Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **39** (2019) no. 2, 71–86. doi:10.1285/i15900932v39n2p71

Derivations of abelian Lie algebra extensions

M. Barati
Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran.
mehdibarati1352@gmail.com

F. Saeedi
Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran.
saeedi@mshdiau.ac.ir

Received: 26.1.2019; accepted: 12.8.2019.

Abstract. Let $0 \to A \to L \to B \to 0$ be an abelian extension of Lie algebras. In this paper, we construct certain exact sequences which relate derivations with the Lie algebra cohomology group $H^2(B, A)$, and apply them to study extending derivations of A and lifting derivations of B to certain derivations of L.

Keywords: Derivation of Lie algebras; extensions of Lie algebras; second cohomology of Lie algebras; semidirect sum

MSC 2000 classification: Primary 17B40, 17B56; Secondary 18G60.

1 Motivation and preliminaries

Cohomology theories of various algebraic structures have been investigated by several authors. The most noteworthy are due to Hochschild, MacLance and Eckmann, Chevalley and Eilenberg, who developed the theory of cohomology groups of associative algebras, abstract groups, and Lie algebras respectively (see [3], [4], [5], and [10] for more information).
The theory of Lie algebras is one of the important parts of algebras. Many papers in the literature make an attempt to generalize the results on finite p-groups to the theory of Lie algebras. On the other hand there are same sporadic results for the Lie algebra that does not coincide with the results for groups. In fact, there are analogies between groups and Lie algebras, but the analogies are not completely identical and most of them should be checked carefully (see [2] and [6] for examples).

The theory of group extensions and their interpretation in terms of cohomology is well known (see [7] and [9] for example).

Over the years there has been an interest in studying the second cohomology of Lie algebras (see [2],[12]).

Let A and B be Lie algebras over a field F, where A is abelian. We say that

A is a left B-module if there is a F-linear transformation $B \otimes A \to A$, written as $b \otimes a \mapsto ba$, such that

$$[b_1, b_2]a = b_1(b_2 a) - b_2(b_1 a) \qquad \text{for all } b_1, b_2 \in B \text{ and } a \in A.$$

Let $\mathrm{Der}(A)$ be the derivations of A equipped with the Lie bracket

$$[f, g] = fg - gf \qquad \text{for all } f, g \in \mathrm{Der}(A).$$

Then a left B-module structure on A is equivalent to the existence of a Lie algebra homomorphism

$$B \to \mathrm{Der}(A).$$

Let A and B be Lie algebras. Then an extension of B by A is a short exact sequence of Lie algebras

$$0 \to A \xrightarrow{i} L \xrightarrow{\pi} B \to 0,$$

where L is a Lie algebra. Without loss of generality, we may assume that i is the inclusion map, and we omit it from the notation. It follows from the exactness that A is an ideal of L. If A is abelian, then such an extension is called an abelian extension. This together with the Jacobi identity gives a left L-module structure on A given by

$$xa := [x, a] \qquad \text{for } x \in L \text{ and } a \in A.$$

Let $t : B \to L$ be a section of π; that is, t is a F-linear map such that $\pi t = 0$. If A is abelian, then this induces a left B-module structure on A given by

$$ba := [t(b), a] \qquad \text{for } \ b \in B \text{ and } \ a \in A.$$

We denote the above B-module structure on A by

$$\alpha : B \to \mathrm{Der}(A).$$

Our aim in this paper is to construct three exact sequences on derivation algebra of abelian Lie algebra extensions which relate derivations with cohomology of Lie algebras.
The sequences resemble well-known Wells exact sequence for group extensions which relate automorphisms with group cohomology (see [11] and [17] for more information).

Let $\mathrm{Der}(A)$, $\mathrm{Der}(L)$, and $\mathrm{Der}(B)$ denote the derivations of A, L, and B, respectively.

Let I and J be two ideals of L. We introduce below some further notation to be used in this paper:

$$\begin{aligned} \mathrm{Der}^I(L) &= \{\gamma \in \mathrm{Der}(L) | \gamma(x) \in I \quad \text{for all } x \in L\} \\ \mathrm{Der}_J(L) &= \{\gamma \in \mathrm{Der}(L) | \gamma(x) = 0 \quad \text{for all } x \in J\} \\ \mathrm{Der}_J^I(L) &= \mathrm{Der}^I(L) \cap \mathrm{Der}_J(L) \\ \mathrm{Der}(L{:}I) &= \{\gamma \in \mathrm{Der}(L) | \gamma(x) \in I \quad \text{for all } x \in I\}\,. \end{aligned}$$

Various aspect of derivations of Lie algebras has been investigated in the literature (see, for example, [13, 14, 15]).
Let $0 \to A \to L \xrightarrow{\pi} B \to 0$ be an abelian extension of Lie algebras over a field F, and let $t : B \to L$ be a section of π; so that every element of L can be written uniquely as $t(b) + a$ for some $b \in B$ and $a \in A$.
Observe that a derivation $\gamma \in \mathrm{Der}(L : A)$ induces derivations $\theta \in \mathrm{Der}(A)$ and $\varphi \in \mathrm{Der}(B)$, given by $\theta(a) = \gamma\big|_A(a)$ for all $a \in A$ and $\varphi(b) = \pi(\gamma(t(b)))$ for all $b \in B$. This gives a Lie homomorphism

$$\tau : \mathrm{Der}(L{:}A) \to \mathrm{Der}(A) \oplus \mathrm{Der}(B)$$

by setting

$$\tau(\gamma) = (\theta, \varphi)$$

We denote the restrictions of τ to $\mathrm{Der}^A(L)$ and $\mathrm{Der}_A(L)$ by τ_1 and τ_2, respectively.

Remark 1.1. To set notation, we briefly recall the definition of cohomology of Lie algebras.
Let B be a Lie algebra, and let A be a left B-module. For each $0 \le k \le \mathrm{dim} B$, define $C^k(B; A) = \mathrm{Hom}(\Lambda^k B, A)$ and $\partial^k : C^k(B; A) \longrightarrow C^{k+1}(B; A)$ by

$$\begin{aligned} \partial^k(\nu)(b_0, \dots, b_k) = &\sum_{i=0}^{k} (-1)^i \ b_i \ \nu(\dots, \hat{b_i}, \dots) \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j} \nu([b_i, b_j], \dots, \hat{b_i}, \dots, \hat{b_j}, \dots) \end{aligned}$$

for all $\nu \in C^k(B; A)$. It is straightforward to verify, using the Jacobi Identity and the B-action on A, that $\partial^{k+1}\partial^k = 0$. Let $Z^k(B; A) = \ker(\partial^k)$ be the group of k-cocycles, and let $B^k(B; A) = \mathrm{image}(\partial^{k-1})$ be the group of k-coboundaries. Then $H^k(B; A) = Z^k(B; A)/B^k(B; A)$ is the *kth* Lie algebra cohomology group of B with values in A.
Recently, Bardakov and Singh[1] gave an explicit description of a certain sequence for automorphisms of Lie algebras. We continue in the present work this

line of investigation for derivations of Lie algebras. In Section 2 we establish our exact sequences.

Theorem 1.2 (Main Theorem)**.** Let $0 \to A \to L \xrightarrow{\pi} B \to 0$ be an abelian extension of Lie algebras. Then there exist the following three exact sequences

$$0 \to \mathrm{Der}_A^A(L) \to \mathrm{Der}^A(L) \xrightarrow{\tau_1} C_1 \xrightarrow{\lambda_1} H^2(B;A), \tag{1.1}$$

$$0 \to \mathrm{Der}_A^A(L) \to \mathrm{Der}_A(L) \xrightarrow{\tau_2} C_2 \xrightarrow{\lambda_2} H^2(B;A), \tag{1.2}$$

$$0 \to \mathrm{Der}_A^A(L) \to \mathrm{Der}(L\colon A) \xrightarrow{\tau} C_\alpha \xrightarrow{\lambda_\varepsilon} H^2(B;A). \tag{1.3}$$

The maps $\lambda_1, \lambda_2, \lambda_\varepsilon$ and notations C_1, C_2 and C_α will be defined in the next section.

2 Description of exact sequences

Let $\alpha : B \to \mathrm{Der}(A)$ be the B-module structure on A, and let $\mathrm{Ext}_\alpha(B, A)$ denote the set of equivalence classes of extensions of B by A inducing α.
Let $\varepsilon : 0 \to A \to L \xrightarrow{\pi} B \to 0$ be an abelian extension of Lie algebras inducing α.
We fix a section $t : B \to L$. For any two elements $b_1, b_2 \in L$, we have

$$\pi(t[b_1, b_2]) = [b_1, b_2] = [\pi t(b_1), \pi t(b_2)] = \pi[t(b_1), t(b_2)].$$

Thus there exists a unique element, say $\delta(b_1, b_2) \in A$, such that $\delta(b_1, b_2) = [t(b_1), t(b_2)] - t([b_1, b_2])$.
Observe that δ is a F-bilinear map from $B \oplus B$ to A such that $\delta(b, b) = 0$ for all $b \in B$.
Then it is easy to see that δ is a two-cocycle and two-coycles corresponding to different sections differ by a two-coboundary. Thus the map $[\varepsilon] \mapsto [\delta]$ gives a bijection $\mathrm{Ext}_\alpha(B, A) \leftrightarrow H^2(B; A)$ (see [8, p. 238]).
In the following we present an important lemma used for proving theorems.

Lemma 2.1. Let $0 \to A \to L \xrightarrow{\pi} B \to 0$ be an abelian extension of Lie algebras over a field F. If $\gamma \in \mathrm{Der}(L : A)$, then there is a triplet $(\theta, \varphi, \chi) \in \mathrm{Der}(A) \oplus \mathrm{Der}(B) \oplus \mathrm{Hom}(B, A)$ such that

(i) $\delta(b, \varphi(b')) + \delta(\varphi(b), b') - \theta(\delta(b, b')) = [t(b'), \chi(b)] - [t(b), \chi(b')] + \chi([b, b'])$

(ii) $\theta([t(b), a]) = [t(\varphi(b)), a] + [t(b), \theta(a)]$ for all $a \in A$ and $b, b' \in B$.

Conversely, if $(\theta, \varphi, \chi) \in \mathrm{Der}(A) \oplus \mathrm{Der}(B) \oplus \mathrm{Hom}(B, A)$ is a triplet satisfying equations (i) and (ii), then, for all $a \in A$ and $b \in B$, γ defined by

$$\gamma(t(b) + a) = t(\varphi(b)) + \chi(b) + \theta(a)$$

is a derivation of L lying in $\mathrm{Der}(L\!:\!A)$.

[Here δ is the two-cocycle corresponding to the extension $0 \to A \to L \xrightarrow{\pi} B \to 0$, $t : B \to L$ is a section of π, and $Hom(B, A)$ is the group of all F-linear maps from B to A].

Proof. Every derivation $\gamma \in \mathrm{Der}(L : A)$ determines a pair$(\theta, \varphi) \in \mathrm{Der}(A) \oplus \mathrm{Der}(B)$ such that γ restricts to θ on A and induces φ on B. For any $b \in B$, we have $\varphi(b) = \pi(\gamma(t(b)))$. Thus $\gamma(t(b)) = t(\varphi(b)) + \chi(b)$ for some element $\chi(b) \in A$. Since all the maps involved are F-linear, it follows that $\chi \in \mathrm{Hom}(B, A)$.
To prove (i), let $b, b' \in B$. We have

$$\delta(b, b') + t([b, b']) = [t(b), t(b')].$$

By applying γ and using the definition of χ, we get

$$\theta(\delta(b, b')) + \chi([b, b']) + t(\varphi([b, b'])) = [\gamma(t(b)), t(b')] + [t(b), \gamma(t(b'))].$$

Since φ is a derivation, we have

$$\begin{aligned}\theta(\delta(b, b')) + \chi([b, b']) + t([\varphi(b), b']) + t([b, \varphi(b')])&\\ = [t(\varphi(b)) + \chi(b), t(b')] + [t(b), t(\varphi(b')) + \chi(b')]&\\ = \delta(\varphi(b), b') + t([\varphi(b), b']) + [\chi(b), t(b')]&\\ + \delta(b, \varphi(b')) + t([b, \varphi(b')]) + [t(b), \chi(b')].&\end{aligned}$$

Thus we obtain condition (i).
To derive condition (ii), we use the fact that γ is a derivation. Let $b \in B$, $a \in A$. Then we have

$$\begin{aligned}\theta([t(b), a]) &= \gamma([t(b), a])\\ &= [\gamma(t(b)), a] + [t(b), \gamma(a)]\\ &= [t(\varphi(b) + \chi(b), a] + [t(b), \theta(a)]\\ &= [t(\varphi(b), a] + [t(b), \theta(a)].\end{aligned}$$

Conversely, let $(\theta, \varphi, \chi) \in \mathrm{Der}(A) \oplus \mathrm{Der}(B) \oplus \mathrm{Hom}(B, A)$ be a triplet satisfying equations of (i) and (ii). To see γ is a derivation of L, first we note that every element of L can be uniquely written as $l = t(b) + a$, where $b = \pi(l)$ and $a \in A$. Let $l_1 = t(b_1) + a_1$ and $l_2 = t(b_2) + a_2$, where $b_1, b_2 \in B$ and $a_1, a_2 \in A$. Then

$$
\begin{aligned}
(1) \quad \gamma([l_1, l_2]) &= \gamma([t(b_1), t(b_2)]) + \gamma([t(b_1), a_2]) + \gamma([a_1, t(b_2)]) \\
&= \theta(\delta(b_1, b_2)) + \chi([b_1, b_2]) + t(\varphi([b_1, b_2])) \\
&\quad + \theta([t(b_1), a_2]) - \theta([t(b_2), a_1]) \\
&= \delta(b_1, \varphi(b_2)) + \delta(\varphi(b_1), b_2) + [t(b_1), \chi(b_2)] \\
&\quad - [t(b_2), \chi(b_1)] + [t(\varphi(b_1)), a_2] + [t(b_1), \theta(a_2)] \\
&\quad - [t(\varphi(b_2)), a_1] - [t(b_2), \theta(a_1)] + t[\varphi(b_1), b_2] \\
&\quad + t[b_1, \varphi(b_2)]
\end{aligned}
$$

and

$$
\begin{aligned}
(2) \quad [\gamma(l_1), l_2] + [l_1, \gamma(l_2)] &= [t(\varphi(b_1)) + \chi(b_1) + \theta(a_1), t(b_2) + a_2] \\
&\quad + [t(b_1) + a_1, t(\varphi(b_2)) + \chi(b_2) + \theta(a_2)] \\
&= [t(\varphi(b_1)), t(b_2)] + [t(\varphi(b_1)), a_2] + [\chi(b_1), t(b_2)] \\
&\quad + [\theta(a_1), t(b_2)] + [t(b_1), t(\varphi(b_2))] + [t(b_1), \chi(b_2)] \\
&\quad + [t(b_1), \theta(a_2)] + [a_1, t(\varphi(b_2))]
\end{aligned}
$$

By comparing (1) and (2), we have $\gamma \in \mathrm{Der}(L)$, and clearly, $\gamma(a) = \theta(a)$ for all $a \in A$. Therefore $\gamma \in \mathrm{Der}(L\!:\!A)$. QED

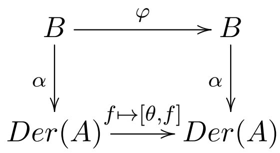
Remark 2.2. If $0 \to A \to L \xrightarrow{\pi} B \to 0$ is a central extension (that is $A \leq Z(L)$), then the action B on A becomes trivial, and therefore Lemma 2.1 takes the following simpler form which we will use in the proof of one of the next theorems.

Lemma 2.3. Let $0 \to A \to L \xrightarrow{\pi} B \to 0$ be a central extension of Lie algebras over a field F. If $\gamma \in \mathrm{Der}(L : A)$, then there is a triplet $(\theta, \varphi, \chi) \in \mathrm{Der}(A) \oplus \mathrm{Der}(B) \oplus \mathrm{Hom}(B, A)$ such that, for all $b, b' \in B$, the following condition is satisfied:

$$\delta(b, \varphi(b')) + \delta(\varphi(b), b') - \theta(\delta(b, b')) = \chi([b, b'])$$

Conversely, if $(\theta, \varphi, \chi) \in \mathrm{Der}(A) \oplus \mathrm{Der}(B) \oplus \mathrm{Hom}(B, A)$ is a triplet satisfying above equation, then γ defined by $\gamma(t(b) + a) = t(\varphi(b)) + \chi(b) + \theta(a)$ is a derivation of L lying in $\mathrm{Der}(L\!:\!A)$.

Remark 2.4. A pair $(\theta, \varphi) \in \mathrm{Der}(A) \oplus \mathrm{Der}(B)$ is called compatible if $\alpha\varphi(b) = [\theta, \alpha(b)]$ for all $b \in B$. Equivalently, the following diagram commutes.



Note that $\alpha\varphi$ also gives a B-module structure on A. Then the compatibility condition is equivalent to saying that $\theta : A \to A$ is a B-module homomorphism from the B-module structure α to the B-module structure $\alpha\varphi$ on A.

Let C_α denote the Lie algebra of all compatible pairs; It is clear that C_α is a subalgebra of $\mathrm{Der}(A) \oplus \mathrm{Der}(B)$. Let

$$C_1 = \{\theta \in \mathrm{Der}(A) | (\theta, 0) \in c_\alpha\}$$

and

$$C_2 = \{\varphi \in \mathrm{Der}(B) | (0, \varphi) \in c_\alpha\}\,.$$

For $\theta \in C_1$ and $\varphi \in C_2$, we define maps $k_\theta, k_\varphi, k_{\theta,\varphi} : B \oplus B \to A$ by setting, for $b, b' \in B$,

$$\begin{aligned}
&k_\theta(b, b') = \theta(\delta(b, b')),\\
&k_\varphi(b, b') = \delta(\varphi(b), b') + \delta(b, \varphi(b')),\\
&k_{\theta,\varphi}(b, b') = \delta(\varphi(b), b') + \delta(b, \varphi(b')) - \theta(\delta(b, b')).
\end{aligned}$$

Lemma 2.5. The maps k_θ, k_φ, and $k_{\theta,\varphi}$ are two-cocycles.

Proof. For $b_0, b_1, b_2 \in B$, by the fact that $\theta \in c_1$ and $\delta \in Z^2(B; A)$, we have

$$\begin{aligned}
&[t(b_0), k_\theta(b_1, b_2)] - [t(b_1), k_\theta(b_0, b_2)] + [t(b_2), k_\theta(b_0, b_1)] - k_\theta([b_0, b_1], b_2)\\
&\quad + k_\theta([b_0, b_2], b_1) - k_\theta([b_1, b_2], b_0) = [t(b_0), \theta\delta(b_1, b_2)] - [t(b_1), \theta\delta(b_0, b_2)]\\
&\qquad + [t(b_2), \theta\delta(b_0, b_1)] - \theta\delta([b_0, b_1], b_2)\\
&\qquad + \theta\delta([b_0, b_2], b_1) - \theta\delta([b_1, b_2], b_0)\\
&\quad = \theta([t(b_0), \delta(b_1, b_2)] - [t(b_1), \delta(b_0, b_2)]\\
&\qquad + [t(b_2), \delta(b_0, b_1)] - \delta([b_0, b_1], b_2)\\
&\qquad + \delta([b_0, b_2], b_1) - \delta([b_1, b_2], b_0))\\
&\quad = \theta(0) = 0
\end{aligned}$$

Hence k_θ is a two-cocycle.
We next show that k_φ is a two-cocycle. For $b_0, b_1, b_2 \in B$, we have

$$\begin{aligned}
&[t(b_0), k_\varphi(b_1, b_2)] - [t(b_1), k_\varphi(b_0, b_2)] + [t(b_2), k_\varphi(b_0, b_1)] - k_\varphi([b_0, b_1], b_2)\\
&\quad + k_\varphi([b_0, b_2], b_1) - k_\varphi([b_1, b_2], b_0) = [t(b_0), \delta(\varphi(b_1), b_2) + \delta(b_1, \varphi(b_2))]\\
&\qquad - [t(b_1), \delta(\varphi(b_0), b_2) + \delta(b_0, \varphi(b_2))]\\
&\qquad + [t(b_2), \delta(\varphi(b_0), b_1) + \delta(b_0, \varphi(b_1))]\\
&\qquad - \delta(\varphi([b_0, b_1], b_2) - \delta([b_0, b_1], \varphi(b_2))\\
&\qquad + \delta(\varphi([b_0, b_2]), b_1)) + \delta([b_0, b_2], \varphi(b_1))
\end{aligned}$$

$$
\begin{aligned}
&- \delta(\varphi([b_1,b_2]),b_0) - \delta([b_1,b_2],\varphi(b_0))\\
&= [t(b_0),\delta(\varphi(b_1),b_2)] + [t(b_0),\delta(b_1,\varphi(b_2))]\\
&- [t(b_1),\delta(\varphi(b_0),b_2)] - [t(b_1),\delta(b_0,\varphi(b_2)]\\
&+ [t(b_2),\delta(\varphi(b_0),b_1)] + [t(b_2),\delta(b_0,\varphi(b_1)]\\
&- \delta([\varphi(b_0),b_1],b_2) - \delta([b_0,\varphi(b_1)],b_2)\\
&+ \delta([b_0,b_1],\varphi(b_2)) + \delta([\varphi(b_0),b_2],b_1)\\
&+ \delta([b_0,\varphi(b_2)],b_1) + \delta([b_0,b_2],\varphi(b_1))\\
&- \delta([\varphi(b_1),b_2],b_0) - \delta([b_1,\varphi(b_2)],b_0)\\
&- \delta([b_1,b_2],\varphi(b_0)) = 0
\end{aligned}
$$

Note that $\varphi \in c_2$; therefore $[t(\varphi(b)),a] = 0$ for $b \in B$ and $a \in A$.
Since $k_{\theta,\varphi} = k_\theta + k_\varphi$, we conclude that $k_{\theta,\varphi} \in Z^2(B;A)$. QED

Define $\lambda_1 : C_1 \to H^2(B;A)$ by setting for $\theta \in C_1$,

$$\lambda_1(\theta) = [k_\theta], \quad \text{the cohomology class of } k_\theta$$

Similarly define $\lambda_2 : C_2 \to H^2(B;A)$ by setting, for $\varphi \in C_2$,

$$\lambda_2(\varphi) = [k_\varphi], \quad \text{the cohomology class of } k_\phi$$

and define $\lambda_\varepsilon(\theta,\varphi) : C_\alpha \to H^2(B;A)$ by setting $\lambda_\varepsilon(\theta,\varphi) = [k_{\theta,\varphi}]$.
To justify these definitions, we need the following.

Lemma 2.6. The maps λ_1, λ_2, and λ_ε are well defined.

Proof. To show that the maps λ_1 and λ_2 are well defined, we need to show that these maps are independent of the choice of sections. Let $t, t^{'} : B \to L$ be two sections. Then there exist maps $\delta, \delta^{'} : B \oplus B \to A$ such that, for $b_1, b_2 \in B$,

$$
\begin{aligned}
\delta(b_1,b_2) &= [t(b_1),t(b_2)] - t([b_1,b_2]),\\
\delta^{'}(b_1,b_2) &= [t^{'}(b_1),t^{'}(b_2)] - t^{'}([b_1,b_2]).
\end{aligned}
$$

For $b \in B$, since $t(b)$ and $t^{'}(b)$ satisfy $\pi(t(b)) = b = \pi(t^{'}(b))$, there exists a unique element $k(b) \in A$ such that $t(b) = k(b) + t^{'}(b)$.
We thus have a map $k : B \to A$ by setting $k(b) = t(b) - t^{'}(b)$ for $b \in B$. For $b_1, b_2 \in B$, $k([b_1,b_2]) + t^{'}([b_1,b_2]) = t([b_1,b_2])$. This gives

$$
\begin{aligned}
\delta(b_1,b_2) - \delta^{'}(b_1,b_2) &= [t(b_1),t(b_2)] - t([b_1,b_2]) - [t^{'}(b_1),t^{'}(b_2)] + t^{'}([b_1,b_2])\\
&= [t(b_1),t(b_2)] - [t(b_1)-k(b_1),t(b_2)-k(b_2)] - k([b_1,b_2])\\
&= [t(b_1),k(b_2)] - [t(b_2),k(b_1)] - k([b_1,b_2]) \in B^2(B;A).
\end{aligned}
$$

Therefore

$$\begin{aligned}\theta(\delta(b_1,b_2))-\theta(\delta^{'}(b_1,b_2)) =&[t(b_1),\theta k(b_2)]-[t(b_2),\theta k(b_1)]\\ &-\theta k([b_1,b_2])[t(b_1),k^{'}(b_2)]\\ &=[t(b_2),k^{'}(b_1)]-k^{'}([b_1,b_2])\in B^2(B;A),\end{aligned}$$

where $k^{'}=\theta k$. This proves that λ_1 is independent of the choice of a section. Next we prove that λ_2 is well defined. It is sufficient to show that

$$\delta(\varphi(b_1),b_2)+\delta(b_1,\varphi(b_2))-\delta'(\varphi(b_1),b_2)-\delta'(b_1,\varphi(b_2))\in B^2(B;A).$$

Proceed just as above and note that $\varphi\in C_2$ and φ is derivation, we have

$$\begin{aligned}&[t(b_1),k\varphi(b_2)]-[t\varphi(b_2),k(b_1)]-k([b_1,\varphi(b_2)])+[t\varphi(b_1),k(b_2)]-[t(b_2),k\varphi(b_1)]\\ &\quad -k([\varphi(b_2),b_1])=[t(b_1),k\varphi(b_2)]-[t(b_2),k\varphi(b_1)]-k\varphi([b_1,b_2]).\end{aligned}$$

Putting $k\varphi=k^{''}$

$$[t(b_1),k^{''}(b_2)]-[t(b_2),k^{''}(b_1)]-k^{''}([b_1,b_2])\in B^2(B;A).$$

This proves that λ_2 is also independent of the choice of a section.
Since λ_1 and λ_2 are well defined, λ_ε is well defined too, and the proof of the lemma is complete. QED

Note that the maps λ_1 and λ_2 are not homomorphisms of Lie algebra.
We are now in position to prove the main theorem.

Proof of Main Theorem. Let $0\to A\to L\overset{\pi}{\to}B\to 0$ be an abelian extension. Clearly all the sequences (1.1), (1.2), and (1.3) are exact at the first two terms. To complete the proof it only remains to show the exactness at the third term of the respective sequences. First consider the sequence (1.1). We show that $\mathrm{Im}\tau_1=\mathrm{Ker}\lambda_1$. Let $\gamma\in\mathrm{Der}^A(L)$. Then $\theta\in C_1$, where θ is the restriction of γ to A. Lemma 2.1 implies that, for $b_1,b_2\in B$,

$$k_\theta(b_1,b_2)=\theta(\delta(b_1,b_2))=[t(b_1),\chi(b_2)]-[t(b_2),\chi(b_1)]-\chi([b_1,b_2])\in B^2(B;A).$$

Thus $k_\theta\in B^2(B;A)$, and hence $\lambda_1(\theta)=0$. Conversely, if $\theta\in C_1$ is such that $\lambda_1(\theta)=0$, then $k_\theta\in B^2(B;A)$. Therefore there exists $\chi\in\mathrm{Hom}(B;A)$ such that

$$\theta(\delta(b_1,b_2))=[t(b_1),\chi(b_2)]-[t(b_2),\chi(b_1)]-\chi(b_1,b_2),$$

since $\theta\in C_1$ and γ, defined by converse Lemma 2.1, is an element of $\mathrm{Der}^A(L)$. Hence the sequence (1.1) is exact.

Next let us consider the sequence (1.2). We show that $\mathrm{Im}\tau_2 = \mathrm{Ker}\lambda_2$. Let $\gamma \in \mathrm{Der}_A(L)$. Then $\varphi \in C_2$, where φ is induced by γ on B. By Lemma 2.1, for $b_1, b_2 \in B$ we have,

$$k_\varphi(b_1, b_2) = \delta(\varphi(b_1), b_2) + \delta(b_1, \varphi(b_2)) = [t(b_2), \chi(b_1)] - [t(b_1), \chi(b_2)] + \chi([b_1, b_2]) \in B^2(B; A).$$

Thus $k_\varphi \in B^2(B; A)$, and hence $\lambda_2(\varphi) = 0$. Conversely, if $\varphi \in C_2$ with $\lambda_2(\varphi) = 0$, then $k_\varphi \in B^2(B; A)$. Therefore there exists $\chi \in \mathrm{Hom}(B, A)$ such that

$$\delta(\varphi(b_1), b_2) + \delta(b_1, \varphi(b_2) = [t(b_1), \chi(b_2)] - [t(b_2), \chi(b_1)] + \chi([b_1, b_2]),$$

since $\varphi \in C_2$ and γ, defined by converse Lemma 2.1, is an element of $\mathrm{Der}_A(L)$. Hence the sequence (1.2) is exact. Similarly one can prove (1.3) . QED

We construct a more general exact sequence.

Theorem 2.7. If $0 \to A \to L \xrightarrow{\pi} B \to 0$ is a central extension, then there exists an exact sequence $0 \to \mathrm{Der}_A^A(L) \to \mathrm{Der}(L : A) \xrightarrow{\tau} \mathrm{Der}(A) \oplus \mathrm{Der}(B) \xrightarrow{\lambda} H^2(B, A)$.

Proof. The sequence is clearly exact at $\mathrm{Der}_A^A(L)$ and $\mathrm{Der}(L : A)$. We construct the map λ and show the exactness at $\mathrm{Der}(A) \oplus \mathrm{Der}(B)$. Consider $(\theta, \varphi) \in \mathrm{Der}(A) \oplus \mathrm{Der}(B)$. Define $k_{\theta,\varphi} : B \oplus B \to A$ by setting, for $b_1, b_2 \in B$,

$$k_{\theta,\varphi}(b_1, b_2) = \delta(b_1, \varphi(b_2)) + \delta(\varphi(b_1), b_2) - \theta(\delta(b_1, b_2)).$$

It is clearly $k_{\theta,\varphi} \in Z^2(B, A)$. Define $\lambda(\theta, \varphi) = [k_{\theta,\varphi}]$, the cohomology class of $k_{\theta,\varphi}$ in $H^2(B, A)$. Similar to Lemma 2.6 one can prove that λ is well defined. If $(\theta, \varphi) \in \mathrm{Der}(A) \oplus \mathrm{Der}(B)$ is induced by some $\gamma \in \mathrm{Der}(L : A)$, then, by Lemma 2.3, we have $k_{\theta,\varphi}(b_1, b_2) = \chi([b_1, b_2])$. Therefore $k_{\theta,\varphi} \in B^2(B, A)$. Hence $\lambda(\theta, \varphi) = 0$. Conversely, if $(\theta, \varphi) \in \mathrm{Der}(A) \oplus \mathrm{Der}(B)$ is such that $[k_{\theta,\varphi}] = 0$, then $k_{\theta,\varphi}(b_1, b_2) = \chi([b_1, b_2])$, for some $\chi : B \to A$. By Lemma 2.3, there exists $\gamma \in \mathrm{Der}(L : A)$ inducing θ and φ. Thus the sequence is exact. QED

3 Splitting of sequences

Let $0 \to A \to L \xrightarrow{\pi} B \to 0$ be an abelian extension. Let $C_1^* = \{\theta \in C_1 | \lambda_1(\theta) = 0\}$, $C_2^* = \{\varphi \in C_2 | \lambda_2(\varphi) = 0\}$, and $C_\alpha^* = \{(\theta, \varphi) \in C_\alpha | \lambda_\varepsilon(\theta, \varphi) = 0\}$. Then it follows from Theorem 1.2 that the sequences

$$0 \to \mathrm{Der}_A^A(L) \to \mathrm{Der}^A(L) \xrightarrow{\tau_1} C_1^* \to 0, \tag{3.1}$$

$$0 \to \mathrm{Der}_A^A(L) \to \mathrm{Der}_A(L) \xrightarrow{\tau_2} C_2^* \to 0, \tag{3.2}$$

and

$$0 \to \mathrm{Der}_A^A(L) \to \mathrm{Der}(L : A) \xrightarrow{\tau} C_\alpha^* \to 0 \tag{3.3}$$

are exact similarly. Let $0 \to A \to L \xrightarrow{\pi} B \to 0$ be a central extension, and let $C^* = \{(\theta, \varphi) \in \mathrm{Der}(A) \oplus \mathrm{Der}(B) | \lambda(\theta, \varphi) = 0\}$. Then it follows from Theorem 2.7 that the sequence

$$0 \to \mathrm{Der}_A^A(L) \to \mathrm{Der}(L\!:\!A) \xrightarrow{\tau} C^* \to 0 \tag{3.4}$$

is exact.

Theorem 3.1. Let L be a finite dimensional Lie algebra, and let A an abelian ideal of L such that the sequence $0 \to A \to L \xrightarrow{\pi} B \to 0$ splits. Then the sequences (3.1),(3.2), and (3.3) split. Furthermore, if $A \leq Z(L)$, then the sequence (3.4) also splits.

Proof. We prove that the sequence (3.3) is split. Let $A \rtimes B = A \oplus B$ be a vector space, equipped with the Lie algebra structure given by $[(a_1, b_1), (a_2, b_2)] = (b_1 a_2 - b_2 a_1, [b_1, b_2])$ for $a_1, a_2 \in A$ and $b_1, b_2 \in B$.
Then the split extension $0 \to A \to L \to B \to 0$ is equivalent to the extension

$$0 \to A \overset{a \to (a,0)}{\longrightarrow} A \rtimes B \overset{(a,b) \to b}{\longrightarrow} B \to 0,$$

and hence $\mathrm{Der}(A \rtimes B : A) \cong \mathrm{Der}(L : A)$. Note that, for a split extension, the corresponding two-cocycle is zero, and hence $C_\alpha^* = C_\alpha$. Now we define a section $\beta : C_\alpha^* \to \mathrm{Der}(A \rtimes B : A)$ by $\beta(\theta, \varphi) = \gamma$, where $\gamma(a, b) = (\theta(a), \varphi(b))$ for $a \in A$ and $b \in B$. So γ is F-linear. Furthermore, for $a_1, a_2 \in A$ and $b_1, b_2 \in B$, we have

$$\begin{aligned} \gamma[(a_1, b_1), (a_2, b_2)] &= \gamma(b_1 a_2 - b_2 a_1, [b_1, b_2]) \\ &= (\theta(b_1 a_2) - \theta(b_2 a_1), \varphi([b_1, b_2])) \\ &= \big(\varphi(b_1)(a_2) + b_1 \theta(a_2) - \varphi(b_2) a_1 - b_2 \theta(a_1), \\ &\qquad [\varphi(b_1), b_2] + [b_1, \varphi(b_2)]\big) \\ &\qquad\qquad \text{(by compatibility of } (\theta, \varphi)) \\ &= [(\theta(a_1), \varphi(b_1)), (a_2, b_2)] + [(a_1, b_1), (\theta(a_2), \varphi(b_2))] \\ &= [\gamma(a_1, b_1), (a_2, b_2)] + [(a_1, b_1), \gamma(a_2, b_2)]; \end{aligned}$$

hence $\gamma \in \mathrm{Der}(A \rtimes B)$.
It is clear that β is a Lie homomorphism, and hence the sequence (3.3) splits. QED

We show that the converse of Theorem 3.1 is not true, in general.

For this purpose we focus on derivation of Lie algebra of free two-step nilpotent the Lie algebras. Let

$$L_{n,2} = \langle x_1, \cdots, x_n | [[x_i, x_j], x_k] = 0 \text{ for all } 1 \leq i, j, k \leq n \rangle$$

be the free two-step nilpotent Lie algebra of rank $n \geq 2$.
Let $L_{n,2}^{(1)} = \langle [x_i, x_j] | 1 \leq j < i \leq n \rangle$ be the derived subalgebra of $L_{n,2}$.
Set $Z = \{z_{i,j} | z_{i,j} = [x_i, x_j] \quad \text{for } 1 \leq j < i \leq n\}$. We see that $L_{n,2}^{(1)}$ is a free abelian Lie algebra with basis Z and rank $\dfrac{n(n-1)}{2}$. If we take the lexicographic order on the basis Z given by

$$z_{2,1} < z_{3,1} < z_{3,2} < \cdots < z_{n,n-1},$$

then $\mathrm{Der}(L_{n,2}^{(1)}) \cong \mathfrak{gl}(\dfrac{n(n-1)}{2}, F)$. Let $\theta \in \mathrm{Der}(L_{n,2}^{(1)})$ be given by

$$\theta : \begin{cases} z_{2,1} \longmapsto b_{2,1;2,1} z_{2,1} + b_{2,1;3,1} z_{3,1} + \cdots + b_{2,1;n,n-1} z_{n,n-1}, \\ \quad \vdots \\ z_{i,j} \longmapsto \sum_{1 \leq l < k \leq n} b_{i,j;k,l} z_{k,l}, \\ \quad \vdots \\ z_{n,n-1} \longmapsto b_{n,n-1;2,1} z_{2,1} + b_{n,n-1;3,1} z_{3,1} + \cdots + b_{n,n-1;n,n-1} z_{n,n-1}. \end{cases}$$

Then the matrix $[\theta] \in \mathfrak{gl}(\frac{n(n-1)}{2}, F)$. Similarly, $L_{n,2}^{ab} = L_{n,2}/L_{n,2}^{(1)} = \langle \bar{x_1}, \cdots, \bar{x_n} \rangle$ is also a free abelian Lie algebra of rank n, and hence $\mathrm{Der}(L_{n,2}^{ab}) \cong \mathfrak{gl}(n, F)$. Let $\varphi \in \mathrm{Der}(L_{2,2}^{ab})$ be given by

$$\varphi : \begin{cases} \bar{x}_1 \longmapsto a_{11}\bar{x}_1 + \cdots + a_{1n}\bar{x}_n, \\ \quad \vdots \\ \bar{x}_i \longmapsto a_{i1}\bar{x}_1 + \cdots + a_{in}\bar{x}_n, \\ \quad \vdots \\ \bar{x}_n \longmapsto a_{n1}\bar{x}_1 + \cdots + a_{nn}\bar{x}_n. \end{cases}$$

Then the matrix $[\varphi] \in \mathfrak{gl}(n, F)$.
Let $0 \to A \to L \xrightarrow{\pi} B \to 0$ be an abelian extension of Lie algebras over a field F. A pair of derivation $(\theta, \varphi) \in \mathrm{Der}(A) \oplus \mathrm{Der}(B)$ is called inducible if there exists $\gamma \in \mathrm{Der}(L : A)$ such that $\tau(\gamma) = (\theta, \varphi)$.
We prove now the following theorem.

Theorem 3.2. Let $(\theta, \varphi) \in \mathrm{Der}(L_{n,2}^{(1)}) \oplus \mathrm{Der}(L_{n,2}^{ab})$. If $[\theta] = (b_{i,j;k,l})$ and $[\varphi] = (a_{ij})$ are the corresponding matrices, then the pair (θ, φ) is inducible if and only if

$$b_{i,j;k,l} = \begin{cases} a_{ii} + a_{jj}, & i = k,\ j = l, \\ a_{jl}, & i = k,\ j \neq l, \\ -a_{il}, & j = k,\ i \neq l, \\ a_{jk}, & i = l,\ j \neq k, \\ a_{ik}, & j = l,\ i \neq k, \\ 0, & \text{otherwise}. \end{cases} \tag{3.5}$$

Proof. For the free nilpotent Lie algebra $L_{n,2}$, we have the following central extension $0 \to L_{n,2}^{(1)} \to L_{n,2} \to L_{n,2}^{ab} \to 0$.
Let $t : L_{n,2}^{ab} \to L_{n,2}$ be the section given by $t(\bar{x_i}) = x_i$ on the generators. Then $\delta(\bar{x_i}, \bar{x_j}) = [x_i, x_j]$ for all $1 \leq i, j \leq n$. Since $L_{n,2}$ is two-step nilpotent, it follows that RHS of Lemma 2.3 is zero. Hence, we have

$$\theta(\delta(\bar{x_i}, \bar{x_j})) = \delta(\bar{x_i}, \varphi(\bar{x_j})) + \delta(\varphi(\bar{x_i}), \bar{x_j}).$$

In what follows, we show that this is precisely the condition (3.5).
$L_{n,2}$ is generated as a Lie algebra by $\{x_1, x_2, \ldots, x_n\}$. Moreover, the set

$$\{x_1, x_2, \ldots, x_n, z_{2,1}, z_{3,1}, \ldots, z_{n,n-1}\}$$

is an ordered basis for $L_{n,2}$ as a vector space over F. Thus, if $\gamma \in \mathrm{Der}(L_{n,2})$, then γ is given by

$$\gamma : \begin{cases} x_1 \longmapsto a'_{11}x_1 + a'_{12}x_2 + \cdots + a'_{1n}x_n + \beta_{1;2,1}z_{2,1} + \beta_{1;3,1}z_{3,1} + \cdots + \beta_{1;n,n-1}z_{n,n-1}, \\ \quad \vdots \\ x_n \longmapsto a'_{n1}x_1 + a'_{n2}x_2 + \cdots + a'_{nn}x_n + \beta_{n;2,1}z_{2,1} + \beta_{n;3,1}z_{3,1} + \cdots + \beta_{n;n,n-1}z_{n,n-1}. \end{cases}$$

for some $a'_{ij}, \beta_{i;k,l} \in F$. Now, suppose that (θ, φ) is inducible by γ. Then

$$\bar{\gamma} : \begin{cases} \bar{x}_1 \longmapsto a'_{11}\bar{x}_1 + a'_{12}\bar{x}_2 + \cdots + a'_{1n}\bar{x}_n, \\ \quad \vdots \\ \bar{x}_n \longmapsto a'_{n1}\bar{x}_1 + a'_{n,2}\bar{x}_2 + \cdots + a'_{n,n}\bar{x}_n. \end{cases}$$

Since $\bar{\gamma} = \varphi$, we obtain

$$a'_{ij} = a_{ij} \qquad \text{for all } 1 \leq i, j \leq n. \tag{3.6}$$

Next, we consider $\gamma|_{L_{n,2}^{(1)}}$ as follows:

$$\begin{aligned}
\gamma(z_{i,j}) &= \gamma([x_i, x_j]) = [\gamma(x_i), x_j] + [x_i, \gamma(x_j)] \\
&= [a'_{i1}x_1 + a'_{i2}x_2 + \cdots + a'_{in}x_n + \beta_{i;2,1}z_{2,1} + \cdots + \beta_{i;n,n-1}z_{n,n-1}, x_j] \\
&\quad + [x_i, \alpha'_{j1}x_1 + a'_{j2}x_2 + \cdots + a'_{jn}x_n + \beta_{j;2,1}z_{2,1} + \cdots + \beta_{j;n,n-1}z_{n,n-1}] \\
&= a'_{i1}[x_1, x_j] + a'_{i2}[x_2, x_j] + \cdots + a'_{in}[x_n, x_j] + a'_{j1}[x_i, x_1] + a'_{j2}[x_i, x_2] \\
&\quad + \cdots + a'_{jn}[x_i, x_n] \qquad \text{(since } L_{n,2} \text{ is two-step nilpotent)} \\
&= -a'_{i1}z_{j,1} - a'_{i2}z_{j,2} - \cdots - a'_{i(j-1)}z_{j,(j-1)} + 0 + a'_{i(i+1)}z_{(j+1),j} + \cdots + a'_{ii}z_{i,j} \\
&\quad + \cdots + a'_{in}z_{n,j} + a'_{j1}z_{i,1} + a'_{i2}z_{i,2} + \cdots + a'_{jj}z_{i,j} + \cdots + a'_{j(i-1)}z_{i,(i-1)} \\
&\quad + 0 - a'_{j(i+1)}z_{(i+1),i} - \cdots - a'_{jn}z_{n,i} \qquad (3.7)
\end{aligned}$$

By explicit computation and combining equations (3.7) and (3.6), we get (3.5). QED

Corollary 3.3. Let $(\theta, \varphi) \in \mathrm{Der}(L^{(1)}_{2,2}) \oplus \mathrm{Der}(L^{ab}_{2,2})$ and $(r, [\varphi])$ be the corresponding pair of matrices. Then the pair (r, φ) is inducible if and only if $r = a_{11} + a_{22}$.

Proof. In this case, the derived subalgebra $L^{(1)}_{2,2}$ is a one-dimensional free abelian algebra generated by $[x_2, x_1]$ and the abelianization $L^{ab}_{2,2}$ is a two-dimensional free abelian algebra with basis $\{\bar{x_1}, \bar{x_2}\}$. The proof now follows from (3.3). QED

We conclude with the following examples to show that the converse of Theorem 3.1 is not true in general.

Recall that a Lie algebra L is called Heisenberg provided that $L^2 = Z(L)$ and dim $L^2 = 1$. A Lie algebra H is called generalized Heisenberg of rank n if $H^2 = Z(H)$ and $\dim H^2 =$ n.

1) Let $0 \to Z(L) \to L \xrightarrow{\pi} L/Z(L) \to 0$ be an exact sequence, where L is a non-abelian finite dimensional generalized Heisenberg Lie algebra. Notice that this sequence does not split under the natural action of $L/Z(L)$ on $Z(L)$. If the sequence splits, then L is a direct sum of $Z(L)$ and $L/Z(L)$. This implies that L is abelian, which is a contraction. In this case, $\mathrm{Der}^{Z(L)}(L) = \mathrm{Der}^{Z(L)}_{Z(L)}(L) = \mathrm{Der}_z(L)$, where $\mathrm{Der}_z(L)$ is the set of all central derivations of L (see [14, 16]). Thus, from the exactness of sequence (3.1), $C^*_1 = 0$ and the sequence splits.

2) For the free nilpotent Lie algebra $L_{n,2}$, we consider the central of Lie algebras

$$0 \to L_{2,2}^{(1)} \to L_{2,2} \to L_{2,2}^{ab} \to 0.$$

Since $L_{2,2}$ is nonabelian, the sequence does not split. We show that the associated short exact sequence

$$0 \to \mathrm{Der}_{L_{2,2}^{(1)}}^{L_{2,2}^{(1)}}(L_{2,2}) \to \mathrm{Der}(L_{2,2} : L_{2,2}^{(1)}) \xrightarrow{\tau} C_{\alpha}^{*} \to 0 \tag{3.8}$$

splits. We define a section $\mu : C_{\alpha}^{*} \to \mathrm{Der}(L_{2,2} : L_{2,2}^{(1)})$, which is a Lie algebra homomorphism, showing that the sequence (3.8) splits. Define $\mu(\theta, \varphi) = \gamma$, where

$$\gamma : \begin{cases} x_1 \longmapsto t(\varphi(\bar{x_1})), \\ x_2 \longmapsto t(\varphi(\bar{x_2})), \\ [x_1, x_2] \longmapsto \theta([x_1, x_2]). \end{cases}$$

Then

$$\begin{aligned}
[\gamma(x_1), x_2] + [x_1, \gamma(x_2)] &= [t(\varphi(\bar{x_1})), x_2] + [x_1, t(\varphi(\bar{x_2}))] \\
&= [t(a_{11}\bar{x_1} + a_{12}\bar{x_2}), x_2] + [x_1, t(a_{21}\bar{x_1} + a_{22}\bar{x_2})] \\
&= [a_{11}x_1 + a_{12}x_2, x_2] + [x_1, a_{21}x_1 + a_{22}x_2] \\
&= (a_{11} + a_{22})[x_1, x_2] \\
&= \theta([x_1, x_2]) \qquad \text{(by Corollary 3.3, since } (\theta, \varphi) \text{ is inducible)} \\
&= \gamma([x_1, x_2])
\end{aligned}$$

It follows that $\gamma \in \mathrm{Der}(L_{2,2} : L_{2,2}^{(1)})$. Since $\tau(\gamma) = (\theta, \varphi), \mu$ is a section. It is easy to see that μ is a Lie algebra homomorphism. This show that the sequence (3.3) splits while $0 \to A \to L \xrightarrow{\pi} B \to 0$ does not split. For the sequence (3.2) is enough let $\theta = 0$ in the exact sequence of example (2) above.

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