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Hall Graph of a Finite Group

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Abstract. The Hall graph of a finite group G is a simple graph whose vertex set is $\pi(G)$, the set of all prime divisors of its order, and two distinct primes p and q are joined by an edge if G has at least one Hall $\{p, q\}$-subgroup. For all primes $p_1 < \cdots < p_k$ of $\pi(G)$, we call the k-tuple $\mathrm{D_H}(G) = (d_{\mathrm{H}}(p_1), \ldots, d_{\mathrm{H}}(p_k))$, the degree pattern of Hall graph of G, where $d_{\mathrm{H}}(p)$ signifies the degree of vertex p. This paper provides some properties of Hall graph. It also gives a characterization for some finite simple groups via order and degree pattern of Hall graph.

Keywords: Hall graph, degree pattern of Hall graph, Hall subgroup, simple group

MSC 2000 classification: primary 20D05, 20D06, 20D08, 20D10, secondary 05C25

Introduction

All groups under consideration of this paper are finite. Given a group G, we denote by $\pi(G)$ the set of all prime divisors of $|G|$ and call this set the *prime spectrum* of G. There are a lot of ways to study the structure of groups. One of the most interesting approaches is to associate a graph with a group. In this way, we get many information about a group via exploring its graph. Studying the prime graph associated with a group is one of such ways. In recent years, the prime graph has played an important role to solve many problems in group theory. This graph is a simple graph and constructs as follows. The vertex set is $\pi(G)$ and two distinct primes p and q form an edge $\{p, q\}$ if G has an element of order pq. In [2], the definition of prime graph was generalized as follows.

Let $\mathcal{P}$ be a group-theoretic property. Given a group G, $S_{\mathcal{P}}(G)$ is the set of all $\mathcal{P}$-subgroups of G. Let σ be a mapping of $S_{\mathcal{P}}(G)$ to the set of natural numbers. The $(\sigma, \mathcal{P})$-graph of G is a simple graph whose vertices are all primes dividing the elements of $\sigma(S_{\mathcal{P}}(G))$ and two vertices p and q are joined by an edge if there is a natural number in $\sigma(S_{\mathcal{P}}(G))$ which can be divided by pq. This graph is denoted by $\Gamma_{\sigma(\mathrm{S}_{\mathcal{P}}(\mathrm{G}))}$. If we consider the mapping "ord" as σ, then we simply call the $(\mathrm{ord}, \mathcal{P})$-graph of G, the $\mathcal{P}$-graph of G and denote it $\Gamma_{\mathcal{P}}(G)$. We illustrate it with some examples.

(1) Let $\mathcal{P}$ stand for "cyclic". Here $S_{\mathcal{P}}(G)$ is the set of all cyclic subgroups of

G and the $\mathcal{P}$-graph of G is called the "*cyclic graph*" of G which is denoted by $\Gamma_{cyc}(G)$. In fact, in the cyclic graph of G, the vertices are all primes dividing the order of G and two distinct vertices p and q are adjacent when G has a cyclic subgroup whose order is divisible by pq. It is good to note that the cyclic graph and the prime graph of a group are exactly one thing. Moreover, if we take "abelian" or "nilpotent" as $\mathcal{P}$, then $\mathcal{P}$-graph of G coincides with the cyclic graph.

(2) Let $\mathcal{P}$ be "solvable". In this case, $S_{\mathcal{P}}(G)$ is the set of all solvable subgroups of G and the $(\mathcal{P}, \sigma)$-graph of G is called the "*solvable graph*" of G which is denoted by $\Gamma_{sol}(G)$. We observe that the solvable graph of G is a generalization of the cyclic graph of G. In fact, the set of vertices is $\pi(G)$, like in the cyclic graph, but two distinct vertices p and q are adjacent when G has a solvable subgroup of order divisible by pq.

(3) Let $\mathcal{P}$ be a group property such that H is a $\mathcal{P}$-subgroup of G if H is a Hall $\pi(H)$-subgroup of G and $|\pi(H)| = 2$. We denote this graph by $\Gamma_{Hall}(G)$. In fact, in the Hall graph of G, the vertices are all primes dividing the order of G and two distinct vertices p and q are adjacent (or we say p and q are joined) when G has a Hall $\{p, q\}$-subgroup. According to this definition, we can see that the solvable graph of G is a generalization of the Hall graph of G.

Many information about the solvable graph associated with a group were found in [1, 2, 3]. In this paper, we are going to focus our attention on the Hall graph. Some results on this graph were obtained in [10].

For a generic group G, it is sometimes convenience to display the graph $\Gamma_{Hall}(G)$ (resp. $\Gamma_{sol}(G)$) in a compact form. The compact form is a graph whose vertices are the disjoint subsets of $\pi(G)$. In more detail, the vertex U represents the complete subgraph of $\Gamma_{Hall}(G)$ (resp. $\Gamma_{sol}(G)$) on U. Moreover, an edge connecting U and W represents the set of edges of $\Gamma_{Hall}(G)$ (resp. $\Gamma_{sol}(G)$) that connect each vertex in U with each vertex in W. We draw the compact form of the Hall and solvable graph of the Suzuki groups $\mathrm{Sz}(q)$ in the example below.

Example. We consider the Suzuki simple group defined over the field with $q = 2^{2m+1}$ $(m \geqslant 1)$ elements of order $q^2(q^2+1)(q-1)$. To drawing the Hall and solvable graph of $\mathrm{Sz}(q)$, we need to know the structure of maximal subgroups of $\mathrm{Sz}(q)$.

Every maximal subgroup of $\mathrm{Sz}(q)$ is isomorphic to one of the following (Suzuki [13]). It is good to mention that $A : B$ denotes a split extension.

$$\mathbb{Z}_{q^2} : \mathbb{Z}_{q-1}, \ \ \mathbb{Z}_{q-1} : \mathbb{Z}_2, \ \ \mathbb{Z}_{q+\sqrt{2q}+1} : \mathbb{Z}_4, \ \ \mathbb{Z}_{q-\sqrt{2q}+1} : \mathbb{Z}_4, \ \ \mathrm{Sz}(q_0), \ q = q_0^{\alpha}, \alpha \in \mathbb{Z}.$$

It is seen that all maximal subgroup of $\mathrm{Sz}(q)$ are solvable. Thus we can draw the solvable graph of $Sz(q)$ as in Figure 1.

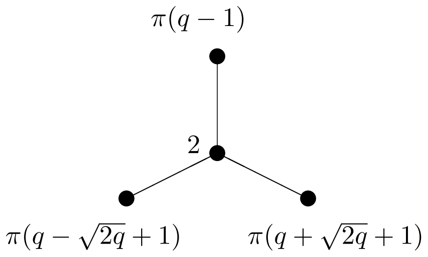

Fig. 1. $\Gamma_{sol}(\mathrm{Sz}(q)), q = 2^{2m+1} > 2^2$.

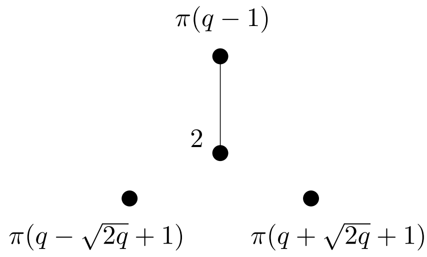

Fig. 2. $\Gamma_{Hall}(\mathrm{Sz}(q)), q = 2^{2m+1} > 2^2$.

Now, we examine the Hall graph of $\mathrm{Sz}(q)$. It is easily seen that the subsets $\pi(q-1)$, $\pi(q-\sqrt{2q}+1)$, $\pi(q+\sqrt{2q}+1)$ and $\{2\}$ are the disjoint subsets of $\pi(\mathrm{Sz}(q))$. Besides, $\mathrm{Sz}(q)$ has the cyclic subgroups of order $q-1$, $q-\sqrt{2q}+1$ and $q+\sqrt{2q}+1$. So we can easily conclude that the subsets above are the complete subgraph of $\Gamma_{Hall}(G)$ and hence they are exactly the vertices in the compact form of the Hall graph of $\mathrm{Sz}(q)$. Next, consider the subgroup $H = \mathbb{Z}_{q^2} : \mathbb{Z}_{q-1}$. Assume that $r \in \pi(q-1)$ and R is a Sylow r-subgroup of H (a Sylow r-subgroup of $\mathrm{Sz}(q)$ too). According to the structure of H, $\mathbb{Z}_{q^2}$ is a normal subgroup of H and so $R \subseteq N_{\mathrm{Sz}(q)}(\mathbb{Z}_{q^2})$ where $N_{\mathrm{Sz}(q)}(\mathbb{Z}_{q^2})$ is the normalizer of $\mathbb{Z}_{q^2}$ in $\mathrm{Sz}(q)$. Therefore, $R\mathbb{Z}_{q^2}$ is a Hall subgroup of $\mathrm{Sz}(q)$. It follows that the vertices 2 and r are adjacent in the Hall graph of $\mathrm{Sz}(q)$. Moreover, the maximal subgroups $\mathbb{Z}_{q-1} : \mathbb{Z}_2$, $\mathbb{Z}_{q+\sqrt{2q}+1} : \mathbb{Z}_4$, $\mathbb{Z}_{q-\sqrt{2q}+1} : \mathbb{Z}_4$ do not contain any Hall subgroups of G with even order because $2^{2n+1} \geqslant 8$. Finally, we can observe that the Hall graph of $\mathrm{Sz}(q)$ is as Figure 2.

More Notation and Terminology. A *simple graph* Γ with vertex set $V = V(\Gamma)$ and edge set $E = E(\Gamma)$ is a graph with no loops or multiple edges. A complete graph with n vertices is denoted by K_n. An empty graph on n vertices consists of n isolated vertices with no edges and denoted by $\overline{K}_n$. The union of simple graphs Γ_1 and Γ_2 is the graph $\Gamma_1 \cup \Gamma_2$ with vertex set $V(\Gamma_1) \cup V(\Gamma_2)$ and edge set $E(\Gamma_1) \cup E(\Gamma_2)$. If Γ_1 and Γ_2 are disjoint (we recall that two graphs are disjoint if they have no vertex in common), we refer to their union as a disjoint union, and generally denote it by $\Gamma_1 \oplus \Gamma_2$.

According to [9], we say a group G satisfies E_π (or we just say G is an E_π-group) if G contains a Hall π-subgroup.Obviously, in this case when $\pi \cap \pi(G)$, for the Hall π-subgroup is trivial. Using our definition, two primes p and q are adjacent in the Hall graph of G if G satisfies $E_{p,q}$. If G satisfies E_π and every two Hall π-subgroups are conjugate, then we say that G satisfies C_π (or just that G is an C_π-group). A prime $p \in \pi(G)$ is called a *complete prime* if p is joined with any other vertices in the Hall graph of G.

The *degree* $d_{\mathrm{H}}(p)$ of a vertex $p \in \pi(G)$ is the number of adjacent vertices to p in $\Gamma_{Hall}(G)$. We suppose that

$$\pi(G) = \{p_1, p_2, \ldots, p_k\},$$

is the prime spectrum of G, where

$$p_1 < p_2 < \cdots < p_k,$$

and define

$$\mathrm{D_H}(G) = \Big(d_{\mathrm{H}}(p_1), d_{\mathrm{H}}(p_2), \ldots, d_{\mathrm{H}}(p_k)\Big),$$

as the *degree pattern of the Hall graph of* G. Of particular interest can be to find finite groups which are determined by order and degree pattern of Hall graph. In general, given a finite group G, we are interested in finding the structure of finite groups H such that $|H| = |G|$ and $\mathrm{D_H}(H) = \mathrm{D_H}(G)$. One of the purposes of this paper is to characterize some simple groups by order and degree pattern of Hall graph. In more detail, we will prove the following theorems.

Theorem A. *The finite simple group G with $|\pi(G)| = 3$ and $G \neq U_4(2)$ is completely determined by order and degree pattern of Hall graph.*

Theorem B. *There are at least* 15 *non-isomorphic groups G with $|G| = |U_4(2)|$ and $\Gamma_{Hall}(G) = \Gamma_{Hall}(U_4(2))$.*

1 Some Results on Hall Graph

In this section, we present some results on the Hall graphs of finite groups. We begin with a fundamental lemma which is taken from [9].

Lemma 1. ([9, Lemma 1]) *Let G be a finite group, and suppose that N is a normal subgroup of G. If H is a Hall π-subgroup of G, then $H \cap N$ is a Hall π-subgroup of N, and HN/N is a Hall π-subgroup of G/N.*

In view of Lemma 1, if G satisfies E_π where $\pi \subseteq \pi(N)$, then N satisfies E_π, too. Moreover, if G satisfies E_π where $\pi \subseteq \pi(G/N)$, then the factor group G/N satisfies E_π.

Lemma 2. *Let G be a finite group and N a normal subgroup of G. Then the following statements hold:*

(1) *For two primes $p, q \in \pi(N)$, if p and q are joined in $\Gamma_{Hall}(G)$, then p and q are joined in $\Gamma_{Hall}(N)$. In particular, $\Gamma_{Hall}(G)$ is a subgraph of $\Gamma_{Hall}(N)$ if $\pi(N) = \pi(G)$.*

(2) *For two primes $p, q \in \pi(G/N)$, if p and q are joined in $\Gamma_{Hall}(G)$, then p and q are joined in $\Gamma_{Hall}(G/N)$. In particular, $\Gamma_{Hall}(G)$ is a subgraph of $\Gamma_{Hall}(G/N)$ if $\pi(G/N) = \pi(G)$.*

(3) *If N is a nilpotent Hall subgroup of G, then for $p \in \pi(N)$ and $q \in \pi(G)\backslash\pi(N)$, p and q are joined in $\Gamma_{Hall}(G)$.*

Proof. Parts (1) and (2) follow immediately from Lemma 1. So we only need to prove part (3). It is seen from Schur-Zassenhaus Theorem that there exists a subgroup K of G such that $G = KN$ and $K \cap N = 1$. On the other hand, any Sylow p-subgroup P of N is a normal Sylow p-subgroup of G because N is nilpotent. Now, assume that Q is a Sylow q-subgroup of K. Then it is easy to see that PQ is a Hall subgroup of G which implies that p and q are joined in $\Gamma_{Hall}(G)$. QED

It is clear that for any subgroup H of finite group G, $\Gamma_{Hall}(H)$ does not need to be a subgraph of $\Gamma_{Hall}(G)$. In the case when H is a Hall subgroup of G, it is easily seen that $\Gamma_{Hall}(H)$ is a subgraph of $\Gamma_{Hall}(G)$.

The following lemma is known.

Lemma 3. ([12, Lemma 3. 1]) *Let G be a finite group, and assume that N is a nilpotent normal subgroup of G. Then G satisfies E_π (resp. C_π) if and only if G/N satisfies E_π (resp. C_π).*

As a result of Lemma 3, we have the following corollary.

Corollary 1. *Let G be a finite group, and let N be a nilpotent normal subgroup of G. Then for two primes $p, q \in \pi(G/N)$, p and q are joined in $\Gamma_{Hall}(G/N)$ if and only if p and q are joined in $\Gamma_{Hall}(G)$.*

Next, we present some results that are needed in Section 3.

Lemma 4. ([9, Corollary E2. 1]) *Let G be a finite group, and suppose that π is a set of primes. If all composition factors of G satisfy C_π, then G satisfies E_π.*

Lemma 5. [7] *Suppose the finite group G has a Hall π-subgroup where π is a set of primes not containing 2. Then all Hall π-subgroups of G are conjugate.*

It is seen from Lemma 5 that in the case when $2 \notin \pi$, the group G satisfies E_π if and only if G satisfies C_π. The following corollary is a conclusion of Lemma 4 and Lemma 5.

Corollary 2. *A finite group G satisfies E_π for a set of primes not containing 2, if and only if every composition factor of G satisfies E_π.*

We now bring one of the main results of [8] which will be used in Section 3.

Lemma 6. ([8]) *Let G be the classical simple group $L_n(q)$ or $S_{2n}(q)$. Assume that G has a Hall π-subgroup with $3 \notin \pi$. Then all Hall π-subgroups of G are conjugate in G.*

It is found from Lemma 6 that for a set of primes not containing 3, the classical groups $L_n(q)$ and $S_{2n}(q)$ satisfy E_π, if and only if they satisfy C_π.

It was shown in [10] that there is no vertex in Hall graph of a finite simple group which is joined with every vertex except for one case. In fact, the following lemma was proved.

Lemma 7. [10] *Let G be a finite non-abelian simple group with $p \in \pi(G)$ which is joined directly with any prime belonging to $\pi(G)$ in $\Gamma_{\mathrm{Hall}}(G)$. Then the only case is $G = L_2(7)$ with $p = 3$.*

We get the following corollary from Lemma 7.

Corollary 3. *Let G be a finite simple group and $G \neq L_2(7)$. Then the Hall graph of G has no complete prime.*

Among non-simple groups, there are some groups whose Hall graphs have complete primes. In the sequel, we consider some case when the non-simple group G has a complete prime.

We first need to recall that a finite group G is p-solvable if each of its composition factors is either a p'-group or is a solvable p-group. Note that a finite group is solvable if and only if it is p-solvable for every prime p.

Du proved the following lemma in [6].

Lemma 8. ([6, Corollary 2]) *Let G be a finite group and $p \in \pi(G)$. Then G is a p-solvable group if and only if*

(1) *G satisfies $E_{p'}$;*

(2) *G satisfies $E_{p,q}$ for all $q \in \pi(G)$.*

The following corollary is a straightforward result of Lemma 8.

Corollary 4. *Let G be a p-solvable group for a prime p. Then p is a complete prime.*

Remark 1. Let G be a finite group and $p \in \pi(G)$. If G has a normal Sylow p-subgroup, then it is obvious that G is a p-solvable group and so p is a complete prime. But, the converse is not always true. In fact, any solvable group has Hall subgroups for any set of primes. The symmetric group $\mathbb{S}_3$,with $p = 2$, is already a counterexample.

There are a lot of ways to study solvable groups. In fact, there are several theorems stating equivalent conditions for solvability. P. Hall proved that G is solvable if and only if G satisfies E_π for all $\pi \subseteq \pi(G)$. More precisely, Hall verified a stronger statement which asserts that G is solvable, if and only if G satisfies $E_{p'}$ for all $p \in \pi(G)$. He also conjectured that G is solvable, if and only if G satisfies $E_{p,q}$ for all $p, q \in \pi(G)$ (see [9]). In the case when G is not a simple group, this conjecture is easy to show. So it is worth to investigate that, when G is a non-abelian simple group. In fact, Hall proved that if G is an alternating group $\mathbb{A}_n$ ($n \geqslant 5$), the conjecture still holds. Finally, Arad and Ward in [4], using the classification of finite simple groups, proved Hall's conjecture.

In our approach, we can express the assertion above in the following.

Lemma 9. *The group G is solvable if and only if $\Gamma_{Hall}(G)$ is complete.*

Proof. It is enough to consider the sufficiency. Suppose that G is a counterexample of minimal order to this statement. Assume first that G is not a simple group. Then there exists a nontrivial normal subgroup N of G. Since $\Gamma_{Hall}(G)$ is complete, we conclude from Lemma 2 that $\Gamma_{Hall}(N)$ and $\Gamma_{Hall}(G/N)$ are complete. Now by the hypothesis, N and G/N are solvable. It follows that G is solvable which contradicts our assumption. Consequently, G must be a non-abelian simple group. In this case, we can get a contradiction from Lemma 7. Therefore, G is a solvable group. QED

2 The Hall Graphs of Some Finite Groups

In this section, we examine the Hall graphs of some groups. We first determine the Hall graph of a Frobenius group. To this aim, it is good to mention that a finite group G is called a *Frobenius group* with kernel N and complement K, if $G = NK$ where N is a normal subgroup and K a subgroup of G, and for all $1 \neq g \in N$, $C_G(g) \subseteq N$.

Lemma 10. *Let G be a Frobenius group. Then, one of the following statements holds:*

(1) *G is solvable and $\Gamma_{Hall}(G) = K_{|\pi(G)|}$.*

(2) *G is non-solvable and $\Gamma_{Hall}(G)$ is obtained from the complete graph on $\pi(G)$ by deleting the edges $\{3,5\}, \{2,5\}$ and $\{2,3\}$ or $\{3,5\}$ and $\{2,5\}$.*

Proof. (1) This is a result obtained by P. Hall in [9]. (2) Suppose that $G = NK$ is a Frobenius group with kernel N and complement K. It is clear that $\pi(N)$ and $\pi(K)$ have no common vertex.

First let $p, q \in \pi(N)$. According to the structure of Frobenius groups, N is a nilpotent group. Thus there exists a Hall $\{p, q\}$-subgroup H of N. On the other hand, N is a Hall subgroup of G. It implies that H is a Hall $\{p, q\}$-subgroup of G. Hence, p and q are joined in $\Gamma_{Hall}(G)$.

Assume next that $p \in \pi(N)$ and $q \in \pi(K)$. In this case, we can see from Lemma 2 (3) that p and q are joined in $\Gamma_{Hall}(G)$.

Finally, suppose that $p, q \in \pi(K)$. Note that K is non-solvable as G is non-solvable. Thus, by the structure of non-solvable complement, K has a normal subgroup K_0 with $|K : K_0| \leqslant 2$ such that $K_0 = \mathrm{SL}(2,5) \times Z$, where every Sylow subgroup of Z is cyclic and $\pi(Z) \cap \pi(30) = \emptyset$ (see Theorem 18.6 in [11]). If $p, q \notin \{2, 3, 5\}$, then p and q are joined in $\Gamma_{Hall}(K_0)$ because obviously every Hall $\{p, q\}$-subgroup of Z is a Hall $\{p, q\}$-subgroup of G. We also easily observe from Lemma 2 (3) that if $p \in \pi(Z)$, then p is joined with 2, 3 and 5. Moreover, 2 and 3 are adjacent if $K = K_0$. This completes the proof. QED

In the sequel, we verify the Hall graphs of the symmetric and alternating groups. All Hall subgroups of symmetric groups have completely found. In more detail, P. Hall determined all solvable Hall subgroups of the symmetric groups in [9, Theorem A4] and G. Thompson found the non-solvable case in [14]. We state them in two following lemmas.

Lemma 11. [9] Let $\mathbb{S}_n$ be the symmetric group on n letters, and assume that p and q are two primes where $p < q \leqslant n$. Then $\mathbb{S}_n$ satisfies $E_{p,q}$ only when $p = 2$, $q = 3$, and $n = 3$, 4, 5, 7, and 8.

Lemma 12. [14] Suppose that H is a non-solvable Hall subgroup of the symmetric group $\mathbb{S}_n$. Then either $H = \mathbb{S}_n$ or $H = \mathbb{S}_{n-1}$ and n is a prime.

According to Lemma 11, we can find the structure of Hall graphs of symmetric groups in the following corollary.

Corollary 5. *The Hall graph of symmetric group* $\mathbb{S}_n$ *is as follows:*

(1) *The disjoint union of graphs* $K_{\{2,3\}}$ *and the empty graph with* $\pi(G)\backslash\{2, 3\}$ *as vertex set if* $3 \leqslant n \leqslant 8$*;*

(2) *The empty graph with* $\pi(G)$ *as vertex set if* $n > 8$.

Besides, the following lemma was proved about alternating groups in [12].

Lemma 13. *Let* $\mathbb{S}_n$ *and* $\mathbb{A}_n$ *be the symmetric and alternating groups on* n *letters, respectively. Then the following statements are equivalent:*

(1) $\mathbb{S}_n$ *satisfies* E_π*;*

(2) $\mathbb{S}_n$ *satisfies* C_π*;*

(3) $\mathbb{A}_n$ *satisfies* E_π*;*

(4) $\mathbb{A}_n$ *satisfies* C_π.

It is found form Lemma 13 that the Hall graph of the alternating group $\mathbb{A}_n$ coincides with the Hall graph of symmetric group $\mathbb{S}_n$.

Finally, we discuss about the Hall graphs of sporadic groups. It is worthwhile to mention that all Hall subgroups of sporadic groups were found in [5]. So the Hall graphs of these groups are completely determined. In fact, the Hall graphs of groups HS, M^cL, Suz, Fi_{22}, He, J_2, J_3, HN, Th are empty graphs. For other sporadic simple groups S, we tabulate $|S|$ and $\mathrm{D_H}(S)$ in Table 1.

Table 1. *The order and degree pattern of Hall graph of a sporadic simple group.*

S	$\|S\|$	$\mathrm{D_H}(S)$
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	$(1, 1, 1, 1)$
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	$(0, 0, 1, 1)$
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	$(0, 0, 1, 0, 1)$
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	$(2, 4, 2, 2, 1, 1)$
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$(1, 1, 1, 0, 2, 1)$
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$(0, 0, 1, 0, 2, 1)$
Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	$(0, 0, 0, 0, 1, 1)$
Co_2	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	$(0, 0, 0, 0, 1, 1)$
Fi_{23}	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$	$(0, 0, 0, 0, 1, , 0, 0, 1)$
Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$	$(0, 0, 0, 0, 1, 0, 1)$
Ru	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$	$(0, 0, 0, 1, 0, 1)$
Fi'_{24}	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$	$(0, 0, 0, 0, 1, 0, 0, 1, 0)$
$O'N$	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$	$(0, 1, 3, 0, 1, 0, 1)$
J_4	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$	$(0, 0, 3, 3, 1, 0, 1, 1, 0, 1)$
B	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$	$(0, 0, 0, 0, 1, 0, 0, 0, 2, 0, 1)$
Ly	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$	$(0, 0, 0, 0, 1, 0, 0, 1)$
M	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot$ $41 \cdot 47 \cdot 59 \cdot 71$	$(0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0,$ $0,\ 1,\ 1,\ 0)$

3 Characterization of Some Simple Groups by Order and Degree Pattern of Hall Graph

In this section, we are going to examine the characterization of simple group S with $|\pi(S)| = 3$ by order and degree pattern of Hall graph. In fact, by the classification of finite non-abelian simple groups, all simple groups whose orders have three primes are found. We can see the list of these groups in [15]. All Hall subgroups of these groups were found in [5]. For convenience, in Table 2, we tabulate $|S|$ and $\mathrm{D_H}(S)$ for simple groups S with $|\pi(S)| = 3$. In the following theorem, we show that any simple group S with $|\pi(S)| = 3$ and $S \neq U_4(2)$ is completely determined by its order and degree Pattern of Hall Graph.

Theorem 1. *Let G be a finite group and S a simple group with $|\pi(S)| = 3$ and $S \neq U_4(2)$. Then G is isomorphic to S if and only if $|G| = |S|$ and $\mathrm{D_H}(G) = \mathrm{D_H}(S)$.*

Proof. We need only prove the sufficiency. Let G be a finite group satisfying the conditions $|G| = |S|$ and $\mathrm{D_H}(G) = \mathrm{D_H}(S)$, where S is one of the groups in Table 2 except for $U_4(2)$. First, it is necessary to mention that G is a non-solvable group since otherwise $\Gamma_{Hall}(G)$ is a complete graph which is false. It follows that G has a non-abelian composition factor H/L where $L \lhd H \leqslant G$. We can observe that H/L is one of the simple groups in Table 2 such that $|H/L| \leqslant |S|$. It will be convenient to consider two cases separately:

Case 1. *Let S be one the following simple groups: $\mathbb{A}_5$, $L_2(7)$, $L_3(3)$, $L_2(17)$.* Then according to Table 2, the only possibility for H/L is S. On the other hand, $|G| = |S|$ which implies that G is isomorphic to S.

Case 2. *Let S be one the following simple groups: $\mathbb{A}_6$, $L_2(8)$, $U_3(3)$.* We verify these groups case by case.

- $S = \mathbb{A}_6$. It is seen from [5] that $\mathbb{A}_6$ has no Hall $\{p, q\}$-subgroup where $p, q \in \{2, 3, 5\}$. It yields that $\mathrm{D_H}(\mathbb{A}_6) = (0, 0, 0)$ (as we see in Table 2). We have by assumption that $\mathrm{D_H}(G) = \mathrm{D_H}(\mathbb{A}_6)$. Considering the order of G, we use from Table 2 and conclude that two possibilities for H/L exist: $\mathbb{A}_5$ or $\mathbb{A}_6$. Suppose that $H/L \cong \mathbb{A}_5$. In this case, it is easy to see that other composition factors of G are abelian simple groups. On the other hand, $\mathbb{A}_4$ is a Hall $\{2, 3\}$-subgroup of $\mathbb{A}_5$ which means that $\mathbb{A}_5$ satisfies $E_{2,3}$. Now, it follows from Lemma 13 that $\mathbb{A}_5$ satisfies $C_{2,3}$. Then we have from Lemma 4 that G satisfies $E_{2,3}$ which implies that the vertices 2 and 3 are adjacent in $\Gamma_{Hall}(G)$, but it is false. Therefore, $H/L \cong \mathbb{A}_6$ which yields that $G \cong \mathbb{A}_6$.

- $S = L_2(8)$. According to [5], the maximal subgroups of $L_2(8)$ are as follows: D_{18}, D_{14} and $2^3 : 7$. Thus, there is no Hall π-subgroup of G where $\pi = \{2, 3\}$ or $\pi = \{3, 7\}$. Moreover, $2^3 : 7$ is a Hall subgroup of G. So we observe that $\mathrm{D_H}(L_2(8)) = (1, 0, 1)$ (see Table 2). We have by the hypothesis that $\mathrm{D_H}(G) = \mathrm{D_H}(L_2(8))$ which implies that $\Gamma_{Hall}(G) = \Gamma_{Hall}(L_2(8))$. Obviously, $3, 7 \in \pi(H/L)$ and we can see from Corollary 2 that 3 is not adjacent to 7 in $\Gamma_{Hall}(H/L)$. Using Table 2, the possibilities for H/L are $L_2(7)$ and $L_2(8)$. Assume that $H/L \cong L_2(7)$. Since $L_2(7)$ has a Hall $\{3,7\}$-subgroup, therefore the vertices 3 and 7 are joined in $\Gamma_{Hall}(L_2(7))$ that is not true. Consequently, $H/L \cong L_2(8)$ which follows that $G \cong L_2(8)$.

- $S = U_3(3)$. We observe from [5] that $\mathrm{D_H}(U_3(3)) = (0,0,0)$. By assuming that $\mathrm{D_H}(G) = \mathrm{D_H}(U_3(3))$, G has no Hall $\{p,q\}$-subgroup where $p,q \in \{2,3,7\}$. On the other hand, it is seen from Table 2 that the possibilities for H/L are $L_2(7)$, $L_2(8)$ and $U_3(3)$. Let first $H/L \cong L_2(7)$. Since $L_2(7)$ satisfies $E_{3,7}$, we can conclude from Corollary 2 that G satisfies $E_{3,7}$. Therefore, 3 and 7 are adjacent in $\Gamma_{Hall}(G)$ which is a contradiction. Assume next that $H/L \cong L_2(8)$. We notice that $L_2(8)$ has a Hall $\{2,7\}$-subgroup. So we have from Lemma 6 that $L_2(8)$ satisfies $C_{2,7}$. Now, it follows from Lemma 4 that G satisfies $E_{2,7}$ that is false. Therefore, $H/L \cong U_3(3)$ which implies that $G \cong U_3(3)$.

Consequently, the proof is complete. *QED*

In the following theorem, we show that the projective special unitary group $U_4(2)$ does not characterize by order and degree pattern of Hall graph. More precisely, there exist some groups G not isomorphic to $U_4(2)$ with $|G| = |U_4(2)|$ and $\Gamma_{Hall}(G) = \Gamma_{Hall}(U_4(2))$.

Theorem 2. *There are at least* 15 *non-isomorphic groups* G *with* $|G| = |U_4(2)|$ *and* $\Gamma_{Hall}(G) = \Gamma_{Hall}(U_4(2))$.

Proof. Let G be a finite group satisfying the conditions $|G| = |U_4(2)|$ and $\Gamma_{Hall}(G) = \Gamma_{Hall}(U_4(2))$. By a similar way to Theorem 1, we can see that G has a non-abelian composition factor H/L where $L \lhd H \leqslant G$. We can observe from Table 2 that the possibility for H/L is one of the simple groups $\mathbb{A}_5$, $\mathbb{A}_6$ and $U_4(2)$.

- If $H/L \cong \mathbb{A}_5$, then other composition factors of G are abelian simple groups. On the other hand, $\mathbb{A}_5$ satisfies $C_{2,3}$. Now we use Lemma 4 and conclude that G satisfies $E_{2,3}$ which follows that the vertices 2 and 3 are adjacent in $\Gamma_{Hall}(G)$ that is impossible.

- Suppose that $H/L \cong \mathbb{A}_6$. Let G be any group containing the symmetric group $\mathbb{S}_6$ as a normal subgroup with index 36. For example, we can take $\mathbb{S}_6 \times K$ where K is any group of order 36. Then $|\mathbb{S}_6 \times K| = |U_4(2)|$. Note that $\mathbb{S}_6$ has no Hall subgroup for any two of the primes 2, 3 and 5. It follows from Lemma 1 that the same is true for $\mathbb{S}_6 \times K$. Therefore, $\Gamma_{Hall}(\mathbb{S}_6 \times K) = \Gamma_{Hall}(U_4(2))$. On the other hand, there exists 4 abelian groups of order 36 and 10 non-abelian solvable groups of order 36.

- If $H/L \cong U_4(2)$, then $G \cong U_4(2)$.

This completes the proof. $\boxed{QED}$

Table 2. *The order and degree pattern of simple groups G with $|\pi(G)| = 3$.*

S	$\lvert S\rvert$	$D_H(S)$
$\mathbb{A}_5 \cong L_2(4) \cong L_2(5)$	$2^2 \cdot 3 \cdot 5$	$(1,1,0)$
$\mathbb{A}_6 \cong L_2(9)$	$2^3 \cdot 3^2 \cdot 5$	$(0,0,0)$
$S_4(3) \cong U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	$(0,0,0)$
$L_2(7) \cong L_3(2)$	$2^3 \cdot 3 \cdot 7$	$(1,2,1)$
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	$(1,0,1)$
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	$(0,0,0)$
$L_3(3)$	$2^4 \cdot 3^3 \cdot 13$	$(0,0,0)$
$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	$(0,0,0)$

References

[1] S. Abe: *A characterization of some finite simple groups by orders of their solvable subgroups*, Hokkaido Math. J., **31**, n. 2, 2002, 349–361.

[2] S. Abe and N. Iiyori: *A generalization of prime graphs of finite groups*, Hokkaido Math. J., **29**, n. 2, 2000, 391–407.

[3] B. Akbari, N. Iiyori and A. R. Moghaddamfar: *A new characterization of some simple groups by order and degree pattern of solvable graph*, Hokkaido Math. J., **45**, n. 3, 2016, 337–363.

[4] Z. Arad and M. B. Ward: *New criteria for the solvability of finite groups*, J. Algebra, **77**, n.1, 1982, 234–246.

[5] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson: Atlas of Finite Groups, Clarendon Press, Oxford, 1985.

[6] Zh. Du: *Hall subgroups and π-separable groups*, J. Algebra, **195**, n. 2, 1997, 501–509.

[7] F. Gross: *Conjugacy of odd order Hall subgroups*, Bull. London Math. Soc., **19**, n. 4, 1987, 311–319.

[8] F. Gross: *Hall subgroups of order not divisible by* 3, Rocky Mountain J. Math., **23**, n. 2, 1993, 569–591.

[9] P. Hall: *Theorem like Sylow's*, Proc. Lond. Math. Soc, **6**, n. 3, 1956, 286–304

[10] N. Iiyori: *p-solvability and a generalization of prime graphs of finite groups*, Comm. Algebra, **30**, n. 4, 2002, 1679–1691.

[11] D. S. Passman: Permutation Groups, W. A. Benjamin Inc., New York, 1968.

[12] D. O. Revin and E. P. Vdovin: *Hall subgroups of finite groups*, Contemp. Math., Israel Math. Conf. Proc., Amer. Math. Soc., Providence, RI, 2006, 229–263.

[13] M. Suzuki: *On a class of doubly transitive groups*, Ann. of Math.(2), **75**, 1962, 105–145.

[14] John G. Thompson: *Hall subgroups of the symmetric groups*, J. Combinatorial Theory 1, 1966, 271–279.

[15] A. V. Zavarnitsine: *Finite simple groups with narrow prime spectrum*, Sib. Elektron. Mat. Izv., **6**, 2009, 1–12.