).

[30] Pierre Pansu, *Cup-products in $L^{q,p}$-cohomology: discretization and quasi-isometry invariance*, Preprint, 2017.

[31] Michel Rumin, *Formes différentielles sur les variétés de contact*, J. Differential Geom. 39 no. 2 (1994), 281–330.

[32] Günter Schwarz, *Hodge decomposition—a method for solving boundary value problems*, Lecture Notes in Mathematics, vol. 1607, Springer-Verlag, Berlin, 1995.

[33] Stephen Semmes, *On the nonexistence of bi-Lipschitz parameterizations and geometric problems about A_∞-weights*, Rev. Mat. Iberoamericana 12 no. 2 (1996), 337–410.

[34] Lesley M. Sibner - Robert J. Sibner, *A non-linear Hodge-de-Rham theorem*, Acta

Math. 125 (1970), 57–73.

[35] Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[36] Bianca Stroffolini, *On weakly A-harmonic tensors*, Studia Math. 114 no. 3 (1995), 289–301.

[37] Nicholas Th. Varopoulos - Laurent Saloff-Coste - Thierry Coulhon, *Analysis and geometry on groups*, Cambridge Tracts in Mathematics, vol. 100, Cambridge University Press, Cambridge, 1992.

[38] Richard L. Wheeden - Antoni Zygmund, *Measure and integral*, second ed., Pure and Applied Mathematics (Boca Raton), CRC Press, Boca Raton, FL, 2015, An introduction to real analysis.

ANNALISA BALDI
Dipartimento di Matematica
Università di Bologna, Italy
e-mail: `annalisa.baldi2@unibo.it`

BRUNO FRANCHI
Dipartimento di Matematica
Università di Bologna, Italy
e-mail: `bruno.franchi@unibo.it`

PIERRE PANSU
Laboratoire de Mathématiques d'Orsay
Université Paris-Sud, CNRS
Université Paris-Saclay, France
e-mail: `pierre.pansu@math.u-psud.fr`