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Isoperimetric Problems on the Line with Density $|x|^p$

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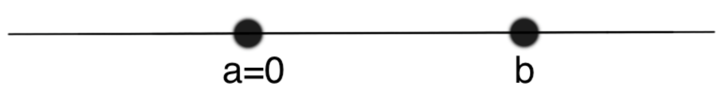
Abstract. On the line with density $|x|^p$, we prove that the best single bubble is an interval with endpoint at the origin and that the best double bubble is two adjacent intervals that meet at the origin.

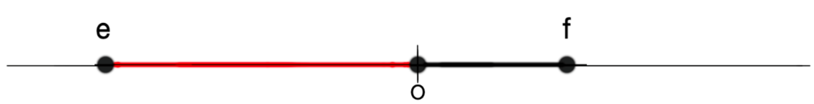
1 Introduction

In the past dozen years there has been a surge of interest in manifolds with density, partly because of their role in Perelman's 2006 proof of the Poincaré Conjecture. We consider isoperimetric problems on the line with density $|x|^p$ and prove the following single and double bubble theorems:

Single Bubble Theorem: On the line with density $|x|^p$, $(p > 0)$, the least-perimeter region with given mass is an interval with one endpoint at the origin.



Double Bubble Theorem: On the line with density $|x|^p$, $(p > 0)$, the least-perimeter way to enclose and separate two given masses is two adjacent intervals that meet at the origin.



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Density. Density is used to weight both perimeter and length. With density $|x|^p$, each boundary point b contributes $|b|^p$ to the perimeter, and the mass of an interval (a, b) is

$$\int_a^b |x|^p dx.$$

For example, the region of Figure 1 has perimeter $a^p + b^p + c^p$ and mass

$$\int_b^a x^p dx + \int_0^c x^p dx = \frac{a^{p+1} - b^{p+1} + c^{p+1}}{p+1}.$$

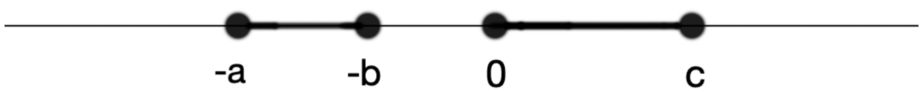

Figure 1: A region with (weighted) perimeter $a^p + b^p + c^p$ and mass $\int_b^a x^p dx + \int_0^c x^p dx$.

The Black and Red regions of Figure 2.1 have perimeter $a^p + b^p + c^p + d^p + e^p$ and Black mass $\int_a^b |x|^p dx$ and Red mass $\int_b^c |x|^p dx + \int_d^e |x|^p dx$. Note that the common boundary point counts once.

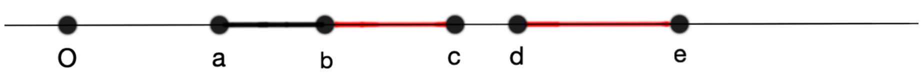

Figure 2: A Black and a Red region with total perimeter $a^p + b^p + c^p + d^p + e^p$, and Black mass $\int_a^b |x|^p dx$ and Red mass $\int_b^c |x|^p dx + \int_d^e |x|^p dx$.

History. The single bubble with our density $|x|^p$ was proved to be a sphere through the origin in $\mathbb{R}^n$ for $n = 2$ by Dahlberg et al. [3] and for $n > 2$ by Boyer et al. [2]. For this density the double bubble is new, although it is known in Euclidean and other spaces (see [4]), including the line with strictly convex density [1, 6]. It was earlier studied at the 2017 Texas State Honors Summer Math Camp, but that work was not completed. The double bubble for density $|x|^p$ remains open for higher dimensions.

Proofs. Our proofs employ elementary comparisons, without even any recourse to stability theory. For the single bubble, it follows from Rosales et al. [[5], Thm. 4.14(iii)] that the minimizer is an interval containing the origin, but our proof does cite that fact.

A finite-perimeter region on the line with unit density must consist of finitely many intervals because it has just finitely many boundary points. With density $|x|^p$, there is the possibility of countably many intervals converging to the origin.

Acknowledgements. This research was conducted during an Institute for Advanced Research winter camp for high school students in Shanghai under the direction of Frank Morgan.

2 The Best Single Region

Our main Theorem 2.3 identifies the least-perimeter single region of given mass on the line with density $|x|^p$, $(p > 0)$. We start with a geometric proof that works only for $p = 1$.

Proposition 2.1. *On the line with density $|x|$, the least-perimeter single interval of given mass is an interval with one endpoint at the origin.*

Proof. We consider two cases according to whether the interval (a, b) contains the origin. If (a, b) does not contain the origin, we may assume that $0 \le a < b$, as in Figure 3.

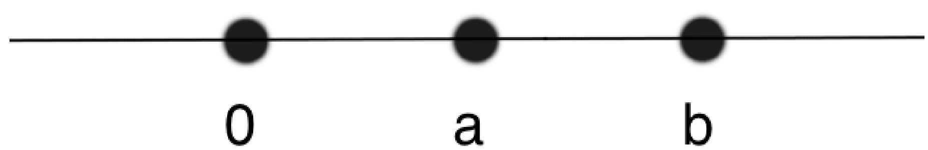

Figure 3: In the first case, the origin lies outside the interval, which has perimeter $a + b$.

The perimeter equals $a + b$. An interval $(0, b')$ of the same mass beginning at the origin, as in Figure 4, has perimeter $b' \le b \le a + b$, with equality only if we had that interval to start with.

Figure 4: An interval of the same mass starting at the origin has perimeter $b' \le b \le a + b$.

On the other hand, as in Figure 5, consider an interval $(-a, b)$ containing the origin and compare it to the interval $(0, b')$ of the same mass, obtained by replacing $(-a, 0)$ with an interval (b, b') of the same mass. Since the latter has larger average density but the same mass, $b' - b < a$, so that $b' < a + b$. Therefore $(0, b')$ has less perimeter than $(-a, b)$. □

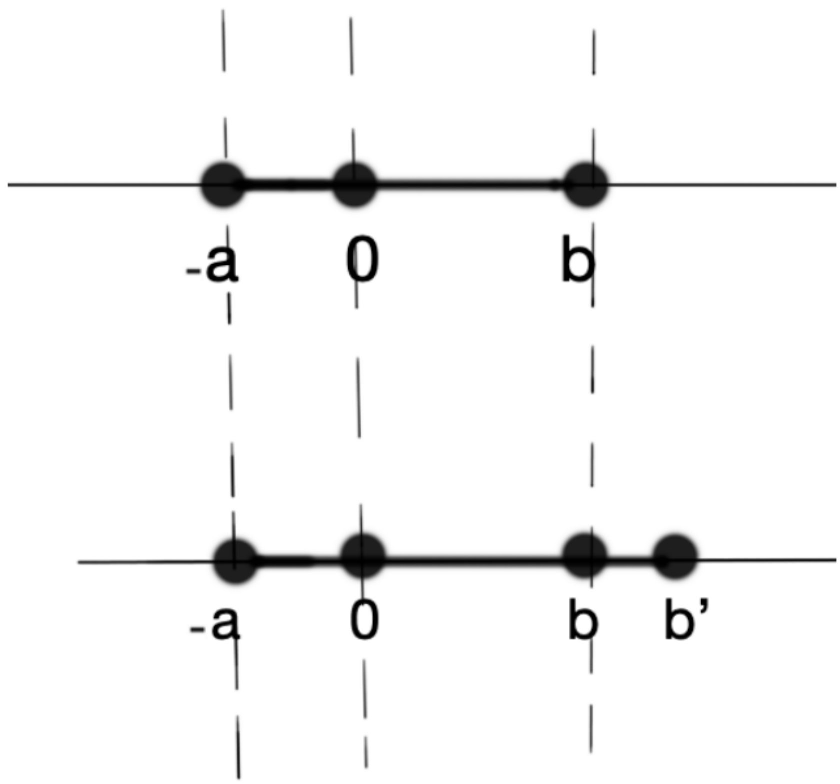

Figure 5: In the second case, the interval $(-a, b)$ contains the origin.

Remark 2.2. We provide another argument for the second case of the preceding proof (Figure 4). The mass of $(-a, 0)$ and $(0, b)$ may be computed as the length times the average density, yielding total mass

$$m = a(a/2) + b(b/2) = \frac{a^2 + b^2}{2},$$

so that

$$a^2 + b^2 = \left(\sqrt{2m}\right)^2.$$

On this quarter circle, as in Figure 6, the perimeter $a + b$, represented by two sides of the pictured triangle, always exceeds the third side $\sqrt{2m}$, with equality only if a or b is 0, i.e., when the interval has an endpoint at the origin.

We now give our main single bubble theorem. Our first, geometric proof works just for density $|x|$. Our second, algebraic proof works for density $|x|^p$.

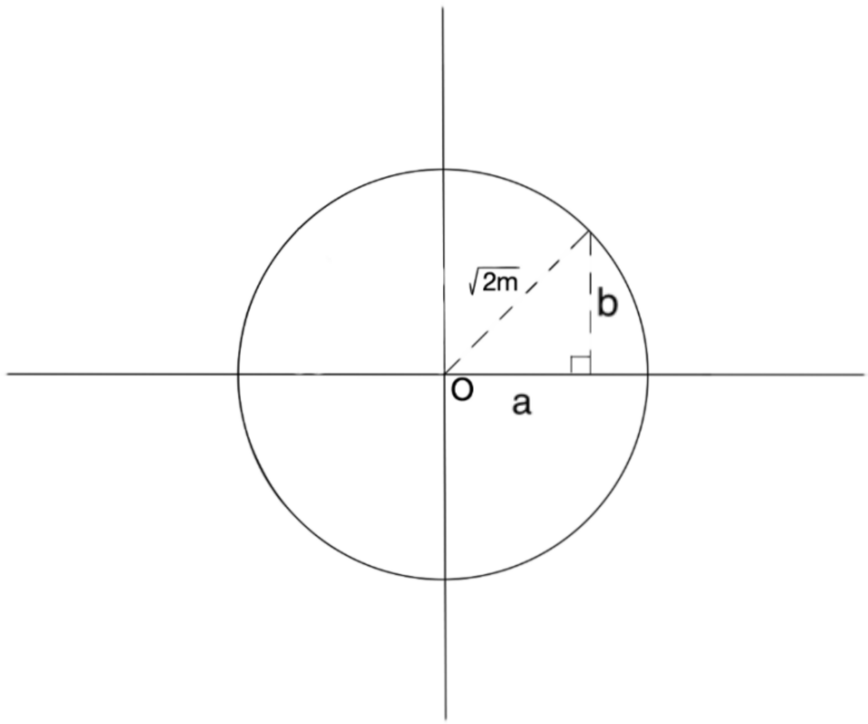

Figure 6: Given $a^2 + b^2 = \left(\sqrt{2m}\right)^2$, $a + b$ is minimized when a or b is 0.

Theorem 2.3 (Single Bubble Theorem)**.** *On the line with density* $|x|^p\,(p > 0)$*, the least-perimeter region with given mass is an interval with one endpoint at the origin.*

Proof for density $|x|$. Consider a region with a given mass and finite perimeter, perhaps consisting of many intervals as shown in Figure 7. Replace the mass to the right of the origin with a single interval $(0, b)$. The new perimeter b no greater than the original, indeed is less than or equal to the cost of the original right-most endpoint. Similarly replace the mass to the left of the origin by a single interval $(-a, 0)$. We have thus produced a single interval $(-a, b)$ of no greater perimeter, with equality only if the original region was that interval. Finally, by Proposition 2.1, a single interval $(-a', 0)$ or $(0, b')$ is best. □

Proof for density $|x|^p$. Consider any region of finite perimeter. It consists of countably many intervals, the origin the only possible limit point. If an interval contains the origin, split it into two intervals. Denote the intervals to the right of the origin by (a_i, b_i) and the intervals to the left of the origin by $(-c_j, -d_j)$. Note that the mass of (a, b) is the integral

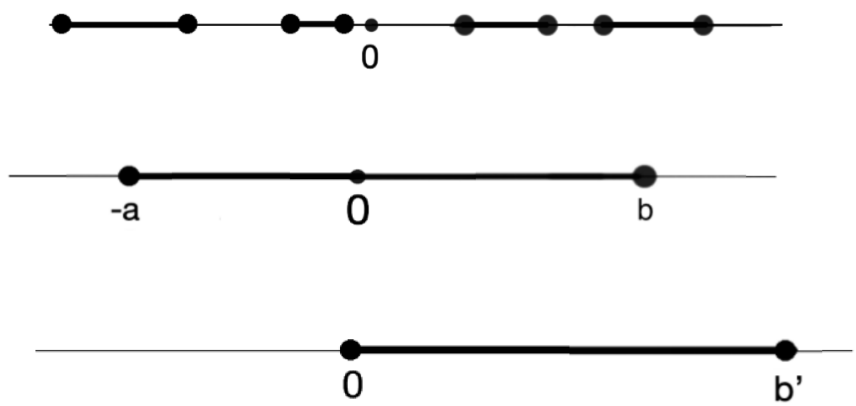

Figure 7: Replacing the portions of a given region to the left and right of the origin with single intervals reduces perimeter. Then by Proposition 2.1 a single interval with an endpoint at the origin is best.

from a to b of x^p, namely $\left(b^{p+1}-a^{p+1}\right)/(p+1)$. Hence the total mass m satisfies

$$(p+1)m = \sum\left(b_i^{p+1}-a_i^{p+1}\right)+\sum\left(c_j^{p+1}-d_j^{p+1}\right),$$

and the total cost is $\sum a_i^p + b_i^p + \sum c_j^p + d_j^p$.

Compare with the single interval $(0,b)$ with the same mass $m = b^{p+1}/(p+1)$ and perimeter b^p. Since they have the same mass,

$$\sum\left(b_i^{p+1}-a_i^{p+1}\right)+\sum\left(c_j^{p+1}-d_j^{p+1}\right) = b^{p+1}.$$

Therefore,

$$\sum\left(b_i^{p+1}-a_i^{p+1}\right)+\sum\left(c_j^{p+1}-d_j^{p+1}\right) \geq b^{p+1}.$$

Now by Lemma 2.4, the total perimeter satisfies

$$\sum\left(b_i^p+a_i^p\right)+\sum\left(c_j^p+d_j^p\right) \geq b^p,$$

the perimeter of $(0,b)$, with equality only if we started with a single interval with endpoint at the origin. □

Lemma 2.4. *For $p>0$, suppose $\sum a_i^{p+1} \geq b^{p+1}$. Then*

$$\sum a_i^p \geq b^p,$$

with equality only if there is just one nonzero a_i.

Proof. Let

$$\mathrm{A}_i = a_i^p,$$
$$u = (p+1)/p > 1.$$

Now the lemma reduces to the standard fact that

$$\left(\sum \mathrm{A}_i\right)^u \geq \sum \mathrm{A}_i^u.$$

Indeed, after normalizing to $\sum \mathrm{A}_i^u = 1$, the result is trivial. □

3 The Best Double Region

Our main Theorem 3.4 will provide the least-perimeter way to enclose and separate two given masses ("Black" and "Red") on the line with density $|x|^p$ $(p > 0)$. Our first proof works just for density $|x|$. It starts with a proposition about the half-line that will reduce candidates on the whole line to those consisting of two, three, or four adjacent intervals, in turn handled by Propositions 3.2 and 3.3.

Proposition 3.1. *On the positive axis $\{x > 0\}$ with density $|x|$, given two regions, there are two adjacent intervals with one endpoint at 0 with the same masses and no more perimeter.*

Proof. Consider two regions such as the Black and Red intervals of Figure 8. We may assume that the last interval is Red. In comparison, take two adjacent Black and Red intervals at the origin with the same masses as the original regions, as in Figure 8. The first boundary point contributes no more perimeter than the last Black boundary point of the original, and the second boundary point contributes no more perimeter than the last Red boundary point of the original. Therefore, those adjacent intervals of the same masses have no more perimeter as claimed. □

Proposition 3.2. *On the line with density $|x|$, when enclosing and separating "Black" and "Red" masses, two intervals Red and Black meeting at the origin require less perimeter than any other adjacent Red-Black-Red or Red-Black with the origin inside Black.*

Proof. First, we reduce the Red-Black-Red case to the Red-Black case. As shown in Figure 9, the perimeter for the Black region is $b + c$. By symmetry, we may assume that $c < b$.

As in Figure 10, move the mass of the interval (c, d) to the left to form an interval $(-g, -a)$. Since we have assumed that $c < b$, therefore $a > c$ and the whole interval $(-g, -a)$ is farther from the origin than (c, d), and hence its length is less: $g - a < d - c$. Therefore, the new Red-Black perimeter, $b + c + g$ is less than the original Red-Black-Red perimeter $a + b + c + d$.

Finally, we compare Red-Black with the origin inside Black with the origin at their meeting point, as in Figure 11. By Proposition 2.1, $f \leq b + c$. Also $e < g$. Therefore, the two intervals that meet at the origin are better as claimed. □

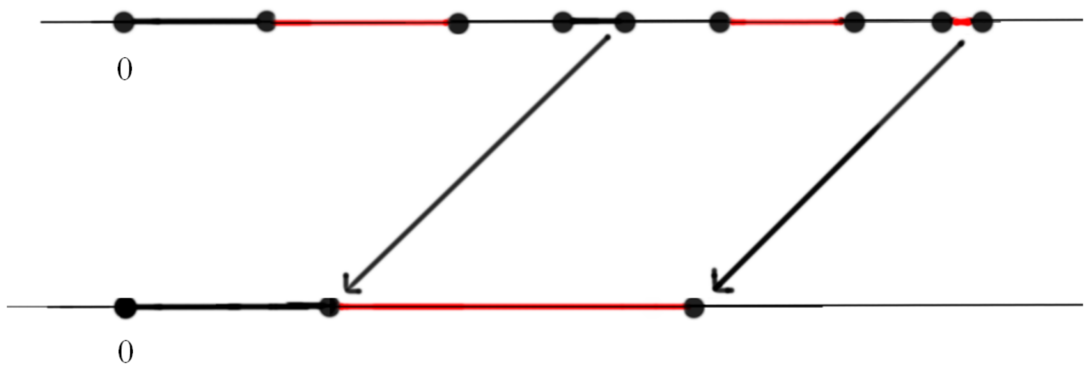

Figure 8: Two intervals are better than many.

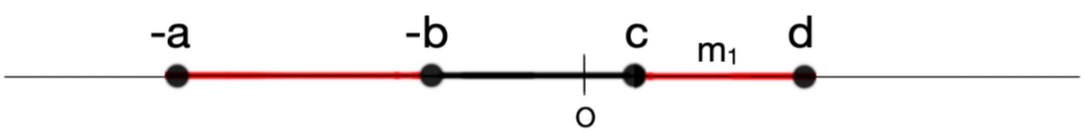

Figure 9: Adjacent Red-Black-Red with the origin inside Black.

Proposition 3.3. *On the line with density $|x|$, when enclosing and separating "Black" and "Red" masses, two intervals meeting at the origin have less perimeter than adjacent Red-Black-origin-Red-Black.*

Proof. By symmetry, we may assume that $c < b$. As in Figure 12, we show that exchanging the outside masses reduces perimeter and yields the desired two intervals meeting at the origin.

Since $c < b$, the whole interval $(-a', -b)$ is farther from the origin than (c, d), and hence the length is less: $a' - b < d–c$. Since $c < b$, the interval $(0, d')$ has less perimeter than $(-a, 0)$: $d' < a$.

As a result, $a' + d' < a + b - c + d$. Therefore, the new perimeter for both regions $a' + d'$ is less than the original perimeter $a + b + c + d$. □

We now give our main double bubble theorem. Our first, geometric proof works just for density $|x|$. Our second, algebraic proof works for density $|x|^p$.

Theorem 3.4 (Double Bubble Theorem)**.** *On the line with density $|x|^p$ $(p > 0)$, the least-perimeter way to enclose and separate two given masses is two adjacent intervals that meet at the origin.*

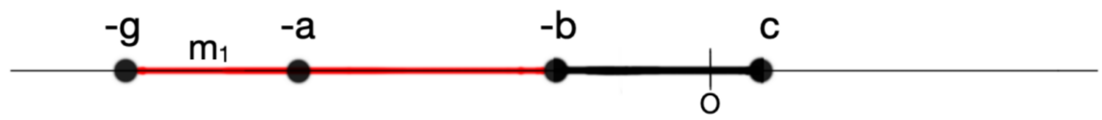

Figure 10: Move the mass of (c,d) to the left of $(-a,-b)$.

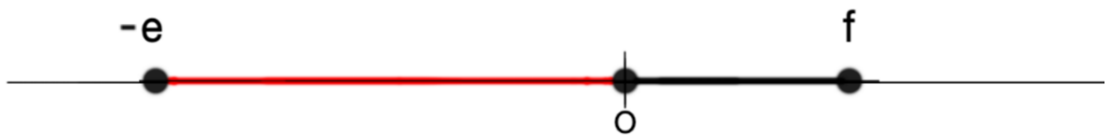

Figure 11: Meeting at the origin is best.

Proof for density $|x|$. Consider two regions of finite perimeter and given masses. To each half of the line meeting both regions, apply Proposition 3.1 to reduce to the case of two adjacent intervals with one endpoint at the origin. To each half of the line meeting just one region, a single interval with one endpoint at the origin is best. We may assume the resulting candidate consists of adjacent intervals of the "Black" and "Red" regions of one of the following three types:

a Red-Black with the origin inside Black,

b Red-Black-Red with the origin inside Black,

c Red-Black-Origin-Red-Black.

By Proposition 3.2 and 3.3, two adjacent intervals that meet at the origin is best. □

Proof for density $|x|^p$. Consider two regions ("Black" and "Red") of finite perimeter. Each consists of countably many intervals, the origin the only possible limit point. If an interval contains the origin, split it into two intervals. Denote the Black intervals by (a_i, b_i) and $(-c_j, -d_j)$, the Red intervals by (e_k, f_k) and $(-g_m, -h_m)$. Note that the mass of (a,b) is the integral from a to b ofx^p, namely $\left(b^{p+1} - a^{p+1}\right)/(p+1)$. Hence the total Black mass m_B satisfies

$$(p+1)m_B = \sum\left(b_i^{p+1} - a_i^{p+1}\right) + \sum\left(c_j^{p+1} - d_j^{p+1}\right),$$

and the Red mass m_R satisfies

$$(p+1)m_R = \sum\left(f_k^{p+1} - e_k^{p+1}\right) + \sum\left(h_m^{p+1} - g_m^{p+1}\right).$$

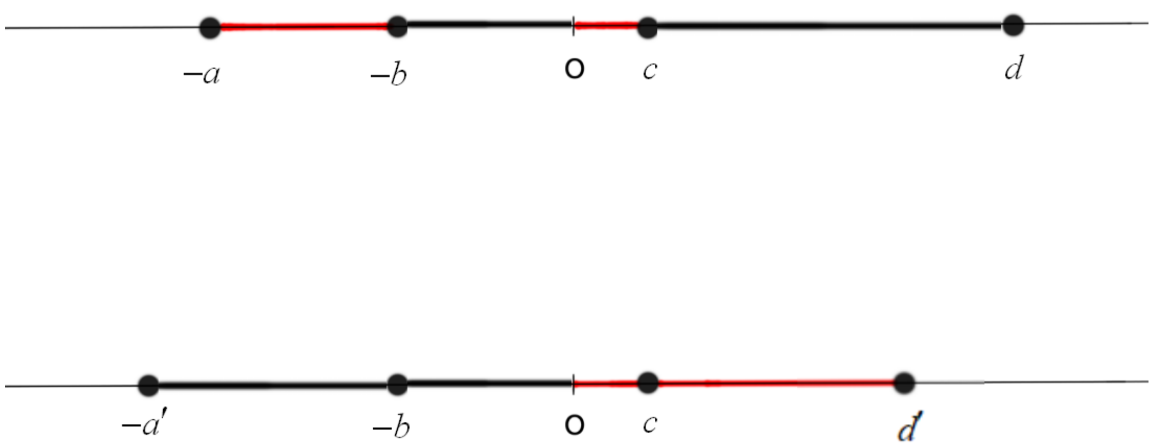

Figure 12: Exchanging the outside masses reduces perimeter.

The total perimeter is at least $\sum b_i^p + \sum c_j^p + \sum f_k^p + \sum g_m^p$, with equality only for two intervals that meet at the origin. Note that by using just the endpoint of each interval farthest from the origin, we avoid double counting endpoints common to adjacent Black and Red intervals.

Compare with two intervals $(-r, 0)$ and $(0, s)$ with the same masses: $r^{p+1}/(p+1) = m_{\mathrm{B}}$ and $s^{p+1}/(p+1) = m_{\mathrm{S}}$. Their boundary cost is $r^p + s^p$. Since they have the same mass,

$$\sum \left(b_i^{p+1} - a_i^{p+1}\right) + \sum \left(c_j^{p+1} - d_j^{p+1}\right) = r^{p+1},$$

$$\sum \left(f_k^{p+1} - e_k^{p+1}\right) + \sum \left(g_m^{p+1} - h_m^{p+1}\right) = s^{p+1}.$$

Therefore,

$$\sum b_i^{p+1} + \sum c_j^{p+1} \geq r^{p+1},$$
$$\sum f_k^{p+1} + \sum g_m^{p+1} \geq s^{p+1}.$$

Now by Lemma 2.4,

$$\sum b_i^p + \sum c_j^p \geq r^p,$$
$$\sum f_k^p + \sum g_m^p \geq s^p.$$

Hence the total cost is at least

$$\sum b_i^p + \sum c_j^p + \sum f_k^p + \sum g_m^p \geq r^p + s^p,$$

with equality if and only if we started with two intervals $(-r, 0)$ and $(0, s)$ meeting at the origin. □

References

[1] Eliot Bongiovanni, Leonardo Di Giosia, Alejandro Diaz, Jahangir Habib, Arjun Kakkar, Lea Kenigsberg, Dylanger Pittman, Nat Sothanaphan, and Weitao Zhu. Double bubbles on the real line with log-convex density. *Anal. Geom. Metr. Spaces*, 6(1):64–88, 2018.

[2] Wyatt Boyer, Bryan Brown, Gregory R. Chambers, Alyssa Loving, and Sarah Tammen. Isoperimetric regions in $\mathbb{R}^n$ with density r^p. *Anal. Geom. Metr. Spaces*, 4(1):236–265, 2016.

[3] Jonathan Dahlberg, Alexander Dubbs, Edward Newkirk, and Hung Tran. Isoperimetric regions in the plane with density r^p. *New York J. Math.*, 16:31–51, 2010.

[4] Frank Morgan. *Geometric measure theory*. Elsevier/Academic Press, Amsterdam, fifth edition, 2016. A beginner's guide, Illustrated by James F. Bredt.

[5] César Rosales, Antonio Cañete, Vincent Bayle, and Frank Morgan. On the isoperimetric problem in Euclidean space with density. *Calc. Var. Partial Differential Equations*, 31(1):27–46, 2008.

[6] Nat Sothanaphan. Double bubbles on the line with log-convex density f with $(\log f)'$ bounded. *Missouri J. Math. Sci.*, 30(2):166–175, 2018.

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