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Decoupling methods for the time-dependent Navier-Stokes-Darcy interface model

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DECOUPLING METHODS FOR THE TIME-DEPENDENT NAVIER-STOKES-DARCY INTERFACE MODEL

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ABSTRACT

In this research, several decoupling methods are developed and analyzed for approximating the solution of time-dependent Navier-Stokes-Darcy (NS-Darcy) interface problems. This research on decoupling methods is motivated to efficiently solve the complex Stokes-Darcy or NS-Darcy type models, which arise from many interesting real world problems involved with or even dominated by the coupled porous media flow and free flow. We first discuss a semi-implicit, multi-step non-iterative domain decomposition (NIDDM) to solve a coupled unsteady NS-Darcy system with Beavers-Joseph-Saffman-Jones (BJSJ) interface condition and obtain optimal error estimates. Second, a parallel NIDDM is developed to solve unsteady NS-Darcy model with Beavers-Joseph (BJ) interface condition, which is much more complicated than BJSJ interface condition. We overcome the major difficulties in the analysis which arise from nonlinear terms and BJ interface condition. Furthermore, a Lagrange multiplier method is proposed under the framework of the domain decomposition method to overcome the difficulty of non-unique solutions arising from the defective boundary condition. Meanwhile, we propose and analyze an efficient ensemble algorithm, which can significantly improve the computational efficiency, for fast computation of multiple realizations of the stochastic Stokes-Darcy model with a random hydraulic conductivity tensor. Furthermore, we utilize the idea of artificial compressibility, which decouples the velocity and pressure, to construct the decoupled ensemble algorithm to improve computational efficiency further. We prove that the proposed ensemble methods offer long time stability and optimal error estimates under a time-step condition and two parameter conditions.

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TABLE OF CONTENTS

Page

LIST OF ILLUSTRATIONS

Figure Page

LIST OF TABLES

1. INTRODUCTION

Many interesting real-world problems are involved with or even dominated by the coupled porous media flow and free flow, which can be accurately described by the so called Stokes-Darcy or Navier-Stokes-Darcy type models. The Stokes-Darcy interface model has attracted significant attention from scientists and engineers due to its wide range of applications, such as interaction between surface and subsurface flows Discacciati (2004a); Discacciati and Quarteroni (2004a); Hoppe *et al.* (2007); Layton *et al.* (2002), industrial filtrations Ervin *et al.* (2009); Hanspal *et al.* (2006), groundwater system in karst aquifers Cao *et al.* (2010a,c); Gao *et al.* (2018); Han *et al.* (2014), and petroleum extraction Arbogast and Brunson (2007); Arbogast and Lehr (2006); Hou *et al.* (2016).

Moreover, many engineering and geological applications require effective simulations of the coupling of groundwater flows (in porous media) and surface flows. Accurate simulations are usually not feasible due to the fact that it is physically impossible to know the exact parameter values, e.g., the hydraulic conductivity tensor, at every point in the domain as the realistic domains are of large scale and natural randomness occurs at a small scale. Consequently, these uncertainties must be taken into account to obtain meaningful results. The usual way is to model the parameter of interest as a stochastic function that is determined by an underlying random field with a prescribed (usually experimentally determined) covariance structure, and then recast the original deterministic system as a stochastic system. As a result, numerical approximations that involve repeated sampling and simulations pose great challenges on the computer resources and capability.

It is not surprising that many different numerical methods have been proposed and analyzed for the Stokes-Darcy model, such as domain decomposition methods Boubendir and Tlupova (2013); Cao *et al.* (2011); Chen *et al.* (2011); Discacciati *et al.* (2002a); Discacciati and Quarteroni (2004a); Discacciati *et al.* (2007); He *et al.* (2015); Vassilev *et al.* (2014), Lagrange multiplier methods Babuška and Gatica (2010); Gatica *et al.* (2009, 2011); Huang *et al.* (2012); Layton *et al.* (2002), discontinuous Galerkin methods Girault and Rivière (2009); Kanschat and Riviére (2010); Lipnikov *et al.* (2014); Rivière (2005); Rivière and Yotov (2005), multigrid methods Arbogast and Gomez (2009); Cai *et al.* (2009); Mu and Xu (2007), partitioned time stepping methods Kubacki and Moraiti (2015a); Mu and Zhu (2010a); Shan and Zheng (2013a), coupled finite element methods Camano *et al.* (2015); Cao *et al.* (2010a); Karper *et al.* (2009); Márquez *et al.* (2015), mortar finite element methods Galvis and Sarkis (2007); Girault *et al.* (2014), boundary integral methods Boubendir and Tlupova (2009); Tlupova and Cortez (2009), least square methods Ervin *et al.* (2014); Lee and Rife (2014); Münzenmaier and Starke (2011), the Lattice Boltzmann method Fattahi *et al.* (2016), and so on. Recently, the Navier-Stokes-Darcy model has attracted scientists' attention, including the steady state problem Badea *et al.* (2010); Cai *et al.* (2009); Chidyagwai and Rivière (2009); Discacciati and Quarteroni (2009); Girault and Rivière (2009); Hadji *et al.* (2015) and the unsteady problem Çeşmelioğlu *et al.* (2013); Çeşmelioğlu and Rivière (2008, 2009). Compared with the extensively studied Stokes-Darcy model, the unsteady Navier-Stokes-Darcy model is still in great need of continued efforts for developing and analyzing efficient, stable, and accurate numerical methods.

In this dissertation, based on the key idea of the non-iterative DDM for the Stokes-Darcy model Cao *et al.* (2014), we first propose and rigorously analyze a semi-implicit, multi-step, non-iterative DDM to solve the unsteady Navier-Stokes-Darcy system by using finite elements for spatial discretization. Robin boundary conditions between the Navier-Stokes domain and the Darcy domain are constructed by directly re-organizing the terms in the three interface conditions, including the BJSJ condition. Compared with the traditional iterative domain decomposition, which applies a domain decomposition iteration at each time step for the interface information, the non-iterative DDM takes advantage of the solutions obtained in previous time steps to directly predict the interface information without any iteration at the current time step. Multi-step backward differentiation formulae, which can improve the accuracy in time with unconditional stability, are used for the temporal discretization, and spatial discretization is effected by using finite element methods. The k-step backward differentiation formulae ($1 \leq k \leq 5$) with finite elements in spatial discretization are analyzed in a general framework using the multiplier technique while mathematical induction proof is utilized to handle the term arising from the nonlinear advection. Compared with the implicit temporal discretization in Cao *et al.* (2014), we use a semi-implicit scheme to linearize the nonlinear term of Navier-Stokes equation. Moreover, we also develop a parallel, non-iterative, multi-physics domain decomposition method to solve the sophisticated time-dependent NS-Darcy system with BJ interface condition and defective boundary condition. Beavers-Joseph interface condition needs special treatments in both the analysis and the construction of the Robin boundary conditions for the domain decomposition. The nonlinear advection also increases the difficulty of the analysis. Therefore, the analysis for the proposed method in this dissertation is much more difficult than that of Cao *et al.* (2014) and thus needs significant extra efforts, which will be illustrated in detail in the analysis section. Based on the solid foundation built for the domain decomposition method of the NS-Darcy system with BJSJ (BJ) interface condition, we further propose the Lagrange multipliers to deal with this model with a defective boundary condition whose solutions are not unique under the same framework of the domain decomposition method. One interesting finding is that the Lagrange multipliers are time dependent functions instead of constants.

On the other hand, the ensemble algorithm has been extensively studied and tested for ensemble simulations to account for uncertainties in initial conditions and forcing terms Jiang (2015); Jiang *et al.* (2015); Jiang and Layton (2014, 2015); Jiang and Schneier (2018); Mohebujjaman and Rebholz (2017); Neda *et al.* (2016). Some recent works include incorporating model reduction techniques to further reduce computational cost Gunzburger *et al.* (2017a, 2018a) and devising ensemble algorithms to account for various model parameters of Navier-Stokes equations Gunzburger *et al.* (2017b, 2018b), Boussinesq equations

Fiordilino (2018), and a simple elliptic equation Luo and Wang (2018). In this dissertation, we will further develop an efficient ensemble algorithm for the fast computation of multiple realizations of the stochastic Stokes-Darcy model with random hydraulic conductivity (including the one in the interface conditions), source terms, and initial conditions. The solutions are found by solving two smaller decoupled subproblems with two common time-independent coefficient matrices for all realizations, which significantly improves the efficiency for both assembling and solving the matrix systems. The fully coupled Stokes-Darcy system can first be decoupled into two smaller sub-physics problems by the idea of partitioned time stepping, which reduces the size of the linear systems and allows parallel computing for each sub-physics problem. We prove that the ensemble method is long time stable and first-order in time convergent under a time-step condition and two parameter conditions. Furthermore, we utilize the idea of artificial compressibility to further decouple the velocity and pressure in the Stokes equation, which further reduces storage requirements and improves computational efficiency. We prove the long time stability and the convergence for this artificial compressibility ensemble method. Numerical examples are presented to support the theoretical results and illustrate the features of the corresponding algorithm, including the convergence, stability, efficiency, and applicability.

1.1. MODEL INTERFACE PROBLEMS

In this section we introduce the NS-Darcy model with interface conditions and defective boundary conditions. For a brief introduction, we consider the following time-dependent NS-Darcy model with interface conditions and a defective boundary condition on a bounded domain $\Omega = \Omega_D \bigcup \Omega_S \subset \mathbb{R}^d$, $(d = 2, 3)$. See Figure 1.1.

In the porous media region Ω_D, let $\overrightarrow{u}_D$ denote the fluid discharge rate in the porous media, $\mathbb{K} = K\mathbb{I}$ denote the hydraulic conductivity tensor, f_D denote the sink/source term, and ϕ_D denote the hydraulic head. Specifically, $\phi_D = z + \frac{p_D}{\rho g}$ where p_D is the dynamic pressure, z is the height, ρ is the density, and g is the gravity constant. Then the porous

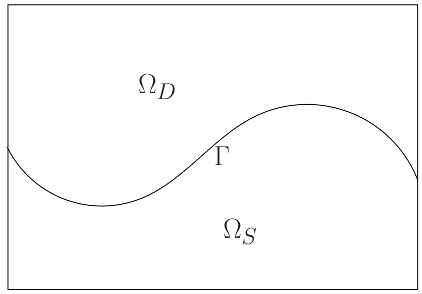

Figure 1.1. Sketch of porous median domain Ω_D, fluid domain Ω_S, and the interface Γ.

media flow is assumed to satisfy the following Darcy equation:

$$\vec{u}_D = -\mathbb{K}\nabla\phi_D, \tag{1.1}$$

$$\frac{\partial \phi_D}{\partial t} + \nabla\cdot\vec{u}_D = f_D,\ t\in[0,T]. \tag{1.2}$$

Eliminating $\vec{u}_D$, we obtain a second-order equation for the Darcy flow:

$$\frac{\partial \phi_D}{\partial t} - \nabla\cdot(\mathbb{K}\nabla\phi_D) = f_D,\ t\in[0,T]. \tag{1.3}$$

In the fluid region Ω_S, let $\vec{u}_S$ denote the fluid velocity, p_S denote the kinematic pressure, $\vec{f}_S$ denote the external body force, and ν denote the kinematic viscosity of the fluid. Before we introduce the complicated Navier-Stokes equation, we first consider the relatively simple Stokes equation.

$$\frac{\partial \vec{u}_S}{\partial t} - \nabla\cdot\mathbb{T}(\vec{u}_S,p_S) = \vec{f}_S,\ t\in[0,T], \tag{1.4}$$

$$\nabla\cdot\vec{u}_S = 0. \tag{1.5}$$

Then we can consider the Navier-Stokes equation:

$$\frac{\partial \vec{u}_S}{\partial t} + (\vec{u}_S\cdot\nabla)\vec{u}_S - \nabla\cdot\mathbb{T}(\vec{u}_S,p_S) = \vec{f}_S,\ t\in[0,T], \tag{1.6}$$

$$\nabla\cdot\vec{u}_S = 0. \tag{1.7}$$

where $\mathbb{T}(\overrightarrow{u}_S, p_S) = 2\nu\mathbb{D}(\overrightarrow{u}_S) - p_S\mathbb{I}$ is the stress tensor and $\mathbb{D}(\overrightarrow{u}_S) = 1/2(\nabla\overrightarrow{u}_S + \nabla^T\overrightarrow{u}_S)$ is the deformation tensor.

Assume that the hydraulic head ϕ_D and the fluid velocity $\overrightarrow{u}_S$ satisfies the homogeneous Dirichlet boundary condition except on Γ , i.e., $\phi_D = 0$ on the boundary $\partial\Omega_D\backslash\Gamma$ and $\overrightarrow{u}_S = 0$ on the boundary $\partial\Omega_S\backslash\Gamma$. Assume that the hydraulic head ϕ_D and the fluid velocity $\overrightarrow{u}_S$ satisfy the following initial conditions:

$$\phi_D(0, x, y) = \phi_0(x, y) \quad and \quad \overrightarrow{u}_S(0, x, y) = \overrightarrow{u}_0(x, y). \tag{1.8}$$

1.1.1. The NS-Darcy System with Interface Conditions. In this section, we consider a coupled Navier-Stokes-Darcy system with interface conditions. Let $\overline{\Gamma} = \overline{\Omega}_D \cap \overline{\Omega}_S$ denote the interface between the fluid and porous media regions. Along the interface Γ, we first impose the following two well-accepted interface conditions:

$$\overrightarrow{u}_S \cdot \overrightarrow{n}_S = -\overrightarrow{u}_D \cdot \overrightarrow{n}_D, \;\; -\overrightarrow{n}_S \cdot (\mathbb{T}(\overrightarrow{u}_S, p_S) \cdot \overrightarrow{n}_S) = g\phi_D, \tag{1.9}$$

where $\overrightarrow{n}_S$ and $\overrightarrow{n}_D$ denote the unit outer normal to the fluid and the porous media regions at the interface Γ, respectively. These two interface conditions are for the continuity of normal velocity and the balance of force normal to the interface. Other equivalent interface conditions are

$$\begin{aligned} &\mathbb{K}\nabla\phi \cdot \overrightarrow{n}_D = \overrightarrow{n}_S \cdot \overrightarrow{n}_S, \\ &-\mathbb{T}(\overrightarrow{u}_S, p_S)\overrightarrow{n}_S = \phi\overrightarrow{n}_S + \frac{\alpha\nu\sqrt{2}}{\sqrt{\text{trace}(\Pi)}}\left[\overrightarrow{u}_S - (\overrightarrow{u}_S \cdot \overrightarrow{n}_S)\overrightarrow{n}_S\right], \end{aligned} \tag{1.10}$$

where Π denotes the permeability of the porous media, α is the Beavers-Joseph constant Beavers and Joseph (1967b), and $\overrightarrow{n}_S = -\overrightarrow{n}_D$ denotes the unit normal vector on Γ towards Ω_D.

Remark 1 *Note that* (1.10) *is equivalent in the following two traditional interface conditions in the literature:*

$$-\mathbb{T}(\overrightarrow{u}_S, p_S)\overrightarrow{n}_S \cdot \overrightarrow{n}_S = \phi$$
$$-\mathbb{T}(\overrightarrow{u}_S, p_S)\overrightarrow{n}_S \cdot \boldsymbol{\tau} = \frac{\alpha\nu\sqrt{2}}{\sqrt{trace(\Pi)}}\overrightarrow{u}_S \cdot \boldsymbol{\tau},$$

where $\boldsymbol{\tau}$ *denotes the unit tangential vector on the interface* Γ.

Moreover, the following boundary conditions are considered:

$$-\mathbb{T}(\overrightarrow{u}_S, p_S)\overrightarrow{n}_S = 0 \quad \text{on} \quad \partial\Omega_S\backslash\overline{\Gamma}, \tag{1.11}$$

$$\mathbb{K}\nabla\phi \cdot \overrightarrow{n}_D = 0 \quad \text{on} \quad \partial\Omega_D\backslash\overline{\Gamma}. \tag{1.12}$$

(1.10) is the Beavers-Joseph-Saffman-Jones (BJSJ) interface condition Jäger and Mikelič (2000); Jones (1973); Saffman (1971a). If we consider the much more complicated Beavers-Joseph (BJ) interface condition Beavers and Joseph (1967b), which is imposed in the tangential direction on the interface, we have

$$-\boldsymbol{\tau}_j \cdot (\mathbb{T}(\overrightarrow{u}_S, p_S) \cdot \overrightarrow{n}_S) \quad = \quad \frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\prod)}}\boldsymbol{\tau}_j \cdot (\overrightarrow{u}_S - \overrightarrow{u}_D), \tag{1.13}$$

where $\boldsymbol{\tau}_j$ $(j = 1, \cdots, d-1)$ denote mutually orthogonal unit vectors tangential to the interface Γ, and $\prod = \frac{\mathbb{K}\nu}{g}$.

1.1.2. The NS-Darcy System with the Defective Boundary and Interface Conditions. Figure 1.2 illustrates a simple system with defective boundary conditions. For example, in a simplified typical karst aquifer system, the free flow is confined in the underground conduit while porous media is surrounding the conduit. The region occupied by the conduit and the porous media are denoted by Ω_S and Ω_D, respectively. On the boundary of Ω_S, we particularly consider $\Gamma_S = \partial\Omega_S\backslash\Gamma = \bigcup\limits_{i=0}^{m} S_i$ for the defective boundary condition.

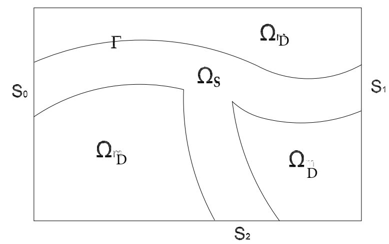

Figure 1.2. Typical components of a karst aquifer.

The governing equations and interface conditions are still (1.3)-(1.13). However, it is often difficult to obtain the velocity data on Γ_S for different applications, but easier to obtain flow rates Q_i on the boundary S_i Formaggia *et al.* (2002); Roscoe *et al.* (1997). Hence we consider the following prescribed flow rate condition on $\Gamma_S = \partial\Omega_S \backslash \Gamma = \bigcup_{i=0}^{m} S_i$:

$$\int_{S_i} \overrightarrow{u}_S \cdot \overrightarrow{n}_S \, \mathrm{d}s = Q_i, \quad \text{for } \ i = 0, 1, \cdots, m, \tag{1.14}$$

where the flow rates Q_i (also called velocity fluxes) are functions of time.

The other boundary and initial conditions we consider for the model are the same as in Section 1.1.1.

1.2. ORGANIZATION OF THE DISSERTATION

In this dissertation, we will discuss the three fundamental aspects for the development of decoupling methods for time-dependent Stokes/Navier-Stokes-Darcy interface model: semi-implicit multi-step non-iterative DDM to solve the unsteady NS-Darcy system with BJSJ interface condition and defective boundary condition; parallel non-iterative multi-physics DDM to solve the time-dependent NS-Darcy system with BJ interface condition and defective boundary condition; and efficient ensemble algorithms for fast computation of multiple realizations of the stochastic Stokes-Darcy interface model. The rest of this dissertation is organized as follows.

In Section 2, we introduce a semi-implicit, multi-step, non-iterative domain decomposition method to solve a coupled time-dependent NS-Darcy system with BJSJ interface condition and the defective boundary condition. Multi-step backward differentiation formulae, which can improve the accuracy in time with unconditional stability, are used for the temporal discretization, and spatial discretization is effected by using finite element methods. A semi-implicit scheme is proposed to linearize the nonlinear convection. In order to prove the convergence of the finite element solution of the proposed method, we derive the error estimate in L^2 norm for the joint Stokes-Darcy Ritz-projection without using H^2 regularity assumption of the elliptic problem corresponding to this joint Ritz-projection. Furthermore, for the NS-Darcy system with defective boundary conditions whose solutions are not unique, a Lagrange multiplier method is proposed under the framework of the semi-implicit, multi-step, non-iterative domain decomposition method. One interesting finding is that the Lagrange multipliers are time-dependent functions instead of constants. Numerical examples are provided to illustrate the proposed methods and verify the theoretical conclusions.

In Section 3, a parallel, non-iterative, multi-physics DDM is prosed to solve a time-dependent NS-Darcy model with BJ interface condition and defective boundary condition. In this method, in order to avoid the traditional iteration for the domain decomposition method at each time step, the interface information, which is needed for the Robin type transmission conditions at the current time step, is directly predicted based on the numerical solution of the previous time steps. We use a series of technical treatments to overcome the major difficulties in the analysis that arise from nonlinear terms and BJ interface condition. Furthermore, we develop a Lagrange multiplier method under the framework of the domain decomposition method to overcome the difficulty of non-unique solutions arising from the defective boundary condition. Numerical examples are provided to illustrate the features of the proposed method.

In Section 4, we first propose and analyze an efficient ensemble algorithm for fast decoupled computation of multiple realizations of the stochastic Stokes-Darcy model with random hydraulic conductivity (including the one in the interface conditions), source terms, and initial conditions. This proposed algorithm results in one common coefficient matrix for all realizations at each time step, which allows the use of efficient iterative or direct methods for solving the linear systems at greatly reduced computational cost. Moreover, it also decouples the original coupled problem into two sub-physics problems, which reduces the size of the linear systems to be solved and allows parallel computation of the two sub-physics problems. Furthermore, based on the idea of the ensemble method, we utilize the idea of artificial compressibility and partitioned time-stepping methods to construct a new decoupled ensemble algorithm to solve the stochastic Stokes-Darcy interface model. We prove these ensemble methods are long time stable and convergent under a time-step condition and two parameter conditions. Numerical examples are presented to support the theoretical results and illustrate the application of these algorithms.

In Section 5, we draw some conclusions for this dissertation. Some future plans are also discussed in this section.

This dissertation consists of material from one published paper Jiang and Qiu (2019). Some minor changes to the preprints have been made in this dissertation in order to increase the readability of the dissertation; no fundamental changes to the preprints have been made in this dissertation.

2. SEMI-IMPLICIT MULTI-STEP NON-ITERATIVE DDM TO SOLVE NS-DARCY MODEL WITH BJSJ INTERFACE CONDITION

In this section, a semi-implicit multi-step non-iterative domain decomposition method (NIDDM) is proposed to solve a coupled unsteady NS-Darcy system. Without any iteration at each time step, the results in the previous time steps are utilized to directly predict the interface information for decomposing the Navier-Stokes and Darcy sub-domains. Multi-step backward differentiation formulae, which can improve the accuracy in time with unconditional stability, are used for the temporal discretization, and spatial discretization is effected by using finite element methods. We use the semi-implicit scheme to linearize the nonlinear convection. As preparation for proving the convergence of the finite element solution of the proposed method, we derive the error estimate in L^2 norm for the joint Stokes-Darcy Ritz-projection without using H^2 regularity assumption of the elliptic problem, corresponding to this joint Ritz-projection. The k-step backward differentiation formulae ($1 \leq k \leq 5$) with finite elements in spatial discretization are analyzed in a general framework of multiplier technique, while a mathematical induction proof is utilized to handle the term arising from the nonlinear advection. Furthermore, a Lagrange multiplier method under the framework of the semi-implicit, multi-step NIDDM is presented to deal with the unsteady NS-Darcy system with BJSJ interface condition and defective boundary (DB) condition. One interesting finding is that the Lagrange multipliers are time dependent functions instead of constants. Numerical examples are provided to illustrate the proposed methods and verify the theoretical conclusions.

2.1. REVIEW OF DDM FOR STOKES/NS-DARCY MODEL

At the beginning of our work, we first review some representative work for solving the Stokes/NS-Darcy model by using domain decomposition methods. As an efficient numerical tool to solve partial differential equations in parallel, non-overlapping domain decomposition methods have a key step which is to define the values on the interface between subdomains. In the literature, two major ideas were utilized for this difficulty: one is to apply the iterative domain decomposition method for elliptic equations at each time step Cai (1994); Daoud and Wade (2001); Dryja (1991); Gander *et al.* (2001); Lions (1988); Qin and Xu (2008); Quarteroni and Valli (1999); Xu and Zou (1998); the other one is to take advantage of information gained in the previous time steps to predict the values on the interface at the current time step, such as the explicit/implicit domain decomposition (EIDD) method Dawson *et al.* (1991); Dawson and Dupont (1994); Du *et al.* (2001/02) and the stabilized EIDD method Zhuang and Sun (2002). In general, the second idea saves on both computation and communication costs because it is non-iterative. The key issue for this idea is how to obtain optimal accuracy and better stability because it uses lagged results from the the previous time steps, and explicit treatment instead of iterations to predict the interface values.

It is well-known that Ritz projection plays a key role in the error estimates of finite element solutions for parabolic problems Thomée (2006); Wheeler (1971). Therefore, error estimates of the joint Stokes-Darcy Ritz-projection, which includes interface terms, have been often utilized as assumptions without proof in the existing articles of the Stokes-Darcy model Cao *et al.* (2014); Çeşmelioğlu and Rivière (2008, 2009); Chen *et al.* (2013b); Kanschat and Riviére (2010); Layton *et al.* (2002); Mu and Zhu (2010a); Rivière and Yotov (2005); Shan and Zheng (2013a); Vassilev and Yotov (2009). In Cao *et al.* (2010a), a brief proof is provided for the joint Ritz-projection error estimate with a H^2 regularity assumption of the elliptic problem corresponding to the joint Ritz-projection.

On the other hand, a parallel, non-iterative, multi-physics domain decomposition method (DDM) was proposed in Cao *et al.* (2014) for the Stokes-Darcy model with BJSJ interface condition. The convergence of the backward Euler scheme of this domain decomposition method for the Stokes-Darcy model was proved in Cao *et al.* (2014) based on some assumptions on the joint Ritz-projection error estimates. The convergence of the multi-step backward differentiation (BDF) schemes of this domain decomposition method for the Stokes-Darcy model was analyzed in Gunzburger *et al.* based on a complete L^∞ error analysis of the separate Ritz-projections for Stokes and Poisson equations. In Chen *et al.* (2013b, doi: 10.1007/s00211-015-0789-3), three two-step and three-step methods are proposed for long term stability, but there is no analysis for their spatial discretization.

2.2. BASIC IDEA OF SEMI-IMPLICIT MULTI-STEP NON-ITERATIVE DDM

In this section, based on the key idea of the non-iterative DDM for the Stokes-Darcy model Cao *et al.* (2014), we propose and rigorously analyze a semi-implicit multi-step non-iterative DDM to solve the unsteady NS-Darcy system by using finite elements for spatial discretization. Compared with the traditional iterative domain decomposition, which applies a domain decomposition iteration at each time step for the interface information, the non-iterative DDM takes advantage of the solutions obtained in previous time steps to directly predict the interface information without any iteration at the current time step. Compared with the implicit temporal discretization in Cao *et al.* (2014), we use a semi-implicit scheme to linearize the nonlinear term of the Navier-Stokes equation. We will also consider the general k-step ($1 \leq k \leq 5$) BDF schemes for the temporal discretization and carry out a rigorous analysis for them under a general framework of the multiplier technique of Nevanlinna and Odeh Nevanlinna and Odeh (1981), which has been used to analyze parabolic problems recently in Akrivis (2015); Akrivis and Lubich (2015). Moreover, in order to analyze the convergence of the finite element solution of the proposed DDM method, as the first work, we prove the approximation properties of the joint Stokes-Darcy

Ritz-projection in L^2 norm rigorously without using H^2 regularity assumption. On the other hand, in order to overcome the difficulty of non-unique solutions arising from the defective boundary conditions of the NS-Darcy model, Lagrange multipliers are utilized under the framework of the semi-implicit multi-step non-iterative DDM to enrich the algorithm for handling the defective boundary conditions. One interesting finding is that the Lagrange multipliers are time-dependent functions instead of constants. Numerical examples are provided to illustrate the optimal convergence, stability, and applicability of the proposed method.

There are three major difficulties in the error estimates of the joint Ritz-projection in L^2 norm and the semi-implicit multi-step non-iterative DDM. Firstly, in practice the H^2 regularity assumption may not be valid for certain types of problems, hence it may be difficult to prove it mathematically. In fact, the interface terms reduce the regularity of the elliptic problem corresponding to the joint Ritz-projection by an arbitrarily small order, δ. Therefore, in this article we rigorously prove this $H^{2-\delta}$ regularity, which is more general in practice, and use it to derive the error estimate for the joint Ritz-projection. Secondly, due to the semi-implicit feature of the temporal discretization and the needs of the joint Ritz-projection, there is a mismatch between the numerical solution of the semi-implicit scheme and the joint Ritz-projection of the analytic solution. In order to overcome this difficulty, we need to utilize the fully implicit scheme to bridge these two key components in the error estimation. Thirdly, the nonlinear advection of the Navier-Stokes equation leads to a term that is difficult to be bounded in the analysis. Hence the mathematical induction is constructed in the proof to deal with the nonlinear advection.

2.3. FORMULATION OF THE SEMI-IMPLICIT MULTI-STEP NON-ITERATIVE DDM

Let the domain Ω be divided into quasi-uniform triangles that fit the boundary $\partial\Omega$ and the interface Γ. Correspondingly, the domains Ω_D and Ω_S are divided into quasi-uniform triangles, respectively.

Let $(\overrightarrow{v}_h^r(\Omega_S), V_h^{r-1}(\Omega_S))$ denote the Taylor-Hood finite element space of $\mathbf{H}^1(\Omega_S) \times L^2(\Omega_S)$ of polynomial degree $r \geq 2$ subject to the triangulation of Ω_S. This finite element space satisfies the inf-sup condition

$$\|q_h\|_{L^2(\Omega_S)} \leq \sup_{\vec{v}_h \in \vec{V}_h^r(\Omega_S)} \frac{C|(q_h, \nabla \cdot \vec{v}_h)|}{\|\nabla \vec{v}_h\|_{L^2(\Omega_S)}}, \quad \forall\, q_h \in V_h^{r-1}(\Omega_S),$$

hence it is stable for solving the Navier-Stokes problem Girault and Raviart (1986b); Gunzburger (1989a). Let $V_h^r(\Omega_D)$ be the finite element subspace of $H^1(\Omega_D)$, consisting of continuous piecewise polynomials of degree r.

Let $t_n := n\tau, n = 0, \ldots, N$, be a uniform partition of the time interval $[0, T]$, with time step $\tau := T/N$. For any given integer $1 \leq k \leq 5$, we let d_j $(j = 0, \ldots, k)$ and γ_j $(j = 0, \ldots, k-1)$ be the coefficients of the polynomials

$$\textstyle\sum_{j=1}^{k} \frac{1}{j}(1-\zeta)^j = \sum\limits_{j=0}^{k} d_j \zeta^j, \quad \frac{1}{\zeta}\left[1-(1-\zeta)^k\right] = \sum_{j=0}^{k-1} \gamma_j \zeta^j.$$

Then the following approximation results hold for all $\varphi \in C^{k+1}([0,T]; L^2(X))$ and $v \in C^k([0,T]; L^2(X))$; see **?**:

$$\left\| \frac{1}{\tau} \sum_{j=0}^{k} d_j \varphi(\cdot, t_{n-j}) - \partial_t \varphi(\cdot, t_n) \right\|_{L^2(X)} \leq C\tau^k \|\partial_t^{k+1} \varphi\|_{C([0,T];L^2(X))} \qquad \text{for } n \geq k, \tag{2.1}$$

$$\left\| \sum_{j=0}^{k-1} \gamma_j v(\cdot, t_{n-1-j}) - v(\cdot, t_n) \right\|_{L^2(X)} \leq C\tau^k \|\partial_t^k v\|_{C([0,T];L^2(X))} \qquad \text{for } n \geq k, \tag{2.2}$$

where the $\|\cdot\|_{C([0,T];L^2(X))} = \max\limits_{0\le t\le T}\|\cdot\|_{L^2(X)}$, $X = \Omega_D$, Ω_S or Γ, and the constant C is independent of τ and n.

2.3.1. Framework of Semi-implicit Multi-step Non-Iterative DDM. First, we define the following function spaces

$$
\begin{aligned}
X_S &= \{\overrightarrow{v} \in [H^1(\Omega_S)]^d \mid \overrightarrow{v} = 0 \text{ on } \partial\Omega_S\backslash\Gamma\} \\
Q_S &= L^2(\Omega_S) \\
X_D &= \{\psi \in H^1(\Omega_D) \mid \psi = 0 \text{ on } \partial\Omega_D\backslash\Gamma\} \\
L^2(0,T;Q_S) &= \{\phi : \phi(t,\cdot) \in Q_S, \ \forall\ t \in [0,T]\} \\
H^1(0,T;X_D,X_D') &= \{\phi : \phi \in L^2(0,T;X_D) \text{ and } \frac{\partial\phi}{\partial t} \in L^2(0,T;X_D')\} \\
H^1(0,T;X_S,X_S') &= \{\phi : \phi \in L^2(0,T;X_S) \text{ and } \frac{\partial\phi}{\partial t} \in L^2(0,T;X_S')\}.
\end{aligned}
$$

Here, X_D' and X_S' are the dual spaces of X_D and X_S. For the domain D ($D = \Omega_S$ or Ω_D), $(\cdot,\cdot)_D$ denotes the L^2 inner product on the domain D, and $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product on the interface Γ or the duality pairing between $(H_{00}^{1/2}(\Gamma))'$ and $H_{00}^{1/2}(\Gamma)$. P_τ denoted the projection onto the tangent space on Γ, i.e. $P_\tau\overrightarrow{u} = \sum_{j=1}^{d-1}(\overrightarrow{u}\cdot\boldsymbol{\tau}_j)\boldsymbol{\tau}_j$. With these notations, the weak formulation of the coupled NS-Darcy model with BJSJ interface condition is defined as follows: find $(\overrightarrow{u}_S, p_S) \in H^1(0,T;X_S,X_S') \times L^2(0,T;Q_S)$ and $\phi_D \in H^1(0,T;X_D,X_D')$ such that

$$
\begin{aligned}
&(\frac{\partial\overrightarrow{u}_S}{\partial t},\overrightarrow{v})_{\Omega_S} + g(\frac{\partial\phi_D}{\partial t},\psi)_{\Omega_D} + C_S(\overrightarrow{u}_S,\overrightarrow{u}_S,\overrightarrow{v}) + a_S(\overrightarrow{u}_S,\overrightarrow{v}) + b_S(\overrightarrow{v},p_S) + g a_D(\phi_D,\psi) \\
&+\langle g\phi_D, \overrightarrow{v}\cdot\overrightarrow{n}_S\rangle - g\langle\overrightarrow{u}_S\cdot\overrightarrow{n}_S,\psi\rangle + \frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\prod)}}\langle P_\tau\overrightarrow{u}_S, P_\tau\overrightarrow{v}\rangle \\
&= g(f_D,\psi)_{\Omega_D} + (\overrightarrow{f}_S,\overrightarrow{v})_{\Omega_S}, \ \forall\,\overrightarrow{v} \in X_S, \ \ \psi \in X_D, \qquad (2.3)\\
&b_S(\overrightarrow{u}_S, q) = 0, \ \forall\ q \in Q_S, \qquad (2.4)
\end{aligned}
$$

where the bilinear forms and the trilinear form are

$$a_D(\phi_D, \psi) = (\mathbb{K}\nabla\phi_D, \nabla\psi)_{\Omega_D}, \;\; a_S(\vec{u}_S, \vec{v}) = 2\nu(\mathbb{D}(\vec{u}_S), \mathbb{D}(\vec{v}))_{\Omega_S},$$
$$b_S(\vec{v}, q) = -(\nabla\cdot\vec{v}, q)_{\Omega_S}, \;\; C_s(\vec{u}_S, \vec{u}_S, \vec{v}) = ((\vec{u}_S\cdot\nabla)\vec{u}_S, \vec{v}).$$

This weak formulation is similar to that of Cao *et al.* (2014), but takes the nonlinear advection.

In order to solve the coupled NS-Darcy problem utilizing the domain decomposition idea, we naturally consider Robin boundary conditions for the Darcy and the Navier-Stokes equations by following the idea in Cao *et al.* (2014). First we consider the following two Robin-type conditions for the Navier-Stokes equations:

$$\vec{n}_S\cdot(\mathbb{T}(\vec{u}_S, p_S)\cdot\vec{n}_S) + \vec{u}_S\cdot\vec{n}_S = \xi_S \qquad \text{on } \Gamma, \tag{2.5}$$

for given function $\xi_S \in L^2(0,T; L^2(\Gamma))$. Then, the corresponding weak formulation for the Navier-Stokes system is to find $\vec{u}_S \in H^1(0,T; X_S, X_S')$ and $p_S \in L^2(0,T; Q_S)$ such that

$$(\frac{\partial\vec{u}_S}{\partial t}, \vec{v})_{\Omega_S} + C_S(\vec{u}_S, \vec{u}_S, \vec{v}) + a_S(\vec{u}_S, \vec{v}) + b_S(\vec{v}, p_S) + \langle\vec{u}_S\cdot\vec{n}_S, \vec{v}\cdot\vec{n}_S\rangle$$
$$+\frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\prod)}}\langle P_\tau\vec{u}_S, P_\tau\vec{v}\rangle \;=\; (\vec{f}_S, \vec{v})_{\Omega_S} + \langle\xi_S, \vec{v}\cdot\vec{n}_S\rangle, \; \forall\,\vec{v}\in X_S, \tag{2.6}$$
$$b_S(\vec{u}_S, q) \;=\; 0, \; \forall\, q\in Q_S \tag{2.7}$$

On the other hand, we consider the following Robin condition for the Darcy system:

$$\mathbb{K}\nabla\phi_D\cdot\vec{n}_D + g\phi_D = \xi_D \qquad \text{on } \Gamma, \tag{2.8}$$

for a given function $\xi_D \in L^2(0,T;L^2(\Gamma))$. Hence, the corresponding weak formulation for the Darcy system is to find $\phi_D \in H^1(0,T;X_D,X_D')$ such that

$$(\frac{\partial \phi_D}{\partial t},\psi)_{\Omega_D} + a_D(\phi_D,\psi) + \langle g\phi_D,\psi\rangle \;=\; (f_D,\psi)_{\Omega_D} + \langle \xi_D,\psi\rangle, \forall \psi \in X_D. \tag{2.9}$$

The Navier-Stokes and Darcy systems with Robin boundary conditions can be combined into one system. Indeed, it is easy to see that if ξ_D and ξ_S are given, then there exists a unique solution $(\phi_D,\vec{u}_S,p_S) \in H^1(0,T;X_D,X_D') \times H^1(0,T;X_S,X_S') \times L^2(0,T;Q_S)$ such that

$$(\frac{\partial \vec{u}_S}{\partial t},\vec{v})_{\Omega_S} + g(\frac{\partial \phi_D}{\partial t},\psi)_{\Omega_D} + C_S(\vec{u}_S,\vec{u}_S,\vec{v}) + a_S(\vec{u}_S,\vec{v}) + b_S(\vec{v},p_S) + g a_D(\phi_D,\psi)$$
$$+\langle \vec{u}_S \cdot \vec{n}_S, \vec{v} \cdot \vec{n}_S\rangle + g\langle \phi_D,\psi\rangle + \frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\prod)}}\langle P_\tau \vec{u}_S, P_\tau \vec{v}\rangle = g(f_D,\psi)_{\Omega_D}$$
$$+(\vec{f}_S,\vec{v})_{\Omega_S} + \langle \xi_S, \vec{v}\cdot\vec{n}_S\rangle + g\langle \xi_D,\psi\rangle,\ \forall \psi \in Q_S, \vec{v} \in X_S, \tag{2.10}$$
$$b_S(\vec{u}_S,q) = 0,\ \forall\, q \in Q_S, \tag{2.11}$$
$$\phi_D(0) \;=\; \phi_0,\;\; \vec{u}_S(0) \;=\; \vec{u}_0. \tag{2.12}$$

Similar to Proposition 3.1 in Cao *et al.* (2014), it is easy to show that the solutions of the coupled NS-Darcy system are equivalent to solutions of the decoupled system if the following compatibility conditions are satisfied:

$$\xi_D = \vec{u}_S \cdot \vec{n}_S + g\phi_D, \quad \xi_S = \vec{u}_S \cdot \vec{n}_S - g\phi_D. \tag{2.13}$$

These compatibility conditions provide the key tool to predict ξ_D and ξ_S on the interface at each time step based on the results from the previous time steps.

Motivated by the Robin-type domain decomposition conditions (2.5) and (3.7), we propose the following full discretization scheme of the semi-implicit multi-step non-iterative domain decomposition method: find $(\vec{u}_h^n, p_h^n) \in \vec{v}_h^r(\Omega_S) \times V_h^{r-1}(\Omega_S)$ and $\phi_h^n \in V_h^r(\Omega_D)$

$(n = k, \cdots, N)$ such that

$$\Big(\frac{1}{\tau}\sum_{j=0}^{k} d_j \vec{u}_h^{\,n-j}, \vec{v}_h\Big)_{\Omega_S} + \Big(\sum_{j=0}^{k-1} \gamma_j \vec{u}_h^{\,n-1-j} \cdot \nabla \vec{u}_h^{\,n}, \vec{v}_h\Big)_{\Omega_S} + a_S(\vec{u}_h^{\,n}, \vec{v}_h) + b_S(\vec{v}_h, p_h^n)$$
$$+\langle \vec{u}_h^{\,n} \cdot \vec{n}_S, \vec{v}_h \cdot \vec{n}_S\rangle + \beta\langle \vec{u}_h^{\,n} - (\vec{u}_h^{\,n} \cdot \vec{n}_S)\vec{n}_S, \vec{v}_h - (\vec{v}_h \cdot \vec{n}_S)\vec{n}_S\rangle \tag{2.14}$$
$$= \big(\mathbf{f}_S^n, \vec{v}_h\big)_{\Omega_S} + \langle \sum_{j=0}^{k-1} \gamma_j \xi_{Sh}^{n-1-j}, \vec{v}_h \cdot \vec{n}_S\rangle,$$
$$\big(\nabla \cdot \vec{u}_h^{\,n}, q_h\big)_{\Omega_S} = 0, \tag{2.15}$$
$$\Big(\frac{1}{\tau}\sum_{j=0}^{k} d_j \phi_h^{n-j}, \varphi_h\Big)_{\Omega_D} + a_D(\phi_h^n, \varphi_h) + \langle \phi_h^n, \varphi_h\rangle \tag{2.16}$$
$$= \big(f_D^n, \varphi_h\big)_{\Omega_D} + \langle \sum_{j=0}^{k-1} \gamma_j \xi_{Dh}^{n-1-j}, \varphi_h\rangle,$$

where $\beta = \frac{\alpha\nu\sqrt{2}}{\sqrt{\mathrm{trace}(\Pi)}}$ and ξ_{Sh}^n, ξ_{Dh}^n are defined as

$$\xi_{Sh}^n := \vec{u}_h^{\,n} \cdot \vec{n}_S - \phi_h^n, \quad \xi_{Dh}^n := \vec{u}_h^{\,n} \cdot \vec{n}_S + \phi_h^n. \tag{2.17}$$

Here, the boundary condition (1.11) and (1.12) are considered. For the approximations $(\vec{u}_h^{\,n}, p_h^n, \phi_h^n)$ $(n = 0, \cdots, k-1)$ of the first $k-1$ time steps, one can utilize the initial condition at $n = 0$, Runge-Kutta method with high enough orders, and the $k-1$ step backward differentiation method to determine them. More details of the example of the three-step method can be found in Cao *et al.* (2014).

2.3.2. Lagrange Multiplier Method under the Framework of Semi-implicit Multi-step Non-iterative DDM. If the defective boundary condition (1.14) is utilized on the boundary of Ω_S, then we propose the following Lagrange multiplier method under the framework of the semi-implicit, multi-step, non-iterative domain decomposition: at the n^{th} step, find $(\vec{u}_h^{\,n}, p_h^n) \in \vec{v}_h^{\,r}(\Omega_S) \times V_h^{r-1}(\Omega_S)$, $\phi_h^n \in V_h^r(\Omega_D)$ $(n = k, \cdots, N)$ and

$\lambda = \{\lambda_i(t)\}_{i=0}^{m} \in L^2(0,T)^{m+1}$ such that

$$\left(\tfrac{1}{\tau}\sum_{j=0}^{k} d_j \overrightarrow{u}_h^{n-j}, \overrightarrow{v}_h\right)_{\Omega_S} + \sum_{i=0}^{m} \lambda_i^{n+1} \int_{S_i} \overrightarrow{v}_h \cdot \overrightarrow{n}_S \, ds + \left(\sum_{j=0}^{k-1} \gamma_j \overrightarrow{u}_h^{n-1-j} \cdot \nabla \overrightarrow{u}_h^{n}, \overrightarrow{v}_h\right)_{\Omega_S}$$
$$+a_S(\overrightarrow{u}_h^{n}, \overrightarrow{v}_h) + b_S(\overrightarrow{v}_h, p_h^n) + \langle \overrightarrow{u}_h^{n} \cdot \overrightarrow{n}_S, \overrightarrow{v}_h \cdot \overrightarrow{n}_S \rangle$$
$$+\beta \langle \overrightarrow{u}_h^{n} - (\overrightarrow{u}_h^{n} \cdot \overrightarrow{n}_S)\overrightarrow{n}_S, \overrightarrow{v}_h - (\overrightarrow{v}_h \cdot \overrightarrow{n}_S)\overrightarrow{n}_S \rangle$$
$$= \left(\mathbf{f}_S^n, \overrightarrow{v}_h\right)_{\Omega_S} + \langle \textstyle\sum_{j=0}^{k-1} \gamma_j \xi_{Sh}^{n-1-j}, \overrightarrow{v}_h \cdot \overrightarrow{n}_S \rangle.$$
$$\sum_{i=0}^{m} \mu_i^{n+1} \int_{S_i} \overrightarrow{v}_h \cdot \overrightarrow{n}_S \, ds + \left(\nabla \cdot \overrightarrow{u}_h^{n}, q_h\right)_{\Omega_S} = \sum_{i=0}^{m} \mu_i^{n+1} Q_i,$$

$$\left(\tfrac{1}{\tau}\sum_{j=0}^{k} d_j \phi_h^{n-j}, \varphi_h\right)_{\Omega_D} + a_D(\phi_h^n, \varphi_h) + \langle \phi_h^n, \varphi_h \rangle = \left(f_D^n, \varphi_h\right)_{\Omega_D} + \langle \textstyle\sum_{j=0}^{k-1} \gamma_j \xi_{Dh}^{n-1-j}, \varphi_h \rangle.$$

where $\forall \overrightarrow{v}_h \in \overrightarrow{v}_h^r(\Omega_S), \forall q_h \in \overrightarrow{v}_h^{r-1}(\Omega_S), \forall \varphi_h \in \overrightarrow{v}_h^r(\Omega_D)$ and $\mu_i^{n+1} \in R^{n+1}$.

2.4. JOINT STOKES-DARCY RITZ-PROJECTION

In this section, we work with the following assumptions, with given integers $1 \leq k \leq 5$ and $r \geq 2$.

(A1) (*Physical parameters*) The physical parameters ν, g and β are positive, and $\mathbb{K}$ is a symmetric and positive definite matrix, satisfying

$$\kappa_0 |\boldsymbol{\xi}|^2 \leq \mathbb{K}\boldsymbol{\xi} \cdot \boldsymbol{\xi} \leq \kappa_0^{-1} |\boldsymbol{\xi}|^2, \quad \forall \boldsymbol{\xi} \in \mathbf{R}^d,$$

for some positive constant κ_0.

(A2) (*Regularity of the domains*) The boundaries of the domains Ω_D and Ω_S are both piecewise smooth, and the interior angles of the corners of the domains Ω_D and Ω_S are all strictly between 0 and π.

(A3) (*Regularity of the solution*) The problem (1.3)-(1.12) has a unique solution that is sufficiently regular, i.e.,

$$\overrightarrow{u} \in C^1([0,T]; H^{r+1}(\Omega_S)) \cap C^{k+1}([0,T]; L^2(\Omega_S)) \cap C^k([0,T]; H^1(\Omega_S)),$$
$$p \in C^1([0,T]; H^r(\Omega_S)) \cap C^k([0,T]; L^2(\Omega_S)),$$
$$\phi \in C^1([0,T]; H^{r+1}(\Omega_D)) \cap C^{k+1}([0,T]; L^2(\Omega_D)) \cap C^k([0,T]; H^1(\Omega_D))$$

(A4) (*Accuracy of the starting values*) The given starting values $(\overrightarrow{u}_h^n, \phi_h^n)$, $n = 0, \ldots, k-1$, satisfy

$$\max_{0 \le n \le k-1} \left(\|\overrightarrow{u}_h^n - \overline{\mathbf{u}}_h^n\|^2_{L^2(\Omega_S)} + \|\phi_h^n - \overline{\phi}_h^n\|^2_{L^2(\Omega_D)} \right)$$
$$+ \tau \sum_{n=0}^{k-1} \left(\|\nabla(\overrightarrow{u}_h^n - \overline{\mathbf{u}}_h^n)\|^2_{L^2(\Omega_S)} + \|\nabla(\phi_h^n - \overline{\phi}_h^n)\|^2_{L^2(\Omega_D)} \right) \le C_\star(\tau^k + h^{r+1-2\delta})^2.$$

for some positive constant $\delta \in (0, 1/2)$ and $C_\star$, where $(\overline{\mathbf{u}}_h^n, \overline{p}_h^n, \overline{\phi}_h^n)$ denotes the joint Stokes-Darcy Ritz-projection of $(\overrightarrow{u}^n, p^n, \phi^n) := (\overrightarrow{u}(\cdot, t_n), p(\cdot, t_n), \phi(\cdot, t_n))$, defined in (2.19).

For $(\overrightarrow{u}, p, \phi)$ in $\mathbf{H}^1(\Omega_S) \times L^2(\Omega_S) \times H^1(\Omega_D)$ and $(\overrightarrow{v}, q, \varphi)$ in $\mathbf{H}^1(\Omega_S) \times L^2(\Omega_S) \times H^1(\Omega_D)$, we define the tangential projection

$$P_{\tan}\overrightarrow{u} := \overrightarrow{u} - (\overrightarrow{u} \cdot \overrightarrow{n}_S)\overrightarrow{n}_S$$

and a bilinear form

$$\begin{aligned} a(\overrightarrow{u}, p, \phi; \overrightarrow{v}, q, \varphi) : \quad &= a_D(\overrightarrow{u}, \overrightarrow{v}) + b_S(\overrightarrow{v}, p_S) + (\nabla \cdot \overrightarrow{u}, q)_{\Omega_S} + \beta \langle P_{\tan}\overrightarrow{u}, P_{\tan}\overrightarrow{v} \rangle \\ &+ \langle \phi, \overrightarrow{v} \cdot \overrightarrow{n}_S \rangle + a_D(\phi, \varphi) - \langle \overrightarrow{u} \cdot \overrightarrow{n}_S, \varphi \rangle + (\overrightarrow{u}, \overrightarrow{v})_{\Omega_S} + (\phi, \varphi)_{\Omega_D}. \end{aligned} \tag{2.18}$$

Then the joint Stokes-Darcy Ritz-projection of $(\overrightarrow{u}, p, \phi) \in \mathbf{H}^1(\Omega_S) \times L^2(\Omega_S) \times H^1(\Omega_D)$, denoted by $R_h(\overrightarrow{u}, p, \phi)$, is defined as the unique element $(\bar{\mathbf{u}}_h, \overline{p}_h, \overline{\phi}_h) \in V_h^r(\Omega_D) \times \overrightarrow{v}_h^r(\Omega_S) \times V_h^{r-1}(\Omega_S)$ satisfying the following equation:

$$a(\overrightarrow{u} - \bar{\mathbf{u}}_h, p - \overline{p}_h, \phi - \overline{\phi}_h; \overrightarrow{v}_h, q_h, \varphi_h) = 0, \tag{2.19}$$

where $\forall\,(\overrightarrow{v}_h, q_h, \varphi_h) \in V_h^r(\Omega_D) \times \overrightarrow{v}_h^r(\Omega_S) \times V_h^{r-1}(\Omega_S)$. Note that (2.19) is equivalent to the following equations:

$$\begin{cases} \big(2\nu\mathbb{D}(\overrightarrow{u} - \bar{\mathbf{u}}_h), \mathbb{D}(\overrightarrow{v}_h)\big)_{\Omega_S} - (p - \overline{p}_h, \nabla\cdot\overrightarrow{v}_h)_{\Omega_S} + \beta\langle P_{\tan}(\overrightarrow{u} - \bar{\mathbf{u}}_h), P_{\tan}\overrightarrow{v}_h\rangle \\ \quad + \langle \phi - \overline{\phi}_h, \overrightarrow{v}_h \cdot \overrightarrow{n}_S\rangle + (\overrightarrow{u} - \bar{\mathbf{u}}_h, \overrightarrow{v}_h)_{\Omega_S} = 0, \quad \forall\,\overrightarrow{v}_h \in \overrightarrow{v}_h^r(\Omega_S), \\ (\nabla\cdot(\overrightarrow{u} - \bar{\mathbf{u}}_h), q_h) = 0, \quad \forall\, q_h \in V_h^{r-1}(\Omega_S), \\ \big(\mathbb{K}\nabla(\phi - \overline{\phi}_h), \nabla\varphi_h\big)_{\Omega_D} - \langle(\overrightarrow{u} - \bar{\mathbf{u}}_h)\cdot\overrightarrow{n}_S, \varphi_h\rangle + (\phi - \overline{\phi}_h, \varphi_h)_{\Omega_D} = 0, \quad \forall\,\varphi_h \in V_h^r(\Omega_D), \end{cases} \tag{2.20}$$

Now we present the conclusion on the approximation property of the joint Stokes-Darcy Ritz-projection.

Theorem 1 *Under assumptions* (A1)-(A2)*, for* $(\overrightarrow{u}, p, \phi) \in \mathbf{H}^{r+1-\delta}(\Omega_S) \times H^{r-\delta}(\Omega_S) \times H^{r+1-\delta}(\Omega_D)$*, the joint Stokes-Darcy Ritz-projection, defined in* (2.19)*, obeys the following estimates:*

$$\|\overrightarrow{u} - \bar{\mathbf{u}}_h\|_{\mathbf{L}^2(\Omega_S)} + \|\phi - \overline{\phi}_h\|_{L^2(\Omega_D)} \tag{2.21}$$
$$\leq C_\delta h^{r+1-2\delta}(\|\overrightarrow{u}\|_{\mathbf{H}^{r+1-\delta}(\Omega_S)} + \|p\|_{H^{r-\delta}(\Omega_S)} + \|\phi\|_{H^{r+1-\delta}(\Omega_D)}),$$
$$\|\overrightarrow{u} - \bar{\mathbf{u}}_h\|_{\mathbf{H}^1(\Omega_S)} + \|p - \overline{p}_h\|_{L^2(\Omega_S)} + \|\phi - \overline{\phi}_h\|_{H^1(\Omega_D)} \tag{2.22}$$
$$\leq C_\delta h^{r-\delta}(\|\overrightarrow{u}\|_{\mathbf{H}^{r+1-\delta}(\Omega_S)} + \|p\|_{H^{r-\delta}(\Omega_S)} + \|\phi\|_{H^{r+1-\delta}(\Omega_D)}),$$
$$\|\bar{\mathbf{u}}_h\|_{\mathbf{H}^1(\Omega_S)} + \|\overline{p}_h\|_{L^2(\Omega_S)} + \|\overline{\phi}_h\|_{H^1(\Omega_D)} \tag{2.23}$$
$$\leq C(\|\overrightarrow{u}\|_{\mathbf{H}^1(\Omega_S)} + \|p\|_{L^2(\Omega_S)} + \|\phi\|_{H^1(\Omega_D)}),$$
$$\|\bar{\mathbf{u}}_h\|_{\mathbf{W}^{1,\infty}(\Omega_S)} \leq C(\|\overrightarrow{u}\|_{\mathbf{H}^{r+1-\delta}(\Omega_S)} + \|p\|_{H^{r-\delta}(\Omega_S)} + \|\phi\|_{H^{r+1-\delta}(\Omega_D)}). \tag{2.24}$$

where $\delta \in (0, 1)$ *can be arbitrarily small.*

As a key preparation step for proving the main theoretical conclusion on the convergence of the proposed semi-implicit multi-step non-iterative domain decomposition method, we will prove the approximation property of the joint Stokes-Darcy Ritz-projection, i.e, Theorem 1. We will first analyze the regularity of the Stokes-Darcy problem.

2.4.1. Regularity of Stokes-Darcy Problem. In this subsection, we consider the coupled Stokes-Darcy problem:

$$\begin{cases} -\nabla \cdot \big(2\nu\mathbb{D}(\mathbf{w}) - \sigma\mathbb{I}\big) + \mathbf{w} = \mathbf{f} & \text{in } \Omega_S, \\ \nabla \cdot \mathbf{w} = 0 & \text{in } \Omega_S, \\ -\big(2\nu\mathbb{D}(\mathbf{w}) - \sigma\mathbb{I}\big) \cdot \overrightarrow{n}_S = 0 & \text{on } \partial\Omega_S\backslash\Gamma, \\ -\big(2\nu\mathbb{D}(\mathbf{w}) - \sigma\mathbb{I}\big) \cdot \overrightarrow{n}_S = -\psi\overrightarrow{n}_S + \beta(\mathbf{w} - (\mathbf{w}\cdot\overrightarrow{n}_S)\overrightarrow{n}_S) & \text{on } \Gamma, \end{cases} \tag{2.25}$$

and

$$\begin{cases} -\nabla \cdot (\mathbb{K}\nabla\psi) + \psi = f & \text{in } \Omega_D, \\ \mathbb{K}\nabla\psi \cdot \overrightarrow{n}_D = 0 & \text{on } \partial\Omega_D\backslash\Gamma, \\ \mathbb{K}\nabla\psi \cdot \overrightarrow{n}_D = \mathbf{w}\cdot\overrightarrow{n}_S & \text{on } \Gamma. \end{cases} \tag{2.26}$$

and prove the following proposition.

Proposition 1 *Under assumption* (A1)-(A2) *and for any given* $\delta \in (0, 1/2)$*, there is weak solution* $(\mathbf{w}, \sigma, \psi)$ *of the coupled system* (2.25)-(2.26) *which satisfies*

$$\|\mathbf{w}\|_{H^{2-\delta}(\Omega_S)} + \|\sigma\|_{H^{1-\delta}(\Omega_S)} + \|\psi\|_{H^{2-\delta}(\Omega_D)} \le C(\|\mathbf{f}\|_{H^{-\delta}(\Omega_S)} + \|f\|_{H^{-\delta}(\Omega_D)}). \tag{2.27}$$

The proof of Proposition 1 is based on the following lemma.

Lemma 1 (Regularity results of the Darcy problem and the Stokes problem) *Under assumption* (A1)-(A2)*, any weak solution of the equation*

$$\begin{cases} -\nabla \cdot (\mathbb{K}\nabla\psi) + \psi = f & in \;\; \Omega_D, \\ \mathbb{K}\nabla\psi \cdot \overrightarrow{n}_D = F & on \;\; \partial\Omega_D \end{cases}$$

satisfies

$$\|\psi\|_{H^{2-\delta}(\Omega_D)} \leq C\|f\|_{H^{-\delta}(\Omega_D)} + C\sum_{j=1}^{J_D} \|F\|_{H^{1/2-\delta}(\Gamma_{D,j})} \quad for \;\; \delta \in (0, 1/2), \tag{2.28}$$

where $\Gamma_{D,j}$, $j = 1, \cdots, J_D$*, denote the smooth pieces of the boundary* $\partial\Omega_D$*.*

Similarly, under assumption (A1)-(A2)*, any weak solution of the stationary Stokes equation:*

$$\begin{cases} -\nabla \cdot (2\nu\mathbb{D}(\mathbf{w}) - \sigma\mathbb{I}) + \mathbf{w} = \mathbf{f} & in \;\; \Omega_S, \\ \nabla \cdot \mathbf{w} = G & in \;\; \Omega_S, \\ 2\nu\mathbb{D}(\mathbf{w})\overrightarrow{n}_S - \sigma\overrightarrow{n}_S = \mathbf{g} & on \;\; \partial\Omega_S, \end{cases}$$

satisfies

$$\begin{aligned} &\|\mathbf{w}\|_{H^{2-\delta}(\Omega_S)} + \|\sigma\|_{H^{1-\delta}(\Omega_S)} \\ &\leq C(\|\mathbf{f}\|_{\mathbf{H}^{-\delta}(\Omega_S)} + \|G\|_{H^{1-\delta}(\Omega_S)}) + C\sum_{j=1}^{J_S} \|\mathbf{g}\|_{\mathbf{H}^{1/2-\delta}(\Gamma_{S,j})} \quad for \;\; \delta \in (0, 1/2), \end{aligned} \tag{2.29}$$

where $\Gamma_{S,j}$, $j = 1, \cdots, J_S$*, denote the smooth pieces of the boundary* $\partial\Omega_S$*.*

The regularity result (2.28) is a consequence of (23.3) in Dauge (1988) and Theorem 5.2 and Lemma 5.3 in Dauge *et al.* (2007), and (2.29) is a consequence of Corollary 4.2(N-N case) in Orlt and Sändig (1995); also see corollary 2.1 in Li.

Remark 2 Due to the existence of corners of Ω_S at the intersection between Γ and $\partial\Omega$, (2.29) does not hold when $\delta = 0$. In the case of $\delta = 0$, (2.29) needs to be changed to the following form:

$$\|\mathbf{w}\|_{H^2(\Omega_S)} + \|\sigma\|_{H^1(\Omega_S)} \le C(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega_S)} + \|G\|_{H^1(\Omega_S)} + \|\frac{G}{\rho}\|_{L^2(\Omega_S)})$$
$$+ C\sum_{j=1}^{J_S} \|\mathbf{g}\|_{\mathbf{H}^{1/2}(\Gamma_{S,j})} + \|\frac{\mathbf{g}}{\sqrt{\rho}}\|_{\mathbf{L}^2(\Gamma_{S,j})})$$

where $\rho = \rho(x)$ denotes the minimal distance from x to the corners of $\partial\Omega_S$; see (Li, corollary 2.1). Due to the existence of the term $\|\mathbf{g}/\sqrt{\rho}\|_{\mathbf{L}^2(\Gamma_{S,j})}$ in the inequality above, Proposition 1 does not hold in the case $\delta = 0$ unless

$$-\psi\overrightarrow{n}_S + \beta(\mathbf{w} - (\mathbf{w}\cdot\overrightarrow{n}_S)\overrightarrow{n}_S) = 0 \quad \text{at the corners of } \partial\Omega_S.$$

Proof of Proposition 1: By applying (2.28) of Lemma 1 to the problem (2.26), we obtain

$$\|\psi\|_{H^{2-\delta}(\Omega_D)} \le C\|f\|_{H^{-\delta}(\Omega_D)} + C\|\mathbf{w}\cdot\mathbf{n}\|_{H^{1/2-\delta}(\Gamma)} \le C\|f\|_{H^{-\delta}(\Omega_D)} + C\|\mathbf{w}\|_{H^{1-\delta}(\Omega_S)}.$$

Choose $\mathbf{g} := -\psi\overrightarrow{n}_S + \beta(\mathbf{w} - (\mathbf{w}\cdot\overrightarrow{n}_S)\overrightarrow{n}_S)$ and apply (2.29) of Lemma 1 to the problem (2.25). We obtain

$$\|\mathbf{w}\|_{H^{2-\delta}(\Omega_S)} + \|\sigma\|_{H^{1-\delta}(\Omega_S)}$$
$$\le C(\|\mathbf{f}\|_{\mathbf{H}^{-\delta}(\Omega_S)} + C\|\mathbf{g}\|_{\mathbf{H}^{1/2-\delta}(\Gamma)}$$
$$\le C(\|\mathbf{f}\|_{\mathbf{H}^{-\delta}(\Omega_S)} + C(\|\psi\|_{H^{1-\delta}(\Omega_D)} + \|\mathbf{w}\|_{\mathbf{H}^{1-\delta}(\Omega_S)}),$$

where the trace inequality is used in the last step. The sum of the last two inequalities yield

$$\begin{aligned}
&\|\mathbf{w}\|_{H^{2-\delta}(\Omega_S)} + \|\sigma\|_{H^{1-\delta}(\Omega_S)} + \|\psi\|_{H^{2-\delta}(\Omega_D)} \\
&\leq C(\|\mathbf{f}\|_{\mathbf{H}^{-\delta}(\Omega_S)} + \|f\|_{H^{-\delta}(\Omega_D)}) + C(\|\psi\|_{H^{1-\delta}(\Omega_S)} + \|\mathbf{w}\|_{\mathbf{H}^{1-\delta}(\Omega_S)}) \\
&\leq C(\|\mathbf{f}\|_{\mathbf{H}^{-\delta}(\Omega_S)} + \|f\|_{H^{-\delta}(\Omega_D)}) + C(\|\psi\|_{H^{1}(\Omega_S)} + \|\mathbf{w}\|_{\mathbf{H}^{1}(\Omega_S)}) \\
&\leq C(\|\mathbf{f}\|_{\mathbf{H}^{-\delta}(\Omega_S)} + \|f\|_{H^{-\delta}(\Omega_D)}) + C(\|\mathbf{f}\|_{\mathbf{H}^{1}(\Omega_S)'} + \|f\|_{H^{1}(\Omega_D)'}) \\
&\leq C(\|\mathbf{f}\|_{\mathbf{H}^{-\delta}(\Omega_S)} + \|f\|_{H^{-\delta}(\Omega_D)}) + C(\|\mathbf{f}\|_{\mathbf{H}^{\delta}(\Omega_S)'} + \|f\|_{H^{\delta}(\Omega_D)'}) \\
&\simeq C(\|\mathbf{f}\|_{\mathbf{H}^{-\delta}(\Omega_S)} + \|f\|_{H^{-\delta}(\Omega_D)}) + C(\|\mathbf{f}\|_{\mathbf{H}_0^{\delta}(\Omega_S)'} + \|f\|_{H_0^{\delta}(\Omega_D)'}) \\
&\simeq C(\|\mathbf{f}\|_{\mathbf{H}^{-\delta}(\Omega_S)} + \|f\|_{H^{-\delta}(\Omega_D)}),
\end{aligned}$$

where "$\simeq$" indicates equivalent norms. The equivalence of norms between $\mathbf{H}^{\delta}(\Omega_S)$ and $\mathbf{H}_0^{\delta}(\Omega_S)$, $\delta \in (0, 1/2)$, is a consequence of (McLean, 2000, Theorem 3.40).

2.4.2. Proof for Theorem 1. Now we prove Theorem 1 by using Proposition 1 and the following two Lemmas. Recall

Lemma 2 (Girault and Scott (2003), Existence of the Fortin projection operator) *There exists a projection operator* $\Pi_h : \mathbf{H}^1(\Omega_S) \to \overrightarrow{v}_h^r(\Omega_S)$*, called the Fortin projection, satisfying*

$$(\nabla \cdot (\overrightarrow{v} - \Pi_h \overrightarrow{v}), q_h)_{\Omega_S} = 0, \qquad \forall \overrightarrow{v} \in \mathbf{H}^1(\Omega_S),\ \ q_h \in V_h^{r-1}(\Omega_S), \tag{2.30}$$

$$\|\overrightarrow{v} - \Pi_h \overrightarrow{v}\|_{H^l(\Omega_S)} \leq Ch^{r+s-l} \|\overrightarrow{v}\|_{H^{r+s}(\Omega_S)}, \quad \forall \overrightarrow{v} \in \mathbf{H}^{r+s}(\Omega_S),\ l = 0, 1,\ s \in [0, 1], \tag{2.31}$$

$$\|\Pi_h \overrightarrow{v}\|_{H^1(\Omega_S)} \leq C \|\overrightarrow{v}\|_{H^1(\Omega_S)}, \qquad \forall \overrightarrow{v} \in \mathbf{H}^1(\Omega_S). \tag{2.32}$$

There exists a projection operator $I_h : H^1(\Omega_D) \to V_h^r(\Omega_D)$, called the L^2 projection, satisfying

$$\|\varphi - I_h \varphi\|_{H^s(\Omega_D)} \leq Ch^{l-s} |\varphi|_{H^l(\Omega_D)}, \quad \forall\, \varphi \in H^l(\Omega_D), \tag{2.33}$$

for all $0 \leq s \leq 1 \leq l \leq r + 1$.

For the simplicity of notations, we use the same notation I_h to denote the L^2 projection from $L^2(\Omega_S)$ onto $V_h^{r-1}(\Omega_S)$, which satisfies

$$\|q - I_h q\|_{H^s(\Omega_S)} \leq C h^{l-s} |q|_{H^l(\Omega_S)}, \quad \forall\, q \in H^l(\Omega_S), \tag{2.34}$$

for all $0 \leq s \leq 1 \leq l \leq r$.

From the definition (2.18), we see that the bilinear form a is coercive:

$$\|\vec{u} - \bar{\mathbf{u}}_h\|^2_{\mathbf{H}^1(\Omega_S)} + \|\phi - \bar{\phi}_h\|^2_{H^1(\Omega_D)} \leq C a(\vec{u} - \bar{\mathbf{u}}_h, p - \bar{p}_h, \phi - \bar{\phi}_h; \vec{u} - \vec{u}_h, p - p_h, \phi - \phi_h).$$

By substituting $\vec{v}_h = \Pi_h \vec{u} - \vec{u}_h$, $q_h = I_h p - p_h$ and $\varphi_h = I_h \phi - \phi_h$ into (2.19), we obtain

$$\begin{aligned}
0 &= a(\vec{u} - \bar{\mathbf{u}}_h, p - \bar{p}_h, \phi - \bar{\phi}_h; \Pi_h \vec{u} - \vec{u}_h, I_h p - p_h, I_h \phi - \phi_h) \\
&= a(\vec{u} - \bar{\mathbf{u}}_h, p - \bar{p}_h, \phi - \bar{\phi}_h; \vec{u} - \vec{u}_h, p - p_h, \phi - \phi_h) \\
&+ a(\vec{u} - \bar{\mathbf{u}}_h, p - \bar{p}_h, \phi - \bar{\phi}_h; \Pi_h \vec{u} - \vec{u}, I_h p - p, I_h \phi - \phi) \\
&= a(\vec{u} - \bar{\mathbf{u}}_h, p - \bar{p}_h, \phi - \bar{\phi}_h; \vec{u} - \vec{u}_h, p - p_h, \phi - \phi_h) \\
&+ a(\vec{u} - \bar{\mathbf{u}}_h, p - I_h p, \phi - \bar{\phi}_h; \Pi_h \vec{u} - \vec{u}, I_h p - p, I_h \phi - \phi),
\end{aligned}$$

where we substitute $q_h = I_h p - \bar{p}_h$ into (2.30) and add it into the above equality at last step.

The equation above implies

$$\begin{aligned}
&\|\vec{u} - \bar{\mathbf{u}}_h\|^2_{\mathbf{H}^1(\Omega_S)} + \|\phi - \bar{\phi}_h\|^2_{H^1(\Omega_D)} \leq C a(\vec{u} - \bar{\mathbf{u}}_h, p - \bar{p}_h, \phi - \bar{\phi}_h; \vec{u} - \vec{u}_h, p - p_h, \phi - \phi_h) \\
&= -C a(\vec{u} - \bar{\mathbf{u}}_h, p - I_h p, \phi - \bar{\phi}_h; \Pi_h \vec{u} - \vec{u}, I_h p - p, I_h \phi - \phi) \\
&\leq C(\|\vec{u} - \bar{\mathbf{u}}_h\|_{\mathbf{H}^1(\Omega_S)} + \|p - I_h p\|_{L^2(\Omega_S)} + \|\phi - \bar{\phi}_h\|_{H^1(\Omega_D)}) \\
&\quad \cdot(\|\Pi_h \vec{u} - \vec{u}\|_{\mathbf{H}^1(\Omega_S)} + \|I_h p - p\|_{L^2(\Omega_S)} + \|I_h \phi - \phi\|_{H^1(\Omega_D)}) \\
&\leq C(\|\vec{u} - \bar{\mathbf{u}}_h\|_{\mathbf{H}^1(\Omega_S)} + \|\phi - \bar{\phi}_h\|_{H^1(\Omega_D)}) \\
&\cdot(\|\Pi_h \vec{u} - \vec{u}\|_{\mathbf{H}^1(\Omega_S)} + \|I_h p - p\|_{L^2(\Omega_S)} + \|I_h \phi - \phi\|_{H^1(\Omega_D)}) \\
&\quad + C(\|\Pi_h \vec{u} - \vec{u}\|^2_{\mathbf{H}^1(\Omega_S)} + \|I_h p - p\|^2_{L^2(\Omega_S)} + \|I_h \phi - \phi\|^2_{H^1(\Omega_D)}),
\end{aligned}$$

Then we can get the estimate immediately

$$\|\overrightarrow{u} - \bar{\mathbf{u}}_h\|_{\mathbf{H}^1(\Omega_S)} + \|\phi - \overline{\phi}_h\|_{H^1(\Omega_D)} \\ \leq C(\|\Pi_h \overrightarrow{u} - \overrightarrow{u}\|_{\mathbf{H}^1(\Omega_S)} + \|I_h p - p\|_{L^2(\Omega_S)} + \|I_h\phi - \phi\|_{H^1(\Omega_D)}). \quad (2.35)$$

Moreover, (2.19) implies that for arbitrary $\overrightarrow{v}_h \in H_0^1(\Omega_S)$,

$$|(p - \overline{p}_h, \nabla \cdot \overrightarrow{v}_h)| \leq C\|\nabla(\overrightarrow{u} - \bar{\mathbf{u}}_h)\|_{\mathbf{L}^2(\Omega_S)}\|\nabla \overrightarrow{v}_h\|_{\mathbf{L}^2(\Omega_S)} + C\|\phi - \overline{\phi}_h\|_{\mathbf{L}^2(\Gamma)}\|\overrightarrow{v}_h\|_{\mathbf{L}^2(\Gamma)} \\ + C\|\overrightarrow{u} - \bar{\mathbf{u}}_h\|_{\mathbf{L}^2(\Gamma)}\|\overrightarrow{v}_h\|_{\mathbf{L}^2(\Gamma)} + C\|\overrightarrow{u} - \bar{\mathbf{u}}_h\|_{\mathbf{L}^2(\Omega_S)}\|\overrightarrow{v}_h\|_{\mathbf{L}^2(\Omega_S)} \\ \leq C\|\overrightarrow{u} - \bar{\mathbf{u}}_h\|_{\mathbf{H}^1(\Omega_S)}\|\nabla \overrightarrow{v}_h\|_{\mathbf{L}^2(\Omega_S)},$$

where we have used (P. F. Antonietti and Smears, 2016, (12)) and Poincaré inequality. By the inf-sup condition, the last inequality implies

$$\|p - \overline{p}_h\|_{\mathbf{L}^2(\Omega_S)} \leq C\|\overrightarrow{u} - \bar{\mathbf{u}}_h\|_{\mathbf{H}^1(\Omega_S)}. \quad (2.36)$$

Combining (2.31), (2.33), (2.35), and (2.36) together, we can get (2.22) and (2.23).

To prove (2.21), we define a dual bilinear form:

$$\widetilde{a}(\mathbf{w}, \sigma, \psi; \overrightarrow{v}, q, \varphi) := a(\overrightarrow{v}, q, \varphi; \mathbf{w}, \sigma, \psi) \\ = \left(2\nu\mathbb{D}(\mathbf{w}), \mathbb{D}(\overrightarrow{v})\right)_{\Omega_S} - (\nabla \cdot \mathbf{w}, q)_{\Omega_S} + (\sigma, \nabla \cdot \overrightarrow{v})_{\Omega_S} + \beta(P_{\tan}\mathbf{w}, P_{\tan}\overrightarrow{v})_\Gamma \\ + \langle \mathbf{w} \cdot \overrightarrow{n}_S, \varphi\rangle + \left(\mathbb{K}\nabla\psi, \nabla\varphi\right)_{\Omega_D} - \langle \psi, \overrightarrow{v} \cdot \overrightarrow{n}_S\rangle + \left(\mathbf{w}, \overrightarrow{v}\right)_{\Omega_S} + \left(\psi, \varphi\right)_{\Omega_D}.$$

Choose $(\mathbf{w}, \sigma, \psi) \in H^1(\Omega_D) \times \mathbf{H}^1(\Omega_S) \times L^2(\Omega_S)$ to be the solution of the dual problem

$$\widetilde{a}(\mathbf{w}, \sigma, \psi; \overrightarrow{v}, q, \varphi) = (\overrightarrow{u} - \bar{\mathbf{u}}_h, \overrightarrow{v})_{\Omega_S} + (\phi - \overline{\phi}_h, \varphi)_{\Omega_D}, \quad (2.37)$$

where $\forall (\overrightarrow{v}, q, \varphi) \in \mathbf{H}^1(\Omega_S) \times L^2(\Omega_S) \times H^1(\Omega_D)$.

Equivalently, $(\mathbf{w}, \sigma, \psi)$ is the solution of (2.25)-(2.26) with $\mathbf{f} = \overrightarrow{u} - \overline{\mathbf{u}}_h$ and $f = \phi - \overline{\phi}_h$, and Proposition 1 implies

$$\begin{aligned} &\|\mathbf{w}\|_{H^{2-\delta}(\Omega_S)} + \|\sigma\|_{H^{1-\delta}(\Omega_S)} + \|\psi\|_{H^{2-\delta}(\Omega_D)} \\ &\leq C(\|\overrightarrow{u} - \overline{\mathbf{u}}_h\|_{\mathbf{H}^{-\delta}(\Omega_S)} + \|\phi - \overline{\phi}_h\|_{H^{-\delta}(\Omega_D)}). \end{aligned} \tag{2.38}$$

Hence, by choosing $\overrightarrow{v} = \overrightarrow{u} - \overline{\mathbf{u}}_h$ and $\varphi = \phi - \overline{\phi}_h$ in (2.37) and using (2.22) and (2.38), we have

$$\begin{aligned} &\|\overrightarrow{u} - \overline{\mathbf{u}}_h\|^2_{L^2(\Omega_S)} + \|\phi - \overline{\phi}_h\|^2_{L^2(\Omega_D)} = \widetilde{a}(\mathbf{w}, \sigma, \psi; \overrightarrow{u} - \overline{\mathbf{u}}_h, p - \overline{p}_h, \phi - \overline{\phi}_h) \\ &= a(\overrightarrow{u} - \overline{\mathbf{u}}_h, p - \overline{p}_h, \phi - \overline{\phi}_h; \mathbf{w}, \sigma, \psi) \\ &= a(\overrightarrow{u} - \overline{\mathbf{u}}_h, p - I_h p, \phi - \overline{\phi}_h; \mathbf{w} - \Pi_h \mathbf{w}, \sigma - I_h \sigma, \psi - I_h \psi) \\ &\leq C(\|\overrightarrow{u} - \overline{\mathbf{u}}_h\|_{H^1(\Omega_S)} + \|p - I_h p\|_{L^2(\Omega_S)} + \|\phi - \overline{\phi}_h\|_{H^1(\Omega_D)}) \\ &\quad \cdot(\|\mathbf{w} - \Pi_h \mathbf{w}\|_{H^1(\Omega_S)} + \|\sigma - I_h \sigma\|_{L^2(\Omega_S)} + \|\psi - I_h \psi\|_{H^1(\Omega_D)}) \\ &\leq C(\|\overrightarrow{u} - \overline{\mathbf{u}}_h\|_{H^1(\Omega_S)} + \|p - I_h p\|_{L^2(\Omega_S)} + \|\phi - \overline{\phi}_h\|_{H^1(\Omega_D)}) \\ &\quad \cdot(\|\mathbf{w}\|_{\mathbf{H}^{2-\delta}(\Omega_S)} + \|\sigma\|_{H^{1-\delta}(\Omega_S)} + \|\psi\|_{H^{2-\delta}(\Omega_D)}) h^{1-\delta} \\ &\leq C(\|\overrightarrow{u} - \overline{\mathbf{u}}_h\|_{H^1(\Omega_S)} + \|p - I_h p\|_{L^2(\Omega_S)} + \phi - \overline{\phi}_h\|_{H^1(\Omega_D)}) \\ &\quad \cdot(\|\overrightarrow{u} - \overline{\mathbf{u}}_h\|_{\mathbf{H}^{-\delta}(\Omega_S)} + \|\phi - \overline{\phi}_h\|_{H^{-\delta}(\Omega_D)}) h^{1-\delta} \\ &\leq C(\|\overrightarrow{u} - \overline{\mathbf{u}}_h\|_{H^1(\Omega_S)} + \|p - I_h p\|_{L^2(\Omega_S)} + \|\phi - \overline{\phi}_h\|_{H^1(\Omega_D)}) \\ &\quad \cdot(\|\overrightarrow{u} - \overline{\mathbf{u}}_h\|_{\mathbf{L}^2(\Omega_S)} + \|\phi - \overline{\phi}_h\|_{L^2(\Omega_D)}) h^{1-\delta}, \end{aligned}$$

which leads to

$$\begin{aligned} \|\overrightarrow{u} - \overline{\mathbf{u}}_h\|_{L^2(\Omega_S)} + \|\phi - \overline{\phi}_h\|_{L^2(\Omega_D)} \leq C h^{1-\delta}(\|\overrightarrow{u} - \overline{\mathbf{u}}_h\|_{H^1(\Omega_S)} &+ \|p - I_h p\|_{L^2(\Omega_S)} \\ &+ \|\phi - \overline{\phi}_h\|_{H^1(\Omega_D)}). \end{aligned}$$

The last inequality, together with (2.22), implies (2.21).

Let L_h denote the Lagrange interpolation operator onto the finite element space $\overrightarrow{v}^{r}_{h}(\Omega_S)$ (with $r \geq 2$). Then (2.22) implies

$$\begin{aligned}\|L_h\overrightarrow{u} - \bar{\mathbf{u}}_h\|_{\mathbf{H}^1(\Omega_S)} &\leq \|L_h\overrightarrow{u} - \overrightarrow{u}\|_{\mathbf{H}^1(\Omega_S)} + \|\overrightarrow{u} - \bar{\mathbf{u}}_h\|_{\mathbf{H}^1(\Omega_S)} \\ &\leq C_\delta h^{r-\delta}(\|\overrightarrow{u}\|_{\mathbf{H}^{r+1-\delta}(\Omega_S)} + \|p\|_{H^{r-\delta}(\Omega_S)} + \|\phi\|_{H^{r+1-\delta}(\Omega_D)}).\end{aligned}$$

By using inverse inequality of the finite element space, we have

$$\begin{aligned}\|L_h\overrightarrow{u} - \bar{\mathbf{u}}_h\|_{\mathbf{W}^{1,\infty}(\Omega_S)} &\leq Ch^{-1}\|L_h\overrightarrow{u} - \bar{\mathbf{u}}_h\|_{\mathbf{H}^1(\Omega_S)} \\ &\leq C_\delta h^{r-1-\delta}(\|\overrightarrow{u}\|_{\mathbf{H}^{r+1-\delta}(\Omega_S)} + \|p\|_{H^{r-\delta}(\Omega_S)} + \|\phi\|_{H^{r+1-\delta}(\Omega_D)}).\end{aligned}$$

Furthermore, the Lagrange interpolation operator satisfies

$$\|L_h\overrightarrow{u} - \overrightarrow{u}\|_{\mathbf{W}^{1,\infty}(\Omega_S)} \leq Ch^{r-1-\delta}\|\overrightarrow{u}\|_{\mathbf{H}^{r+1-\delta}(\Omega_S)} \leq Ch^{1-\delta}\|\overrightarrow{u}\|_{\mathbf{H}^{r+1-\delta}(\Omega_S)}.$$

Hence the last two estimates imply

$$\begin{aligned}\|\bar{\mathbf{u}}_h\|_{\mathbf{W}^{1,\infty}(\Omega_S)} &\leq \|L_h\overrightarrow{u} - \bar{\mathbf{u}}_h\|_{\mathbf{W}^{1,\infty}(\Omega_S)} + \|L_h\overrightarrow{u} - \overrightarrow{u}\|_{\mathbf{W}^{1,\infty}(\Omega_S)} + \|\overrightarrow{u}\|_{\mathbf{W}^{1,\infty}(\Omega_S)} \\ &\leq C(\|\overrightarrow{u}\|_{\mathbf{H}^{r+1-\delta}(\Omega_S)} + \|p\|_{H^{r-\delta}(\Omega_S)} + \|\phi\|_{H^{r+1-\delta}(\Omega_D)}),\end{aligned}$$

which proves (2.24). Hence, the proof of Theorem 1 is completed.

2.5. CONVERGENCE ANALYSIS

In this section, we prove our main theorem of the convergence for the finite element solution of the proposed semi-implicit multi-step non-iterative DDM. Now we state the main theoretical result for the convergence of the semi-implicit multi-step non-iterative domain decomposition method.

Theorem 2 *Under assumptions* (A1)-(A4)*, there exist positive constants* τ_0*,* h_0 *and* C_* *such that when* $\tau \leq \tau_0$ *and* $h \leq h_0$*, the system* (2.14)-(2.16) *has a unique solution, which satisfies the following estimate:*

$$\begin{aligned}
&\max_{k\leq n\leq N}\big(\|\overrightarrow{u}_h^n - \overline{\mathbf{u}}_h^n\|^2_{L^2(\Omega_S)} + \|\phi_h^n - \overline{\phi}_h^n\|^2_{L^2(\Omega_D)}\big)\\
&+\tau\sum_{n=k}^{N}\big(\|\nabla(\overrightarrow{u}_h^n - \mathbf{u}_h^n)\|^2_{L^2(\Omega_S)} + \|\nabla(\phi_h^n - \overline{\phi}_h^n)\|^2_{L^2(\Omega_D)}\big)\\
&\leq C(\tau^k + h^{r+1-2\delta})^2.
\end{aligned}$$

In the following analysis, we will prove Theorem 2 with δ denoting the fixed constant in assumption (A4). To simplify our notations, we let C denote a generic positive constant, ε denote an arbitrary positive constant such that $\varepsilon \in (0, 1/2)$, and C_ε denote a generic positive constant which depends on ε.

Before we prove the main theorem, we need to present some lemmas. First, based on (2.2) and (2.23), one can obtain the following conclusion, which will be used in the proof of Theorem 2 in the next section.

Lemma 3 *Under assumption* (A1)-(A3)*, we have*

$$\Big\|\sum_{j=0}^{k-1}\gamma_j\overline{\mathbf{u}}_h(\cdot, t_{n-1-j}) - \overline{\mathbf{u}}_h(\cdot, t_n)\Big\|_{L^2(\Gamma)} \leq C\tau^k\,, \tag{2.39}$$

$$\Big\|\sum_{j=0}^{k-1}\gamma_j\overline{\phi}_h(\cdot, t_{n-1-j}) - \overline{\phi}_h(\cdot, t_n)\Big\|_{L^2(\Gamma)} \leq C\tau^k\,. \tag{2.40}$$

Proof of Lemma 3: Since the joint Stokes-Darcy Ritz-projection is a linear map from $(\overrightarrow{u}, p, \phi)$ to $(\overline{\mathbf{u}}_h, \overline{p}_h, \overline{\phi}_h)$, it follows that $\partial_t^k\overline{\mathbf{u}}_h = \overline{(\partial_t^k\overrightarrow{u})_h}$, where $(\overline{(\partial_t^k\overrightarrow{u})_h}, \overline{(\partial_t^k p)_h}, \overline{(\partial_t^k\phi)_h})$ denotes the joint Stokes-Darcy Ritz-projection of $(\partial_t^k\overrightarrow{u}, \partial_t^k p, \partial_t^k\phi)$.

By applying (2.2), we have the following estimate

$$\Big\|\sum_{j=0}^{k-1}\gamma_j\overline{\mathbf{u}}_h(\cdot,t_{n-1-j})-\overline{\mathbf{u}}_h(\cdot,t_n)\Big\|_{L^2(\Gamma)} \le C\tau^k\|\partial_t^k\overline{\mathbf{u}}_h\|_{C([0,T];L^2(\Gamma))} = C\tau^k\|\overline{(\partial_t^k\overrightarrow{u})_h}\|_{C([0,T];L^2(\Gamma))}$$

$$\le C\tau^k\big(\|\partial_t^k\overrightarrow{u}\|_{C([0,T];H^1(\Omega_S))}+\|\partial_t^k p\|_{C([0,T];L^2(\Omega_S))} + \|\partial_t^k\phi\|_{C([0,T];H^1(\Omega_D))}\big)$$

where we have used (2.23) in the last step above. This completes the proof of (2.39), and the proof of (2.40) is similar.

Second, we recall the following result of BDF methods in Lemma 1.1 of Akrivis (2015) and Lemma 2 of Akrivis and Lubich (2015); see also Dahlquist (1978); Nevanlinna and Odeh (1981).

Lemma 4 *For any given integer* $1 \le k \le 5$*, there exists a constant* $\eta_k \in (0,1)$ *such that*

$$\sum_{j=0}^{k}\frac{1}{\tau}d_j\varphi^{n-j}(\varphi^n-\eta_k\varphi^{n-1}) \ge D_\tau\left(\sum_{i,j=0}^{k-1}g_{ij}\varphi^{n-i}\varphi^{n-j}\right) \qquad n \ge k,$$

for any sequence φ^n*,* $n = 0,1,\dots,N$*, where* $D_\tau\varphi^n := \frac{1}{\tau}(\varphi^n-\varphi^{n-1})$ *and* (g_{ij}) *is a symmetric positive definite* $k \times k$ *matrix such that*

$$\kappa\sum_{i=0}^{k-1}\xi_i^2 \le \sum_{i,j=0}^{k-1}g_{ij}\xi_i\xi_j \le \kappa^{-1}\sum_{i=0}^{k-1}\xi_i^2, \quad \forall\,\xi_i \in \mathbf{R},\ \ i = 0,\dots,k-1$$

for some positive constant κ*.*

Besides Lemma 4, we shall also use Korn's inequality (c.f. John and Layton (2002)):

$$\|\mathbb{D}(\overrightarrow{u})\|_{L^2(\Omega_S)} \ge \theta\|\nabla\overrightarrow{u}\|_{L^2(\Omega_S)}, \quad \text{for some constant } \theta > 0, \tag{2.41}$$

Young's inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \forall\, p, q \in [1, \infty] \text{ such that } \frac{1}{p} + \frac{1}{q} = 1, \tag{2.42}$$

interpolation inequality (cf. Proposition 1.1.14 and Exercise 1.1.16 in Grafakos (2008)):

$$\|v\|_{L^p(\Gamma)} \leq \|v\|^{1-\theta}_{L^2(\Gamma)} \|v\|^{\theta}_{L^q(\Gamma)}, \quad \text{with } \frac{1-\theta}{2} + \frac{\theta}{q} = \frac{1}{p}, \ \theta \in [0,1], \ 2 \leq p \leq q, \tag{2.43}$$

and the trace inequalities:

$$\|\mathbf{w}\|_{L^2(\Gamma)} \leq \varepsilon \|\nabla \mathbf{w}\|_{L^2(\Omega_S)} + C\varepsilon^{-1} \|\mathbf{w}\|_{L^2(\Omega_S)}, \qquad \forall\, \mathbf{w} \in \mathbf{H}^1(\Omega_S), \tag{2.44}$$

$$\|\varphi\|_{L^2(\Gamma)} \leq \varepsilon \|\nabla \varphi\|_{L^2(\Omega_D)} + C\varepsilon^{-1} \|\varphi\|_{L^2(\Omega_D)}, \qquad \forall\, \varphi \in H^1(\Omega_S), \tag{2.45}$$

where $\varepsilon \in (0,1)$ can be arbitrarily small.

Then we evaluate the weak formulation (2.6)-(2.7) and (2.9) at t_n and rewrite it into the following equations:

$$\Big(\frac{1}{\tau} \sum_{j=0}^{k} d_j \vec{u}^{\,n-j}, \vec{v}_h\Big)_{\Omega_S} + \Big(\sum_{j=0}^{k-1} \gamma_j \vec{u}^{\,n-1-j} \cdot \nabla \vec{u}^{\,n}, \vec{v}_h\Big)_{\Omega_S} + a_S(\vec{u}^{\,n}, \vec{v}_h) \tag{2.46}$$

$$+ b_S(\vec{v}_h, p_S^n) + \langle \vec{u}^{\,n} \cdot \vec{n}_S, \vec{v}_h \cdot \vec{n}_S \rangle + \beta \langle \vec{u}^{\,n} - (\vec{u}^{\,n} \cdot \vec{n}_S)\vec{n}_S, \vec{v}_h - (\vec{v}_h \cdot \vec{n}_S)\vec{n}_S \rangle$$

$$= \big(\mathbf{f}_S^n, \vec{v}_h\big)_{\Omega_S} + \langle \xi_S^n, \vec{v}_h \cdot \vec{n}_S \rangle + \big(E_u^n, \vec{v}_h\big)_{\Omega_S} + \big(E_N^n, \vec{v}_h\big)_{\Omega_S}, \quad \forall\, \vec{v}_h \in \vec{v}_h^{\,r}(\Omega_S),$$

$$\big(\nabla \cdot \vec{u}^{\,n}, q_h\big)_{\Omega_S} = 0, \quad \forall\, q_h \in V_h^{r-1}(\Omega_S). \tag{2.47}$$

$$\Big(\frac{1}{\tau} \sum_{j=0}^{k} d_j \phi^{n-j}, \varphi_h\Big)_{\Omega_D} + a_D(\phi^n, \varphi_h) + \langle \phi^n, \varphi_h \rangle \tag{2.48}$$

$$= \big(f_D^n, \varphi_h\big)_{\Omega_D} + \langle \xi_D^n, \varphi_h \rangle + \big(E_\phi^n, \varphi_h\big)_{\Omega_D}, \quad \forall\, \varphi_h \in V_h^r(\Omega_D),$$

where $\xi_S^n := \vec{u}^{\,n} \cdot \vec{n}_S - \phi^n, \quad \xi_D^n := \vec{u}^{\,n} \cdot \vec{n}_S + \phi^n.$

The truncation errors of temporal discretization are

$$E_u^n := \frac{1}{\tau}\sum_{j=0}^k d_j \overrightarrow{u}^{n-j} - \partial_t \overrightarrow{u}(\cdot, t_n), \;\; E_\phi^n := \frac{1}{\tau}\sum_{j=0}^k d_j \phi^{n-j} - \partial_t \phi(\cdot, t_n),$$
$$E_N^n := (\textstyle\sum_{j=0}^k \gamma_j \overrightarrow{u}^{n-1-j} - \overrightarrow{u}^n) \cdot \nabla \overrightarrow{u}^n,$$

In view of (2.1), we have

$$\|E_u^n\|_{L^2(\Omega_S)} + \|E_\phi^n\|_{L^2(\Omega_D)} \leq C\tau^k. \tag{2.49}$$

As mentioned in the last section, the joint Stokes-Darcy Ritz-projection is a key to our error analysis. Substituting (2.20) into (2.46)-(2.48), we obtain

$$\begin{aligned}
&\Big(\frac{1}{\tau}\sum_{j=0}^{k} d_j \overline{\mathbf{u}}_h^{n-j}, \overrightarrow{v}_h\Big)_{\Omega_S} + \Big(\sum_{j=0}^{k-1} \gamma_j \overline{\mathbf{u}}_h^{n-1-j} \cdot \nabla \overline{\mathbf{u}}_h^n, \overrightarrow{v}_h\Big)_{\Omega_S} + a_S(\overline{\mathbf{u}}_h^n, \overrightarrow{v}_h) \qquad (2.50)\\
&\quad + b_S(\overrightarrow{v}_h, \overline{p}_h^n) + \langle \overline{\mathbf{u}}_h^n \cdot \overrightarrow{n}_S, \overrightarrow{v}_h \cdot \overrightarrow{n}_S\rangle + \beta\langle \overline{\mathbf{u}}_h^n - (\overline{\mathbf{u}}_h^n \cdot \overrightarrow{n}_S)\overrightarrow{n}_S, \overrightarrow{v}_h - (\overrightarrow{v}_h \cdot \overrightarrow{n}_S)\overrightarrow{n}_S\rangle\\
&= \big(\mathbf{f}_S^n, \overrightarrow{v}_h\big)_{\Omega_S} + \langle \overline{\xi}_{Sh}^n, \overrightarrow{v}_h \cdot \overrightarrow{n}_S\rangle + \big(E_u^n, \overrightarrow{v}_h\big)_{\Omega_S} + \big(E_N^n, \overrightarrow{v}_h\big)_{\Omega_S}\\
&\quad + \Big(\frac{1}{\tau}\sum_{j=0}^{k} d_j (\overline{\mathbf{u}}_h^{n-j} - \overrightarrow{u}^{n-j}), \overrightarrow{v}_h\Big)_{\Omega_S} + \Big(\sum_{j=0}^{k-1} \gamma_j \overrightarrow{u}^{n-1-j} \cdot \nabla(\overline{\mathbf{u}}_h^n - \overrightarrow{u}^n), \overrightarrow{v}_h\Big)_{\Omega_S}\\
&\quad + \Big(\sum_{j=0}^{k-1} \gamma_j (\overline{\mathbf{u}}_h^{n-1-j} - \overrightarrow{u}^{n-1-j}) \cdot \nabla \overline{\mathbf{u}}_h^n, \overrightarrow{v}_h\Big)_{\Omega_S} + \big(\overrightarrow{u} - \overline{\mathbf{u}}_h, \overrightarrow{v}_h\big)_{\Omega_S}, \quad \forall\, \overrightarrow{v}_h \in \overrightarrow{v}_h^r(\Omega_S),\\
&\big(\nabla \cdot \overline{\mathbf{u}}_h^n, q_h\big)_{\Omega_S} = 0, \quad \forall\, q_h \in V_h^{r-1}(\Omega_S). \qquad (2.51)\\
&\Big(\frac{1}{\tau}\sum_{j=0}^{k} d_j \overline{\phi}_h^{n-j}, \varphi_h\Big)_{\Omega_D} + a_D(\overline{\phi}_h^n, \varphi_h) + (\overline{\phi}_h^n, \varphi_h)_\Gamma \qquad (2.52)\\
&= \big(f_D^n, \varphi_h\big)_{\Omega_D} + \langle \overline{\xi}_{Dh}^n, \varphi_h\rangle + \big(E_\phi^n, \varphi_h\big)_{\Omega_D} + \Big(\frac{1}{\tau}\sum_{j=0}^{k} d_j (\overline{\phi}_h^{n-j} - \phi^{n-j}), \varphi_h\Big)_{\Omega_D}\\
&\quad + \big(\phi - \overline{\phi}_h, \varphi_h\big)_{\Omega_D} \quad \forall\, \varphi_h \in V_h^r(\Omega_D),
\end{aligned}$$

where

$$\overline{\xi}_{Sh}^n := \overline{\mathbf{u}}_h^n \cdot \overrightarrow{n}_S - \overline{\phi}_h^n, \quad \overline{\xi}_{Dh}^n := \overline{\mathbf{u}}_h^n \cdot \overrightarrow{n}_S + \overline{\phi}_h^n. \tag{2.53}$$

Let $\mathbf{e}_{h,u}^n := \overrightarrow{u}_h^n - \overline{\mathbf{u}}_h^n$, $\quad e_{h,p}^n := p_h^n - \overline{p}_h^n$ $\quad$ and $\quad e_{h,\phi}^n := \phi_h^n - \overline{\phi}_h^n$ denote the error between the finite element solution and the joint Stokes-Darcy Ritz-projection of the solution. Then the difference between (2.14)-(2.16) and (2.50)-(2.52) gives

$$\begin{aligned}
&\Big(\frac{1}{\tau}\sum_{j=0}^{k} d_j \mathbf{e}_{h,u}^{n-j}, \overrightarrow{v}_h\Big)_{\Omega_S} + 2\nu\big(\mathbb{D}(\mathbf{e}_{h,u}^n), \mathbb{D}(\overrightarrow{v}_h)\big)_{\Omega_S} - \big(e_{h,p}^n, \nabla\cdot\overrightarrow{v}_h\big)_{\Omega_S} \qquad (2.54)\\
&\quad + \langle \mathbf{e}_{h,u}^n \cdot \overrightarrow{n}_S, \overrightarrow{v}_h \cdot \overrightarrow{n}_S\rangle + \beta\langle P_{\tan}\mathbf{e}_{h,u}^n, P_{\tan}\overrightarrow{v}_h\rangle\\
&= \langle \sum_{j=0}^{k-1} \gamma_j(\xi_{Sh}^{n-1-j} - \overline{\xi}_{Sh}^{n-1-j}), \overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle + \langle \sum_{j=0}^{k-1}\gamma_j \overline{\xi}_{Sh}^{n-1-j} - \overline{\xi}_{Sh}^n, \overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle\\
&\quad - \Big(\frac{1}{\tau}\sum_{j=0}^{k} d_j(\overline{\mathbf{u}}_h^{n-j} - \overrightarrow{u}^{n-j}), \overrightarrow{v}_h\Big)_{\Omega_S} - \big(E_u^n, \overrightarrow{v}_h\big)_{\Omega_S} - \big(E_N^n, \overrightarrow{v}_h\big)_{\Omega_S} - \big(\overrightarrow{u} - \overline{\mathbf{u}}_h, \overrightarrow{v}_h\big)_{\Omega_S}\\
&\quad - \Big(\sum_{j=0}^{k-1}\gamma_j \mathbf{e}_{h,u}^{n-1-j}\cdot\nabla\overline{\mathbf{u}}_h^n, \overrightarrow{v}_h\Big)_{\Omega_S} - \Big(\sum_{j=0}^{k-1}\gamma_j \overrightarrow{u}_h^{n-1-j}\cdot\nabla\mathbf{e}_{h,u}^n, \overrightarrow{v}_h\Big)_{\Omega_S}, \quad \forall\, \overrightarrow{v}_h \in \overrightarrow{v}_h^r(\Omega_S),\\
&\big(\nabla\cdot\mathbf{e}_{h,u}^n, q_h\big)_{\Omega_S} = 0, \quad \forall\, q_h \in V_h^{r-1}(\Omega_S). \qquad (2.55)\\
&\Big(\frac{1}{\tau}\sum_{j=0}^{k} d_j e_{h,\phi}^{n-j}, \varphi_h\Big)_{\Omega_D} + \big(\mathbb{K}\nabla e_{h,\phi}^n, \nabla\varphi_h\big)_{\Omega_D} + \langle e_{h,\phi}^n, \varphi_h\rangle \qquad (2.56)\\
&= \langle \sum_{j=0}^{k-1}\gamma_j(\xi_{Dh}^{n-1-j} - \overline{\xi}_{Dh}^{n-1-j}), \varphi_h\rangle + \langle\sum_{j=0}^{k-1}\gamma_j\overline{\xi}_{Dh}^{n-1-j} - \overline{\xi}_{Dh}^n, \varphi_h\rangle\\
&\quad - \Big(\frac{1}{\tau}\sum_{j=0}^{k} d_j(\overline{\phi}_h^{n-j} - \phi^{n-j}), \varphi_h\Big)_{\Omega_D} - \big(E_\phi^n, \varphi_h\big)_{\Omega_D} - \big(\phi - \overline{\phi}_h, \varphi_h\big)_{\Omega_D}, \quad \forall\,\varphi_h \in V_h^r(\Omega_D).
\end{aligned}$$

From (2.55), we can get

$$(e_{h,p}^n, \nabla\cdot(\mathbf{e}_{h,u}^n - \eta_k\mathbf{e}_{h,u}^{n-1}))_{\Omega_S} = 0.$$

Using this identity and substituting $\overrightarrow{v}_h = \mathbf{e}_{h,u}^n - \eta_k\mathbf{e}_{h,u}^{n-1}$ into (2.54), we obtain

$$\begin{aligned}
&\Big(\frac{1}{\tau}\sum_{j=0}^{k} d_j\mathbf{e}_{h,u}^{n-j}, \mathbf{e}_{h,u}^n - \eta_k\mathbf{e}_{h,u}^{n-1}\Big)_{\Omega_S} + 2\nu\big(\mathbb{D}(\mathbf{e}_{h,u}^n), \mathbb{D}(\mathbf{e}_{h,u}^n - \eta_k\mathbf{e}_{h,u}^{n-1})\big)_{\Omega_S} \qquad (2.57)\\
&\quad + \langle\mathbf{e}_{h,u}^n\cdot\overrightarrow{n}_S, (\mathbf{e}_{h,u}^n - \eta_k\mathbf{e}_{h,u}^{n-1})\cdot\overrightarrow{n}_S\rangle + \beta\langle P_{\tan}\mathbf{e}_{h,u}^n, P_{\tan}\mathbf{e}_{h,u}^n - \eta_k P_{\tan}\mathbf{e}_{h,u}^{n-1}\rangle\\
&=: J_1 + J_2 - J_3 - J_4 - J_5 - J_6 - J_7.
\end{aligned}$$

The corresponding terms of $J_i (i = 1, \ldots, 7)$ are

$$
\begin{aligned}
J_1 &:= \big(\textstyle\sum_{j=0}^{k-1} \gamma_j (\xi_{Sh}^{n-1-j} - \overline{\xi}_{Sh}^{n-1-j}), (\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}) \cdot \overrightarrow{n}_S \big)_\Gamma, \\
J_2 &:= \big(\textstyle\sum_{j=0}^{k-1} \gamma_j \overline{\xi}_{Sh}^{n-1-j} - \overline{\xi}_{Sh}^n, (\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}) \cdot \overrightarrow{n}_S \big)_\Gamma, \\
J_3 &:= \big(\tfrac{1}{\tau} \textstyle\sum_{j=0}^{k} d_j (\overline{\mathbf{u}}_h^{n-j} - \overrightarrow{u}^{n-j}), (\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}) \big)_{\Omega_S}, \\
J_4 &:= \big(E_u^n, (\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}) \big)_{\Omega_S}, \quad J_5 := \big(E_N^n, (\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}) \big)_{\Omega_S}, \\
J_6 &:= \big(\textstyle\sum_{j=0}^{k-1} \gamma_j \overrightarrow{u}_h^{n-1-j} \cdot \nabla \mathbf{e}_{h,u}^n, (\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}) \big)_{\Omega_S}, \\
J_7 &:= \big(\textstyle\sum_{j=0}^{k-1} \gamma_j \mathbf{e}_{h,u}^{n-1-j} \cdot \nabla \overline{\mathbf{u}}_h^n, (\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}) \big)_{\Omega_S}, \quad J_8 := \big(\overrightarrow{u} - \overline{\mathbf{u}}_h, \mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1} \big)_{\Omega_S}.
\end{aligned}
$$

The left side of (2.57) will be estimated by using the multiplier technique of (Nevanlinna and Odeh, 1981, see Lemma 4). The details are carried out below for the convenience of the readers. Note that

$$
\begin{aligned}
\big(\mathbb{D}(\mathbf{e}_{h,u}^n), \mathbb{D}(\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}) \big)_{\Omega_S} &= \big(\mathbb{D}(\mathbf{e}_{h,u}^n), \mathbb{D}(\mathbf{e}_{h,u}^n) \big)_{\Omega_S} - \eta_k \big(\mathbb{D}(\mathbf{e}_{h,u}^n), \mathbb{D}(\mathbf{e}_{h,u}^{n-1}) \big)_{\Omega_S} \\
&\geq \|\mathbb{D}(\mathbf{e}_{h,u}^n)\|_{L^2(\Omega_S)}^2 - \frac{\eta_k}{2} \|\mathbb{D}(\mathbf{e}_{h,u}^n)\|_{L^2(\Omega_S)}^2 - \frac{\eta_k}{2} \|\mathbb{D}(\mathbf{e}_{h,u}^{n-1})\|_{L^2(\Omega_S)}^2 \\
&= (1 - \eta_k) \|\mathbb{D}(\mathbf{e}_{h,u}^n)\|_{L^2(\Omega_S)}^2 + D_\tau \big(\frac{\eta_k \tau}{2} \|\mathbb{D}(\mathbf{e}_{h,u}^n)\|_{L^2(\Omega_S)}^2 \big), \\
\langle \mathbf{e}_{h,u}^n \cdot \overrightarrow{n}_S, (\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}) \cdot \overrightarrow{n}_S \rangle &= \langle \mathbf{e}_{h,u}^n \cdot \overrightarrow{n}_S, \mathbf{e}_{h,u}^n \cdot \overrightarrow{n}_S \rangle - \eta_k \langle \mathbf{e}_{h,u}^n \cdot \overrightarrow{n}_S, \mathbf{e}_{h,u}^{n-1} \cdot \overrightarrow{n}_S \rangle \\
&\geq \|\mathbf{e}_{h,u}^n \cdot \overrightarrow{n}_S\|_{L^2(\Gamma)}^2 - \frac{\eta_k}{2} \|\mathbf{e}_{h,u}^n \cdot \overrightarrow{n}_S\|_{L^2(\Gamma)}^2 - \frac{\eta_k}{2} \|\mathbf{e}_{h,u}^{n-1} \cdot \overrightarrow{n}_S\|_{L^2(\Gamma)}^2 \\
&= (1 - \eta_k) \|\mathbf{e}_{h,u}^n \cdot \overrightarrow{n}_S\|_{L^2(\Gamma)}^2 + D_\tau \big(\frac{\eta_k \tau}{2} \|\mathbf{e}_{h,u}^n \cdot \overrightarrow{n}_S\|_{L^2(\Gamma)}^2 \big), \\
\langle P_{\tan} \mathbf{e}_{h,u}^n, P_{\tan} \mathbf{e}_{h,u}^n - \eta_k P_{\tan} \mathbf{e}_{h,u}^{n-1} \rangle &= \langle P_{\tan} \mathbf{e}_{h,u}^n, P_{\tan} \mathbf{e}_{h,u}^n \rangle - \eta_k \langle P_{\tan} \mathbf{e}_{h,u}^n, P_{\tan} \mathbf{e}_{h,u}^{n-1} \rangle \\
&\geq \|P_{\tan} \mathbf{e}_{h,u}^n\|_{L^2(\Gamma)}^2 - \frac{\eta_k}{2} \|P_{\tan} \mathbf{e}_{h,u}^n\|_{L^2(\Gamma)}^2 - \frac{\eta_k}{2} \|P_{\tan} \mathbf{e}_{h,u}^{n-1}\|_{L^2(\Gamma)}^2 \\
&= (1 - \eta_k) \|P_{\tan} \mathbf{e}_{h,u}^n\|_{L^2(\Gamma)}^2 + D_\tau \big(\frac{\eta_k \tau}{2} \|P_{\tan} \mathbf{e}_{h,u}^n\|_{L^2(\Gamma)}^2 \big).
\end{aligned}
$$

By using Lemma 4 and the last three estimates, (2.57) reduces to

$$
\begin{aligned}
&D_\tau\Big(\sum_{i,j=0}^{k-1} g_{ij}\big(\mathbf{e}_{h,u}^{n-i},\mathbf{e}_{h,u}^{n-j}\big)_{\Omega_S} + \frac{\eta_k\tau}{2}\|\mathbb{D}(\mathbf{e}_{h,u}^{n})\|^2_{L^2(\Omega_S)}\Big)\\
&+D_\tau\Big(\frac{\eta_k\tau}{2}\|\mathbf{e}_{h,u}^{n}\cdot\overrightarrow{n}_S\|^2_{L^2(\Gamma)} + \frac{\eta_k\tau}{2}\|P_{\tan}\mathbf{e}_{h,u}^{n}\|^2_{L^2(\Gamma)}\Big)\\
&+(1-\eta_k)\big(2\nu\|\mathbb{D}(\mathbf{e}_{h,u}^{n})\|^2_{L^2(\Omega_S)} + \|\mathbf{e}_{h,u}^{n}\cdot\overrightarrow{n}_S\|^2_{L^2(\Gamma)} + \beta\|P_{\tan}\mathbf{e}_{h,u}^{n}\|^2_{L^2(\Gamma)}\big)\\
&\le |J_1|+|J_2|+|J_3|+|J_4|+|J_5|+|J_6|+|J_7|+|J_8|.
\end{aligned}
\tag{2.58}
$$

Similarly, substituting $\varphi_h = e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1}$ into (2.56), we obtain

$$
\begin{aligned}
&\Big(\frac{1}{\tau}\sum_{j=0}^{k} d_j e_{h,\phi}^{n-j}, e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1}\Big)_{\Omega_D} + \big(K\nabla e_{h,\phi}^n, \nabla(e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1})\big)_{\Omega_D} + \langle e_{h,\phi}^n, e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1}\rangle\\
&=: J_9 + J_{10} - J_{11} - J_{12} - J_{13}
\end{aligned}
\tag{2.59}
$$

where

$$
\begin{aligned}
&J_9 = \langle\sum_{j=0}^{k-1}\gamma_j(\xi_{Dh}^{n-1-j} - \overline{\xi}_{Dh}^{n-1-j}), e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1}\rangle, J_{10} = \langle\sum_{j=0}^{k-1}\gamma_j\overline{\xi}_{Dh}^{n-1-j} - \overline{\xi}_{Dh}^{n}, e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1}\rangle,\\
&J_{11} = \Big(\frac{1}{\tau}\sum_{j=0}^{k} d_j(\overline{\phi}_h^{n-j} - \phi^{n-j}), e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1}\Big)_{\Omega_D}, \quad J_{12} = \big(E_\phi^n, e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1}\big)_{\Omega_D},\\
&J_{13} = \big(\phi - \overline{\phi}_h, e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1}\big)_{\Omega_D}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\big(\mathbb{K}\nabla e_{h,\phi}^n, \nabla(e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1})\big)_{\Omega_D} = \big(\mathbb{K}\nabla e_{h,\phi}^n, \nabla e_{h,\phi}^n\big)_{\Omega_D} - \eta_k\big(\mathbb{K}\nabla e_{h,\phi}^n, \nabla e_{h,\phi}^{n-1}\big)_{\Omega_D}\\
&\ge \big(\mathbb{K}\nabla e_{h,\phi}^n, \nabla e_{h,\phi}^n\big)_{\Omega_D} - \frac{\eta_k}{2}\big(\mathbb{K}\nabla e_{h,\phi}^n, \nabla e_{h,\phi}^n\big)_{\Omega_D} - \frac{\eta_k}{2}\big(\mathbb{K}\nabla e_{h,\phi}^{n-1}, \nabla e_{h,\phi}^{n-1}\big)_{\Omega_D}\\
&= (1-\eta_k)\big(\mathbb{K}\nabla e_{h,\phi}^n, \nabla e_{h,\phi}^n\big)_{\Omega_D} + \frac{\eta_k\tau}{2}D_\tau\big(\mathbb{K}\nabla e_{h,\phi}^n, \nabla e_{h,\phi}^n\big)_{\Omega_D},
\end{aligned}
$$

$$
\begin{aligned}
&\langle e_{h,\phi}^n, e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1}\rangle = \big(e_{h,\phi}^n, e_{h,\phi}^n\big)_\Gamma - \eta_k\big(e_{h,\phi}^n, e_{h,\phi}^{n-1}\big)_\Gamma\\
&\ge \|e_{h,\phi}^n\|^2_{L^2(\Gamma)} - \frac{\eta_k}{2}\|e_{h,\phi}^n\|^2_{L^2(\Gamma)} - \frac{\eta_k}{2}\|e_{h,\phi}^{n-1}\|^2_{L^2(\Gamma)}\\
&= (1-\eta_k)\|e_{h,\phi}^n\|^2_{L^2(\Gamma)} + D_\tau\big(\frac{\eta_k\tau}{2}\|e_{h,\phi}^n\|^2_{L^2(\Gamma)}\big).
\end{aligned}
$$

By using Lemma 4 and the last two estimates, (2.59) reduces to

$$D_\tau\Big(\sum_{i,j=0}^{k-1} g_{ij}\big(e_{h,\phi}^{n-i}, e_{h,\phi}^{n-j}\big)_{\Omega_D} + \frac{\eta_k\tau}{2}\big(\mathbb{K}\nabla e_{h,\phi}^n, \nabla e_{h,\phi}^n\big)_{\Omega_D} + \frac{\eta_k\tau}{2}\|e_{h,\phi}^n\|^2_{L^2(\Gamma)}\Big) \tag{2.60}$$
$$+(1-\eta_k)\big(\mathbb{K}\nabla e_{h,\phi}^n, \nabla e_{h,\phi}^n\big)_{\Omega_D} + (1-\eta_k)\|e_{h,\phi}^n\|^2_{L^2(\Gamma)} \le |J_9| + |J_{10}| + |J_{11}| + |J_{12}| + |J_{13}|.$$

Define

$$\Phi_h^n := \textstyle\sum_{i,j=0}^{k-1} g_{ij}\big(\mathbf{e}_{h,u}^{n-i}, \mathbf{e}_{h,u}^{n-j}\big)_{\Omega_S} + \sum_{i,j=0}^{k-1} g_{ij}\big(e_{h,\phi}^{n-i}, e_{h,\phi}^{n-j}\big)_{\Omega_D} + \frac{\eta_k\tau}{2}\|\mathbb{D}(\mathbf{e}_{h,u}^n)\|^2_{L^2(\Omega_S)}$$
$$+\tfrac{\eta_k\tau}{2}\|\mathbf{e}_{h,u}^n\cdot\overrightarrow{n}_S\|^2_{L^2(\Gamma)} + \tfrac{\eta_k\tau}{2}\|P_{\tan}\mathbf{e}_{h,u}^n\|^2_{L^2(\Gamma)} + \tfrac{\eta_k\tau}{2}\big(\mathbb{K}\nabla e_{h,\phi}^n, \nabla e_{h,\phi}^n\big)_{\Omega_D} + \tfrac{\eta_k\tau}{2}\|e_{h,\phi}^n\|^2_{L^2(\Gamma)}.$$

By Lemma 4, we can obtain

$$\begin{aligned}\Phi_h^n &\ge \sum_{i,j=0}^{k-1} g_{ij}\big(\mathbf{e}_{h,u}^{n-i}, \mathbf{e}_{h,u}^{n-j}\big)_{\Omega_S} + \sum_{i,j=0}^{k-1} g_{ij}\big(e_{h,\phi}^{n-i}, e_{h,\phi}^{n-j}\big)_{\Omega_D}\\ &\ge \kappa\sum_{j=0}^{k-1}\|\mathbf{e}_{h,u}^{n-j}\|^2_{L^2(\Omega_S)} + \kappa\sum_{j=0}^{k-1}\|\mathbf{e}_{h,\phi}^{n-j}\|^2_{L^2(\Omega_D)}.\end{aligned} \tag{2.61}$$

Then summing up (2.58) and (2.60) yields

$$D_\tau\Phi_h^n + 2\nu(1-\eta_k)\|\mathbb{D}(\mathbf{e}_{h,u}^n)\|^2_{L^2(\Omega_S)} + (1-\eta_k)\big(\mathbb{K}\nabla e_{h,\phi}^n, \nabla e_{h,\phi}^n\big)_{\Omega_D} \le \textstyle\sum_{j=1}^{13}|J_j|. \tag{2.62}$$

In the rest of the proof, we estimate the right side of (2.62), in which the domain integrals will be estimated in a standard way by using Hölder's inequality and the interface integrals on the right side of (2.57) will be estimated by using the trace inequalities (2.44)-(2.45) in terms of the norms of $L^2(\Omega_S)$ and $H^1(\Omega_S)$. The extrapolation errors $\sum_{j=0}^{k-1}\gamma_j\overline{\xi}_{Sh}^{n-1-j} - \overline{\xi}_{Sh}^n$ and $\sum_{j=0}^{k-1}\gamma_j\overline{\xi}_{Dh}^{n-1-j} - \overline{\xi}_{Dh}^n$ will be estimated by Lemma 3.

To estimate the nonlinear terms involved in the right side of (2.62), we need to use a mathematical induction assumption:

$$\|\overrightarrow{u}_h^n\|_{L^4(\Omega_S)} \le \|\overrightarrow{u}\|_{C([0,T];L^4(\Omega_S))} + 2. \tag{2.63}$$

We shall assume that (2.63) holds for $0 \le n \le m-1$ with some $k \le m \le N$. Such an induction assumption will help to estimate the product of three terms involved in the right side of (2.62), provided one of the three terms would be from the last time step (using explicit scheme). We will use this mathematical induction in the estimation of (2.62) for $n = m$ and use the estimated results to prove that (2.63) also holds for $n = m$. Using (2.17), (2.53), (2.61), the Cauchy-Schwarz inequality, trace inequalities (2.44)-(2.45), the triangle inequality, and Young's inequality, we have, for $k \le n \le m$,

$$\begin{aligned}
|J_1| &= \left|\langle \textstyle\sum_{j=0}^{k-1} \gamma_j (\xi_{Sh}^{n-1-j} - \overline{\xi}_{Sh}^{n-1-j}), (\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}) \cdot \overrightarrow{n}_S \rangle\right| \\
&\le \textstyle\sum_{j=0}^{k-1} |\gamma_j| \, \|\mathbf{e}_{h,u}^{n-1-j} \cdot \overrightarrow{n}_S - g e_{h,\phi}^{n-1-j}\|_{L^2(\Gamma)} \|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\|_{L^2(\Gamma)} \\
&\le C \textstyle\sum_{j=0}^{k} \left(\|\mathbf{e}_{h,u}^{n-j}\|_{L^2(\Gamma)}^2 + \|e_{h,\phi}^{n-j}\|_{L^2(\Gamma)}^2\right) \\
&\le \varepsilon \textstyle\sum_{j=0}^{k} \left(\|\nabla \mathbf{e}_{h,u}^{n-j}\|_{L^2(\Omega_S)}^2 + \|\nabla e_{h,\phi}^{n-j}\|_{L^2(\Omega_D)}^2\right) \\
&\quad + C_\varepsilon \textstyle\sum_{j=0}^{k} \left(\|\mathbf{e}_{h,u}^{n-j}\|_{L^2(\Omega_S)}^2 + \|e_{h,\phi}^{n-j}\|_{L^2(\Omega_D)}^2\right) \\
&\le \varepsilon \textstyle\sum_{j=0}^{k} \left(\|\nabla \mathbf{e}_{h,u}^{n-j}\|_{L^2(\Omega_S)}^2 + \|\nabla e_{h,\phi}^{n-j}\|_{L^2(\Omega_D)}^2\right) + C_\varepsilon(\Phi_h^n + \Phi_h^{n-1}),
\end{aligned} \tag{2.64}$$

$$\begin{aligned}
|J_2| &= \left|\langle \textstyle\sum_{j=0}^{k-1} \gamma_j \overline{\xi}_{Sh}^{n-1-j} - \overline{\xi}_{Sh}^n, (\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}) \cdot \overrightarrow{n}_S \rangle\right| \\
&\le \left\| \textstyle\sum_{j=0}^{k-1} \gamma_j \overline{\xi}_{Sh}^{n-1-j} - \overline{\xi}_{Sh}^n \right\|_{L^2(\Gamma)} \left\|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\right\|_{L^2(\Gamma)} \\
&\le C\tau^k \left\|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\right\|_{L^2(\Gamma)} \qquad \text{[using Lemma 3]} \\
&\le C\tau^k (\varepsilon \left\|\nabla(\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1})\right\|_{L^2(\Omega_S)} + C_\varepsilon \left\|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\right\|_{L^2(\Omega_S)}) \\
&\le C\tau^{2k} + \varepsilon \left\|\nabla(\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1})\right\|_{L^2(\Omega_S)}^2 + C_\varepsilon \left\|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\right\|_{L^2(\Omega_S)}^2 \\
&\le C\tau^{2k} + \varepsilon \textstyle\sum_{j=0}^{k} \left\|\nabla \mathbf{e}_{h,u}^{n-j}\right\|_{L^2(\Omega_S)}^2 + C_\varepsilon(\Phi_h^n + \Phi_h^{n-1}),
\end{aligned} \tag{2.65}$$

$$\begin{aligned}
|J_4| &= \left|\left(E_u^n, (\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1})\right)_{\Omega_S}\right| \le \left\|E_u^n\right\|_{L^2(\Omega_S)} \left\|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\right\|_{L^2(\Omega_S)} \\
&\le C\tau^k \left\|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\right\|_{L^2(\Omega_S)} \qquad \text{[using (2.49)]} \\
&\le C\tau^{2k} + C(\Phi_h^n + \Phi_h^{n-1}),
\end{aligned} \tag{2.66}$$

$$\begin{aligned}
|J_5| &= \left|\left(E_N^n, (\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1})\right)_{\Omega_S}\right| \\
&\le \| \textstyle\sum_{j=0}^{k-1} \gamma_j \overrightarrow{u}^{n-1-j} - \overrightarrow{u}^n \|_{L^2(\Omega_S)} \|\nabla \overrightarrow{u}^n\|_{L^\infty(\Omega_S)} \|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\|_{L^2(\Omega_S)} \\
&\le C\tau^k \left\|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\right\|_{L^2(\Omega_S)} \qquad \text{[using (2.2)]} \\
&\le C\tau^{2k} + C(\Phi_h^n + \Phi_h^{n-1}),
\end{aligned} \tag{2.67}$$

$$
\begin{aligned}
|J_6| &= \Big|\big(\textstyle\sum_{j=0}^{k-1} \gamma_j \overrightarrow{u}_h^{n-1-j} \cdot \nabla \mathbf{e}_{h,u}^n, (\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1})\big)_{\Omega_S}\Big| \\
&\le \textstyle\sum_{j=0}^{k-1} |\gamma_j| \, \big\|\overrightarrow{u}_h^{n-1-j}\big\|_{L^4(\Omega_S)} \big\|\nabla \mathbf{e}_{h,u}^n\big\|_{L^2(\Omega_S)} \big\|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\big\|_{L^4(\Omega_S)} \\
&\le C\big\|\nabla \mathbf{e}_{h,u}^n\big\|_{L^2(\Omega_S)} \big\|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\big\|_{L^4(\Omega_S)} \qquad \text{[using (2.63)]} \\
&\le C\big\|\nabla \mathbf{e}_{h,u}^n\big\|_{L^2(\Omega_S)} \big\|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\big\|_{L^2(\Omega_S)}^{1/4} \big\|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\big\|_{L^6(\Omega_S)}^{3/4} \\
&\le C\big\|\nabla \mathbf{e}_{h,u}^n\big\|_{L^2(\Omega_S)} \big\|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\big\|_{L^2(\Omega_S)}^{1/4} \big\|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\big\|_{H^1(\Omega_S)}^{3/4} \\
&\qquad\qquad \text{[using Sobolev embedding } H^1(\Omega_S) \hookrightarrow L^6(\Omega_S)] \\
&\le \big\|\nabla \mathbf{e}_{h,u}^n\big\|_{L^2(\Omega_S)} (C_\varepsilon \big\|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\big\|_{L^2(\Omega_S)} + \varepsilon \big\|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\big\|_{H^1(\Omega_S)}) \\
&\le \varepsilon \big\|\nabla \mathbf{e}_{h,u}^n\big\|_{L^2(\Omega_S)}^2 + \varepsilon \big\|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\big\|_{H^1(\Omega_S)}^2 + C_\varepsilon \big\|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\big\|_{L^2(\Omega_S)}^2 \\
&\le 3\varepsilon \textstyle\sum_{j=0}^{1} \big\|\nabla \mathbf{e}_{h,u}^{n-j}\big\|_{L^2(\Omega_S)}^2 + C_\varepsilon \textstyle\sum_{j=0}^{1} \big\|\mathbf{e}_h^{n-j}\big\|_{L^2(\Omega_S)}^2 \\
&\le 3\varepsilon \textstyle\sum_{j=0}^{1} \big\|\nabla \mathbf{e}_{h,u}^{n-j}\big\|_{L^2(\Omega_S)}^2 + C_\varepsilon (\Phi_h^n + \Phi_h^{n-1}),
\end{aligned} \tag{2.68}
$$

$$
\begin{aligned}
|J_7| &= \Big|\big(\textstyle\sum_{j=0}^{k-1} \gamma_j \mathbf{e}_{h,u}^{n-1-j} \cdot \nabla \overline{\mathbf{u}}_h^n, (\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1})\big)_{\Omega_S}\Big| \\
&\le \|\mathbf{e}_{h,u}^{n-1-j}\|_{L^2(\Omega_S)} \|\nabla \overline{\mathbf{u}}_h^n\|_{L^\infty(\Omega_S)} \|\mathbf{e}_{h,u}^n\|_{L^2(\Omega_S)} \\
&\le C \textstyle\sum_{j=0}^{k} \|\mathbf{e}_{h,u}^{n-j}\|_{L^2(\Omega_S)}^2 \qquad \text{[using (2.24)]} \\
&\le C(\Phi_h^n + \Phi_h^{n-1}),
\end{aligned} \tag{2.69}
$$

$$
\begin{aligned}
|J_8| &= \big|(\overrightarrow{u} - \overline{\mathbf{u}}_h, \mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1})_{\Omega_S}\big| \le \|\overrightarrow{u} - \overline{\mathbf{u}}_h\|_{L^2(\Omega_S)} \|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\|_{L^2(\Omega_S)} \\
&\le C h^{r+1-2\delta} \|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\|_{L^2(\Omega_S)} \qquad \text{[using (2.21)]} \\
&\le C h^{2r+2-4\delta} + C\|\mathbf{e}_{h,u}^n - \eta_k \mathbf{e}_{h,u}^{n-1}\|_{L^2(\Omega_S)}^2 \\
&\le C h^{2r+2-4\delta} + C(\Phi_h^n + \Phi_h^{n-1}),
\end{aligned} \tag{2.70}
$$

$$
\begin{aligned}
|J_9| &= \big|\langle \textstyle\sum_{j=0}^{k-1} \gamma_j (\xi_{Dh}^{n-1-j} - \overline{\xi}_{Dh}^{n-1-j}), e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1}\rangle\big| \quad \text{[similar to } J_1] \\
&\le \textstyle\sum_{j=0}^{k-1} |\gamma_j| \, \big\|\mathbf{e}_{h,u}^{n-1-j} \cdot \overrightarrow{n}_S + g e_{h,\phi}^{n-1-j}\|_{L^2(\Gamma)} \|e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1}\|_{L^2(\Gamma)} \\
&\le \varepsilon \textstyle\sum_{j=0}^{k} (\big\|\nabla \mathbf{e}_{h,u}^{n-j}\|_{L^2(\Omega_S)}^2 + \big\|\nabla e_{h,\phi}^{n-j}\|_{L^2(\Omega_D)}^2) + C_\varepsilon (\Phi_h^n + \Phi_h^{n-1}),
\end{aligned} \tag{2.71}
$$

$$
\begin{aligned}
|J_{10}| &= \big|\langle \textstyle\sum_{j=0}^{k-1} \gamma_j \overline{\xi}_{Dh}^{n-1-j} - \overline{\xi}_{Dh}^n, e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1}\rangle\big| \\
&\le \big\| \textstyle\sum_{j=0}^{k-1} \gamma_j \overline{\xi}_{Dh}^{n-1-j} - \overline{\xi}_{Dh}^n\big\|_{L^2(\Gamma)} \|e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1}\|_{L^2(\Gamma)} \\
&\le C_\varepsilon \tau^{2k} + \varepsilon \textstyle\sum_{j=0}^{k} \|\nabla e_{h,\phi}^{n-j}\|_{L^2(\Omega_D)}^2 + C_\varepsilon (\Phi_h^n + \Phi_h^{n-1}), \quad \text{[similar to } J_2]
\end{aligned} \tag{2.72}
$$

$$
\begin{aligned}
|J_{12}| &= \big|(E_\phi^n, e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1})_{\Omega_D}\big| \le \|E_\phi^n\|_{L^2(\Omega_D)} \|e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1}\|_{L^2(\Omega_D)} \\
&\le C\tau^{2k} + C(\Phi_h^n + \Phi_h^{n-1}), \quad \text{[similar to } J_4]
\end{aligned} \tag{2.73}
$$

$$
|J_{13}| = \big|(\phi - \overline{\phi}_h, e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1})_{\Omega_D}\big| \le C h^{2r+2-4\delta} + C(\Phi_h^n + \Phi_h^{n-1}).\text{[similar to } J_8] \tag{2.74}
$$

Now for (2.62), it remains to estimate $|J_3|$ and $|J_{11}|$. Since $\frac{1}{\tau}\sum_{j=0}^{k} d_j \overline{\mathbf{u}}_h^{n-j} = \overline{(\frac{1}{\tau}\sum_{j=0}^{k} d_j \overrightarrow{u}^{n-j})_h}$ is the first component of the joint Stokes-Darcy Ritz-projection of

$$\left(\tfrac{1}{\tau}\textstyle\sum_{j=0}^{k} d_j \overrightarrow{u}^{n-j}, \tfrac{1}{\tau}\sum_{j=0}^{k} d_j p^{n-j}, \tfrac{1}{\tau}\sum_{j=0}^{k} d_j \phi^{n-j}\right).$$

Therefore, under assumption (A3), Theorem 1 implies

$$\begin{aligned}
&\|\frac{1}{\tau}\sum_{j=0}^{k} d_j(\overline{\mathbf{u}}_h^{n-j} - \overrightarrow{u}^{n-j})\|_{L^2(\Omega_S)} = \|\overline{(\frac{1}{\tau}\sum_{j=0}^{k} d_j \overrightarrow{u}^{n-j})_h} - \frac{1}{\tau}\sum_{j=0}^{k} d_j \overrightarrow{u}^{n-j}\|_{L^2(\Omega_S)}\\
&\le Ch^{r+1-2\delta}\big(\|\frac{1}{\tau}\sum_{j=0}^{k} d_j \overrightarrow{u}^{n-j}\|_{H^{r+1}(\Omega_S)} + \|\frac{1}{\tau}\sum_{j=0}^{k} d_j p^{n-j}\|_{H^{r}(\Omega_S)}\big)\\
&+Ch^{r+1-2\delta}\|\frac{1}{\tau}\sum_{j=0}^{k} d_j \phi^{n-j}\|_{H^{r+1}(\Omega_D)}\\
&\le Ch^{r+1-2\delta}\big(\|\partial_t \overrightarrow{u}\|_{C([0,T];H^{r+1}(\Omega_S))} + \|\partial_t p\|_{C([0,T];H^{r}(\Omega_S))}\big) + Ch^{r+1-2\delta}\|\partial_t \phi\|_{C([0,T];H^{r+1}(\Omega_D))}\big)\\
&\le Ch^{r+1-2\delta},
\end{aligned}$$

and similarly, $\left\|\frac{1}{\tau}\sum_{j=0}^{k} d_j(\overline{\phi}_h^{n-j} - \phi^{n-j})\right\|_{L^2(\Omega_D)} \le Ch^{r+1-2\delta}$.

Using the Cauchy-Schwarz inequality, the above two inequalities, Young's inequality, and (2.61), we have

$$\begin{aligned}
|J_3| &= \left|\left(\tfrac{1}{\tau}\textstyle\sum_{j=0}^{k} d_j(\overline{\mathbf{u}}_h^{n-j} - \overrightarrow{u}^{n-j}), (\mathbf{e}_{h,u}^{n} - \eta_k \mathbf{e}_{h,u}^{n-1})\right)_{\Omega_S}\right|\\
&\le \left\|\tfrac{1}{\tau}\textstyle\sum_{j=0}^{k} d_j(\overline{\mathbf{u}}_h^{n-j} - \overrightarrow{u}^{n-j})\right\|_{L^2(\Omega_S)}\left\|\mathbf{e}_{h,u}^{n} - \eta_k \mathbf{e}_{h,u}^{n-1}\right\|_{L^2(\Omega_S)}\\
&\le Ch^{r+1-2\delta}\left\|\mathbf{e}_{h,u}^{n} - \eta_k \mathbf{e}_{h,u}^{n-1}\right\|_{L^2(\Omega_S)} \qquad (2.75)\\
&\le Ch^{2r+2-4\delta} + \left\|\mathbf{e}_{h,u}^{n} - \eta_k \mathbf{e}_{h,u}^{n-1}\right\|_{L^2(\Omega_S)}^2\\
&\le Ch^{2r+2-4\delta} + C(\Phi_h^n + \Phi_h^{n-1}),
\end{aligned}$$

$$
\begin{aligned}
|J_{11}| &= \big|\big(\textstyle\sum_{j=0}^{k} d_j(\overline{\phi}_h^{n-j} - \phi^{n-j}), e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1}\big)_{\Omega_D}\big| \\
&\leq \big\|\textstyle\sum_{j=0}^{k} d_j(\overline{\phi}_h^{n-j} - \phi^{n-j})\|_{L^2(\Omega_D)}\|e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1}\|_{L^2(\Omega_D)} \\
&\leq Ch^{r+1-2\delta}\|e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1}\|_{L^2(\Omega_D)} \\
&\leq Ch^{2r+2-4\delta} + C\|e_{h,\phi}^n - \eta_k e_{h,\phi}^{n-1}\|_{L^2(\Omega_D)}^2 \\
&\leq Ch^{2r+2-4\delta} + C(\Phi_h^n + \Phi_h^{n-1}).
\end{aligned}
\tag{2.76}
$$

Substituting the estimates (2.64)-(2.76) into (2.62), we obtain: for $k \leq n \leq m$,

$$
\begin{aligned}
&D_\tau \Phi_h^n + 2\nu(1-\eta_k)\|\mathbb{D}(\mathbf{e}_{h,u}^n)\|_{L^2(\Omega_S)}^2 + (1-\eta_k)\big(K\nabla e_{h,\phi}^n, \nabla e_{h,\phi}^n\big)_{\Omega_D} \\
&\leq C_\varepsilon(\tau^{2k} + h^{2r+2-4\delta}) + C_\varepsilon(\Phi_h^n + \Phi_h^{n-1}) + \varepsilon \textstyle\sum_{j=0}^{k}(\big\|\nabla \mathbf{e}_{h,u}^{n-j}\|_{L^2(\Omega_S)}^2 + \big\|\nabla e_{h,\phi}^{n-j}\|_{L^2(\Omega_D)}^2).
\end{aligned}
$$

Summing up the equations above for $n = k, \ldots, \ell$, and using assumption (A4), we have

$$
\begin{aligned}
&\Phi_h^\ell + \tau\sum_{n=k}^{\ell}(1-\eta_k)\Big(2\nu\|\mathbb{D}(\mathbf{e}_{h,u}^n)\|_{L^2(\Omega_S)}^2 + \big(K\nabla e_{h,\phi}^n, \nabla e_{h,\phi}^n\big)_{\Omega_D}\Big) \\
&\leq C_\varepsilon(\tau^{2k} + h^{2r+2-4\delta}) + \tau\sum_{n=k}^{\ell} C_\varepsilon(\Phi_h^n + \Phi_h^{n-1}) \\
&+\varepsilon\tau\sum_{n=k}^{\ell}\sum_{j=0}^{k}(\big\|\nabla \mathbf{e}_{h,u}^{n-j}\|_{L^2(\Omega_S)}^2 + \big\|\nabla e_{h,\phi}^{n-j}\|_{L^2(\Omega_D)}^2) \\
&\leq C_\varepsilon(\tau^{2k} + h^{2r+2-4\delta}) + \tau\sum_{n=k}^{\ell} C_\varepsilon(\Phi_h^n + \Phi_h^{n-1}) + \varepsilon C_\delta^\star(\tau^k + h^{r+1-2\delta})^2 \\
&\quad +\varepsilon\tau k\sum_{n=k}^{\ell}(\big\|\nabla \mathbf{e}_{h,u}^{n}\|_{L^2(\Omega_S)}^2 + \big\|\nabla e_{h,\phi}^{n}\|_{L^2(\Omega_D)}^2), \qquad \forall\, k \leq \ell \leq m,
\end{aligned}
$$

By using assumption (A2) and Korn's inequality (2.41), the last inequality reduces to

$$
\begin{aligned}
&\Phi_h^\ell + \tau\sum_{n=k}^{\ell}(1-\eta_k)\big(2\nu\theta\|\nabla \mathbf{e}_{h,u}^n\|_{L^2(\Omega_S)}^2 + \kappa_0\|\nabla e_{h,\phi}^n\|_{L^2(\Omega_D)}^2\big) \\
&\leq (C_\varepsilon + 2\varepsilon C_\delta^\star)(\tau^{2k} + h^{2r+2-4\delta}) + \tau\sum_{n=k}^{\ell} C_\varepsilon(\Phi_h^n + \Phi_h^{n-1}) \\
&\quad +\varepsilon\tau k\sum_{n=k}^{\ell}(\big\|\nabla \mathbf{e}_{h,u}^{n}\|_{L^2(\Omega_S)}^2 + \big\|\nabla e_{h,\phi}^{n}\|_{L^2(\Omega_D)}^2), \qquad \forall\, k \leq \ell \leq m,
\end{aligned}
$$

By choosing sufficiently small ε, the last term on the right-hand side above can be controlled by part of the left-hand side,

$$
\begin{aligned}
&\varepsilon\tau k\sum_{n=k}^{\ell}\big(\big\|\nabla\mathbf{e}_{h,u}^{n}\big\|_{L^2(\Omega_S)}^2+\big\|\nabla e_{h,\phi}^{n}\big\|_{L^2(\Omega_D)}^2\big)\\
&\le\frac{\tau}{2}\sum_{n=k}^{\ell}(1-\eta_k)\big(2\nu\theta\|\nabla\mathbf{e}_{h,u}^{n}\|_{L^2(\Omega_S)}^2+\kappa_0\|\nabla e_{h,\phi}^{n}\|_{L^2(\Omega_D)}^2\big).
\end{aligned}
$$

Then the above two inequalities lead to

$$
\begin{aligned}
&\Phi_h^{\ell}+\frac{\tau}{2}\sum_{n=k}^{\ell}(1-\eta_k)\big(2\nu\theta\|\nabla\mathbf{e}_{h,u}^{n}\|_{L^2(\Omega_S)}^2+\kappa_0\|\nabla e_{h,\phi}^{n}\|_{L^2(\Omega_D)}^2\big)\\
&\le C(\tau^{2k}+h^{2r+2-4\delta})+\tau\sum_{n=k}^{\ell}C(\Phi_h^{n}+\Phi_h^{n-1}),\qquad\forall\, k\le\ell\le m.
\end{aligned}
\tag{2.77}
$$

Then Gronwall's inequality implies the existence of a positive constant τ_1 (independent of m) such that when $\tau\le\tau_1$ we have

$$
\max_{k\le n\le m}\Phi_h^{n}+\frac{\tau}{2}\sum_{n=k}^{m}(1-\eta_k)\big(2\nu\theta\|\nabla\mathbf{e}_{h,u}^{n}\|_{L^2(\Omega_S)}^2+\kappa_0\|\nabla e_{h,\phi}^{n}\|_{L^2(\Omega_D)}^2\big)\le C(\tau^{2k}+h^{2r+2-4\delta}).
$$

In other words, we have

$$
\begin{aligned}
&\max_{k\le n\le m}(\|\mathbf{e}_{h,u}^{n}\|_{L^2(\Omega_S)}^2+\|e_{h,\phi}^{n}\|_{L^2(\Omega_D)}^2)+\tau\sum_{n=k}^{m}\big(\|\nabla\mathbf{e}_{h,u}^{n}\|_{L^2(\Omega_S)}^2+\|\nabla e_{h,\phi}^{n}\|_{L^2(\Omega_D)}^2\big)\\
&\le C(\tau^{2k}+h^{2r+2-4\delta}).
\end{aligned}
\tag{2.78}
$$

In the case of $\tau\le h$, the inverse inequality and (2.78) imply

$$
\begin{aligned}
\|\mathbf{e}_{h,u}^{m}\|_{L^4(\Omega_S)}&\le Ch^{-1/2}\|\mathbf{e}_{h,u}^{m}\|_{L^2(\Omega_S)}\le Ch^{-1/2}(\tau^{k}+h^{r+1-2\delta})\le Ch^{-1/2}(h^{k}+h^{r+1-2\delta})\\
&\le C(h^{k-1/2}+h^{r+1/2-2\delta}).
\end{aligned}
$$

In the case of $\tau \geq h$, the Sobolev embedding $H^1(\Omega_S) \hookrightarrow L^4(\Omega_S)$ and (2.78) imply

$$\begin{aligned}\|\mathbf{e}_{h,u}^m\|_{L^4(\Omega_S)} &\leq C\|\mathbf{e}_{h,u}^m\|_{H^1(\Omega_S)} \leq C(\|\mathbf{e}_{h,u}^m\|_{L^2(\Omega_S)} + \|\nabla\mathbf{e}_{h,u}^m\|_{L^2(\Omega_S)}) \leq C\tau^{-1/2}(\tau^k + h^{r+1-2\delta}) \\ &\leq C(\tau^{k-1/2} + \tau^{r+1/2-2\delta}).\end{aligned}$$

For both cases, there exist positive constants τ_2 and h_1 (independent of m) such that when $\tau \leq \tau_2$ and $h \leq h_1$, we have $\|\mathbf{e}_{h,u}^m\|_{L^4(\Omega_S)} \leq 1$, which implies that

$$\begin{aligned}&\|\overrightarrow{u}_h^m - \overrightarrow{u}^m\|_{L^4(\Omega_S)} \leq \|\mathbf{e}_{h,u}^m\|_{L^4(\Omega_S)} + \|\overline{\mathbf{u}}_h^m - \overrightarrow{u}^m\|_{L^4(\Omega_S)} \leq 1 + \|\overline{\mathbf{u}}_h^m - \overrightarrow{u}^m\|_{L^4(\Omega_S)} \\ &\leq 1 + C\|\overline{\mathbf{u}}_h^m - \overrightarrow{u}^m\|_{H^1(\Omega_S)} \quad \text{[using the Sobolev embedding } H^1(\Omega_S) \hookrightarrow L^4(\Omega_S)] \\ &\leq 1 + Ch^{r-\delta}. \qquad \text{[using (2.22)]}\end{aligned}$$

Hence there exists a positive constant h_2 (independent of m) such that when $h \leq h_2$, we have $\|\overrightarrow{u}_h^m - \overrightarrow{u}^m\|_{L^4(\Omega_S)} \leq 2$. This completes the mathematical induction on (2.63) in the case $\tau \leq \tau_0 := \min(\tau_1, \tau_2)$ and $h \leq h_0 := \min(h_1, h_2)$. In this case, (2.63) holds for all $0 \leq n \leq N$. Since all the generic constants in our proof are independent of m, it follows that (2.78) holds for $m = N$. This completes the proof of theorem (2).

2.6. NUMERICAL EXAMPLES

In this section, we will present examples to illustrate the features of the proposed method. The Taylor-Hood element pair will be used for the Navier-Stokes equations, and the quadratic finite element will be used for the second order formulations of the Darcy equation. All numerical results below are for $T = 1$.

2.6.1. Numerical Examples for Semi-implicit Multi-step NIDDM with Interface Conditions. Consider the model problem on $\Omega = [0,\pi]\times[-1,1]$, where $\Omega_D = [0,\pi]\times[-1,0]$ and $\Omega_S = [0,\pi]\times[0,1]$. $\alpha = 1$, $\upsilon = 1$, $g = 1$, $z = 0$. The boundary condition functions and the source terms are chosen such that the following functions are the exact solutions.

$$\phi_D = (e^y - e^{-y})sin(x)e^t,\ \overrightarrow{u}_S = [\frac{1}{\pi}sin(2\pi y)cos(x), (-2 + \frac{1}{\pi^2}sin^2(\pi y))sin(x)]^T e^t,\ p_S = 0.$$

First,we will list the results of the semi-implicit multi-step non-iterative DDM with $\mathbb{K} = KI, K = 1$, which is a single-step method in Table 2.1, by setting $\tau = 8h^3$. These results match the regular expectations of accuracy order arising from backward Euler method and quadratic elements, i.e, $O(h^3 + \tau)$ for L^2 norm and $O(h^2 + \tau)$ for H^1 norm and hence verify the theoretical results in Theorem 2.

Table 2.1. Errors of the single-step method for $\tau = 8h^3$.

h	$\left\|\overrightarrow{u}_h - \overrightarrow{u}\right\|_0$	$rate$	$\left\|\overrightarrow{u}_h - \overrightarrow{u}\right\|_1$	$rate$	$\|p_h - p\|_0$	$rate$	$\|\phi_h - \phi\|_0$	$rate$	$\|\phi_h - \phi\|_1$	$rate$
1/4	2.9838×10^{-2}	–	6.3157×10^{-1}	–	2.5485×10^{-1}	–	2.6561×10^{-1}	–	6.2370×10^{-1}	–
1/8	3.6977×10^{-3}	3.01	1.6725×10^{-1}	1.92	2.8500×10^{-2}	3.16	3.7390×10^{-2}	2.83	1.0422×10^{-1}	2.58
1/16	4.5806×10^{-4}	3.01	4.2874×10^{-2}	1.96	3.2192×10^{-3}	3.14	4.7400×10^{-3}	3.06	1.9654×10^{-2}	2.40
1/32	5.6751×10^{-5}	3.01	1.0813×10^{-2}	1.99	3.7633×10^{-4}	3.09	5.9341×10^{-4}	2.91	4.4095×10^{-3}	2.15
1/64	7.0311×10^{-6}	3.01	2.7270×10^{-3}	1.99	4.4746×10^{-5}	3.07	7.4290×10^{-5}	3.00	1.1663×10^{-3}	1.92

For the two-step method, we choose $\tau = h$, and Table 2.2 provides errors for different choices of h for the semi-implicit non-iterative domain decomposition method with $k = 2$. Since the accuracy order $O(\tau^2)$ is expected for the temporal discretization and h is chosen to be the same as τ, the second order convergence rates in Table 2.2 are expected for the utilized quadratic and linear finite elements.

Table 2.2. Errors of the two-step method for $\tau = h$.

h	$\left\|\overrightarrow{u}_h - \overrightarrow{u}\right\|_0$	$rate$	$\left\|\overrightarrow{u}_h - \overrightarrow{u}\right\|_1$	$rate$	$\|p_h - p\|_0$	$rate$	$\|\phi_h - \phi\|_0$	$rate$	$\|\phi_h - \phi\|_1$	$rate$
1/4	3.0606×10^{-2}	$rate$	6.3617×10^{-1}	$rate$	1.9401×10^{-1}	$rate$	1.1446×10^{-1}	$rate$	3.5789×10^{-1}	$rate$
1/8	4.6203×10^{-3}	2.72	1.6846×10^{-1}	1.92	3.1171×10^{-2}	2.33	3.2467×10^{-2}	1.82	9.6824×10^{-2}	1.89
1/16	8.5377×10^{-4}	2.43	4.3164×10^{-2}	1.96	6.3608×10^{-3}	2.29	8.5831×10^{-3}	1.92	2.5123×10^{-2}	1.95
1/32	1.9327×10^{-4}	2.14	1.0886×10^{-2}	1.99	1.5366×10^{-3}	2.04	2.2051×10^{-3}	1.96	6.3961×10^{-3}	1.97
1/64	4.7286×10^{-5}	2.03	2.7299×10^{-3}	2.00	3.8463×10^{-4}	2.00	5.5648×10^{-4}	1.99	1.6107×10^{-3}	1.99

For the three-step method, by setting $\tau = h$ we can get the Table 2.3 with $k = 3$. We can observe the third order convergence rates for $\overrightarrow{u}$ and ϕ in L^2 norms, which are consistent with the theoretical expectation in Theorem 2 for the accuracy order, i.e., $O(h^3 + \tau^3)$ for quadratic finite elements in L^2 norms.

Table 2.3. Errors of the three-step method for $\tau = h$.

h	$\|\overrightarrow{u}_h - \overrightarrow{u}\|_0$	$rate$	$\|\overrightarrow{u}_h - \overrightarrow{u}\|_1$	$rate$	$\|p_h - p\|_0$	$rate$	$\|\phi_h - \phi\|_0$	$rate$	$\|\phi_h - \phi\|_1$	$rate$
1/4	2.9397×10^{-2}	–	6.3190×10^{-1}	–	1.7011×10^{-1}	–	5.0992×10^{-2}	–	2.7626×10^{-1}	–
1/8	3.7079×10^{-3}	2.99	1.6730×10^{-1}	1.92	2.2059×10^{-2}	2.95	4.7639×10^{-3}	3.42	6.7329×10^{-2}	2.04
1/16	4.5978×10^{-4}	3.01	4.2876×10^{-2}	1.96	2.3572×10^{-3}	3.23	6.3881×10^{-4}	2.99	1.6863×10^{-2}	2.00
1/32	5.6955×10^{-5}	3.01	1.0813×10^{-2}	1.99	2.6322×10^{-4}	3.16	7.7080×10^{-5}	3.05	4.2241×10^{-3}	2.00
1/64	7.0773×10^{-6}	3.01	2.7112×10^{-3}	2.00	3.1520×10^{-5}	3.06	6.7330×10^{-6}	3.21	1.0574×10^{-3}	2.00

Second, we list the numerical results of the semi-implicit multi-step NIDDM with a more realistic hydraulic conductivity $\mathbb{K} = KI$, $K = 10^{-3}$. Since the results of single-step and two-step methods are similar to those of the three-step method, we will only illustrate the three-step method due to the page limitation. We choose $\tau = h$ and list the corresponding errors in Table 2.4. From this table, we can find that the proposed method is still optimally convergent with respect to the chosen finite elements. That is, for quadratic finite elements, we see the third order accuracy in L^2 norm and second order accuracy in H^1 norm while the expected accuracy order in temporal discretization is the third order.

Table 2.4. Errors of the three-step method for $K = 10^{-3}$ and $\tau = h$.

h	$\|\overrightarrow{u}_h - \overrightarrow{u}\|_0$	$rate$	$\|\overrightarrow{u}_h - \overrightarrow{u}\|_1$	$rate$	$\|p_h - p\|_0$	$rate$	$\|\phi_h - \phi\|_0$	$rate$	$\|\phi_h - \phi\|_1$	$rate$
1/4	1.0171×10^{-4}	–	1.1041×10^{-3}	–	1.6843×10^{-3}	–	2.3789×10^{-2}	–	2.9996×10^{-1}	–
1/8	1.2895×10^{-5}	2.98	2.4518×10^{-4}	2.17	2.1424×10^{-4}	2.97	3.1274×10^{-3}	2.93	6.9957×10^{-2}	2.10
1/16	1.6356×10^{-6}	2.98	6.1825×10^{-5}	1.99	2.6457×10^{-5}	3.01	4.0371×10^{-4}	2.95	1.7106×10^{-2}	2.03
1/32	2.1445×10^{-7}	2.93	1.5494×10^{-5}	2.00	3.3029×10^{-6}	3.00	5.0419×10^{-5}	3.00	4.2734×10^{-3}	2.00
1/64	2.6834×10^{-8}	3.00	3.8737×10^{-6}	2.00	4.1235×10^{-7}	3.00	6.3102×10^{-6}	3.00	1.0691×10^{-3}	1.99

2.6.2. Results of the Lagrange Multiplier Method for Defective Boundary Conditions under the Framework of Semi-implicit Multi-step Non-iterative DDM. In this example, we apply the proposed Lagrange multiplier method for defective boundary conditions to a local simulation of the subsurface flow merging from two conduits into one while communicating with the surrounding porous media flows. The computational domain is a unit square divided into the porous media domain Ω_D and the free flow Ω_S. Let Ω_S be the polygon $\overline{ABCDEFGHIJ}$ where $A = (0,1), B = (0,3/4), C = (1/2,1/4), D = (1/2,0), E = (3/4,0), F = (3/4,1/4), G = (1,1/4), H = (1,1/2), I = (3/4,1/2)$, and $J = (1/4,1)$. Let $\Omega_D = \Omega/\Omega_S$, $S_0 = \overline{AB} \cup \overline{JA}$, $S_1 = \overline{DE}$, and $S_2 = \overline{GH}$. Choose $T = 1$, $\alpha = 1$, $\nu = 1$, $g = 1$, $z = 0$, and $\mathbb{K} = KI$. The boundary condition data and source terms are chosen to be 0 except Q_i on S_i $(i = 0,1,2)$. We subdivide Ω into a rectangle whose height and width equal $h = 1/M$, where M denotes a positive integer, and then subdivide each rectangle into two triangles by drawing a diagonal. For this numerical experiment, we choose $M = 32$ and $\tau = h$. Three-step backward differentiation is selected for the temporal discretization. Next, we will provide the numerical results at $T = 1$ for the algorithm.

Figures 2.1 and 2.2 illustrate the numerical solutions at the end time $T = 1$ by setting $K = 1$, $K = 10^{-3}$ for three tests. In the first test, we set $Q_1 = Q_2 = -1$ and $Q_0 = 2$ so that the total inflow rate is equal to the total outflow rate. In the second test, we keep the same Q_1 and Q_2 but set $Q_0 = 1$ so that the total inflow rate is larger than the total outflow rate. This causes more water to be pushed out of the conduits into the porous media, which happens during a rain season. In the third test, we keep the same Q_1 and Q_2 but set $Q_0 = 3$ so that the total inflow rate is smaller than the total outflow rate. More outflow causes more water to flow into the conduits from the porous media, which is what happens during a dry season. These phenomena are expected due to the chosen unbalanced inflow and outflow rates for the conduit.

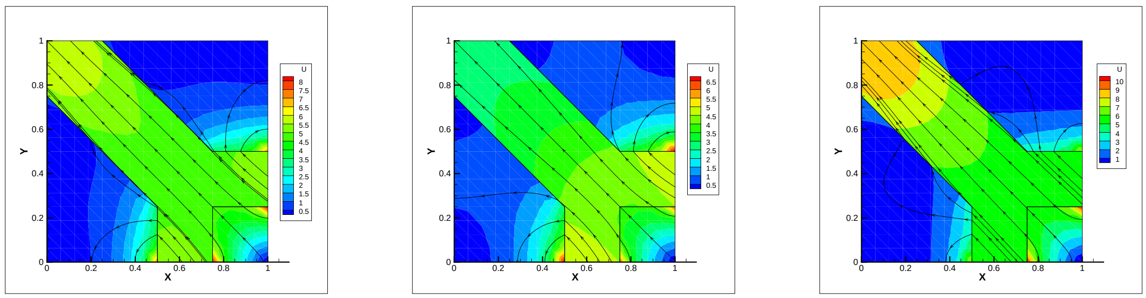

Figure 2.1. Streamlines when $K = 1$ for $Q_1 = -1$, $Q_2 = -1$, and different Q_0: $Q_0 = 2$ (left), $Q_0 = 1$ (middle), and $Q_0 = 3$ (right).

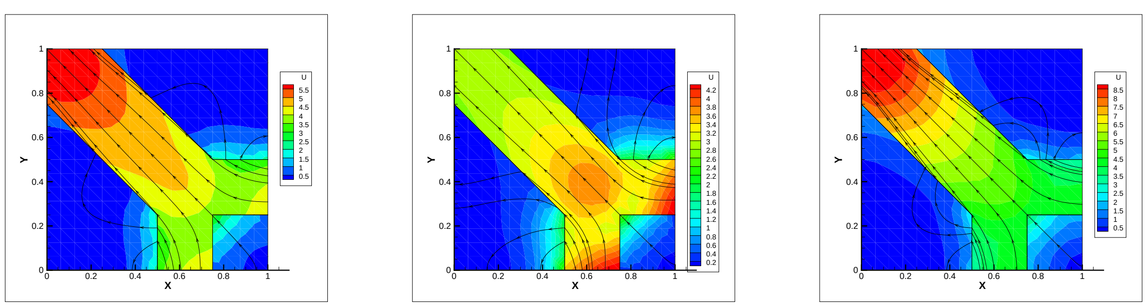

Figure 2.2. Streamlines when $K = 10^{-3}$ for $Q_1 = -1$, $Q_2 = -1$, and different Q_0: $Q_0 = 2$ (left), $Q_0 = 1$ (middle) and $Q_0 = 3$ (right).

3. PARALLEL NON-ITERATIVE MULTI-PHYSICS DDM TO SOLVE NS-DARCY MODEL WITH BJ INTERFACE CONDITION

In this section, a parallel non-iterative multi-physics DDM is proposed to solve a time-dependent NS-Darcy model with Beavers-Joseph interface condition and defective boundary condition. In order to deal with the Beavers-Joseph interface condition, we need some special treatments in both the analysis and the construction of the Robin boundary conditions for the domain decomposition. The backward Euler scheme is first utilized for the temporal discretization while finite elements are used for the spatial discretization. The convergences of this domain decomposition method are rigorously analyzed for the time-dependent NS-Darcy model with the BJ interface condition. Based on the above preparation, we further develop a Lagrange multiplier method under the framework of the DDM to overcome the difficulty of non-unique solutions arising from the defective boundary condition. In order to improve the accuracy for the temporal discretization, a three-step backward differentiation scheme is used to replace the backward Euler scheme. Compared with the first scheme, the second one allows us to use the relatively larger time step to reduce the computational cost while keeping the same accuracy. Numerical examples are provided to illustrate the features of the proposed method.

3.1. BASIC IDEA OF DDM TO SOLVE NS-DARCY SYSTEM WITH BJ INTERFACE CONDITION

Recently, the attention of scientists has been attracted to the NS-Darcy model, including the steady state problem Badea *et al.* (2010); Cai *et al.* (2009); Chidyagwai and Rivière (2009); Discacciati and Quarteroni (2009); Girault and Rivière (2009); Hadji *et al.* (2015); He *et al.* (2015) and the unsteady problem Çeşmelioğlu *et al.* (2013); Çeşmelioğlu and Rivière (2008, 2009). Compared with the extensively studied Stokes-Darcy model, the more difficult time-dependent NS-Darcy model is still in great need of continued efforts to develop and analyze stable, accurate, and efficient numerical methods, especially for the model with more realistic and difficult boundary/interface conditions. In fact, it is difficult or expensive in many applications to measure the fluid flow velocity for the boundary conditions but much easier and more cost-efficient to obtain flow rates on the boundary Heywood *et al.* (1996); Roscoe *et al.* (1997). Therefore, the corresponding defective boundary conditions were considered for the Navier-Stokes equation Formaggia *et al.* (2002). More recent development for defective boundary problems can be found in Ervin *et al.* (2014); Ervin and Lee (2007); Formaggia and Vergara (2012).

Furthermore, there are two choices for the interface condition in the tangential direction: the original Beavers-Joseph (BJ) interface condition Beavers and Joseph (1967b) and the simplified Beavers-Joseph-Saffman-Jones (BJSJ) interface condition Jäger and Mikelič (2000); Jones (1973); Saffman (1971a). It is true for some cases that the contribution of the Darcy flow in the tangential direction is heuristically much smaller than that of Stokes flow on the interface, and hence the BJSJ simplification can be used. There are related theoretical works in Chen *et al.* (2010) using the Brinkman-Stokes model as the starting point and periodicity in the horizontal (along the interface) direction. They demonstrated that the BJ interface condition is more accurate than the BJSJ interface condition or its further simplifications. The error is not necessarily small for all parameters, and it could be of order 1 for the lower values of the hydraulic conductivity/permeability/porosity.

In this section, based on the key ideas of Cao *et al.* (2014), which were a fundamental development for the simple Stokes-Darcy model with BJSJ interface condition, we first develop a parallel non-iterative multi-physics domain decomposition method to solve the sophisticated time-dependent NS-Darcy system with BJ interface condition. In order to avoid the traditional iteration for the domain decomposition at each time step, the interface information at the current time step is directly predicted based on the numerical solution of the previous time steps. The major difficulties in both the analysis and the construction of the Robin boundary conditions arise from nonlinear terms and BJ interface condition, including a series of technical treatments and the final special norm used in the discrete Gronwall's inequality for the analysis of full discretization. Therefore, the analysis for the proposed method in this article is much more difficult than that of Cao *et al.* (2014), and thus needs significant extra efforts, which will be illustrated in detail in the analysis section. Finite elements are used for the spatial discretization. Backward Euler and three-step backward differentiation schemes are used for the temporal discretization. Based on the solid foundation built for the domain decomposition method of the NS-Darcy system with BJ interface condition, we further propose the Lagrange multipliers to deal with the defective boundary conditions under the same framework of the domain decomposition method. One interesting finding of this section is that the Lagrange multipliers are time dependent functions instead of constants.

3.2. THE PARALLEL NON-ITERATIVE MULTI-PHYSICS DOMAIN DECOMPOSITION METHOD

In this section, we consider the time-dependent NS-Darcy with BJ interface condition in Section 1.1.1. We will first present the coupled weak formulation and introduce Robin boundary conditions of the Darcy and Navier-Stokes systems on the interface Γ for the domain decomposition. Then we will present the parallel non-iterative multi-physics domain decomposition method with a backward Euler scheme in temporal discretization, whose stability and convergence will be analyzed in Section 3.3.

3.2.1. Formulation for the NS-Darcy with BJ Interface Condition. First, by recalling the function spaces and notations in Section 2.3.1, we can get the weak formulation of the coupled NS-Darcy model with the BJ interface condition as follows:

find $(\overrightarrow{u}_S, p_S) \in H^1(0,T;X_S,X_S') \times L^2(0,T;Q_S)$ and $\phi_D \in H^1(0,T;X_D,X_D')$ such that

$$(\frac{\partial \overrightarrow{u}_S}{\partial t}, \overrightarrow{v})_{\Omega_S} + g(\frac{\partial \phi_D}{\partial t}, \psi)_{\Omega_D} + C_S(\overrightarrow{u}_S, \overrightarrow{u}_S, \overrightarrow{v}) + a_S(\overrightarrow{u}_S, \overrightarrow{v}) + b_S(\overrightarrow{v}, p_S) + g a_D(\phi_D, \psi)$$
$$+\langle g\phi_D, \overrightarrow{v} \cdot \overrightarrow{n}_S\rangle - g\langle \overrightarrow{u}_S \cdot \overrightarrow{n}_S, \psi\rangle + \frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\prod)}}\langle P_\tau(\overrightarrow{u}_S + \mathbb{K}\nabla\phi_D), P_\tau\overrightarrow{v}\rangle$$
$$= g(f_D, \psi)_{\Omega_D} + (\overrightarrow{f}_S, \overrightarrow{v})_{\Omega_S}, \ \forall \overrightarrow{v} \in X_S, \ \ \psi \in X_D, \tag{3.1}$$
$$b_S(\overrightarrow{u}_S, q) = 0, \ \forall \ q \in Q_S. \tag{3.2}$$

This weak formulation is similar to that of (3.1) and (3.2), but takes the nonlinear advection and Beavers-Joseph interface condition into account.

In order to solve the coupled NS-Darcy problem utilizing the domain decomposition idea, we naturally consider Robin boundary conditions for the Darcy and Navier-Stokes equations by following the ideas in (2.5) and (2.8). However, the Robin boundary conditions need to be modified according to the Beavers-Joseph interface condition.

First we consider the following two Robin-type conditions for the Navier-Stokes equations

$$-P_\tau(\mathbb{T}(\overrightarrow{u}_S, p_S) \cdot \overrightarrow{n}_S) - \frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\prod)}} P_\tau \overrightarrow{u}_S = \xi_{S\tau} \quad \text{on } \Gamma, \tag{3.3}$$
$$\overrightarrow{n}_S \cdot (\mathbb{T}(\overrightarrow{u}_S, p_S) \cdot \overrightarrow{n}_S) + \overrightarrow{u}_S \cdot \overrightarrow{n}_S = \xi_S \qquad \text{on } \Gamma, \tag{3.4}$$

for two given functions $\xi_{S\tau}$, $\xi_S \in L^2(0,T;L^2(\Gamma))$. Then, the corresponding weak formulation for the Navier-Stokes system is to find $\overrightarrow{u}_S \in H^1(0,T;X_S,X_S')$ and $p_S \in L^2(0,T;Q_S)$ such that $\forall \overrightarrow{v} \in X_S$, $\forall\, q \in Q_S$:

$$(\frac{\partial \overrightarrow{u}_S}{\partial t}, \overrightarrow{v})_{\Omega_S} + C_S(\overrightarrow{u}_S, \overrightarrow{u}_S, \overrightarrow{v}) + a_S(\overrightarrow{u}_S, \overrightarrow{v}) + b_S(\overrightarrow{v}, p_S) + \langle \overrightarrow{u}_S \cdot \overrightarrow{n}_S, \overrightarrow{v} \cdot \overrightarrow{n}_S\rangle$$
$$+\frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\prod)}}\langle P_\tau\overrightarrow{u}_S, P_\tau\overrightarrow{v}\rangle \ = \ (\overrightarrow{f}_S, \overrightarrow{v})_{\Omega_S} + \langle \xi_S, \overrightarrow{v} \cdot \overrightarrow{n}_S\rangle - \langle \xi_{S\tau}, P_\tau\overrightarrow{v}\rangle, \tag{3.5}$$
$$b_S(\overrightarrow{u}_S, q) \ = \ 0. \tag{3.6}$$

On the other hand, we have the same Robin condition for the Darcy system:

$$\mathbb{K}\nabla\phi_D \cdot \overrightarrow{n}_D + g\phi_D = \xi_D \qquad \text{on } \Gamma, \tag{3.7}$$

for a given function $\xi_D \in L^2(0,T;L^2(\Gamma))$. Hence, the corresponding weak formulation for the Darcy system is to find $\phi_D \in H^1(0,T;X_D,X'_D)$ such that

$$(\frac{\partial\phi_D}{\partial t},\psi)_{\Omega_D} + a_D(\phi_D,\psi) + \langle g\phi_D,\psi\rangle \;=\; (f_D,\psi)_{\Omega_D} + \langle\xi_D,\psi\rangle, \forall\psi \in X_D. \tag{3.8}$$

The Navier-Stokes and Darcy systems with Robin boundary conditions can be combined into one system. There exists a unique solution $(\phi_D,\overrightarrow{u}_S,p_S) \in H^1(0,T;X_D,X'_D) \times H^1(0,T;X_S,X'_S) \times L^2(0,T;Q_S)$ such that

$$\begin{aligned}
&(\frac{\partial\overrightarrow{u}_S}{\partial t},\overrightarrow{v})_{\Omega_S} + g(\frac{\partial\phi_D}{\partial t},\psi)_{\Omega_D} + C_S(\overrightarrow{u}_S,\overrightarrow{u}_S,\overrightarrow{v}) + a_S(\overrightarrow{u}_S,\overrightarrow{v}) + b_S(\overrightarrow{v},p_S) + ga_D(\phi_D,\psi)\\
&+\langle\overrightarrow{u}_S\cdot\overrightarrow{n}_S,\overrightarrow{v}\cdot\overrightarrow{n}_S\rangle + g\langle\phi_D,\psi\rangle + \frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\prod)}}\langle P_\tau\overrightarrow{u}_S,P_\tau\overrightarrow{v}\rangle = g(f_D,\psi)_{\Omega_D}\\
&+(\overrightarrow{f}_S,\overrightarrow{v})_{\Omega_S} + \langle\xi_S,\overrightarrow{v}\cdot\overrightarrow{n}_S\rangle + g\langle\xi_D,\psi\rangle - \langle\xi_{S\tau},P_\tau\overrightarrow{v}\rangle,\ \forall\,\psi\in Q_S,\overrightarrow{v}\in X_S,
\end{aligned} \tag{3.9}$$

$$b_S(\overrightarrow{u}_S,q) = 0,\ \forall\ q\in Q_S, \tag{3.10}$$

$$\phi_D(0) \;=\; \phi_0,\ \ \overrightarrow{u}_S(0) \;=\; \overrightarrow{u}_0. \tag{3.11}$$

Similar to Proposition 3.1 in Cao *et al.* (2014), it is easy to show that the solutions of the coupled NS-Darcy system are equivalent to solutions of the decoupled system if the following compatibility conditions are satisfied:

$$\xi_D = \overrightarrow{u}_S\cdot\overrightarrow{n}_S + g\phi_D,\ \xi_S = \overrightarrow{u}_S\cdot\overrightarrow{n}_S - g\phi_D,\ \xi_{S\tau} = \frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\prod)}}P_\tau(\mathbb{K}\nabla\phi_D). \tag{3.12}$$

These compatibility conditions provide the key tool to predict ξ_D,ξ_S, and $\xi_{S\tau}$ on the interface at each time step based on the results from the previous time steps.

3.2.2. The Non-iterative Domain Decomposition Method. Recalling the function spaces in Section 2.3.1, suppose we have finite element spaces $X_{Dh} \subset X_D$, $X_{Sh} \subset X_S$, and $Q_{Sh} \subset Q_S$. Here, we assume that $X_{Sh} \subset X_S$ and $Q_{Sh} \subset Q_S$ satisfy the following inf-sup condition: there exists a constant $\gamma > 0$ such that

$$\inf_{0\neq q\in Q_{sh}} \sup_{0\neq \overrightarrow{v}\in X_{Sh}} \frac{b_S(\overrightarrow{v},q)}{\left\|\overrightarrow{v}\right\|_1 \|q\|_0} \quad > \quad \gamma. \tag{3.13}$$

Define $P_h : X_D \to X_{Dh}$ and $\mathbb{P}_h : X_S \to X_{Sh}$ to be the regular orthogonal projections. We have the following regular approximation capability for them:

$$\|P_h\phi - \phi\|_0 \ \leq \ Ch^r \|\phi\|_r, \ \forall\ \phi \in H^r(\Omega_D), \tag{3.14}$$

$$\left\|\mathbb{P}_h\overrightarrow{u} - \overrightarrow{u}\right\|_0 \ \leq \ Ch^r \left\|\overrightarrow{u}\right\|_r, \ \forall\, \overrightarrow{u} \in [H^r(\Omega_S)]^d. \tag{3.15}$$

Then the semi-discretization of the decoupled systems (3.8) and (3.5)-(3.6) is to find $\phi_h \in H^1(0,T;X_{Dh})$, $\overrightarrow{u}_h \in H^1(0,T;X_{Sh})$ and $p_h \in L^2(0,T;Q_{Sh})$ such that

$$(\frac{\partial\phi_h}{\partial t},\psi_h)_{\Omega_D} + a_D(\phi_h,\psi_h) + \langle g\phi_h,\psi_h\rangle = (f_D,\psi_h)_{\Omega_D} + \langle\xi_{Dh},\psi_h\rangle, \tag{3.16}$$

$$\begin{aligned}&(\frac{\partial\overrightarrow{u}_h}{\partial t},\overrightarrow{v}_h)_{\Omega_S} + C_S(\overrightarrow{u}_h,\overrightarrow{u}_h,\overrightarrow{v}) + \langle\overrightarrow{u}_h\cdot\overrightarrow{n}_S,\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle + a_S(\overrightarrow{u}_h,\overrightarrow{v}_h) + b_S(\overrightarrow{v}_h,p_h)\\ &+\frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\prod)}}\langle P_\tau\overrightarrow{u}_h,P_\tau\overrightarrow{v}_h\rangle = (\overrightarrow{f}_S,\overrightarrow{v}_h)_{\Omega_S} + \langle\xi_{Sh},\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle - \langle\xi_{S\tau h},P_\tau\overrightarrow{v}_h\rangle,\end{aligned} \tag{3.17}$$

$$b_S(\overrightarrow{u}_h,q_h) = 0, \tag{3.18}$$

$$\phi_h(0) \ = \ P_h\phi_0, \ \ \overrightarrow{u}_h(0) \ = \ \mathbb{P}_h\overrightarrow{u}_0, \tag{3.19}$$

where $\forall\ \psi_h \in X_{Dh}$, $\forall\,\overrightarrow{v}_h \in X_{Sh}$, $\forall\ q_h \in Q_{Sh}$, and

$$\xi_{Dh} = \overrightarrow{u}_h\cdot\overrightarrow{n}_S + g\phi_h, \quad \xi_{Sh} = \overrightarrow{u}_h\cdot\overrightarrow{n}_S - g\phi_h, \quad \xi_{S\tau h} = \frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\prod)}}P_\tau(\mathbb{K}\nabla\phi_h).$$

Based on the compatibility conditions above and the backward Euler scheme in temporal discretization, now we present the following full discretization for the parallel non-iterative multi-physics domain decomposition method. At the n^{th} $(n = 0, 1, 2, \cdots, N-1)$ time iteration step,

1. compute

$$\xi_D^n = \overrightarrow{u}_h^n \cdot \overrightarrow{n}_S + g\phi_h^n, \quad \xi_S^n = \overrightarrow{u}_h^n \cdot \overrightarrow{n}_S - g\phi_h^n, \quad \xi_{S\tau}^n = \frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\prod)}} P_\tau(\mathbb{K}\nabla\phi_h^n) \tag{3.20}$$

by using the initial conditions $\phi_h^0 = P_h\phi_0$ and $\overrightarrow{u}_h^0 = \mathbb{P}_h\overrightarrow{u}_0$, and the numerical solutions ϕ_h^n and $\overrightarrow{u}_h^n$ at t_n.

2. independently solve

$$(\frac{\phi_h^{n+1} - \phi_h^n}{\Delta t}, \psi_h)_{\Omega_D} + a_D(\phi_h^{n+1}, \psi_h) + \langle g\phi_h^{n+1}, \psi_h\rangle = (f_D^{n+1}, \psi_h)_{\Omega_D} + \langle \xi_D^n, \psi_h\rangle,$$
$$\forall\ \psi_h \in X_{Dh} \tag{3.21}$$
$$(\frac{\overrightarrow{u}_h^{n+1} - \overrightarrow{u}_h^n}{\Delta t}, \overrightarrow{v}_h)_{\Omega_S} + C_S(\overrightarrow{u}_h^{n+1}, \overrightarrow{u}_h^{n+1}, \overrightarrow{v}_h) + a_S(\overrightarrow{u}_h^{n+1}, \overrightarrow{v}_h) + b_S(\overrightarrow{v}_h, p_h^{n+1})$$
$$+\langle \overrightarrow{u}_h^{n+1} \cdot \overrightarrow{n}_S, \overrightarrow{v}_h \cdot \overrightarrow{n}_S\rangle + \frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\prod)}}\langle P_\tau\overrightarrow{u}_h^{n+1}, P_\tau\overrightarrow{v}_h\rangle$$
$$= (\overrightarrow{f}_S^{n+1}, \overrightarrow{v}_h)_{\Omega_S} + \langle \xi_S^n, \overrightarrow{v}_h \cdot \overrightarrow{n}_S\rangle - \langle \xi_{S\tau}^n, P_\tau\overrightarrow{v}_h\rangle, \ \forall\ \overrightarrow{v}_h \in X_{Sh}, \tag{3.22}$$
$$b_S(\overrightarrow{u}_h^{n+1}, q_h) = 0, \ \forall\ q_h \in Q_{Sh}, \tag{3.23}$$

for ϕ_h^{n+1}, $\overrightarrow{u}_h^{n+1}$ and p_h^{n+1}.

3.3. CONVERGENCE ANALYSIS FOR THE DECOUPLED SYSTEM

In this section, we will analyze the convergence for the parallel, non-iterative, multi-physics domain decomposition method proposed above. The major difficulty is to bound the terms arising from the nonlinear advection and the BJ interface condition. In order to deal with the nonlinear terms, we recall the following inequlities Cesmelioglu *et al.* (2013); Gao *et al.* (2018): there exists constants C_1 and C_2 depending only on Ω_S, such that

$$|v| \leq C_1 \|\nabla v\|_0, \quad \|v\|_{L^4} \leq C_2 |v|, \tag{3.24}$$

where $v \in X_S$. Based on the work in Cesmelioglu *et al.* (2013); Gao *et al.* (2018), we have the following lemma.

Lemma 5 *Assume that both $\overrightarrow{u}_S$ and $\overrightarrow{u}_h$ satisfy the following smallness condition:*

$$\|\nabla \overrightarrow{u}\|_{L^2} < \frac{\nu}{8C_1^3 C_2^2} \; \forall \; t \in [0, T]. \tag{3.25}$$

Then, we have the estimate

$$|((\overrightarrow{u} \cdot \nabla)v, \omega)| \leq \frac{\nu}{8} \|\nabla v\|_0 \|\nabla \omega\|_0 \; \forall \; v, w \in X_S. \tag{3.26}$$

Proof 1 *By using $H\ddot{o}lder's$ inequality and* (3.24)*, we have*

$$\begin{aligned} |((\overrightarrow{u} \cdot \nabla)v, \omega)| &\leq \|\overrightarrow{u}\|_{L^4} |v| \|\omega\|_{L^4} \leq C_2^2 |\overrightarrow{u}| |v| |\omega| \leq C_1^3 C_2^2 \|\nabla \overrightarrow{u}\|_0 \|\nabla v\|_0 \|\nabla \omega\|_0 \\ &\leq \tfrac{\nu}{8} \|\nabla v\|_0 \|\nabla \omega\|_0. \end{aligned}$$

3.3.1. Convergence Analysis for the Semi-discrete Solution. We will follow the well-known framework of the energy method to analyze the convergence of the semi-discrete solution Douglas and Dupont (1970); Thomée (2006); Wheeler (1973). The major difficulties in the analysis are caused by the nonlinear advection and the Beavers-Joseph interface condition. Let C be a generic constant independent of h and $\triangle t$, whose value might be different from line to line.

Assume X_{Dh} and X_{Sh} consist of piece-wise polynomial of degree k and Q_{Sh} consists of piece-wise polynomial of degree $k-1$. For the analysis in the NS-Darcy system, we introduce the projection operator $\mathbb{P} = (\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3) : X_D \times X_S \times Q_S \to X_{Dh} \times X_{Sh} \times Q_{Sh}$ such that for any $\phi \in X_D$, $\overrightarrow{u} \in X_S$, $p \in Q_S$ and a rescaling constant η, the projection satisfies

$$\begin{aligned}
&\eta a_D(\mathbb{P}_1\phi - \phi, \psi_h) + a_S(\mathbb{P}_2\overrightarrow{u} - \overrightarrow{u}, \overrightarrow{v}_h) + \langle g(\mathbb{P}_1\phi - \phi), \overrightarrow{v}_h \cdot \overrightarrow{n}_S\rangle \\
&-\eta\langle(\mathbb{P}_2\overrightarrow{u} - \overrightarrow{u}) \cdot \overrightarrow{n}_S, \psi_h\rangle + \beta\langle P_\tau((\mathbb{P}_2\overrightarrow{u} - \overrightarrow{u}) + \mathbb{K}\nabla(\mathbb{P}_1\phi - \phi)), P_\tau\overrightarrow{v}_h\rangle \\
&+b_S(\overrightarrow{v}_h, \mathbb{P}_3 p - p) = 0, \ \forall\, \psi_h \in X_{Dh}, \overrightarrow{v}_h \in X_{Sh}. \qquad (3.27)\\
&b_S(\mathbb{P}_2\overrightarrow{u} - \overrightarrow{u}, q_h) \ = \ 0, \ \forall\, q_h \in Q_{Sh}. \qquad (3.28)
\end{aligned}$$

where $\beta = \frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\prod)}}$. Similar to Proposition 4.1 and Proposition 4.3 in Cao *et al.* (2010a), we have the following properties for the projection operator $\mathbb{P}$.

Lemma 6 *For any $\overrightarrow{u} \in X_S$, $p \in Q_S$, we have*

$$\begin{aligned}
&\|\mathbb{P}_1\phi - \phi\|_{L^q(0,T;H^1)} + \left\|\mathbb{P}_2\overrightarrow{u} - \overrightarrow{u}\right\|_{L^q(0,T;H^1)} + \|\mathbb{P}_3 p - p\|_{L^q(0,T;L^2)} \\
\leq\ & Ch^{r-1}(\|\phi\|_{L^q(0,T;H^r)} + \left\|\overrightarrow{u}\right\|_{L^q(0,T;H^r)} + \|p\|_{L^q(0,T;H^{r-1})}),\ q \geq 1, \qquad (3.29)\\
&\|\mathbb{P}_1\phi - \phi\|_{L^q(0,T;L^2)} + \left\|\mathbb{P}_2\overrightarrow{u} - \overrightarrow{u}\right\|_{L^q(0,T;L^2)} + h\,\|\mathbb{P}_3 p - p\|_{L^q(0,T;L^2)} \\
\leq\ & Ch^{r}(\|\phi\|_{L^q(0,T;H^r)} + \left\|\overrightarrow{u}\right\|_{L^q(0,T;H^r)} + \|p\|_{L^q(0,T;H^{r-1})}),\ q \geq 1, \qquad (3.30)\\
&\left\|\mathbb{P}_1\frac{\partial^m\phi}{\partial t^m} - \frac{\partial^m\phi}{\partial t^m}\right\|_{L^2(0,T;H^1)} + \left\|\mathbb{P}_2\frac{\partial^m\overrightarrow{u}}{\partial t^m} - \frac{\partial^m\overrightarrow{u}}{\partial t^m}\right\|_{L^2(0,T;H^1)} + \left\|\mathbb{P}_3\frac{\partial^m p}{\partial t^m} - \frac{\partial^m p}{\partial t^m}\right\|_{L^2(0,T;L^2)} \\
\leq\ & Ch^{r-1}\left(\|\phi\|_{H^m(0,T;H^r)} + \left\|\overrightarrow{u}\right\|_{H^m(0,T;H^r)} + \|p\|_{H^m(0,T;H^{r-1})}\right),\ m \geq 0, \qquad (3.31)\\
&\left\|\mathbb{P}_1\frac{\partial^m\phi}{\partial t^m} - \frac{\partial^m\phi}{\partial t^m}\right\|_{L^2(0,T;L^2)} + \left\|\mathbb{P}_2\frac{\partial^m\overrightarrow{u}}{\partial t^m} - \frac{\partial^m\overrightarrow{u}}{\partial t^m}\right\|_{L^2(0,T;L^2)} + h\left\|\mathbb{P}_3\frac{\partial^m p}{\partial t^m} - \frac{\partial^m p}{\partial t^m}\right\|_{L^2(0,T;L^2)} \\
\leq\ & Ch^{r}\left(\|\phi\|_{H^m(0,T;H^r)} + \left\|\overrightarrow{u}\right\|_{H^m(0,T;H^r)} + \|p\|_{H^m(0,T;H^{r-1})}\right),\ m \geq 0. \qquad (3.32)
\end{aligned}$$

Then, the error estimates for the semi-discrete approximations are given as follows:

Theorem 3 *Assume that* $\phi_D \in H^1(0,T;H^{r+1}(\Omega_D))$ *and* $\overrightarrow{u}_S \in H^1(0,T;[H^{r+1}(\Omega_S)]^d)$. *Then*

$$\|\phi_h - \phi_D\|_0 + \left\|\overrightarrow{u}_h - \overrightarrow{u}_S\right\|_0 \leq Ch^r \left(\|\phi_D\|_{H^1(0,T;H^{r+1}(\Omega_D))} + \left\|\overrightarrow{u}_S\right\|_{H^1(0,T;[H^{r+1}(\Omega_S)]^d)}\right),$$

where $0 < r \leq k+1$. *Here* k *is the degree of piece-wise polynomial of* X_{Dh} *and* X_{Sh}.

Proof 2 *Taking* $\psi = \psi_h \in X_{Dh}$ *in* (3.8), *plugging* ξ_D *into* (3.8), *and subtracting* (3.16) *from* (3.8), *we have*

$$\begin{aligned}
&(\frac{\partial\phi_D - \partial\phi_h}{\partial t},\psi_h)_{\Omega_D} + a_D(\phi_D - \phi_h,\psi_h) + \langle g(\phi_D - \phi_h),\psi_h\rangle \\
= \;& \langle(\overrightarrow{u}_S - \overrightarrow{u}_h)\cdot\overrightarrow{n}_S + g(\phi_D - \phi_h),\psi_h\rangle,\ \forall\ \psi_h \in X_{Dh}. \qquad (3.33)
\end{aligned}$$

Taking $\overrightarrow{v} = \overrightarrow{v}_h \in X_{Sh}$ *and* $q = q_h \in Q_{Sh}$ *in* (3.5) *and* (3.6), *plugging* ξ_S *and* $\xi_{S\tau}$ *into* (3.5), *and subtracting* (3.17)-(3.18) *from* (3.5)-(3.6), *we have*

$$\begin{aligned}
&(\frac{\partial(\overrightarrow{u}_S - \overrightarrow{u}_h)}{\partial t},\overrightarrow{v}_h)_{\Omega_S} + C_S(\overrightarrow{u}_S - \overrightarrow{u}_h,\overrightarrow{u}_S,\overrightarrow{v}_h) + C_S(\overrightarrow{u}_h,\overrightarrow{u}_S - \overrightarrow{u}_h,\overrightarrow{v}_h) \\
&+a_S(\overrightarrow{u}_S - \overrightarrow{u}_h,\overrightarrow{v}_h) + b_S(\overrightarrow{v}_h,p_S - p_h) - b_S(\overrightarrow{u}_S - \overrightarrow{u}_h,q_h) + \langle(\overrightarrow{u}_S - \overrightarrow{u}_h)\cdot\overrightarrow{n}_S,\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle \\
&+\beta\langle P_\tau\left(\overrightarrow{u}_S - \overrightarrow{u}_h\right),P_\tau\overrightarrow{v}_h\rangle \\
&= \langle(\overrightarrow{u}_S - \overrightarrow{u}_h)\cdot\overrightarrow{n}_S - g(\phi_D - \phi_h),\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle - \langle\beta P_\tau(\mathbb{K}\nabla(\phi_D - \phi_h)),P_\tau\overrightarrow{v}_h\rangle. \qquad (3.34)
\end{aligned}$$

Define

$$\theta = \mathbb{P}_1\phi_D - \phi_h,\ \ \rho = \phi_D - \mathbb{P}_1\phi_D. \qquad (3.35)$$

Then we can split the error $\phi_D - \phi_h = \theta + \rho$. *Define*

$$\overrightarrow{\theta}_1 = \mathbb{P}_2\overrightarrow{u}_S - \overrightarrow{u}_h,\ \ \overrightarrow{\rho}_1 = \overrightarrow{u}_S - \mathbb{P}_2\overrightarrow{u}_S,\ \ \theta_2 = \mathbb{P}_3 p_S - p_h,\ \ \rho_2 = p_S - \mathbb{P}_3 p_S. \qquad (3.36)$$

Then $\overrightarrow{u}_S - \overrightarrow{u}_h = \overrightarrow{\theta}_1 + \overrightarrow{\rho}_1$ *and* $p_S - p_h = \theta_2 + \rho_2$.

Plugging (3.35) *and* (3.36) *into* (3.33) *and* (3.34)*, we have*

$$
\begin{aligned}
&(\frac{\partial(\overrightarrow{\theta}_1+\overrightarrow{\rho}_1)}{\partial t},\overrightarrow{v}_h)_{\Omega_S}+\eta(\frac{\partial(\theta+\rho)}{\partial t},\psi_h)_{\Omega_D}+a_S(\overrightarrow{\theta}_1+\overrightarrow{\rho}_1,\overrightarrow{v}_h)+\eta a_D(\theta+\rho,\psi_h)\\
&+C_S(\overrightarrow{\theta}_1+\overrightarrow{\rho}_1,\overrightarrow{u}_S,\overrightarrow{v}_h)+C_S(\overrightarrow{u}_h,\overrightarrow{\theta}_1+\overrightarrow{\rho}_1,\overrightarrow{v}_h)+\eta\langle g(\theta+\rho),\psi_h\rangle\\
&+\langle(\overrightarrow{\theta}_1+\overrightarrow{\rho}_1)\cdot\overrightarrow{n}_S,\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle+\beta\langle P_\tau\left(\overrightarrow{\theta}_1+\overrightarrow{\rho}_1\right),P_\tau\overrightarrow{v}_h\rangle+b_S(\overrightarrow{v}_h,\theta_2+\rho_2)\\
&-b_S(\overrightarrow{\theta}_1+\overrightarrow{\rho}_1,q_h)\\
&=\eta\langle(\overrightarrow{\theta}_1+\overrightarrow{\rho}_1)\cdot\overrightarrow{n}_S+g(\theta+\rho),\psi_h\rangle+\langle(\overrightarrow{\theta}_1+\overrightarrow{\rho}_1)\cdot\overrightarrow{n}_S-g(\theta+\rho),\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle\\
&-\langle\beta P_\tau(\mathbb{K}\nabla(\theta+\rho)),P_\tau\overrightarrow{v}_h\rangle.
\end{aligned}
\tag{3.37}
$$

where η is the rescaling parameter. By using (3.27) *and* (3.28)*, we obtain*

$$
\begin{aligned}
&(\frac{\partial(\overrightarrow{\theta}_1+\overrightarrow{\rho}_1)}{\partial t},\overrightarrow{v}_h)_{\Omega_S}+\eta(\frac{\partial(\theta+\rho)}{\partial t},\psi_h)_{\Omega_D}+a_S(\overrightarrow{\theta}_1,\overrightarrow{v}_h)+\eta a_D(\theta,\psi_h)\\
&+C_S(\overrightarrow{\theta}_1+\overrightarrow{\rho}_1,\overrightarrow{u}_S,\overrightarrow{v}_h)+C_S(\overrightarrow{u}_h,\overrightarrow{\theta}_1+\overrightarrow{\rho}_1,\overrightarrow{v}_h)+\langle g\theta,\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle\\
&-\eta\langle\overrightarrow{\theta}_1\cdot\overrightarrow{n}_S,\psi_h\rangle+\beta\langle P_\tau\overrightarrow{\theta}_1,P_\tau\overrightarrow{v}_h\rangle+b_S(\overrightarrow{v}_h,\theta_2)-b_S(\overrightarrow{\theta}_1,q_h)\\
&=-\langle\beta P_\tau(\mathbb{K}\nabla\theta,P_\tau\overrightarrow{v}_h\rangle.
\end{aligned}
\tag{3.38}
$$

Choosing $\psi_h=\theta$, $\overrightarrow{v}_h=\overrightarrow{\theta}_1$ and $q_h=\theta_2$ in (3.38)*, we can get*

$$
\begin{aligned}
&(\frac{\partial\overrightarrow{\theta}_1}{\partial t},\overrightarrow{\theta}_1)_{\Omega_S}+\eta(\frac{\partial\theta}{\partial t},\theta)_{\Omega_D}+a_S(\overrightarrow{\theta}_1,\overrightarrow{\theta}_1)+\eta a_D(\theta,\theta)+C_S(\overrightarrow{\theta}_1+\overrightarrow{\rho}_1,\overrightarrow{u}_S,\overrightarrow{\theta}_1)\\
&+C_S(\overrightarrow{u}_h,\overrightarrow{\theta}_1+\overrightarrow{\rho}_1,\overrightarrow{\theta}_1)+\langle g\theta,\overrightarrow{\theta}_1\cdot\overrightarrow{n}_S\rangle-\eta\langle\overrightarrow{\theta}_1\cdot\overrightarrow{n}_S,\theta\rangle+\beta\langle P_\tau(\overrightarrow{\theta}_1+\mathbb{K}\nabla\theta),P_\tau\overrightarrow{\theta}_1\rangle\\
&=-(\frac{\partial\overrightarrow{\rho}_1}{\partial t},\overrightarrow{\theta}_1)_{\Omega_S}-\eta(\frac{\partial\rho}{\partial t},\theta)_{\Omega_D}.
\end{aligned}
$$

Since the estimates of ρ and $\overrightarrow{\rho}_1$ are given by Lemma 6, θ and $\overrightarrow{\theta}_1$ are the main objects of the analysis.

Hence, considering the given estimates of ρ *and* $\overrightarrow{\rho}_1$*, we have*

$$\begin{aligned}
&\frac{\eta}{2}\frac{d\,\|\theta\|_0^2}{dt} + \frac{1}{2}\frac{d\left\|\overrightarrow{\theta}_1\right\|_0^2}{dt} + \eta a_D(\theta,\theta) + a_S(\overrightarrow{\theta}_1,\overrightarrow{\theta}_1) - \eta\langle\overrightarrow{\theta}_1\cdot\overrightarrow{n}_S,\theta\rangle + \langle g\theta,\overrightarrow{\theta}_1\cdot\overrightarrow{n}_S\rangle \\
&+\beta\langle P_\tau(\overrightarrow{\theta}_1+\mathbb{K}\nabla\theta),P_\tau\overrightarrow{\theta}_1\rangle \\
= \;& -C_S(\overrightarrow{\theta}_1+\overrightarrow{\rho}_1,\overrightarrow{u}_S,\overrightarrow{\theta}_1) - C_S(\overrightarrow{u}_h,\overrightarrow{\theta}_1+\overrightarrow{\rho}_1,\overrightarrow{\theta}_1) - \eta(\frac{\partial\rho}{\partial t},\theta)_{\Omega_D} - (\frac{\partial\overrightarrow{\rho}_1}{\partial t},\overrightarrow{\theta}_1)_{\Omega_S} \qquad (3.39)\\
\leq \;& |C_S(\overrightarrow{\theta}_1+\overrightarrow{\rho}_1,\overrightarrow{u}_S,\overrightarrow{\theta}_1)| + |C_S(\overrightarrow{u}_h,\overrightarrow{\theta}_1+\overrightarrow{\rho}_1,\overrightarrow{\theta}_1)| + \eta\left\|\frac{\partial\rho}{\partial t}\right\|_0\|\theta\|_0 + \left\|\frac{\partial\overrightarrow{\rho}_1}{\partial t}\right\|_0\left\|\overrightarrow{\theta}_1\right\|_0.
\end{aligned}$$

By using (3.25) *and* (3.26)*, the Holder inequality, and the Young's inequality, we can obtain the following conclusions for the nonlinear terms:*

$$\begin{aligned}
|C_S(\overrightarrow{\rho}_1,\overrightarrow{u}_S,\overrightarrow{\theta}_1) + C_S(\overrightarrow{u}_h,\overrightarrow{\rho}_1,\overrightarrow{\theta}_1)| &\leq \frac{\nu}{8}\|\nabla\overrightarrow{\rho}_1\|_0\|\nabla\overrightarrow{\theta}_1\|_0 + \frac{\nu}{8}\|\nabla\overrightarrow{\rho}_1\|_0\|\nabla\overrightarrow{\theta}_1\|_0 \\
&= \frac{\nu}{4}\|\nabla\overrightarrow{\rho}_1\|_0\|\nabla\overrightarrow{\theta}_1\|_0 \\
&\leq \frac{\nu}{8}\|\nabla\overrightarrow{\theta}_1\|_0^2 + \frac{\nu}{8}\|\nabla\overrightarrow{\rho}_1\|_0^2, \qquad (3.40)\\
|C_S(\overrightarrow{\theta}_1,\overrightarrow{u}_S,\overrightarrow{\theta}_1) + C_S(\overrightarrow{u}_h,\overrightarrow{\theta}_1,\overrightarrow{\theta}_1)| &\leq \frac{\nu}{8}\|\nabla\overrightarrow{\theta}_1\|_0^2 + \frac{\nu}{8}\|\nabla\overrightarrow{\theta}_1\|_0^2 \\
&= \frac{\nu}{4}\|\nabla\overrightarrow{\theta}_1\|_0^2. \qquad (3.41)
\end{aligned}$$

By plugging (3.40)-(3.41) *into* (3.39) *and using the Young's inequality and Poincaré inequality, we have*

$$\begin{aligned}
&\eta\frac{d\,\|\theta\|_0^2}{dt} + \frac{d\left\|\overrightarrow{\theta}_1\right\|_0^2}{dt} + 2\eta a_D(\theta,\theta) + 2a_S(\overrightarrow{\theta}_1,\overrightarrow{\theta}_1) - 2\eta\langle\overrightarrow{\theta}_1\cdot\overrightarrow{n}_S,\theta\rangle + 2\langle g\theta,\overrightarrow{\theta}_1\cdot\overrightarrow{n}_S\rangle \\
&+2\beta\langle P_\tau(\overrightarrow{\theta}_1+\mathbb{K}\nabla\theta),P_\tau\overrightarrow{\theta}_1\rangle \\
\leq \;& \frac{3\nu}{4}\|\nabla\overrightarrow{\theta}_1\|_0^2 + \eta\|\theta\|_0^2 + \|\overrightarrow{\theta}_1\|_0^2 + \frac{\nu}{4}\|\nabla\overrightarrow{\rho}_1\|_0^2 + \eta\|\frac{\partial\rho}{\partial t}\|_0^2 + \|\frac{\partial\overrightarrow{\rho}_1}{\partial t}\|_0^2.
\end{aligned}$$

By using (4.2) in Cao et al. *(2010c), we can obtain the following conclusion to treat the BJ interface condition for small enough ν and large enough η:*

$$
\begin{aligned}
& 2\eta a_D(\theta,\theta) + 2a_S(\overrightarrow{\theta}_1, \overrightarrow{\theta}_1) - 2\eta\langle \overrightarrow{\theta}_1 \cdot \overrightarrow{n}_S, \theta\rangle + 2\langle g\theta, \overrightarrow{\theta}_1 \cdot \overrightarrow{n}_S\rangle \\
& +2\beta\langle P_\tau(\overrightarrow{\theta}_1 + \mathbb{K}\nabla\theta), P_\tau \overrightarrow{\theta}_1\rangle \\
\geq\ & \frac{3\nu}{4}\left\|\nabla \overrightarrow{\theta}_1\right\|_0^2 - C_3\|\theta\|_0^2 - C_3\|\overrightarrow{\theta}_1\|_0^2,
\end{aligned}
$$

where $C_3 > 0$ is a constant. Hence,

$$
\begin{aligned}
& \eta\frac{d\,\|\theta\|_0^2}{dt} + \frac{d\left\|\overrightarrow{\theta}_1\right\|_0^2}{dt} \\
\leq\ & C(\eta\,\|\theta\|_0^2 + \left\|\overrightarrow{\theta}_1\right\|_0^2 + \left\|\frac{\partial\rho}{\partial t}\right\|_0^2 + \left\|\frac{\partial\overrightarrow{\rho}_1}{\partial t}\right\|_0^2 + \|\nabla\overrightarrow{\rho}_1\|_0^2).
\end{aligned}
\tag{3.42}
$$

Integrating (3.42) *from* 0 *to t and applying Gronwall's inequaltiy, we get*

$$
\begin{aligned}
& \eta\,\|\theta(t)\|_0^2 + \left\|\overrightarrow{\theta}_1(t)\right\|_0^2 \\
\leq\ & C\Big[\eta\,\|\theta(0)\|_0^2 + \left\|\overrightarrow{\theta}_1(0)\right\|_0^2 + \int_0^t \left(\left\|\frac{\partial\rho}{\partial t}\right\|_0^2 + \left\|\frac{\partial\overrightarrow{\rho}_1}{\partial t}\right\|_0^2 + \|\nabla\overrightarrow{\rho}_1\|_0^2\right) ds\Big].
\end{aligned}
$$

Then by Lemma 6, we finish the proof.

3.3.2. Convergence Analysis of the Fully Discrete Approximate Solution. The following theorem states that the first parallel non-iterative domain decomposition method is unconditionally stable and has optimal rates of convergence. The major difficulties in the analysis arise from nonlinear terms and BJ interface condition which need a series of technical treatments and the final special norm used in the discrete Gronwall's inequality for the analysis of full discretization.

Theorem 4 *If* $\phi_D \in H^1(0,T;H^2(\Omega_D)) \cap L^\infty(0,T;H^2(\Omega_D)) \cap H^2(0,T;L^2(\Omega_D))$, $\overrightarrow{u}_S \in H^1(0,T;H^2(\Omega_S)) \cap L^\infty(0,T;H^2(\Omega_S)) \cap H^2(0,T;L^2(\Omega_S))$, $\xi_D \in H^1(0,T;L^2(\Gamma))$, *and* $\xi_S \in H^1(0,T;L^2(\Gamma))$, *then*

$$
\begin{aligned}
&\left\|\phi_h^n - \phi_D(t_n)\right\|_0 + \left\|\overrightarrow{u}_h^n - \overrightarrow{u}_S(t_n)\right\|_0 \\
&\leq Ce^{CT} \triangle t \Big[\int_0^{t_n} \left\|\frac{\partial^2 \phi_D}{\partial t^2}\right\|_0 dt + \int_0^{t_n} \left\|\frac{\partial \xi_D}{\partial t}\right\|_{0,\Gamma} dt \\
&+ \int_0^{t_n} \left\|\frac{\partial^2 \overrightarrow{u}_S}{\partial t^2}\right\|_0 dt + \int_0^{t_n} \left\|\frac{\partial \xi_S}{\partial t}\right\|_{0,\Gamma} dt + \int_0^{t_n} \left\|\frac{\partial \phi_D}{\partial t}\right\|_1 dt \Big] \\
&+Ce^{CT} h^2 \Big[\int_0^{t_n} \left\|\frac{\partial \phi_D}{\partial t}\right\|_2 dt + \int_0^{t_n} \left\|\frac{\partial \overrightarrow{u}_S}{\partial t}\right\|_2 dt \\
&+ \max_{0\leq s\leq t_n} \|\phi_D(s)\|_2 + \max_{0\leq s\leq t_n} \left(\left\|\overrightarrow{u}_S(s)\right\|_2 + \|p_S(s)\|_1 \right) \Big].
\end{aligned}
\tag{3.43}
$$

Proof 3 *We follow the standard energy method framework Douglas and Dupont (1970); Thomée (2006); Wheeler (1973) to analyze the error of fully discrete approximations. For the Darcy part, taking* $\psi = \psi_h \in X_{Dh}$ *in* (3.8) *and subtracting* (3.8) *from* (3.21), *we have*

$$
\begin{aligned}
&\left(\frac{\phi_h^{n+1} - \phi_h^n}{\Delta t} - \frac{\partial \phi_D(t_{n+1})}{\partial t}, \psi_h\right)_{\Omega_D} + a_D(\phi_h^{n+1} - \phi_D(t_{n+1}), \psi_h) \\
&\quad +\langle g(\phi_h^{n+1} - \phi_D(t_{n+1})), \psi_h\rangle \;=\; \langle \xi_D^n - \xi_D(t_{n+1}), \psi_h\rangle \quad \forall \psi_h \in X_{Dh}.
\end{aligned}
\tag{3.44}
$$

We can define θ^n *and* ρ^n *as follows:*

$$
\theta^n = \phi_h^n - \mathbb{P}_1\phi_D(t_n) \qquad and \qquad \rho^n = \mathbb{P}_1\phi_D(t_n) - \phi_D(t_n). \tag{3.45}
$$

so we can get $\phi_h^n - \phi_D(t_n) = \theta^n + \rho^n$. *Here,* ρ^n *is bounded because of Lemma 2 and we can get the similar estimates like (6.29) in Cao* et al. *(2014).*

$$
\|\rho^n\|_0 \leq Ch^2 \|\phi_D(t_n)\|_2 \,. \tag{3.46}
$$

We also need to define some notations for Navier-Stokes equation as follows:

$$\overrightarrow{\theta}_1^n = \overrightarrow{u}_h^n - \mathbb{P}_2 \overrightarrow{u}_S(t_n), \qquad \overrightarrow{\rho}_1^n = \mathbb{P}_2 \overrightarrow{u}_S(t_n) - \overrightarrow{u}_S(t_n), \tag{3.47}$$

$$\theta_2^n = p_h^n - \mathbb{P}_3 p_S(t_n), \qquad \rho_2^n = \mathbb{P}_3 p_S(t_n) - p_S(t_n), \tag{3.48}$$

then $\overrightarrow{u}_h^n - \overrightarrow{u}_S(t_n) = \overrightarrow{\theta}_1^n + \overrightarrow{\rho}_1^n$ *and* $p_h^n - p_S(t_n) = \theta_2^n + \rho_2^n$*. From (6.35)-(6.36) in Cao* et al. *(2014), we can get the estimates about* $\overrightarrow{\rho}_1^n$ *and* ρ_2^n*.*

$$\left\|\overrightarrow{\rho}_1^n\right\|_0 + h\left\|\overrightarrow{\rho}_1^n\right\|_1 \leq Ch^2(\left\|\overrightarrow{u}_S(t_n)\right\|_2 + \|p_S(t_n)\|_1) \tag{3.49}$$

$$\left\|\rho_2^n\right\|_0, \leq Ch^2(\left\|\overrightarrow{u}_S(t_n)\right\|_2 + \|p_S(t_n)\|_1). \tag{3.50}$$

Also, we have the following relations for the approximations of the coupling functions. Subtracting ξ_D *in* (3.12) *from* (3.20)*, we have*

$$\xi_D^n - \xi_D(t_n) = \left(\overrightarrow{\theta}_1^n + \overrightarrow{\rho}_1^n\right) \cdot \overrightarrow{n}_S + g\left(\theta^n + \rho^n\right), \tag{3.51}$$

Define

$$w_1^{n+1} = \mathbb{P}_1\left(\frac{\phi_D(t_{n+1}) - \phi_D(t_n)}{\triangle t}\right) - \frac{\partial \phi_D(t_{n+1})}{\partial t}, \;\; w_2^{n+1} = \xi_D(t_{n+1}) - \xi_D(t_n) \tag{3.52}$$

Then, (3.44) *becomes,* $\forall\, \psi_h \in X_{Dh}$

$$\begin{aligned}&(\frac{\theta^{n+1} - \theta^n}{\Delta t}, \psi_h)_{\Omega_D} + a_D(\theta^{n+1} + \rho^{n+1}, \psi_h) + \langle g(\theta^{n+1} + \rho^{n+1}), \psi_h\rangle \\ &= -(w_1^{n+1}, \psi_h) + \langle(\overrightarrow{\theta}_1^n + \overrightarrow{\rho}_1^n) \cdot \overrightarrow{n}_S + g(\theta^n + \rho^n), \psi_h\rangle - \langle w_2^{n+1}, \psi_h\rangle.\end{aligned} \tag{3.53}$$

For the Navier-Stokes part, choosing $\overrightarrow{v} = \overrightarrow{v}_h \in X_{Sh}$ *in* (3.5) *and* $q = q_h \in Q_{Sh}$ *in* (3.6)*, then subtracting* (3.5) *and* (3.6) *from* (3.22) *and* (3.23) *separately, we obtain*

$$
\begin{aligned}
&(\frac{\overrightarrow{u}_h^{n+1} - \overrightarrow{u}_h^{n}}{\Delta t} - \frac{\partial \overrightarrow{u}_S(t_{n+1})}{\partial t}, \overrightarrow{v}_h)_{\Omega_S} + C_S(\overrightarrow{u}_h^{n+1} - \overrightarrow{u}_S(t_{n+1}), \overrightarrow{u}_h^{n+1} - \overrightarrow{u}_S(t_{n+1}), \overrightarrow{v}_h) \\
&+a_S(\overrightarrow{u}_h^{n+1} - \overrightarrow{u}_S(t_{n+1}), \overrightarrow{v}_h) + b_S(\overrightarrow{v}_h, p_h^{n+1} - p_S(t_{n+1})) \\
&+\langle\left(\overrightarrow{u}_h^{n+1} - \overrightarrow{u}_S(t_{n+1})\right) \cdot \overrightarrow{n}_S, \overrightarrow{v}_h \cdot \overrightarrow{n}_S\rangle + \beta\langle P_\tau \left(\overrightarrow{u}_h^{n+1} - \overrightarrow{u}_S(t_{n+1})\right), P_\tau \overrightarrow{v}_h\rangle \\
&= \langle \xi_S^n - \xi_S(t_{n+1}), \overrightarrow{v}_h \cdot \overrightarrow{n}_S\rangle - \langle \xi_{S\tau}^n - \xi_{S\tau}(t_{n+1}), P_\tau \overrightarrow{v}_h\rangle, \qquad (3.54)
\end{aligned}
$$

$$
b_S(\overrightarrow{u}_h^{n+1} - \overrightarrow{u}_S(t_{n+1}), q_h) = 0. \qquad (3.55)
$$

Subtracting ξ_S *in* (3.12) *from* (3.20)*, we have*

$$
\xi_{Sh}^n - \xi_S(t_n) = \left(\overrightarrow{\theta}_1^n + \overrightarrow{\rho}_1^n\right) \cdot \overrightarrow{n}_S - g\left(\theta^n + \rho^n\right). \qquad (3.56)
$$

For the special treatment for BJ interface condition, which is one of the major difficulties, we have

$$
\begin{aligned}
& \xi_{S\tau h}^n - \xi_{S\tau}(t_{n+1}) \qquad (3.57) \\
= \;& \beta P_\tau\left(\mathbb{K}\nabla\phi_h^n\right) - \beta P_\tau\left(\mathbb{K}\nabla\phi_D(t_{n+1})\right) \\
= \;& \beta P_\tau\left(\mathbb{K}\nabla\left(\phi_h^n - \phi_D(t_{n+1})\right)\right) \\
= \;& \beta P_\tau\left(\mathbb{K}\nabla\left(\phi_h^n - \mathbb{P}_1\phi_D(t_n) + \mathbb{P}_1\phi_D(t_{n+1}) - \phi_D(t_{n+1}) - \mathbb{P}_1\phi_D(t_{n+1}) + \mathbb{P}_1\phi_D(t_n)\right)\right) \\
= \;& \beta P_\tau\left(\mathbb{K}\nabla(\theta^n + \rho^{n+1} - (\mathbb{P}_1\phi_D(t_{n+1}) - \mathbb{P}_1\phi_D(t_n))\right).
\end{aligned}
$$

We can define $\overrightarrow{w}_3^{n+1}$ *and* w_4^{n+1} *as follows:*

$$
\begin{aligned}
\overrightarrow{w}_3^{n+1} &= \mathbb{P}_2\left(\frac{\overrightarrow{u}_S(t_{n+1}) - \overrightarrow{u}_S(t_n)}{\Delta t}\right) - \frac{\partial \overrightarrow{u}_S(t_{n+1})}{\partial t}, \qquad (3.58) \\
w_4^{n+1} &= \xi_S(t_{n+1}) - \xi_S(t_n). \qquad (3.59)
\end{aligned}
$$

Then, (3.54) *and* (3.55) *become*

$$
\begin{aligned}
&(\frac{\overrightarrow{\theta}_1^{n+1}-\overrightarrow{\theta}_1^{n}}{\Delta t},\overrightarrow{v}_h)_{\Omega_S}+C_S(\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1},\overrightarrow{u}_h^{n+1},\overrightarrow{\theta}_1^{n+1})+C_S(\overrightarrow{u}_S(t_{n+1}),\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1},\overrightarrow{\theta}_1^{n+1})\\
&+a_S(\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1},\overrightarrow{v}_h)+b_S(\overrightarrow{v}_h,\theta_2^{n+1}+\rho_2^{n+1})+\langle\left(\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1}\right)\cdot\overrightarrow{n}_S,\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle\\
&+\beta\langle P_\tau\left(\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1}\right),P_\tau\overrightarrow{v}_h\rangle\\
&=-(\overrightarrow{w}_3^{n+1},\overrightarrow{v}_h)-\langle w_4^{n+1},\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle+\langle\left(\overrightarrow{\theta}_1^{n}+\overrightarrow{\rho}_1^{n}\right)\cdot\overrightarrow{n}_S-g\left(\theta^n+\rho^n\right),\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle\\
&-\langle\beta P_\tau\left(\mathbb{K}\nabla(\theta^n+\rho^{n+1}-(\mathbb{P}_1\phi_D(t_{n+1})-\mathbb{P}_1\phi_D(t_n))\right),P_\tau\overrightarrow{v}_h\rangle, \qquad (3.60)\\
&b_S(\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1},q_h)=0. \qquad (3.61)
\end{aligned}
$$

Combining (3.53) *and* (3.60)-(3.61)*, we get*

$$
\begin{aligned}
&\eta(\frac{\theta^{n+1}-\theta^n}{\Delta t},\psi_h)_{\Omega_D}+(\frac{\overrightarrow{\theta}_1^{n+1}-\overrightarrow{\theta}_1^{n}}{\Delta t},\overrightarrow{v}_h)_{\Omega_S}+C_S(\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1},\overrightarrow{u}_h^{n+1},\overrightarrow{\theta}_1^{n+1})\\
&+C_S(\overrightarrow{u}_S(t_{n+1}),\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1},\overrightarrow{\theta}_1^{n+1})+\eta a_D(\theta^{n+1}+\rho^{n+1},\psi_h)+a_S(\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1},\overrightarrow{v}_h)\\
&+b_S(\overrightarrow{v}_h,\theta_2^{n+1}+\rho_2^{n+1})+b_S(\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1},q_h)+\eta\langle g(\theta^{n+1}+\rho^{n+1}),\psi_h\rangle\\
&+\langle\left(\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1}\right)\cdot\overrightarrow{n}_S,\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle+\beta\langle P_\tau\left(\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1}\right),P_\tau\overrightarrow{v}_h\rangle \qquad (3.62)\\
&=-\eta(w_1^{n+1},\psi_h)-(\overrightarrow{w}_3^{n+1},\overrightarrow{v}_h)-\langle w_4^{n+1},\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle-\eta\langle w_2^{n+1},\psi_h\rangle\\
&+\eta\langle(\overrightarrow{\theta}_1^{n}+\overrightarrow{\rho}_1^{n})\cdot\overrightarrow{n}_S+g(\theta^n+\rho^n),\psi_h\rangle+\langle\left(\overrightarrow{\theta}_1^{n}+\overrightarrow{\rho}_1^{n}\right)\cdot\overrightarrow{n}_S-g\left(\theta^n+\rho^n\right),\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle\\
&-\langle\beta P_\tau\mathbb{K}\nabla(\theta^n+\rho^{n+1}),P_\tau\overrightarrow{v}_h\rangle+\langle\beta P_\tau\mathbb{K}\nabla(\mathbb{P}_1\phi_D(t_{n+1})-\mathbb{P}_1\phi_D(t_n)),P_\tau\overrightarrow{v}_h\rangle.
\end{aligned}
$$

Before using (3.27)-(3.28)*, we need to add some terms on both sides in* (3.62) *as follows:*

$$
\begin{aligned}
&\eta(\frac{\theta^{n+1}-\theta^n}{\Delta t},\psi_h)_{\Omega_D}+(\frac{\overrightarrow{\theta}_1^{n+1}-\overrightarrow{\theta}_1^{n}}{\Delta t},\overrightarrow{v}_h)_{\Omega_S}+C_S(\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1},\overrightarrow{u}_h^{n+1},\overrightarrow{\theta}_1^{n+1})\\
&+C_S(\overrightarrow{u}_S(t_{n+1}),\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1},\overrightarrow{\theta}_1^{n+1})+\eta a_D(\theta^{n+1}+\rho^{n+1},\psi_h)+a_S(\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1},\overrightarrow{v}_h)\\
&+b_S(\overrightarrow{v}_h,\theta_2^{n+1}+\rho_2^{n+1})+b_S(\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1},q_h)+\langle g\rho^{n+1},\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle-\eta\langle\overrightarrow{\rho}_1^{n+1}\cdot\overrightarrow{n}_S,\psi_h\rangle\\
&+\eta\langle\overrightarrow{\rho}_1^{n+1}\cdot\overrightarrow{n}_S+g(\theta^{n+1}+\rho^{n+1}),\psi_h\rangle+\langle\left(\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1}\right)\cdot\overrightarrow{n}_S-g\rho^{n+1},\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle\\
&+\beta\langle P_\tau\left(\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1}\right),P_\tau\overrightarrow{v}_h\rangle=-\eta(w_1^{n+1},\psi_h)-(\overrightarrow{w}_3^{n+1},\overrightarrow{v}_h)-\langle w_4^{n+1},\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle-\eta\langle w_2^{n+1},\psi_h\rangle\\
&+\eta\langle(\overrightarrow{\theta}_1^{n}+\overrightarrow{\rho}_1^{n})\cdot\overrightarrow{n}_S+g(\theta^n+\rho^n),\psi_h\rangle+\langle\left(\overrightarrow{\theta}_1^{n}+\overrightarrow{\rho}_1^{n}\right)\cdot\overrightarrow{n}_S-g\left(\theta^n+\rho^n\right),\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle\\
&-\langle\beta P_\tau\mathbb{K}\nabla(\theta^n+\rho^{n+1}),P_\tau\overrightarrow{v}_h\rangle+\langle\beta P_\tau\mathbb{K}\nabla(\mathbb{P}_1\phi_D(t_{n+1})-\mathbb{P}_1\phi_D(t_n)),P_\tau\overrightarrow{v}_h\rangle.
\end{aligned}
$$

By using (3.27)-(3.28) *and setting* $\psi_h = \theta^{n+1}$, $\overrightarrow{v}_h = \overrightarrow{\theta}_1^{n+1}$ *and* $q_h = \theta_2^{n+1}$, *we have*

$$\begin{aligned}
&\eta(\frac{\theta^{n+1}-\theta^n}{\Delta t},\theta^{n+1})_{\Omega_D} + (\frac{\overrightarrow{\theta}_1^{n+1}-\overrightarrow{\theta}_1^{n}}{\Delta t},\overrightarrow{\theta}_1^{n+1})_{\Omega_S} + C_S(\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1},\overrightarrow{u}_h^{n+1},\overrightarrow{\theta}_1^{n+1})\\
&+C_S(\overrightarrow{u}_S(t_{n+1}),\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1},\overrightarrow{\theta}_1^{n+1}) + \eta a_D(\theta^{n+1},\theta^{n+1}) + a_S(\overrightarrow{\theta}_1^{n+1},\overrightarrow{\theta}_1^{n+1})\\
&+\eta\langle\overrightarrow{\rho}_1^{n+1}\cdot\overrightarrow{n}_S + g(\theta^{n+1}+\rho^{n+1}),\theta^{n+1}\rangle + \beta\langle P_\tau\overrightarrow{\theta}_1^{n+1},P_\tau\overrightarrow{\theta}_1^{n+1}\rangle \qquad (3.63)\\
&+\langle(\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1})\cdot\overrightarrow{n}_S - g\rho^{n+1},\overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S\rangle\\
&= -\eta(w_1^{n+1},\theta^{n+1}) - (\overrightarrow{w}_3^{n+1},\overrightarrow{\theta}_1^{n+1}) - \langle w_4^{n+1},\overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S\rangle - \eta\langle w_2^{n+1},\theta^{n+1}\rangle\\
&+\eta\langle(\overrightarrow{\theta}_1^{n}+\overrightarrow{\rho}_1^{n})\cdot\overrightarrow{n}_S + g(\theta^n+\rho^n),\theta^{n+1}\rangle + \langle\left(\overrightarrow{\theta}_1^{n}+\overrightarrow{\rho}_1^{n}\right)\cdot\overrightarrow{n}_S - g\left(\theta^n+\rho^n\right),\overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S\rangle\\
&-\langle\beta P_\tau\mathbb{K}\nabla\theta^n,P_\tau\overrightarrow{\theta}_1^{n+1}\rangle + \langle\beta P_\tau\mathbb{K}\nabla(\mathbb{P}_1\phi_D(t_{n+1})-\mathbb{P}_1\phi_D(t_n)),P_\tau\overrightarrow{\theta}_1^{n+1}\rangle.
\end{aligned}$$

In order to make use of (4.2) in Cao et al. *(2010c) to deal with the difficulties from the BJ interface condition, we need to add some terms on both sides of the above inequality to re-write it as follows:*

$$\begin{aligned}
&\eta(\frac{\theta^{n+1}-\theta^n}{\Delta t},\theta^{n+1})_{\Omega_D} + (\frac{\overrightarrow{\theta}_1^{n+1}-\overrightarrow{\theta}_1^{n}}{\Delta t},\overrightarrow{\theta}_1^{n+1})_{\Omega_S} + \eta a_D(\theta^{n+1},\theta^{n+1}) + a_S(\overrightarrow{\theta}_1^{n+1},\overrightarrow{\theta}_1^{n+1})\\
&+\langle g\theta^{n+1},\overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S\rangle - \eta\langle\overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S,\theta^{n+1}\rangle + \beta\langle P_\tau(\overrightarrow{\theta}_1^{n+1}+\mathbb{K}\nabla\theta^{n+1}),P_\tau\overrightarrow{\theta}_1^{n+1}\rangle\\
\leq\;& -\eta\langle\overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S + g\theta^{n+1},\theta^{n+1}\rangle - \eta\langle\overrightarrow{\rho}_1^{n+1}\cdot\overrightarrow{n}_S + g\rho^{n+1},\theta^{n+1}\rangle\\
&-\langle\overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S - g\theta^{n+1},\overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S\rangle - \langle\overrightarrow{\rho}_1^{n+1}\cdot\overrightarrow{n}_S - g\rho^{n+1},\overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S\rangle\\
&+|C_S(\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1},\overrightarrow{u}_h^{n+1},\overrightarrow{\theta}_1^{n+1})| + |C_S(\overrightarrow{u}_S(t_{n+1}),\overrightarrow{\theta}_1^{n+1}+\overrightarrow{\rho}_1^{n+1},\overrightarrow{\theta}_1^{n+1})|\\
&-\eta(w_1^{n+1},\theta^{n+1}) - \eta\langle w_2^{n+1},\theta^{n+1}\rangle - (\overrightarrow{w}_3^{n+1},\overrightarrow{\theta}_1^{n+1}) - \langle w_4^{n+1},\overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S\rangle\\
&+\eta\langle\overrightarrow{\theta}_1^{n}\cdot\overrightarrow{n}_S + g\theta^n,\theta^{n+1}\rangle + \eta\langle\overrightarrow{\rho}_1^{n}\cdot\overrightarrow{n}_S + g\rho^n,\theta^{n+1}\rangle\\
&+\langle\overrightarrow{\theta}_1^{n}\cdot\overrightarrow{n}_S - g\theta^n,\overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S\rangle + \langle\overrightarrow{\rho}_1^{n}\cdot\overrightarrow{n}_S - g\rho^n,\overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S\rangle\\
&+\beta\langle P_\tau\mathbb{K}\nabla\theta^{n+1},P_\tau\overrightarrow{\theta}_1^{n+1}\rangle - \langle\beta P_\tau\mathbb{K}\nabla\theta^n,P_\tau\overrightarrow{\theta}_1^{n+1}\rangle\\
&+\langle\beta P_\tau\mathbb{K}\nabla(\mathbb{P}_1\phi_D(t_{n+1})-\mathbb{P}_1\phi_D(t_n)),P_\tau\overrightarrow{\theta}_1^{n+1}\rangle.
\end{aligned}$$

By using the Schwarz inequalities and Young inequalities, we can get

$$\eta\langle \overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S + g\theta^{n+1}, \theta^{n+1}\rangle \le \eta\left[\frac{1}{\epsilon_3}\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + 2\epsilon_3\|\theta^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_3}\|\theta^{n+1}\|_{0,\Gamma}^2\right],$$
$$\eta\langle \overrightarrow{\rho}_1^{n+1}\cdot\overrightarrow{n}_S + g\rho^{n+1}, \theta^{n+1}\rangle \le \eta\left[\frac{1}{\epsilon_2}\|\overrightarrow{\rho}_1^{n+1}\|_{0,\Gamma}^2 + 2\epsilon_2\|\theta^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_2}\|\rho^{n+1}\|_{0,\Gamma}^2\right],$$
$$\langle \overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S - g\theta^{n+1}, \overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S\rangle \le \frac{1}{\epsilon_3}\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + 2\epsilon_3\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_3}\|\theta^{n+1}\|_{0,\Gamma}^2,$$
$$\langle \overrightarrow{\rho}_1^{n+1}\cdot\overrightarrow{n}_S - g\rho^{n+1}, \overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S\rangle \le \frac{1}{\epsilon_2}\|\overrightarrow{\rho}_1^{n+1}\|_{0,\Gamma}^2 + 2\epsilon_2\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_2}\|\rho^{n+1}\|_{0,\Gamma}^2,$$
$$\eta\langle \overrightarrow{\theta}_1^{n}\cdot\overrightarrow{n}_S + g\theta^{n}, \theta^{n+1}\rangle \le \eta\left[\frac{1}{\epsilon_3}\|\overrightarrow{\theta}_1^{n}\|_{0,\Gamma}^2 + 2\epsilon_3\|\theta^{n}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_3}\|\theta^{n}\|_{0,\Gamma}^2\right],$$
$$\eta\langle \overrightarrow{\rho}_1^{n}\cdot\overrightarrow{n}_S + g\rho^{n}, \theta^{n+1}\rangle \le \eta\left[\frac{1}{\epsilon_2}\|\overrightarrow{\rho}_1^{n}\|_{0,\Gamma}^2 + 2\epsilon_2\|\theta^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_2}\|\rho^{n}\|_{0,\Gamma}^2\right],$$
$$\langle \overrightarrow{\theta}_1^{n}\cdot\overrightarrow{n}_S - g\theta^{n}, \overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S\rangle \le \frac{1}{\epsilon_3}\|\overrightarrow{\theta}_1^{n}\|_{0,\Gamma}^2 + 2\epsilon_3\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_3}\|\theta^{n}\|_{0,\Gamma}^2,$$
$$\langle \overrightarrow{\rho}_1^{n}\cdot\overrightarrow{n}_S - g\rho^{n}, \overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S\rangle \le \frac{1}{\epsilon_2}\|\overrightarrow{\rho}_1^{n}\|_{0,\Gamma}^2 + 2\epsilon_2\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_2}\|\rho^{n+1}\|_{0,\Gamma}^2.$$

By using the above inequalities, we have

$$\begin{aligned}
&\eta\left(\frac{\theta^{n+1}-\theta^n}{\Delta t}, \theta^{n+1}\right)_{\Omega_D} + \left(\frac{\overrightarrow{\theta}_1^{n+1}-\overrightarrow{\theta}_1^{n}}{\Delta t}, \overrightarrow{\theta}_1^{n+1}\right)_{\Omega_S} + \eta a_D(\theta^{n+1},\theta^{n+1}) + a_S(\overrightarrow{\theta}_1^{n+1},\overrightarrow{\theta}_1^{n+1}) \\
&+\langle g\theta^{n+1}, \overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S\rangle - \eta\langle \overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S, \theta^{n+1}\rangle + \beta\langle P_\tau(\overrightarrow{\theta}_1^{n+1} + \mathbb{K}\nabla\theta^{n+1}), P_\tau\overrightarrow{\theta}_1^{n+1}\rangle \qquad (3.64) \\
\le\ & |C_S(\overrightarrow{\theta}_1^{n+1} + \overrightarrow{\rho}_1^{n+1}, \overrightarrow{u}_h^{n+1}, \overrightarrow{\theta}_1^{n+1})| + |C_S(\overrightarrow{u}_S(t_{n+1}), \overrightarrow{\theta}_1^{n+1} + \overrightarrow{\rho}_1^{n+1}, \overrightarrow{\theta}_1^{n+1})| \\
&+\eta\left[\frac{1}{\epsilon_3}\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + 2\epsilon_3\|\theta^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_3}\|\theta^{n+1}\|_{0,\Gamma}^2\right] \\
&+\eta\left[\frac{1}{\epsilon_2}\|\overrightarrow{\rho}_1^{n+1}\|_{0,\Gamma}^2 + 2\epsilon_2\|\theta^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_2}\|\rho^{n+1}\|_{0,\Gamma}^2\right] \\
&+\left[\frac{1}{\epsilon_3}\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + 2\epsilon_3\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_3}\|\theta^{n+1}\|_{0,\Gamma}^2\right] \\
&+\left[\frac{1}{\epsilon_2}\|\overrightarrow{\rho}_1^{n+1}\|_{0,\Gamma}^2 + 2\epsilon_2\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_2}\|\rho^{n+1}\|_{0,\Gamma}^2\right] + \eta\left[\frac{1}{\epsilon_1}\|w_1^{n+1}\|_0^2 + \epsilon_1\|\theta^{n+1}\|_0^2\right] \\
&+\frac{1}{\epsilon_1}\|\overrightarrow{w}_3^{n+1}\|_0^2 + \epsilon_1\|\overrightarrow{\theta}_1^{n+1}\|_0^2 + \eta\left[\frac{1}{\epsilon_2}\|w_2^{n+1}\|_{0,\Gamma}^2 + \epsilon_2\|\theta^{n+1}\|_{0,\Gamma}^2\right] \\
&+\frac{1}{\epsilon_2}\|w_4^{n+1}\|_{0,\Gamma}^2 + \epsilon_2\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + \eta\left[\frac{1}{\epsilon_3}\|\overrightarrow{\theta}_1^{n}\|_{0,\Gamma}^2 + 2\epsilon_3\|\theta^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_3}\|\theta^{n}\|_{0,\Gamma}^2\right] \\
&+\eta\left[\frac{1}{\epsilon_2}\|\overrightarrow{\rho}_1^{n}\|_{0,\Gamma}^2 + 2\epsilon_2\|\theta^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_2}\|\rho^{n}\|_{0,\Gamma}^2\right] + \left[\frac{1}{\epsilon_3}\|\overrightarrow{\theta}_1^{n}\|_{0,\Gamma}^2 + 2\epsilon_3\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_3}\|\theta^{n}\|_{0,\Gamma}^2\right] \\
&+\left[\frac{1}{\epsilon_2}\|\overrightarrow{\rho}_1^{n}\|_{0,\Gamma}^2 + 2\epsilon_2\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_2}\|\rho^{n}\|_{0,\Gamma}^2\right] + \beta\langle P_\tau\mathbb{K}\nabla\theta^{n+1}, P_\tau\overrightarrow{\theta}_1^{n+1}\rangle \\
&-\langle\beta P_\tau\mathbb{K}\nabla\theta^{n}, P_\tau\overrightarrow{\theta}_1^{n+1}\rangle + \langle\beta P_\tau\mathbb{K}\nabla(\mathbb{P}_1\phi_D(t_{n+1}) - \mathbb{P}_1\phi_D(t_n)), P_\tau\overrightarrow{\theta}_1^{n+1}\rangle.
\end{aligned}$$

Now we need to handle the difficulties arising from the nonlinear terms and interface terms. For the nonlinear terms in the above inequality, we follow the idea of (3.40)-(3.41) *in semi-discretization analysis to obtain the inequalities as follows:*

$$\begin{aligned}
& |C_S(\overrightarrow{\rho}_1^{n+1}, \overrightarrow{u}_h^{n+1}, \overrightarrow{\theta}_1^{n+1}) + C_S(\overrightarrow{u}_S(t_{n+1}), \overrightarrow{\rho}_1^{n+1}, \overrightarrow{\theta}_1^{n+1})| \\
\leq & \frac{\nu}{8}\|\nabla\overrightarrow{\rho}_1^{n+1}\|_0\|\nabla\overrightarrow{\theta}_1^{n+1}\|_0 + \frac{\nu}{8}\|\nabla\overrightarrow{\rho}_1^{n+1}\|_0\|\nabla\overrightarrow{\theta}_1^{n+1}\|_0 \\
= & \frac{\nu}{4}\|\nabla\overrightarrow{\rho}_1^{n+1}\|_0\|\nabla\overrightarrow{\theta}_1^{n+1}\|_0 \leq \frac{\nu}{8}\|\nabla\overrightarrow{\theta}_1^{n+1}\|_0^2 + \frac{\nu}{8}\|\nabla\overrightarrow{\rho}_1^{n+1}\|_0^2, \qquad (3.65) \\
& |C_S(\overrightarrow{\theta}_1^{n+1}, \overrightarrow{u}_h^{n+1}, \overrightarrow{\theta}_1^{n+1}) + C_S(\overrightarrow{u}_S(t_{n+1}), \overrightarrow{\theta}_1^{n+1}, \overrightarrow{\theta}_1^{n+1})| \\
\leq & \frac{\nu}{8}\|\nabla\overrightarrow{\theta}_1^{n+1}\|_0^2 + \frac{\nu}{8}\|\nabla\overrightarrow{\theta}_1^{n+1}\|_0^2 = \frac{\nu}{4}\|\nabla\overrightarrow{\theta}_1^{n+1}\|_0^2. \qquad (3.66)
\end{aligned}$$

For interface terms, by trace theory and Young's inequality, we can get

$$\begin{aligned}
\left\|\theta^{n+1}\right\|_{0,\Gamma}^2 &\leq C\left\|\theta^{n+1}\right\|_0\left\|\nabla\theta^{n+1}\right\|_0 \leq \frac{C}{2}\left[\frac{1}{\varepsilon_4}\left\|\theta^{n+1}\right\|_0^2 + \varepsilon_4\left\|\nabla\theta^{n+1}\right\|_0^2\right], \\
\left\|\overrightarrow{\theta}_1^{n+1}\right\|_{0,\Gamma}^2 &\leq C\left\|\overrightarrow{\theta}_1^{n+1}\right\|_0\left\|\nabla\overrightarrow{\theta}_1^{n+1}\right\|_0 \leq \frac{C}{2}\left[\frac{1}{\varepsilon_4}\left\|\overrightarrow{\theta}_1^{n+1}\right\|_0^2 + \varepsilon_4\left\|\nabla\overrightarrow{\theta}_1^{n+1}\right\|_0^2\right], \\
\|\theta^n\|_{0,\Gamma}^2 &\leq C\|\theta^n\|_0\|\nabla\theta^n\|_0 \leq \frac{C}{2}\left[\frac{1}{\varepsilon_4}\|\theta^n\|_0^2 + \varepsilon_4\|\nabla\theta^n\|_0^2\right], \\
\left\|\overrightarrow{\theta}_1^{n}\right\|_{0,\Gamma}^2 &\leq C\left\|\overrightarrow{\theta}_1^{n}\right\|_0\left\|\nabla\overrightarrow{\theta}_1^{n}\right\|_0 \leq \frac{C}{2}\left[\frac{1}{\varepsilon_4}\left\|\overrightarrow{\theta}_1^{n}\right\|_0^2 + \varepsilon_4\left\|\nabla\overrightarrow{\theta}_1^{n}\right\|_0^2\right]. \qquad (3.67)
\end{aligned}$$

Similar to the proof of (4.2) in Cao et al. *(2010c), we have*

$$\begin{aligned}
& |\beta\langle P_\tau\mathbb{K}\nabla\theta^{n+1}, P_\tau\overrightarrow{\theta}_1^{n+1}\rangle| + |\beta\langle P_\tau(\mathbb{K}\nabla\theta^n), P_\tau\overrightarrow{\theta}_1^{n+1}\rangle| \\
& +|\langle\beta P_\tau\mathbb{K}\nabla(\mathbb{P}_1\phi_D(t_{n+1}) - \mathbb{P}_1\phi_D(t_n)), P_\tau\overrightarrow{\theta}_1^{n+1}\rangle| \\
\leq\ & C_k\nu\|\nabla\theta^{n+1}\|_0\|\nabla\overrightarrow{\theta}_1^{n+1}\|_0 + C_k\nu\|\nabla\theta^n\|_0\|\nabla\overrightarrow{\theta}_1^{n+1}\|_0 \\
& +C_k\nu\|\nabla(\mathbb{P}_1\phi_D(t_{n+1}) - \mathbb{P}_1\phi_D(t_n))\|_0\|\nabla\overrightarrow{\theta}_1^{n+1}\|_0 \\
\leq\ & \frac{C_k\nu}{2}\|\nabla\theta^{n+1}\|_0^2 + \frac{3C_k\nu}{2}\|\nabla\overrightarrow{\theta}_1^{n+1}\|_0^2 + \frac{C_k\nu}{2}\|\nabla\theta^n\|_0^2 + \frac{\Delta t C_k\nu}{2}\int_{t_n}^{t_{n+1}}\|\nabla\phi_{D,t}\|_0^2\,dt
\end{aligned}$$

where C_k is proportional to $\sqrt{k}$ and (4.13) in Shan and Zheng (2013a) is used in the last step.

Hence, multiplying (3.64) *with* $2\Delta t$ *and using the above inequalities, we have*

$$
\begin{aligned}
& \eta\|\theta^{n+1}\|_0^2 - \eta\|\theta^n\|_0^2 + \|\overrightarrow{\theta}_1^{n+1}\|_0^2 - \|\overrightarrow{\theta}_1^{n}\|_0^2 + 2\eta\Delta t a_D(\theta^{n+1},\theta^{n+1}) \\
& +2\Delta t a_S(\overrightarrow{\theta}_1^{n+1},\overrightarrow{\theta}_1^{n+1}) - 2\eta\Delta t\langle \overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S,\theta^{n+1}\rangle + 2\Delta t\langle g\theta^{n+1},\overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S\rangle \\
& +2\Delta t\beta\langle P_\tau(\overrightarrow{\theta}_1^{n+1} + \mathbb{K}\nabla\theta^{n+1}), P_\tau\overrightarrow{\theta}_1^{n+1}\rangle \\
\leq\ & C\Delta t\left[\|w_1^{n+1}\|_0^2 + \|w_3^{n+1}\|_0^2 + \|w_2^{n+1}\|_{0,\Gamma}^2 + \|w_4^{n+1}\|_{0,\Gamma}^2\right] \\
& +\Delta t\left[\frac{3\nu}{4} + C(5\epsilon_2+4\epsilon_3)\epsilon_4 + \frac{C(1+\eta)\epsilon_4}{\epsilon_3} + 3C_k\nu\right]\|\nabla\overrightarrow{\theta}_1^{n+1}\|_0^2 \\
& +\Delta t\left[C\eta(5\epsilon_2+4\epsilon_3)\epsilon_4 + \frac{C(1+\eta)g^2\epsilon_4}{\epsilon_3} + C_k\nu\right]\|\nabla\theta^{n+1}\|_0^2 \\
& \Delta t\left[2\epsilon_1 + \frac{C(5\epsilon_2+4\epsilon_3)}{\epsilon_4} + \frac{C(1+\eta)}{\epsilon_3\epsilon_4}\right]\|\overrightarrow{\theta}_1^{n+1}\|_0^2 \\
& +\Delta t\left[2\eta\epsilon_1 + \frac{C\eta(5\epsilon_2+4\epsilon_3)}{\epsilon_4} + \frac{C(1+\eta)g^2}{\epsilon_3\epsilon_4}\right]\|\theta^{n+1}\|_0^2 \\
& +\Delta t\frac{C(1+\eta)\epsilon_4}{\epsilon_3}\|\nabla\overrightarrow{\theta}_1^{n}\|_0^2 + \Delta t\left[\frac{C(1+\eta)g^2\epsilon_4}{\epsilon_3} + C_k\nu\right]\|\nabla\theta^n\|_0^2 \\
& +\Delta t\frac{C(1+\eta)}{\epsilon_3\epsilon_4}\|\overrightarrow{\theta}_1^{n}\|_0^2 + \Delta t\frac{C(1+\eta)g^2}{\epsilon_3\epsilon_4}\|\theta^n\|_0^2 + \Delta t^2 C_k\nu\int_{t_n}^{t_{n+1}}\|\nabla\phi_{D,t}\|_0^2\,dt \\
& +C\Delta t\left[\|\overrightarrow{\rho}_1^{n+1}\|_{0,\Gamma}^2 + \|\rho^{n+1}\|_{0,\Gamma}^2 + \|\overrightarrow{\rho}_1^{n}\|_{0,\Gamma}^2 + \|\rho^n\|_{0,\Gamma}^2 + \|\nabla\overrightarrow{\rho}_1^{n+1}\|_0^2\right] \\
\leq\ & C\Delta t\left[\|w_1^{n+1}\|_0^2 + \|w_3^{n+1}\|_0^2 + \|w_2^{n+1}\|_{0,\Gamma}^2 + \|w_4^{n+1}\|_{0,\Gamma}^2\right] \qquad (3.68) \\
& +\Delta t\left[\frac{3\nu}{4} + C(5\epsilon_2+4\epsilon_3)\epsilon_4 + \frac{C(1+\eta)\epsilon_4}{\epsilon_3} + 3C_k\nu\right]\|\nabla\overrightarrow{\theta}_1^{n+1}\|_0^2 \\
& +\Delta t\left[C\eta(5\epsilon_2+4\epsilon_3)\epsilon_4 + \frac{C(1+\eta)\epsilon_4 + \epsilon_3 C_k\nu}{\epsilon_3}\right]\|\nabla\theta^{n+1}\|_0^2 \\
& \Delta t\left[2\epsilon_1 + \frac{C(5\epsilon_2+4\epsilon_3)}{\epsilon_4} + \frac{C(1+\eta)}{\epsilon_3\epsilon_4}\right]\|\overrightarrow{\theta}_1^{n+1}\|_0^2 \\
& +\Delta t\left[2\eta\epsilon_1 + \frac{C\eta(5\epsilon_2+4\epsilon_3)}{\epsilon_4} + \frac{C(1+\eta)g^2}{\epsilon_3\epsilon_4}\right]\|\theta^{n+1}\|_0^2 \\
& +\Delta t\frac{C(1+\eta)\epsilon_4}{\epsilon_3}\|\nabla\overrightarrow{\theta}_1^{n}\|_0^2 + \Delta t\left[\frac{C(1+\eta)\epsilon_4 + \epsilon_3 C_k\nu}{\epsilon_3}\right]\|\nabla\theta^n\|_0^2 \\
& +\Delta t\frac{C(1+\eta)}{\epsilon_3\epsilon_4}\|\overrightarrow{\theta}_1^{n}\|_0^2 + \Delta t\frac{C(1+\eta)g^2}{\epsilon_3\epsilon_4}\|\theta^n\|_0^2 + \Delta t^2 C_k\nu\int_{t_n}^{t_{n+1}}\|\nabla\phi_{D,t}\|_0^2\,dt \\
& +C\Delta t\left[\|\overrightarrow{\rho}_1^{n+1}\|_{0,\Gamma}^2 + \|\rho^{n+1}\|_{0,\Gamma}^2 + \|\overrightarrow{\rho}_1^{n}\|_{0,\Gamma}^2 + \|\rho^n\|_{0,\Gamma}^2 + \|\nabla\overrightarrow{\rho}_1^{n+1}\|_0^2\right]
\end{aligned}
$$

Based on the above preparation for the treatment of BJ interface condition, we can utilize (4.2) in Cao et al. *(2010c) to advance the proof as follows. For small enough ν, we can choose ϵ_2, ϵ_3, and ϵ_4 to obtain*

$$\begin{aligned}
& 2\eta a_D(\theta^{n+1}, \theta^{n+1}) + 2a_S(\overrightarrow{\theta}_1^{n+1}, \overrightarrow{\theta}_1^{n+1}) - 2\eta\langle \overrightarrow{\theta}_1^{n+1} \cdot \overrightarrow{n}_S, \theta^{n+1}\rangle + 2\langle g\theta^{n+1}, \overrightarrow{\theta}_1^{n+1} \cdot \overrightarrow{n}_S\rangle \\
& +2\beta\langle P_\tau(\overrightarrow{\theta}_1^{n+1} + \mathbb{K}\nabla\theta^{n+1}), P_\tau \overrightarrow{\theta}_1^{n+1}\rangle \qquad (3.69) \\
\geq \;& \left[C\eta(5\epsilon_2 + 4\epsilon_3)\epsilon_4 + \frac{2C(1+\eta)\epsilon_4 + 2C_k\nu\epsilon_3}{\epsilon_3}\right] \|\nabla\theta^{n+1}\|_0^2 \\
& + \left[\frac{3\nu}{4} + C(5\epsilon_2 + 4\epsilon_3)\epsilon_4 + \frac{2C(1+\eta)\epsilon_4 + 3C_k\nu\epsilon_3}{\epsilon_3}\right] \|\nabla\overrightarrow{\theta}_1^{n+1}\|_0^2 \\
& -C_3\|\theta^{n+1}\|_0^2 - C_3\|\overrightarrow{\theta}_1^{n+1}\|_0^2
\end{aligned}$$

Then, substituting (3.69) *into* (3.68)*, we have*

$$\begin{aligned}
& \eta\left\|\theta^{n+1}\right\|_0^2 + \left\|\overrightarrow{\theta}_1^{n+1}\right\|_0^2 + \Delta t\left[\frac{C(1+\eta)\epsilon_4 + C_k\nu\epsilon_3}{\epsilon_3}\right]\left\|\nabla\theta^{n+1}\right\|_0^2 \\
& +\Delta t\left[\frac{C(1+\eta)\epsilon_4}{\epsilon_3}\right]\left\|\nabla\overrightarrow{\theta}_1^{n+1}\right\|_0^2 \\
\leq \;& (1 + C\Delta t)(\eta\left\|\theta^n\right\|_0^2 + \left\|\overrightarrow{\theta}_1^n\right\|_0^2) + \Delta t\left[\frac{C(1+\eta)\epsilon_4 + C_k\nu\epsilon_3}{\epsilon_3}\right]\left\|\nabla\theta^n\right\|_0^2 \\
& +\Delta t\left[\frac{C(1+\eta)\epsilon_4}{\epsilon_3}\right]\|\nabla\overrightarrow{\theta}_1^n\|_0^2 \\
& +C\Delta t(\|w_1^{n+1}\|_0^2 + \|w_3^{n+1}\|_0^2 + \|w_2^{n+1}\|_{0,\Gamma}^2 + \|w_4^{n+1}\|_{0,\Gamma}^2) \\
& +C\Delta t(\|\overrightarrow{\rho}_1^{n+1}\|_{0,\Gamma}^2 + \|\rho^{n+1}\|_{0,\Gamma}^2 + \|\overrightarrow{\rho}_1^n\|_{0,\Gamma}^2 + \|\rho^n\|_{0,\Gamma}^2 + \|\nabla\overrightarrow{\rho}_1^{n+1}\|_0^2) \\
& +C\Delta t(\|\theta^{n+1}\|_0^2 + \|\overrightarrow{\theta}_1^{n+1}\|_0^2) + C\Delta t^2\int_{t_n}^{t_{n+1}}\|\nabla\phi_{D,t}\|_0^2\,dt
\end{aligned}$$

Setting $\epsilon = \frac{C(1+\eta)\epsilon_4 + C_k\nu\epsilon_3}{\epsilon_3}$ and $\tilde{\epsilon} = \frac{C(1+\eta)\epsilon_4}{\epsilon_3}$, then we can consider the special norm $\eta\left\|\theta^n\right\|_0^2 + \left\|\overrightarrow{\theta}_1^n\right\|_0^2 + \epsilon\Delta t\left\|\nabla\theta^n\right\|_0^2 + \tilde{\epsilon}\Delta t\left\|\nabla\overrightarrow{\theta}_1^n\right\|_0^2$ $(n = 1, \ldots, N)$ for the distcrete Gronwall's inequality as follows.

Summing above inequality from $n = 0$ *to* $N - 1$, *we have*

$$
\begin{aligned}
& \eta \left\|\theta^N\right\|_0^2 + \left\|\overrightarrow{\theta}_1^N\right\|_0^2 + \epsilon\Delta t \left\|\nabla\theta^N\right\|_0^2 + \tilde{\epsilon}\Delta t \left\|\nabla\overrightarrow{\theta}_1^N\right\|_0^2 \\
\leq\ & (1 + C\Delta t)^N[\eta \left\|\theta^0\right\|_0^2 + \left\|\overrightarrow{\theta}_1^0\right\|_0^2 + \epsilon\Delta t \left\|\nabla\theta^0\right\|_0^2 + \tilde{\epsilon}\Delta t \left\|\nabla\overrightarrow{\theta}_1^0\right\|_0^2] \\
& + C\Delta t \sum_{j=0}^{N-1}(1 + C\Delta t)^j[\left\|w_1^{j+1}\right\|_0^2 + \left\|w_2^{j+1}\right\|_{0,\Gamma}^2 + \left\|w_3^{j+1}\right\|_0^2 + \left\|w_4^{j+1}\right\|_{0,\Gamma}^2 \\
& + \|\overrightarrow{\rho}_1^{j+1}\|_{0,\Gamma}^2 + \|\rho^{j+1}\|_{0,\Gamma}^2 + \|\overrightarrow{\rho}_1^{j}\|_{0,\Gamma}^2 + \|\rho^{j}\|_{0,\Gamma}^2 + \|\nabla\overrightarrow{\rho}_1^{j+1}\|_0^2] \\
& + C\Delta t \sum_{j=0}^{N-1}(1 + C\Delta t)^j[\|\theta^{j+1}\|_0^2 + \|\overrightarrow{\theta}_1^{j+1}\|_0^2] \\
& + C\Delta t^2 \sum_{j=0}^{N-1}(1 + C\Delta t)^j \int_{t_j}^{t_{j+1}} \|\nabla\phi_{D,t}\|_0^2 \, dt \\
\leq\ & Ce^{CT}\Big[\eta \left\|\theta^0\right\|_0^2 + \left\|\overrightarrow{\theta}_1^0\right\|_0^2 + \tilde{\epsilon}\Delta t \left\|\overrightarrow{\theta}_1^0\right\|_1^2 + \epsilon\Delta t \left\|\theta^0\right\|_1^2 \\
& + \Delta t \sum_{j=0}^{N-1}\Big(\left\|w_1^{j+1}\right\|_0^2 + \left\|w_2^{j+1}\right\|_{0,\Gamma}^2 + \left\|w_3^{j+1}\right\|_0^2 + \left\|w_4^{j+1}\right\|_{0,\Gamma}^2 + \Delta t \int_{t_j}^{t_{j+1}} \|\nabla\phi_{D,t}\|_0^2 \, dt \\
& + \|\overrightarrow{\rho}_1^{j+1}\|_{0,\Gamma}^2 + \|\rho^{j+1}\|_{0,\Gamma}^2 + \|\overrightarrow{\rho}_1^{j}\|_{0,\Gamma}^2 + \|\rho^{j}\|_{0,\Gamma}^2 + \|\nabla\overrightarrow{\rho}_1^{j+1}\|_0^2\Big)\Big]. \qquad (3.70)
\end{aligned}
$$

Then by Lemma 6, we complete the proof of (3.43).

3.4. THE LAGRANGE MULTIPLIER METHOD UNDER THE FRAMEWORK OF DOMAIN DECOMPOSITION FOR NS-DARCY SYSTEM WITH THE DEFECTIVE BOUNDARY

With the foundation built up in the previous sections, in this section we propose the Lagrange multiplier method under the framework of the parallel non-iterative multi-physics domain decomposition for the NS-Darcy system with BJ interface condition and the defective boundary condition. Based on the weak formulation with Lagrange multipliers, we utilize the three-step backward differentiation scheme for the temporal discretization and finite elements for the spatial discretization.

First, we present the following coupled weak formulation with Lagrange multipliers for the NS-Darcy system with the defective boundary, which is defined in Section 1.1.2: find $(\overrightarrow{u}_S, p_S) \in H^1(0,T; X_S, X_S') \times L^2(0,T; Q_S)$, $\phi_D \in H^1(0,T; X_D, X_D')$ and $\lambda = \{\lambda_i(t)\}_{i=0}^m \in L^2(0,T)^{m+1}$ such that

$$\begin{aligned}
&(\frac{\partial \overrightarrow{u}_S}{\partial t}, \overrightarrow{v})_{\Omega_S} + g(\frac{\partial \phi_D}{\partial t}, \psi)_{\Omega_D} + \sum_{i=0}^m \lambda_i(t) \int_{S_i} \overrightarrow{v} \cdot \overrightarrow{n}_S \, ds + C_S(\overrightarrow{u}_S, \overrightarrow{u}_S, \overrightarrow{v}) + a_S(\overrightarrow{u}_S, \overrightarrow{v}) \\
&+ b_S(\overrightarrow{v}, p_S) + g a_D(\phi_D, \psi) + \langle g\phi_D, \overrightarrow{v} \cdot \overrightarrow{n}_S \rangle - g\langle \overrightarrow{u}_S \cdot \overrightarrow{n}_S, \psi \rangle \\
&+ \frac{\alpha \nu \sqrt{d}}{\sqrt{\mathrm{trace}(\prod)}} \langle P_\tau(\overrightarrow{u}_S + \mathbb{K}\nabla\phi_D), P_\tau \overrightarrow{v} \rangle \\
= \;& g(f_D, \psi)_{\Omega_D} + (\overrightarrow{f}_S, \overrightarrow{v})_{\Omega_S}, \; \forall \overrightarrow{v} \in X_S, \;\; \psi \in X_D, \qquad (3.71) \\
&\sum_{i=0}^m \mu_i(t) \int_{S_i} \overrightarrow{v} \cdot \overrightarrow{n}_S \, ds + b_S(\overrightarrow{u}_S, q) = \sum_{i=0}^m \mu_i Q_i, \; \forall \; q \in Q_S, \mu = \{\mu_i(t)\}_{i=0}^m \in L^2(0,T)^{m+1}.
\end{aligned}$$

For problem (3.71) to be solvable, the following compatibility conditions must be satisfied Heywood *et al.* (1996):

$$\int_{S_i} \overrightarrow{u}_0 \cdot \overrightarrow{n}_S ds = Q_i(0), \qquad i = 0, 1, \cdots, n. \qquad (3.72)$$

Second, based on the Robin boundary conditions (3.3)-(3.4) and (3.7), we propose the following decoupled weak formulation with Lagrange multipliers for the NS-Darcy system with BJ interface condition and the defective boundary: for $\forall \; \psi \in X_D$, $\forall \; \overrightarrow{v} \in X_S, \forall \; q \in Q_S$,

$$(\frac{\partial \phi_D}{\partial t}, \psi)_{\Omega_D} + a_D(\phi_D, \psi) + \langle g\phi_D, \psi \rangle = (f_D, \psi)_{\Omega_D} + \langle \xi_D, \psi \rangle, \qquad (3.73)$$

$$\begin{aligned}
&(\frac{\partial \overrightarrow{u}_S}{\partial t}, \overrightarrow{v})_{\Omega_S} + \sum_{i=0}^m \lambda_i(t) \int_{S_i} \overrightarrow{v} \cdot \overrightarrow{n}_S \, ds + C_S(\overrightarrow{u}_S, \overrightarrow{u}_S, \overrightarrow{v}) + a_S(\overrightarrow{u}_S, \overrightarrow{v}) + b_S(\overrightarrow{v}, p_S) \\
&+ \langle \overrightarrow{u}_S \cdot \overrightarrow{n}_S, \overrightarrow{v} \cdot \overrightarrow{n}_S \rangle + \frac{\alpha \nu \sqrt{d}}{\sqrt{\mathrm{trace}(\prod)}} \langle P_\tau \overrightarrow{u}_S, P_\tau \overrightarrow{v} \rangle \\
= \;& (\overrightarrow{f}_S, \overrightarrow{v})_{\Omega_S} + \langle \xi_S, \overrightarrow{v} \cdot \overrightarrow{n}_S \rangle - \langle \xi_{S\tau}, P_\tau \overrightarrow{v} \rangle, \qquad (3.74)
\end{aligned}$$

$$\sum_{i=0}^m \mu_i(t) \int_{S_i} \overrightarrow{v} \cdot \overrightarrow{n}_S \, ds + b_S(\overrightarrow{u}_S, q) = \sum_{i=0}^m \mu_i Q_i, \;\; \mu = \{\mu_i(t)\}_{i=0}^m \in L^2(0,T)^{m+1}. \qquad (3.75)$$

Then, the semi-discretization is to find $\phi_h \in H^1(0,T;X_{Dh})$, $\overrightarrow{u}_h \in H^1(0,T;X_{Sh})$, $p_h \in L^2(0,T;Q_{Sh})$, and $\lambda = \{\lambda_i(t)\}_{i=0}^m \in L^2(0,T)^{m+1}$ such that

$$(\frac{\partial \phi_h}{\partial t},\psi_h)_{\Omega_D} + a_D(\phi_h,\psi_h) + \langle g\phi_h,\psi_h\rangle = (f_D,\psi_h)_{\Omega_D} + \langle \xi_D,\psi_h\rangle, \tag{3.76}$$

$$\begin{aligned}
&(\frac{\partial \overrightarrow{u}_h}{\partial t},\overrightarrow{v}_h)_{\Omega_S} + \sum_{i=0}^{m}\lambda_i(t)\int_{S_i}\overrightarrow{v}_h\cdot\overrightarrow{n}_S\,ds + C_S(\overrightarrow{u}_h,\overrightarrow{u}_h,\overrightarrow{v}_h) + a_S(\overrightarrow{u}_h,\overrightarrow{v}_h) + b_S(\overrightarrow{v}_h,p_h)\\
&+\langle \overrightarrow{u}_h\cdot\overrightarrow{n}_S,\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle + \frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\prod)}}\langle P_\tau\overrightarrow{u}_h,P_\tau\overrightarrow{v}_h\rangle\\
&= (\overrightarrow{f}_S,\overrightarrow{v}_h)_{\Omega_S} + \langle \xi_S,\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle - \langle \xi_{S\tau},P_\tau\overrightarrow{v}_h\rangle,
\end{aligned} \tag{3.77}$$

$$\sum_{i=0}^{m}\mu_i(t)\int_{S_i}\overrightarrow{v}_h\cdot\overrightarrow{n}_S\,ds + b_S(\overrightarrow{u}_h,q_h) = \sum_{i=0}^{m}\mu_iQ_i,\ \ \mu = \{\mu_i(t)\}_{i=0}^m \in L^2(0,T)^{m+1}. \tag{3.78}$$

Furthermore, compared with the backward Euler scheme for the temporal discretization in Section 3.2.2, we utilize three-step backward differentiation to improve the accuracy from the first order to the third order. Then, the full discretization scheme of the Lagrange multiplier method is defined as follows: at the $n+1^{th}$ step, set

$$\xi_{Dh}^{n+1} = 3\xi_{Dh}^{n} - 3\xi_{Dh}^{n-1} + \xi_{Dh}^{n-2},\ \ \xi_{Sh}^{n+1} = 3\xi_{Sh}^{n} - 3\xi_{Sh}^{n-1} + \xi_{Sh}^{n-2},\ \ \xi_{S\tau h}^{n+1} = 3\xi_{S\tau h}^{n} - 3\xi_{S\tau h}^{n-1} + \xi_{S\tau h}^{n-2},$$

and then for $\forall\, \psi_h \in X_{Dh}, \overrightarrow{v}_h \in X_{Sh}, q_h \in Q_{Sh}$ and $\forall\, \mu_i^{n+1} \in R^{n+1}$, independently solve

$$\begin{aligned}
&(\frac{11\phi_h^{n+1} - 18\phi_h^{n} + 9\phi_h^{n-1} - 2\phi_h^{n-2}}{6\Delta t},\psi_h)_{\Omega_D} + a_D(\phi_h^{n+1},\psi_h) + \langle g\phi_h^{n+1},\psi_h\rangle\\
&= (f_D^{n+1},\psi_h)_{\Omega_D} + \langle \xi_{Dh}^{n+1},\psi_h\rangle,
\end{aligned} \tag{3.79}$$

$$\begin{aligned}
&(\frac{11\overrightarrow{u}_h^{n+1} - 18\overrightarrow{u}_h^{n} + 9\overrightarrow{u}_h^{n-1} - 2\overrightarrow{u}_h^{n-2}}{6\Delta t},\overrightarrow{v}_h)_{\Omega_S} + \sum_{i=0}^{m}\lambda_i^{n+1}\int_{S_i}\overrightarrow{v}_h\cdot\overrightarrow{n}_S\,ds\\
&+C_S(\overrightarrow{u}_h^{n+1},\overrightarrow{u}_h^{n+1},\overrightarrow{v}_h) + a_S(\overrightarrow{u}_h^{n+1},\overrightarrow{v}_h) + b_S(\overrightarrow{v}_h,p_h^{n+1}) + \langle \overrightarrow{u}_h^{n+1}\cdot\overrightarrow{n}_S,\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle\\
&+\frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\prod)}}\langle P_\tau\overrightarrow{u}_h^{n+1},P_\tau\overrightarrow{v}_h\rangle = (\overrightarrow{f}_S^{n+1},\overrightarrow{v}_h)_{\Omega_S} + \langle \xi_{Sh}^{n+1},\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle - \langle \xi_{S\tau h}^{n+1},P_\tau\overrightarrow{v}_h\rangle,
\end{aligned} \tag{3.80}$$

$$\sum_{i=0}^{m}\mu_i^{n+1}\int_{S_i}\overrightarrow{v}_h\cdot\overrightarrow{n}_S\,ds + b_S(\overrightarrow{u}_h^{n+1},q_h) = \sum_{i=0}^{m}\mu_i^{n+1}Q_i, \tag{3.81}$$

for λ_i^{n+1} $(i = 0,\cdots,m)$, ϕ_h^{n+1}, $\overrightarrow{u}_h^{n+1}$, p_h^{n+1} and μ_h^{n+1}.

3.5. NUMERICAL EXAMPLES

In this section, we will present two examples to illustrate the features of the two algorithms proposed in Section 3.2 and Section 3.4.

Example 1: Consider the NS-Darcy model in Section 1.1.1 on the domain $\Omega = [0,1]\times[-0.25,0.75]$, where $\Omega_D = [0,1]\times[0,0.75]$ and $\Omega_S = [0,1]\times[-0.25,0]$. Choose $\alpha = 1$, $\nu = 1$, $g = 1$, $z = 0$, and $\mathbb{K} = kI$ where I is the identity matrix and $k = 1$. The boundary condition functions and the source terms are chosen such that the exact solutions are

$$\phi_D = [2-\pi sin(\pi x)][-y + cos(\pi(1-y))]cos(2\pi t), \tag{3.82}$$

$$\overrightarrow{u}_S = [x^2y^2 + e^{-y}, -\frac{2}{3}xy^3 + 2 - \pi sin(\pi x)]^T cos(2\pi t), \tag{3.83}$$

$$p_S = -[2-\pi sin(\pi x)]cos(2\pi y)cos(2\pi t), \tag{3.84}$$

which satisfy the interface conditions (1.9)-(1.13), including the Beavers-Joseph interface condition. The Taylor-Hood elements are used for the Navier-Stokes equations, and the quadratic finite elements are used for the second order formulations of the Darcy equation. Newton iteration is used to deal with the nonlinear advection. Next, we will provide the numerical results at $T = 1$ for the algorithm in Section 3.2. Table 3.1 provides the numerical solution errors with $\Delta t = 8h^3$. Using linear regression, these errors satisfy

$$\left\|\overrightarrow{u}_h - \overrightarrow{u}\right\|_0 \approx 1.9847h^{3.0179},\ \left|\overrightarrow{u}_h - \overrightarrow{u}\right|_1 \approx 10.219h^{1.9758}$$

$$\|p_h - p\|_0 \approx 18.967h^{3.1255},\ \|\phi_h - \phi\|_0 \approx 18.835h^{2.9899},\ |\phi_h - \phi|_1 \approx 11.213h^{2.2741}.$$

Table 3.2 provides the numerical solution errors with $\Delta t = h$. From this table, we can see that the non-iterative domain decomposition algorithm is still stable, but the accuracy order is about first order. These results numerically verify the expected optimal accuracy orders arising from backward Euler scheme, Taylor-Hood elements and quadratic elements.

Table 3.1. Errors of the first non-iterative DDM with backward Euler for $\Delta t = 8h^3$.

h	$\|u_h - u\|_0$	$\|u_h - u\|_1$	$\|p_h - p\|_0$	$\|\phi_h - \phi\|_0$	$\|\phi_h - \phi\|_1$
1/4	3.0184×10^{-2}	6.2839×10^{-1}	2.6888×10^{-1}	2.6574×10^{-1}	6.2396×10^{-1}
1/8	3.7366×10^{-3}	1.6705×10^{-1}	2.8863×10^{-2}	3.7408×10^{-2}	1.0425×10^{-1}
1/16	4.6077×10^{-4}	4.2861×10^{-2}	3.2376×10^{-3}	4.7426×10^{-3}	1.9657×10^{-2}
1/32	5.6977×10^{-5}	1.0812×10^{-2}	3.7882×10^{-4}	5.9374×10^{-4}	4.4097×10^{-3}

Table 3.2. Errors of the first non-iterative DDM with backward Euler for $\Delta t = h$.

h	$\|u_h - u\|_0$	$\|u_h - u\|_1$	$\|p_h - p\|_0$	$\|\phi_h - \phi\|_0$	$\|\phi_h - \phi\|_1$
1/8	2.8012×10^{-2}	4.0061×10^{-1}	5.0047×10^{-1}	2.5251×10^{-2}	1.4670×10^{-1}
1/16	1.0208×10^{-2}	1.2970×10^{-1}	1.5062×10^{-1}	1.5081×10^{-2}	7.5080×10^{-2}
1/32	4.1315×10^{-3}	4.6703×10^{-2}	5.6933×10^{-2}	8.4446×10^{-3}	3.9880×10^{-2}
1/64	1.8419×10^{-3}	1.9169×10^{-2}	2.5420×10^{-2}	4.4881×10^{-3}	2.0805×10^{-2}

In order to improve the accuracy order of the temporal discretization, we use the three-step backward differentiation scheme to replace the backward Euler scheme in the algorithm of Section 3.2. Then, we list the corresponding numerical solution errors for $\Delta t = h$ in Table 3.3. Using linear regression, the errors in Table 3.3 satisfy

$$\left\|\vec{u}_h - \vec{u}\right\|_0 \approx 118.04h^{3.7902}, \; \left|\vec{u}_h - \vec{u}\right|_1 \approx 1233.32h^{3.6311}$$
$$\|p_h - p\|_0 \approx 158.94h^{3.7014}, \; \|\phi_h - \phi\|_0 \approx 1.7227h^{2.7446}, \; |\phi_h - \phi|_1 \approx 5.4419h^{2.0414}.$$

These results are consistent with the expected optimal accuracy orders arising from the three-step backward differentiation scheme, Taylor-Hood elements, and quadratic elements. In particular, we see the optimal accuracy order of $O(h^3 + \Delta t^3) = O(h^3)$ with respect to L^2 norms for $\vec{u}$ and ϕ since we use quadratic finite elements for them.

Table 3.3. Errors of the non-iterative DDM with three-step BDF for $\Delta t = h$.

h	$\|\vec{u}_h - \vec{u}\|_0$	$\|\vec{u}_h - \vec{u}\|_1$	$\|p_h - p\|_0$	$\|\phi_h - \phi\|_0$	$\|\phi_h - \phi\|_1$
1/8	4.8057×10^{-2}	7.4713×10^{-1}	8.0323×10^{-1}	6.2662×10^{-3}	7.9365×10^{-2}
1/16	2.9286×10^{-3}	4.5974×10^{-2}	5.1411×10^{-2}	7.2423×10^{-4}	1.8559×10^{-2}
1/32	2.2501×10^{-4}	3.5787×10^{-3}	3.6113×10^{-3}	1.3492×10^{-4}	4.5594×10^{-3}
1/64	1.7783×10^{-5}	3.9761×10^{-4}	3.7601×10^{-4}	1.9332×10^{-5}	1.1335×10^{-3}

Example 2: Consider the time-dependent NS-Darcy model with defective boundary conditions and BJ interface condition on the following domain. Let $\Omega = [0,1] \times [0,1]$. We choose Ω_S to be the polygon $\overline{ABCDEFGHIJ}$ where $A = (0,1), B = (0,3/4), C = (1/2,1/4), D = (1/2,0), E = (3/4,0), F = (3/4,1/4), G = (1,1/4), H = (1,1/2), I = (3/4,1/2)$ and $J = (1/4,1)$. Let $\Omega_D = \Omega/\Omega_S$, $S_0 = \overline{AB} \cup \overline{JA}$, $S_1 = \overline{DE}$, and $S_2 = \overline{GH}$.

Set $T = 1$, $\alpha = 1$, $\nu = 1$, $g = 1$, $z = 0$, and $\mathbb{K} = K\mathbb{I}$ where $\mathbb{I}$ denotes the identity matrix and $K = 1$. The boundary condition data and source terms are chosen to be 0 except Q_i on S_i $(i = 0,1,2)$. We subdivide Ω into a rectangle of height and width $h = 1/M$, where M denotes a positive integer, and then subdivide each rectangle into two triangles by drawing a diagonal. For this numerical experiment, we choose $M = 32$ and $\Delta t = h$. The Taylor-Hood elements are used for the Navier-Stokes equations and the quadratic finite elements are used for the second order formulations of the Darcy equation. We will provide the numerical results at $T = 1$ for the algorithm in Section 3.4. In the first test, we set $Q_1 = Q_2 = -1$ and $Q_0 = 2$ so that the total inflow rate is equal to the total outflow rate. In the second test, we keep the same Q_1 and Q_2, but set $Q_0 = 1$ so that the total inflow rate is larger than the total outflow rate. In the third test, we keep the same Q_1 and Q_2, but set $Q_0 = 3$ so that the total inflow rate is smaller than the total outflow rate.

Figure 3.1-3.3 illustrate the numerical solutions at the end time of $T = 1$ for these three tests. These physically valid velocity fields verify the effectiveness of the proposed Lagrange multiplier method under the framework of the parallel non-iterative multi-physics

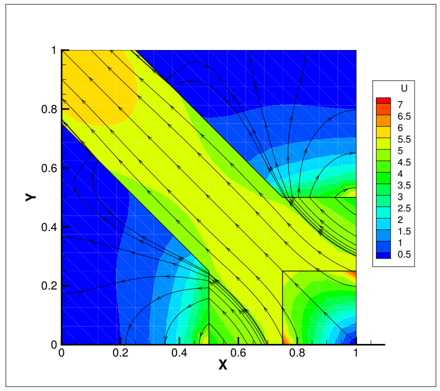

Figure 3.1. Velocity streamlines for $Q_0 = 2$, $Q_1 = -1$, and $Q_2 = -1$.

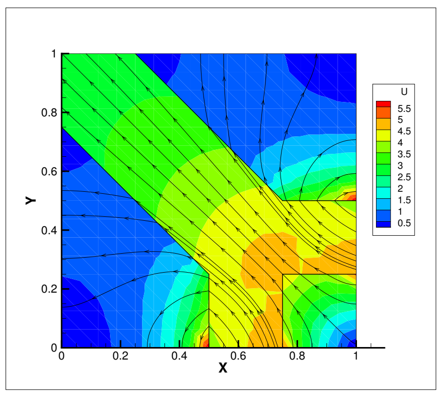

Figure 3.2. Velocity streamlines for $Q_0 = 1$, $Q_1 = -1$, and $Q_2 = -1$.

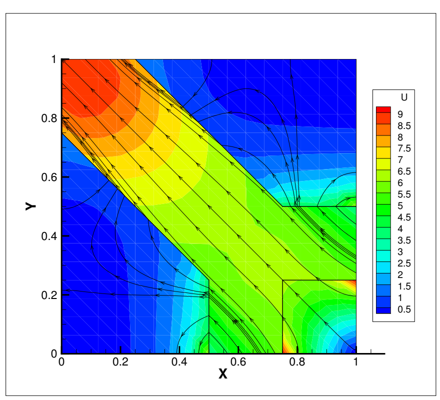

Figure 3.3. Velocity streamlines for $Q_0 = 3$, $Q_1 = -1$, and $Q_2 = -1$.

domain decomposition. Specifically, when compared with Figure 3.1, we observe more flow from the conduit to the porous media in Figure 3.2 and more flow from the porous media to the conduit in Figure 3.3, especially in the area around the outflow boundary S_0. These phenomena are expected due to the chosen unbalanced inflow and outflow rates for the conduit.

4. EFFICIENT ENSEMBLE ALGORITHMS FOR THE STOCHASTIC STOKES-DARCY INTERFACE MODEL

In this section, we first propose and analyze an efficient ensemble algorithm for the fast computation of multiple realizations of the stochastic Stokes-Darcy model with random hydraulic conductivity (including the one in the interface conditions), source terms, and initial conditions. This algorithm results in a common coefficient matrix for all realizations at each time step, making solving the linear systems much less expensive while maintaining comparable accuracy to traditional methods that compute each realization separately. Moreover, it decouples the Stokes-Darcy system into two smaller sub-physics problems, which reduces the size of the linear systems and allows parallel computation of the two sub-physics problems. We prove the long time stability and error estimate for this ensemble method under a time step condition and two parameter conditions. Numerical examples are presented to support the theoretical results and illustrate the application of the algorithm.

Second, we utilize the idea of artificial compressibility and partitioned time stepping methods to construct the decoupled ensemble algorithm for efficiently computing multiple realizations of the stochastic Stokes-Darcy interface model with a random hydraulic conductivity tensor, source terms, and initial conditions. The solutions are found by solving three smaller decoupled subproblems with two common time-independent coefficient matrices for all realizations, which significantly improves the efficiency for both assembling and solving the matrix systems. The fully coupled Stokes-Darcy system can first be decoupled into two smaller sub-physics problems by the idea of partitioned time stepping, which reduces the size of the linear systems and allows parallel computing for each sub-physics problem. The artificial compressibility further decouples the velocity and pressure, which further reduces storage requirements and improves computational efficiency. We prove the long time stability and the convergence for this new ensemble method. Three numerical examples are presented to support the theoretical results and illustrate the features of the algorithm, including the convergence, stability, efficiency, and applicability.

4.1. BACKGROUND FOR ENSEMBLE METHOD

In this section, we review the literature of Stokes-Darcy interface model and some recent works of ensemble method. Based on the foundamental work in Jiang and Layton (2014), we develop the efficient ensemble method for the numerical approximation of Stokes-Darcy interface model.

4.1.1. Model Problem. We consider a linear Stokes-Darcy model for the coupling of the surface and porous media flows, where the Stokes equations describe the incompressible surface fluid flow and the Darcy model describes the groundwater flow in porous media. For derivation and more detailed discussions of the Stokes-Darcy model, see Boubendir and Tlupova (2013); Cao *et al.* (2011, 2014); Chidyagwai and Rivière (2009); Discacciati *et al.* (2002a, 2007); Ervin *et al.* (2014); Galvis and Sarkis (2007); Gatica *et al.* (2011); Girault and Rivière (2009); Girault *et al.* (2014); Kanschat and Riviére (2010); Layton *et al.* (2002); Mu and Xu (2007); Tlupova and Cortez (2009); Wang and Xu (2014). By Figure 1.1, let Ω_S denote the surface fluid flow region and Ω_D denote the porous media flow region, where $\Omega_S, \Omega_D \subset R^d (d = 2, 3)$ are both open, bounded domains. These two domains lie across an interface, Γ, from each other, and $\Omega_S \cap \Omega_D = \emptyset, \bar{\Omega}_S \cap \bar{\Omega}_D = \Gamma$.

In order to simplify the following formulation and analysis, we denote fluid velocity $\overrightarrow{u}_S(x,t)$, fluid pressure $p_S(x,t)$, and hydraulic head $\phi_D(x,t)$ by $\overrightarrow{u}$, p and ϕ that satisfy

$$\begin{aligned}
&\overrightarrow{u}_t - \nu\Delta\overrightarrow{u} + \nabla p = f_f(x,t), \nabla \cdot \overrightarrow{u} = 0, \text{ in } \Omega_S,\\
&S_0\phi_t - \nabla \cdot (\mathcal{K}(x)\nabla\phi) = f_p(x,t), \text{ in } \Omega_D,\\
&\phi(x,0) = \phi_0(x), \text{ in } \Omega_D \text{ and } \overrightarrow{u}(x,0) = \overrightarrow{u}_0(x), \text{ in } \Omega_S,\\
&\phi(x,t) = 0, \text{ in } \partial\Omega_D \backslash \Gamma \text{ and } \overrightarrow{u}(x,t) = 0, \text{ in } \partial\Omega_S \backslash \Gamma.
\end{aligned} \tag{4.1}$$

Let $\widehat{n}_{f/p}$ denote the outward unit normal vector on Γ associated with $\Omega_{S/D}$, where $\widehat{n}_f = -\widehat{n}_p$. The coupling conditions across Γ are conservation of mass, balance of forces, and the BJSJ condition on the tangential velocity:

$$\overrightarrow{u} \cdot \widehat{n}_f - \mathcal{K}\nabla\phi \cdot \widehat{n}_p = 0 \text{ and } p - \nu\, \widehat{n}_f \cdot \nabla\overrightarrow{u} \cdot \widehat{n}_f = g\phi \text{ on } \Gamma,$$

$$-\nu\, \nabla\overrightarrow{u} \cdot \widehat{n}_f = \frac{\alpha_{\text{BJS}}}{\sqrt{\widehat{\tau}_i \cdot \mathcal{K}\widehat{\tau}_i}}\overrightarrow{u} \cdot \widehat{\tau}_i \text{ on } \Gamma, \text{ for any tangential vector } \widehat{\tau}_i \text{ on } \Gamma.$$

see Beavers and Joseph (1967a), Saffman (1971b), Jager and Mikelic (2000). Here, g, $\mathcal{K}$, ν, and S_0 are the gravitational acceleration constant, hydraulic conductivity tensor, kinematic viscosity, and specific mass storativity coefficient, respectively, which are all positive. $\mathcal{K}$ is assumed to be symmetric positive definite (SPD).

In simulations of porous media flows, the major difficulty is the determination of the hydraulic conductivity tensor $\mathcal{K}$. In the simplest case of isotropic homogeneous media, the hydraulic conductivity tensor is diagonal and constant. However, in most geophysical and engineering applications, the media are usually randomly heterogeneous, and each component $k_{ij}(x, w)$ of the hydraulic conductivity tensor is a random function that depends on spatial coordinates. Then the problem becomes solving a stochastic PDE system instead of a deterministic PDE system, and the goal of mathematical analysis and computer simulations is the prediction of statistical moments of the solution, such as the mean and variance. The most popular approach in solving a PDE system with random inputs is the Monte Carlo method, which is easy to implement and allows the use of existing deterministic codes. The main disadvantage of the Monte Carlo method is its very slow convergence rate ($1/\sqrt{J}$), which inevitably requires computation of a large number of realizations to obtain useful statistical information from the solutions. Other ensemble-based methods have been devised to produce faster convergence rates and reduce numerical efforts, including the multilevel Monte Carlo method Barth and Lang (2012), quasi-Monte Carlo sequences Kuo *et al.* (2012), Latin hypercube sampling Helton and Davis (2003), centroidal Voronoi tessellations Burkardt *et al.* (2006), and more recently developed stochastic collocation methods Babuška *et al.* (2007); Xiu and Hesthaven (2005) and non-intrusive polynomial chaos methods Hosder *et al.* (2006); Reagan *et al.* (2003). All these methods are non-intrusive in the sense that the stochastic and spatial degrees of freedom are decoupled and deterministic codes can be used directly without any modification. However, repetitive runs of an existing deterministic solver can be prohibitively costly when the governing equations take complicated forms.

A recent ensemble algorithm aimed at significantly reducing the computational cost of the ensemble simulations and consequently improving the performance of the aforementioned ensemble-based stochastic approaches was proposed in Jiang and Layton (2014). This ensemble algorithm solves all realizations simultaneously instead of solving them individually. It utilizes the mean of the solutions at each time step to form a coefficient matrix that is independent of the realization index j; that is, all realizations have the same coefficient matrix at each time step. Then the problem is reduced to solving one linear system with multiple right-hand sides, for which the computational cost can be significantly reduced. This ensemble algorithm has been extensively studied and tested for ensemble simulations to account for uncertainties in initial conditions and forcing terms Jiang (2015); Jiang *et al.* (2015); Jiang and Layton (2014, 2015); Jiang and Schneier (2018); Mohebujjaman and Rebholz (2017); Neda *et al.* (2016). Some recent work include strategies such as incorporating model reduction techniques to further reduce computational cost Gunzburger *et al.* (2017a, 2018a) and devising ensemble algorithms to account for various model parameters of Navier-Stokes equations Gunzburger *et al.* (2017b, 2018b), Boussinesq equations Fiordilino (2018), and a simple elliptic equation Luo and Wang (2018). In this paper, we will further develop the ensemble algorithm for computing an ensemble of the Stokes-Darcy systems to account for uncertainties in initial conditions, forcing terms, and the hydraulic conductivity tensor. Herein we consider computing an ensemble of J Stokes-

Darcy systems corresponding to J different parameter sets $(\overrightarrow{u}_j^0, \phi_j^0, f_{fj}, f_{pj}, \mathcal{K}_j)$, $j = 1, ..., J$,

$$
\begin{aligned}
\overrightarrow{u}_{j,t} - \nu_j \Delta \overrightarrow{u}_j + \nabla p_j = f_{f,j}(x,t),\ \nabla \cdot \overrightarrow{u}_j = 0, \text{ in } \Omega_S, \\
S_0 \phi_{j,t} - \nabla \cdot \big(\mathcal{K}_j(x) \nabla \phi_j\big) = f_{p,j}(x,t), \text{ in } \Omega_D, \\
\phi_j(x,t) = 0, \text{ in } \partial\Omega_D \backslash \Gamma \text{ and } \overrightarrow{u}_j(x,t) = 0, \text{ in } \partial\Omega_S \backslash \Gamma.
\end{aligned}
\tag{4.2}
$$

Here we assume that there are uncertainties in initial conditions $\overrightarrow{u}^0(x), \phi^0(x)$, forcing terms $f_f(x,t), f_p(x,t)$, and the hydraulic conductivity tensor $\mathcal{K}(x)$, and $(\overrightarrow{u}_j^0, \phi_j^0, f_{fj}, f_{pj}, \mathcal{K}_j)$ is one of the samples drawn from the respective probabilistic distributions. J is the number of total samples.

4.1.2. Notation and Preliminaries. We denote the $L^2(\Gamma)$ norm by $\|\cdot\|_\Gamma$ and the $L^2(\Omega_{S/D})$ norms by $\|\cdot\|_{f/p}$; the corresponding inner products are denoted by $(\cdot,\cdot)_{f/p}$. Furthermore, we denote the $H^k(\Omega_{S/D})$ norm by $\|\cdot\|_{H^k(D_{f/p})}$. The following inequalities will be used in the proofs Layton *et al.* (2013):

$$
\|\phi\|_I \leq C(D_p)\sqrt{\|\phi\|_p \|\nabla\phi\|_p}, \tag{4.3}
$$

$$
\|\overrightarrow{u}\|_I \leq C(D_f)\sqrt{\|\overrightarrow{u}\|_f \|\nabla\overrightarrow{u}\|_f}, \tag{4.4}
$$

where $C(D_{f/p}) = O(\sqrt{L_{f/p}}), L_{f/p} = diameter(\Omega_{S/D})$.

The function spaces are defined in Section 2.3.1. To discretize the Stokes-Darcy problem in space by the finite element method, we choose conforming velocity, pressure, and hydraulic head finite element spaces based on an edge to edge triangulation ($d = 2$) or tetrahedralization ($d = 3$) of the domain $\Omega(S/D)$ with maximum element diameter h:

$$
X_{Sh} \subset X_S\ ,\ Q_{Sh} \subset Q_S,\ X_{Dh} \subset X_D.
$$

The continuity across the interface Γ between the finite element meshes in the two subdomains is not assumed. The finite element spaces (X_{Sh}, Q_{Sh}) are assumed to satisfy the usual discrete inf-sup $/LBB^h$ condition for stability of the discrete pressure; see Gunzburger (1989b) for more on this condition. Taylor-Hood elements Gunzburger (1989b) are one such choice used in the section of numerical tests. We will also consider the discretely divergence-free space:

$$V_f^h := \{v_h \in X_f^h : (q_h, \nabla \cdot v_h)_f = 0, \ \forall q_h \in Q_f^h\}.$$

For functions $v(x,t)$ defined on $(0,T)$, we define the continuous norm

$$\|v\|_{m,k,r} \ := \|v\|_{L^m(0,T;H^k(D_r))}, \ r \in \{f,p\}.$$

Given a time step Δt, let $t_n = n\Delta t, T = N\Delta t$, $v^n = v(x,t_n)$ and define the discrete norms

$$\begin{aligned} |||v|||_{\infty,k,r} &= \max_{0 \le n \le N} \|v^n\|_{H^k(D_r)} \quad \text{and} \\ |||v|||_{m,k,r} &:= \left(\sum_{n=0}^{N} \|v^n\|_{H^k(D_r)}^m \Delta t \right)^{1/m}, \ \ r \in \{f,p\}. \end{aligned}$$

4.2. ENSEMBLE ALGORITHM FOR THE STOCHASTIC STOKES-DARCY INTERFACE MODEL

In this section, we propose and analyze an efficient ensemble algorithm for the fast computation of multiple realizations of the stochastic Stokes-Darcy model with random hydraulic conductivity (including the one in the interface conditions), source terms, and initial conditions.

4.2.1. Formulation of Ensemble Method. In this section we study a partitioned, ensemble time-stepping method to compute the Stokes-Darcy models with different parameter sets. The ensemble algorithm is

Algorithm 1 *Find* $(\overrightarrow{u}_j^{n+1}, p_j^{n+1}, \phi_j^{n+1}) \in X_S \times Q_f \times X_D$ *satisfying* $\forall\, (v, q, \psi) \in X_S \times Q_f \times X_D$,

$$\begin{aligned}
&\left(\frac{\overrightarrow{u}_j^{n+1} - \overrightarrow{u}_j^{n}}{\Delta t}, v\right)_f + \nu(\nabla \overrightarrow{u}_j^{n+1}, \nabla v)_f + \sum_i \int_\Gamma \bar{\eta}_i (\overrightarrow{u}_j^{n+1} \cdot \widehat{\tau}_i)(v \cdot \widehat{\tau}_i)\, ds \\
&+ \sum_i \int_\Gamma (\eta_{i,j} - \bar{\eta}_i)(\overrightarrow{u}_j^{n} \cdot \widehat{\tau}_i)(v \cdot \widehat{\tau}_i)\, ds - \left(p_j^{n+1}, \nabla \cdot v\right)_f + c_\Gamma(v, \phi_j^n) = (f_{f,j}^{n+1}, v)_f, \\
&(q, \nabla \cdot \overrightarrow{u}_j^{n+1})_f = 0, \qquad\qquad (4.5)\\
&gS_0 \left(\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t}, \psi\right)_p + g(\bar{\mathcal{K}} \nabla \phi_j^{n+1}, \nabla \psi)_p + g((\mathcal{K}_j - \bar{\mathcal{K}}) \nabla \phi_j^n, \nabla \psi)_p \\
&\qquad - c_\Gamma(\overrightarrow{u}_j^{n}, \psi) = g(f_{p,j}^{n+1}, \psi)_p.
\end{aligned}$$

where

$$\begin{aligned}
&\bar{\mathcal{K}} = \frac{1}{J} \sum_{j=1}^{J} \mathcal{K}_j, \quad \eta_{i,j} = \frac{\alpha_{BJS}}{\sqrt{\widehat{\tau}_i \cdot \mathcal{K}_j \widehat{\tau}_i}} \quad \textit{and} \quad \bar{\eta}_i = \frac{1}{J} \sum_{j=1}^{J} \eta_{i,j}, \\
&c_\Gamma(\overrightarrow{u}, \phi) = g \int_\Gamma \phi \overrightarrow{u} \cdot \widehat{n}_f \, ds.
\end{aligned}$$

This algorithm decouples the original problem into two sub-physics problems, which can be run in parallel. Moreover, at each time step, all realizations share the same coefficient matrix, which allows the use of efficient block solvers, e.g, block CG Feng *et al.* (1995), block GMRES Gallopulos and Simoncini (1996), or direct solvers such as LU factorization, to reduce both storage and computation time. The fully discrete approximation is:

Algorithm 2 *Find* $(\overrightarrow{u}_{j,h}^{n+1}, p_{j,h}^{n+1}, \phi_{j,h}^{n+1}) \in X_{Sh} \times Q_{Sh} \times X_{Dh}$ *satisfying* $\forall\, (v_h, q_h, \psi_h) \in X_{Sh} \times Q_{Sh} \times X_{Dh}$,

$$\begin{aligned}
&\left(\frac{\overrightarrow{u}_{j,h}^{n+1} - \overrightarrow{u}_{j,h}^{n}}{\Delta t}, v_h\right)_f + \nu(\nabla \overrightarrow{u}_{j,h}^{n+1}, \nabla v_h)_f + \sum_i \int_\Gamma \bar{\eta}_i (\overrightarrow{u}_{j,h}^{n+1} \cdot \widehat{\tau}_i)(v_h \cdot \widehat{\tau}_i)\, ds \\
&+ \sum_i \int_\Gamma (\eta_{i,j} - \bar{\eta}_i)(\overrightarrow{u}_{j,h}^{n} \cdot \widehat{\tau}_i)(v_h \cdot \widehat{\tau}_i)\, ds - \left(p_{j,h}^{n+1}, \nabla \cdot v_h\right)_f + c_\Gamma(v_h, \phi_{j,h}^n) = (f_{f,j}^{n+1}, v_h)_f, \\
&(q_h, \nabla \cdot \overrightarrow{u}_{j,h}^{n+1})_f = 0, \qquad\qquad (4.6)\\
&gS_0 \left(\frac{\phi_{j,h}^{n+1} - \phi_{j,h}^n}{\Delta t}, \psi_h\right)_p + g(\bar{\mathcal{K}} \nabla \phi_{j,h}^{n+1}, \nabla \psi_h)_p + g((\mathcal{K}_j - \bar{\mathcal{K}}) \nabla \phi_{j,h}^n, \nabla \psi_h)_p \\
&\qquad - c_\Gamma(\overrightarrow{u}_{j,h}^{n}, \psi_h) = g(f_{p,j}^{n+1}, \psi_h)_p.
\end{aligned}$$

Moving all the known quantities to the right-hand side, the algorithm is as follows.

$$
\begin{aligned}
&\left(\frac{\overrightarrow{u}_{j,h}^{n+1}-\overrightarrow{u}_{j,h}^{n}}{\Delta t}, v_h\right)_f + \nu(\nabla \overrightarrow{u}_{j,h}^{n+1}, \nabla v)_f + \sum_i \int_\Gamma \bar{\eta}_i (\overrightarrow{u}_{j,h}^{n+1}\cdot \widehat{\tau}_i)(v\cdot \widehat{\tau}_i)\, ds \\
&- \left(p_{j,h}^{n+1}, \nabla\cdot v_h\right)_f = (f_{f,j}^{n+1}, v_h)_f - \sum_i \int_\Gamma (\eta_{i,j}-\bar{\eta}_i)(\overrightarrow{u}_{j,h}^{n}\cdot \widehat{\tau}_i)(v\cdot \widehat{\tau}_i)\, ds - c_\Gamma(v_h, \phi_{j,h}^n), \\
&(q_h, \nabla\cdot \overrightarrow{u}_{j,h}^{n+1})_f = 0, \qquad (4.7)\\
&gS_0\left(\frac{\phi_{j,h}^{n+1}-\phi_{j,h}^{n}}{\Delta t}, \psi_h\right)_p + g(\bar{\mathcal{K}}\nabla\phi_{j,h}^{n+1}, \nabla\psi)_p \\
&\qquad = g(f_{p,j}^{n+1}, \psi_h)_p - g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla\phi_{j,h}^n, \nabla\psi)_p + c_\Gamma(u_h^n, \psi_h).
\end{aligned}
$$

At each time step, we have the same, shared coefficient matrix for two sets of linear systems (for $(\overrightarrow{u}_j, p_j)$ and ϕ_j respectively):

$$
A\left[\begin{array}{c|c|c} u_1 & \cdots & u_J \\ p_1 & \cdots & p_J \end{array}\right] = [RHS_1 \,|\cdots|\, RHS_J], \qquad (4.8)
$$

$$
B\left[\begin{array}{c|c|c} \phi_1 & \cdots & \phi_J \end{array}\right] = \left[RHS_1^* \,|\cdots|\, RHS_J^*\right]. \qquad (4.9)
$$

This structure of the linear systems allows the use of efficient iterative solvers or direct solvers such as LU factorization for fast calculation. The two sets of linear systems (4.8) and (4.9) can also be run in parallel to reduce the computation time.

4.2.2. Long-time Stability and Error Analysis. Let $C_{P,f}$ and $C_{P,p}$ be the Poincar$\acute{e}$ constants of the indicated domains and $\bar{k}_{min}(x)$ be the minimum eigenvalue of the mean hydraulic conductivity tensor $\bar{\mathcal{K}}(x)$. Define $\bar{k}_{min} = \min_{x\in\Omega_p} \bar{k}_{min}(x)$ and two parameter-dependent constants

$$
C_1 = \frac{C_{P,f}^2[gC(D_f)C(D_p)]^4}{4\nu^2}, \qquad C_2 = \frac{C_{P,p}^2 g^2[C(D_f)C(D_p)]^4}{4\bar{k}_{min}^2}.
$$

For the coupling term $c_I(\overrightarrow{u},\phi)$, we have

Lemma 7 *For any $(\overrightarrow{u},\phi) \in X_S \times X_D$ and any $\epsilon_1, \epsilon_2, \alpha_1, \beta_1 > 0$,*

$$|c_I(\overrightarrow{u},\phi)| \le \frac{1}{4\epsilon_1}\|\phi\|_p^2 + \frac{\epsilon_1}{\alpha_1^2}C_1\|\nabla\phi\|_p^2 + \alpha_1\nu\|\nabla\overrightarrow{u}\|_f^2, \tag{4.10}$$

$$|c_I(\overrightarrow{u},\phi)| \le \frac{1}{4\epsilon_2}\|\overrightarrow{u}\|_f^2 + \frac{\epsilon_2}{\beta_1^2}C_2\|\nabla\overrightarrow{u}\|_f^2 + \beta_1 g\bar{k}_{min}\|\nabla\phi\|_p^2. \tag{4.11}$$

Proof 4 *The proof is similar to that in Layton* et al. *(2013). Using inequalities* (4.3) *and* (4.4) *as well as the inequality $abc \le \frac{1}{4}a^4 + \frac{1}{4}b^4 + c^2$, we have*

$$\begin{aligned}
c_\Gamma(\overrightarrow{u},\phi) &= g\int_\Gamma \phi\overrightarrow{u}\cdot\widehat{n}_f\, ds \le gC(D_f)C(D_p)\sqrt{\|\phi\|_p\|\nabla\phi\|_p}\sqrt{\|\overrightarrow{u}\|_f\|\nabla\overrightarrow{u}\|_f} \\
&\le \left(\frac{1}{\epsilon_1^{1/4}}\|\phi\|_p^{1/2}\right)\left(gC(D_f)C(D_p)\epsilon_1^{1/4}\frac{1}{\alpha_1^{1/2}\nu^{1/2}}C_{P,f}^{1/2}\|\nabla\phi\|_p^{1/2}\right)\left(\alpha_1^{1/2}\nu^{1/2}\|\nabla\overrightarrow{u}\|_f\right) \\
&\le \frac{1}{4\epsilon_1}\|\phi\|_p^2 + \frac{\epsilon_1}{\alpha_1^2}C_1\|\nabla\phi\|_p^2 + \alpha_1\nu\|\nabla\overrightarrow{u}\|_f^2,
\end{aligned}$$

and

$$\begin{aligned}
c_\Gamma(\overrightarrow{u},\phi) &= g\int_\Gamma \phi\overrightarrow{u}\cdot\widehat{n}_f\, ds \le gC(D_f)C(D_p)\sqrt{\|\phi\|_p\|\nabla\phi\|_p}\sqrt{\|\overrightarrow{u}\|_f\|\nabla\overrightarrow{u}\|_f} \\
&\le \left(\frac{1}{\epsilon_2^{1/4}}\|\overrightarrow{u}\|_f^{1/2}\right)\left(gC(D_f)C(D_p)\epsilon_2^{1/4}\frac{1}{\beta_1^{1/2}(g\bar{k}_{min})^{1/2}}C_{P,p}^{1/2}\|\nabla\overrightarrow{u}\|_f^{1/2}\right) \\
&\left(\beta_1^{1/2}(g\bar{k}_{min})^{1/2}\|\nabla\phi\|_p\right) \\
&\le \frac{1}{4\epsilon_2}\|\overrightarrow{u}\|_f^2 + \frac{\epsilon_2}{\beta_1^2}C_2\|\nabla\overrightarrow{u}\|_f^2 + \beta_1 g\bar{k}_{min}\|\nabla\phi\|_p^2.
\end{aligned}$$

Let $k_{j,min}(x)$, $\bar{k}_{min}(x)$ be the minimum eigenvalue of the hydraulic conductivity tensor $\mathcal{K}_j(x)$, $\bar{\mathcal{K}}(x)$, respectively, and $\rho'_j(x)$ be the spectral radius of the fluctuation of hydraulic conductivity tensor $\mathcal{K}_j(x) - \bar{\mathcal{K}}(x)$. Since both $\mathcal{K}_j(x)$ and $\bar{\mathcal{K}}(x)$ are symmetric, $|\mathcal{K}_j(x) - \bar{\mathcal{K}}(x)|_2 = \rho'_j(x)$.

We then define the following quantities that will be used in our proof:

$$\eta_{i,j}'^{max} = \max_{x\in\Gamma} \left|\eta_{i,j}(x) - \bar{\eta}_i(x)\right|, \qquad \eta_i'^{max} = \max_j \eta_{i,j}'^{max}, \qquad \bar{\eta}_i^{min} = \min_{x\in\Gamma} \bar{\eta}_i(x),$$

$$k_{j,min} = \min_{x\in D_p} k_{j,min}(x), \qquad k_{min} = \min_j k_{j,min}, \qquad \bar{k}_{min} = \min_{x\in D_p} \bar{k}_{min}(x),$$

$$\rho'_{j,max} = \max_{x\in D_p} \rho'_{j,max}(x), \qquad \rho'_{max} = \max_j \rho'_{j,max}.$$

We prove the long time stability of Algorithm 2 under a time-step condition and two parameter conditions

$$\Delta t \leq \min\left\{\frac{2(1-\alpha_1-\alpha_2)\beta_1^2}{[C(D_f)C(D_p)]^4 C_{P,p}^2}\frac{\nu\bar{k}_{min}^2}{g^2}, \frac{2(1-\beta_1-\beta_2-\frac{\rho'_{max}}{\bar{k}_{min}})\alpha_1^2}{[C(D_f)C(D_p)]^4 C_{P,f}^2}\frac{\nu^2\bar{k}_{min}S_0}{g^2}\right\}, \tag{4.12}$$

$$\eta_i'^{max} \leq \bar{\eta}_i^{min}, \tag{4.13}$$

$$\rho'_{max} < \bar{k}_{min}. \tag{4.14}$$

Remark 3 *The two parameter conditions* (4.13) *and* (4.14) *relate to the probability distribution of the random hydraulic conductivity tensor. They require that the magnitude of the fluctuations be smaller than the magnitude of the mean. In many applications, this can be easily achieved by dividing the ensemble of samples into smaller ensembles.*

Based on the above preparation, we will show the theories of long-time stability and error estimate for the proposed ensemble method. We will also propose an alternative ensemble method which is much more easier and prove the long-time stability.

Theorem 5 (Long time stability of Algorithm 2) *If the two parameter conditions* (4.13), (4.14) *hold, and there exist* $\alpha_1, \alpha_2, \beta_1, \beta_2$ *in* $(0, 1)$ *such that the time-step condition* (4.12) *also holds, then the Algorithm 2 is long time stable: for any* $N > 0$,

$$\begin{aligned}
&\frac{1}{2}\|\overrightarrow{u}^N_{j,h}\|_f^2 + \frac{gS_0}{2}\|\phi^N_{j,h}\|_p^2 + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla\overrightarrow{u}^N_{j,h}\|_f^2 + \Delta t\sum_i \frac{\bar{\eta}_i^{min}}{2}\int_\Gamma(\overrightarrow{u}^N_{j,h}\cdot\widehat{\tau}_i)^2\,ds \\
&+\left(\Delta t^2\frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \Delta t\frac{g\rho'_{max}}{2}\right)\|\nabla\phi^N_{j,h}\|_p^2 \\
&\le \frac{1}{2}\|\overrightarrow{u}^0_{j,h}\|_f^2 + \frac{gS_0}{2}\|\phi^0_{j,h}\|_p^2 + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla\overrightarrow{u}^0_{j,h}\|_f^2 + \Delta t\sum_i \frac{\bar{\eta}_i^{min}}{2}\int_\Gamma(\overrightarrow{u}^0_{j,h}\cdot\widehat{\tau}_i)^2\,ds \\
&+\left(\Delta t^2\frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \Delta t\frac{g\rho'_{max}}{2}\right)\|\nabla\phi^0_{j,h}\|_p^2 + \Delta t\sum_{n=0}^{N-1}\frac{C_{P,f}^2}{4\alpha_2\nu}\|f^{n+1}_{f,j}\|_f^2 + \Delta t\sum_{n=0}^{N-1}\frac{gC_{P,p}^2}{4\beta_2\bar{k}_{min}}\|f^{n+1}_{p,j}\|_p^2.
\end{aligned} \tag{4.15}$$

Proof 5 *Setting* $v_h = \overrightarrow{u}^{n+1}_{j,h}$, $\psi_h = \phi^{n+1}_{j,h}$ *in Algorithm 2 and adding all three equations yields*

$$\begin{aligned}
&\frac{1}{2\Delta t}\|\overrightarrow{u}^{n+1}_{j,h}\|_f^2 - \frac{1}{2\Delta t}\|\overrightarrow{u}^{n}_{j,h}\|_f^2 + \frac{1}{2\Delta t}\|\overrightarrow{u}^{n+1}_{j,h} - \overrightarrow{u}^{n}_{j,h}\|_f^2 + \nu\|\nabla\overrightarrow{u}^{n+1}_{j,h}\|_f^2 \\
&+\sum_i\int_\Gamma \bar{\eta}_i(\overrightarrow{u}^{n+1}_{j,h}\cdot\widehat{\tau}_i)(\overrightarrow{u}^{n+1}_{j,h}\cdot\widehat{\tau}_i)\,ds + \frac{gS_0}{2\Delta t}\|\phi^{n+1}_{j,h}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi^{n}_{j,h}\|_p^2 \\
&+\frac{gS_0}{2\Delta t}\|\phi^{n+1}_{j,h} - \phi^{n}_{j,h}\|_p^2 + g(\bar{\mathcal{K}}\nabla\phi^{n+1}_{j,h}, \nabla\phi^{n+1}_{j,h})_p + c_I(\overrightarrow{u}^{n+1}_{j,h}, \phi^n_{j,h}) - c_I(\overrightarrow{u}^{n}_{j,h}, \phi^{n+1}_{j,h}) \\
&= (f^{n+1}_{f,j}, \overrightarrow{u}^{n+1}_{j,h})_f + g(f^{n+1}_{p,j}, \phi^{n+1}_{j,h})_p - \sum_i\int_\Gamma(\eta_{i,j} - \bar{\eta}_i)(\overrightarrow{u}^{n}_{j,h}\cdot\widehat{\tau}_i)(\overrightarrow{u}^{n+1}_{j,h}\cdot\widehat{\tau}_i)\,ds \\
&\quad - g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla\phi^n_{j,h}, \nabla\phi^{n+1}_{j,h})_p.
\end{aligned} \tag{4.16}$$

Note that

$$\begin{aligned}
&c_I(\overrightarrow{u}^{n+1}_{j,h}, \phi^n_{j,h}) - c_I(\overrightarrow{u}^{n}_{j,h}, \phi^{n+1}_{j,h}) \\
&\quad = \left[c_I(\overrightarrow{u}^{n+1}_{j,h}, \phi^n_{j,h}) - c_I(\overrightarrow{u}^{n+1}_{j,h}, \phi^{n+1}_{j,h})\right] - \left[c_I(\overrightarrow{u}^{n}_{j,h}, \phi^{n+1}_{j,h}) - c_I(\overrightarrow{u}^{n+1}_{j,h}, \phi^{n+1}_{j,h})\right] \\
&\quad = -c_I(\overrightarrow{u}^{n+1}_{j,h}, \phi^{n+1}_{j,h} - \phi^n_{j,h}) + c_I(\overrightarrow{u}^{n+1}_{j,h} - \overrightarrow{u}^{n}_{j,h}, \phi^{n+1}_{j,h}).
\end{aligned} \tag{4.17}$$

Applying estimates (4.10) *and* (4.11) *with* $\epsilon_1 = \frac{\Delta t}{2gS_0}$, $\epsilon_2 = \frac{\Delta t}{2}$, *and if the time-step condition* (4.12) *holds, we have*

$$\begin{aligned}
&c_I(\overrightarrow{u}_{j,h}^{n+1} - \overrightarrow{u}_{j,h}^{n}, \phi_{j,h}^{n+1}) - c_I(\overrightarrow{u}_{j,h}^{n+1}, \phi_{j,h}^{n+1} - \phi_{j,h}^{n}) \\
&\geq -\frac{1}{2\Delta t}\|\overrightarrow{u}_{j,h}^{n+1} - \overrightarrow{u}_{j,h}^{n}\|_f^2 - \frac{\Delta t}{2}\frac{C_2}{\beta_1^2}\|\nabla(\overrightarrow{u}_{j,h}^{n+1} - \overrightarrow{u}_{j,h}^{n})\|_f^2 - \beta_1 g\bar{k}_{min}\|\nabla\phi_{j,h}^{n+1}\|_p^2 \\
&\quad - \frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n+1} - \phi_{j,h}^{n}\|_p^2 - \frac{\Delta t}{2gS_0}\frac{C_1}{\alpha_1^2}\|\nabla(\phi_{j,h}^{n+1} - \phi_{j,h}^{n})\|_p^2 - \alpha_1\nu\|\nabla\overrightarrow{u}_{j,h}^{n+1}\|_f^2 \\
&\geq -\frac{1}{2\Delta t}\|\overrightarrow{u}_{j,h}^{n+1} - \overrightarrow{u}_{j,h}^{n}\|_f^2 - \Delta t\frac{C_2}{\beta_1^2}\left(\|\nabla\overrightarrow{u}_{j,h}^{n+1}\|_f^2 + \|\nabla\overrightarrow{u}_{j,h}^{n}\|_f^2\right) - \beta_1 g\bar{k}_{min}\|\nabla\phi_{j,h}^{n+1}\|_p^2 \\
&\quad - \frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n+1} - \phi_{j,h}^{n}\|_p^2 - \frac{\Delta t}{gS_0}\frac{C_1}{\alpha_1^2}\left(\|\nabla\phi_{j,h}^{n+1}\|_p^2 + \|\nabla\phi_{j,h}^{n}\|_p^2\right) - \alpha_1\nu\|\nabla\overrightarrow{u}_{j,h}^{n+1}\|_f^2.
\end{aligned} \tag{4.18}$$

Applying Cauchy-Schwarz and Young's inequalities to the source terms, for any $\alpha_2 > 0, \beta_2 > 0$ *we have*

$$\begin{aligned}
&(f_{f,j}^{n+1}, \overrightarrow{u}_{j,h}^{n+1})_f + g(f_{p,j}^{n+1}, \phi_{j,h}^{n+1})_p \\
&\qquad \leq \|f_{f,j}^{n+1}\|_f\|\overrightarrow{u}_{j,h}^{n+1}\|_f + g\|f_{p,j}^{n+1}\|_p\|\phi_{j,h}^{n+1}\|_p \\
&\qquad \leq C_{P,f}\|f_{f,j}^{n+1}\|_f\|\nabla\overrightarrow{u}_{j,h}^{n+1}\|_f + gC_{P,p}\|f_{p,j}^{n+1}\|_p\|\nabla\phi_{j,h}^{n+1}\|_p \\
&\qquad \leq \frac{C_{P,f}^2}{4\alpha_2\nu}\|f_{f,j}^{n+1}\|_f^2 + \alpha_2\nu\|\nabla\overrightarrow{u}_{j,h}^{n+1}\|_f^2 + \frac{gC_{P,p}^2}{4\beta_2\bar{k}_{min}}\|f_{p,j}^{n+1}\|_p^2 + \beta_2 g\bar{k}_{min}\|\nabla\phi_{j,h}^{n+1}\|_p^2.
\end{aligned} \tag{4.19}$$

By Young's inequalities, we have

$$\begin{aligned}
&-\sum_i\int_\Gamma(\eta_{i,j} - \bar{\eta}_i)(\overrightarrow{u}_{j,h}^{n}\cdot\widehat{\tau}_i)(\overrightarrow{u}_{j,h}^{n+1}\cdot\widehat{\tau}_i)\,ds \\
&\qquad\qquad \leq \sum_i\int_\Gamma\left|\eta_{i,j} - \bar{\eta}_i\right|\left|(\overrightarrow{u}_{j,h}^{n}\cdot\widehat{\tau}_i)(\overrightarrow{u}_{j,h}^{n+1}\cdot\widehat{\tau}_i)\right|ds \\
&\qquad\qquad \leq \sum_i\eta_{i,j}'^{max}\int_\Gamma\left|(\overrightarrow{u}_{j,h}^{n}\cdot\widehat{\tau}_i)(\overrightarrow{u}_{j,h}^{n+1}\cdot\widehat{\tau}_i)\right|\,ds, \\
&\qquad\qquad \leq \sum_i\left[\frac{\eta_i'^{max}}{2}\int_\Gamma(\overrightarrow{u}_{j,h}^{n}\cdot\widehat{\tau}_i)^2\,ds + \frac{\eta_i'^{max}}{2}\int_\Gamma(\overrightarrow{u}_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2\,ds\right],
\end{aligned} \tag{4.20}$$

For the right-hand side of (4.16)*, the last term can be bounded as follows:*

$$
\begin{aligned}
-g\left((\mathcal{K}_j-\bar{\mathcal{K}})\nabla\phi^n_{j,h},\nabla\phi^{n+1}_{j,h}\right)_p &\le g\int_{D_p}|\nabla\phi^{n+1}_{j,h}|_2|\mathcal{K}_j-\bar{\mathcal{K}}|_2|\nabla\phi^n_{j,h}|_2\,dx\\
&\le g\int_{D_p}\rho'_j(x)|\nabla\phi^{n+1}_{j,h}|_2|\nabla\phi^n_{j,h}|_2\,dx\\
&\le g\rho'_{j,max}\int_{D_p}|\nabla\phi^{n+1}_{j,h}|_2|\nabla\phi^n_{j,h}|_2\,dx\\
&\le g\rho'_{j,max}\|\nabla\phi^n_{j,h}\|_p\|\nabla\phi^{n+1}_{j,h}\|_p\\
&\le \frac{g\rho'_{max}}{2}\|\nabla\phi^n_{j,h}\|_p^2+\frac{g\rho'_{max}}{2}\|\nabla\phi^{n+1}_{j,h}\|_p^2.
\end{aligned}
$$

Using the above estimates, equation (4.16) *becomes*

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|\overrightarrow{u}^{n+1}_{j,h}\|_f^2-\frac{1}{2\Delta t}\|\overrightarrow{u}^{n}_{j,h}\|_f^2+\left(1-\alpha_1-\alpha_2-\Delta t\frac{2C_2}{\beta_1^2\nu}\right)\nu\|\nabla\overrightarrow{u}^{n+1}_{j,h}\|_f^2 \qquad (4.21)\\
&+\Delta t\frac{C_2}{\beta_1^2}\left(\|\nabla\overrightarrow{u}^{n+1}_{j,h}\|_f^2-\|\nabla\overrightarrow{u}^{n}_{j,h}\|_f^2\right)+\sum_i\left[\frac{\bar{\eta}_i^{min}}{2}-\frac{\eta_i'^{max}}{2}\right]\int_\Gamma(\overrightarrow{u}^{n+1}_{j,h}\cdot\widehat{\tau}_i)^2\,ds\\
&+\sum_i\frac{\bar{\eta}_i^{min}}{2}\left[\int_\Gamma(\overrightarrow{u}^{n+1}_{j,h}\cdot\widehat{\tau}_i)^2\,ds-\int_\Gamma(\overrightarrow{u}^{n}_{j,h}\cdot\widehat{\tau}_i)^2\,ds\right]\\
&+\sum_i\left[\frac{\bar{\eta}_i^{min}}{2}-\frac{\eta_i'^{max}}{2}\right]\int_\Gamma(\overrightarrow{u}^{n}_{j,h}\cdot\widehat{\tau}_i)^2\,ds+\frac{gS_0}{2\Delta t}\|\phi^{n+1}_{j,h}\|_p^2-\frac{gS_0}{2\Delta t}\|\phi^{n}_{j,h}\|_p^2\\
&+(1-\beta_1-\beta_2-\Delta t\frac{1}{g^2S_0\bar{k}_{min}}\frac{2C_1}{\alpha_1^2}-\frac{\rho'_{max}}{\bar{k}_{min}})g\bar{k}_{min}\|\nabla\phi^{n+1}_{j,h}\|_p^2\\
&+\left(\Delta t\frac{1}{gS_0}\frac{C_1}{\alpha_1^2}+\frac{g\rho'_{max}}{2}\right)\left(\|\nabla\phi^{n+1}_{j,h}\|_p^2-\|\nabla\phi^{n}_{j,h}\|_p^2\right)\\
&\le\frac{C_{P,f}^2}{4\alpha_2\nu}\|f^{n+1}_{f,j}\|_f^2+\frac{gC_{P,p}^2}{4\beta_2\bar{k}_{min}}\|f^{n+1}_{p,j}\|_p^2.
\end{aligned}
$$

To obtain stability, we need

$$1-\alpha_1-\alpha_2-\Delta t\frac{2C_2}{\beta_1^2\nu}\ge 0,\quad \frac{\bar{\eta}_i^{min}}{2}-\frac{\eta_i'^{max}}{2}\ge 0, \qquad (4.22)$$

$$1-\beta_1-\beta_2-\Delta t\frac{1}{g^2S_0\bar{k}_{min}}\frac{2C_1}{\alpha_1^2}-\frac{\rho'_{max}}{\bar{k}_{min}}\ge 0. \qquad (4.23)$$

Recall that $\alpha_1, \alpha_2, \beta_1, \beta_2, \Delta t, \eta_i'^{max}, \rho'_{max}$ *are all positive; we then have the following constraints on these parameters.*

$$0 < \alpha_1 < 1, \quad 0 < \alpha_2 < 1, \qquad 0 < \beta_1 < 1, \quad 0 < \beta_2 < 1, \tag{4.24}$$

$$\frac{\rho'_{max}}{\bar{k}_{min}} < 1, \quad \eta_i'^{max} \le \bar{\eta}_i^{min}, \tag{4.25}$$

$$\begin{aligned} \Delta t &\le \min\left\{ \frac{(1-\alpha_1-\alpha_2)\beta_1^2 \nu}{2C_2}, \frac{(1-\beta_1-\beta_2-\frac{\rho'_{max}}{\bar{k}_{min}})\alpha_1^2 g^2 S_0 \bar{k}_{min}}{2C_1} \right\} \\ &\le \min\left\{ \frac{2(1-\alpha_1-\alpha_2)\beta_1^2}{[C(D_f)C(D_p)]^4 C_{P,p}^2}\frac{\nu \bar{k}_{min}^2}{g^2}, \frac{2(1-\beta_1-\beta_2-\frac{\rho'_{max}}{\bar{k}_{min}})\alpha_1^2}{[C(D_f)C(D_p)]^4 C_{P,f}^2}\frac{\nu^2 \bar{k}_{min} S_0}{g^2} \right\}. \end{aligned} \tag{4.26}$$

(4.25) *leads to the two parameter conditions* (4.13) *and* (4.14)*, and* (4.26) *leads to the time step condition* (4.12) *required for stability. Now if the time step condition* (4.12) *and the two parameter conditions* (4.13) *and* (4.14) *all hold,* (4.21) *reduces to*

$$\begin{aligned} &\frac{1}{2\Delta t}\|\vec{u}_{j,h}^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|\vec{u}_{j,h}^{n}\|_f^2 + \Delta t \frac{C_2}{\beta_1^2}\left(\|\nabla \vec{u}_{j,h}^{n+1}\|_f^2 - \|\nabla \vec{u}_{j,h}^{n}\|_f^2\right) \\ &+ \sum_i \frac{\bar{\eta}_i^{min}}{2}\left[\int_\Gamma (\vec{u}_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2\,ds - \int_\Gamma (\vec{u}_{j,h}^{n}\cdot\widehat{\tau}_i)^2\,ds\right] + \frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n}\|_p^2 \\ &+ \left(\Delta t \frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \frac{g\rho'_{max}}{2}\right)\left(\|\nabla\phi_{j,h}^{n+1}\|_p^2 - \|\nabla\phi_{j,h}^{n}\|_p^2\right) \\ &\le \frac{C_{P,f}^2}{4\alpha_2\nu}\|f_{f,j}^{n+1}\|_f^2 + \frac{gC_{P,p}^2}{4\beta_2\bar{k}_{min}}\|f_{p,j}^{n+1}\|_p^2. \end{aligned} \tag{4.27}$$

Sum (4.27) *from* $n = 0$ *to* $N-1$ *and multiply through by* Δt *to get*

$$\begin{aligned} &\frac{1}{2}\|\vec{u}_{j,h}^{N}\|_f^2 + \frac{gS_0}{2}\|\phi_{j,h}^{N}\|_p^2 + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla\vec{u}_{j,h}^{N}\|_f^2 + \Delta t\sum_i \frac{\bar{\eta}_i^{min}}{2}\int_\Gamma (\vec{u}_{j,h}^{N}\cdot\widehat{\tau}_i)^2\,ds \\ &+ \left(\Delta t^2\frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \Delta t\frac{g\rho'_{max}}{2}\right)\|\nabla\phi_{j,h}^{N}\|_p^2 \\ &\le \frac{1}{2}\|\vec{u}_{j,h}^{0}\|_f^2 + \frac{gS_0}{2}\|\phi_{j,h}^{0}\|_p^2 + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla\vec{u}_{j,h}^{0}\|_f^2 + \Delta t\sum_i \frac{\bar{\eta}_i^{min}}{2}\int_\Gamma (\vec{u}_{j,h}^{0}\cdot\widehat{\tau}_i)^2\,ds \\ &+ \left(\Delta t^2\frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \Delta t\frac{g\rho'_{max}}{2}\right)\|\nabla\phi_{j,h}^{0}\|_p^2 + \Delta t\sum_{n=0}^{N-1}\frac{C_{P,f}^2}{4\alpha_2\nu}\|f_{f,j}^{n+1}\|_f^2 + \Delta t\sum_{n=0}^{N-1}\frac{gC_{P,p}^2}{4\beta_2\bar{k}_{min}}\|f_{p,j}^{n+1}\|_p^2. \end{aligned} \tag{4.28}$$

The error analysis for Algorithm 2 can be done similarly with minor modification. We assume the finite element spaces satisfy the approximation properties of piecewise polynomials on quasiuniform meshes:

$$\inf_{v_h \in X_{Sh}} \|v - v_h\|_f \leq Ch^{k+1} \|\overrightarrow{u}\|_{H^{k+1}(D_f)} \qquad \forall v \in [H^{k+1}(D_f)]^d, \tag{4.29}$$

$$\inf_{v_h \in X_{Sh}} \|\nabla(v - v_h)\|_f \leq Ch^{k} \|v\|_{H^{k+1}(D_f)} \qquad \forall v \in [H^{k+1}(D_f)]^d, \tag{4.30}$$

$$\inf_{q_h \in Q_{Sh}} \|q - q_h\|_f \leq Ch^{s+1} \|q\|_{H^{s+1}(D_f)} \qquad \forall q \in H^{s+1}(D_f), \tag{4.31}$$

$$\inf_{\psi_h \in X_{Dh}} \|\psi - \psi_h\|_p \leq Ch^{m+1} \|\psi\|_{H^{m+1}(D_p)} \qquad \forall \psi \in H^{m+1}(D_p), \tag{4.32}$$

$$\inf_{\psi_h \in X_{Dh}} \|\nabla(\psi - \psi_h)\|_p \leq Ch^{m} \|\psi\|_{H^{m+1}(D_p)} \qquad \forall \psi \in H^{m+1}(D_p), \tag{4.33}$$

where the generic constant $C > 0$ is independent of the mesh size h. An example for which both the LBB^h stability condition and the approximation properties are satisfied is the finite elements $(P_{l+1}–P_l–P_{l+1})$, $l \geq 1$; see Girault and Raviart (1979); Gunzburger (1989b); Layton (2008) for more details.

We also assume the following regularity on the true solution of the Stokes-Darcy equations:

$$\begin{aligned}
&\overrightarrow{u}_j \in L^\infty(0,T;H^{k+1}(D_f)), u_{j,t} \in L^2(0,T;H^{k+1}(D_f)), u_{j,tt} \in L^2(0,T;L^2(D_f)),\\
&\phi_j \in L^\infty(0,T;H^{m+1}(D_p)), \phi_{j,t} \in L^2(0,T;H^{m+1}(D_p)), \phi_{j,tt} \in L^2(0,T;L^2(D_p)),\\
&p_j \in L^2(0,T;H^{s+1}(D_f)).
\end{aligned}$$

Let $\overrightarrow{u}_j^n = \overrightarrow{u}_j(x,t_n), p_j^n = p_j(x,t_n), \phi_j^n = \phi_j(x,t_n)$. Denote the errors by $e_{j,\overrightarrow{u}}^n := \overrightarrow{u}_j^n - \overrightarrow{u}_{j,h}^n, e_{j,\phi}^n := \phi_j^n - \phi_{j,h}^n$.

We prove the convergence of Algorithm 2 under a time-step condition and two parameter conditions:

$$\Delta t \leq \min\left\{\frac{(1-\alpha_1-\alpha_2)\beta_1^2\bar{k}_{min}}{C_{P,p}^2}, \frac{(1-\sigma_4-\beta_1-\beta_2-(1+\sigma_3)\frac{\rho'_{max}}{\bar{k}_{min}})\alpha_1^2 S_0 \nu}{C_{P,f}^2}\right\} \quad (4.34)$$

$$\frac{2\nu\bar{k}_{min}}{g^2[C(D_f)C(D_p)]^4},$$

$$\eta_i^{\prime max} \leq \bar{\eta}_i^{min}, \quad (4.35)$$

$$\rho'_{max} < \bar{k}_{min}. \quad (4.36)$$

Note that the two parameter conditions are the same as those for stability. The time step condition is slightly different with two extra constants, $\sigma_3 > 0$ and $\sigma_4 \in (0, 1)$.

Theorem 6 (Error Estimate) *For any $j = 1, \ldots, J$, if the two parameter conditions* (4.35) *and* (4.36) *hold, and there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \sigma_4 \in (0, 1)$ and $\sigma_3 > 0$ such that the time step condition* (4.34) *also holds, then there is a positive constant C independent of the time step Δt and mesh size h such that*

$$\begin{aligned}
&\frac{1}{2}\|e^N_{j,\vec{u}}\|_f^2 + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla e^N_{j,\vec{u}}\|_f^2 + \frac{gS_0}{2}\|e^N_{j,\phi}\|_p^2 + \Delta t\left(\frac{1}{2}g\rho'_{max} + \frac{\Delta t C_1}{gS_0\alpha_1^2}\right)\|\nabla e^N_{j,\phi}\|_p^2 \quad (4.37)\\
&\leq \frac{1}{2}\|e^0_{j,\vec{u}}\|_f^2 + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla e^0_{j,\vec{u}}\|_f^2 + \Delta t\sum_i \frac{1}{2}\eta_i^{\prime max}\int_\Gamma (e^0_{j,\vec{u}}\cdot\widehat{\tau}_i)^2\, ds + \frac{gS_0}{2}\|e^0_{j,\phi}\|_p^2\\
&+ \Delta t\left(\frac{1}{2}g\rho'_{max} + \frac{\Delta t C_1}{gS_0\alpha_1^2}\right)\|\nabla e^0_{j,\phi}\|_p^2 + C\Delta t^2\|u_{j,t}\|_{2,1,f}^2 + Ch^{2k}\|\!|\vec{u}_j|\!\|_{2,k+1,f}^2 + C\Delta t^2\|u_{j,tt}\|_{2,0,f}^2\\
&+ C\Delta t^2\|\phi_{j,t}\|_{2,1,p}^2 + C\Delta t^2\|\phi_{j,tt}\|_{2,0,p}^2 + Ch^{2k+2}\|u_{j,t}\|_{2,k+1,f} + Ch^{2k+2}\|\phi_{j,t}\|_{2,m+1,p}\\
&+ Ch^{2k}\|\!|\phi_j|\!\|_{2,m+1,p}^2 + Ch^{2s+2}\|\!|p_j|\!\|_{2,s+1,f}^2 + Ch^{2k+2}\|\!|\vec{u}_j|\!\|_{\infty,k+1,f} + C\Delta t^2h^{2k}\|\!|\vec{u}_j|\!\|_{\infty,k+1,f}\\
&+ Ch^{2k+2}\|\!|\phi_j|\!\|_{\infty,m+1,p} + C\Delta t h^{2k}\|\!|\phi_j|\!\|_{\infty,m+1,p}.
\end{aligned}$$

In particular, if Taylor-Hood elements ($k = 2$, $s = 1$) are used for approximating $(\overrightarrow{u}_j, p_j)$, i.e., the C^0 piecewise-quadratic velocity space X_{Sh} and the C^0 piecewise-linear pressure space Q_S^h, and P_2 element ($m = 2$) is used for X_{Dh}, we have the following estimate.

Corollary 1 *Assume that $\|e^0_{j,\overrightarrow{u}}\|$, $\|\nabla e^0_{j,\overrightarrow{u}}\|$, $\|e^0_{j,\phi}\|$ and $\|\nabla e^0_{j,\phi}\|$ are all $O(h^2)$ accurate or better. Then, if (X_{Sh}, Q_S^h, X_{Dh}) are chosen as the (P_2, P_1, P_2) elements, we have*

$$\frac{1}{2}\|e^N_{j,\overrightarrow{u}}\|_f^2 + \Delta t^2 \frac{C_2}{\beta_1^2}\|\nabla e^N_{j,\overrightarrow{u}}\|_f^2 + \frac{gS_0}{2}\|e^N_{j,\phi}\|_p^2 + \Delta t\left(\frac{1}{2}g\rho'_{max} + \frac{\Delta t C_1}{gS_0\alpha_1^2}\right)\|\nabla e^N_{j,\phi}\|_p^2 \leq C(h^4 + \Delta t^2)\,.$$

Proof 6 *(of Theorem 6) For $\forall v_h \in V_f^h, \forall \psi_h \in X_{Dh}, \forall \lambda_h^{n+1} \in Q_{Sh}$, the true solution $(\overrightarrow{u}_j, p_j, \phi_j)$ satisfies*

$$\begin{aligned}
&\left(\frac{\overrightarrow{u}_j^{n+1} - \overrightarrow{u}_j^n}{\Delta t}, v_h\right)_f + \nu(\nabla \overrightarrow{u}_j^{n+1}, \nabla v_h)_f + \sum_i \int_\Gamma \eta_{i,j}(\overrightarrow{u}_j^{n+1}\cdot\widehat{\tau_i})(v_h\cdot\widehat{\tau_i})\,ds \\
&-\left(p_j^{n+1} - \lambda_h^{n+1}, \nabla\cdot v_h\right)_f + c_\Gamma(v_h, \phi_j^n) = (f_{f,j}^{n+1}, v_h)_f + \epsilon_{j,f}^{n+1}(v_h), \\
&gS_0\left(\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t}, \psi_h\right)_p + g(\mathcal{K}_j\nabla\phi_j^{n+1}, \nabla\psi_h)_p - c_\Gamma(\overrightarrow{u}_j^n, \psi_h) \\
&= g(f_{p,j}^{n+1}, \psi_h)_p + \epsilon_{j,p}^{n+1}(\psi_h).
\end{aligned} \tag{4.38}$$

The consistency errors $\epsilon_{j,f}^{n+1}(v_h), \epsilon_{j,p}^{n+1}(\psi_h)$ are defined by

$$\begin{aligned}
\epsilon_{j,f}^{n+1}(v_h) &:= \left(\frac{\overrightarrow{u}_j^{n+1} - \overrightarrow{u}_j^n}{\Delta t} - u_{j,t}^{n+1}, v_h\right)_f - c_\Gamma(v_h, \phi_j^{n+1} - \phi_j^n), \\
\epsilon_{j,p}^{n+1}(\psi_h) &:= gS_0\left(\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} - \phi_{j,t}^{n+1}, \psi_h\right)_p + c_\Gamma(\overrightarrow{u}_j^{n+1} - \overrightarrow{u}_j^n, \psi_h).
\end{aligned}$$

Subtracting (4.6) *from* (4.38) *gives, for* $\forall v_h \in V_f^h, \forall \psi_h \in X_{Dh}, \forall \lambda_h^{n+1} \in Q_{Sh}$,

$$\begin{aligned}
&\left(\frac{e^{n+1}_{j,\overrightarrow{u}} - e^{n}_{j,\overrightarrow{u}}}{\Delta t}, v_h\right)_f + \nu(\nabla e^{n+1}_{j,\overrightarrow{u}}, \nabla v_h)_f + \sum_i \int_\Gamma \bar{\eta}_i (e^{n+1}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)(v_h \cdot \widehat{\tau}_i)\, ds \\
&+ \sum_i \int_\Gamma \left(\eta_{i,j} - \bar{\eta}_i\right) (e^{n}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)(v_h \cdot \widehat{\tau}_i)\, ds - \left(p_j^{n+1} - \lambda_h^{n+1}, \nabla \cdot v_h\right)_f + c_\Gamma(v_h, e^n_{j,\phi}) \\
&= -\sum_i \int_\Gamma \left(\eta_{i,j} - \bar{\eta}_i\right) ((\overrightarrow{u}_j^{n+1} - \overrightarrow{u}_j^{n}) \cdot \widehat{\tau}_i)(v_h \cdot \widehat{\tau}_i)\, ds + \epsilon^{n+1}_{j,f}(v_h), \\
&g S_0 \left(\frac{e^{n+1}_{j,\phi} - e^{n}_{j,\phi}}{\Delta t}, \psi_h\right)_p + g(\bar{\mathcal{K}} \nabla e^{n+1}_{j,\phi}, \nabla \psi_h)_p + g((\mathcal{K}_j - \bar{\mathcal{K}}) \nabla e^{n}_{j,\phi}, \nabla \psi_h)_p \qquad (4.39) \\
&- c_\Gamma(e^n_{j,\overrightarrow{u}}, \psi_h) = -g((\mathcal{K}_j - \bar{\mathcal{K}}) \nabla(\phi_j^{n+1} - \phi_j^n), \nabla \psi_h)_p + \epsilon^{n+1}_{j,p}(\psi_h).
\end{aligned}$$

Let U_j^{n+1}, Φ_j^{n+1} *be an interpolation of* $\overrightarrow{u}_j^{n+1}$ *and* ϕ_j^{n+1} *in* V_f^h *and* X_p^h *correspondingly. Denote*

$$\begin{aligned}
e^{n+1}_{j,\overrightarrow{u}} &= \left(\overrightarrow{u}_j^{n+1} - U_j^{n+1}\right) + \left(U_j^{n+1} - \overrightarrow{u}_{j,h}^{n+1}\right) =: \mu^{n+1}_{j,\overrightarrow{u}} + \xi^{n+1}_{j,\overrightarrow{u}}, \\
e^{n+1}_{j,\phi} &= \left(\phi_j^{n+1} - \Phi_j^{n+1}\right) + \left(\Phi_j^{n+1} - \phi_{j,h}^{n+1}\right) =: \mu^{n+1}_{j,\phi} + \xi^{n+1}_{j,\phi}.
\end{aligned}$$

Then we can rewrite the (4.39) *as two equations. The first one is*

$$\begin{aligned}
&\left(\frac{\xi^{n+1}_{j,\overrightarrow{u}} - \xi^{n}_{j,\overrightarrow{u}}}{\Delta t}, v_h\right)_f + \nu(\nabla \xi^{n+1}_{j,\overrightarrow{u}}, \nabla v_h)_f + \sum_i \int_\Gamma \bar{\eta}_i (\xi^{n+1}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)(v_h \cdot \widehat{\tau}_i)\, ds \\
&\quad + \sum_i \int_\Gamma \left(\eta_{i,j} - \bar{\eta}_i\right) (\xi^{n}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)(v_h \cdot \widehat{\tau}_i)\, ds - \left(p_j^{n+1} - \lambda_h^{n+1}, \nabla \cdot v_h\right)_f + c_\Gamma(v_h, \xi^n_{j,\phi}) \\
&= -\sum_i \int_\Gamma \left(\eta_{i,j} - \bar{\eta}_i\right) ((\overrightarrow{u}_j^{n+1} - \overrightarrow{u}_j^{n}) \cdot \widehat{\tau}_i)(v_h \cdot \widehat{\tau}_i)\, ds + \epsilon^{n+1}_{j,f}(v_h), \\
&\quad - \left(\frac{\mu^{n+1}_{j,\overrightarrow{u}} - \mu^{n}_{j,\overrightarrow{u}}}{\Delta t}, v_h\right)_f - \nu(\nabla \mu^{n+1}_{j,\overrightarrow{u}}, \nabla v_h)_f - \sum_i \int_\Gamma \bar{\eta}_i (\mu^{n+1}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)(v_h \cdot \widehat{\tau}_i)\, ds \\
&\quad - \sum_i \int_\Gamma \left(\eta_{i,j} - \bar{\eta}_i\right) (\mu^{n}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)(v_h \cdot \widehat{\tau}_i)\, ds - c_\Gamma(v_h, \mu^n_{j,\phi}). \qquad (4.40)
\end{aligned}$$

And the second equation in the (4.39) *can be rewritten as*

$$
\begin{aligned}
&gS_0\left(\frac{\xi_{j,\phi}^{n+1}-\xi_{j,\phi}^{n}}{\Delta t},\psi_h\right)_p + g(\bar{\mathcal{K}}\nabla\xi_{j,\phi}^{n+1},\nabla\psi_h)_p + g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla\xi_{j,\phi}^{n},\nabla\psi_h)_p - c_\Gamma(\xi_{j,\overrightarrow{u}}^{n},\psi_h)\\
&= -g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla(\phi_j^{n+1}-\phi_j^{n}),\nabla\psi_h)_p + \epsilon_{j,p}^{n+1}(\psi_h) - gS_0\left(\frac{\mu_{j,\phi}^{n+1}-\mu_{j,\phi}^{n}}{\Delta t},\psi_h\right)_p\\
&\quad - g(\bar{\mathcal{K}}\nabla\mu_{j,\phi}^{n+1},\nabla\psi_h)_p - g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla\mu_{j,\phi}^{n},\nabla\psi_h)_p + c_\Gamma(\mu_{j,\overrightarrow{u}}^{n},\psi_h).
\end{aligned}
$$

Letting $v_h = \xi_{j,\overrightarrow{u}}^{n+1}, \psi_h = \xi_{j,\phi}^{n+1}$ *in* (4.40) *and adding the two equations yields*

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|\xi_{j,\overrightarrow{u}}^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|\xi_{j,\overrightarrow{u}}^{n}\|_f^2 + \frac{1}{2\Delta t}\|\xi_{j,\overrightarrow{u}}^{n+1}-\xi_{j,\overrightarrow{u}}^{n}\|_f^2 + \nu\|\nabla\xi_{j,\overrightarrow{u}}^{n+1}\|_f^2\\
&+\sum_i\int_\Gamma \bar{\eta}_i(\xi_{j,\overrightarrow{u}}^{n+1}\cdot\widehat{\tau}_i)^2\,ds + \frac{gS_0}{2\Delta t}\|\xi_{j,\phi}^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\xi_{j,\phi}^{n}\|_p^2 + \frac{gS_0}{2\Delta t}\|\xi_{j,\phi}^{n+1}-\xi_{j,\phi}^{n}\|_p^2\\
&+g(\bar{\mathcal{K}}\nabla\xi_{j,\phi}^{n+1},\nabla\xi_{j,\phi}^{n+1})_p + c_I(\xi_{j,\overrightarrow{u}}^{n+1},\xi_{j,\phi}^{n}) - c_I(\xi_{j,\overrightarrow{u}}^{n},\xi_{j,\phi}^{n+1})\\
&= -\sum_i\int_\Gamma\left(\eta_{i,j}-\bar{\eta}_i\right)(\xi_{j,\overrightarrow{u}}^{n}\cdot\widehat{\tau}_i)(\xi_{j,\overrightarrow{u}}^{n+1}\cdot\widehat{\tau}_i)\,ds\\
&-\sum_i\int_\Gamma\left(\eta_{i,j}-\bar{\eta}_i\right)((\overrightarrow{u}_j^{n+1}-\overrightarrow{u}_j^{n})\cdot\widehat{\tau}_i)(\xi_{j,\overrightarrow{u}}^{n+1}\cdot\widehat{\tau}_i)\,ds\\
&+\epsilon_{j,f}^{n+1}(\xi_{j,\overrightarrow{u}}^{n+1}) - \left(\frac{\mu_{j,\overrightarrow{u}}^{n+1}-\mu_{j,\overrightarrow{u}}^{n}}{\Delta t},\xi_{j,\overrightarrow{u}}^{n+1}\right)_f - \nu(\nabla\mu_{j,\overrightarrow{u}}^{n+1},\nabla\xi_{j,\overrightarrow{u}}^{n+1})_f\\
&-\sum_i\int_\Gamma\bar{\eta}_i(\mu_{j,\overrightarrow{u}}^{n+1}\cdot\widehat{\tau}_i)(\xi_{j,\overrightarrow{u}}^{n+1}\cdot\widehat{\tau}_i)\,ds - \sum_i\int_\Gamma\left(\eta_{i,j}-\bar{\eta}_i\right)(\mu_{j,\overrightarrow{u}}^{n}\cdot\widehat{\tau}_i)(\xi_{j,\overrightarrow{u}}^{n+1}\cdot\widehat{\tau}_i)\,ds\\
&-c_\Gamma(\xi_{j,\overrightarrow{u}}^{n+1},\mu_{j,\phi}^{n}) + \left(p_j^{n+1}-\lambda_h^{n+1},\nabla\cdot\xi_{j,\overrightarrow{u}}^{n+1}\right)_f - g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla(\phi_j^{n+1}-\phi_j^{n}),\nabla\xi_{j,\phi}^{n+1})_p\\
&+\epsilon_{j,p}^{n+1}(\xi_{j,\phi}^{n+1}) - gS_0\left(\frac{\mu_{j,\phi}^{n+1}-\mu_{j,\phi}^{n}}{\Delta t},\xi_{j,\phi}^{n+1}\right)_p - g(\bar{\mathcal{K}}\nabla\mu_{j,\phi}^{n+1},\nabla\xi_{j,\phi}^{n+1})_p\\
&-g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla\mu_{j,\phi}^{n},\nabla\xi_{j,\phi}^{n+1})_p + c_\Gamma(\mu_{j,\overrightarrow{u}}^{n},\xi_{j,\phi}^{n+1}) - g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla\xi_{j,\phi}^{n},\nabla\xi_{j,\phi}^{n+1})_p.
\end{aligned}
\tag{4.41}
$$

Using the same techniques in the stability proof (see (4.17) *and* (4.18)*), we have*

$$
\begin{aligned}
&c_I(\xi^{n+1}_{j,\vec{u}}, \xi^{n}_{j,\phi}) - c_I(\xi^{n}_{j,\vec{u}}, \xi^{n+1}_{j,\phi}) \qquad (4.42)\\
&= c_I(\xi^{n+1}_{j,\vec{u}} - \xi^{n}_{j,\vec{u}}, \xi^{n+1}_{j,\phi}) - c_I(\xi^{n+1}_{j,\vec{u}}, \xi^{n+1}_{j,\phi} - \xi^{n}_{j,\phi})\\
&\geq -\frac{1}{2\Delta t}\|\xi^{n+1}_{j,\vec{u}} - \xi^{n}_{j,\vec{u}}\|_f^2 - \Delta t\frac{C_2}{\beta_1^2}\left(\|\nabla\xi^{n+1}_{j,\vec{u}}\|_f^2 + \|\nabla\xi^{n}_{j,\vec{u}}\|_f^2\right) - \beta_1 g\bar{k}_{min}\|\nabla\xi^{n+1}_{j,\phi}\|_p^2\\
&\quad - \frac{gS_0}{2\Delta t}\|\xi^{n+1}_{j,\phi} - \xi^{n}_{j,\phi}\|_p^2 - \frac{\Delta t}{gS_0}\frac{C_1}{\alpha_1^2}\left(\|\nabla\xi^{n+1}_{j,\phi}\|_p^2 + \|\nabla\xi^{n}_{j,\phi}\|_p^2\right) - \alpha_1\nu\|\nabla\xi^{n+1}_{j,\vec{u}}\|_f^2.
\end{aligned}
$$

Next we bound the terms on the right-hand side of (4.41).

$$
\begin{aligned}
&-\left(\frac{\mu^{n+1}_{j,\vec{u}} - \mu^{n}_{j,\vec{u}}}{\Delta t}, \xi^{n+1}_{j,\vec{u}}\right)_f - gS_0\left(\frac{\mu^{n+1}_{j,\phi} - \mu^{n}_{j,\phi}}{\Delta t}, \xi^{n+1}_{j,\phi}\right)_p \qquad (4.43)\\
&\leq \frac{5C_{P,f}^2}{4\alpha_2\nu}\|\frac{\mu^{n+1}_{j,\vec{u}} - \mu^{n}_{j,\vec{u}}}{\Delta t}\|_f^2 + \frac{\alpha_2}{5}\nu\|\nabla\xi^{n+1}_{j,\vec{u}}\|_f^2 + \frac{C_{P,p}^2 gS_0^2}{\beta_2\bar{k}_{min}}\|\frac{\mu^{n+1}_{j,\phi} - \mu^{n}_{j,\phi}}{\Delta t}\|_p^2 + \frac{\beta_2}{4}g\bar{k}_{min}\|\nabla\xi^{n+1}_{j,\phi}\|_p^2\\
&\leq \frac{5C_{P,f}^2}{4\alpha_2\nu}\left\|\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\mu_{j,\vec{u},t}\,dt\right\|_f^2 + \frac{\alpha_2}{5}\nu\|\nabla\xi^{n+1}_{j,\vec{u}}\|_f^2 + \frac{C_{P,p}^2 gS_0^2}{\beta_2\bar{k}_{min}}\left\|\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\mu_{j,\phi,t}\,dt\right\|_p^2\\
&\quad + \frac{\beta_2}{4}g\bar{k}_{min}\|\nabla\xi^{n+1}_{j,\phi}\|_p^2\\
&\leq \frac{5C_{P,f}^2}{4\alpha_2\nu}\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\|\mu_{j,\vec{u},t}\|_f^2\,dt + \frac{\alpha_2}{5}\nu\|\nabla\xi^{n+1}_{j,\vec{u}}\|_f^2 + \frac{C_{P,p}^2 gS_0^2}{\beta_2\bar{k}_{min}}\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\|\mu_{j,\phi,t}\|_p^2\,dt\\
&\quad + \frac{\beta_2}{4}g\bar{k}_{min}\|\nabla\xi^{n+1}_{j,\phi}\|_p^2.
\end{aligned}
$$

$$
\begin{aligned}
&-\nu(\nabla\mu^{n+1}_{j,\vec{u}}, \nabla\xi^{n+1}_{j,\vec{u}})_f - g(\bar{\mathcal{K}}\nabla\mu^{n+1}_{j,\phi}, \nabla\xi^{n+1}_{j,\phi})_p \qquad (4.44)\\
&\quad\leq C\left(\|\nabla\mu^{n+1}_{j,\vec{u}}\|_f^2 + \|\nabla\mu^{n+1}_{j,\phi}\|_p^2\right) + \frac{\alpha_2}{5}\nu\|\nabla\xi^{n+1}_{j,\vec{u}}\|_f^2 + \frac{\beta_2}{4}g\bar{k}_{min}\|\nabla\xi^{n+1}_{j,\phi}\|_p^2.
\end{aligned}
$$

By trace theorem, we have the following estimates

$$
\begin{aligned}
&-c_\Gamma(\xi^{n+1}_{j,\vec{u}}, \mu^{n}_{j,\phi}) + c_\Gamma(\mu^{n}_{j,\vec{u}}, \xi^{n+1}_{j,\phi}) \qquad (4.45)\\
&\quad\leq C\left(\|\nabla\mu^{n}_{j,\vec{u}}\|_f^2 + \|\nabla\mu^{n}_{j,\phi}\|_p^2\right) + \frac{\alpha_2}{5}\nu\|\nabla\xi^{n+1}_{j,\vec{u}}\|_f^2 + \frac{\beta_2}{4}g\bar{k}_{min}\|\nabla\xi^{n+1}_{j,\phi}\|_p^2.
\end{aligned}
$$

The pressure term can be bounded as follows:

$$\left(p_j^{n+1} - \lambda_h^{n+1}, \nabla \cdot \xi_{j,\vec{u}}^{n+1}\right)_f \leq C\|p_j^{n+1} - \lambda_h^{n+1}\|_f^2 + \frac{\alpha_2}{5}\nu\|\nabla \xi_{j,\vec{u}}^{n+1}\|_f^2. \tag{4.46}$$

Next, we bound the consistency errors:

$$\begin{aligned}
\epsilon_{j,f}^{n+1}(\xi_{j,\vec{u}}^{n+1}) &\leq C\left\|\frac{\vec{u}_j^{n+1} - \vec{u}_j^n}{\Delta t} - u_{j,t}^{n+1}\right\|_f^2 + C\|\nabla(\phi_j^{n+1} - \phi_j^n)\|_p^2 + \frac{\alpha_2}{5}\nu\|\nabla \xi_{j,\vec{u}}^{n+1}\|_f^2 \qquad (4.47)\\
&\leq C\Delta t \int_{t^n}^{t^{n+1}} \|u_{j,tt}\|_f^2\, dt + C\Delta t \int_{t^n}^{t^{n+1}} \|\nabla \phi_{j,t}\|_p^2\, dt + \frac{\alpha_2}{5}\nu\|\nabla \xi_{j,\vec{u}}^{n+1}\|_f^2.\\
\epsilon_{j,p}^{n+1}(\xi_{j,\phi}^{n+1}) &\leq C\left\|\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} - \phi_{j,t}^{n+1}\right\|_p^2 + C\|\nabla(\vec{u}_j^{n+1} - \vec{u}_j^n)\|_f^2 + \frac{\beta_2}{4} g\bar{k}_{min}\|\nabla \xi_{j,\phi}^{n+1}\|_p^2 \qquad (4.48)\\
&\leq C\Delta t \int_{t^n}^{t^{n+1}} \|\phi_{j,tt}\|_p^2\, dt + C\Delta t \int_{t^n}^{t^{n+1}} \|\nabla u_{j,t}\|_f^2\, dt + \frac{\beta_2}{4} g\bar{k}_{min}\|\nabla \xi_{j,\phi}^{n+1}\|_p^2.
\end{aligned}$$

The rest of the terms on the right-hand side of (4.41) *can be bounded as follows:*

$$\begin{aligned}
&-\sum_i \int_\Gamma \left(\eta_{i,j} - \bar{\eta}_i\right)(\xi_{j,\vec{u}}^n \cdot \widehat{\tau}_i)(\xi_{j,\vec{u}}^{n+1} \cdot \widehat{\tau}_i)\, ds \qquad (4.49)\\
&\leq \sum_i \eta_{i,j}'^{max} \int_\Gamma |(\xi_{j,\vec{u}}^n \cdot \widehat{\tau}_i)(\xi_{j,\vec{u}}^{n+1} \cdot \widehat{\tau}_i)|\, ds\\
&\leq \sum_i \left[\frac{\eta_i'^{max}}{2} \int_\Gamma (\xi_{j,\vec{u}}^n \cdot \widehat{\tau}_i)^2\, ds + \frac{\eta_i'^{max}}{2} \int_\Gamma (\xi_{j,\vec{u}}^{n+1} \cdot \widehat{\tau}_i)^2\, ds\right].
\end{aligned}$$

By (4.4) *and Poincaré inequality, we have for any* $\sigma_1 > 0$

$$\begin{aligned}
&-\sum_i \int_\Gamma \left(\eta_{i,j} - \bar{\eta}_i\right)((\vec{u}_j^{n+1} - \vec{u}_j^n) \cdot \widehat{\tau}_i)(\xi_{j,\vec{u}}^{n+1} \cdot \widehat{\tau}_i)\, ds \qquad (4.50)\\
&\leq \sum_i \eta_{i,j}'^{max} \int_\Gamma |((\vec{u}_j^{n+1} - \vec{u}_j^n) \cdot \widehat{\tau}_i)(\xi_{j,\vec{u}}^{n+1} \cdot \widehat{\tau}_i)|\, ds\\
&\leq \sum_i \eta_i'^{max} \left[\frac{1}{2\sigma_1} \int_\Gamma \left((\vec{u}_j^{n+1} - \vec{u}_j^n) \cdot \widehat{\tau}_i\right)^2 ds + \frac{\sigma_1}{2} \int_\Gamma \left(\xi_{j,\vec{u}}^{n+1} \cdot \widehat{\tau}_i\right)^2 ds\right]\\
&\leq \sum_i \left[\frac{\eta_i'^{max}}{2\sigma_1}\|\vec{u}_j^{n+1} - \vec{u}_j^n\|_\Gamma^2 + \frac{\sigma_1}{2}\eta_i'^{max} \int_\Gamma (\xi_{j,\vec{u}}^{n+1} \cdot \widehat{\tau}_i)^2\, ds\right]\\
&\leq \sum_i \left[\frac{C_{P,f}C^2(D_f)}{2\sigma_1}\eta_i'^{max}\|\nabla(\vec{u}_j^{n+1} - \vec{u}_j^n)\|_f^2 + \frac{\sigma_1}{2}\eta_i'^{max} \int_\Gamma (\xi_{j,\vec{u}}^{n+1} \cdot \widehat{\tau}_i)^2\, ds\right]\\
&\leq \sum_i \left[\frac{C_{P,f}C^2(D_f)}{2\sigma_1}\eta_i'^{max}\Delta t \int_{t^n}^{t^{n+1}} \|\nabla u_{j,t}\|_f^2\, dt + \frac{\sigma_1}{2}\eta_i'^{max} \int_\Gamma (\xi_{j,\vec{u}}^{n+1} \cdot \widehat{\tau}_i)^2\, ds\right].
\end{aligned}$$

Similarly, for any $\sigma_2 > 0$

$$
\begin{aligned}
&-\sum_i \int_\Gamma \bar{\eta}_i(\mu^{n+1}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)(\xi^{n+1}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)\, ds \\
&\quad \le \sum_i \left[\frac{1}{4\sigma_2} \int_\Gamma \bar{\eta}_i(\mu^{n+1}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)^2\, ds + \sigma_2 \int_\Gamma \bar{\eta}_i(\xi^{n+1}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)^2\, ds \right] \\
&\quad \le \sum_i \left[\frac{1}{4\sigma_2} \bar{\eta}_i^{max} \|\mu^{n+1}_{j,\overrightarrow{u}}\|^2_\Gamma + \sigma_2 \int_\Gamma \bar{\eta}_i(\xi^{n+1}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)^2\, ds \right] \\
&\quad \le \sum_i \left[\frac{C^2(D_f)C_{P,f}}{4\sigma_2} \bar{\eta}_i^{max} \|\nabla \mu^{n+1}_{j,\overrightarrow{u}}\|^2_f + \sigma_2 \int_\Gamma \bar{\eta}_i(\xi^{n+1}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)^2\, ds \right].
\end{aligned} \tag{4.51}
$$

$$
\begin{aligned}
&-\sum_i \int_\Gamma \left(\eta_{i,j} - \bar{\eta}_i\right) (\mu^{n}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)(\xi^{n+1}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)\, ds \\
&\quad \le \sum_i \eta'^{max}_{i,j} \int_\Gamma |(\mu^{n}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)(\xi^{n+1}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)|\, ds \\
&\quad \le \sum_i \eta'^{max}_{i} \left[\frac{1}{2\sigma_1} \int_\Gamma (\mu^{n}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)^2\, ds + \frac{\sigma_1}{2} \int_\Gamma (\xi^{n+1}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)^2 \right] ds \\
&\quad \le \sum_i \left[\frac{1}{2\sigma_1} \eta'^{max}_{i} \|\mu^{n}_{j,\overrightarrow{u}}\|^2_\Gamma + \frac{\sigma_1}{2} \eta'^{max}_{i} \int_\Gamma (\xi^{n+1}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)^2\, ds \right] \\
&\quad \le \sum_i \left[\frac{C^2(D_f)C_{P,f}}{2\sigma_1} \eta'^{max}_{i,j} \|\nabla \mu^{n}_{j,\overrightarrow{u}}\|^2_f + \frac{\sigma_1}{2} \eta'^{max}_{i} \int_\Gamma (\xi^{n+1}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)^2\, ds \right].
\end{aligned} \tag{4.52}
$$

The hydraulic conductivity tensor terms are estimated as follows:

$$
\begin{aligned}
&-g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla \xi^n_{j,\phi}, \nabla \xi^{n+1}_{j,\phi})_p \\
&\quad \le g \int_{D_p} |\nabla \xi^n_{j,\phi}|_2 |\mathcal{K}_j - \bar{\mathcal{K}}|_2 |\nabla \xi^{n+1}_{j,\phi}|_2\, dx \\
&\quad \le g \int_{D_p} \rho'_j(x) |\nabla \xi^n_{j,\phi}|_2 |\nabla \xi^{n+1}_{j,\phi}|_2\, dx \\
&\quad \le g \rho'_{j,max} \int_{D_p} |\nabla \xi^n_{j,\phi}|_2 |\nabla \xi^{n+1}_{j,\phi}|_2\, dx \\
&\quad \le g \rho'_{max} \|\nabla \xi^n_{j,\phi}\|_p \|\nabla \xi^{n+1}_{j,\phi}\|_p \\
&\quad \le \frac{g\rho'_{max}}{2} \|\nabla \xi^n_{j,\phi}\|^2_p + \frac{g\rho'_{max}}{2} \|\nabla \xi^{n+1}_{j,\phi}\|^2_p.
\end{aligned} \tag{4.53}
$$

For any $\sigma_3 > 0$, we have

$$\begin{aligned}
-g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla(\phi_j^{n+1} - \phi_j^n), \nabla\xi_{j,\phi}^{n+1})_p &\le g\int_{D_p} |\nabla(\phi_j^{n+1} - \phi_j^n)|_2 |\mathcal{K}_j - \bar{\mathcal{K}}|_2 |\nabla\xi_{j,\phi}^{n+1}|_2 \, dx \\
&\le g\int_{D_p} \rho_j'(x)|\nabla(\phi_j^{n+1} - \phi_j^n)|_2 |\nabla\xi_{j,\phi}^{n+1}|_2 \, dx \\
&\le g\rho_{j,max}'\int_{D_p} |\nabla(\phi_j^{n+1} - \phi_j^n)|_2 |\nabla\xi_{j,\phi}^{n+1}|_2 \, dx \\
&\le g\rho_{max}' \|\nabla(\phi_j^{n+1} - \phi_j^n)\|_p \|\nabla\xi_{j,\phi}^{n+1}\|_p \qquad (4.54) \\
&\le \frac{g\rho_{max}'}{2\sigma_3}\|\nabla(\phi_j^{n+1} - \phi_j^n)\|_p^2 + \frac{\sigma_3}{2} g\rho_{max}' \|\nabla\xi_{j,\phi}^{n+1}\|_p^2 \\
&\le \frac{g\rho_{max}'}{2\sigma_3}\|\int_{t^n}^{t^{n+1}} \nabla\phi_{j,t} \, dt\|_p^2 + \frac{\sigma_3}{2} g\rho_{max}' \|\nabla\xi_{j,\phi}^{n+1}\|_p^2 \\
&\le \frac{g\rho_{max}'}{2\sigma_3}\Delta t\int_{t^n}^{t^{n+1}} \|\nabla\phi_{j,t}\|_p^2 \, dt + \frac{\sigma_3}{2} g\rho_{max}' \|\nabla\xi_{j,\phi}^{n+1}\|_p^2.
\end{aligned}$$

$$\begin{aligned}
-g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla\mu_{j,\phi}^n, \nabla\xi_{j,\phi}^{n+1})_p &\le g\int_{D_p} |\nabla\mu_{j,\phi}^n|_2 |\mathcal{K}_j - \bar{\mathcal{K}}|_2 |\nabla\xi_{j,\phi}^{n+1}|_2 \, dx \qquad (4.55) \\
&\le g\int_{D_p} \rho_j'(x)|\nabla\mu_{j,\phi}^n|_2 |\nabla\xi_{j,\phi}^{n+1}|_2 \, dx \\
&\le g\rho_{j,max}'\int_{D_p} |\nabla\mu_{j,\phi}^n|_2 |\nabla\xi_{j,\phi}^{n+1}|_2 \, dx \\
&\le g\rho_{max}' \|\nabla\mu_{j,\phi}^n\|_p \|\nabla\xi_{j,\phi}^{n+1}\|_p \\
&\le \frac{g\rho_{max}'}{2\sigma_3}\|\nabla\mu_{j,\phi}^n\|_p^2 + \frac{\sigma_3}{2} g\rho_{max}' \|\nabla\xi_{j,\phi}^{n+1}\|_p^2.
\end{aligned}$$

Since $\bar{\mathcal{K}}$ is SPD, for any $\sigma_4 > 0$

$$\begin{aligned}
-g(\bar{\mathcal{K}}\nabla\mu_{j,\phi}^{n+1}, \nabla\xi_{j,\phi}^{n+1})_p &= -g(\bar{\mathcal{K}}^{\frac{1}{2}}\nabla\mu_{j,\phi}^{n+1}, \bar{\mathcal{K}}^{\frac{1}{2}}\nabla\xi_{j,\phi}^{n+1})_p \qquad (4.56) \\
&\le g\|\bar{\mathcal{K}}^{\frac{1}{2}}\nabla\mu_{j,\phi}^{n+1}\|_p \|\bar{\mathcal{K}}^{\frac{1}{2}}\nabla\xi_{j,\phi}^{n+1}\|_p \\
&\le \frac{1}{4\sigma_4} g\|\bar{\mathcal{K}}^{\frac{1}{2}}\nabla\mu_{j,\phi}^{n+1}\|_p^2 + \sigma_4 g\|\bar{\mathcal{K}}^{\frac{1}{2}}\nabla\xi_{j,\phi}^{n+1}\|_p^2 \\
&\le \frac{1}{4\sigma_4} g\bar{k}^{max}\|\nabla\mu_{j,\phi}^{n+1}\|_p^2 + \sigma_4 g\|\bar{\mathcal{K}}^{\frac{1}{2}}\nabla\xi_{j,\phi}^{n+1}\|_p^2.
\end{aligned}$$

Combining all these estimates, we have the following inequality:

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|\xi^{n+1}_{j,\vec{u}}\|_f^2 - \frac{1}{2\Delta t}\|\xi^{n}_{j,\vec{u}}\|_f^2 + \left(1-\alpha_1-\alpha_2-\Delta t\frac{2C_2}{\beta_1^2\nu}\right)\nu\|\nabla\xi^{n+1}_{j,\vec{u}}\|_f^2 \\
&+\Delta t\frac{C_2}{\beta_1^2}\left(\|\nabla\xi^{n+1}_{j,\vec{u}}\|_f^2-\|\nabla\xi^{n}_{j,\vec{u}}\|_f^2\right)+\sum_i\left((1-\sigma_2)\bar{\eta}_i^{min}-(1+\sigma_1)\eta_i'^{max}\right)\int_\Gamma(\xi^{n+1}_{j,\vec{u}}\cdot\widehat{\tau}_i)^2\,ds \\
&+\sum_i\frac{1}{2}\eta_i'^{max}\left(\int_\Gamma(\xi^{n+1}_{j,\vec{u}}\cdot\widehat{\tau}_i)^2\,ds-\int_\Gamma(\xi^{n}_{j,\vec{u}}\cdot\widehat{\tau}_i)^2\,ds\right)+\frac{gS_0}{2\Delta t}\|\xi^{n+1}_{j,\phi}\|_p^2-\frac{gS_0}{2\Delta t}\|\xi^{n}_{j,\phi}\|_p^2 \\
&+\left((1-\sigma_4-\beta_1-\beta_2-\Delta t\frac{2C_1}{g^2S_0\bar{k}_{min}\alpha_1^2})-(1+\sigma_3)\frac{\rho'_{max}}{\bar{k}_{min}}\right)g\bar{k}_{min}\|\nabla\xi^{n+1}_{j,\phi}\|_p^2 \\
&+\left(\frac{1}{2}g\rho'_{max}+\frac{\Delta t C_1}{gS_0\alpha_1^2}\right)\left(\|\nabla\xi^{n+1}_{j,\phi}\|_p^2-\|\nabla\xi^{n}_{j,\phi}\|_p^2\right) \\
&\le\sum_i\frac{C_{P,f}C^2(D_f)}{4\sigma_1}\eta_i'^{max}\Delta t\int_{t^n}^{t^{n+1}}\|\nabla u_{j,t}\|_f^2\,dt+\sum_i\frac{C^2(D_f)C_{P,f}}{4\sigma_1}\bar{\eta}_i^{max}\|\nabla\mu^{n+1}_{j,\vec{u}}\|_f^2 \\
&+\sum_i\frac{C^2(D_f)C_{P,f}}{4\sigma_1}\eta_{i,j}'^{max}\|\nabla\mu^{n}_{j,\vec{u}}\|_f^2+C\Delta t\int_{t^n}^{t^{n+1}}\|u_{j,tt}\|_f^2\,dt+C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}\|_p^2\,dt \\
&+C\Delta t\int_{t^n}^{t^{n+1}}\|\phi_{j,tt}\|_p^2\,dt+C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla u_{j,t}\|_f^2\,dt+\frac{5C_{P,f}^2}{4\alpha_2\nu}\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\|\mu_{j,\vec{u},t}\|_f^2\,dt \\
&+\frac{C_{P,g}^2gS_0^2}{\beta_2\bar{k}_{min}}\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\|\mu_{j,\phi,t}\|_p^2\,dt+C\left(\|\nabla\mu^{n+1}_{j,\vec{u}}\|_f^2+\|\nabla\mu^{n+1}_{j,\phi}\|_p^2\right)+C\left(\|\nabla\mu^{n}_{j,\vec{u}}\|_f^2+\|\nabla\mu^{n}_{j,\phi}\|_p^2\right) \\
&+C\|p_j^{n+1}-\lambda_h^{n+1}\|_f^2+\frac{g\rho'_{max}}{4\sigma_2}\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}\|_p^2\,dt+\frac{1}{4\sigma_2}g\bar{k}^{max}\|\nabla\mu^{n+1}_{j,\phi}\|_p^2+\frac{g\rho'_{max}}{4\sigma_2}\|\nabla\mu^{n}_{j,\phi}\|_p^2.
\end{aligned}
\tag{4.57}
$$

To make sure the third, fifth and ninth term on the left-hand side are non-negative, we need $0<\alpha_1,\alpha_2,\sigma_2,\sigma_4,\beta_1,\beta_2<1$, *and*

$$\frac{\eta_i'^{max}}{\bar{\eta}_i^{min}}\le\frac{1-\sigma_2}{1+\sigma_1},\qquad\frac{\rho'_{max}}{\bar{k}_{min}}<\frac{1}{1+\sigma_3}. \tag{4.58}$$

For $\forall\sigma_2\in(0,1),\forall\sigma_1>0,\forall\sigma_3>0$, *we can derive that* $\frac{1-\sigma_2}{1+\sigma_1},\frac{1}{1+\sigma_3}\in(0,1)$. *Now if the two parameter conditions* (4.13) *and* (4.14) *are satisfied, we have* $\frac{\eta_i'^{max}}{\bar{\eta}_i^{min}},\frac{\rho'_{max}}{\bar{k}_{min}}\in(0,1)$. *Then we can easily find* $\sigma_2\in(0,1),\sigma_1>0$ *such that* $\frac{\eta_i'^{max}}{\bar{\eta}_i^{min}}=\frac{1-\sigma_2}{1+\sigma_1}$, *and* $\sigma_3>0$ *such that* $\frac{\rho'_{max}}{\bar{k}_{min}}<\frac{1}{1+\sigma_3}$.

Then under two the parameter conditions (4.35) *and* (4.36)*, and the time step condition* (4.34)*,* (4.57) *reduces to*

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|\xi^{n+1}_{j,\vec{u}}\|_f^2 - \frac{1}{2\Delta t}\|\xi^{n}_{j,\vec{u}}\|_f^2 + \Delta t\frac{C_2}{\beta_1^2}\left(\|\nabla\xi^{n+1}_{j,\vec{u}}\|_f^2 - \|\nabla\xi^{n}_{j,\vec{u}}\|_f^2\right) + \frac{gS_0}{2\Delta t}\|\xi^{n+1}_{j,\phi}\|_p^2 - \frac{gS_0}{2\Delta t}\|\xi^{n}_{j,\phi}\|_p^2 \\
&+\sum_i \frac{1}{2}\eta_i'^{max}\left(\int_\Gamma(\xi^{n+1}_{j,\vec{u}}\cdot\widehat{\tau}_i)^2\,ds - \int_\Gamma(\xi^{n}_{j,\vec{u}}\cdot\widehat{\tau}_i)^2\,ds\right) + \left(\frac{1}{2}g\rho'_{max} + \frac{\Delta t C_1}{gS_0\alpha_1^2}\right)\left(\|\nabla\xi^{n+1}_{j,\phi}\|_p^2 - \|\nabla\xi^{n}_{j,\phi}\|_p^2\right) \\
&\le \sum_i \frac{C_{P,f}C^2(D_f)}{4\sigma_1}\eta_i'^{max}\Delta t\int_{t^n}^{t^{n+1}}\|\nabla u_{j,t}\|_f^2\,dt + \sum_i\frac{C^2(D_f)C_{P,f}}{4\sigma_2}\bar{\eta}_i^{max}\|\nabla\mu^{n+1}_{j,\vec{u}}\|_f^2 \qquad (4.59) \\
&+\sum_i\frac{C^2(D_f)C_{P,f}}{4\sigma_1}\eta_{i,j}'^{max}\|\nabla\mu^{n}_{j,\vec{u}}\|_f^2 + C\Delta t\int_{t^n}^{t^{n+1}}\|u_{j,tt}\|_f^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}\|_p^2\,dt \\
&+C\Delta t\int_{t^n}^{t^{n+1}}\|\phi_{j,tt}\|_p^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla u_{j,t}\|_f^2\,dt + \frac{5C_{P,f}^2}{4\alpha_2\nu}\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\|\mu_{j,\vec{u},t}\|_f^2\,dt \\
&+\frac{C_{P.g}^2 gS_0^2}{\beta_2\bar{k}_{min}}\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\|\mu_{j,\phi,t}\|_p^2\,dt + C\left(\|\nabla\mu^{n+1}_{j,\vec{u}}\|_f^2 + \|\nabla\mu^{n+1}_{j,\phi}\|_p^2\right) \\
&+C\left(\|\nabla\mu^{n}_{j,\vec{u}}\|_f^2 + \|\nabla\mu^{n}_{j,\phi}\|_p^2\right) + C\|p_j^{n+1} - \lambda_h^{n+1}\|_f^2 + \frac{g\rho'_{max}}{4\sigma_3}\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}\|_p^2\,dt \\
&+\frac{1}{4\sigma_3}g\bar{k}^{max}\|\nabla\mu^{n+1}_{j,\phi}\|_p^2 + \frac{g\rho'_{max}}{4\sigma_4}\|\nabla\mu^{n}_{j,\phi}\|_p^2.
\end{aligned}
$$

Summing up from $n = 0$ *to* $n = N-1$ *and multiplying through by* Δt *yields*

$$
\begin{aligned}
&\frac{1}{2}\|\xi^{N}_{j,\vec{u}}\|_f^2 + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla\xi^{N}_{j,\vec{u}}\|_f^2 + \Delta t\sum_i\frac{1}{2}\eta_i'^{max}\int_\Gamma(\xi^{N}_{j,\vec{u}}\cdot\widehat{\tau}_i)^2\,ds \qquad (4.60) \\
&+\frac{gS_0}{2}\|\xi^{N}_{j,\phi}\|_p^2 + \Delta t\left(\frac{1}{2}g\rho'_{max} + \frac{\Delta t C_1}{gS_0\alpha_1^2}\right)\|\nabla\xi^{N}_{j,\phi}\|_p^2 \\
&\le \frac{1}{2}\|\xi^{0}_{j,\vec{u}}\|_f^2 + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla\xi^{0}_{j,\vec{u}}\|_f^2 + \Delta t\sum_i\frac{1}{2}\eta_i'^{max}\int_\Gamma(\xi^{0}_{j,\vec{u}}\cdot\widehat{\tau}_i)^2\,ds + \frac{gS_0}{2}\|\xi^{0}_{j,\phi}\|_p^2 \\
&+\Delta t\left(\frac{1}{2}g\rho'_{max} + \frac{\Delta t C_1}{gS_0\alpha_1^2}\right)\|\nabla\xi^{0}_{j,\phi}\|_p^2 + \Delta t\sum_{n=1}^{N-1}\Big\{\sum_i\frac{C_{P,f}C^2(D_f)}{4\sigma_1}\eta_i'^{max}\Delta t\int_{t^n}^{t^{n+1}}\|\nabla u_{j,t}\|_f^2\,dt \\
&+\sum_i\frac{C^2(D_f)C_{P,f}}{4\sigma_2}\bar{\eta}_i^{max}\|\nabla\mu^{n+1}_{j,\vec{u}}\|_f^2 + \sum_i\frac{C^2(D_f)C_{P,f}}{4\sigma_1}\eta_{i,j}'^{max}\|\nabla\mu^{n}_{j,\vec{u}}\|_f^2 \\
&+C\Delta t\int_{t^n}^{t^{n+1}}\|u_{j,tt}\|_f^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}\|_p^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}}\|\phi_{j,tt}\|_p^2\,dt \\
&+C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla u_{j,t}\|_f^2\,dt + \frac{5C_{P,f}^2}{4\alpha_2\nu}\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\|\mu_{j,\vec{u},t}\|_f^2\,dt + \frac{C_{P.g}^2 gS_0^2}{\beta_2\bar{k}_{min}}\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\|\mu_{j,\phi,t}\|_p^2\,dt \\
&+C\left(\|\nabla\mu^{n+1}_{j,\vec{u}}\|_f^2 + \|\nabla\mu^{n+1}_{j,\phi}\|_p^2\right) + C\left(\|\nabla\mu^{n}_{j,\vec{u}}\|_f^2 + \|\nabla\mu^{n}_{j,\phi}\|_p^2\right) + C\|p_j^{n+1} - \lambda_h^{n+1}\|_f^2 \\
&+\frac{g\rho'_{max}}{4\sigma_3}\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}\|_p^2\,dt + \frac{1}{4\sigma_3}g\bar{k}^{max}\|\nabla\mu^{n+1}_{j,\phi}\|_p^2 + \frac{g\rho'_{max}}{4\sigma_4}\|\nabla\mu^{n}_{j,\phi}\|_p^2\Big\}.
\end{aligned}
$$

Using interpolation inequalities, we obtain

$$
\begin{aligned}
&\frac{1}{2}\|\xi^N_{j,\overrightarrow{u}}\|^2_f + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla\xi^N_{j,\overrightarrow{u}}\|^2_f + \Delta t\sum_i \frac{1}{2}\eta_i'^{max}\int_\Gamma(\xi^N_{j,\overrightarrow{u}}\cdot\widehat{\tau}_i)^2\,ds \\
&+\frac{gS_0}{2}\|\xi^N_{j,\phi}\|^2_p + \Delta t\left(\frac{1}{2}g\rho'_{max}+\frac{\Delta t C_1}{gS_0\alpha_1^2}\right)\|\nabla\xi^N_{j,\phi}\|^2_p \\
&\leq \frac{1}{2}\|\xi^0_{j,\overrightarrow{u}}\|^2_f + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla\xi^0_{j,\overrightarrow{u}}\|^2_f + \Delta t\sum_i \frac{1}{2}\eta_i'^{max}\int_\Gamma(\xi^0_{j,\overrightarrow{u}}\cdot\widehat{\tau}_i)^2\,ds + \frac{gS_0}{2}\|\xi^0_{j,\phi}\|^2_p \\
&+\Delta t\left(\frac{1}{2}g\rho'_{max}+\frac{\Delta t C_1}{gS_0\alpha_1^2}\right)\|\nabla\xi^0_{j,\phi}\|^2_p + C\Delta t^2\|u_{j,t}\|^2_{2,1,f} + Ch^{2k}|||\overrightarrow{u}_j|||^2_{2,k+1,f} + C\Delta t^2\|u_{j,tt}\|^2_{2,0,f} \\
&+C\Delta t^2\|\phi_{j,t}\|^2_{2,1,p} + C\Delta t^2\|\phi_{j,tt}\|^2_{2,0,p} + Ch^{2k+2}\|u_{j,t}\|_{2,k+1,f} + Ch^{2k+2}\|\phi_{j,t}\|_{2,m+1,p} \\
&+Ch^{2k}|||\phi_j|||^2_{2,m+1,p} + Ch^{2s+2}|||p_j|||^2_{2,s+1,f}.
\end{aligned} \tag{4.61}
$$

Using the triangle inequality on the error equations yields (4.37) *and completes the proof.*

4.2.3. Stochastic Stokes-Darcy Equations. In this section, we consider using the presented ensemble algorithm for solving stochastic Stokes-Darcy equations with a random hydraulic conductivity tensor $\mathcal{K}(x,w)$. Note that the ensemble algorithm can also deal with uncertainties in initial conditions and the forcing terms. Here, for simplicity of presentation, we only consider the example that has a random hydraulic conductivity tensor. Let $(\Theta,\mathcal{F},\mathcal{P})$ be a complete probability space. Here, Θ is the set of outcomes, $\mathcal{F}\in 2^\Theta$ is the $\sigma-$algebra of events, and $\mathcal{P}:\mathcal{F}\to[0,1]$ is a probability measure. The stochastic Stokes-Darcy system considered reads: Find the functions $\overrightarrow{u}:\Omega_S\times[0,T]\times\Theta\to\mathbb{R}^d$ $(d=2,3)$, $p:\Omega_S\times[0,T]\times\Theta\to\mathbb{R}$, and $\phi:\Omega_D\times[0,T]\times\Theta\to\mathbb{R}$, such that it holds $\mathcal{P}-a.e.$ in Ω.

So we have the stochastic Stokes-Darcy system as follows:

$$
\begin{aligned}
u_t(x,t,\omega) - \nu\Delta\vec{u}(x,t,\omega) + \nabla p(x,t,\omega) = f_f(x,t), \quad \nabla\cdot\vec{u}(x,t,\omega) = 0, \ \text{in } \Omega_S\times\Theta, \\
S_0\phi_t(x,t,\omega) - \nabla\cdot(\mathcal{K}(x,\omega)\nabla\phi(x,t,\omega)) = f_p(x,t), \ \text{in } \ \Omega_D\times\Theta, \qquad (4.62) \\
\phi(x,0) = \phi_0(x), \ \text{in } \Omega_D, \ \text{and } \vec{u}(x,0) = u_0(x), \ \text{in } \Omega_S, \\
\phi(x,t,\omega) = 0, \ \text{in } \partial\Omega_D\backslash\Gamma \text{ and } \vec{u}(x,t,\omega) = 0, \ \text{in } \partial\Omega_S\backslash\Gamma,
\end{aligned}
$$

where $f_f(x,t) \in L^2(\Omega_S), f_p(x,t) \in L^2(\Omega_D)$. The hydraulic conductivity $\mathcal{K}(x,\omega)$ is a stochastic function, which is assumed to have a continuous and bounded correlation function.

The Monte Carlo method is one of the most classical approaches for solving stochastic PDEs. It consists of repeated sampling of the input parameter and solving the corresponding deterministic PDEs using standard numerical methods, which generates identically distributed approximations of the solution. Then, the approximate solutions are further analyzed to yield statistical moments or distributions. The Monte Carlo method is known to be computationally expensive, as it usually requires a large number of sample points at a high resolution level. Herein we investigate incorporating the proposed ensemble algorithm with the Monte Carlo method to solve the stochastic Stokes-Darcy equations at reduced computational cost. The computation procedure is as follows:

(1) Generate a number of independently, identically distributed (i.i.d.) samples for the random hydraulic conductivity $\mathcal{K}(x,\omega_j)$, $j = 1,\cdots,J$;

(2) Apply either Algorithm 2 or Algorithm 3 to solve for approximate solutions $\vec{u}_{j,h}^{n+1}(x), p_{j,h}^{n+1}(x), \phi_{j,h}^{n+1}(x)$, $j = 1,\cdots,J$;

(3) Output required statistical information such as the expectation of $\vec{u}(x,t_n,\omega)$: $E[\vec{u}(x,t_n,\omega)] \approx \frac{1}{J}\sum_{j=1}^{J}\vec{u}_{j,h}^{n}(x)$.

Remark 4 *Similar procedures can also be carried out for other ensemble-based UQ methods, such as sparse grid collocation methods and non-intrusive polynomial chaos methods. The Monte Carlo method is chosen here for a simple demonstration of the effectiveness and efficiency of our ensemble algorithm.*

Let $\overrightarrow{u}^n(x,\omega) = \overrightarrow{u}(x,t_n,\omega)$. The error for approximating $E[\overrightarrow{u}(x,t_n,\omega)]$ is then

$$E[\overrightarrow{u}^n] - \frac{1}{J}\sum_{j=1}^{J}\overrightarrow{u}^n_{j,h} = \left(E[\overrightarrow{u}^n] - \frac{1}{J}\sum_{j=1}^{J}\overrightarrow{u}^n_j\right) + \left(\frac{1}{J}\sum_{j=1}^{J}\overrightarrow{u}^n_j - \frac{1}{J}\sum_{j=1}^{J}\overrightarrow{u}^n_{j,h}\right) \qquad (4.63)$$

$$= \mathcal{E}^n_{MC,\overrightarrow{u}} + \mathcal{E}^n_{EN,\overrightarrow{u}},$$

where $\mathcal{E}^n_{MC,\overrightarrow{u}}$ represents the numerical error from using the Monte Carlo method while $\mathcal{E}^n_{EN,\overrightarrow{u}}$ is the error due to using the ensemble algorithm for the numerical solution.

Theorem 7 *If the time step condition* (4.92) *holds and the two parameter conditions* (4.35), (4.36) *all hold, then for any* $N > 0$*, there holds*

$$E\left[\|E[\overrightarrow{u}^n] - \frac{1}{J}\sum_{j=1}^{J}\overrightarrow{u}^n_{j,h}\|^2\right] \leq \frac{1}{J}E[\|\overrightarrow{u}^n\|_f^2] + C(h^4 + \Delta t^2). \qquad (4.64)$$

Proof 7

$$E[\|\mathcal{E}^n_{MC,\overrightarrow{u}}\|^2] = E\left[\left(E[\overrightarrow{u}^n] - \frac{1}{J}\sum_{j=1}^{J}\overrightarrow{u}^n_j, E[\overrightarrow{u}^n] - \frac{1}{J}\sum_{j=1}^{J}\overrightarrow{u}^n_j\right)_f\right] \qquad (4.65)$$

$$= E\left[\left(\frac{1}{J}\sum_{j=1}^{J}(E[\overrightarrow{u}^n] - \overrightarrow{u}^n_j), \frac{1}{J}\sum_{j=1}^{J}(E[\overrightarrow{u}^n] - \overrightarrow{u}^n_j)\right)_f\right]$$

$$= \frac{1}{J^2}\sum_{i=1}^{J}\sum_{j=1}^{J} E\left[\left(E[\overrightarrow{u}^n] - u^n_i, E[\overrightarrow{u}^n] - \overrightarrow{u}^n_j\right)_f\right].$$

Since $u^n_1(x), u^n_2(x), \cdots, u^n_J(x)$ *are i.i.d., we have*

$$E\left[\left(E[\overrightarrow{u}^n] - u^n_i, E[\overrightarrow{u}^n] - \overrightarrow{u}^n_j\right)_f\right] = 0, \qquad \textit{if } i \neq j. \qquad (4.66)$$

Therefore,

$$E[\|\mathcal{E}^n_{MC,\overrightarrow{u}}\|^2] = \frac{1}{J^2}\sum_{j=1}^{J} E\left[\left(E[\overrightarrow{u}^n] - \overrightarrow{u}^n_j, E[\overrightarrow{u}^n] - \overrightarrow{u}^n_j\right)_f\right] \tag{4.67}$$
$$= \frac{1}{J^2}\sum_{j=1}^{J} E\left[\|E[\overrightarrow{u}^n]\|^2_f - 2(\overrightarrow{u}^n_j, E[\overrightarrow{u}^n])_f + \|\overrightarrow{u}^n_j\|^2_f\right]$$
$$= \frac{1}{J^2}\sum_{j=1}^{J} \|E[\overrightarrow{u}^n]\|^2_f - \frac{2}{J^2}\sum_{j=1}^{J}\|E[\overrightarrow{u}^n]\|^2_f + \frac{1}{J^2}\sum_{j=1}^{J} E[\|\overrightarrow{u}^n_j\|^2_f]$$
$$= \frac{1}{J}E[\|\overrightarrow{u}^n\|^2_f] - \frac{1}{J}\|E[\overrightarrow{u}^n]\|^2_f$$
$$\leq \frac{1}{J}E[\|\overrightarrow{u}^n\|^2_f].$$

$$\|\mathcal{E}^n_{EN_u}\|^2_f = \left\|\frac{1}{J}\sum_{j=1}^{J}\overrightarrow{u}^n_j - \frac{1}{J}\sum_{j=1}^{J}\overrightarrow{u}^n_{j,h}\right\|^2_f = \left\|\frac{1}{J}\sum_{j=1}^{J}\left(\overrightarrow{u}^n_j - \overrightarrow{u}^n_{j,h}\right)\right\|^2_f = \left\|\frac{1}{J}\sum_{j=1}^{J} e^n_{j,\overrightarrow{u}}\right\|^2_f \tag{4.68}$$
$$\leq \frac{1}{J}\sum_{j=1}^{J}\left\|e^n_{j,\overrightarrow{u}}\right\|^2_f \leq C(h^4 + \Delta t^2).$$

Then we have the following estimate on the expectation of the L^2 norm of the error for approximating $E[\overrightarrow{u}(x, t_n, \omega)]$:

$$E\left[\|E[\overrightarrow{u}^n] - \frac{1}{J}\sum_{j=1}^{J}\overrightarrow{u}^n_{j,h}\|^2\right] = E\left[\|\mathcal{E}^n_{MC,\overrightarrow{u}} + \mathcal{E}^n_{EN,\overrightarrow{u}}\|^2\right] \tag{4.69}$$
$$\leq E\left[\|\mathcal{E}^n_{MC,\overrightarrow{u}}\|^2\right] + E\left[\|\mathcal{E}^n_{EN,\overrightarrow{u}}\|^2\right]$$
$$\leq \frac{1}{J}E[\|\overrightarrow{u}^n\|^2_f] + C(h^4 + \Delta t^2).$$

4.2.4. An Alternative Approach. Let $k_{j,max}(x)$ be the maximum eigenvalue of the hydraulic conductivity tensor $\mathcal{K}_j(x)$, and we define

$$\eta^{max}_{i,j} = \max_{x\in\Gamma}\eta_{i,j}(x),\ \eta^{max}_i = \max_j \eta^{max}_{i,j},\ k_{j,max} = \max_{x\in D_p} k_{j,max}(x),\ k_{max} = \max_j k_{j,max}.$$

Then the following algorithm can be used, which removes one of the parameter conditions for stability.

Algorithm 3 *Find* $(\overrightarrow{u}_{j,h}^{n+1}, p_{j,h}^{n+1}, \phi_{j,h}^{n+1}) \in X_{Sh} \times Q_{Sh} \times X_{Dh}$ *satisfying* $\forall\ (v_h, q_h, \psi_h) \in X_{Sh} \times Q_{Sh} \times X_{Dh}$,

$$\left(\frac{\overrightarrow{u}_{j,h}^{n+1} - \overrightarrow{u}_{j,h}^{n}}{\Delta t}, v_h\right)_f + \nu(\nabla \overrightarrow{u}_{j,h}^{n+1}, \nabla v)_f + \sum_i \int_\Gamma \eta_i^{max} (\overrightarrow{u}_{j,h}^{n+1} \cdot \widehat{\tau_i})(v \cdot \widehat{\tau_i})\, ds$$
$$+ \sum_i \int_\Gamma (\eta_{i,j} - \eta_i^{max})(\overrightarrow{u}_{j,h}^{n} \cdot \widehat{\tau_i})(v \cdot \widehat{\tau_i})\, ds - \left(p_{j,h}^{n+1}, \nabla \cdot v_h\right)_f + c_\Gamma(v_h, \phi_{j,h}^n) = (f_{f,j}^{n+1}, v_h)_f,$$
$$(q_h, \nabla \cdot \overrightarrow{u}_{j,h}^{n+1})_f = 0, \tag{4.70}$$
$$gS_0\left(\frac{\phi_{j,h}^{n+1} - \phi_{j,h}^{n}}{\Delta t}, \psi_h\right)_p + k_{max} g(\nabla \phi_{j,h}^{n+1}, \nabla \psi)_p + g((\mathcal{K}_j - k_{max}\mathcal{I})\nabla \phi_{j,h}^n, \nabla \psi)_p$$
$$- c_\Gamma(\overrightarrow{u}_{j,h}^{n}, \psi_h) = g(f_{p,j}^{n+1}, \psi_h)_p.$$

For this approach, since $\mathcal{K}_j(x)$ and $k_{max}\mathcal{I}$ are both symmetric, we have $|\mathcal{K}_j(x) - k_{max}\mathcal{I}|_2 \leq k_{max} - k_{min}$. Next, we will prove the long time stability of Algorithm 3 under a similar time-step condition, *without* any parameter conditions.

$$\Delta t \leq \min\left\{\frac{2(1-\alpha_1-\alpha_2)\beta_1^2}{[C(D_f)C(D_p)]^4 C_{P,p}^2}\frac{\nu k_{max}^2}{g^2}, \frac{2(1-\beta_1-\beta_2-\frac{k_{max}-k_{min}}{k_{max}})\alpha_1^2}{[C(D_f)C(D_p)]^4 C_{P,f}^2}\frac{\nu^2 k_{max} S_0}{g^2}\right\}. \tag{4.71}$$

Next, we will prove the theorem of long-time stability for the proposed alternative ensemble method. Since the error estimate is very similar to the prood of Theorem 6, we will not show this work again.

Theorem 8 (Long time stability of Algorithm 3) *If there exist $\alpha_1, \alpha_2, \beta_1, \beta_2$ in $(0,1)$ such that the time-step condition (4.71) holds, then the Algorithm 3 is long time stable: for any $N > 0$,*

$$\begin{aligned}
&\frac{1}{2}\|\overrightarrow{u}^{N}_{j,h}\|_f^2 + \frac{gS_0}{2}\|\phi^N_{j,h}\|_p^2 + \frac{\Delta t\nu}{8}\|\nabla\overrightarrow{u}^{N}_{j,h}\|_f^2 + \Delta t\sum_i \frac{\eta_i^{max}}{2}\int_\Gamma(\overrightarrow{u}^{N}_{j,h}\cdot\widehat{\tau}_i)^2\,ds \\
&+ \frac{\Delta t}{8}gk_{max}\|\nabla\phi^N_{j,h}\|_p^2 + \Delta t\sum_{n=0}^{N-1}\frac{\nu}{4}\|\nabla\overrightarrow{u}^{n+1}_{j,h}\|_f^2 \\
&\leq \frac{1}{2}\|\overrightarrow{u}^{0}_{j,h}\|_f^2 + \frac{gS_0}{2}\|\phi^0_{j,h}\|_p^2 + \frac{\Delta t\nu}{8}\|\nabla\overrightarrow{u}^{0}_{j,h}\|_f^2 + \Delta t\sum_i \frac{\eta_i^{max}}{2}\int_\Gamma(\overrightarrow{u}^{0}_{j,h}\cdot\widehat{\tau}_i)^2\,ds \\
&+ \frac{\Delta t}{8}gk_{max}\|\nabla\phi^0_{j,h}\|_p^2 + \Delta t\sum_{n=0}^{N-1}\frac{C_{P,f}^2}{\nu}\|f^{n+1}_{f,j}\|_f^2 + \Delta t\sum_{n=0}^{N-1}\frac{gC_{P,p}^2}{k_{max}}\|f^{n+1}_{p,j}\|_p^2.
\end{aligned} \tag{4.72}$$

Proof 8 *Setting $v_h = \overrightarrow{u}^{n+1}_{j,h}$, $\psi_h = \phi^{n+1}_{j,h}$ in Algorithm 3 and adding all three equations yields*

$$\begin{aligned}
&\frac{1}{2\Delta t}\|\overrightarrow{u}^{n+1}_{j,h}\|_f^2 - \frac{1}{2\Delta t}\|\overrightarrow{u}^{n}_{j,h}\|_f^2 + \frac{1}{2\Delta t}\|\overrightarrow{u}^{n+1}_{j,h} - \overrightarrow{u}^{n}_{j,h}\|_f^2 + \nu\|\nabla\overrightarrow{u}^{n+1}_{j,h}\|_f^2 \\
&+ \sum_i\int_\Gamma \eta_i^{max}(\overrightarrow{u}^{n+1}_{j,h}\cdot\widehat{\tau}_i)(\overrightarrow{u}^{n+1}_{j,h}\cdot\widehat{\tau}_i)\,ds + \frac{gS_0}{2\Delta t}\|\phi^{n+1}_{j,h}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi^{n}_{j,h}\|_p^2 \\
&+ \frac{gS_0}{2\Delta t}\|\phi^{n+1}_{j,h} - \phi^n_{j,h}\|_p^2 + gk_{max}(\nabla\phi^{n+1}_{j,h}, \nabla\phi^{n+1}_{j,h})_p + c_I(\overrightarrow{u}^{n+1}_{j,h}, \phi^n_{j,h}) - c_I(\overrightarrow{u}^{n}_{j,h}, \phi^{n+1}_{j,h}) \\
&= (f^{n+1}_{f,j}, \overrightarrow{u}^{n+1}_{j,h})_f + g(f^{n+1}_{p,j}, \phi^{n+1}_{j,h})_p - \sum_i\int_\Gamma(\eta_{i,j} - \eta_i^{max})(\overrightarrow{u}^{n}_{j,h}\cdot\widehat{\tau}_i)(\overrightarrow{u}^{n+1}_{j,h}\cdot\widehat{\tau}_i)\,ds \\
&\quad - g((\mathcal{K}_j - k_{max}\mathcal{I})\nabla\phi^n_{j,h}, \nabla\phi^{n+1}_{j,h})_p.
\end{aligned} \tag{4.73}$$

Comparing with the proof of Theorem (5)*, we just have two main difference at the estimates as follows. The first one is*

$$\begin{aligned}
&-\sum_i\int_\Gamma(\eta_{i,j} - \eta_i^{max})(\overrightarrow{u}^{n}_{j,h}\cdot\widehat{\tau}_i)(\overrightarrow{u}^{n+1}_{j,h}\cdot\widehat{\tau}_i)\,ds \leq \sum_i\int_\Gamma\left|\eta_{i,j} - \eta_i^{max}\right|(\overrightarrow{u}^{n}_{j,h}\cdot\widehat{\tau}_i)(\overrightarrow{u}^{n+1}_{j,h}\cdot\widehat{\tau}_i)\,ds \\
&\leq \sum_i\eta_i^{max}\int_\Gamma(\overrightarrow{u}^{n}_{j,h}\cdot\widehat{\tau}_i)(\overrightarrow{u}^{n+1}_{j,h}\cdot\widehat{\tau}_i)\,ds \\
&\leq \sum_i\left[\frac{\eta_i^{max}}{2}\int_\Gamma(\overrightarrow{u}^{n}_{j,h}\cdot\widehat{\tau}_i)^2\,ds + \frac{\eta_i^{max}}{2}\int_\Gamma(\overrightarrow{u}^{n+1}_{j,h}\cdot\widehat{\tau}_i)^2\,ds\right],
\end{aligned}$$

The second term can be bounded as follows:

$$
\begin{aligned}
-g\left((\mathcal{K}_j - k_{max}\mathcal{I})\nabla\phi^n_{j,h}, \nabla\phi^{n+1}_{j,h}\right)_p &\le g\int_{D_p} |\nabla\phi^{n+1}_{j,h}|_2|\mathcal{K}_j - k_{max}\mathcal{I}|_2|\nabla\phi^n_{j,h}|_2\, dx \\
&\le g(k_{max} - k_{min})\int_{D_p} |\nabla\phi^{n+1}_{j,h}|_2|\nabla\phi^n_{j,h}|_2\, dx \\
&\le g(k_{max} - k_{min})\|\nabla\phi^n_{j,h}\|_p\|\nabla\phi^{n+1}_{j,h}\|_p \\
&\le \frac{g(k_{max} - k_{min})}{2}\|\nabla\phi^n_{j,h}\|_p^2 + \frac{g(k_{max} - k_{min})}{2}\|\nabla\phi^{n+1}_{j,h}\|_p^2.
\end{aligned}
$$

Then we have the following inequality

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|\overrightarrow{u}^{n+1}_{j,h}\|_f^2 - \frac{1}{2\Delta t}\|\overrightarrow{u}^{n}_{j,h}\|_f^2 + \left(1 - \alpha_1 - \alpha_2 - \Delta t\frac{2C_2}{\beta_1^2\nu}\right)\nu\|\nabla\overrightarrow{u}^{n+1}_{j,h}\|_f^2 \\
&+ \Delta t\frac{C_2}{\beta_1^2}\left(\|\nabla\overrightarrow{u}^{n+1}_{j,h}\|_f^2 - \|\nabla\overrightarrow{u}^{n}_{j,h}\|_f^2\right) + \sum_i \frac{\eta_i^{max}}{2}\left[\int_\Gamma(\overrightarrow{u}^{n+1}_{j,h}\cdot\widehat{\tau}_i)^2\, ds - \int_\Gamma(\overrightarrow{u}^{n}_{j,h}\cdot\widehat{\tau}_i)^2\, ds\right] \\
&+ \frac{gS_0}{2\Delta t}\|\phi^{n+1}_{j,h}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi^{n}_{j,h}\|_p^2 + \left(\Delta t\frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \frac{g(k_{max} - k_{min})}{2}\right)\left(\|\nabla\phi^{n+1}_{j,h}\|_p^2 - \|\nabla\phi^{n}_{j,h}\|_p^2\right) \\
&+ (1 - \beta_1 - \beta_2 - \Delta t\frac{1}{g_0^2 k_{max}}\frac{2C_1}{\alpha_1^2} - \frac{k_{max} - k_{min}}{k_{max}})gk_{max}\|\nabla\phi^{n+1}_{j,h}\|_p^2 \\
&\le \frac{C_{P,f}^2}{4\alpha_2\nu}\|f^{n+1}_{f,j}\|_f^2 + \frac{gC_{P,p}^2}{4\beta_2 k_{max}}\|f^{n+1}_{p,j}\|_p^2.
\end{aligned}
\tag{4.74}
$$

Since we assume $\mathcal{K}_j$ *is SPD, and any two ensemble members have different hydraulic conductivity tensor* $\mathcal{K}$*, we have* $k_{max} > k_{min} > 0$ *and thus* $0 < \frac{k_{max}-k_{min}}{k_{max}} < 1$*. So we do not need any constraints on these parameters. Now if the time step condition* (4.71) *holds,* (4.74) *reduces to*

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|\overrightarrow{u}^{n+1}_{j,h}\|_f^2 - \frac{1}{2\Delta t}\|\overrightarrow{u}^{n}_{j,h}\|_f^2 + \Delta t\frac{C_2}{\beta_1^2}\left(\|\nabla\overrightarrow{u}^{n+1}_{j,h}\|_f^2 - \|\nabla\overrightarrow{u}^{n}_{j,h}\|_f^2\right) \\
&+ \sum_i \frac{\eta_i^{max}}{2}\left[\int_\Gamma(\overrightarrow{u}^{n+1}_{j,h}\cdot\widehat{\tau}_i)^2\, ds - \int_\Gamma(\overrightarrow{u}^{n}_{j,h}\cdot\widehat{\tau}_i)^2\, ds\right] + \frac{gS_0}{2\Delta t}\|\phi^{n+1}_{j,h}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi^{n}_{j,h}\|_p^2 \\
&+ \left(\Delta t\frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \frac{g(k_{max} - k_{min})}{2}\right)\left(\|\nabla\phi^{n+1}_{j,h}\|_p^2 - \|\nabla\phi^{n}_{j,h}\|_p^2\right) \le \frac{C_{P,f}^2}{4\alpha_2\nu}\|f^{n+1}_{f,j}\|_f^2 + \frac{gC_{P,p}^2}{4\beta_2 k_{max}}\|f^{n+1}_{p,j}\|_p^2.
\end{aligned}
\tag{4.75}
$$

Sum (4.75) *from* $n = 0$ *to* $N - 1$ *and multiply through by* Δt *to get*

$$
\begin{aligned}
&\frac{1}{2}\|\overrightarrow{u}_{j,h}^{N}\|_f^2 + \frac{gS_0}{2}\|\phi_{j,h}^{N}\|_p^2 + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla\overrightarrow{u}_{j,h}^{N}\|_f^2 + \Delta t\sum_i \frac{\eta_i^{max}}{2}\int_\Gamma(\overrightarrow{u}_{j,h}^{N}\cdot\widehat{\tau}_i)^2\, ds \\
&+\left(\Delta t^2\frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \Delta t\frac{g(k_{max}-k_{min})}{2}\right)\|\nabla\phi_{j,h}^{N}\|_p^2 \\
&\le \frac{1}{2}\|\overrightarrow{u}_{j,h}^{0}\|_f^2 + \frac{gS_0}{2}\|\phi_{j,h}^{0}\|_p^2 + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla\overrightarrow{u}_{j,h}^{0}\|_f^2 + \Delta t\sum_i \frac{\eta_i^{max}}{2}\int_\Gamma(\overrightarrow{u}_{j,h}^{0}\cdot\widehat{\tau}_i)^2\, ds \\
&+\left(\Delta t^2\frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \Delta t\frac{g(k_{max}-k_{min})}{2}\right)\|\nabla\phi_{j,h}^{0}\|_p^2 + \Delta t\sum_{n=0}^{N-1}\frac{C_{P,f}^2}{4\alpha_2\nu}\|f_{f,j}^{n+1}\|_f^2 \\
&+\Delta t\sum_{n=0}^{N-1}\frac{gC_{P,p}^2}{4\beta_2 k_{max}}\|f_{p,j}^{n+1}\|_p^2.
\end{aligned}
$$

4.2.5. Numerical Illustrations. We will use numerical examples to test the theoretical results, including (1) convergence of the ensemble algorithm; (2) combination with the Monte Carlo method to efficiently simulate the Stokes-Darcy system with a random hydraulic conductivity tensor.

4.2.5.1. Convergence test. In this section, we will check the error estimate for the ensemble method by using a known exact solution. Specifically, we consider the model problem on $\Omega = [0,\pi]\times[-1,1]$, where $\Omega_D = [0,\pi]\times[-1,0]$, and $\Omega_S = [0,\pi]\times[0,1]$. We take $\alpha_{BJS} = 1$, $\nu = 1$, $g = 1$, $S_0 = 1$, and

$$
\mathcal{K} = \mathcal{K}_j = \begin{bmatrix} k_{11}^j & 0 \\ 0 & k_{22}^j \end{bmatrix}, \quad j = 1,\ldots,J,
$$

where $\mathcal{K}$ is the random hydraulic conductivity tensor and $\mathcal{K}_j$ is one of the samples of $\mathcal{K}$. Here we only consider k_{11}, k_{22} are random variables that are independent of spatial coordinates.

We use the boundary condition functions and the source terms as follows such that for $t = T = 1$,

$$\begin{aligned} \phi_D &= (e^y - e^{-y}) sin(x) e^t, \\ \overrightarrow{u}_S &= [\frac{k_{11}^j}{\pi} sin(2\pi y) cos(x), (-2k_{22}^j + \frac{k_{22}^j}{\pi^2} sin^2(\pi y)) sin(x)]^T e^t, \\ p_S &= 0. \end{aligned}$$

We consider a group of simulations with $J = 3$ members. The three members are corresponding to different hydraulic conductivity tensors, i.e. $k_{11}^1 = k_{22}^1 = 2.21, k_{11}^2 = k_{22}^2 = 4.11, k_{11}^3 = k_{22}^3 = 6.21$. As $\mathcal{K}$ is diagonal, we use Algorithm 3 for computation, and thus there are no parameter conditions for both stability and convergence. In order to check the convergence order in time, we uniformly refine the mesh size h and time step size Δt from the initial mesh size $1/4$ and time step size $\Delta t = h^3$. The approximation errors of the ensemble method are listed in Table 4.1, Table 4.2, and Table 4.3, for the velocity $\overrightarrow{u}$, the hydraulic head ϕ, and the pressure p, respectively. From these tables, we can find that the rate of convergence is $O(h^3 + \triangle t) = O(h^3) = O(\Delta t)$ with respect to L^2 norms for $\overrightarrow{u}$ and ϕ, which confirms that our ensemble algorithm is first order in time convergent in both fluid velocity and hydraulic head. In this test, the time step seems to be small enough, as we did not observe any instabilies. The convergence rate for the pressure p is somehow better than expected, which may be because the exact solution for the pressure vanishes Cao *et al.* (2014).

4.2.5.2. Random hydraulic conductivity tensor. Next, we consider approximating the stochastic Stokes-Darcy equations with a random hydraulic conductivity tensor $\mathcal{K}(\vec{x}, \omega)$ that depends on spatial coordinates, using the Monte Carlo method for sampling and our ensemble algorithm for numerical simulations, as described in Section 4.2.3.

Table 4.1. Errors and convergence rates of the ensemble algorithm ($J = 3$) for $\triangle t = h^3$.

h	$\|\overrightarrow{u}_h - \overrightarrow{u}\|_0^{E,1}$	rate	$\|\overrightarrow{u}_h - \overrightarrow{u}\|_0^{E,2}$	rate	$\|\overrightarrow{u}_h - \overrightarrow{u}\|_0^{E,3}$	rate
1/4	6.0818×10^{-2}	–	1.1996×10^{-1}	–	1.7971×10^{-1}	–
1/8	7.5907×10^{-3}	3.00	1.4960×10^{-2}	3.00	2.2409×10^{-2}	3.00
1/16	9.3433×10^{-4}	3.02	1.8431×10^{-3}	3.02	2.7611×10^{-3}	3.02
1/32	1.1534×10^{-4}	3.01	2.3009×10^{-4}	3.00	3.4513×10^{-4}	3.00
h	$\|\overrightarrow{u}_h - \overrightarrow{u}\|_1^{E,1}$	rate	$\|\overrightarrow{u}_h - \overrightarrow{u}\|_1^{E,2}$	rate	$\|\overrightarrow{u}_h - \overrightarrow{u}\|_1^{E,3}$	rate
1/4	1.2578×10^{0}	–	2.5143×10^{0}	–	3.7713×10^{0}	–
1/8	3.3416×10^{-1}	1.91	6.6823×10^{-1}	1.91	1.0023×10^{0}	1.91
1/16	8.5725×10^{-2}	1.96	1.7144×10^{-1}	1.96	2.5717×10^{-1}	1.96
1/32	2.1431×10^{-2}	2.00	4.2861×10^{-2}	2.00	6.4292×10^{-2}	2.00

We first consider the case that the hydraulic conductivity tensor is diagonal and Algorithm 3 will be used for computation. We construct the random hydraulic conductivity tensor that varies in the vertical direction as follows:

$$\mathcal{K}(\vec{x},\omega) = \begin{bmatrix} k_{11}(\vec{x},\omega) & 0 \\ 0 & k_{22}(\vec{x},\omega) \end{bmatrix}, \quad \text{and}$$

$$k_{11}(\vec{x},\omega) = k_{22}(\vec{x},\omega) = a_0 + \sigma\sqrt{\lambda_0}Y_0(\omega) + \sum_{i=1}^{n_f} \sigma\sqrt{\lambda_i}[Y_i(\omega)cos(i\pi y) + Y_{n_f+i}(\omega)sin(i\pi y)],$$

where $\vec{x} = (x,y)^T, \lambda_0 = \frac{\sqrt{\pi L_c}}{2}$, $\lambda_i = \sqrt{\pi}L_c e^{-\frac{(i\pi L_c)^2}{4}}$ for $i = 1,\ldots,n_f$ and $Y_0,\ldots,Y_{2n_f}$ are uncorrelated random variables with zero mean and unit variance. In the following numerical test, we take the desired physical correlation length $L_c = 0.25$ for the random field and $a_0 = 1$, $\sigma = 0.15$, $n_f = 3$. We assume the random variables $Y_0,\ldots,Y_{2n_f}$ are independent and uniformly distributed in the interval $[-\sqrt{3},\sqrt{3}]$. Note that in this setting, the random functions $k_{11}(\vec{x},w), k_{22}(\vec{x},w)$ are guaranteed to be positive, and the corresponding $\mathcal{K}(\vec{x},\omega)$ is SPD.

Table 4.2. Errors and convergence rates of the ensemble algorithm ($J = 3$) for $\triangle t = h^3$.

h	$\|\phi_h - \phi\|_0^{E,1}$	rate	$\|\phi_h - \phi\|_0^{E,2}$	rate	$\|\phi_h - \phi\|_0^{E,3}$	rate
1/4	1.1563×10^{-1}	–	4.5348×10^{-2}	–	2.2165×10^{-2}	–
1/8	1.4786×10^{-2}	2.97	5.6293×10^{-3}	3.01	2.6717×10^{-3}	3.05
1/16	1.8504×10^{-3}	2.99	6.9932×10^{-4}	3.00	3.3003×10^{-4}	3.01
1/32	2.3132×10^{-4}	3.00	8.7305×10^{-5}	3.00	3.7074×10^{-5}	3.1
h	$\lvert\phi_h - \phi\rvert_1^{E,1}$	rate	$\lvert\phi_h - \phi\rvert_1^{E,2}$	rate	$\lvert\phi_h - \phi\rvert_1^{E,3}$	rate
1/4	3.5679×10^{-1}	–	2.7501×10^{-1}	–	2.6241×10^{-1}	–
1/8	7.3695×10^{-2}	2.08	6.7565×10^{-2}	2.03	6.6760×10^{-2}	1.99
1/16	1.7274×10^{-2}	2.09	1.6874×10^{-2}	2.00	1.6824×10^{-2}	2.00
1/32	4.1129×10^{-3}	2.07	4.1156×10^{-3}	2.03	4.1061×10^{-3}	2.03

Table 4.3. Errors and convergence rates of the ensemble algorithm ($J = 3$) for $\triangle t = h^3$.

h	$\|p_h - p\|_0^{E,1}$	rate	$\|p_h - p\|_0^{E,2}$	rate	$\|p_h - p\|_0^{E,3}$	rate
1/4	4.4572×10^{-1}	–	7.2784×10^{-1}	–	1.0725×10^{0}	–
1/8	5.5340×10^{-2}	3.00	9.0644×10^{-2}	3.00	1.3392×10^{-1}	3.00
1/16	6.2909×10^{-3}	3.03	9.7592×10^{-3}	3.12	1.4333×10^{-2}	3.13
1/32	7.7665×10^{-4}	3.01	1.2048×10^{-3}	3.02	1.7479×10^{-3}	3.03

The domain and parameters are the same as those in the first test. But in this test, the problem is associated with the forcing terms as follows:

$$\begin{aligned}
f_p &= (e^y - e^{-y})sin(x)e^t, \\
f_{f_1} &= [(1 + \nu + 4\nu\pi^2)\frac{k(\vec{x},\omega)}{\pi}]sin(2\pi y)cos(x)e^t, \\
f_{f_2} &= -2\nu k(\vec{x},\omega)cos(2\pi y)sin(x)e^t + (1+\nu)[-2k(\vec{x},\omega) + \frac{k(\vec{x},\omega)}{\pi^2}sin^2(\pi y)]sin(x)e^t.
\end{aligned}$$

The Dirichlet boundary condition:

$$\begin{aligned}
\phi &= (e^y - e^{-y})sin(x)e^t, \\
\overrightarrow{u} &= [\frac{k(\vec{x},\omega)}{\pi}sin(2\pi y)cos(x), (-2k(\vec{x},\omega) + \frac{k(\vec{x},\omega)}{\pi^2}sin^2(\pi y))sin(x)]^T e^t,
\end{aligned}$$

will be used on the boundary of the domain, and the initial conditions are chosen by

$$\begin{aligned} \phi &= (e^y - e^{-y}) sin(x), \\ \overrightarrow{u} &= [\frac{k(\vec{x},\omega)}{\pi} sin(2\pi y) cos(x), (-2k(\vec{x},\omega) + \frac{k(\vec{x},\omega)}{\pi^2} sin^2(\pi y)) sin(x)]^T. \end{aligned}$$

We simulate the system over the time interval $[0, 0.5]$, and the uniform triangulation with mesh size $h = 1/32$ and the uniform time partition with time step size $\triangle t = h^3$ are used.

We generate a set of J random samples of $\mathcal{K}$ by the Monte Carlo sampling and run our code for simulating the ensemble of the system associated with the J realizations. We use Algorithm 3 for ensemble computation, since $\mathcal{K}$ is diagonal, and multifrontal LU factorization as the linear solver. We first check the rate of convergence with respect to the number of samples, J. As the exact solution to the stochastic Stokes-Darcy system is unknown, we take the ensemble mean of numerical solutions of $J_0 = 1000$ realizations as our exact solution (expectation) and then evaluate the approximation errors based on this. The numerical results with $J = 10, 20, 40, 80, 160$ realizations are listed in Table 4.4. Using linear regression, the errors in Table 4.4 satisfy

$$\begin{aligned} &\|\overrightarrow{u}_h - \overrightarrow{u}\|_0 \approx 0.0291 J^{-0.5074}, \ \ |\overrightarrow{u}_h - \overrightarrow{u}|_1 \approx 0.2534 J^{-0.4870}, \\ &\|p_h - p\|_0 \approx 0.0267 J^{-0.5199}, \ \ \|\phi_h - \phi\|_0 \approx 0.0540 J^{-0.4996}. \end{aligned}$$

The values of $\|\cdot\|_0$ and $|\cdot|_1$ together with their linear regression models are plotted in Figure 4.1. It is seen that the rate of convergence with respect to J is close to -0.5, which coincides with our theoretical results.

We next test the efficiency of our ensemble algorithm. By setting $J = 1000$ and $h = 1/64$, we run the simulations again with the same parameter samples using the traditional approach, i.e., solving each realization individually. A comparison of the elapsed

Table 4.4. Errors of ensemble simulations.

J	10	20	40	80	100
$\Vert \overrightarrow{u}_h - \overrightarrow{u} \Vert_0^E$	9.0319×10^{-3}	6.2865×10^{-3}	4.5452×10^{-3}	3.2131×10^{-3}	2.1762×10^{-3}
$\vert \overrightarrow{u}_h - \overrightarrow{u} \vert_1^E$	8.2725×10^{-2}	5.7495×10^{-2}	4.3362×10^{-2}	3.0247×10^{-2}	2.1095×10^{-2}
$\Vert \phi_h - \phi \Vert_0^E$	8.0585×10^{-3}	5.5982×10^{-3}	3.9585×10^{-3}	2.8010×10^{-3}	1.8792×10^{-3}
$\vert \phi_h - \phi \vert_1^E$	1.7074×10^{-2}	1.2073×10^{-2}	8.5392×10^{-3}	6.1365×10^{-3}	4.2391×10^{-3}

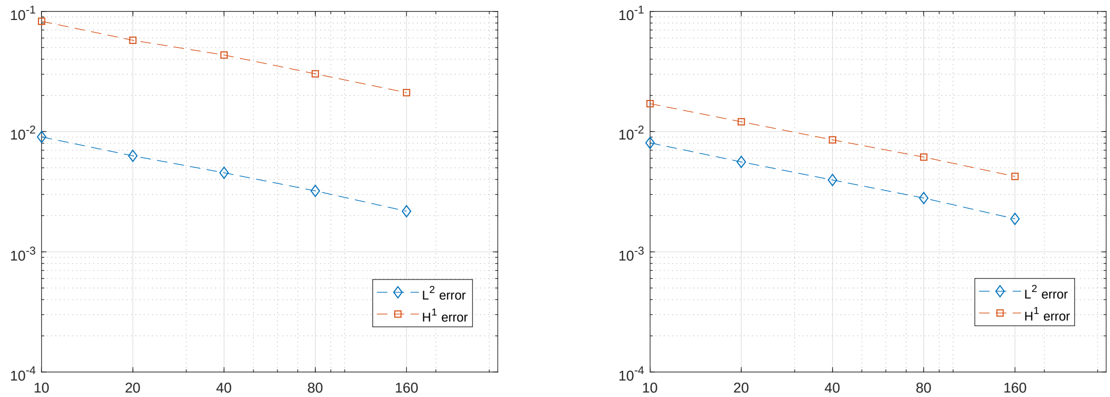

Figure 4.1. The rate of Ensemble simulations errors is $O(1/\sqrt{J})$ for $\overrightarrow{u}$ (left) and ϕ (right).

costs is presented in Table 4.5, from which one can clearly see that our ensemble algorithm is much faster than the traditional approach. The ensemble algorithm saves about 88% of the computation time.

Table 4.5. Solver comparison of ensemble with traditional method.

	matrix size	*Assembly*	*solving*
individual	54148×54148	1000	1000
ensemble	37407×37407	1	1000
	16641×16641	1	1000

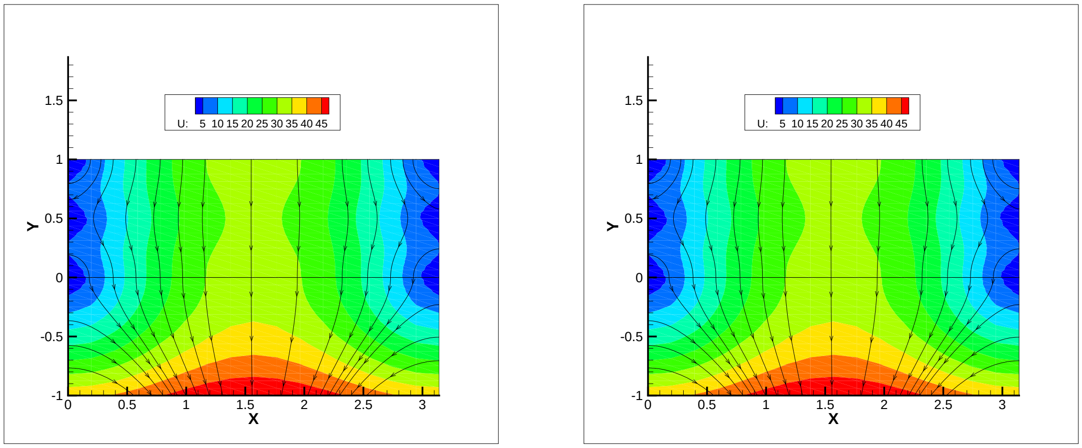

Figure 4.2. Streamlines of the ensemble mean obtained from individual runs (left) and ensemble algorithm (right) with $J = 80$ at $T = 0.5$.

We also plot numerical results of our ensemble algorithm and those of individual runs for comparison. The speed contours and velocity streamlines of the ensemble mean computed from both approaches at $T = 0.5$ with $J = 80$ realizations are presented in Figure 4.2. It is observed that both approaches capture the same general behavior of the flow, while our ensemble algorithm saves 89% of the computation time.

Next we consider the more realistic case where the hydraulic conductivity tensor is non-diagonal, for which we need to use Algorithm 2 for ensemble computation. Let

$$\mathcal{K}(\vec{x},\omega) = \begin{bmatrix} k_{11}(\vec{x},\omega) & k_{12}(\vec{x},\omega) \\ k_{21}(\vec{x},\omega) & k_{22}(\vec{x},\omega) \end{bmatrix},$$

where $k_{11}(\vec{x},\omega) = k_{22}(\vec{x},\omega) \neq 0$ and $k_{21}(\vec{x},\omega) = k_{12}(\vec{x},\omega) \neq 0$; i.e. $\mathcal{K}(\vec{x},\omega)$ is not diagonal but symmetric and

$$k_{11}(\vec{x},\omega) = k_{22}(\vec{x},\omega) = a_1 + \sigma\sqrt{\lambda_0}Y_0(\omega) + \sum_{i=1}^{n_f} \sigma\sqrt{\lambda_i}[Y_i(\omega)cos(i\pi y) + Y_{n_f+i}(\omega)sin(i\pi y)],$$

$$k_{21}(\vec{x},\omega) = k_{12}(\vec{x},\omega) = a_2 + \sigma\sqrt{\lambda_0}Y_0(\omega) + \sum_{i=1}^{n_f} \sigma\sqrt{\lambda_i}[Y_i(\omega)cos(i\pi y) + Y_{n_f+i}(\omega)sin(i\pi y)].$$

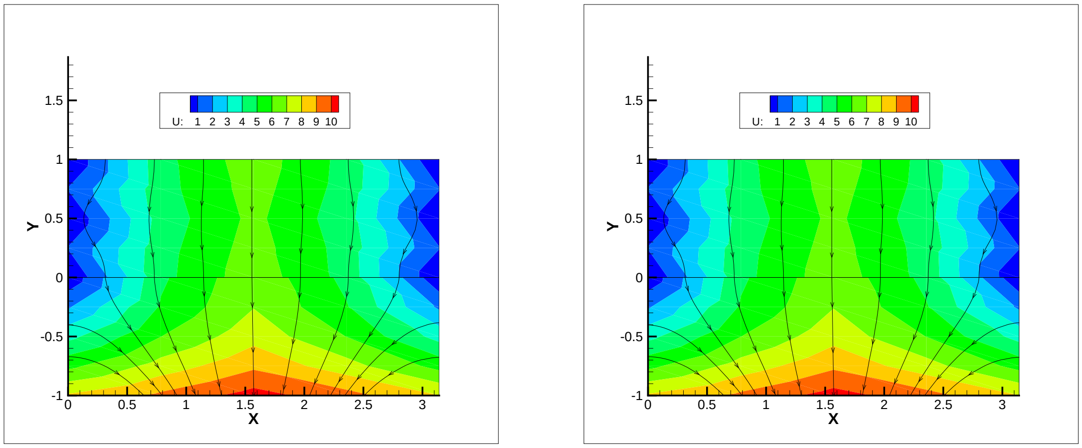

Figure 4.3. Streamlines of the ensemble mean obtained from individual runs (left) and computation using our ensemble algorithm (right) with $J = 100$ at $T = 0.5$.

The corresponding forcing term for the Darcy equation is $f_p = (1+k_{11}(\vec{x},\omega)-k_{22}(\vec{x},\omega))(e^y - e^{-y})sin(x)e^t - (k_{12}(\vec{x},\omega) + k_{21}(\vec{x},\omega))(e^y - e^{-y})cos(x)e^t$; for the Stokes equations, f_{f_1} and f_{f_2} are the same as those in Section 4.3.3.2. The boundary conditions and initial conditions are also the same as those in Section 4.3.3.2.

We take $a_1 = 10$ and $a_2 = 1$ so that the random hydraulic conductivity tensor $\mathcal{K}(\vec{x},\omega)$ is SPD. We consider a group of simulations with $J = 100$, using the Monte Carlo method for sampling. We plot the numerical results of our ensemble algorithm (Algorithm 2) and those of individual runs for comparison. The speed contours and velocity streamlines of the ensemble mean, computed from both approaches at $T = 0.5$ with $J = 100$ realizations, are presented in Figure 4.3. It can be seen that both approaches capture the same general behavior of the flow while our ensemble algorithm is much faster.

4.3. ARTIFICIAL COMPRESSIBILITY ENSEMBLE ALGORITHM

The artificial compressibility (AC) methods were first studied in the 1960s to decouple the velocity and pressure by a regularization of the divergence-free constraint for incompressible fluid flow equations. The idea is to add a small perturbation, e.g., ϵp or

ϵp_t, to the mass conservation equation and then eliminate the pressure from the momentum equation, so that one avoids 1) solving a saddle point problem at each time step, and 2) spurious boundary layers for the pressure caused by imposing artificial boundary conditions. Some of the first AC methods in the literature were proposed by Chorin Chorin (1967), Temam Temam (1968, 1969a,b), Vladimirova, Kuznetsov, and Yanenko Kuznetsov *et al.* (1966). The AC methods can have severe time step restrictions if not discretized carefully Chorin (1967). In a recent work, Guermond and Minev introduced a bootstrapping technique to design unconditionally stable, higher order AC methods in Guermond and Minev (2015). The proposed methods, unlike the popular projection methods, which cannot exceed second-order accuracy in time without out losing unconditional stability, can reach any order in time while being unconditionally stable (for the unsteady Stokes equations). DeCaria, Layton, and McLaughlin DeCaria *et al.* (2017) studied an unconditionally stable AC method based on a Crank-Nicolson Leapfrog time discretization for the Navier-Stokes equations. The AC methods have also been recently applied to MHD flows for efficiency, Rong *et al.* (2018). In Philippe and Pierre (2012), the artificial compressibility splitting method, which is extended from the penalty-projection method for the unsteady Navier-Stokes equations, was viewed as a hybrid two-step prediction-correction method by combining the artificial compressibility method and an augmented Lagrangian method without inner iteration. Error analysis of some variants of the AC method for computing the solutions of the Navier-Stokes problems can be found in Shen (1995, 1996). Compared with the extensively studied Stokes/Navier-Stokes equations, the coupled time-dependent Stokes-Darcy equation is still in need of continued efforts to develop efficient methods in this area.

Based on the key ideas of Jiang and Layton (2014); Jiang and Qiu (2019), which was a fundamental development of the efficient ensemble algorithm for flow equations, in this article we utilize the idea of artificial compressibility and partitioned time stepping methods to construct the decoupled ensemble algorithm for efficiently computing multiple realizations of the stochastic Stokes-Darcy interface model with a random hydraulic conductivity tensor, source terms, and initial conditions. In this algorithm, the originally coupled multi-physics model is decoupled by a two-level technique. The first level is to decouple the Stokes flow from the Darcy flow by the physical interface conditions and the partitioned time stepping method. The second level is to decouple the velocity and pressure by the artificial compressibility method in the Stokes equation. Hence, the Stokes-Darcy model is decoupled into three subproblems. One of the three subproblems is a straightforward update for the pressure. For each of the other two subproblems, all the realizations share the same coefficient matrix, which is independent of time. The common coefficient matrix feature eliminates many redundant matrix operations, such as matrix assembly and matrix preprocess, before solving the system. Hence, the efficiency can be increased by a significant amount, which will be further explained in more details in the second numerical experiment. These features of the proposed algorithm significantly reduce the storage requirements and computational costs. Furthermore, compared with the previous works on non-interface problems in this area, extra efforts are required in order to deal with the randomness in the interface conditions of the Stokes-Darcy system.

4.3.1. Formulation of Artificial Compressibility Ensemble Algorithm. For the artificial compressibility ensemble algorithm, the originally coupled Stokes-Darcy interface model is decoupled into three subproblems by a two-level technique. We can use the first level to decouple the Stokes flow from the Darcy flow by the physical interface conditions and the partitioned time stepping method to get subproblem 1. In order to get the subproblem 3, the second level is to decouple the velocity and pressure by the artificial compressibility method in the Stokes equation. Subproblem 2 is a straightforward update for the pressure.

We can propose the artificial compressibility ensemble algorithm as follows:

Algorithm 4 *Find* $(\overrightarrow{u}_j^{n+1}, p_j^{n+1}, \phi_j^{n+1}) \in X_S \times Q_f \times X_D$ *satisfying* $\forall\, (v, \psi) \in X_S \times X_D$,

$$\begin{cases} \left(\dfrac{\overrightarrow{u}_j^{n+1} - \overrightarrow{u}_j^{n}}{\Delta t}, v \right)_f + \nu(\nabla \overrightarrow{u}_j^{n+1}, \nabla v)_f + \sum_i \int_\Gamma \bar{\eta}_i (\overrightarrow{u}_j^{n+1} \cdot \widehat{\tau}_i)(v \cdot \widehat{\tau}_i)\, ds \\ + \sum_i \int_\Gamma (\eta_{i,j} - \bar{\eta}_i)(\overrightarrow{u}_j^{n} \cdot \widehat{\tau}_i)(v \cdot \widehat{\tau}_i)\, ds + \gamma(\nabla \cdot \overrightarrow{u}_j^{n+1}, \nabla \cdot v)_f - \left(p_j^n, \nabla \cdot v \right)_f \\ \qquad + c_\Gamma(v, \phi_j^n) = (f_{f,j}^{n+1}, v)_f, \end{cases} \tag{subproblem 1}$$

$$p_j^{n+1} = p_j^n - \gamma \nabla \cdot \overrightarrow{u}_j^{n+1}, \tag{subproblem 2}$$

$$\begin{cases} g S_0 \left(\dfrac{\phi_j^{n+1} - \phi_j^n}{\Delta t}, \psi \right)_p + g(\bar{\mathcal{K}} \nabla \phi_j^{n+1}, \nabla \psi)_p + g((\mathcal{K}_j - \bar{\mathcal{K}}) \nabla \phi_j^n, \nabla \psi)_p \\ \qquad - c_\Gamma(\overrightarrow{u}_j^n, \psi) = g(f_{p,j}^{n+1}, \psi)_p, \end{cases} \tag{subproblem 3}$$

where $\bar{\mathcal{K}} = \frac{1}{J} \sum_{j=1}^{J} \mathcal{K}_j$, $\quad \eta_{i,j} = \frac{\alpha_{\mathrm{BJS}}}{\sqrt{\widehat{\tau}_i \cdot \mathcal{K}_j \widehat{\tau}_i}}$ $\quad$ and $\quad \bar{\eta}_i = \frac{1}{J} \sum_{j=1}^{J} \eta_{i,j}$.

Subproblem 2 can be rewritten as $\frac{\Delta t}{\gamma} \frac{p_j^{n+1} - p_j^n}{\Delta t} + \nabla \cdot \overrightarrow{u}_j^{n+1} = 0$. Thus γ should chosen to be $O(1)$ or larger for the method to be first order convergent. Moving all the known quantities to the right hand side, the algorithm is as follows:

$$\begin{cases} \left(\dfrac{\overrightarrow{u}_j^{n+1} - \overrightarrow{u}_j^{n}}{\Delta t}, v \right)_f + \nu(\nabla \overrightarrow{u}_j^{n+1}, \nabla v)_f + \sum_i \int_\Gamma \bar{\eta}_i (\overrightarrow{u}_j^{n+1} \cdot \widehat{\tau}_i)(v \cdot \widehat{\tau}_i)\, ds \\ + \gamma(\nabla \cdot \overrightarrow{u}_j^{n+1}, \nabla \cdot v)_f = \left(p_j^n, \nabla \cdot v \right)_f + (f_{f,j}^{n+1}, v)_f \\ - \sum_i \int_\Gamma (\eta_{i,j} - \bar{\eta}_i)(\overrightarrow{u}_j^{n} \cdot \widehat{\tau}_i)(v \cdot \widehat{\tau}_i)\, ds - c_\Gamma(v, \phi_j^n), \end{cases} \tag{subproblem 1}$$

$$p_j^{n+1} = p_j^n - \gamma \nabla \cdot \overrightarrow{u}_j^{n+1}, \tag{subproblem 2}$$

$$\begin{cases} g S_0 \left(\dfrac{\phi_j^{n+1} - \phi_j^n}{\Delta t}, \psi_h \right)_p + g(\bar{\mathcal{K}} \nabla \phi_j^{n+1}, \nabla \psi)_p \\ \quad = g(f_{p,j}^{n+1}, \psi)_p - g((\mathcal{K}_j - \bar{\mathcal{K}}) \nabla \phi_j^n, \nabla \psi)_p + c_\Gamma(\overrightarrow{u}^n, \psi). \end{cases} \tag{subproblem 3}$$

Remark 5 *From the above algorithm, it is easy to see that the original coupled system is decoupled into three subproblems, i.e., the (subproblem 1), (subproblem 2), and (subproblem 3) by the two-level technique. For all time steps and realizations, (subproblem 1) can be solved by the linear systems with one common coefficient matrix, since the coefficients of the unknowns are independent of both time and the ensemble index. The coefficients of the unknown interface term are always consistent at every time step because of the commonly used mean* $\bar{\eta}_i$*. Similarly, all realizations in (subproblem 3) also share the common coefficient matrix for solving the linear systems. The second equation (subproblems 2) is a straightforward update that does not require solving any linear systems.*

4.3.2. Long-time Stability and Error Analysis. We prove the long time stability of Algorithm 4 under a time step condition and two parameter conditions:

$$\Delta t \le \min\left\{\frac{(1-\alpha_1-\alpha_2)\beta_1^2\bar{k}_{min}}{C_{P,p}^2}, \frac{(1-\beta_1-\beta_2-\frac{\rho'_{max}}{\bar{k}_{min}})\alpha_1^2 S_0 \nu}{C_{P,f}^2}\right\}\frac{2\nu\bar{k}_{min}}{g^2[C(D_f)C(D_p)]^4}, \quad (4.76)$$

$$\eta_i'^{max} \le \bar{\eta}_i^{min}, \qquad \text{and} \qquad \rho'_{max} < \bar{k}_{min}. \quad (4.77)$$

Theorem 9 (Long time stability of Algorithm 4) *If the two parameter conditions in* (4.77) *both hold, and there exist* $\alpha_1, \alpha_2, \beta_1, \beta_2$ *in* $(0,1)$ *such that the time-step condition* (4.76) *also holds, then Algorithm 4 is long time stable: for any* $N > 0$,

$$\begin{aligned}
&\frac{1}{2}\|\overrightarrow{u}_j^N\|_f^2 + \frac{gS_0}{2}\|\phi_j^N\|_p^2 + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla\overrightarrow{u}_j^N\|_f^2 + \frac{\Delta t}{2\gamma}\|p_j^N\|_f^2 + \Delta t\sum_i \frac{\bar{\eta}_i^{min}}{2}\int_\Gamma (\overrightarrow{u}_j^N\cdot\widehat{\tau}_i)^2\,ds \\
&+\left(\Delta t^2\frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \Delta t\frac{g\rho'_{max}}{2}\right)\|\nabla\phi_j^N\|_p^2 + \frac{\Delta t}{2\gamma}\sum_{n=0}^{N-1}\|p_j^{n+1}-p_j^n\|_f^2 \qquad (4.78)\\
&\le \frac{1}{2}\|\overrightarrow{u}_j^0\|_f^2 + \frac{gS_0}{2}\|\phi_j^0\|_p^2 + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla\overrightarrow{u}_j^0\|_f^2 + \frac{\Delta t}{2\gamma}\|p_j^0\|_f^2 + \Delta t\sum_i \frac{\bar{\eta}_i^{min}}{2}\int_\Gamma (\overrightarrow{u}_j^0\cdot\widehat{\tau}_i)^2\,ds \\
&+\left(\Delta t^2\frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \Delta t\frac{g\rho'_{max}}{2}\right)\|\nabla\phi_j^0\|_p^2 + \Delta t\sum_{n=0}^{N-1}\frac{C_{P,f}^2}{4\alpha_2\nu}\|f_{f,j}^{n+1}\|_f^2 + \Delta t\sum_{n=0}^{N-1}\frac{gC_{P,p}^2}{4\beta_2\bar{k}_{min}}\|f_{p,j}^{n+1}\|_p^2.
\end{aligned}$$

Proof 9 *Setting* $v = \overrightarrow{u}_j^{n+1}$, $\psi = \phi_j^{n+1}$ *in Algorithm 4, replacing* $\gamma\nabla\cdot\overrightarrow{u}_j^{n+1}$ *in the momentum equation by* $p_j^{n+1} - p_j^n$, *taking inner product of the mass conservation equation by* $\gamma^{-1}p_j^{n+1}$, *using* $a^2 - ab = 1/2[a^2 - b^2 + (a-b)^2]$ *and adding all three equations yields*

$$\begin{aligned}
&\frac{1}{2\Delta t}\|\overrightarrow{u}_j^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|\overrightarrow{u}_j^{n}\|_f^2 + \frac{1}{2\Delta t}\|\overrightarrow{u}_j^{n+1} - \overrightarrow{u}_j^{n}\|_f^2 + \nu\|\nabla\overrightarrow{u}_j^{n+1}\|_f^2 \\
&+ \sum_i \int_\Gamma \bar{\eta}_i(\overrightarrow{u}_j^{n+1}\cdot\widehat{\tau}_i)(\overrightarrow{u}_j^{n+1}\cdot\widehat{\tau}_i)\,ds + \frac{1}{2\gamma}\left(\|p_j^{n+1}\|_f^2 - \|p_j^n\|_f^2 + \|p_j^{n+1} - p_j^n\|_f^2\right) \\
&+ \frac{gS_0}{2\Delta t}\|\phi_j^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi_j^{n}\|_p^2 + \frac{gS_0}{2\Delta t}\|\phi_j^{n+1} - \phi_j^n\|_p^2 + g(\bar{\mathcal{K}}\nabla\phi_j^{n+1}, \nabla\phi_j^{n+1})_p \\
&+ c_I(\overrightarrow{u}_j^{n+1}, \phi_j^n) - c_I(\overrightarrow{u}_j^{n}, \phi_j^{n+1}) \\
&= (f_{f,j}^{n+1}, \overrightarrow{u}_j^{n+1})_f + g(f_{p,j}^{n+1}, \phi_j^{n+1})_p - \sum_i \int_\Gamma (\eta_{i,j} - \bar{\eta}_i)(\overrightarrow{u}_j^{n}\cdot\widehat{\tau}_i)(\overrightarrow{u}_j^{n+1}\cdot\widehat{\tau}_i)\,ds \\
&- g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla\phi_j^n, \nabla\phi_j^{n+1})_p.
\end{aligned} \tag{4.79}$$

Applying estimates (4.10) *and* (4.11) *with* $\epsilon_1 = \frac{\Delta t}{2gS_0}$, $\epsilon_2 = \frac{\Delta t}{2}$, *we have the same result as (3.7) in Jiang and Qiu (2019), as follows:*

$$\begin{aligned}
&c_I(\overrightarrow{u}_j^{n+1}, \phi_j^n) - c_I(\overrightarrow{u}_j^{n}, \phi_j^{n+1}) = c_I(\overrightarrow{u}_j^{n+1} - \overrightarrow{u}_j^{n}, \phi_j^{n+1}) - c_I(\overrightarrow{u}_j^{n+1}, \phi_j^{n+1} - \phi_j^n) \\
&\geq -\frac{1}{2\Delta t}\|\overrightarrow{u}_j^{n+1} - \overrightarrow{u}_j^{n}\|_f^2 - \Delta t\frac{C_2}{\beta_1^2}\left(\|\nabla\overrightarrow{u}_j^{n+1}\|_f^2 + \|\nabla\overrightarrow{u}_j^{n}\|_f^2\right) - \beta_1 g\bar{k}_{min}\|\nabla\phi_j^{n+1}\|_p^2 \\
&\quad - \frac{gS_0}{2\Delta t}\|\phi_j^{n+1} - \phi_j^n\|_p^2 - \frac{\Delta t}{gS_0}\frac{C_1}{\alpha_1^2}\left(\|\nabla\phi_j^{n+1}\|_p^2 + \|\nabla\phi_j^n\|_p^2\right) - \alpha_1\nu\|\nabla\overrightarrow{u}_j^{n+1}\|_f^2.
\end{aligned} \tag{4.80}$$

By using the Cauchy-Schwarz inequalities and Young's inequalities, for any $\alpha_2 > 0, \beta_2 > 0$ *we have*

$$\begin{aligned}
&(f_{f,j}^{n+1}, \overrightarrow{u}_j^{n+1})_f + g(f_{p,j}^{n+1}, \phi_j^{n+1})_p \\
&\qquad \leq \frac{C_{P,f}^2}{4\alpha_2\nu}\|f_{f,j}^{n+1}\|_f^2 + \alpha_2\nu\|\nabla\overrightarrow{u}_j^{n+1}\|_f^2 + \frac{gC_{P,p}^2}{4\beta_2\bar{k}_{min}}\|f_{p,j}^{n+1}\|_p^2 + \beta_2 g\bar{k}_{min}\|\nabla\phi_j^{n+1}\|_p^2.
\end{aligned} \tag{4.81}$$

By recalling (3.9) in Jiang and Qiu (2019), the other two terms on the right hand side of (4.79) *can be bounded as follows:*

$$-\sum_i \int_\Gamma (\eta_{i,j} - \bar{\eta}_i)(\overrightarrow{u}_j^n \cdot \widehat{\tau}_i)(\overrightarrow{u}_j^{n+1} \cdot \widehat{\tau}_i)\, ds \tag{4.82}$$
$$\leq \sum_i \left[\frac{\eta_i'^{max}}{2} \int_\Gamma (\overrightarrow{u}_j^n \cdot \widehat{\tau}_i)^2\, ds + \frac{\eta_i'^{max}}{2} \int_\Gamma (\overrightarrow{u}_j^{n+1} \cdot \widehat{\tau}_i)^2\, ds \right],$$

$$-g\left((\mathcal{K}_j - \bar{\mathcal{K}})\nabla\phi_j^n, \nabla\phi_j^{n+1}\right)_p \leq g\int_{D_p} |\nabla\phi_j^{n+1}|_2 |\mathcal{K}_j - \bar{\mathcal{K}}|_2 |\nabla\phi_j^n|_2\, dx \tag{4.83}$$
$$\leq \frac{g\rho'_{max}}{2}\|\nabla\phi_j^n\|_p^2 + \frac{g\rho'_{max}}{2}\|\nabla\phi_j^{n+1}\|_p^2.$$

Using the above estimates, equation (4.79) *becomes*

$$\begin{aligned}
&\frac{1}{2\Delta t}\|\overrightarrow{u}_j^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|\overrightarrow{u}_j^n\|_f^2 + \left(1 - \alpha_1 - \alpha_2 - \Delta t\frac{2C_2}{\beta_1^2\nu}\right)\nu\|\nabla\overrightarrow{u}_j^{n+1}\|_f^2 \\
&+\Delta t\frac{C_2}{\beta_1^2}\left(\|\nabla\overrightarrow{u}_j^{n+1}\|_f^2 - \|\nabla\overrightarrow{u}_j^n\|_f^2\right) + \sum_i\left[\frac{\bar{\eta}_i^{min}}{2} - \frac{\eta_i'^{max}}{2}\right]\int_\Gamma(\overrightarrow{u}_j^{n+1}\cdot\widehat{\tau}_i)^2\, ds \\
&+\sum_i \frac{\bar{\eta}_i^{min}}{2}\left[\int_\Gamma(\overrightarrow{u}_j^{n+1}\cdot\widehat{\tau}_i)^2\, ds - \int_\Gamma(\overrightarrow{u}_j^n\cdot\widehat{\tau}_i)^2\, ds\right] \\
&+\sum_i\left[\frac{\bar{\eta}_i^{min}}{2} - \frac{\eta_i'^{max}}{2}\right]\int_\Gamma(\overrightarrow{u}_j^n\cdot\widehat{\tau}_i)^2\, ds + \frac{1}{2\gamma}\left(\|p_j^{n+1}\|_f^2 - \|p_j^n\|_f^2 + \|p_j^{n+1} - p_j^n\|_f^2\right) \\
&+\frac{gS_0}{2\Delta t}\|\phi_j^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi_j^n\|_p^2 + (1 - \beta_1 - \beta_2 - \Delta t\frac{1}{g^2S_0\bar{k}_{min}}\frac{2C_1}{\alpha_1^2} - \frac{\rho'_{max}}{\bar{k}_{min}})g\bar{k}_{min}\|\nabla\phi_j^{n+1}\|_p^2 \\
&+\left(\Delta t\frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \frac{g\rho'_{max}}{2}\right)\left(\|\nabla\phi_j^{n+1}\|_p^2 - \|\nabla\phi_j^n\|_p^2\right) \leq \frac{C_{P,f}^2}{4\alpha_2\nu}\|f_{f,j}^{n+1}\|_f^2 + \frac{gC_{P,p}^2}{4\beta_2\bar{k}_{min}}\|f_{p,j}^{n+1}\|_p^2.
\end{aligned} \tag{4.84}$$

The stability holds if

$$1 - \alpha_1 - \alpha_2 - \Delta t\frac{2C_2}{\beta_1^2\nu} \geq 0, \quad \frac{\bar{\eta}_i^{min}}{2} - \frac{\eta_i'^{max}}{2} \geq 0, \tag{4.85}$$

$$1 - \beta_1 - \beta_2 - \Delta t\frac{1}{g^2S_0\bar{k}_{min}}\frac{2C_1}{\alpha_1^2} - \frac{\rho'_{max}}{\bar{k}_{min}} \geq 0. \tag{4.86}$$

Recall that if $\alpha_1, \alpha_2, \beta_1, \beta_2, \Delta t, \eta_i'^{max}, \rho'_{max}$ are all positive, we then have the following constraints on these parameters:

$$0 < \alpha_1 < 1, \quad 0 < \alpha_2 < 1, \qquad 0 < \beta_1 < 1, \quad 0 < \beta_2 < 1, \tag{4.87}$$

$$\frac{\rho'_{max}}{\bar{k}_{min}} < 1, \quad \eta_i'^{max} \leq \bar{\eta}_i^{min}, \tag{4.88}$$

$$\Delta t \leq \min\left\{ \frac{(1-\alpha_1-\alpha_2)\beta_1^2 \nu}{2C_2}, \frac{(1-\beta_1-\beta_2-\frac{\rho'_{max}}{\bar{k}_{min}})\alpha_1^2 g^2 S_0 \bar{k}_{min}}{2C_1} \right\}. \tag{4.89}$$

(4.88) *leads to the two parameter conditions listed in* (4.77)*, and* (4.89) *leads to the time step condition* (4.76) *required for stability. Now if the time step condition* (4.76) *and the two parameter conditions in* (4.77) *all hold,* (4.84) *reduces to*

$$\begin{aligned} &\frac{1}{2\Delta t}\|\overrightarrow{u}_j^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|\overrightarrow{u}_j^{n}\|_f^2 + \Delta t \frac{C_2}{\beta_1^2}\left(\|\nabla \overrightarrow{u}_j^{n+1}\|_f^2 - \|\nabla \overrightarrow{u}_j^{n}\|_f^2\right) \\ &+ \sum_i \frac{\bar{\eta}_i^{min}}{2}\left[\int_\Gamma (\overrightarrow{u}_j^{n+1}\cdot\widehat{\tau}_i)^2\, ds - \int_\Gamma (\overrightarrow{u}_j^{n}\cdot\widehat{\tau}_i)^2\, ds\right] + \frac{1}{2\gamma}\left(\|p_j^{n+1}\|_f^2 - \|p_j^n\|_f^2 + \|p_j^{n+1}-p_j^n\|_f^2\right) \\ &+ \frac{gS_0}{2\Delta t}\|\phi_j^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi_j^{n}\|_p^2 + \left(\Delta t \frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \frac{g\rho'_{max}}{2}\right)\left(\|\nabla\phi_j^{n+1}\|_p^2 - \|\nabla\phi_j^{n}\|_p^2\right) \\ &\leq \frac{C_{P,f}^2}{4\alpha_2\nu}\|f_{f,j}^{n+1}\|_f^2 + \frac{gC_{P,p}^2}{4\beta_2\bar{k}_{min}}\|f_{p,j}^{n+1}\|_p^2. \end{aligned} \tag{4.90}$$

Sum (4.90) *from $n = 0$ to $N-1$ and multiply it by Δt to get*

$$\begin{aligned} &\frac{1}{2}\|\overrightarrow{u}_j^{N}\|_f^2 + \frac{gS_0}{2}\|\phi_j^N\|_p^2 + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla\overrightarrow{u}_j^{N}\|_f^2 + \frac{\Delta t}{2\gamma}\|p_j^N\|_f^2 + \Delta t\sum_i \frac{\bar{\eta}_i^{min}}{2}\int_\Gamma(\overrightarrow{u}_j^{N}\cdot\widehat{\tau}_i)^2\, ds \\ &+ \left(\Delta t^2\frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \Delta t\frac{g\rho'_{max}}{2}\right)\|\nabla\phi_j^N\|_p^2 + \frac{\Delta t}{2\gamma}\sum_{n=0}^{N-1}\|p_j^{n+1}-p_j^n\|_f^2 \\ &\leq \frac{1}{2}\|\overrightarrow{u}_j^{0}\|_f^2 + \frac{gS_0}{2}\|\phi_j^0\|_p^2 + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla\overrightarrow{u}_j^{0}\|_f^2 + \frac{\Delta t}{2\gamma}\|p_j^0\|_f^2 + \Delta t\sum_i \frac{\bar{\eta}_i^{min}}{2}\int_\Gamma(\overrightarrow{u}_j^{0}\cdot\widehat{\tau}_i)^2\, ds \\ &+ \left(\Delta t^2\frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \Delta t\frac{g\rho'_{max}}{2}\right)\|\nabla\phi_j^0\|_p^2 + \Delta t\sum_{n=0}^{N-1}\frac{C_{P,f}^2}{4\alpha_2\nu}\|f_{f,j}^{n+1}\|_f^2 + \Delta t\sum_{n=0}^{N-1}\frac{gC_{P,p}^2}{4\beta_2\bar{k}_{min}}\|f_{p,j}^{n+1}\|_p^2, \end{aligned} \tag{4.91}$$

which completes the proof.

Now we start to give a detailed error analysis for Algorithm 4. Let $e^n_{j,\overrightarrow{u}} := \overrightarrow{u}_j(t_n) - \overrightarrow{u}^n_j, e^n_{j,p} := p_j(t_n) - p^n_j, e^n_{j,\phi} := \phi_j(t_n) - \phi^n_{j,h}$ denote the errors at t_n between the true solution $(\overrightarrow{u}_j(t_n), p_j(t_n), \phi_j(t_n))$ of (4.2) and the approximation $(\overrightarrow{u}^n_j, p^n_j, \phi^n_j)$ obtained using the AC ensemble Algorithm 4. We prove the convergence of Algorithm 4 under a time step condition and two parameter conditions.

$$\Delta t \leq \min\left\{\frac{(1-\alpha_1-\alpha_2)\beta_1^2\bar{k}_{min}}{C^2_{P,p}}, \frac{(1-\beta_1-\beta_2-(1+\beta_3)\frac{\rho'_{max}}{\bar{k}_{min}})\alpha_1^2 S_0 \nu}{C^2_{P,f}}\right\}\frac{2\nu\bar{k}_{min}}{g^2[C(D_f)C(D_p)]^4}, \tag{4.92}$$

$$\eta_i'^{max} \leq \bar{\eta}_i^{min} \qquad \text{and} \qquad \rho'_{max} < \bar{k}_{min}. \tag{4.93}$$

Theorem 10 (Error Estimate) *For any $j = 1, \ldots, J$, if the two parameter conditions in* (4.93) *hold, and there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0,1)$ and $\beta_3 > 0$ such that the time step condition* (4.92) *also holds, then there is a positive constant C independent of the time step Δt such that*

$$\begin{aligned}&\frac{1}{2}\|e^N_{j,\overrightarrow{u}}\|_f^2 + (\frac{\alpha_2}{3}\nu\Delta t + \Delta t^2\frac{C_2}{\beta_1^2})\|\nabla e^N_{j,\overrightarrow{u}}\|_f^2 + \sum_i(\sigma_2 + \frac{1}{2})\Delta t\eta_i'^{max}\int_\Gamma(e^N_{j,\overrightarrow{u}}\cdot\widehat{\tau}_i)^2\,ds\\ &+\frac{gS_0}{2}\|e^N_{j,\phi}\|_p^2 + \left(\frac{\beta_2}{2}g\bar{k}_{min} + \frac{1}{2}g\rho'_{max} + \frac{\Delta t C_1}{gS_0\alpha_1^2}\right)\Delta t\|\nabla e^N_{j,\phi}\|_p^2 + \frac{1}{2\gamma}\Delta t\|e^N_{j,p}\|_f^2\\ &+\sum_{n=0}^{N-1}\frac{1}{2\gamma}\Delta t\|e^{n+1}_{j,p} - e^n_{j,p}\|_f^2 \leq C\Delta t^2.\end{aligned} \tag{4.94}$$

Proof 10 *For $\forall v \in X_S, \forall\psi \in X_D, \forall q \in Q_S$, the true solution $(\overrightarrow{u}_j, p_j, \phi_j)$ satisfies*

$$\begin{aligned}&\left(\frac{\overrightarrow{u}_j(t_{n+1}) - \overrightarrow{u}_j(t_n)}{\Delta t}, v\right)_f + \nu(\nabla\overrightarrow{u}_j(t_{n+1}), \nabla v)_f + \sum_i\int_\Gamma \eta_{i,j}(\overrightarrow{u}_j(t_{n+1})\cdot\widehat{\tau}_i)(v\cdot\widehat{\tau}_i)\,ds\\ &\qquad - \left(p_j(t_{n+1}), \nabla\cdot v\right)_f + c_\Gamma(v, \phi_j(t_n)) = (f^{n+1}_{f,j}, v)_f + \epsilon^{n+1}_{j,f}(v),\end{aligned} \tag{4.95}$$

$$\left(p_j(t_{n+1}) - p_j(t_n), q\right)_f + \gamma(\nabla\cdot\overrightarrow{u}_j(t_{n+1}), q)_f = \left(p_j(t_{n+1}) - p_j(t_n), q\right)_f, \tag{4.96}$$

$$\begin{aligned}&gS_0\left(\frac{\phi_j(t_{n+1}) - \phi_j(t_n)}{\Delta t}, \psi\right)_p + g(\mathcal{K}_j\nabla\phi_j(t_{n+1}), \nabla\psi_h)_p - c_\Gamma(\overrightarrow{u}_j(t_n), \psi)\\ &= g(f^{n+1}_{p,j}, \psi)_p + \epsilon^{n+1}_{j,p}(\psi).\end{aligned} \tag{4.97}$$

The consistency errors $\epsilon_{j,f}^{n+1}(v), \epsilon_{j,p}^{n+1}(\psi)$ *are defined by*

$$\epsilon_{j,f}^{n+1}(v) := \left(\frac{\overrightarrow{u}_j(t_{n+1}) - \overrightarrow{u}_j(t_n)}{\Delta t} - u_{j,t}(t_{n+1}), v \right)_f - c_\Gamma(v, \phi_j(t_{n+1}) - \phi_j(t_n)),$$

$$\epsilon_{j,p}^{n+1}(\psi) := gS_0 \left(\frac{\phi_j(t_{n+1}) - \phi_j(t_n)}{\Delta t} - \phi_{j,t}(t_{n+1}), \psi \right)_p + c_\Gamma(\overrightarrow{u}_j(t_{n+1}) - \overrightarrow{u}_j(t_n), \psi).$$

Subtracting Algorithm 4 from (4.95)-(4.97)*, then for* $\forall v \in X_S$*,* $\forall \psi \in X_D$*, and* $\forall q \in Q_S$*,*

$$\begin{aligned}
&\left(\frac{e_{j,\overrightarrow{u}}^{n+1} - e_{j,\overrightarrow{u}}^{n}}{\Delta t}, v \right)_f + \nu(\nabla e_{j,\overrightarrow{u}}^{n+1}, \nabla v)_f + \sum_i \int_\Gamma \bar{\eta}_i (e_{j,\overrightarrow{u}}^{n+1} \cdot \widehat{\tau}_i)(v \cdot \widehat{\tau}_i)\, ds \\
&\quad + \sum_i \int_\Gamma \left(\eta_{i,j} - \bar{\eta}_i\right) (e_{j,\overrightarrow{u}}^{n} \cdot \widehat{\tau}_i)(v \cdot \widehat{\tau}_i)\, ds - \left(e_{j,p}^{n+1}, \nabla \cdot v\right)_f + c_\Gamma(v, e_{j,\phi}^{n}) \\
&\quad = -\sum_i \int_\Gamma \left(\eta_{i,j} - \bar{\eta}_i\right) ((\overrightarrow{u}_j^{n+1} - \overrightarrow{u}_j^{n}) \cdot \widehat{\tau}_i)(v \cdot \widehat{\tau}_i)\, ds + \epsilon_{j,f}^{n+1}(v), \qquad (4.98)\\
&\frac{1}{\gamma}\left(e_{j,p}^{n+1} - e_{j,p}^{n}, q\right)_f + (\nabla \cdot e_{j,\overrightarrow{u}}^{n+1}, q)_f = \frac{1}{\gamma}\left(p_j(t_{n+1}) - p_j(t_n), q\right)_f, \qquad (4.99)\\
&gS_0 \left(\frac{e_{j,\phi}^{n+1} - e_{j,\phi}^{n}}{\Delta t}, \psi \right)_p + g(\bar{\mathcal{K}} \nabla e_{j,\phi}^{n+1}, \nabla \psi)_p + g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla e_{j,\phi}^{n}, \nabla\psi)_p - c_\Gamma(e_{j,\overrightarrow{u}}^{n}, \psi) \\
&\quad = -g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla(\phi_j^{n+1} - \phi_j^{n}), \nabla\psi)_p + \epsilon_{j,p}^{n+1}(\psi). \qquad (4.100)
\end{aligned}$$

Setting $v = e_{j,\overrightarrow{u}}^{n+1}, q = e_{j,p}^{n+1}, \psi = e_{j,\phi}^{n+1}$ *in* (4.98)-(4.100) *and adding the three equations yields*

$$\begin{aligned}
&\frac{1}{2\Delta t}\|e_{j,\overrightarrow{u}}^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|e_{j,\overrightarrow{u}}^{n}\|_f^2 + \frac{1}{2\Delta t}\|e_{j,\overrightarrow{u}}^{n+1} - e_{j,\overrightarrow{u}}^{n}\|_f^2 + \nu\|\nabla e_{j,\overrightarrow{u}}^{n+1}\|_f^2 + \sum_i \int_\Gamma \bar{\eta}_i (e_{j,\overrightarrow{u}}^{n+1} \cdot \widehat{\tau}_i)^2\, ds \\
&+\frac{gS_0}{2\Delta t}\|e_{j,\phi}^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|e_{j,\phi}^{n}\|_p^2 + \frac{gS_0}{2\Delta t}\|e_{j,\phi}^{n+1} - e_{j,\phi}^{n}\|_p^2 + g(\bar{\mathcal{K}}\nabla e_{j,\phi}^{n+1}, \nabla e_{j,\phi}^{n+1})_p + c_I(e_{j,\overrightarrow{u}}^{n+1}, e_{j,\phi}^{n}) \\
&-c_I(e_{j,\overrightarrow{u}}^{n}, e_{j,\phi}^{n+1}) + \frac{1}{2\gamma}\left(\|e_{j,p}^{n+1}\|_f^2 - \|e_{j,p}^{n}\|^2\right) + \frac{1}{2\gamma}\|e_{j,p}^{n+1} - e_{j,p}^{n}\|_f^2 \qquad (4.101)\\
&= -\sum_i \int_\Gamma \left(\eta_{i,j} - \bar{\eta}_i\right)(e_{j,\overrightarrow{u}}^{n} \cdot \widehat{\tau}_i)(e_{j,\overrightarrow{u}}^{n+1} \cdot \widehat{\tau}_i)\, ds + \epsilon_{j,f}^{n+1}(e_{j,\overrightarrow{u}}^{n+1}) + \frac{1}{\gamma}\left(p_j(t_{n+1}) - p_j(t_n), e_{j,p}^{n+1}\right)_f \\
&\quad -\sum_i \int_\Gamma \left(\eta_{i,j} - \bar{\eta}_i\right)((\overrightarrow{u}_j^{n+1} - \overrightarrow{u}_j^{n}) \cdot \widehat{\tau}_i)(e_{j,\overrightarrow{u}}^{n+1} \cdot \widehat{\tau}_i)\, ds - g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla(\phi_j^{n+1} - \phi_j^{n}), \nabla e_{j,\phi}^{n+1})_p \\
&\quad -g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla e_{j,\phi}^{n}, \nabla e_{j,\phi}^{n+1})_p + \epsilon_{j,p}^{n+1}(e_{j,\phi}^{n+1}).
\end{aligned}$$

Using the same technique in the stability proof (see (4.80)*), we have for any* $\alpha_1, \beta_1 > 0$

$$
\begin{aligned}
c_I(e^{n+1}_{j,\overrightarrow{u}}, e^n_{j,\phi}) - c_I(e^n_{j,\overrightarrow{u}}, e^{n+1}_{j,\phi}) &= c_I(e^{n+1}_{j,\overrightarrow{u}} - e^n_{j,\overrightarrow{u}}, e^{n+1}_{j,\phi}) - c_I(e^{n+1}_{j,\overrightarrow{u}}, e^{n+1}_{j,\phi} - e^n_{j,\phi}) \qquad (4.102)\\
&\geq -\frac{1}{2\Delta t}\|e^{n+1}_{j,\overrightarrow{u}} - e^n_{j,\overrightarrow{u}}\|_f^2 - \Delta t \frac{C_2}{\beta_1^2}\left(\|\nabla e^{n+1}_{j,\overrightarrow{u}}\|_f^2 + \|\nabla e^n_{j,\overrightarrow{u}}\|_f^2\right) - \beta_1 g \bar{k}_{min}\|\nabla e^{n+1}_{j,\phi}\|_p^2\\
&\quad - \frac{gS_0}{2\Delta t}\|e^{n+1}_{j,\phi} - e^n_{j,\phi}\|_p^2 - \frac{\Delta t}{gS_0}\frac{C_1}{\alpha_1^2}\left(\|\nabla e^{n+1}_{j,\phi}\|_p^2 + \|\nabla e^n_{j,\phi}\|_p^2\right) - \alpha_1 \nu \|\nabla e^{n+1}_{j,\overrightarrow{u}}\|_f^2.
\end{aligned}
$$

Next, we bound the terms on the right-hand side of (4.101) *one by one. First,*

$$
\begin{aligned}
-\sum_i \int_\Gamma (\eta_{i,j} - \bar{\eta}_i)\,(e^n_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)(e^{n+1}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)\, ds &\leq \sum_i \eta'^{max}_{i,j} \int_\Gamma |(e^n_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)(e^{n+1}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)|\, ds\\
&\leq \sum_i \left[\frac{\eta'^{max}_i}{2} \int_\Gamma (e^n_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)^2\, ds + \frac{\eta'^{max}_i}{2}\int_\Gamma (e^{n+1}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)^2\, ds\right]. \qquad (4.103)
\end{aligned}
$$

By Poincaré inequality and (4.4)*, for any* $\sigma_1 > 0$*, we can get*

$$
\begin{aligned}
&-\sum_i \int_\Gamma (\eta_{i,j} - \bar{\eta}_i)\,((\overrightarrow{u}^{n+1}_j - \overrightarrow{u}^n_j) \cdot \widehat{\tau}_i)(e^{n+1}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)\, ds \qquad (4.104)\\
&\leq \sum_i \left[\frac{C}{\sigma_1}\eta'^{max}_i \Delta t \int_{t_n}^{t_{n+1}} \|\nabla u_{j,t}\|_f^2\, dt + \sigma_1 \eta'^{max}_i \int_\Gamma (e^{n+1}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)^2\, ds\right].
\end{aligned}
$$

Next, we bound the consistency errors:

$$
\begin{aligned}
&\epsilon^{n+1}_{j,f}(e^{n+1}_{j,\overrightarrow{u}})\\
&\leq C\left\|\frac{\overrightarrow{u}_j(t_{n+1}) - \overrightarrow{u}_j(t_n)}{\Delta t} - u_{j,t}(t_{n+1})\right\|_f^2 + C\|\nabla(\phi_j(t_{n+1}) - \phi_j(t_n))\|_p^2 + \alpha_2 \nu \|\nabla e^{n+1}_{j,\overrightarrow{u}}\|_f^2\\
&\leq C\Delta t \int_{t_n}^{t_{n+1}} \|u_{j,tt}(t)\|_f^2\, dt + C\Delta t \int_{t_n}^{t_{n+1}} \|\nabla \phi_{j,t}(t)\|_p^2\, dt + \frac{\alpha_2}{3}\nu\|\nabla e^{n+1}_{j,\overrightarrow{u}}\|_f^2.\\
&\epsilon^{n+1}_{j,p}(e^{n+1}_{j,\phi})\\
&\leq C\left\|\frac{\phi_j(t_{n+1}) - \phi_j(t_n)}{\Delta t} - \phi_{j,t}(t_{n+1})\right\|_p^2 + C\|\nabla(\overrightarrow{u}_j(t_{n+1}) - \overrightarrow{u}_j(t_n))\|_f^2 + \beta_2 g \bar{k}_{min}\|\nabla e^{n+1}_{j,\phi}\|_p^2\\
&\leq C\Delta t \int_{t_n}^{t_{n+1}} \|\phi_{j,tt}(t)\|_p^2\, dt + C\Delta t \int_{t_n}^{t_{n+1}} \|\nabla u_{j,t}(t)\|_f^2\, dt + \frac{\beta_2}{2} g \bar{k}_{min}\|\nabla e^{n+1}_{j,\phi}\|_p^2.
\end{aligned}
$$

The hydraulic conductivity tensor terms are estimated as follows: for any $\beta_3 > 0$,

$$-g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla e^n_{j,\phi}, \nabla e^{n+1}_{j,\phi})_p \leq \frac{g\rho'_{max}}{2}\|\nabla e^n_{j,\phi}\|^2_p + \frac{g\rho'_{max}}{2}\|\nabla e^{n+1}_{j,\phi}\|^2_p. \tag{4.105}$$

$$-g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla(\phi^{n+1}_j - \phi^n_j), \nabla e^{n+1}_{j,\phi})_p \leq \frac{Cg\rho'_{max}}{\beta_3}\Delta t\int_{t_n}^{t_{n+1}}\|\nabla\phi_{j,t}\|^2_p\,dt + \beta_3 g\rho'_{max}\|\nabla e^{n+1}_{j,\phi}\|^2_p. \tag{4.106}$$

Lastly, we bound the pressure term. Consider the decomposition $H^1_0(\Omega) = Y \oplus Y^\perp$, where $Y^\perp = \{(-\Delta)^{-1}\nabla q : q \in L^2(\Omega)\}$; see Girault and Raviart (1986a); Shen (1995). For $p_{j,t}(t), p_{j,tt}(t) \in L^2(\Omega)/\mathbb{R}$, there exists a unique $\mu_j(t)$ such that $\nabla\cdot\mu_j(t) = p_{j,t}(t), \nabla\cdot\mu_{j,t}(t) = p_{j,tt}(t)$, and we have

$$\|\nabla\mu_j(t)\| \leq C\|p_{j,t}(t)\|, \quad \|\nabla\mu_{j,t}(t)\| \leq C\|p_{j,tt}(t)\|, \qquad \forall t\in[0,T].$$

The pressure term can be rewritten using (4.98) *as*

$$\begin{aligned}
&\frac{1}{\gamma}\left(p_j(t_{n+1}) - p_j(t_n), e^{n+1}_{j,p}\right)_f = \frac{1}{\gamma}\left(\int_{t_n}^{t_{n+1}} p_{j,t}(t)\,dt, e^{n+1}_{j,p}\right)_f \qquad (4.107)\\
&= \frac{1}{\gamma}\left(\int_{t_n}^{t_{n+1}}\nabla\cdot\mu_j(t)\,dt, e^{n+1}_{j,p}\right)_f = -\frac{1}{\gamma}\left(\int_{t_n}^{t_{n+1}}\mu_j(t)\,dt, \nabla e^{n+1}_{j,p}\right)_f\\
&= \frac{1}{\gamma}\left(\frac{e^{n+1}_{j,\overrightarrow{u}} - e^n_{j,\overrightarrow{u}}}{\Delta t}, \int_{t_n}^{t_{n+1}}\mu_j(t)\,dt\right)_f + \frac{\nu}{\gamma}(\nabla e^{n+1}_{j,\overrightarrow{u}}, \int_{t_n}^{t_{n+1}}\nabla\mu_j(t)\,dt)_f\\
&\quad + \frac{1}{\gamma}\sum_i\int_\Gamma \bar{\eta}_i(e^{n+1}_{j,\overrightarrow{u}}\cdot\widehat{\tau}_i)(\int_{t_n}^{t_{n+1}}\mu_j(t)\,dt\cdot\widehat{\tau}_i)\,ds\\
&\quad + \frac{1}{\gamma}\sum_i\int_\Gamma\left(\eta_{i,j} - \bar{\eta}_i\right)(e^n_{j,\overrightarrow{u}}\cdot\widehat{\tau}_i)(\int_{t_n}^{t_{n+1}}\mu(t)\,dt\cdot\widehat{\tau}_i)\,ds\\
&+ \frac{1}{\gamma}\sum_i\int_\Gamma\left(\eta_{i,j} - \bar{\eta}_i\right)((\overrightarrow{u}_j(t_{n+1}) - \overrightarrow{u}_j(t_n))\cdot\widehat{\tau}_i)(\int_{t_n}^{t_{n+1}}\mu(t)\,dt\cdot\widehat{\tau}_i)\,ds\\
&\quad + \frac{1}{\gamma}c_\Gamma(\int_{t_n}^{t_{n+1}}\mu_j(t)\,dt, e^n_{j,\phi}) - \frac{1}{\gamma}\epsilon^{n+1}_{j,f}(\int_{t_n}^{t_{n+1}}\mu_j(t)\,dt).
\end{aligned}$$

We need to bound each term on the right hand side of (4.107)*:*

$$
\begin{aligned}
&-\frac{1}{\gamma}\epsilon_{j,f}^{n+1}\Big(\int_{t_n}^{t_{n+1}}\mu(t)\,dt\Big) && (4.108)\\
&\le \frac{C}{\gamma}\Big\|\frac{\overrightarrow{u}_j(t_{n+1})-\overrightarrow{u}_j(t_n)}{\Delta t}-u_{j,t}(t_{n+1})\Big\|_f^2+\frac{C}{\gamma}\|\nabla(\phi_j(t_{n+1})-\phi_j(t_n))\|_p^2+\frac{C}{\gamma}\Big\|\int_{t_n}^{t_{n+1}}\nabla\mu_j(t)\,dt\Big\|_f^2\\
&\le \frac{C}{\gamma}\Delta t\int_{t^n}^{t^{n+1}}\|u_{j,tt}(t)\|_f^2\,dt+\frac{C}{\gamma}\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}(t)\|_p^2\,dt+\frac{C}{\gamma}\nu\Delta t\int_{t_n}^{t_{n+1}}\|\nabla\mu_j(t)\|_f^2dt\\
&\le C\Delta t\int_{t^n}^{t^{n+1}}\|u_{j,tt}(t)\|_f^2\,dt+C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}(t)\|_p^2\,dt+\frac{C}{\gamma}\nu\Delta t\int_{t_n}^{t_{n+1}}\|p_{j,t}(t)\|_f^2dt.
\end{aligned}
$$

$$
\frac{\nu}{\gamma}(\nabla e_{j,\overrightarrow{u}}^{n+1},\int_{t_n}^{t_{n+1}}\nabla\mu_j(t)\,dt)_f\le\frac{\alpha_2}{3}\nu\Delta t\|\nabla e_{j,\overrightarrow{u}}^{n+1}\|_f^2+\frac{C\nu\Delta t}{\alpha_2\gamma^2}\int_{t_n}^{t_{n+1}}\|p_{j,t}(t)\|_f^2dt. \qquad (4.109)
$$

$$
\begin{aligned}
&\frac{1}{\gamma}\left(\frac{e_{j,\overrightarrow{u}}^{n+1}-e_{j,\overrightarrow{u}}^{n}}{\Delta t},\int_{t_n}^{t_{n+1}}\mu_j(t)\,dt\right)_f && (4.110)\\
&=\frac{1}{\gamma\Delta t}\left[\left(e_{j,\overrightarrow{u}}^{n+1},\int_{t_n}^{t_{n+1}}\mu_j(t)\,dt\right)_f-\left(e_{j,\overrightarrow{u}}^{n},\int_{t_{n-1}}^{t_n}\mu_j(t)\,dt\right)_f\right]\\
&-\frac{1}{\gamma\Delta t}\left(e_{j,\overrightarrow{u}}^{n},\int_{t_n}^{t_{n+1}}\mu_j(t)\,dt-\int_{t_{n-1}}^{t_n}\mu_j(t)\,dt\right)_f\\
&\le\frac{1}{\gamma\Delta t}\left[\left(e_{j,\overrightarrow{u}}^{n+1},\int_{t_n}^{t_{n+1}}\mu_j(t)\,dt\right)_f-\left(e_{j,\overrightarrow{u}}^{n},\int_{t_{n-1}}^{t_n}\mu_j(t)\,dt\right)_f\right]+\frac{2\Delta t}{\gamma}\left|\left(e_{j,\overrightarrow{u}}^{n},\mu_{j,t}(\xi_n)\right)_f\right|\\
&\le\frac{1}{\gamma\Delta t}\left[\left(e_{j,\overrightarrow{u}}^{n+1},\int_{t_n}^{t_{n+1}}\mu_j(t)\,dt\right)_f-\left(e_{j,\overrightarrow{u}}^{n},\int_{t_{n-1}}^{t_n}\mu_j(t)\,dt\right)_f\right]+\frac{2\Delta t}{\gamma}\|e_{j,\overrightarrow{u}}^{n}\|_f\|\mu_{j,t}(\xi_n)\|_f\\
&\le\frac{1}{\gamma\Delta t}\left[\left(e_{j,\overrightarrow{u}}^{n+1},\int_{t_n}^{t_{n+1}}\mu_j(t)\,dt\right)_f-\left(e_{j,\overrightarrow{u}}^{n},\int_{t_{n-1}}^{t_n}\mu_j(t)\,dt\right)_f\right]+\frac{\alpha_2}{3}\nu\|\nabla e_{j,\overrightarrow{u}}^{n}\|_f^2\\
&+\frac{C\Delta t^2}{\alpha_2\gamma^2}\|p_{j,tt}(\xi_n)\|_f^2,\ \ \xi_n\in(t_{n-1},t_{n+1}).
\end{aligned}
$$

We also need to bound the following terms.

$$\begin{aligned}
&\frac{1}{\gamma}\sum_i \int_\Gamma \left(\eta_{i,j} - \bar{\eta}_i\right)(e^n_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)(\int_{t_n}^{t_{n+1}} \mu_j(t)\,dt \cdot \widehat{\tau}_i)\,ds \\
&\leq \frac{1}{\gamma}\sum_i \eta'^{max}_{i,j} \int_\Gamma |(e^n_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)(\int_{t_n}^{t_{n+1}} \mu_j(t)\,dt \cdot \widehat{\tau}_i)|\,ds \\
&\leq \sum_i \left[\sigma_2 \eta'^{max}_i \int_\Gamma (e^n_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)^2\,ds + \frac{C\eta'^{max}_i}{\sigma_2\gamma^2} \int_\Gamma (\int_{t_n}^{t_{n+1}} \mu_j(t)\,dt \cdot \widehat{\tau}_i)^2\,ds\right] \qquad (4.111) \\
&\leq \sum_i \left[\sigma_2 \eta'^{max}_i \int_\Gamma (e^n_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)^2\,ds + \frac{C\eta'^{max}_i}{\sigma_2\gamma^2}\Delta t \int_{t_n}^{t_{n+1}} \|\nabla \mu_j(t)\|_f^2\,dt\right] \\
&\leq \sum_i \left[\sigma_2 \eta'^{max}_i \int_\Gamma (e^n_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)^2\,ds + \frac{C\eta'^{max}_i}{\sigma_2\gamma^2}\Delta t \int_{t_n}^{t_{n+1}} \|p_{j,t}(t)\|_f^2\,dt\right].
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\gamma}\sum_i \int_\Gamma \left(\eta_{i,j} - \bar{\eta}_i\right)((\overrightarrow{u}_j(t_{n+1}) - \overrightarrow{u}_j(t_n)) \cdot \widehat{\tau}_i)(\int_{t_n}^{t_{n+1}} \mu_j(t)\,dt \cdot \widehat{\tau}_i)\,ds \qquad (4.112) \\
&\leq \frac{1}{\gamma}\sum_i \eta'^{max}_{i,j} \int_\Gamma |((\overrightarrow{u}_j(t_{n+1}) - \overrightarrow{u}_j(t_n)) \cdot \widehat{\tau}_i)(\int_{t_n}^{t_{n+1}} \mu_j(t)\,dt \cdot \widehat{\tau}_i)|\,ds \\
&\leq \sum_i \left[\frac{C}{\gamma}\eta'^{max}_i \Delta t \int_{t_n}^{t_{n+1}} \|\nabla u_{j,t}\|_f^2\,dt + \frac{C}{\gamma}\eta'^{max}_i \Delta t \int_{t_n}^{t_{n+1}} \|p_{j,t}(t)\|_f^2\,dt\right].
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\gamma}\sum_i \int_\Gamma \bar{\eta}_i (e^{n+1}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)(\int_{t_n}^{t_{n+1}} \mu_j(t)\,dt \cdot \widehat{\tau}_i)\,ds \qquad (4.113) \\
&\leq \sum_i \left[\sigma_3 \int_\Gamma \bar{\eta}_i (e^{n+1}_{j,\overrightarrow{u}} \cdot \widehat{\tau}_i)^2 ds + \frac{C\bar{\eta}_i}{\sigma_3\gamma^2}\Delta t \int_{t_n}^{t_{n+1}} \|p_{j,t}(t)\|_f^2\,dt\right].
\end{aligned}$$

$$\frac{1}{\gamma} c_\Gamma(\int_{t_n}^{t_{n+1}} \mu_j(t)\,dt, e^n_{j,\phi}) \leq \frac{C\Delta t}{\beta_2\gamma^2 g\bar{k}_{min}} \int_{t_n}^{t_{n+1}} \|p_{j,t}(t)\|_f^2 + \frac{\beta_2}{2} g\bar{k}_{min}\|\nabla e^n_{j,\phi}\|^2. \qquad (4.114)$$

Combining all these estimates, we have the following inequality

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|e^{n+1}_{j,\vec{u}}\|_f^2 - \frac{1}{2\Delta t}\|e^{n}_{j,\vec{u}}\|_f^2 + \left(1-\alpha_1-\alpha_2-\Delta t\frac{2C_2}{\beta_1^2\nu}\right)\nu\|\nabla e^{n+1}_{j,\vec{u}}\|_f^2 \qquad (4.115)\\
&+(\frac{\alpha_2}{3}\nu+\Delta t\frac{C_2}{\beta_1^2})\left(\|\nabla e^{n+1}_{j,\vec{u}}\|_f^2-\|\nabla e^{n}_{j,\vec{u}}\|_f^2\right)+\frac{gS_0}{2\Delta t}\|e^{n+1}_{j,\phi}\|_p^2-\frac{gS_0}{2\Delta t}\|e^{n}_{j,\phi}\|_p^2\\
&+\sum_i\left((1-\sigma_3)\bar{\eta}_i^{min}-(1+\sigma_1+\sigma_2)\eta_i'^{max}\right)\int_\Gamma(e^{n+1}_{j,\vec{u}}\cdot\widehat{\tau}_i)^2\,ds\\
&+\sum_i(\sigma_2+\frac{1}{2})\eta_i'^{max}\left(\int_\Gamma(e^{n+1}_{j,\vec{u}}\cdot\widehat{\tau}_i)^2\,ds-\int_\Gamma(e^{n}_{j,\vec{u}}\cdot\widehat{\tau}_i)^2\,ds\right)\\
&+\left((1-\beta_1-\beta_2-\Delta t\frac{2C_1}{g^2S_0\bar{k}_{min}\alpha_1^2})-(1+\beta_3)\frac{\rho'_{max}}{\bar{k}_{min}}\right)g\bar{k}_{min}\|\nabla e^{n+1}_{j,\phi}\|_p^2+\frac{1}{2\gamma}\|e^{n+1}_{j,p}-e^{n}_{j,p}\|_f^2\\
&+\left(\frac{\beta_2}{2}g\bar{k}_{min}+\frac{1}{2}g\rho'_{max}+\frac{\Delta tC_1}{gS_0\alpha_1^2}\right)\left(\|\nabla e^{n+1}_{j,\phi}\|_p^2-\|\nabla e^{n}_{j,\phi}\|_p^2\right)+\frac{1}{2\gamma}\left(\|e^{n+1}_{j,p}\|_f^2-\|e^{n}_{j,p}\|_f^2\right)\\
&\le\sum_i\frac{C}{\sigma_1}\eta_i'^{max}\Delta t\int_{t^n}^{t^{n+1}}\|\nabla u_{j,t}\|_f^2\,dt+C\Delta t\int_{t^n}^{t^{n+1}}\|u_{j,tt}\|_f^2\,dt+C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}\|_p^2\,dt\\
&+C\Delta t\int_{t^n}^{t^{n+1}}\|\phi_{j,tt}\|_p^2\,dt+C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla u_{j,t}\|_f^2\,dt+\frac{Cg\rho'_{max}}{\beta_3}\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}\|_p^2\,dt\\
&+C\Delta t\int_{t^n}^{t^{n+1}}\|u_{j,tt}(t)\|_f^2\,dt+C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}(t)\|_p^2\,dt+\frac{C}{\gamma}\nu\Delta t\int_{t_n}^{t_{n+1}}\|p_{j,t}(t)\|_f^2dt\\
&+\frac{C\nu\Delta t}{\alpha_2\gamma^2}\int_{t_n}^{t_{n+1}}\|p_{j,t}(t)\|_f^2dt+\frac{1}{\gamma\Delta t}\left[\left(e^{n+1}_{j,\vec{u}},\int_{t_n}^{t_{n+1}}\mu_j(t)\,dt\right)_f-\left(e^{n}_{j,\vec{u}},\int_{t_{n-1}}^{t_n}\mu_j(t)\,dt\right)_f\right]\\
&+\frac{C\Delta t^2}{\alpha_2\gamma^2}\|p_{j,tt}(\xi_n)\|_f^2+\sum_i\left(\frac{C\eta_i'^{max}}{\sigma_2\gamma^2}+\frac{C\bar{\eta}_i}{\sigma_3\gamma^2}\right)\Delta t\int_{t_n}^{t_{n+1}}\|p_{j,t}(t)\|_f^2\,dt\\
&+\sum_i\left[\frac{C}{\gamma}\eta_i'^{max}\Delta t\int_{t_n}^{t_{n+1}}\|\nabla u_{j,t}\|_f^2\,dt+\frac{C}{\gamma}\eta_i'^{max}\Delta t\int_{t_n}^{t_{n+1}}\|p_{j,t}(t)\|_f^2\,dt\right]\\
&+\frac{C\Delta t}{\beta_2\gamma^2g\bar{k}_{min}}\int_{t_n}^{t_{n+1}}\|p_{j,t}(t)\|_f^2.
\end{aligned}
$$

To make sure the third, sixth and eighth term on the left-hand side are non-negative, we need $0<\alpha_1,\alpha_2,\sigma_3,\beta_1,\beta_2<1$*, and*

$$
\frac{\eta_i'^{max}}{\bar{\eta}_i^{min}}\le\frac{1-\sigma_3}{1+\sigma_1+\sigma_2},\qquad\frac{\rho'_{max}}{\bar{k}_{min}}<\frac{1}{1+\beta_3}. \qquad (4.116)
$$

For $\forall\sigma_3 \in (0,1), \forall\sigma_1 > 0, \forall\sigma_2 > 0, \forall\beta_3 > 0$*, we can derive that* $\frac{1-\sigma_3}{1+\sigma_1+\sigma_2}, \frac{1}{1+\beta_3} \in (0,1)$*. Now if the two parameter conditions in* (4.77) *are satisfied, we have* $\frac{\eta_i'^{max}}{\bar{\eta}_i^{min}}, \frac{\rho'_{max}}{\bar{k}_{min}} \in (0,1)$*. Then we can easily find* $\sigma_3 \in (0,1), \sigma_1 > 0, \sigma_2 > 0$ *such that* $\frac{\eta_i'^{max}}{\bar{\eta}_i^{min}} = \frac{1-\sigma_3}{1+\sigma_1+\sigma_2}$*, and* $\beta_3 > 0$ *such that* $\frac{\rho'_{max}}{\bar{k}_{min}} < \frac{1}{1+\beta_3}$*.*

Then under the two parameter conditions in (4.93) *and the time-step condition* (4.92)*,* (4.115) *reduces to*

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|e_{j,\vec{u}}^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|e_{j,\vec{u}}^{n}\|_f^2 + (\frac{\alpha_2}{3}\nu + \Delta t\frac{C_2}{\beta_1^2})\left(\|\nabla e_{j,\vec{u}}^{n+1}\|_f^2 - \|\nabla e_{j,\vec{u}}^{n}\|_f^2\right) \qquad (4.117)\\
&+ \sum_i (\sigma_2 + \frac{1}{2})\eta_i'^{max}\left(\int_\Gamma (e_{j,\vec{u}}^{n+1}\cdot\widehat{\tau}_i)^2\,ds - \int_\Gamma (e_{j,\vec{u}}^{n}\cdot\widehat{\tau}_i)^2\,ds\right) + \frac{gS_0}{2\Delta t}\|e_{j,\phi}^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|e_{j,\phi}^{n}\|_p^2\\
&+ \left(\frac{\beta_2}{2}g\bar{k}_{min} + \frac{1}{2}g\rho'_{max} + \frac{\Delta t C_1}{gS_0\alpha_1^2}\right)\left(\|\nabla e_{j,\phi}^{n+1}\|_p^2 - \|\nabla e_{j,\phi}^{n}\|_p^2\right) + \frac{1}{2\gamma}\left(\|e_{j,p}^{n+1}\|_f^2 - \|e_{j,p}^{n}\|_f^2\right)\\
&+ \frac{1}{2\gamma}\|e_{j,p}^{n+1} - e_{j,p}^{n}\|_f^2\\
&\leq \sum_i \frac{C}{\sigma_1}\eta_i'^{max}\Delta t\int_{t^n}^{t^{n+1}}\|\nabla u_{j,t}\|_f^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}}\|u_{j,tt}\|_f^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}\|_p^2\,dt\\
&+ C\Delta t\int_{t^n}^{t^{n+1}}\|\phi_{j,tt}\|_p^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla u_{j,t}\|_f^2\,dt + \frac{Cg\rho'_{max}}{\beta_3}\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}\|_p^2\,dt\\
&+ C\Delta t\int_{t^n}^{t^{n+1}}\|u_{j,tt}(t)\|_f^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}(t)\|_p^2\,dt + \frac{C}{\gamma}\nu\Delta t\int_{t_n}^{t_{n+1}}\|p_{j,t}(t)\|_f^2 dt\\
&+ \frac{C\nu\Delta t}{\alpha_2\gamma^2}\int_{t_n}^{t_{n+1}}\|p_{j,t}(t)\|_f^2 dt + \frac{1}{\gamma\Delta t}\left[\left(e_{j,\vec{u}}^{n+1}, \int_{t_n}^{t_{n+1}}\mu_j(t)\,dt\right)_f - \left(e_{j,\vec{u}}^{n}, \int_{t_{n-1}}^{t_n}\mu_j(t)\,dt\right)_f\right]\\
&+ \frac{C\Delta t^2}{\alpha_2\gamma^2}\|p_{j,tt}(\xi_n)\|_f^2 + \sum_i\left(\frac{C\eta_i'^{max}}{\sigma_2\gamma^2} + \frac{C\bar{\eta}_i}{\sigma_3\gamma^2}\right)\Delta t\int_{t_n}^{t_{n+1}}\|p_{j,t}(t)\|_f^2\,dt\\
&+ \sum_i\left[\frac{C}{\gamma}\eta_i'^{max}\Delta t\int_{t_n}^{t_{n+1}}\|\nabla u_{j,t}\|_f^2\,dt + \frac{C}{\gamma}\eta_i'^{max}\Delta t\int_{t_n}^{t_{n+1}}\|p_{j,t}(t)\|_f^2\,dt\right]\\
&+ \frac{C\Delta t}{\beta_2\gamma^2 g\bar{k}_{min}}\int_{t_n}^{t_{n+1}}\|p_{j,t}(t)\|_f^2.
\end{aligned}
$$

Since $e^0_{j,\overrightarrow{u}} = 0, e^0_{j,p} = 0$*, and* $e^0_{j,\phi} = 0$*, summing up* (4.117) *from* $n = 0$ *to* $n = N-1$ *and multiplying through by* Δt *yields*

$$\begin{aligned}
&\frac{1}{2}\|e^N_{j,\overrightarrow{u}}\|^2_f + (\frac{\alpha_2}{3}\nu\Delta t + \Delta t^2\frac{C_2}{\beta_1^2})\|\nabla e^N_{j,\overrightarrow{u}}\|^2_f + \sum_i(\sigma_2 + \frac{1}{2})\Delta t\eta_i'^{max}\int_\Gamma (e^N_{j,\overrightarrow{u}}\cdot\widehat{\tau}_i)^2\,ds \\
&+\frac{gS_0}{2}\|e^N_{j,\phi}\|^2_p + \left(\frac{\beta_2}{2}g\bar{k}_{min} + \frac{1}{2}g\rho'_{max} + \frac{\Delta t C_1}{gS_0\alpha_1^2}\right)\Delta t\|\nabla e^N_{j,\phi}\|^2_p + \frac{1}{2\gamma}\Delta t\|e^N_{j,p}\|^2_f \\
&+\sum_{n=0}^{N-1}\frac{1}{2\gamma}\Delta t\|e^{n+1}_{j,p} - e^n_{j,p}\|^2_f \qquad (4.118)\\
&\le \Delta t\sum_{n=0}^{N-1}\left\{\sum_i\frac{C}{\sigma_1}\eta_i'^{max}\Delta t\int_{t^n}^{t^{n+1}}\|\nabla u_{j,t}\|^2_f\,dt + C\Delta t\int_{t^n}^{t^{n+1}}\|u_{j,tt}\|^2_f\,dt\right.\\
&+C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}\|^2_p\,dt + C\Delta t\int_{t^n}^{t^{n+1}}\|\phi_{j,tt}\|^2_p\,dt + C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla u_{j,t}\|^2_f\,dt\\
&+\frac{Cg\rho'_{max}}{\beta_3}\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}\|^2_p\,dt + C\Delta t\int_{t^n}^{t^{n+1}}\|u_{j,tt}(t)\|^2_f\,dt + C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}(t)\|^2_p\,dt\\
&+\frac{C}{\gamma}\nu\Delta t\int_{t_n}^{t_{n+1}}\|p_{j,t}(t)\|^2_f dt + \frac{C\nu\Delta t}{\alpha_2\gamma^2}\int_{t_n}^{t_{n+1}}\|p_{j,t}(t)\|^2_f dt + \frac{C\Delta t^2}{\alpha_2\gamma^2}\|p_{j,tt}(\xi_n)\|^2_f\\
&+\sum_i\left(\frac{C\eta_i'^{max}}{\sigma_2\gamma^2} + \frac{C\bar{\eta}_i}{\sigma_3\gamma^2}\right)\Delta t\int_{t_n}^{t_{n+1}}\|p_{j,t}(t)\|^2_f\,dt\\
&+\sum_i\left[\frac{C}{\gamma}\eta_i'^{max}\Delta t\int_{t_n}^{t_{n+1}}\|\nabla u_{j,t}\|^2_f\,dt + \frac{C}{\gamma}\eta_i'^{max}\Delta t\int_{t_n}^{t_{n+1}}\|p_{j,t}(t)\|^2_f\,dt\right]\\
&\left.+\frac{C\Delta t}{\beta_2\gamma^2 g\bar{k}_{min}}\int_{t_n}^{t_{n+1}}\|p_{j,t}(t)\|^2_f\right\} + \frac{1}{\gamma}\left(e^N_{j,\overrightarrow{u}}, \int_{t_{N-1}}^{t_N}\mu_j(t)\,dt\right)_f.
\end{aligned}$$

The last term in (4.118) *can be bounded as*

$$\begin{aligned}
\frac{1}{\gamma}\left(e^N_{j,\overrightarrow{u}}, \int_{t_{N-1}}^{t_N}\mu_j(t)\,dt\right)_f &= \frac{\Delta t}{\gamma}\left(e^N_{j,\overrightarrow{u}}, \mu_j(\theta_N)\right)_f \le \frac{1}{4}\|e^N_{j,\overrightarrow{u}}\|^2_f + \frac{\Delta t^2}{\gamma^2}\|\mu_j(\theta_N)\|^2_f \qquad (4.119)\\
&\le \frac{1}{4}\|e^N_{j,\overrightarrow{u}}\|^2_f + C\Delta t^2\|p_{j,t}(\theta_N)\|^2_f \le \frac{1}{4}\|e^N_{j,\overrightarrow{u}}\|^2_f + C\Delta t^2\|p_{j,t}\|^2_{\infty,0,f},
\end{aligned}$$

where $\theta_N \in (t_{N-1}, t_N)$.

Moreover, for functions $v(x,t)$ defined on $D_f \times (-T,T)$, we define the norm

$$\|v\|_{\infty^*,0,f} \ := \|v\|_{L^\infty(-T,T;L^2(D_f))}.$$

Then the second term at the sixth line of (4.118) *can be bounded as follows:*

$$\Delta t \sum_{n=1}^{N-1} \frac{C\Delta t^2}{\alpha_2 \gamma^2} \|p_{j,tt}(\xi_n)\|_f^2 \leq C\Delta t^2 \|p_{j,tt}\|^2_{\infty^*,0,f}. \tag{4.120}$$

Then (4.118) *reduces to*

$$\begin{aligned}
&\frac{1}{2}\|e^N_{j,\vec{u}}\|^2_f + (\frac{\alpha_2}{3}\nu\Delta t + \Delta t^2 \frac{C_2}{\beta_1^2})\|\nabla e^N_{j,\vec{u}}\|^2_f + \sum_i (\sigma_2 + \frac{1}{2})\Delta t \eta_i'^{max} \int_\Gamma (e^N_{j,\vec{u}} \cdot \widehat{\tau}_i)^2 \, ds + \frac{gS_0}{2}\|e^N_{j,\phi}\|^2_p \\
&+ \left(\frac{\beta_2}{2} g\bar{k}_{min} + \frac{1}{2} g\rho'_{max} + \frac{\Delta t C_1}{gS_0 \alpha_1^2}\right)\Delta t\|\nabla e^N_{j,\phi}\|^2_p + \frac{1}{2\gamma}\Delta t\|e^N_{j,p}\|^2_f + \sum_{n=0}^{N-1} \frac{1}{2\gamma}\Delta t\|e^{n+1}_{j,p} - e^n_{j,p}\|^2_f \\
&\leq C\Delta t^2\|u_{j,t}\|^2_{2,1,f} + C\Delta t^2\|u_{j,tt}\|^2_{2,0,f} + C\Delta t^2\|\phi_{j,t}\|^2_{2,1,p} + C\Delta t^2\|\phi_{j,tt}\|^2_{2,0,p} + C\Delta t^2\|p_{j,t}(t)\|^2_{2,0,f} \\
&+ C\Delta t^2\|p_{j,tt}\|^2_{\infty^*,0,f} + C\Delta t^2\|p_{j,t}\|^2_{\infty,0,f} \leq C\Delta t^2.
\end{aligned}$$

4.3.3. Numerical Illustrations. In this section, the features of the proposed AC ensemble scheme for the Stokes-Darcy system are shown by three examples of numerical experiments.

4.3.3.1. Stability and convergence test. First, we set the model problem on $\Omega = [0,\pi] \times [-1,1]$, where $\Omega_D = [0,\pi] \times [-1,0]$ and $\Omega_S = [0,\pi] \times [0,1]$. We take $\alpha_{BJS} = 1$, $\nu = 1$, $g = 1$, and $S_0 = 1$. The boundary condition functions and the source terms are chosen such that the following functions are the exact solutions:

$$\begin{aligned}
\phi &= (e^y - e^{-y}) sin(x) e^t, \\
\vec{u} &= [\frac{k^j_{11}}{\pi} sin(2\pi y) cos(x), (-2k^j_{22} + \frac{k^j_{22}}{\pi^2} sin^2(\pi y)) sin(x)]^T e^t, \\
p &= sin(\pi x y) e^t.
\end{aligned}$$

For the hydraulic conductivity tensor, we set

$$\mathcal{K} = \mathcal{K}_j = \begin{bmatrix} k_{11}^j & 0 \\ 0 & k_{22}^j \end{bmatrix}, \quad j = 1, \ldots, J,$$

where $\mathcal{K}_j$ is one of the samples of $\mathcal{K}$. In this simple test, we only consider the case that k_{11}, k_{22} are random variables that are independent of spatial coordinates. All the numerical results below are for $t = T = 1$.

We consider a group of simulations with $J = 3$ members. The three members are corresponding to different hydraulic conductivity tensors, i.e. $k_{11}^1 = k_{22}^1 = 1e^{-3}, k_{11}^2 = k_{22}^2 = 0.9e^{-3}, k_{11}^3 = k_{22}^3 = 1.1e^{-3}$. As $\mathcal{K}$ is diagonal, we use Algorithm 5 for computation, and thus there are no parameter conditions for both stability and convergence. In order to check the convergence order in time, we uniformly refine the mesh size h and time step size Δt from the initial mesh size $1/4$ and time step size $\Delta t = 0.1h$. The approximation errors of the AC ensemble method are listed in Table 4.6, Table 4.7, and Table 4.8, for the velocity $\overrightarrow{u}$, the hydraulic head ϕ, and the pressure p, respectively. From these tables, we can find that our ensemble algorithm is first order convergence in time. Moreover, from Table 4.9, we find the results are not convergent to the exact solution when $\Delta t = 3h$, which is consistent with our theoretical result that a time step condition must be satisfied to ensure stability and convergence.

4.3.3.2. Convergence and efficiency test for *J* random samples. We next consider using the presented ensemble algorithm for approximating stochastic Stokes-Darcy equations with a random hydraulic conductivity tensor $\mathcal{K}(x, w)$ that depends on spatial coordinates. Let $(\Theta, \mathcal{F}, \mathcal{P})$ be a complete probability space. Here Θ is the set of outcomes, $\mathcal{F} \in 2^{\Theta}$ is the $\sigma-$algebra of events, and $\mathcal{P} : \mathcal{F} \to [0, 1]$ is a probability measure. The stochastic Stokes-Darcy system considered reads: Find the functions $\overrightarrow{u} : \Omega_S \times [0, T] \times \Theta \to \mathbb{R}^d$ $(d = 2, 3)$, $p : \Omega_S \times [0, T] \times \Theta \to \mathbb{R}$, and $\phi : \Omega_D \times [0, T] \times \Theta \to \mathbb{R}$, such that it holds $\mathcal{P} - a.e.$

Table 4.6. Errors and convergence rates of the AC ensemble algorithm ($J = 3$) for $\Delta t = 0.1h$.

h	$\|u_h - \overrightarrow{u}\|_0^{E,1}$	rate	$\|u_h - \overrightarrow{u}\|_0^{E,2}$	rate	$\|u_h - \overrightarrow{u}\|_0^{E,3}$	rate
1/4	6.199×10^{-2}	–	6.189×10^{-2}	–	6.200×10^{-2}	–
1/8	2.944×10^{-2}	1.07	2.906×10^{-2}	1.09	2.924×10^{-2}	1.08
1/16	1.408×10^{-2}	1.06	1.377×10^{-2}	1.07	1.469×10^{-2}	0.99
1/32	6.935×10^{-3}	1.02	6.784×10^{-3}	1.02	7.348×10^{-3}	1.00
h	$\|u_h - \overrightarrow{u}\|_1^{E,1}$	rate	$\|u_h - \overrightarrow{u}\|_1^{E,2}$	rate	$\|u_h - \overrightarrow{u}\|_1^{E,3}$	rate
1/4	1.259×10^{-1}	–	1.248×10^{-1}	–	1.260×10^{-1}	–
1/8	5.246×10^{-2}	1.26	5.403×10^{-2}	1.20	5.612×10^{-2}	1.21
1/16	2.385×10^{-2}	1.13	2.573×10^{-2}	1.07	2.647×10^{-2}	0.96
1/32	1.135×10^{-2}	1.07	1.169×10^{-2}	1.13	1.260×10^{-2}	1.07

Table 4.7. Errors and convergence rates of the AC ensemble algorithm ($J = 3$) for $\triangle t = 0.1h$.

h	$\|\phi_h - \phi\|_0^{E,1}$	rate	$\|\phi_h - \phi\|_0^{E,2}$	rate	$\|\phi_h - \phi\|_0^{E,3}$	rate
1/4	1.799×10^{-1}	–	1.780×10^{-1}	–	1.800×10^{-1}	–
1/8	8.177×10^{-2}	1.13	8.091×10^{-2}	1.13	8.372×10^{-2}	1.10
1/16	3.894×10^{-2}	1.07	3.799×10^{-2}	1.09	3.987×10^{-2}	1.07
1/32	1.928×10^{-2}	1.01	1.809×10^{-2}	1.07	1.954×10^{-2}	1.03
h	$\|\phi_h - \phi\|_1^{E,1}$	rate	$\|\phi_h - \phi\|_1^{E,2}$	rate	$\|\phi_h - \phi\|_1^{E,3}$	rate
1/4	4.620×10^{-1}	–	4.599×10^{-1}	–	4.625×10^{-1}	–
1/8	2.090×10^{-1}	1.14	2.169×10^{-1}	1.08	2.171×10^{-1}	1.09
1/16	9.955×10^{-2}	1.07	1.033×10^{-1}	1.07	4.3370×10^{-2}	1.07
1/32	4.953×10^{-2}	1.11	5.139×10^{-2}	1.00	1.0858×10^{-3}	1.09

in Θ, or in other words, almost surely

$$\overrightarrow{u}_t(x,t,\omega) - \nu\Delta\overrightarrow{u}(x,t,\omega) + \nabla p(x,t,\omega) = f_f(x,t), \quad \nabla\cdot\overrightarrow{u}(x,t,\omega) = 0, \text{ in } \Omega_S \times \Theta$$

$$S_0\phi_t(x,t,\omega) - \nabla\cdot(\mathcal{K}(x,\omega)\nabla\phi(x,t,\omega)) = f_p(x,t), \text{ in } \Omega_D \times \Theta, \tag{4.121}$$

$$\phi(x,0) = \phi_0(x), \text{ in } \Omega_D, \text{ and } \overrightarrow{u}(x,0) = u_0(x), \text{ in } \Omega_S,$$

$$\phi(x,t,\omega) = 0, \text{ in } \partial\Omega_D\backslash\Gamma \text{ and } \overrightarrow{u}(x,t,\omega) = 0, \text{ in } \partial\Omega_S\backslash\Gamma.$$

Table 4.8. Errors and convergence rates of the AC ensemble algorithm ($J = 3$) for $\triangle t = 0.1h$.

h	$\|p_h - p\|_0^{E,1}$	rate	$\|p_h - p\|_0^{E,2}$	rate	$\|p_h - p\|_0^{E,3}$	rate
1/4	5.558×10^{-1}	–	5.578×10^{-1}	–	5.577×10^{-1}	–
1/8	2.316×10^{-1}	1.25	2.425×10^{-1}	1.20	2.403×10^{-1}	1.21
1/16	1.007×10^{-1}	1.20	1.097×10^{-1}	1.14	1.145×10^{-1}	1.06
1/32	4.536×10^{-2}	1.15	4.989×10^{-2}	1.13	5.451×10^{-2}	1.07

Table 4.9. Errors of the AC ensemble algorithm for $\triangle t = 3h$.

h	$\|u_h - \overrightarrow{u}\|_0$	$\|u_h - \overrightarrow{u}\|_1$	$\|\phi_h - \phi\|_0$	$\|\phi_h - \phi\|_1$	$\|p_h - p\|_0$
1/4	8.850×10^{0}	1.158×10^{1}	8.619×10^{-1}	2.906×10^{0}	3.334×10^{0}
1/8	8.852×10^{0}	1.102×10^{1}	8.692×10^{-1}	2.439×10^{0}	3.453×10^{0}
1/16	2.417×10^{0}	3.023×10^{0}	2.345×10^{-1}	7.896×10^{-1}	2.160×10^{0}
1/32	2.423×10^{0}	3.009×10^{0}	2.377×10^{-1}	6.713×10^{-1}	2.115×10^{0}

where $f_f(x,t) \in L^2(\Omega_S), f_p(x,t) \in L^2(\Omega_D)$. The hydraulic conductivity $\mathcal{K}(x,\omega)$ is a stochastic function, which is assumed to have continuous and bounded correlation function.

We construct the random hydraulic conductivity tensor that varies in the vertical direction as follows

$$\mathcal{K}(\vec{x},\omega) = \begin{bmatrix} k_{11}(\vec{x},\omega) & 0 \\ 0 & k_{22}(\vec{x},\omega) \end{bmatrix}, \quad \text{and}$$

$$k(\vec{x},\omega) = a_0 + \sigma\sqrt{\lambda_0}Y_0(\omega) + \sum_{i=1}^{n_f} \sigma\sqrt{\lambda_i}[Y_i(\omega)cos(i\pi y) + Y_{n_f+i}(\omega)sin(i\pi y)],$$

where $\vec{x} = (x,y)^T, \lambda_0 = \frac{\sqrt{\pi}L_c}{2}$, $\lambda_i = \sqrt{\pi}L_c e^{-\frac{(i\pi L_c)^2}{4}}$ for $i = 1,\ldots,n_f$ and $Y_0,\ldots,Y_{2n_f}$ are uncorrelated random variables with zero mean and unit variance. In the following numerical test, we take the desired physical correlation length $L_c = 0.25$ for the random field and $a_0 = 1$, $\sigma = 0.15$, $n_f = 3$. We assume the random variables $Y_0,\ldots,Y_{2n_f}$ are independent and uniformly distributed in the interval $[-\sqrt{3}, \sqrt{3}]$. Note that in this setting, the random functions $k_{11}(\vec{x}, w), k_{22}(\vec{x}, w)$ are guaranteed to be positive, and the corresponding $\mathcal{K}(\vec{x}, \omega)$ is SPD.

The domain and parameters are the same as those in the first test. But in this test, the problem is associated with the forcing terms as follows:

$$\begin{aligned}
f_p &= (e^y - e^{-y})sin(x)e^t, \\
f_{f_1} &= [(1 + \nu + 4\nu\pi^2)\frac{k(\vec{x}, \omega)}{\pi}]sin(2\pi y)cos(x)e^t + \pi y cos(\pi x y)e^t, \\
f_{f_2} &= -2\nu k(\vec{x}, \omega)cos(2\pi y)sin(x)e^t + (1 + \nu)[-2k(\vec{x}, \omega) + \frac{k(\vec{x}, \omega)}{\pi^2}sin^2(\pi y)]sin(x)e^t \\
&+ \pi x cos(\pi x y)e^t.
\end{aligned}$$

The Dirichlet boundary condition:

$$\begin{aligned}
\phi &= (e^y - e^{-y})sin(x)e^t, \\
\overrightarrow{u} &= [\frac{k(\vec{x}, \omega)}{\pi}sin(2\pi y)cos(x), (-2k(\vec{x}, \omega) + \frac{k(\vec{x}, \omega)}{\pi^2}sin^2(\pi y))sin(x)]^T e^t
\end{aligned}$$

will be used on the boundary of the domain, and the initial conditions are chosen by

$$\begin{aligned}
\phi &= (e^y - e^{-y})sin(x), \\
\overrightarrow{u} &= [\frac{k(\vec{x}, \omega)}{\pi}sin(2\pi y)cos(x), (-2k(\vec{x}, \omega) + \frac{k(\vec{x}, \omega)}{\pi^2}sin^2(\pi y))sin(x)]^T, \\
p &= sin(\pi x y).
\end{aligned}$$

We simulate the system over the time interval $[0, 0.5]$ and the uniform triangulation with mesh size $h = 1/32$ and the uniform time partition with time step size $\triangle t = 8h^3$ are used. We generate a set of J random samples of $\mathcal{K}$ by the Monte Carlo sampling and run our code for simulating the ensemble of the system associated with the J realizations. First, we need to check the rate of convergence with respect to the number of samples, J. Since

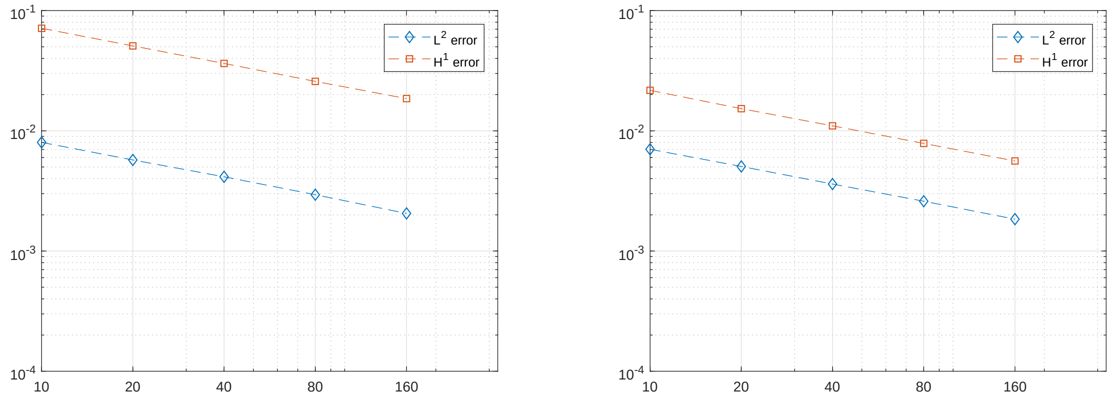

Figure 4.4. The ensemble simulations errors are $O(1/\sqrt{J})$ for $\overrightarrow{u}$ (left) and ϕ (right).

$\mathcal{K}$ is diagonal, Algorithm 5 is used for ensemble computation. As the exact solution to the stochastic Stokes-Darcy system is unknown, we take the ensemble mean of numerical solutions of $J_0 = 1000$ realizations as our exact solution (expectation), which is denoted by u_{J_0}. We also define u_h as the ensemble mean of J realizations. The numerical results with $J = 10, 20, 40, 80, 160$ realizations are listed in Table 4.10. Using linear regression, the errors in Table 4.10 satisfy

$$\|u_h - u_{J_0}\|_0 \approx 0.0248 J^{-0.4985}, \quad \|u_h - u_{J_0}\|_1 \approx 0.2183 J^{-0.4864},$$
$$\|p_h - p_{J_0}\|_0 \approx 0.0214 J^{-0.5033}, \quad \|\phi_h - \phi_{J_0}\|_0 \approx 0.0650 J^{-0.4825}.$$

The values of $\|\cdot\|_0$ and $\|\cdot\|_1$ together with their linear regression models are plotted in Figure 4.4. It is seen that the rate of convergence with respect to J is close to -0.5.

Table 4.10. Errors of ensemble simulations.

J	10	20	40	80	160
$\|u_h - u_{J_0}\|_0^E$	8.0121×10^{-3}	5.7229×10^{-3}	4.1470×10^{-3}	2.9411×10^{-3}	2.0559×10^{-3}
$\|u_h - u_{J_0}\|_1^E$	7.1225×10^{-2}	5.0867×10^{-2}	3.6343×10^{-2}	2.5775×10^{-2}	1.8543×10^{-2}
$\|\phi_h - \phi_{J_0}\|_0^E$	7.0235×10^{-3}	5.0528×10^{-3}	3.6091×10^{-3}	2.5964×10^{-3}	1.8414×10^{-3}
$\|\phi_h - \phi_{J_0}\|_1^E$	2.1714×10^{-2}	1.5291×10^{-2}	1.1001×10^{-2}	7.8578×10^{-3}	5.6127×10^{-3}

Next, we briefly discuss the efficiency of our AC ensemble algorithm compared with the traditional method that runs the simulations individually, based on a test with $J = 1000$ ensemble members and the mesh size $h = 1/64$. A comparison between the matrix systems of these two methods is presented in Table 4.11. First, because all the realizations in our ensemble method share the two common matrices, which are assembled only once, the cost for the matrix assembly is significantly reduced. This can be easily observed in the third column of Table 4.11. Secondly, the common matrix can provide opportunities to preprocess the matrix systems for all the realizations in a unified way, depending on the chosen matrix solver, such as the LU decomposition discussed above. This may lead to a significant reduction of the computational cost for solving the matrix systems. Thirdly, even though each realization in our method has two matrix systems to solve, each of these two matrices, which arise from the two-level decoupling technique in the proposed method, is much smaller than the only one matrix in the traditional coupled method. Therefore, our method saves a lot of computational cost for solving the linear matrix systems.

Table 4.11. Solver comparison of ensemble simulations with a traditional method.

	matrix size	*number of matrix assembly*	*number of matrix solving*
individual	54148×54148	1000	1000
$ensemble_{AC}$	33282×33282	1	1000
	16641×16641	1	1000

4.3.4. An Alternative Approach. If $\mathcal{K}_j(x)$ is diagonal, an alternative artificial compressiblity ensemble algorithm can be devised to remove the parameter conditions for stability.

Algorithm 5 *Find* $(\overrightarrow{u}_j^{n+1}, p_j^{n+1}, \phi_j^{n+1}) \in X_S \times Q_f \times X_D$ *satisfying* $\forall\ (v,\psi) \in X_S \times X_D$,

$$\begin{cases} \left(\dfrac{\overrightarrow{u}_j^{n+1} - \overrightarrow{u}_j^{n}}{\Delta t}, v\right)_f + \nu(\nabla \overrightarrow{u}_j^{n+1}, \nabla v)_f + \sum_i \int_\Gamma \eta_i^{max}(\overrightarrow{u}_j^{n+1} \cdot \widehat{\tau}_i)(v \cdot \widehat{\tau}_i)\, ds \\ + \sum_i \int_\Gamma (\eta_{i,j} - \eta_i^{max})(\overrightarrow{u}_j^{n} \cdot \widehat{\tau}_i)(v \cdot \widehat{\tau}_i)\, ds + \gamma(\nabla \cdot \overrightarrow{u}_j^{n+1}, \nabla \cdot v)_f - \left(p_j^n, \nabla \cdot v\right)_f \\ \qquad + c_\Gamma(v, \phi_j^n) = (f_{f,j}^{n+1}, v)_f, \end{cases} \quad \text{(subproblem 1)}$$

$$p_j^{n+1} = p_j^n - \gamma \nabla \cdot \overrightarrow{u}_j^{n+1}, \quad \text{(subproblem 2)}$$

$$\begin{cases} gS_0\left(\dfrac{\phi_j^{n+1} - \phi_j^n}{\Delta t}, \psi_h\right)_p + k_{max} g(\nabla \phi_j^{n+1}, \nabla \psi)_p + g((\mathcal{K}_j - k_{max}\mathcal{I})\nabla \phi_j^n, \nabla \psi)_p \\ \qquad - c_\Gamma(\overrightarrow{u}_j^n, \psi_h) = g(f_{p,j}^{n+1}, \psi)_p, \end{cases} \quad \text{(subproblem 3)}$$

4.3.4.1. Analysis of long-time stability. We can prove long time stability of Algorithm 5 under a similar time step condition, *without* any parameter conditions.

$$\Delta t \le \min\left\{ \frac{(1-\alpha_1-\alpha_2)\beta_1^2 k_{max}}{C_{P,p}^2}, \frac{(1-\beta_1-\beta_2-\frac{k_{max}-k_{min}}{k_{max}})\alpha_1^2 S_0 \nu}{C_{P,f}^2} \right\} \frac{2\nu k_{max}}{g^2[C(D_f)C(D_p)]^4}. \quad (4.122)$$

Theorem 11 (Long time stability of Algorithm 5) *If there exist* $\alpha_1, \alpha_2, \beta_1, \beta_2$ *in* $(0,1)$ *such that the time step condition* (4.122) *holds, then the Algorithm 5 is long time stable: for any* $N > 0$,

$$\begin{aligned} &\frac{1}{2}\|\overrightarrow{u}_j^N\|_f^2 + \frac{gS_0}{2}\|\phi_j^N\|_p^2 + \Delta t^2 \frac{C_2}{\beta_1^2}\|\nabla \overrightarrow{u}_j^N\|_f^2 + \frac{\Delta t}{2\gamma}\|p_j^N\|_f^2 + \Delta t \sum_i \frac{\eta_i^{max}}{2}\int_\Gamma (\overrightarrow{u}_j^N \cdot \widehat{\tau}_i)^2\, ds \\ &+ \left(\Delta t^2 \frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \Delta t \frac{g(k_{max}-k_{min})}{2}\right)\|\nabla \phi_j^N\|_p^2 + \frac{\Delta t}{2\gamma}\sum_{n=0}^{N-1}\|p_j^{n+1} - p_j^n\|_f^2 \\ &\le \frac{1}{2}\|\overrightarrow{u}_j^0\|_f^2 + \frac{gS_0}{2}\|\phi_j^0\|_p^2 + \Delta t^2 \frac{C_2}{\beta_1^2}\|\nabla \overrightarrow{u}_j^0\|_f^2 + \frac{\Delta t}{2\gamma}\|p_j^0\|_f^2 + \Delta t \sum_i \frac{\eta_i^{max}}{2}\int_\Gamma (\overrightarrow{u}_j^0 \cdot \widehat{\tau}_i)^2\, ds \\ &+ \left(\Delta t^2 \frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \Delta t \frac{g(k_{max}-k_{min})}{2}\right)\|\nabla \phi_j^0\|_p^2 + \Delta t \sum_{n=0}^{N-1}\frac{C_{P,f}^2}{4\alpha_2 \nu}\|f_{f,j}^{n+1}\|_f^2 \\ &+ \Delta t \sum_{n=0}^{N-1}\frac{gC_{P,p}^2}{4\beta_2 k_{max}}\|f_{p,j}^{n+1}\|_p^2. \end{aligned} \quad (4.123)$$

Proof: Setting $v_h = \overrightarrow{u}_j^{n+1}, \psi = \phi_j^{n+1}$ in Algorithm 5, replacing $\gamma\nabla\cdot\overrightarrow{u}_j^{n+1}$ in the momentum equation by $p_j^{n+1} - p_j^n$, taking inner product of the mass conservation equation by $\gamma^{-1}p_j^{n+1}$ and adding all three equations yields

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|\overrightarrow{u}_j^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|\overrightarrow{u}_j^{n}\|_f^2 + \frac{1}{2\Delta t}\|\overrightarrow{u}_j^{n+1} - \overrightarrow{u}_j^{n}\|_f^2 + \nu\|\nabla\overrightarrow{u}_j^{n+1}\|_f^2 \\
&+ \sum_i \int_\Gamma \eta_i^{max}(\overrightarrow{u}_j^{n+1}\cdot\widehat{\tau}_i)(\overrightarrow{u}_j^{n+1}\cdot\widehat{\tau}_i)\,ds + \frac{1}{2\gamma}\left(\|p_j^{n+1}\|_f^2 - \|p_j^n\|_f^2 + \|p_j^{n+1} - p_j^n\|_f^2\right) \\
&+ \frac{gS_0}{2\Delta t}\|\phi_j^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi_j^{n}\|_p^2 + \frac{gS_0}{2\Delta t}\|\phi_j^{n+1} - \phi_j^n\|_p^2 + gk_{max}(\nabla\phi_j^{n+1}, \nabla\phi_j^{n+1})_p \qquad (4.124)\\
&+ c_I(\overrightarrow{u}_j^{n+1}, \phi_j^n) - c_I(\overrightarrow{u}_j^{n}, \phi_j^{n+1}) \\
&= (f_{f,j}^{n+1}, \overrightarrow{u}_j^{n+1})_f + g(f_{p,j}^{n+1}, \phi_j^{n+1})_p - \sum_i \int_\Gamma (\eta_{i,j} - \eta_i^{max})(\overrightarrow{u}_j^{n}\cdot\widehat{\tau}_i)(\overrightarrow{u}_j^{n+1}\cdot\widehat{\tau}_i)\,ds \\
&\quad - g((\mathcal{K}_j - k_{max}\mathcal{I})\nabla\phi_j^n, \nabla\phi_j^{n+1})_p.
\end{aligned}
$$

Since $\mathcal{K}_j(x)$ and $k_{max}\mathcal{I}$ are both symmetric, we have $|\mathcal{K}_j(x) - k_{max}\mathcal{I}|_2 \le k_{max} - k_{min}$. The main difference from the proof of Theorem 9 is on the estimates of the following two terms:

$$
\begin{aligned}
&-\sum_i \int_\Gamma (\eta_{i,j} - \eta_i^{max})(\overrightarrow{u}_j^{n}\cdot\widehat{\tau}_i)(\overrightarrow{u}_j^{n+1}\cdot\widehat{\tau}_i)\,ds \\
&\le \sum_i \left[\frac{\eta_i^{max}}{2}\int_\Gamma (\overrightarrow{u}_j^{n}\cdot\widehat{\tau}_i)^2\,ds + \frac{\eta_i^{max}}{2}\int_\Gamma (\overrightarrow{u}_j^{n+1}\cdot\widehat{\tau}_i)^2\,ds\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&-g\left((\mathcal{K}_j - k_{max}\mathcal{I})\nabla\phi_j^n, \nabla\phi_j^{n+1}\right)_p \\
&\le \frac{g(k_{max} - k_{min})}{2}\|\nabla\phi_j^n\|_p^2 + \frac{g(k_{max} - k_{min})}{2}\|\nabla\phi_j^{n+1}\|_p^2.
\end{aligned}
$$

The estimates of other terms are similar to those in the proof of Theorem 9. Combining all estimates, we then have the following inequality:

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|\overrightarrow{u}_j^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|\overrightarrow{u}_j^{n}\|_f^2 + \left(1 - \alpha_1 - \alpha_2 - \Delta t\frac{2C_2}{\beta_1^2\nu}\right)\nu\|\nabla\overrightarrow{u}_j^{n+1}\|_f^2 \\
&+\Delta t\frac{C_2}{\beta_1^2}\left(\|\nabla\overrightarrow{u}_j^{n+1}\|_f^2 - \|\nabla\overrightarrow{u}_j^{n}\|_f^2\right) + \sum_i \frac{\eta_i^{max}}{2}\left[\int_\Gamma (\overrightarrow{u}_j^{n+1}\cdot\widehat{\tau}_i)^2\, ds - \int_\Gamma (\overrightarrow{u}_j^{n}\cdot\widehat{\tau}_i)^2\, ds\right] \\
&+\frac{1}{2\gamma}\left(\|p_j^{n+1}\|_f^2 - \|p_j^n\|_f^2 + \|p_j^{n+1} - p_j^n\|_f^2\right) + \frac{gS_0}{2\Delta t}\|\phi_j^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi_j^n\|_p^2 \\
&+(1 - \beta_1 - \beta_2 - \Delta t\frac{1}{g_0^2 k_{max}}\frac{2C_1}{\alpha_1^2} - \frac{k_{max}-k_{min}}{k_{max}})gk_{max}\|\nabla\phi_j^{n+1}\|_p^2 \\
&+\left(\Delta t\frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \frac{g(k_{max}-k_{min})}{2}\right)\left(\|\nabla\phi_j^{n+1}\|_p^2 - \|\nabla\phi_j^n\|_p^2\right) \le \frac{C_{P,f}^2}{4\alpha_2\nu}\|f_{f,j}^{n+1}\|_f^2 \\
&+\frac{gC_{P,p}^2}{4\beta_2 k_{max}}\|f_{p,j}^{n+1}\|_p^2.
\end{aligned}
\tag{4.125}
$$

Since we assume $\mathcal{K}_j$ is SPD, and any two ensemble members have different hydraulic conductivity tensor $\mathcal{K}$, we have $k_{max} > k_{min} > 0$ and thus $0 < \frac{k_{max}-k_{min}}{k_{max}} < 1$. Thus, no constraints on the parameters are required. Now if the time step condition (4.122) holds, (4.125) reduces to

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|\overrightarrow{u}_j^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|\overrightarrow{u}_j^{n}\|_f^2 + \Delta t\frac{C_2}{\beta_1^2}\left(\|\nabla\overrightarrow{u}_j^{n+1}\|_f^2 - \|\nabla\overrightarrow{u}_j^{n}\|_f^2\right) \\
&+\sum_i \frac{\eta_i^{max}}{2}\left[\int_\Gamma (\overrightarrow{u}_j^{n+1}\cdot\widehat{\tau}_i)^2\, ds - \int_\Gamma (\overrightarrow{u}_j^{n}\cdot\widehat{\tau}_i)^2\, ds\right] + \frac{1}{2\gamma}\left(\|p_j^{n+1}\|_f^2 - \|p_j^n\|_f^2 + \|p_j^{n+1} - p_j^n\|_f^2\right) \\
&+\frac{gS_0}{2\Delta t}\|\phi_j^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi_j^n\|_p^2 + \left(\Delta t\frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \frac{g(k_{max}-k_{min})}{2}\right)\left(\|\nabla\phi_j^{n+1}\|_p^2 - \|\nabla\phi_j^n\|_p^2\right) \\
&\le \frac{C_{P,f}^2}{4\alpha_2\nu}\|f_{f,j}^{n+1}\|_f^2 + \frac{gC_{P,p}^2}{4\beta_2 k_{max}}\|f_{p,j}^{n+1}\|_p^2.
\end{aligned}
\tag{4.126}
$$

Summing up (4.126) from $n = 0$ to $N - 1$ and multiplying through by Δt yields (4.123).

4.3.4.2. Applicational simulation. Next, we apply our AC ensemble algorithm to a simplified simulation of the subsurface flow in a karst aquifer. The computational domain is a unit square divided into the porous media domain Ω_D and the free flow domain Ω_S. Let Ω_S be the polygon $\overline{ABCDEFGHIJ}$ where $A = (0,1), B = (0,3/4), C = (1/2,1/4), D = (1/2,0), E = (3/4,0), F = (3/4,1/4), G = (1,1/4), H = (1,1/2), \Gamma = (3/4,1/2)$ and $J = (1/4,1)$. Let $\Omega_D = \Omega/\Omega_S$, $S_0 = \overline{AB} \cup \overline{JA}$, $S_1 = \overline{DE}$, and $S_2 = \overline{GH}$.

Set $T = 1$, $\alpha = 1$, $\nu = 1$, $g = 1$, $z = 0$. The boundary condition data and source terms are chosen to be 0 and let

$$\overrightarrow{u} = \begin{cases} (U_0, 0)^T on & S_0 \\ (0, U_1)^T on & S_1 \\ (U_3, 0)^T on & S_2 \end{cases}$$

where U_i are constants. We subdivide Ω into a rectangle of height and width $h = 1/M$, where M denotes a positive integer, and then subdivide each rectangle into two triangles by drawing a diagonal. For this numerical experiment, we choose $M = 32$ and $\triangle t = h$. In the following, we will provide the numerical results at $T = 1$ for the algorithm. We construct the random hydraulic conductivity tensor as follows:

$$\begin{aligned} &k(\vec{x}, \omega) \\ &= a_0 + exp\left\{[Y_1(\omega)cos(\pi y) + Y_3(\omega)sin(\pi y)]\, e^{-\frac{1}{8}} + [Y_2(\omega)cos(\pi x) + Y_4(\omega)sin(\pi x)]\, e^{-\frac{1}{8}}\right\}. \end{aligned}$$

where $\vec{x} = (x, y)^T, a_0 = 1/100$, and $Y_1, \ldots, Y_4$ are independent and identically distributed with zero mean and unit variance.

Figure 4.5 shows some realizations of the logarithm of the hydraulic conductivity coefficient. The three graphs in Figure 4.6 illustrate the numerical solutions at the end time $T = 1$ for these three tests. These phenomena are expected due to the chosen unbalanced inflow and outflow rates for the conduit. In the first test, we set $U_1 = U_2 = -1$ and $U_0 = 2$

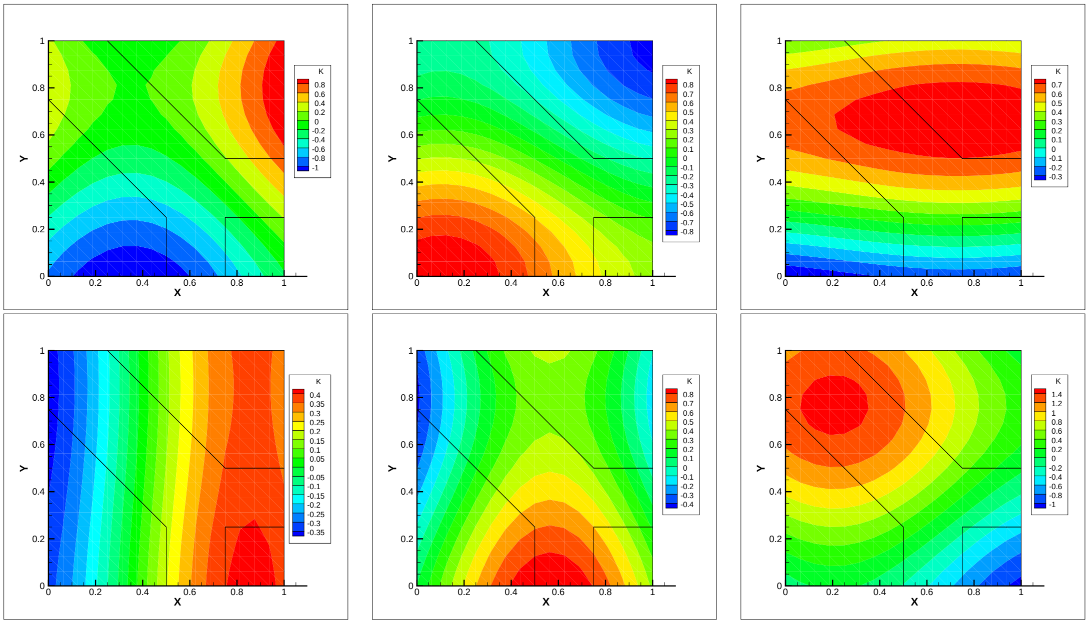

Figure 4.5. Some realizations of log(k).

so that the total inflow rate is equal to the total outflow rate. In the second test, we keep the same U_1 and U_2 but set $U_0 = 1$ so that the total inflow rate is larger than the total outflow rate. This causes more water to be pushed out of the conduits into the porous media, which happens during a rainy season. In the third test, we keep the same U_1 and U_2 but set $U_0 = 3$ so that the total inflow rate is smaller than the total outflow rate. More outflow causes more water to flow into the conduits from the porous media, which is what happens during a dry season. Compared to the solutions of the traditional method (Figure 4.7), we can find they have the same general behavior of the flow while our AC ensemble algorithm is much more efficient.

At last, we consider the more realistic case where the hydraulic conductivity tensor is non-diagonal, for which we need to use Algorithm 4 for ensemble computation. Let

$$\mathcal{K}(\vec{x},\omega) = \begin{bmatrix} k_{11}(\vec{x},\omega) & k_{12}(\vec{x},\omega) \\ k_{21}(\vec{x},\omega) & k_{22}(\vec{x},\omega) \end{bmatrix},$$

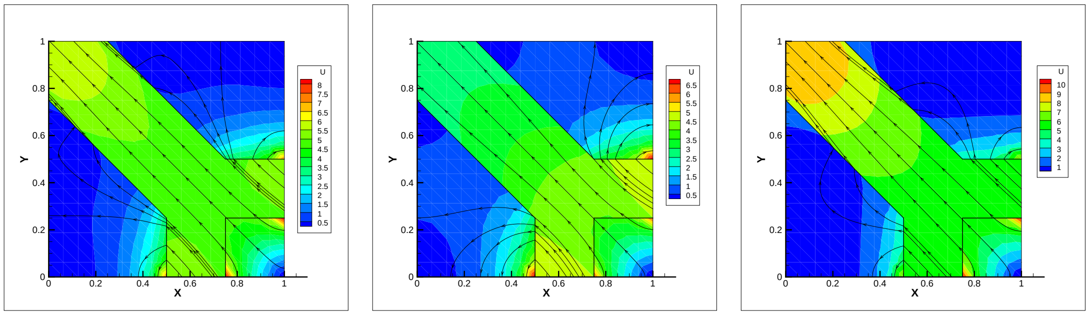

Figure 4.6. Plots of the ensemble mean for the ensemble AC method for $U_1 = -1$, $U_2 = -1$, and different U_0: $U_0 = 2$ in the left graph, $U_0 = 1$ in the middle graph, and $U_0 = 3$ in the right graph.

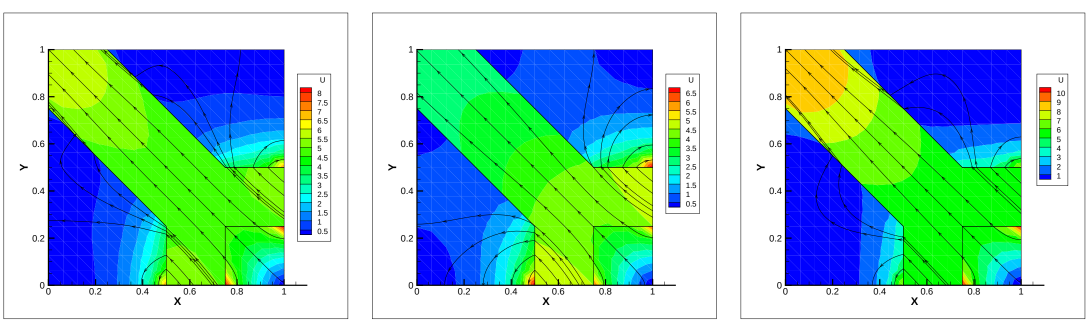

Figure 4.7. Plots of the ensemble mean for the traditional method for $U_1 = -1$, $U_2 = -1$, and different U_0: $U_0 = 2$ in the left graph, $U_0 = 1$ in the middle graph, and $U_0 = 3$ in the right graph.

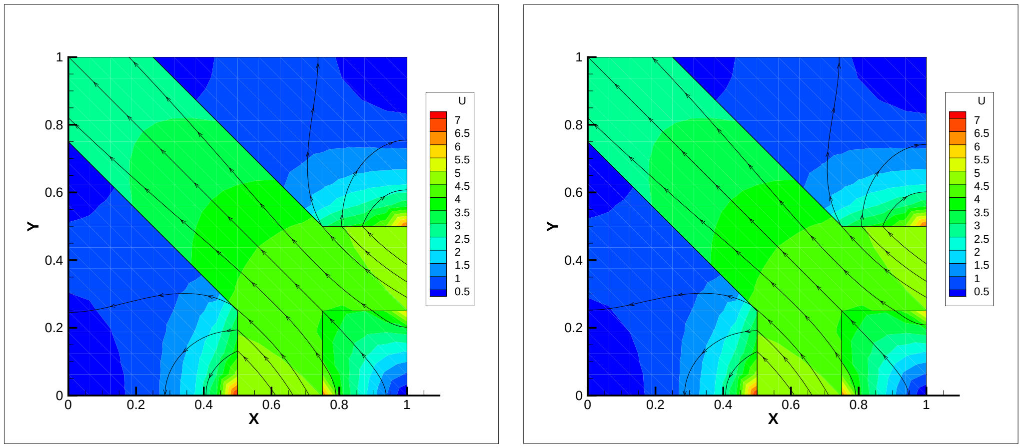

Figure 4.8. Speed contours and velocity streamlines for $U_0 = 1$, $U_1 = -1$, $U_2 = -1$, based on the ensemble mean obtained from our ensemble algorithm (left) and the traditional method (right) and with $J = 100$ at $T = 0.5$.

where $k_{11}(\vec{x},\omega) = k_{22}(\vec{x},\omega) \neq 0$ and $k_{21}(\vec{x},\omega) = k_{12}(\vec{x},\omega) \neq 0$, i.e. $\mathcal{K}(\vec{x},\omega)$ is not diagonal but symmetric.

$$k_{11}(\vec{x},\omega) = k_{22}(\vec{x},\omega) = a_1 + \sigma\sqrt{\lambda_0}Y_0(\omega) + \sum_{i=1}^{n_f} \sigma\sqrt{\lambda_i}[Y_i(\omega)cos(i\pi y) + Y_{n_f+i}(\omega)sin(i\pi y)],$$

$$k_{21}(\vec{x},\omega) = k_{12}(\vec{x},\omega) = a_2 + \sigma\sqrt{\lambda_0}Y_0(\omega) + \sum_{i=1}^{n_f} \sigma\sqrt{\lambda_i}[Y_i(\omega)cos(i\pi y) + Y_{n_f+i}(\omega)sin(i\pi y)].$$

We take $a_1 = 10$ and $a_2 = 1$ such that the random hydraulic conductivity tensor $\mathcal{K}(\vec{x},\omega)$ is SPD. The corresponding forcing term for the Darcy equation is $f_p = (1 + k_{11}(\vec{x},\omega) - k_{22}(\vec{x},\omega))(e^y - e^{-y})sin(x)e^t - (k_{12}(\vec{x},\omega) + k_{21}(\vec{x},\omega))(e^y - e^{-y})cos(x)e^t$; for the Stokes equations, f_{f_1} and f_{f_2} are the same as those in Section 4.3.3.2. The boundary conditions and initial conditions are also the same as those in Section 4.3.3.2.

We consider a group of simulations with $J = 100$, using the Monte Carlo method for sampling. Figure 4.8 shows the numerical results of our ensemble algorithm (Algorithm 4) and those of individual runs for comparison. The speed contours and velocity streamlines of the ensemble mean are computed for both approaches at $T = 0.5$ with $J = 100$ realizations, and then presented in Figure 4.8. It can be seen that both approaches capture the same general behavior of the flow while our AC ensemble algorithm is much more efficient.

5. CONCLUSIONS

In this dissertation, a semi-implicit non-iterative domain decomposition method, which uses k-step backward differentiation formulae ($1 \leq k \leq 5$) for the temporal discretization and finite elements for the spatial discretization, is proposed to solve a coupled unsteady NS-Darcy system with BJSJ interface conditions. Due to the explicit treatment of the interface terms, this domain decomposition does not need an iteration to identify the interface information. The nonlinear convection is handled by a multi-step semi-implicit scheme. For the first time in the literature, we derive the error estimate in L^2 norm for the joint Stokes-Darcy Ritz-projection without using H^2 regularity assumption of the elliptic problem corresponding to this joint Ritz-projection. Then the finite element solution of the k-step backward differentiation formulae ($1 \leq k \leq 5$) of the proposed method is analyzed for its convergence. Next, we first develop and analyze a parallel non-iterative multi-physics DDM for the NS-Darcy system with BJ interface conditions which is much more complicated than BJSJ. Furthermore, in order to handle the defective boundary conditions in NS-Darcy interface system, a Lagrange multiplier method is proposed under the framework of the proposed domain decomposition method. The numerical examples are consistent with the theoretical conclusions, and they validate the proposed method.

For the stochastic model, ensemble calculation is essential in uncertainty quantification, numerical weather prediction, sensitivity analysis, and so on. We proposed an efficient, decoupled ensemble algorithm for fast computation of the stochastic Stokes-Darcy systems. In this algorithm, the linear systems with two shared common coefficient matrices for all realizations at each time step can be solved by efficient iterative or direct methods at a greatly reduced computational cost. Moreover, the fully coupled Stokes-Darcy system can be decoupled into two smaller sub-physics problems, which reduces the size of the linear systems to be solved and allows parallel computation of the two sub-physics problems.

Furthermore, in order to further reduce the storage requirements, an efficient, artificial compressibility ensemble algorithm is proposed, which decouples the velocity and pressure to result in a simple updating step for the pressure. The long time stability and first order accuracy in time under a time step condition and two parameter conditions are proved for both algorithms.

In my future work, I plan to continue my efforts to contribute to several research topics related to the NS-Darcy model, including the long time stability of non-iterative domain decomposition methods with special attention to the difficulty arising from the nonlinear term, and the analysis of the Lagrange multiplier method for the defective boundary conditions. For the efficient ensemble algorithms in the application of uncertainty quantification (UQ) for surface-groundwater flows, higher order ensemble methods and their applications in UQ are our natural next step.

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