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Dendrites or lambda-dendroids as generalized inverse limits

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DENDRITES OR LAMBDA-DENDROIDS AS GENERALIZED INVERSE LIMITS

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ABSTRACT

This research centers on the study of generalized inverse limits. We show that all members of an infinite family of inverse limit spaces are homeomorphics to one particularly complicated inverse limit space known as "The Monster". Further, properties of factor spaces and graphs of bonding functions which are preserved in generalized inverse limit spaces with upper semi-continuous bonding functions with appropriate restrictions are investigated. Some of the properties are locally connectedness, hereditary decomposability, hereditary indecomposability, hereditary unicoherence, arc-likeness, and tree-likeness. The theorems are illustrated by several examples.

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1. INTRODUCTION

The study of inverse limits date back to the 1920's. In 1954, C. E. Capel. published a paper about investigating inverse limits [6]. He showed that the inverse limit space of arcs with monotone bonding maps is an arc. In 1959 R. D. Anderson and Gustave Choquet showed that inverse limits can be used in describing complicated examples [2]. They created, using inverse limits, an example of a planer tree-like continuum no two of whose nondegenerate subcontinua are homeomorphic. G. W. Henderson showed that the pseudo-arc can be constructed as an inverse limit on the interval with a single bonding map [15]. Because of example like these, inverse limits became a powerful tool for constructing complicated continua in the study of continua. Today people know a lot about inverse limits when bonding functions are single valued mapping. In 2004, W. S. Mahavier introduced generalized inverse limits which is the inverse limit with set valued upper semi-continuous functions as bonding functions [39]. The publication of the book by W. T. Ingram and W. S. Mahavier in 2010 [21], helped create a great deal of interest in the study of generalized inverse limits. One such area of research is to understand what properties of the factor spaces or graphs of the bonding functions are preserved in inverse limit spaces having upper semi-continuous bonding functions.

We start with the definition of a continuum and some subtypes. Most of these definitions can be found in [33] and [21]. A *continuum* X is non-empty, compact, connected, metric space X. Some researchers do not require a continuum be metric only Hausdorff. We will not use this more general meaning for continuum in this dissertation. A *sub-continuum* of a continuum X is a continuum subset of X. A continuum is said to be *decomposable* if it is the union of two proper sub-continua otherwise, it is *indecomposable*. An *arc* is a continuum that is homeomorphic to a closed interval. A simple closed curve is a continuum that is homeomorphic to a circle. A continuum M is a *triod* if M contains a subcontinuum K such that $M - K$ has at least three components. The subcontinuum K is called a *core* of the triod. An *atriodic* continuum is one which does not contain a triod. A continuum X is said to be *irreducible* if there are two points p and q in X such that no proper subcontinuum of X contains both p and q. A continuum X is said to be irreducible between closed subsets A and B if X intersects each of A and B but no proper subcontinuum of X does. A continuum X is *unicoherent* if any two subcontinua of X whose union is X have a connected intersection. A continuum is called *hereditarily unicoherent* if each of its subcontinua is unicoherent. A continuum X is said to be a *dendrite* if it is locally connected and contains no simple closed curve, it is a *dendroid* if it is arcwise connected and hereditarily unicoherent. A hereditarily decomposable and hereditarily unicoherent continuum is said to be a *λ-dendroid*. A subcontinua K of a compact metric space X is called a *terminal continuum* if every subcontinua of X which intersects K and its complement contains K. Note this definition of a terminal continuum agrees with T. Maćkowiak's usage [31]. W. T. Ingram allows K to be a subset, not necessarily a subcontinua, of X and calls such sets C-sets. For him terminal continua have a different meaning he says that: A subcontinua K of a compact metric space X is called a *terminal continuum in the notion of Ingram* if A and B are subcontinua of X each intersecting K then $A \subseteq A \cup K \quad \text{or} \quad B \subseteq A \cup K$ see [17].

We start with the necessary definiton of mapping and its type. A continuous function from a topological space X to a topological space Y is called a *map* or *mapping*. A mapping f from X onto Y is said to be *monotone (atomic)* if the inverse image of any point of Y is a subcontinuum (terminal subcontinuum) of X. Given continua X and Y and $\epsilon > 0$, a mapping $f : X \longrightarrow Y$ is called an *ϵ-map* if for each $y \in Y$, $\operatorname{diam}(f^{-1}(y)) < \epsilon$. A continuum X is said to be *arc-like (chainable)* if for every $\epsilon > 0$, there exists an ϵ-map $f_\epsilon : X \longrightarrow [0, 1]$.

Let X be a topological space, we define the following hyperspaces of X: $2^X = \{A \subseteq X : A \neq \emptyset$, closed and compact$\}$ and $C(X) = \{A \in 2^X : A$ connected$\}$. If X is a continuum with a metric d, $\epsilon > 0$, and $A \in 2^X$ then we define the *ϵ-neighborhood* of A by $N_d(\epsilon, A) = \{x \in X, \quad d(x,a) < \epsilon$ for some $a \in A\}$. If $A, B \in 2^X$, then define the *Hausdorff distance* H_d by the formula, $H_d(A,B) = \inf\{\epsilon > 0, \quad A \subset N_d(\epsilon, B) \quad \text{and} \quad B \subset N_d(\epsilon, A)\}$. Let X and Y be continua and $x \in X$, a function $f : X \to 2^Y$ is an *upper semi-continuous at x* provided that for all open sets V in Y which contain $f(x)$, there exist an open set U in X with $x \in U$ such that if $t \in U$, then $f(t) \subseteq V$. If a function $f : X \to 2^Y$ is upper semi-continuous at x for each $x \in X$, we say that f is upper semi-continuous (usc). In the case where both X and Y are compact metric spaces then f is an upper semi-continuous (usc) if and only if the graph of f, $\text{Graph}(f) = \{(x,y) : y \in f(x)\}$, is closed in $X \times Y$ [16, p. 3].

Given a sequence of continua X_n and upper semi-continuous functions $F_n : X_{n+1} \to 2^{X_n}$, the *inverse limit* of F_n is defined by $\varprojlim \{X_n, F_n\} = \{(x_1, x_2, \ldots) : x_n \in F_n(x_{n+1})\}$ where the topology is the subspace topology of the product topology on ΠX_n. Similarly, when X_n and F_n are known, and $i \leq j$, we denote by $G_{i,j} = \{(x_i, x_{i+1}, \ldots, x_j) : x_k \in F_k(x_{k+1})$ for $k \in \{i, i+1, \ldots, j-1\}\}$. If $i = j$, we identify $G_{i,i}$ with X_i and $G_{1,\infty}$ denotes $\varprojlim \{X_n, F_n\}$. For a natural number n let $\alpha_n : \text{Graph}(F_n) \to X_n$ and $\beta_n : \text{Graph}(F_n) \to X_{n+1}$ be the projections. For m, i, j, n such that $m \leq i \leq j \leq n \leq \infty$ we denote by $\rho_{i,j}^{m,n}$ the projection $\rho_{i,j}^{m,n} : G_{m,n} \to G_{i,j}$. If $i = j$ we write $\rho_i^{m,n}$ in place of $\rho_{i,i}^{m,n}$. Note that for each natural number i, the set $G_{i,i+1}$ is equal to the $\text{Graph}(F_i^{-1})$. We also denote by π_k the projection from $\varprojlim\{X_n, F_n\}$ to X_k.

In this dissertation, we studied generalized inverse limits in three papers. In the first paper, We are interested in the family of upper semi-continuous functions $f_a : [0,1] \to [0,1]$ and the corresponding inverse limit spaces $X_a = \varprojlim\{[0,1], f_a\}$, where the graph, $\text{Graph}(f_a)$ is the union of the line segments from $(0,0)$ to $(a,1)$ to $(1,a)$ to $(1,0)$ for $a \in [0,1]$. For $a \in (0,1)$, f_a is a generalized upper semi-continuous (usc) Markov function and it follows from results of Banić and Lunder [4] that if $a, b \in (0,1)$ then X_a homeomorphism to X_b. But for $a \in (0,1)$, X_a and X_1 are not homeomorphic since the first contains the topologists sine curve as a subcontinuum and the second is the harmonic fan. The functions f_a, $a \neq 0$ and f_0 do not satisfy the hypothesis of Banić and Lunder's theorem so we may ask, are $X_{1/2}$ and X_0 homeomorphic? In his Master's thesis Jacobsen [23] studied $X_{1/2}$ where he showed that it contained $2^{\aleph_0}$ arc components and each arc component is dense. The space X_0 is often referred to as 'the Monster', a name reportedly coined by Banić. There are several other authors who have results showing when families of functions have homeomorphic inverse limit spaces. For example, Ingram and Mahavier, [21] have shown that if f and g are (usc) functions which are topological conjugate then the corresponding inverse limit spaces are homeomorphic. Smith and Varagona [36] have shown that N-type (usc) functions which follow the same pattern have homeomorphic inverse limits. Again f_a, $a \in (0,1)$ and f_0 do not satisfy hypothesis of their theorem. Kelly and Meddaugh [26] examine when is it the case that a sequence of (usc) functions f_i converging to an (usc) function f implies that $\varprojlim\{[0,1], f_i\}$ converges to $\varprojlim\{[0,1], f\}$ in the Housdorff metric. If we let $a_i \in (0,1)$ with $a_i \to 0$ then X_{a_i} are all homeomorphic by Banić and Lunder's theorem but again the functions f_{a_i} and f_0 does not satisfy Kelly and Meddaugh's hypothesis. Thus it seems somewhat surprising that it is the case that $X_{1/2}$ (and hence X_a for $a \in (0,1)$) and X_0 are homeomorphic as we show in the main theorem of the first paper [32].

Some topological properties are known to be preserved by (single valued) inverse limits. For example arc-likeness, tree-likeness, dimension, trivial shape etc. are such. Also some properties are preserved if the bonding mappings are members of some classes, for example a theorem by Capel says that local connectedness is preserved by inverse limits if bonding functions are monotone (see [6, Theorem 4.3, p. 241]). Similarly, the property of Kelley is preserved if the bonding mappings are confluent (see [8, Theorem 2, p. 190]). Much less is known about which properties are preserved under inverse limits with multivalued bonding functions. We have some theorems about connectedness of these inverse limits, (see e.g. [21, Theorem 125, p. 89 and Theorem 151, p. 112], [35, Theorem 3.1, p. 170]) and [14]. Also trivial shape is preserved if the bonding functions have trivial shape images (see [12]). J. P. Kelly showed that the inverse limit of intervals is locally connected if the bonding functions have connected images and preimage of points. In the second paper [10], a generalization of Kelly's theorem showing that the inverse limit space of locally connected continua with bonding functions whose graphs are locally connected and preimage of points are connected is locally connected is obtained. We give several examples of applications of the above theorem, in particular we use topological characterizations of some dendrites to show that particular inverse limits are homeomorphic.

Finally, generalized inverse limits of continua with bonding functions F_n such that the projection β_n (α_n) of Graph(F_n) onto the second (first) factor space are atomic and images (pre-image) of points are zero-dimensional are studied [11]. For some properties it is shown that if the first (all) factor space(s) has a certain property then the inverse limit space must have this property. Properties considered include; hereditary decomposability, hereditary indecomposability, hereditary unicoherence, arc-likeness, and tree-likeness. In [17] W. T. Ingram showed that if F is a linearized version of the $\sin(1/x)$ function is used as the single bonding function in an inverse limit system on $[0,1]$, so Graph(F) is chainable, then the resulting inverse limit space was chainable. He followed this up, answering a question from one of us, by showing [19] that the inverse system consisting of a sequence of sinusoids as bonding functions has a chainable inverse limit. In a related result [25] J. P. Kelly considered inverse systems with a single irreducible function as it's bonding function. Kelly's work generalized Ingram's earlier result on the linearized $\sin(1/x)$. Sinusoids do not necessarily satisfy the conditions to be an irreducible function. In this paper we consider a sequence of upper semi-continuous bonding functions, F_n, with the property that for each n the projection α_n (β_n) is an atomic map and for every $x \in X_{n+1}$ the image $F_n(x)$ (every $x \in X_n$ the preimage $F^{-1}(x)$) is zero-dimensional. All of the examples in [17] as well as the sinusoids in [19] satisfy these conditions however Kelly's irreducible functions may not. A number of examples are given to illustrate how the theorems may be used to understand some of the properties of the generalized inverse limit spaces.

PAPER

I. A FAMILY OF GENERALIZED INVERSE LIMITS HOMEOMORPHIC TO 'THE MONSTER'

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ABSTRACT

We show that two generalized inverse limit spaces that one might suspect are not homeomorphic are in fact homeomorphic. **Keywords:** inverse limits, generalized inverse limit, set valued functions, upper semi-continuous

1. INTRODUCTION AND DEFINITIONS

We are interested in the family of upper semi-continuous functions $f_a : [0,1] \to [0,1]$ and the corresponding inverse limits $X_a = \varprojlim\{[0,1], f_a\}$, where the graph, $\gamma(f_a)$, of f_a is the union of the line segments from $(0,0)$ to $(a,1)$ to $(1,a)$ to $(1,0)$ for $a \in [0,1]$. For $a \in (0,1)$, f_a is a generalized upper semi-continuous (usc) Markov function and it follows from results of Banić and Lunder [1] that if $a, b \in (0,1)$ then X_a homeomorphism to X_b. But for $a \in (0,1)$, X_a and X_1 are not homeomorphic since the first contains the topologists sine curve as a subcontinuum and the second is the harmonic fan. The functions f_a, $a \neq 0$

and f_0 do not satisfy the hypothesis of Banić and Lunder's theorem so we may ask, are $X_{1/2}$ and X_0 homeomorphic? In his Master's thesis Jacobsen [3] studied $X_{1/2}$ where he showed that it contained $2^{\aleph_0}$ arc components and each arc component is dense. The space X_0 is often referred to as 'the monster', a name reportedly coined by Banić. There are several other authors who have results showing when families of functions have homeomorphic inverse limits. For example Ingram and Mahavier, [2] have shown that if f and g are usc functions which are topological conjugate then the corresponding inverse limit spaces are homeomorphic. Smith and Varagona [5] have shown that N-type usc functions which follow the same pattern have homeomorphic inverse limits. Again f_a, $a \in (0,1)$ and f_0 do not satisfy hypothesis of their theorem. Kelly and Meddaugh [4] examine when is it the case that a sequence of usc functions f_i converging to an usc function f implies that $\varprojlim\{[0,1], f_i\}$ converges to $\varprojlim\{[0,1], f\}$. If we let $a_i \in (0,1)$ with $a_i \to 0$ then X_{a_i} are all homeomorphic by Banić and Lunder's theorem but again the functions f_{a_i} and f_0 does not satisfy their hypothesis. Thus it seems somewhat surprising that it is the case that $X_{1/2}$ (and hence X_a for $a \in (0,1)$) and X_0 are homeomorphic as we show in our theorem.

A topological space X is a continuum if it is a non-empty, compact, connected, metric space. A continuum subset of the space X is called a subcontinuum of X. Let X and Y be topological spaces, a function $f : X \to 2^Y$ is upper semi-continuous at x provided that for all open sets V in Y which contain $f(x)$, there exist an open set U in X with $x \in U$ such that if $t \in U$, then $f(t) \subseteq V$. If a function $f : X \to 2^Y$ is upper semi-continuous at x for each $x \in X$, we say that f is upper semi-continuous (usc). Let X and Y be compact metric spaces and $f : X \to 2^Y$ a function. It is well known that f is usc if and only if the graph of f, $\gamma(f) = \{(x, y) : x \in X \text{ and } y \in f(x)\}$ is closed in $X \times Y$. Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of continua and for each $i \in \mathbb{N}$, let $f_i : X_{i+1} \to 2^{X_i}$ be an upper semi-continuous function. The inverse limit of $\{X_i, f_i\}$ is denoted as $\varprojlim\{X_i, f_i\}$ and defined by $\varprojlim\{X_i, f_i\} = \{(x_i)_{i=1}^{\infty} : x_i \in f_i(x_{i+1}), x_i \in X_i \text{ for all } i \in \mathbb{N}\}$.

2. MAIN THEOREM

Theorem 2.1. *X_0 is homeomorphic to $X_{1/2}$.*

Proof. Let $f : [0,1] \longrightarrow 2^{[0,1]}$ be given by $f(x) = 2x$ for $x \in [0,1/2]$, $f(x) = 3/2 - x$ for $x \in [1/2,1]$, $f(1) = [0,1/2]$.

$$f(x) = \begin{cases} 2x & \text{if } x \in [0,\frac{1}{2}] \\ \frac{3}{2} - x & \text{if } x \in [\frac{1}{2},1) \\ [0,\frac{1}{2}] & \text{if } x = 1 \end{cases}$$

Let $g : [0,1] \longrightarrow 2^{[0,1]}$ be given by

$$g(x) = \begin{cases} [0,1] & \text{if } x = 0 \\ 1 - x & \text{if } x \in (0,1] \end{cases}$$

Let $A = \{(a_1,\ldots,a_i,\ldots) : a_i \in \{0,1\}$ and $a_i = 1 \Rightarrow a_{i+1} = 0\}$. Let $B = \{(b_1,\ldots,b_i,\ldots) : b_i \in \{0,(1/2)^n\}$ and $b_i = 0 \Rightarrow b_{i+1} \in \{0,1\}$ and $b_i = (1/2)^n \Rightarrow b_{i+1} \in \{(1/2)^{n+1},1\}\}$. It is clear that A and B are subsets of $\varprojlim\{[0,1],g\}$ and $\varprojlim\{[0,1],f\}$ respectively. Two points x and y in A are said to be adjacent if there is a positive integer n such that $\pi_i(x) = \pi_i(y)$ for $i \geq n+1$, $\pi_{n+1}(x) = 0 = \pi_{n+1}(y)$, and $\pi_i(x) = 1 - \pi_i(y)$ for $i \leq n$. Define $r^A_{xy} : [0,1] \rightarrow \varprojlim\{[0,1],g\}$ by $r_{xy}(t) = (t,1-t,t,\ldots,1-t,t,0,x_{n+2},\ldots)$. We say r^A_{xy} is a straight line in $\varprojlim\{[0,1],g\}$ connecting x and y. Notice that any two distinct straight lines can only intersect at endpoints. Two points z and w in B are said to be adjacent if there exist a positive integer n such that $\pi_i(z) = \pi_i(w)$ for $i \geq n+1$, $\pi_{n+1}(z) = 1 = \pi_{n+1}(w)$, and there is a positive integer m such that $\pi_n(z) = 1/2^{m-1}$, $\pi_n(w) = 1/2^m$, $\pi_i(w) = 2\pi_{i+1}(w)$ for $n-m \leq i < n$ and $\pi_i(w) = \frac{3}{2} - \pi_{i+1}(w)$ for $1 \leq i < n-m$, $\pi_i(z) = 2\pi_i(w)$ for $n-m \leq i < n+1$ and $\pi_i(z) = \frac{3}{2} - \pi_{i+1}(w)$ for $1 \leq i < n-m$. Define $r^B_{zw} : [1/2^m, 1/2^{m-1}] \rightarrow \varprojlim\{[0,1],f\}$ where $r^B_{zw}(t) = (\frac{3}{2} - x_2,\ldots,\frac{3}{2} - x_{n-m}, x_{n-m},\ldots,4t,2t,t,1,x_{n+2},\ldots)$ where $x_{n-m} = 2^{n-m}t$ and $\frac{1}{2} \leq 2^{n-m}t \leq 1$. As before, we say r^B_{zw} is a straight line in $\varprojlim\{[0,1],f\}$ connecting z and w.

Again, any two distinct straight lines can only intersect at endpoints. Define $H : B \longrightarrow A$, such that $H(b_1, b_2, ...) = (h_1(b_1), h_2(b_2), , ..)$ and $h_i(b_i) = 1$ for $b_i = 1/2$ and $h_i(b_i) = 0$ otherwise. Define $S : A \longrightarrow B$, such that $S(a_1, a_2, ...) = (s_1(a_1), s_2(a_2), , ..)$ where

$$s_1(a_1) = \begin{cases} \frac{1}{2} & \text{if } a_1 = 1 \\ 1 & \text{if } a_1 = 0 \text{ and } a_2 = 1 \\ 0 & \text{if } a_1 = a_2 = 0 \end{cases}$$

and if $s_k(a_k)$ has been defined for $1 \leq k < i$ let

$$s_i(a_i) = \begin{cases} \frac{1}{2} & \text{if } a_i = 1 \\ 1 & \text{if } a_i = 0 \text{ and } a_{i+1} = 1 \\ \frac{1}{2} s_{i-1}(a_{i-1}) & \text{otherwise} \end{cases}$$

From the definitions of S it can be seen that S is one-to-one and onto. Since all component functions, s_i, are continuous, S is continuous, hence S is a homeomorphism between A and B. Further one can see $H = S^{-1}$. Let a and c be adjacent points in A and r^A_{ac} be a straight line in $\varprojlim\{[0, 1], g\}$ so there is n such that $\pi_i(a) = \pi_i(c)$ for all $i \geq n + 1$ and $\pi_{n+1}(a) = \pi_{n+1}(c) = 0$ and one of $\pi_n(a)$ and $\pi_n(c)$ is zero and the other is 1. Suppose without loss of generality $\pi_n(a) = 0$ and $\pi_n(c) = 1$. We wish to show that there is a unique corresponding straight line $r^B_{S(a)S(c)}$ in $\varprojlim\{[0, 1], f\}$ connecting $S(a)$ and $S(c)$. By definition of S, $s_n(a_n) = 1/4$, $s_{n-1}(a_{n-1}) = s_n(c_n) = 1/2$ and $s_{j-1}(a_{j-1}) = \frac{3}{2} - s_j(a_j)$ for all $j < n$ and $s_{j-1}(c_{j-1}) = \frac{3}{2} - s_j(c_j)$ for all $j \leq n$. Let $l = \min\{k : k > n + 1 \text{ and } a_k = 1\}$. So there is a positive integer m such that $l = n + m$. Since $a_{l-1} = c_{l-1} = 0$ and $a_l = c_l = 1$ so $s_{l-1}(a_{l-1}) = s_{l-1}(c_{l-1}) = 1$ and $s_l(a_l) = s_l(c_l) = 1/2$. $s_{l-2}(a_{l-2}) = s_{n+m-2}(a_{n+m-2}) = \frac{1}{2^m}$ and $s_{l-2}(c_{l-2}) = s_{n+m-2}(c_{n+m-2}) = \frac{1}{2^{m-1}}$. Hence $S(a)$ and $S(c)$ are adjacent in B. So r^A_{ac} is homeomorphic to the corresponding straight line $r^B_{S(a)S(c)}$ in $\varprojlim\{[0, 1], f\}$. Let p and q be adjacent points in B and r^B_{pq} be a straight line in $\varprojlim\{[0, 1], f\}$ so there is n such that $\pi_{n+1}(p) = \pi_{n+1}(q) = 1$ and $\pi_i(p) = \pi_i(q)$ for all $i \geq n+1$ and one of $\pi_n(p)$ and $\pi_n(q)$ is $\frac{1}{2^m}$ and the other is $\frac{1}{2^{m+1}}$ $m \geq 1$. Suppose without lost of generality $\pi_n(p) = \frac{1}{2^m}$ and $\pi_n(q) = \frac{1}{2^{m+1}}$. So $\pi_{n-m+2}(p) = \frac{1}{4}$, $\pi_{n-m+2}(q) = \frac{1}{8}$ by definition of H, $h(\pi_i(p))$ and $h(\pi_i(q))$ equal to zero for $n - m + 2 < i \leq n + 1$, so $n - m + 2$ is the least positive integer such that the image of $h(\pi_{n-m+2}(p))$ and $h(\pi_{n-m+2}(q))$ are zero and $h(\pi_{n-m+1}(p)) = 1$ and $h(\pi_{n-m+1}(q)) = 0$, This means that $H(p)$ and $H(q)$ are adjacent points in A. Thus the set of straight lines in $\varprojlim\{[0,1], g\}$ is mapped one-to-one and onto the set of straight lines in $\varprojlim\{[0,1], f\}$ Hence S (or H) can be piecewise linearly extended to a homeomorphism between $\varprojlim\{[0,1], g\}$ and $\varprojlim\{[0,1], f\}$ completing the proof of the theorem. ■

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II. LOCAL CONNECTEDNESS OF INVERSE LIMITS

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ABSTRACT

We prove a theorem that under some conditions local connectedness is preserved under set-valued inverse limits. The theorem generalizes Capel's theorem that local connectedness is preserved under (single-valued) inverse limits with monotone bonding functions and its set-valued analogue by James Kelly see([12]). As a consequence we can characterize some set-valued inverse limits on intervals. **Keywords:** generalized inverse limit, local connectedness

1. INTRODUCTION AND DEFINITIONS

Some topological properties are known to be preserved by (single valued) inverse limits. For example arc-likeness, tree-likeness, dimension, trivial shap etc. are such. Also some properties are preserved if the bonding mappings are members of some classes, for example a theorem by Capel says that local connectedness is preserved by inverse limits if bonding functions are monotone (see [3, Theorem 4.3, p. 241]). Similarly, the property of Kelley is preserved if the bonding mappings are confluent (see [5, Theorem 2, p. 190]). Much less is known about which properties are preserved under inverse limits with multivalued bonding functions. We have some theorems about connectedness of these inverse limits, (see e.g. [11, Theorem 125, p. 89 and Theorem 151, p. 112], [15, Theorem 3.1, p. 170]) and [8]. Also trivial shape is preserved if the bonding functions have trivial shape images (see [7]). James P. Kelley showed that the inverse limit of intervals is locally connected if the bonding functions have connected images and preimages of points. Here we generalize his theorem showing that the inverse limits of locally connected continua with bonding functions whose graphs are locally connected and preimages of points are connected is locally connected. We give several examples of applications of the above theorem, in particular we use topological characterizations of some dendrites to show that particular inverse limits are homeomorphic. A set X is continuum if it is a non-empty, compact, connected, metric space. A continuum subset of the space X is called a subcontinuum of X. Let X and Y be continua. A function $f : X \longrightarrow 2^Y$ us upper semi-continuous at x provided that for all open sets V in Y which contain $f(x)$, there exist an open set U in X with $x \in U$ such that in $t \in U$, then $f(t) \subseteq V$. If A function $f : X \longrightarrow 2^Y$ us upper semi-continuous at x for each $x \in X$, we say that f is upper semi-continuous (USC). Let X and Y be compact metric spaces and $f : X \longrightarrow 2^Y$ a function. Then f is an upper semi-continuous (USC) if and only if its graph $G(f)$ is closed in $X \times Y$ [9, p. 3]. Let X and Y be compact Hausdorff spaces, and let $f : X \longrightarrow Y$ be a continuous function we say that f is monotone if $f^{-1}(y)$ is a continuum for al $y \in Y$. Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of continua and for each $i \in \mathbb{N}$, let $f_i : X_{i+1} \longrightarrow 2^{X_i}$ be an upper semi-continuous function. The inverse limit of $\{X_i, f_i\}$ is denoted by $\varprojlim \{X_i, f_i\}$ and defined by $\varprojlim \{X_i, f_i\} = \left\{(x_i)_{i=1}^{\infty}, x_i \in f_i\,(x_{i+1}) \text{ for all } i \in \mathbb{N}\right\}$. For $1 \leq i \leq j$ we denote by $G(f_i, f_{i+1}, ..., f_j)$ the set $\{(x_i, x_{i+1}, ...x_j, x_{j+1}) : x_n \in f_n(x_{n+1}) \text{ for } n \in \{i, i+1, ..., j\}\}$. More information about inverse limits of continua with multivalued, USC bonding functions can be found in [11] and [9].

2. MAIN THEOREM

Let us recall that a *dendrite* is a locally connected continuum that contains no simple closed curve. It is known that a subcontinuum of a dendrite is a dendrite and that dendrites are hereditarily unicoherent, i.e. the intersection of any two subcontinua of a dendrite is a continuum. The main theorem of the article is Theorem 2.1 below. It generalizes a theorem by C. E. Capel, see [3, Theorem 4.3, p. 241], stating that the inverse limit of locally connected continua with monotone (single-valued) bonding mappings is locally connected.

Theorem 2.1. *If, for $i \in \{1, 2, \dots\}$, $f_i : X_{i+1} \to 2^{X_i}$ is an upper semi-continuous function, X_i is a dendrite, $G(f_i)$ is a locally connected continuum, and, for each $t \in X_i$, the preimage $f_i^{-1}(t)$ is connected, then $\varprojlim \{X_i, f_i\}$ is a locally connected continuum.*

Proof. First, note that by [11, Theorem 126, p. 90] the inverse limit is a continuum. We will prove by induction that $G(f_1, \ldots, f_n)$ is locally connected. To start, observe that $G(f_1)$ is a locally connected continuum by our assumption. Thus assume that $G(f_1, \ldots, f_{n-1})$ is a locally connected continuum and we will show that $G(f_1, \ldots, f_n)$ is a locally connected continuum. Take a point $(a_1, \ldots, a_{n+1}) \in G(f_1, \ldots, f_n)$, and let U be an open neighborhood of $(a_1, .., a_{n+1})$ in $G(f_1, \ldots, f_n)$. We may assume $U = (U_1 \times \ldots \times U_{n+1}) \cap G(f_1, \ldots, f_n)$, where $U_1, \ldots, U_{n+1}$ are open subsets of $X_1, \ldots, X_{n+1}$ respectively. By the induction hypothesis of local connectedness of $G(f_1, \ldots, f_{n-1})$, there is a continuum K such that $(a_1, \ldots, a_n) \in \operatorname{int}(K) \subseteq (U_1 \times \ldots \times U_n) \cap G(f_1, \ldots, f_{n-1})$. Define $r_n : G(f_1, \ldots, f_n) \longrightarrow G(f_1, \ldots, f_{n-1})$ by $r_n(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n)$ and observe that r_n is a monotone map, because $r_n^{-1}(x_1, \ldots, x_n)$ is homeomorphic to $f_n^{-1}(x_n)$, which is connected by assumption. Let V be a continuum in X_{n+1} satisfying $a_{n+1} \in \operatorname{int}(V) \subseteq V \subseteq U_{n+1}$. Then $f_n^{-1}(a_n) \cap V$ is a continuum, because of hereditary unicohrence of X_{n+1}. Define $E = \left\{(x_1, \ldots, x_{n+1}) \in G(f_1, \ldots, f_n) : (x_1, \ldots, x_n) \in K \text{ and } x_{n+1} \in f_n^{-1}(x_n) \cap V\right\}$. We will show that E is the needed connected neighborhood of $(a_1, \ldots, a_{n+1})$. The condition $(a_1, \ldots, a_{n+1}) \in \operatorname{int}(E) \subseteq E \subseteq U$ follows from the definitions, so we only need to show the connectedness of E. To do this aim suppose on the contrary that $E = P \cup Q$. where P and Q are two disjoint closed nonempty subsets of E. Since $K = r_n(E) = r_n(P) \cup r_n(Q)$, the connectedness of K yields that $r_n(P) \cap r_n(Q) \neq \emptyset$. Taking $(x_1, ..., x_n) \in r_n(P) \cap r_n(Q)$. We have that $r_n^{-1}(x_1, ..., x_n)$ is a subset of P or subset of Q, which is contradiction. This finishes the inductive proof of connectedness of $G(f_1, ..., f_n)$ for every n. Note that $\varprojlim \{X_n, f_n\}$ is homeomorphic to $\varprojlim \{G(f_1, ..., f_n), r_n\}$. The later is an inverse limit of locally connected continua $G(f_1, ..., f_n)$ with the monotone bonding (single-valued) mappings r_n, so the inverse limit is locally connected by a theorem by C. E. Capel, see [3, Theorem 4.3, p. 241]. ■

Now we want to show a theorem about trivial shape of some inverse limits. Let us recall the necessary definitions. The theory of shape is well developed, see for example Borsuk [2]. In this paper we are only concerned with continua having trivial shape. The following are equivalent for a continuum X (see [13, Proposition 1.6, p. 82]):

1. X has trivial shape,
2. X can be written as $X = \bigcap X_n$ where X_n's are contractible continua,
3. X can be written as an inverse limit of contractible continua,
4. For all $\varepsilon > 0$ there exists a contractible continuum Y_ε and an ε-map f_ϵ from X onto Y_ε.

Recall that a mapping $f\colon X \to Y$ is cell-like if for each $y \in Y$, $f^{-1}(y)$ has trivial shape. We will also need basic properties of continua with trivial shape: as a consequence of (2) we know that a decreasing intersection of continua with trivial shape has trivial shape; and as consequence of (3) any inverse limit of continua with trivial shape has trivial shape. To prove properties of our examples we will need the following theorem about trivial shape of inverse limits. It is closely related to [7, Theorem 2]. The authors thank Dr. Robert P. Roe for suggesting the proof of it; it is analogous to the proof of the main theorem of [7], i.e. [7, Theorem 2]. We repeat the idea of that proof with the necessary changes. First of all we need to recall a theorem by R.B. Sher, see [2, Theorem 9.3 p. 325]. We do not need it in full generality, we use it only for continua of trivial shape.

Theorem 2.2. *If X and Y are finite-dimensional compact metric spaces, $f : X \to Y$ is a cell-like map, then the shape of X equals the shape of Y.*

Theorem 2.3. *Let $X_1, X_2, \ldots$ be a sequence of finite dimensional continua with trivial shape and let $f_n : X_{n+1} \to 2^{X_n}$ be upper semi-continuous functions such that $f_n^{-1}(x_n)$ is a continuum with trivial shape for each $x_n \in X_n$, then $\varprojlim \{X_i, f_i\}$ has trivial shape.*

Proof. Following the notation of Ingram and Mahavier, we define $G_n = G(f_1, \ldots, f_n) \times \prod_{i>n+1} X_i$ then $G_{n+1} \subseteq G_n$ and $\varprojlim \{X_i, f_i\} = \bigcap_{n \geq 1} G_n$, thus to show $\varprojlim \{X_i, f_i\}$ has trivial shape, it is enough to prove that each G_n has trivial shape. To do this we will show that $G(f_1, \ldots, f_n)$ has trivial shape and therefore G_n is the product of continua with trivial shape. Define, for arbitrary i and j and k with $i \leq j \leq k+1$, the projection $\pi_j^{i,k} : G(f_i, \ldots, f_k) \to X_j$ by $\pi_j^{i,k}\left(x_i, x_{i+1}, \ldots, x_j, x_{k+1}\right) = x_j$. To show that $G(f_1, \ldots, f_n)$ has trivial shape we will use mathematical induction. We start by observing that $G(f_n)$ has trivial shape by assumption, and that $\pi_n^{n,n}$ is topologically equivalent to f_n, so it is cell-like by the assumption. So assume $G(f_k, f_{k+1}, \ldots, f_n)$ has trivial shape and we will prove that $G(f_{k-1}, f_k, \ldots, f_n)$ has trivial shape. We use Theorem 2.2. Let $\rho : G(f_{k-1}, f_k, \ldots, f_n) \to G(f_k, f_{k+1}, \ldots, f_n)$ be the natural projection. Note that $\rho^{-1}(x_k, x_{k+1}, \ldots, x_n, x_{n+1}) = \{(a, x_k, x_{k+1}, \ldots, x_n, x_{n+1}) : f_k(x_k) = a\}$ is homeomorphic to $f_k^{-1}(x_k)$ so it has trivial shape by assumption. Since $G(f_k, f_{k+1}, \ldots, f_n)$ has trivial shape by the inductive hypothesis, the set $G(f_{k-1}, f_k, \ldots, f_n)$ has trivial shape by Theorem 2.2. This show that $G(f_k, f_{k+1}, \ldots, f_n)$ has trivial shape for all $k \leq n$. Finally, if we denote $g_n : G(f_1, f_2, \ldots, f_n) \to G(f_1, f_2, \ldots, f_{n-1})$ we note that $\varprojlim \{X_n, f_n\}$ is homeomorphic to

$\varprojlim \{G(f_1, f_2, ..., f_n), g_n\}$ and the later is the inverse limit of trivial shape continua with single valued cell-like bonding functions, so the limit has trivial shape. This finishes the proof. ■

3. EXAMPLES

In this section we will show several examples of applications of the theorems shows in sections 2. Before starting over examples, we need to recall some notions and terminology. A dendrite is a locally connected continuum that contains no simple closed curve. It is known that each subcontinuum of a dendrite is a dendrite. Dendrites can be characterized as locally connected one-dimensional continuum with trivial shape. We will use the notion of an order of a point in a dendrite X. A point $p \in X$ has order (in the classical sense) at least n, in symbols $\mathrm{ord}_X(p) \geq n$ if there are n arcs $A_1, A_2, ..., A_n$ such that $A_i \cap A_j = \{p\}$ for $i \neq j, i, j \in \{1, 2, ..., n\}$. Then $\mathrm{ord}_X(p) = n$ if $\mathrm{ord}_X(p) \geq n$ and $\mathrm{ord}_X(p) \geq n + 1$ is not true. If $\mathrm{ord}_X(p) \geq n$ is true for all $n \in \mathbb{N}$, then we write $\mathrm{ord}_X(p) = \omega$. Points of order one are called end-points and are denoted by $E(X)$, points of order two are called ordinary points, and points of order three or more are called ramification points and denoted by $R(X)$.

3.1. THE DENDRITE $\mathcal{P}$

For the next characterization we need to define a special dendrite called W. Its the one pictured in Figure 1; its formal description can be found in [1, p. 3]. To start, let us recall a characterization of a certain dendrite that is shown in [1, Theorem 4.5, p. 9]. This dendrite is pictured in Figure 2, it can be also found in [9, Figure 2.18, p. 34].

Theorem 3.1. *There is only one (up to homeomorphisms) dendrite $\mathcal{P}$ with the following properties:*

1. *$\mathcal{P}$ does not contain a copy of W.*

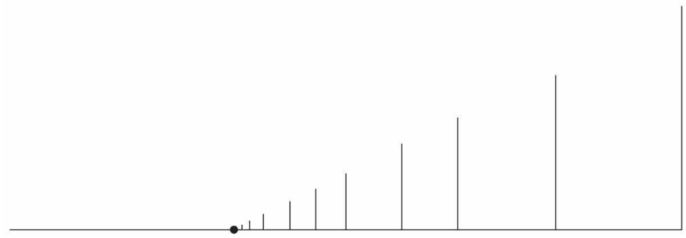


Figure 1. The dendrite W

2. $E(\mathcal{P}) \subseteq \mathrm{cl}(R(\mathcal{P}))$.

3. *each ramification point is of order ω.*

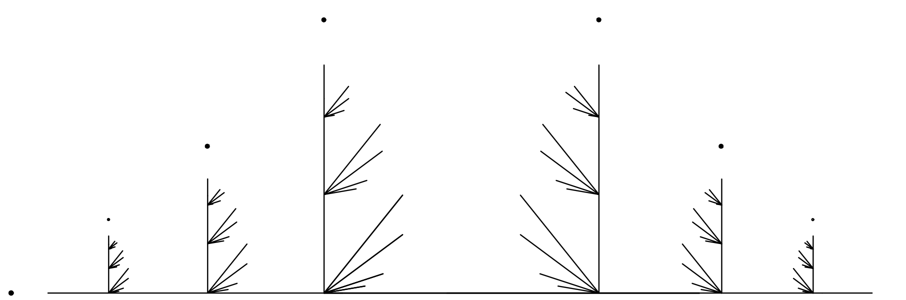


Figure 2. The dendrite $\mathcal{P}$

The following example appears in [9, Example 2.16, p. 34]

Example 3.2. Let $f_1 : [0,1] \to 2^{[0,1]}$ be the multivalued function whose graph is pictured in Figure 3. Its graph is the union of two segments joining $\langle 0,0\rangle$ to $\langle 1,0\rangle$ and joining $\langle 1,0\rangle$ to $\langle 0,1\rangle$. Then the inverse limit $\varprojlim \{[0,1], f_1\}$ is homeomorphic to the dendrite $\mathcal{P}$ of Theorem 3.1. Figure 4 presents the inverse limit with some coordinates of points. The bold digits mean that the pattern is repeated. Denote $X = \varprojlim \{[0,1], f_1\}$ and observe that by Theorem 2.1 X is locally connected, by Theorem 2.3 it has trivial shape. Since the graph of f_1 has no vertical segments, X is one dimensional by Van Nall's Theorem [14, Theorem 5.3, p. 1330], see also [9, Theorem 5.3, p. 69]. One dimensional trivial

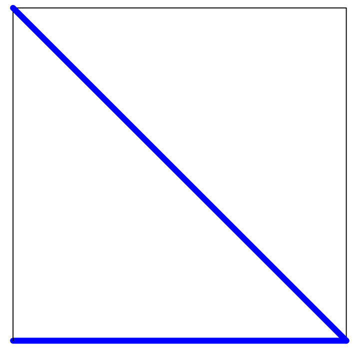


Figure 3. The graph of f_1

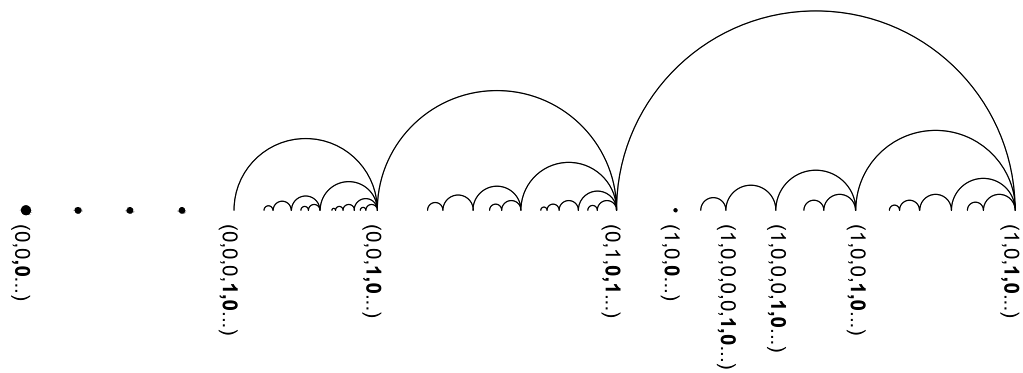


Figure 4. The inverse limit of f_1

shape continua are tree-like, and locally connected tree-like continua are dendrites, so X is a dendrite. Next, observe that if a point $x = (x_1, x_2, \ldots)$ in the inverse limit $X = \varprojlim \{[0,1], f_1\}$ contains a coordinate not in $\{0,1\}$, then $\operatorname{ord}(X, x) = 2$ Really, such point has to be of the form $(a_1, a_2, \ldots, a_n, t, 1-t, t, 1-t, \ldots)$ where $a_n = 0$, $t \notin \{0,1\}$ and $a_i \in \{0,1\}$. Then the neighborhood of such point consists of points $(a_1, a_2, \ldots, a_n, t', 1-t', t', 1-t', \ldots)$ for $t' \in (t-\varepsilon, t+\varepsilon)$ for some $\varepsilon > 0$. Points whose all coordinates are in $\{0,1\}$ are either end-points or points of order ω. Points of order ω have the form $(a_1, a_2, \ldots, a_n, 1, 0, 1, 0, \ldots)$ that is ones and zeros are repeated interchangeably starting from some index. In this case we have arcs A_k of the form $(a_1, a_2, \ldots, a_n, 1, 0, 1, 0, \ldots, t, 1-t, t, 1-t, \ldots.)$ where the k-th coordinate is t and all coordinates a_i for $i \leq k$ are in $\{0,1\}$. Then $A_k \cap A_{k'} = \{(a_1, a_2, \ldots, a_n, 1, 0, 1, 0, \ldots)\}$, where $a_n = 0$. End points have the form $(a_1, a_2, \ldots)$, where $a_i \in \{0, 1\}$ and are not of the form of points of order ω described above. Therefore conditions (2) and (3) of Theorem 3.1 are satisfied. Since $E(X) \cup R(X)$ is closed, Condition (1) is also satisfied. Thus X is homeomorphic to dendrite $\mathcal{P}$ of Theorem 3.1.

Example 3.3. Let $f_2 : [0, 1] \to 2^{[0,1]}$ be the multivalued function whose graph is pictured in Figure 6. Its graph is the union of two segments joining $\langle 1, 0\rangle$ to $\langle 0, 1\rangle$ and joining $\langle 0, 1\rangle$ to $\langle 1, 1\rangle$. As before by interchanging roles of 0 and 1, the inverse limit $\varprojlim \{[0, 1], f_2\}$ is also homeomorphic to the dendrite $\mathcal{P}$ of Theorem 3.1. Thus $\varprojlim \{[0, 1], f_2\}$ is homeomorphic to $\varprojlim \{[0, 1], f_2\}$.

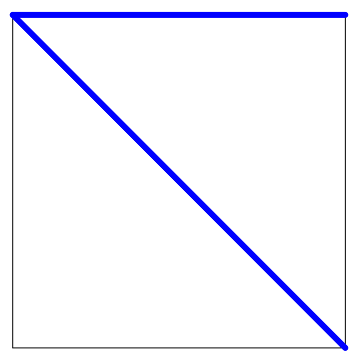


Figure 5. The graph of f_2

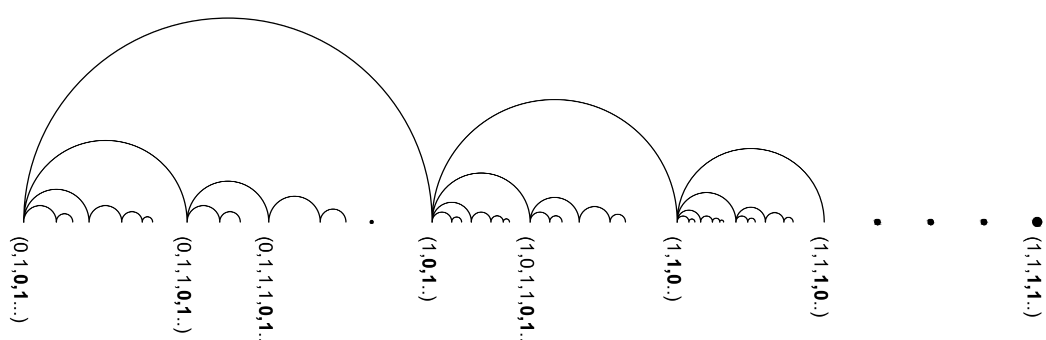


Figure 6. The inverse limits of f_2

Example 3.4. Let $f_3 : [0,1] \to 2^{[0,1]}$ be the multivalued function whose graph is pictured in Figure 2. Its graph is the union of three segments joining $\langle 0,0\rangle$ to $\langle 1,0\rangle$; $\langle 1,0\rangle$ to $\langle 0,1\rangle$ and $\langle 0,1\rangle$ to $\langle 1,1\rangle$. Again points that contain coordinate not in $\{0,1\}$ are ordinary points; points whose all coordinates are zeros and ones are either end-points or points of order ω. This time every sequence of zeros and ones is in the inverse limit. As before inverse limit $\varprojlim\{[0,1], f_3\}$ is dendrite and it is homeomorphic to $\varprojlim\{[0,1], f_1\}$ and $\varprojlim\{[0,1], f_2\}$ by Theorem 3.1. The inverse limit is pictured in Figure 8.

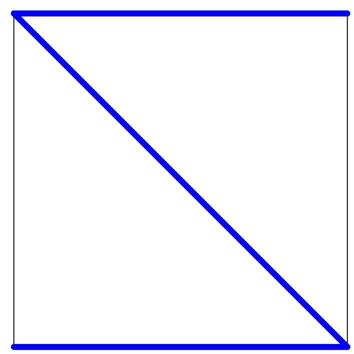


Figure 7. The graph of f_3

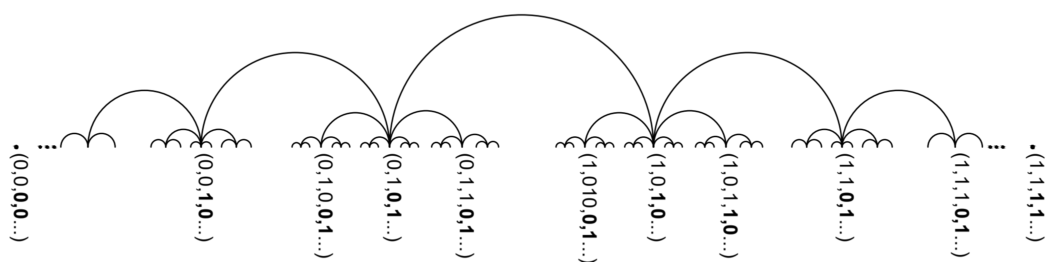


Figure 8. The inverse limit of f_3

Example 3.5. Let $f_4 : [0,1] \to 2^{[0,1]}$ be the multivalued function whose graph is pictured in Figure 9. Its graph is the union of three segments joining $\langle 1,0\rangle$ to $\langle 0,0\rangle$; $\langle 0,0\rangle$ to $\langle 1,1\rangle$ and $\langle 1,1\rangle$ to $\langle 0,1\rangle$. As before, the inverse limits $\varprojlim\{[0,1], f_4\}$ is a dendrite homeomorphic to $\varprojlim\{[0,1], f_3\}$, $\varprojlim\{[0,1], f_2\}$ and $\varprojlim\{[0,1], f_1\}$. Again points that contain coordinate not in $\{0,1\}$ are ordinary points. Points whose all coordinates are zeros and ones are either end-points or points of order ω. This time every sequence of zeros and ones is in the inverse limit. The inverse limit is pictured in Figure 10.

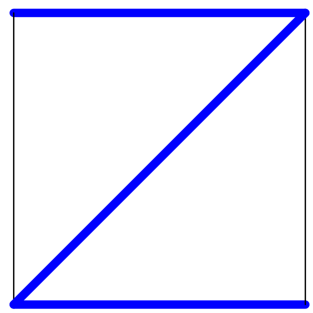


Figure 9. The graph of f_4

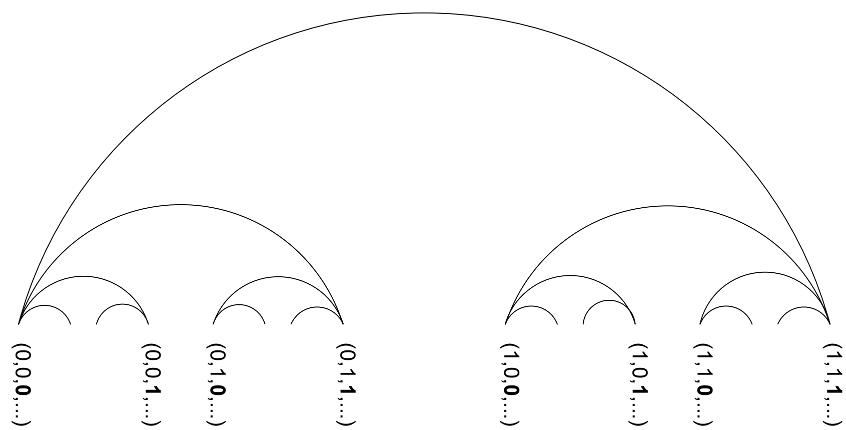


Figure 10. The iverse limit of f_4

3.2. UNIVERSAL DENDRITES

Let us start by recalling a characterization of dendrites D_A, for $A \subseteq \{3,4,\ldots,\omega\}$ due to W. J. Charatonik and A. Dilks (see [6, Theorem 6.2, p. 229]).

Theorem 3.6. *For every $A \subseteq \{3,4,\ldots,\omega\}$ there is one (up to homeomorphism) dendrite D_A with the following properties:*

1. *for every $n \in A$ and every arc $B \subseteq D_A$ there is a ramification point in B of order n;*

2. *each ramification point has order in A.*

If A has only one element n, the we will use the symbol D_n for $D_{\{n\}}$. The dendrites D_n were introduced and investigated in [4]. The picture of D_3 is provided in Figure 11.

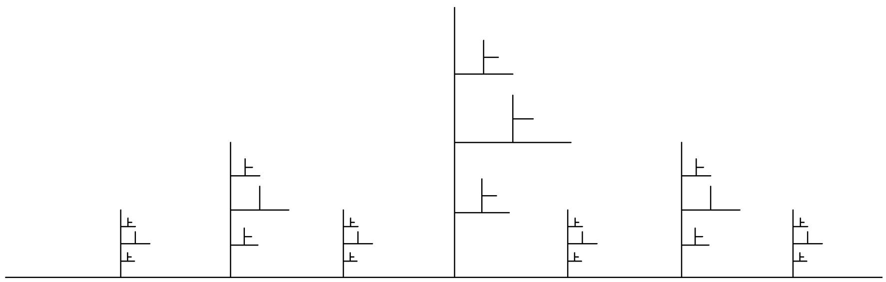


Figure 11. D_3 – the inverse limit of f_5

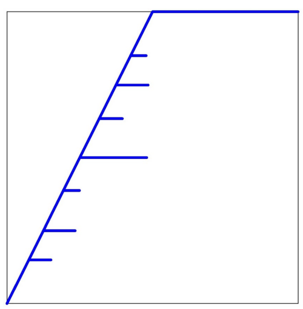
Example 3.7. Let $f_5 : [0,1] \to 2^{[0,1]}$ be the multivalued function whose graph is pictured in Figure 12. To construct the graph let us start with a dense countable subset D of $(0,1)$ such that if $p \in D$, then $p/2^n \notin D$ for all $n \in \{2,3,\ldots\}$. The graph of f_5 contains two arcs $\{(x,2x) : x \in [0,1/2]\}$ and $\{(x,1) : x \in [1/2,1]\}$. For any $p \in D$ we add an interval $\{(x,2p) : x \in [p,q(p)]\}$ where $q(p)$ satisfy the following conditions:

1. $p < q(p) < 2p$;

2. for every integer k we have $2^k q(p) \notin D$;

3. for every sequence p_n of different elements of D we have $\lim_{n\to\infty} q(p_n) - p_n = 0$.

We will show that $\varprojlim \{[0,1], f_5\}$ is homeomorphic to the dendrite D_3 of Theorem 3.6 pictured at Figure 11. As in previous examples one can see that the inverse limit is a dendrite. We show that all points $x = (x_1, x_2, \ldots)$ that have finitely many, but not zero coordinates in D and such that if $x_m \in D$ and $x_n \notin D$ for $n > m$ and $x_{m+1} \neq q(x_m)$ are ramification points of order 3. Let k be a coordinate of x such that $x_k \in D$ and $x_n \notin D$ for $n > k$, and $x_{k+1} \neq q(x_n)$. If x_k is the only coordinate in D, we let $m = 0$, otherwise let m be a coordinate of x sush that $x_m \in D$ but $x_n \notin D$ and $x_{n+1} = \frac{1}{2}x_n$ for $m < n < k$. The arc $l = \left(x_1, x_2, ..., x_m, t, \frac{t}{2}, ..., \frac{t}{2^{k-m}}, \frac{t}{2^{k-m}}, \frac{x_k}{2}, ...\right)$ where $t \in [\frac{x_m}{2}, q(x_m)]$ containing x as its interior because $x_n \in \left\{\frac{t}{2^{k-m}}, t \in [\frac{x_m}{2}, q(x_m)]\right\}$. The arc $M = \left(x_2, x_2, ..., x_m, \frac{x_m}{2}, ..., \frac{x_m}{2^{k-m-1}}, t, \frac{t}{2}, \frac{t}{4}, ...\right)$ where $t \in [\frac{x_k}{2}, q(x_k)]$ contains x as one of its end point. Since x is the intersection of the line containing x as its interior and another line as its end point and it doesn't have any other adjacent lines, so x is of order 3. The density of D implies that the set of ramification points is dense in the inverse limit. Points that have infinitely many of coordinates in D or finitely many of coordinate of D but there is a maximum positive integer k such that $x_k \in D$ and $x_{k+1} \in q(x_k)$ are end points.



D- a dense subset of [0,1]
If p $\in D$, then $\frac{p}{2^k} \notin D$.

Figure 12. The graphs of f_5

Example 3.8. Let $f_6 : [0,1] \to 2^{[0,1]}$ be the multivalued function whose graph is pictured in Figure 13. To construct the graph let us start with a dense countable set $D = \left\{\frac{n}{2^k} \in (0,1) : n, k \in \mathbb{N}\right\}$. The graph of f_6 contains, as before, two arcs $\{(x, 2x) : x \in [0, 1/2]\}$ and $\{(x, 1) : x \in [1/2, 1]\}$. For any $p \in D$ we add an interval $\{(x, 2p) : x \in [p, q(p)]\}$ where $q(p)$ satisfy the following conditions:

1. $p < q(p) < 2p$;

2. for every integer k we have $2^k q(p) \notin D$;

3. for every sequence p_n of different elements of D we have $\lim_{n\to\infty} q(p_n) - p_n = 0$.

We will show that $\varprojlim \{[0,1], f_6\}$ is homeomorphic to the dendrite D_ω of Theorem 3.6 pictured at Figure 14. As in previous examples one can see that the inverse limit is a dendrite. We will classify all points of $\varprojlim \{[0,1], f_6\}$. First observe that each point of $\varprojlim \{[0,1], f_6\}$ has one of the following three forms:

1. $\left(d_1, d_2, ..., d_n, x, \frac{x}{2}, \frac{x}{4}, ...\right)$, where $n \in \{1, 2, ...\}$, $d_1, d_2, ..., d_n$ are in D, and $x \in \left(\frac{d_n}{2}, q(d_n)\right) \setminus D$;

2. $\left(x, \frac{x}{2}, \frac{x}{4}, ...\right)$, where $x \in [0,1]$;

3. $(d_1, d_2, ...)$ where $d_1, d_2, ..., d_n \in D$ and $n \in \{1, 2, ...\}$.

Point of the forms (1) and (2) are ordinary points. To see this observe that $\left(d_1, d_2, ..., d_n, x, \frac{x}{2}, \frac{x}{4}, ...\right) \in \left\{\left(d_1, d_2, ..., d_n, t, \frac{t}{2}, \frac{t}{4}, ...\right) : t \in (x-\epsilon, x+\epsilon)\right\} \subset \left(\frac{d_n}{2}, q(d_n)\right)$ for some $\epsilon > 0$. If $x = q(d_n)$, the point $\left(d_1, d_2, ..., d_n, x, \frac{x}{2}, \frac{x}{4}, ...\right)$ is an end-point. Now, we examine points of the form (3). If there is an index n_0 such that $(d_1, d_2, ...) = \left(d_1, d_2, ...d_{n_0}, \frac{d_{n_0}}{2}, \frac{d_{n_0}}{4}, ...\right)$ then $(d_1, d_2, ...,)$ is a ramification point of order ω. To see that fix an index $i \geq n_0$. Then

$$(d_1, d_2, ...,) \in \left\{\left(d_1, d_2, ...d_{n_0}, \frac{d_{n_0}}{2}, ..., d_{i-1}, t, \frac{t}{2}, \frac{t}{4}, ...\right) : t \in \left(\frac{d_{i-1}}{2}, q(d_{i-1})\right)\right\}.$$

The later set is an open interval containing $(d_1, d_2, ...,)$. Since we have infinitely many such intervals (for infinitely many indices i), the order of $(d_1, d_2, ...,)$ is ω. If there is no index n_0 such that $(d_1, d_2, ...,) = \left(d_1, d_2, ...d_{n_0}, \frac{d_{n_0}}{2}, \frac{d_{n_0}}{4}, ...\right)$, we have that $d_{i+1} \in \left(\frac{d_i}{2}, q(d_i)\right)$ for all $i \in \{1, 2, ...\}$. Then $(d_1, d_2, ...)$ is an end-point. In fact it is an end-point of the interval which goes through points $\left(d_1, \frac{d_1}{2}, \frac{d_1}{4}, \ldots\right), \left(d_1, d_2, \frac{d_2}{2}, \frac{d_2}{4}, ...\right), \left(d_1, d_2, d_3, \frac{d_3}{2}, \frac{d_3}{4}, ...\right), \ldots$

Example 3.9. Let $f_7 : [0,1] \to 2^{[0,1]}$ be the multivalued function whose graph is pictured in Figure 15. Its graph consists of four segments joining $\langle 0,0\rangle$ to $\langle 1,0\rangle$; $\langle 0,1/3\rangle$ to $\langle 1,1/3\rangle$; $\langle 0,1\rangle$ to $\langle 1,1\rangle$; $\langle 1/2,0\rangle$ to $\langle 1/2,1\rangle$. By the previous theorems $\varprojlim \{[0,1], f_7\}$ is a locally

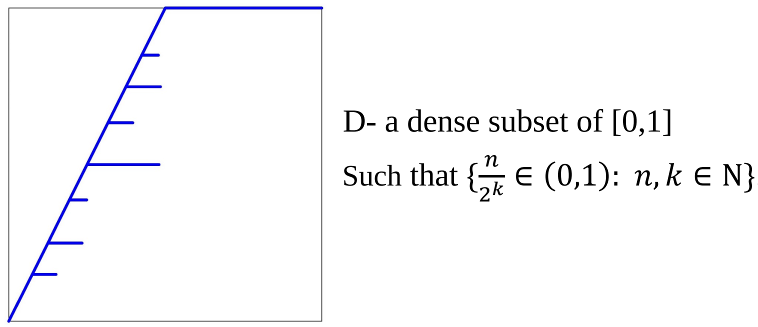


Figure 13. The graphs f_6

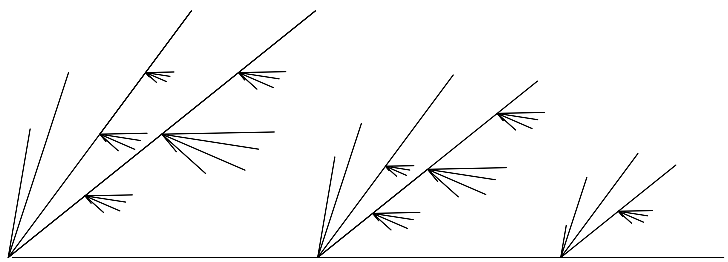


Figure 14. Inverse limit of f_6

connected continuum having trivial shape. Since images of points are not necessarily zero-dimensional, we can not use Nall's theorem. To show 1-dimensionality of $\varprojlim \{[0,1], f_7\}$ instead, we use Ingram's theorem that says that if graphs of compositions of functions are 1-dimensional, then the inverse limit is one-dimensional (see [10, Theorem 4.3, p. 249]). In our case $f_7 \circ f_7 = f_7$, so all compositions have the same one dimensional graph f_7. The inverse limit has a closed set of end-points. In fact the end-points are exactly point will all coordinates in $\{0,1\}$, so it is the Cantor set. Ramification points can be charecterized as points whose coordinates are of the form $\left(x_1, x_2, ..., x_n, \frac{1}{2}, \frac{1}{2}, ...\right)$ where $x_i \in \left\{0, \frac{1}{3}, 1\right\}$ for $i \in \{1, 2, ..., n\}$. $\varprojlim \{[0,1], f_7\}$ pictured in Figure 16.

Example 3.10. Let $f_8 : [0,1] \to 2^{[0,1]}$ be the multivalued function whose graph is pictured in Figure 17. Its graph consists of three segments joining $\langle 0, 1/3\rangle$ to $\langle 1, 1/3\rangle$; $\langle 0, 2/3\rangle$ to $\langle 1, 2/3\rangle$ and $\langle 1/2, 0\rangle$ to $\langle 1/2, 1\rangle$. As before the inverse limit $\varprojlim \{[0,1], f_8\}$ is a dendrite. This

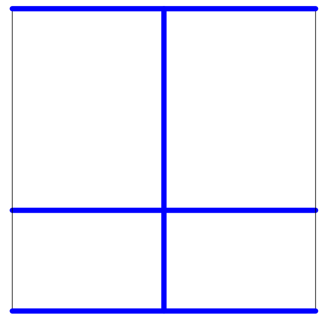


Figure 15. The graph of f_7

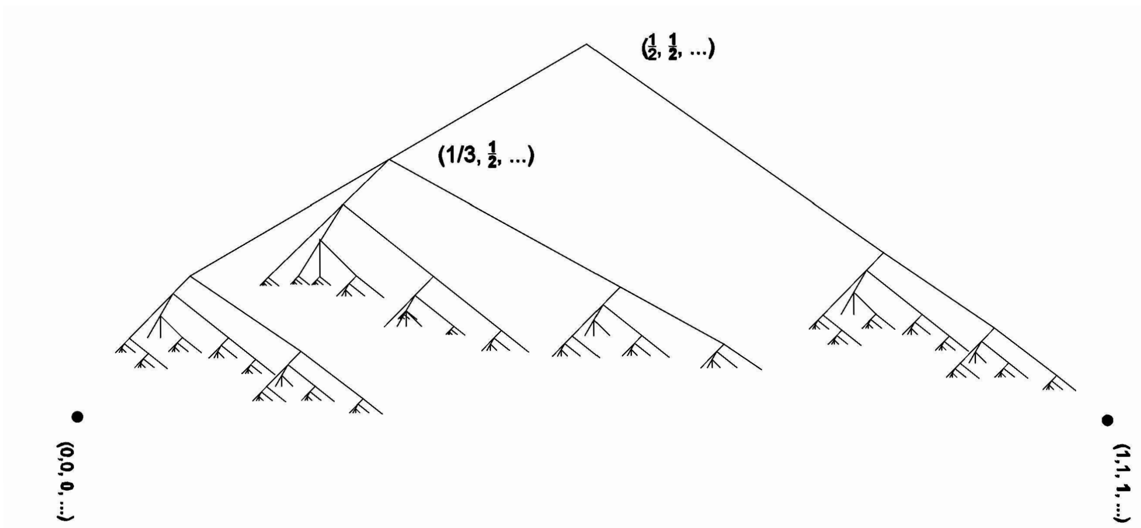


Figure 16. Inverse limit of f_7

time all ramification points are of order 4, but we have isolated end-points. those are points of the form $\left(x_1, x_2, ..., x_n, 0, \frac{1}{2}, \frac{1}{2}, ...\right)$ or $\left(x_1, x_2, ..., x_n, 1, \frac{1}{2}, \frac{1}{2}, ...\right)$, where $x_1, x_2, \ldots, x_n \in \left\{\frac{1}{3}, \frac{2}{3}\right\}$. They form a dense subset of the set of end-points. The inverse limit is pictured in Figure 18.

Example 3.11. Let $f_9 : [0,1] \to 2^{[0,1]}$ be the multivalued function whose graph is on the left side of Figure 19.Its graph consists of four segments joining $\langle 0,0\rangle$ to $\langle 1,0\rangle$; $\langle 1,0\rangle$ to $\langle 1/2,1/2\rangle$; $\langle 1/2,1/2\rangle$ to $\langle 1/2,1\rangle$; $\langle 1/2,1\rangle$ to $\langle 0,1\rangle$. To show that $\varprojlim\{[0,1], f_9\}$ is a dendrite, we use an argument similar to that in Example 3.9. First observe that $f_9^2 = f_9 \circ f_9$

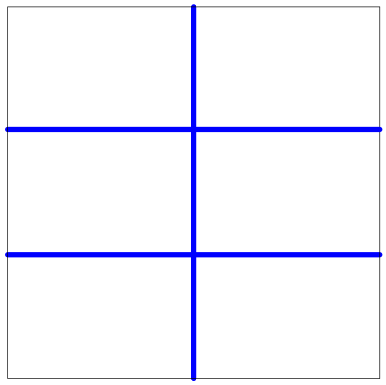


Figure 17. The graph of a f_8

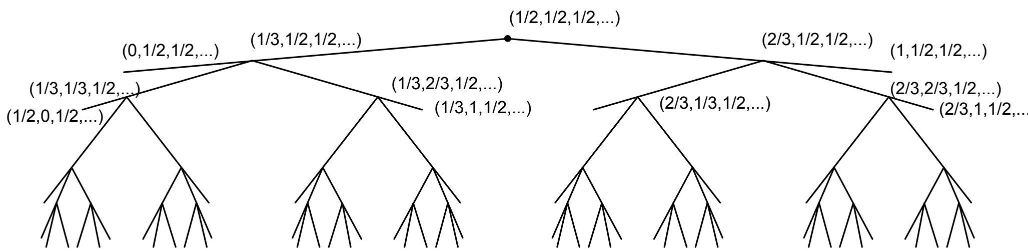


Figure 18. Inverse limit of f_8

is a multivalued function whose graph is on the right side of Figure 19. Next observe that $f_9^n = f_9^2$, so all compositions f_9^n have 1-dimensional graphs, therefore $\varprojlim \{[0,1], f_9\}$ is one dimensional. The inverse limit $\varprojlim \{[0,1], f_9\}$ is homeomorphic to the dendrite G_3 since all ramification points are of order 3 and the set of end-points is closed with no isolated end-points. Note that $\varprojlim \left\{[0,1], f_9^2\right\}$ is homeomorphic to the dendrite G_3 as observed by Ingram (see [9, Example 2.22, p. 39]). Let us recall that in general $\varprojlim \{[0,1], f_9\}$ and $\varprojlim \left\{[0,1], f_9^2\right\}$ do not have to be homeomorphic (see [11, Example 133, p. 107]).

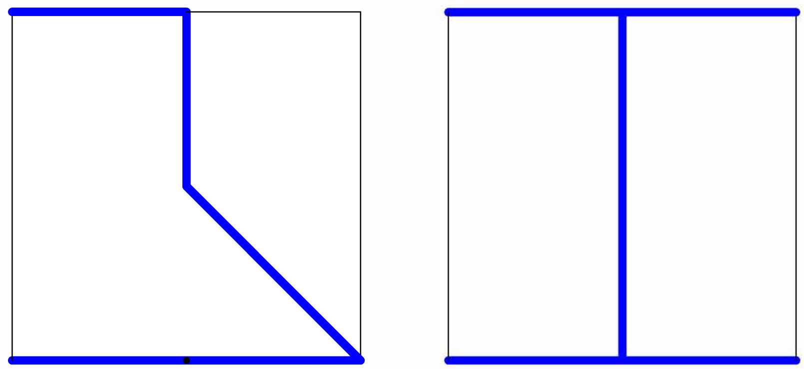


Figure 19. The graph of a f_9 and f_9^2

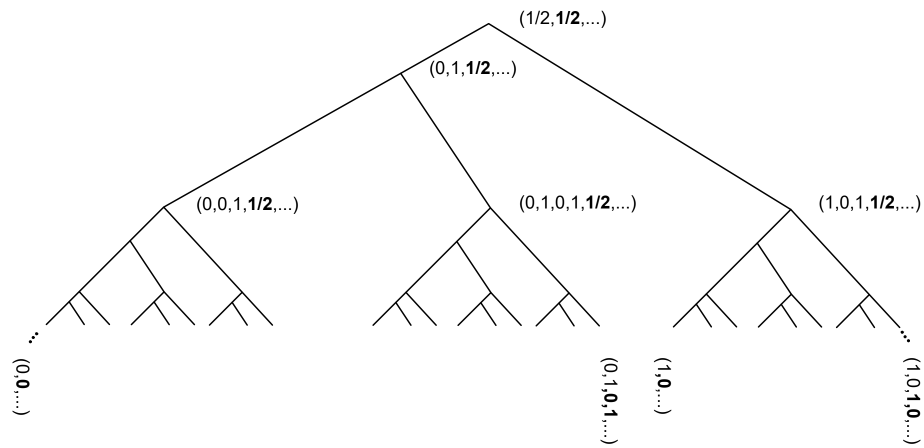


Figure 20. Inverse limit of f_9

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III. INVERSE LIMITS AND ATOMIC PROJECTIONS

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ABSTRACT

We consider generalized inverse limits of continua with bonding functions F_n that have the projection of Graph(F_n) onto the second (first) factor atomic and images (pre-image) of points are zero-dimensional. For such bonding functions we show that under some easily verified conditions that if the first (all) factor space(s) has a certain property then the inverse limit space must have this property. The properties considered include; hereditary decomposability, hereditary indecomposability, hereditary unicoherence, arc-likeness, and tree-likeness. We illustrate the theorems by several examples. **Keywords:** atomic map, arc-like, generalized inverse limit, hereditarily decomposable, hereditarily unicoherent, tree-like

1. INTRODUCTION AND DEFINITIONS

We began the investigation herein thinking about chainability of inverse limit spaces having set valued bonding functions. In [2] W. T. Ingram showed that if a linearized version of the $\sin(1/x)$ functions is used as the single bonding function in an inverse limit system on $[0, 1]$ the resulting inverse limit space was chainable. He followed this up, answering a question from one of us, by showing [3] that the inverse system consisting of a sequence of sinusoids as bonding functions has a chainable inverse limit. In a related result [6] James P. Kelly considered inverse systems with a single irreducible function as it's bonding function. Kelly's work generalized Ingram's earlier result on the linearized $\sin(1/x)$. Sinusoids do not necessarily satisfy the conditions to be an irreducible function. We need to introduce some notations concerning inverse limits. Given a sequence of continua X_n and upper semi-continuous functions $F_n : X_{n+1} \to 2^{X_n}$, we denote by $\varprojlim \{X_n, F_n\} = \{(x_1, x_2, ...) : x_n \in F_n(x_{n+1})\}$. Similarly, when X_n and F_n are known, and $i \leq j$, we denote by $G_{i,j} = \{(x_i, x_{i+1}, ..., x_j) : x_k \in F_k(x_{k+1})$ for $k \in \{i, i+1, ..., j-1\}\}$. If $i = j$, we identify $G_{i,i}$ with X_i. For a natural number n let $\alpha_n : \text{Graph}(F_n) \to X_n$ and $\beta_n : \text{Graph}(F_n) \to X_{n+1}$ be the projections. For m, i, j, n such that $m \leq i \leq j \leq n \leq \infty$ we denote by $\rho_{i,j}^{m,n}$ the projection $\rho_{i,j}^{m,n} : G_{m,n} \to G_{i,j}$, where $G_{1,\infty}$ denotes $\varprojlim \{X_n, F_n\}$. If $i = j$ we write $\rho_i^{m,n}$ in place of $\rho_{i,i}^{m,n}$. Note that for each natural number i, the set $G_{i,i+1}$ is equal to the Graph(F_i^{-1}). We also denote by π_k the projection from $\varprojlim\{X_n, F_n\}$ to the k^{th} coordinate. In this paper we consider a sequence of upper semi-continuous bonding functions, F_n, with the property that for each n the projection α_n (β_n) is an atomic map and for every $x \in X_{n+1}$ the image $F_n(x)$ (every $x \in X_n$ the preimage $F^{-1}(x)$) is zero-dimensional. All of the examples in [2] as well as the sinusoids in [3] satisfy these conditions however Kelly's irreducible functions may not. The key step in our work is showing that these properties of the bonding functions F_n imply that the single valued projection mappings $\rho_{1,n}^{1,n+1}$ must be atomic mappings, see Theorem 3.3 and Corollary 3.4. Combining this with the work of T. Maćkowiak ([11]) on atomic mappings we are able to see when the inverse limit space $\varprojlim\{G_{1,n}, \rho_{1,n}^{1,n+1}\} = \varprojlim\{X_n, F_n\}$ is chainable as well as having other atomic pre-invariant properties. Moreover, factor spaces need not be limited to being an arc.

2. ATOMIC PRE-INVARIANTS

Definition 2.1. *A subcontinua K of a compact metric space X is called a terminal continuum if every subcontinua of X which intersects K and its complement contains K.*

Note this definition of a terminal continuum agrees with T. Maćkowiak's usage [11]. W. T. Ingram allows K to be a subset, not necessarily a subcontinua, of X and calls such sets C-sets. For him terminal continua have a different meaning, see [2].

Definition 2.2. *A mapping $f : X \to Y$ from a continuum X onto a continuum Y is said to be atomic if for each subcontinuum K of X such that the set $f(K)$ is not degenerate we have $K = f^{-1}(f(K))$.*

A basic fact connecting terminal continua and atomic mappings is the following.

Theorem 2.3. *Let $f : X \to Y$ be a mapping. Then f is atomic if and only if for each $y \in Y$, $f^{-1}(y)$ is a terminal continuum.*

Definition 2.4. *A topological property $\mathcal{P}$ of continua is called an atomic pre-invariant if for any atomic map $f : X \to Y$ between continua X and Y, if $Y \in \mathcal{P}$ and for every $y \in Y$ the preimage $f^{-1}(y) \in \mathcal{P}$, then $X \in \mathcal{P}$.*

The following was shown by T. Maćkowiak ([11], Proposition 11 (ii-iii), p. 537).

Theorem 2.5. *The following properties are atomic pre-invariants:*

1. *atriodicity;*
2. *hereditary indecomposability;*
3. *hereditary decomposability;*
4. *hereditary unicoherence;*
5. *hereditary decomposability and arc-likeness;*
6. *hereditary decomposability and tree-likeness;*
7. *hereditary decomposability and acyclicity.*

Definition 2.6. *A topological property $\mathcal{P}$ of continua is called a strong atomic pre-invariant if for any atomic map $f : X \to Y$ between continua X and Y, if $Y \in \mathcal{P}$, then $X \in \mathcal{P}$.*

The following was shown by T. Maćkowiak ([11], Proposition 11 (i), p. 537).

Theorem 2.7. *The following properties are strong atomic pre-invariants:*

1. *decomposability;*
2. *indecomposability;*
3. *unicoherence;*
4. *discoherence;*
5. *irreducibility;*

3. MAIN THEOREMS

Before we formulate and prove the main theorems we need to show the following Observation, Lemma and Theorem. The observation is probably well-known, see [4], but we include it for completeness.

Observation 3.1. *The inverse limit $\varprojlim\{X_n, F_n\}$ is homeomorphic to $\varprojlim\left\{G_{1,n}, \rho_{1,n}^{1,n+1}\right\}$. Note that the bonding functions $\rho_{1,n}^{1,n+1} : G_{1,n+1} \to G_{1,n}$ are single-valued.*

Proof. The function $h : \varprojlim\{X_n, F_n\} \to \varprojlim\left\{G_{1,n}, \rho_{1,n}^{1,n+1}\right\}$ defined by $h(x_1, x_2, x_3, \dots) = ((x_1), (x_1, x_2), (x_1, x_2, x_3), \dots)$ is the required homeomorphism. ∎

Lemma 3.2. *Let $\{X_n, F_n\}$ be an inverse system of continua such that for every index n and for every $x \in X_{n+1}$, the image $F_n(x)$ is zero-dimensional. If C is a subcontinuum of $G_{1,n}$ for some n such that $\rho_i^{1,n}(C)$ is degenerate for some $i \leq n$, then $\rho_j^{1,n}(C)$ is degenerate for $j < i$.*

Proof. Note that $\rho^{1,n}_{i-1}(C) \subseteq F_{i-1}(\rho^{1,n}_{i}(C))$ is a zero-dimensional continuum hence degenerate. The result follows by induction. ■

Theorem 3.3. *Let $\{X_n, F_n\}$ be an inverse system of continua with upper semi-continuous bonding functions such that for each n the projection $\alpha_n : \mathrm{Graph}(F_n) \to X_n$, is atomic and for every index n and for every $x \in X_{n+1}$, the image $F_n(x)$ is zero-dimensional. Then the projections $\rho^{1,n+1}_{1,n} : G_{1,n+1} \to G_{1,n}$ are also atomic.*

Proof. Let $\alpha_n : \mathrm{Graph}(F_n) \to X_n$ be the projection. Note that $\mathrm{Graph}(F_n) \subseteq X_{n+1} \times X_n$. We also need the projection $\alpha'_n : G_{n,n+1} \to X_n$. Note that α_n and α'_n are essentially the same function, the only difference is the order of coordinates: $\alpha'_n(x_n, x_{n+1}) = \alpha_n(x_{n+1}, x_n) = x_n$. In particular, α'_n is an atomic function. By Theorem 2.3 we need to show that $(\rho^{1,n+1}_{1,n})^{-1}(a_1, a_2, ..., a_n)$ is terminal subcontinuum of $G_{1,n+1}$. Note that $(\rho^{1,n+1}_{1,n})^{-1}(a_1, a_2, ..., a_n) = \{(a_1, a_2, ..., a_n)\} \times F^{-1}_n(a_n)$ and denote this set by A. Suppose that A is not terminal. Thus there exists a continuum C in $G_{1,n+1}$ that intersects A and its complement, and does not contain A. Choose points $\mathbf{a} = (a_1, a_2, ..., a_n, a_{n+1}) \in A \cap C$ and $\mathbf{b} = (a_1, a_2, ..., a_n, b_{n+1}) \in A \setminus C$. Let $\mathcal{A}$ be an order arc in the hyperspaces of subcontinua of $G_{1,n+1}$ from $\{\mathbf{a}\}$ to C. Let B be the maximal point of the order arc $\mathcal{A}$ such that $B \subseteq A$. Let B' be a point of $\mathcal{A}$ a little bigger than B. Precisely let B' be a point of $\mathcal{A}$ such that $B \subsetneq B'$ and $b_{n+1} \notin \rho^{1,n+1}_{1,n+1}(B')$. Then $\rho^{1,n+1}_{n,n+1}(\mathcal{A})$ is an order arc in $G_{n,n+1}$ starting from $\{(a_n, a_{n+1})\}$. Denote $P = \rho^{1,n+1}_{n,n+1}(A \cup B') \subseteq G_{n,n+1}$. Note that $(a_n, a_{n+1}) \in P$, while P does not contain a point with the second coordinate b_{n+1}. We claim that $\alpha'_n(P)$, which is equal to $\rho^{1,n+1}_{n}(A \cup B')$, is nondegenerate. Otherwise, by Lemma 3.2 all projection $\rho^{1,n+1}_{i}(A \cup B')$ for $i \leq n$ would be degenerate, contrary to the fact that B' is not a subset of A. Then $(a_n, b_{n+1}) \in (\alpha')^{-1}_n(\alpha'_n(P))$ and $(a_n, b_{n+1}) \notin P$, contrary to α'_n being atomic. ■

Note that one can reverse the order and obtain the following corollary which we will need later.

Corollary 3.4. *Let $\{X_n, F_n\}$ be an inverse system of continua with upper semi-continuous bonding functions such that for each n the projection $\beta_n : \mathrm{Graph}(F_n) \to X_{n+1}$, is atomic and for every index n and for every $x \in X_n$, the preimage $F_n^{-1}(x)$ is zero-dimensional. Then, for $1 \leq i \leq n$, the projections $\rho^{i,n+1}_{i+1,n+1} : G_{i,n+1} \to G_{i+1,n+1}$ are atomic.*

Theorem 3.5. *Let $\{X_n, F_n\}$ be an inverse system of continua and $\mathcal{P}$ be a topological property such that:*

1. *for each n the projection $\alpha_n : \mathrm{Graph}(F_n) \to X_n$ is atomic;*
2. *the space X_1 has the property $\mathcal{P}$;*
3. *for every n, for every $x \in X_n$ the preimage $F_n^{-1}(x)$ has property $\mathcal{P}$;*
4. *property $\mathcal{P}$ is an atomic pre-invariant;*
5. *property $\mathcal{P}$ is preserved under (single valued) inverse limits with atomic bonding mappings;*
6. *for every index n and for every $x \in X_{n+1}$, the image $F_n(x)$ is zero-dimensional;*

Then the inverse limit $\varprojlim \{X_n, F_n\}$ has the property $\mathcal{P}$.

Proof. By Theorem 3.3 all the projections $\rho^{1,n+1}_{1,n}$ are atomic. Since $X_1 = G_{1,1}$ has property $\mathcal{P}$, all preimages $(\rho^{1,2}_{1,1})^{-1}(x)$, which are homeomorphic to $F_1^{-1}(x)$, have property $\mathcal{P}$. Since $\mathcal{P}$ is an atomic preinvariant, we conclude that $G_{1,2}$ has properly $\mathcal{P}$. Similarly, since $\rho^{1,3}_{1,2}$ is atomic, the preimages $(\rho^{1,3}_{1,2})^{-1}(x_1, x_2)$ which are homeomorphic to $F_2^{-1}(x_2)$ have property $\mathcal{P}$. Continuing inductively, all continua $G_{1,3}, G_{1,4}, \ldots, G_{1,n}, \ldots$ have property $\mathcal{P}$. Finally, the inverse limit $\varprojlim \{X_n, F_n\}$ is homeomorphic to the inverse limit $\varprojlim \left\{G_n, \rho^{1,n+1}_{1,n}\right\}$ by Observation 3.1. Thus the conclusion follows from (5). ■

The proof of our next Theorem is analogous to one of Theorem 3.5 and it is left to the reader.

Theorem 3.6. *Let $\{X_n, F_n\}$ be an inverse system of continua and $\mathcal{P}$ be a topological property such that:*

1. *for each n the projection $\alpha_n : \mathrm{Graph}(F_n) \to X_n$ is atomic;*

2. *the space X_1 has the property $\mathcal{P}$;*

3. *property $\mathcal{P}$ is a strong atomic pre-invariant;*

4. *property $\mathcal{P}$ is preserved under (single valued) inverse limits with atomic bonding mappings;*

5. *for every index n and for every $x \in X_{n+1}$, the image $F_n(x)$ is zero-dimensional;*

Then the inverse limit $\varprojlim \{X_n, F_n\}$ has the property $\mathcal{P}$.

Theorem 3.7. *Let $\{X_n, F_n\}$ be an inverse system of continua and $\mathcal{P}$ be a topological property such that:*

1. *for each n the projection $\beta_n : \mathrm{Graph}(F_n) \to X_{n+1}$ is atomic;*

2. *all spaces X_n have the property $\mathcal{P}$;*

3. *for every n, for every $x \in X_{n+1}$ the image $F_n(x)$ has property $\mathcal{P}$;*

4. *property $\mathcal{P}$ is an atomic pre-invariant;*

5. *property $\mathcal{P}$ is preserved under (single valued) inverse limits;*

6. *for every index n and for every $x \in X_n$, the preimage $F_n^{-1}(x)$ is zero-dimensional;*

Then the inverse limit $\varprojlim \{X_n, F_n\}$ has the property $\mathcal{P}$.

Proof. Since $X_{n+1} = G_{n+1,n+1}$ has property $\mathcal{P}$, all preimages $(\rho_{n+1}^{n,n+1})^{-1}(x)$ which are homeomorphic to $F_n(x)$ have property $\mathcal{P}$, and, by Corollary 3.4, $\rho_{n+1}^{n,n+1}$ is atomic. Since $\mathcal{P}$ is an atomic preinvariant, we conclude that $G_{n,n+1}$ has properly $\mathcal{P}$. Again, $\rho_{n,n+1}^{n-1,n+1} : G_{n-1,n+1} \to$

$G_{n,n+1}$ is atomic by Corollary 3.4, so $G_{n_1,n+1}$ has property $\mathcal{P}$. Continuing inductively, we conclude that $G_{i,n+1}$has property $\mathcal{P}$ for all $i \leq n+1$, so in particular $G_{1,n}$ has property $\mathcal{P}$. Finally, the inverse limit $\varprojlim \{X_n, F_n\}$ is homeomorphic to the inverse limit $\varprojlim \left\{G_n, \rho_{1,n}^{1,n+1}\right\}$ by Observation 3.1. Thus the conclusion follows from (5). ■

4. APPLICATIONS

Recall α_n and β_n denoted projections from Graph F_n to X_n and X_{n+1} respectively.

4.1. EXAMPLES WHEN α_N ARE ATOMIC

Example 4.1. W. T. Ingram considered in [2, Example 5.5, p. 339] the inverse limits $X_1 = \varprojlim \{[0,1], f_1\}$ with the bonding function f_1 whose graph is pictured in Figure 1 (see also [7, First part of Example 5.3, p. 74]). Here the graph of f_1 is the union of a segment joining $\langle 0,0\rangle$ with $\langle 1,0\rangle$, and sequence of segments joining $\langle 0,a_n\rangle$ with $\langle 1,b_n\rangle$ and segments joining $\langle 1,b_n\rangle$ with $\langle 0,a_{n+1}\rangle$ for $n \in \mathbb{N}$ such that:

1. $b_1 = 1$;
2. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$;
3. $b_n > a_n > b_{n+1} > 0$ for all $n \in \mathbb{N}$.

He proved that the inverse limit is chainable and decomposable. It follows from Theorem 3.5 that X_1 is a hereditarily decomposable arc-like continuum. We will show that each nondegenerate subcontinuum of X_1 contains a copy of the whole continuum X_1, in particular, X_1 contain no arcs. To this aim, let L be a nondegenerate proper subcontinuum of X_1. Then there is an index n_0 such that $\pi_{n_0}(L)$ is nondegenerate. Since slopes of the segments in the graph of f_1 are strictly between -1 and 1, there is an index n such that $\pi_n(L)$ contains 0. Then L contains a point of the form $\langle c_1, c_2, ..., c_n, c_{n+1}, ...\rangle$, where $c_n = 0$, and thus L contains $(\{\langle c_1, c_2, ..., c_n\rangle\} \times [0,1] \times [0,1] \times ...) \cap X_1$ which is homeomorphic to X_1. The condition that each subcontinuum of X_1 contains a copy of X_1 implies that the continuum is pointwise self-homeomorphic as defined in [1, Definition 2.1]. The set of end-points of X_1 is $\{\langle 1,1,1\ldots\rangle\} \cup \{0,1\} \times \{0,1\} \times \ldots$, so it is homeomorphic a point unioned with the Cantor set.

Example 4.2. Now consider the bonding function f_2 shown in Figure 1 which is the reflection of the image of f_1 about the line $x = 1/2$. Again, from Theorem 3.5 we see that $X_2 = \varprojlim \{[0,1], f_2\}$ is a hereditarily decomposable arc-like continuum. Similarly, X_2 is pointwise self-homeomorphic, the set of end-points is the Cantor set and X_2 contains no arcs. But X_1 and X_2 are not homeomorphic. To see this, we take a small detour. The following theorem summarizes Kuratowski's theory of irreducible continua in the case of hereditarily decomposable continua, see [8], [9], and [10, Chap 48, Sec VIII, p 219]. Note that in the quoted articles Kuratowski used the language of decompositions, rather than functions.

Theorem 4.3. *If X is a hereditarily decomposable continuum, then there is a monotone function $m : X \to [0,1]$ such that:*

1. *X is irreducible between any pair of points from $m^{-1}(0)$ and $m^{-1}(1)$;*
2. *the function m is the finest possible, i.e. for any monotone function $f : X \to [0,1]$ there is a monotone function $g : [0,1] \to [0,1]$ such that $f = g \circ m$.*

As a consequence, for hereditarily decomposable continua X, the set $I(X) = \{x \in X :$ there exists $y \in X$ such that X is irreducible between x and $y\}$ is the union of two disjoint continua: $m^{-1}(0)$ and $m^{-1}(1)$. Here $I(X_1)$ is the union of $\{\langle 1,1,\ldots\rangle\}$ and $\{0\} \times [0,1] \times [0,1] \times \ldots \cap X_1$ which is homeomorphic to X_1 while $I(X_2)$ is the union of the two subcontinua $\{0\} \times [0,1] \times [0,1] \times \ldots \cap X_2$ and $\{1\} \times \{0\} \times [0,1] \times [0,1] \times \ldots \cap X_2$, each of which is homeomorphic to X_2.

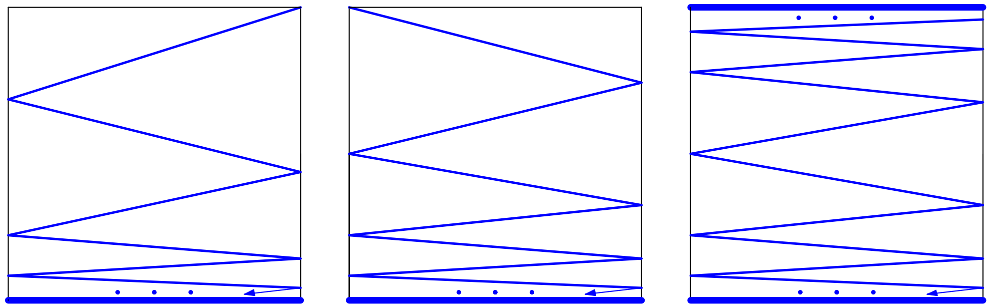

Figure 1. Graphs of f_1, f_2, f_3 respectively

Theorem 4.4. *If $h_i[0,1] \to 2^{[0,1]}$ is a sequence of surjective usc functions and there is a point $c \in [0,1]$ such that $h_1(c)$ is not zero dimensional then $H = \varprojlim \{[0,1], h_i\}$ contains an arc.*

Proof. Let $[a,b] \subseteq h_1(c)$. Since $\langle c, x_2, x_3, \ldots \rangle$ is a point in H, $\{\langle t, c, x_2, x_3, \ldots \rangle : t \in [a,b]\}$ is an arc in H. ■

Example 4.5. Let $f_3 : [0,1] \to 2^{[0,1]}$ be the multivalued function whose graph is pictured in Figure 1. Here the graph of f_3 is the union of a segment joining $\langle 0,0 \rangle$ with $\langle 1,0 \rangle$, and a segment joining $\langle 0,1 \rangle$ with $\langle 1,1 \rangle$, and a sequence of segments joining $\langle 0, a_n \rangle$ with $\langle 1, b_n \rangle$ and segments joining $\langle 1, b_n \rangle$ with $\langle 0, a_{n+1} \rangle$ for $n \in \mathbb{Z}$ such that:

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 1$;
2. $\lim_{n\to-\infty} a_n = \lim_{n\to-\infty} b_n = 0$;
3. $0 < a_n < b_n < a_{n+1} < b_{n+1} < 1$ for all $n \in \mathbb{Z}$.

Again, the inverse limit $X_3 = \varprojlim \{[0,1], f_3\}$ is a hereditarily decomposable arc-like pointwise self-homeomorphic continuum and the set of end-points is the Cantor set. Note that $I(X_3)$ is the union of the two subcontinua $\{0\} \times [0,1] \times [0,1] \times \ldots \cap X_3$ and $\{1\} \times [0,1] \times [0,1] \times \ldots \cap X_3$. Therefore, we may ask the following.

Problem 4.6. *Are the continua X_2 and X_3 homeomorphic?*

Question 4.7. *Are any or all of X_1, X_2 and X_3 Kelley continua?*

Example 4.8. Let $f_4 : [0,1] \to 2^{[0,1]}$ be the multivalued function whose graph is pictured in Figure 2. Here the graph of f_4 is the union of a segment joining $\langle 0,0\rangle$ with $\langle 1,0\rangle$, and a segment joining $\langle 0,1\rangle$ with $\langle 1,1\rangle$ and sequence of segments joining $\langle a_n, b_n\rangle$ with $\langle c_n, d_n\rangle$ and segments joining $\langle c_n, d_n\rangle$ with $\langle a_{n+1}, b_{n+1}\rangle$ for $n \in \mathbb{Z}$ such that:

1. for all $n \in \mathbb{Z}$, $b_n < d_n < b_{n+1} < d_{n+1}$;
2. $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} d_n = 1$ and $\lim\limits_{n\to-\infty} b_n = \lim\limits_{n\to-\infty} d_n = 0$;
3. for all $n \geq 0$, $1/2 > a_n > a_{n+1} > 0$ and $1/2 < c_n < c_{n+1} < 1$;
4. $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} c_n = 1$;
5. for all $n < 0$, $1/2 > a_{n+1} > a_n > 0$ and $1/2 < c_{n+1} < c_n < 1$;
6. $\lim\limits_{n\to-\infty} a_n = 0$ and $\lim\limits_{n\to-\infty} c_n = 1$;

Then the inverse limit $X_4 = \varprojlim\{[0,1], f_4\}$ is hereditarily decomposable arc-like continuum by Theorem 3.7. Since f_4^{-1} is one-to-one on $(0,1)$, $((0,1)\times(0,1)\times\ldots)\cap X_4$ is a ray limiting on two disjoint copies of X_4, namely on $(\{0\}\times[0,1]\times[0,1]\times\ldots)\cap X_4$ and $(\{1\}\times[0,1]\times[0,1]\times\ldots)\cap X_4$.

Example 4.9. Let $f_5 : [0,1] \to 2^{[0,1]}$ be the multivalued function whose graph is pictured in Figure 3. Here the graph of f_5 consists of $[0,1]\times C$, where C is a Cantor set, together with rays in each rectangle $[0,1]\times D$, where D is an open interval in the complement of C, which converges to the horizontal boundaries of the rectangle. As before, the inverse limit $X_5 = \varprojlim\{[0,1], f_5\}$ is a hereditarily decomposable arc-like continuum. Note that for each $c \in C$, $(\{c\}\times[0,1]\times[0,1]\times\ldots)\cap X_5$ is a copy of X_5 contained in X_5. If the rays are like the ray in the graph of f_3 then X_5 contains no arcs while if the rays are like the ray in the graph of f_4 then we have a ray in X_5 converging to the copies of X_5.

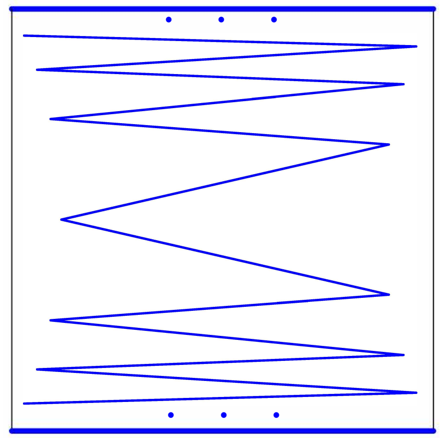

Figure 2. Graph of f_4

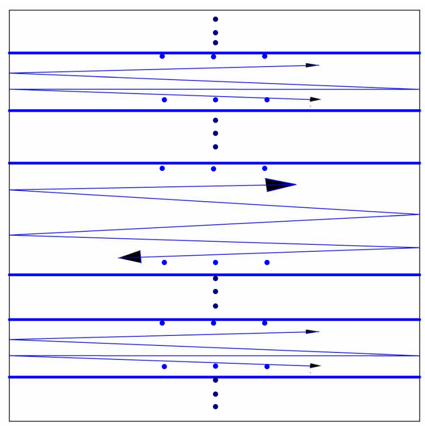

Figure 3. Graph of f_5

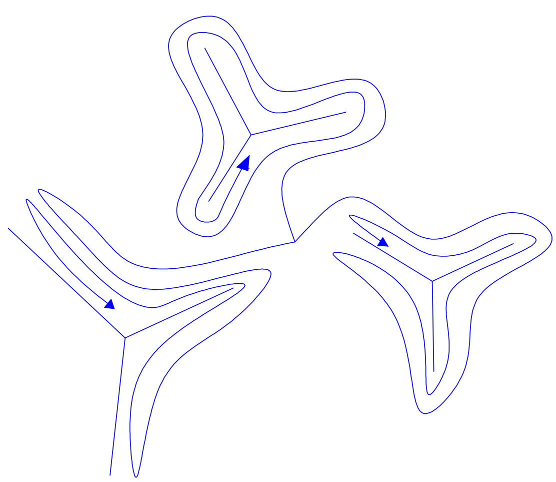

Figure 4. Graph of f_6

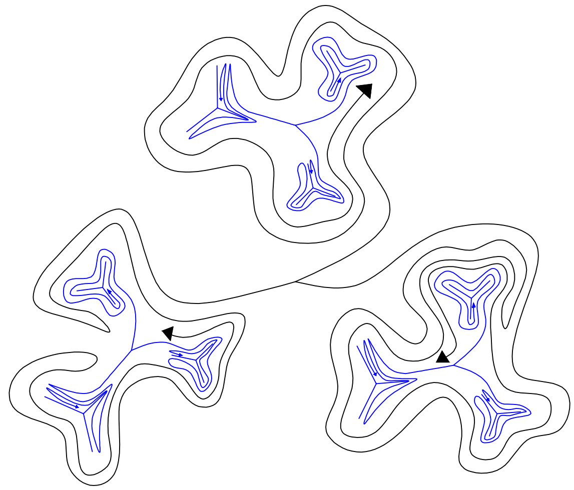

Figure 5. $G_{1,3}$ for f_6

Example 4.10. In this example we want to show an inverse limit of spaces different than an arc. Let T denote the simple triod, and let $f_6 : T \to 2^T$ be the multivalued function whose graph is pictured in Figure 4. The image under f_6 of a point in T is a single point of T or the whole T. Let a, b, c be end points of T. Then f_6 restricted to $T \setminus \{a, b, c\}$ is a single valued one-to-one and continuous, while $f_6(a) = f_6(b) = f_6(c) = T$. The map α_n shrinks all three limiting triods to points. Then the inverse limit $X_6 = \varprojlim \{[0, 1], f_6\}$ is a hereditarily decomposable tree-like continuum by Theorem 3.7. It contains three rays approximating (in different ways) copies of X_6. The picture shown in Figure 5 is $G_{1,3}$ for $X_n = T$ and $F_n = f_6$. The inverse limit X_6 is not a Kelley continuum because $\rho^{1,\infty}_{1,2} : X_6 \to \mathrm{Graph}(f_6)$ is a monotone mapping onto Graph(f_6) which is not a Kelley continuum, contrary to theorem [13] saying that a confluent image of a Kelley continuum is Kelley.

4.2. EXAMPLES WHEN β_N ARE ATOMIC

Observation 4.11. *For $i = 1, ..., 6$, let $g_i = f_i^{-1}$ and $Y_i = \varprojlim \{[0, 1], g_i\}$. Then by Theorem 3.7, Y_i is arc-like if $i < 6$ and tree-like if $i = 6$. Moreover, by [12, Theorem 3.2, p. 1024] the continua $Y_1, Y_2, Y_3,$ andY_5 are indecomposable. Continua Y_4 and Y_6 are also*

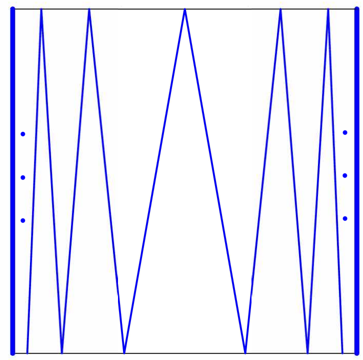

Figure 6. Graph of g_3

indecomposable, but they do not satisfy the assumptions of [12, Theorem 3.2, p. 1024]. To show their indecomposability one may use [5, Theorem 212, p. 147]. In fact the later Theorem can be used to show indecomposability of all continua $Y_1 \ldots Y_6$.

Since all of these inverse limit spaces are indecomposable one might wonder if any are hereditarily indecomposable. The following observation shows that they cannot be.

Observation 4.12. *If $X = \varprojlim \{X_n, h_n\}$ with $h_n : X_{n+1} \to X_n$ surjective and there exists $p \in X_2$ such that $h_1(p)$ contains a nondegenerate arc, then $\varprojlim \{X_n, h_n\}$ contains a nondegenerate arc. To see this, let A be a nondegenerate arc contained in $h_1(p)$. Let $p_2 = p$ and $p_{n+1} \in h_n(p_n)$, then $A \times \{\langle p_2, p_3, \ldots \rangle\}$ is the required arc.*

Proposition 4.13. *If C is an nondegenerate subcontinuum of any the inverse limit spaces $Y_1, \ldots, Y_6$ then C contains a nondegenerate arc.*

Proof. For $Y_1, \ldots, Y_5$ let C be a nondegenerate subcontinuum of Y_i. Note that $\pi_1(C)$ is a nondegenerate subcontinuum of $[0, 1]$. We define, by induction, a sequence of possibly degenerate arcs A_n such that:

1. $A_n \subseteq \pi_n(C)$;

2. A_1 is nondegenerate;

3. $g_i|A_{n+1} : A_{n+1} \longrightarrow A_n$ is a homeomorphism if A_{n+1} is nondegenerate.

Then $(A_1 \times A_2 \times \dots) \cap X$ is an arc contained in C. For Y_6 note that for any nondegenerate arc A in T that does not contain an end point of T and for any point $p \in T$ such that $g_6(p) \in A$ there exists an arc B in T that contains p such that $g_6|_B$ is a homeomorphism. Hence we can obtain an arc contained in C as previously. ■

Proposition 4.14. *For Y_i, $i \in \{3, \dots, 6\}$ contains a subset homeomorphic to* Graph$(g_i) \times$ *Cantor set.*

Proof. Note that for an arbitrary sequence $\langle c_1, c_2, \dots \rangle$ with c_i being end points of $[0, 1]$ or T, the set $(X \times X \times \{\langle c_1, c_2, \dots \rangle\}) \cap Y_i$, where X is $[0, 1]$ or T as appropriate, is homeomorphic to Graph(g_i). ■

Corollary 4.15. *The inverse limit spaces Y_i, $i \in \{3, \dots, 6\}$ are non-Suslinean.*

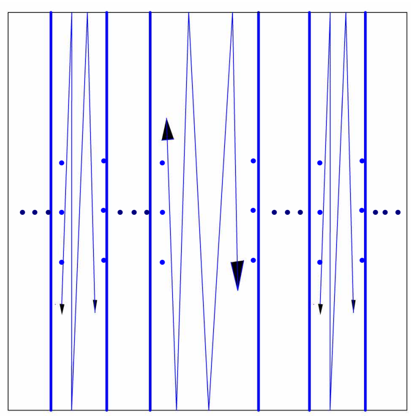

Figure 7. Graph of g_5 or f_5^{-1}

Question 4.16. *Are $Y_1, \dots, Y_5$ Kelley continua?*

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SECTION

2. SUMMARY AND CONCLUSION

After submitting the third paper we learned, by private correspondence, that Ingram and Marsh have submitted a paper [22] that is very similar to ours. In their work they have a condition which is equivalent to our condition that the projection maps β_n or α_n are atomic and a second condition that is weaker than our condition that value of f_n or f_n^{-1} be 0-dimensional. Their result are weaker than ours in that they show only the chainability of the inverse limit space and is limited to having factor spaces being intervals. One possibility for future work would be to see if we could obtain our results using Ingram and Marsh's weaker hypothesis.

Also, a rercent paper by L. Alvin and J. P. Kelly [1] expands the definition of Markov set-valued functions of Črepnjak, M. and Lunder, T. [13] and proves that for similar Markov functions the associated inverse limit spaces are homeomorphic. It may be possible to use their results to make our theorems on atomic projection map apply to a larger class of functions.

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