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U.S. students have consistently demonstrated poor performance in spatial reasoning in standardized testing (e.g., National Assessment of Educational Progress). One possible reason is students' lack of conceptual understanding of measurement concepts (length, area, volume, capacity). This paper describes different ways that mathematics textbooks written for future elementary teachers define meanings of measurement concepts, especially the meaning of *measure, area*, and the measurement process (generally and for area). We base the analysis of definitions and construction of *complete* definitions using several definitions of each concept from mathematics textbooks written for future elementary teachers (e.g., Beckmann, 2012; Sowder, Sowder, & Nickerson, 2010). Although not one mathematics textbook provided a complete definition, together the definitions present a detailed and in-depth look at the measurement process and area measurement.

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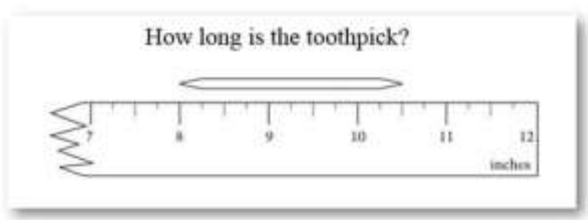
Introduction

Deficiencies in elementary students' conceptual understanding of spatial measurement have persisted, emerging through educational research (e.g., Kamii & Kysh, 2006) and national assessments (e.g., National Assessment of Educational Progress). Investigating several decades of results from the National Assessment of Educational Progress (NAEP), Kloosterman, Rutledge, and Kenney (2009) described particular examples of persistent measurement deficiencies. When asked to measure a toothpick in nonstandard position (shown in Figure 1), only 20% of fourth grade students (NAEP, 2004) determined the correct length. Their incorrect responses indicated that students were unable to adapt typical length measurement strategies to this unfamiliar situation. Their struggle indicated that they did not have a deep understanding of the meaning of length measurement nor the measurement procedure. In the same year, 40% of eighth grade students also answered incorrectly, which indicated that the lack of understanding persisted across grade levels. Similar difficulties in understanding area and volume measurement meanings and procedures can be seen in other NAEP questions.

Figure 1. "Toothpick Problem" from NAEP (2003)



More recently, a 2017 NAEP item asked 144,900 eighth grade students, "Casey goes running every morning. Which of the following units can Casey use to measure the distance he runs: cubic feet, grams, liters, meters, or square meters?" Of all eighth grade students, over half (54%) answered correctly (meters) but 46% answered with square miles (38%), cubic feet (6%), grams (1%), or liters (1%). These results---choosing units that would be used to measure area or volume, rather than length---indicate that students still do not have a deep understanding of fundamental concepts of spatial measurement. Future elementary teachers must be supported in developing deep understanding of these concepts in order to support their future students in their own development.

Definitions are centrally important to mathematics, providing one way to communicate mathematical concepts. Definitions allow for precise communication. One perspective on mathematical definitions focuses on using extremely precise language to accurately describe a concept. Although Vinner (1991) argued that definitions alone are not sufficient in helping students develop a mental image of the concept behind the definition, learning to use definitions to support arguments is important for all students in mathematics courses. McCrory

(2006) explored definitions of fractions in mathematics textbooks written for future elementary teachers and, finding inconsistency in definitions across textbooks, argued:

> These differences in [definitions] may seem insignificant to some, and it is possible, perhaps likely, that each of these mathematicians would judge the others' approaches as correct even if not ideal. Yet the details and how they are addressed represent sophisticated mathematical issues and point to a fundamental mathematical problem that is replayed across the curriculum: How do we create definitions and starting assumptions that are both mathematically correct and at the same time comprehensible and unambiguous to this population of students (prospective elementary teachers)?

This literature review focused on: *What opportunities do future teachers have to encounter key aspects of length, area, or volume measurement in textbook definitions?*

After examining length, area, volume, and capacity definitions in eleven mathematics for elementary teacher textbooks, we found that definitions vary greatly. Different aspects emerge in definitions across different textbooks; no textbook definition included all aspects. Authors likely wrote the definitions, anticipating the needs of making concepts understandable to future teachers, while also attending to the length and focus of the textbook. We present here an analysis of the definitions, connecting and building on the multiple definitions to construct a potentially complete definition. Although length, volume, and capacity, are also interesting, due to space limitations, this paper focuses on definitions of a general measurement process, using the results to then organize and analyze definitions of area measurement.

Method

Building on McCrory's (2006) exploration of 20 mathematics textbooks written for elementary teachers, we searched for updated editions of the textbooks, excluding those that are now out of print. We ended with 11 mathematics textbooks written for elementary teachers. We looked in the index of each textbook for "measure(ment)," "length," "area," "volume," and "capacity."

Some textbooks included multiple definitions: we counted a definition as separate from another definition if they were on different pages (and not connected on consecutive pages). we did not capture associated student activities or illustrative examples because our focus was on the way the textbook *told* the PSTs, through definitions or descriptions, the mathematical meaning of each concept.

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We printed definitions on cards: one stack for each concept (e.g., measurement, area). We sorted the cards multiple times, according to aspects included in the descriptions and definitions. First, writing individual aspects, as expressed in each definition. Then, allowing themes to emerge by grouping and regrouping those aspects (e.g., Creswell, 2007; Machi & McEvoy, 2016). For example, one definition of *area* by Parker and Baldridge (2004) was Definition 1.2 (School Definition). The area of a region tiled by unit squares is the number of squares it contains.

For this definition, we noted ideas of the area *action* (e.g., tiling), *unit* (e.g., unit squares), *space* (e.g., region), and *quantification* of area (e.g., number of squares [the tiled region] contains). Comparing aspects expressed in one card to other cards created many potential themes. As we compared cards and aspects, we found overarching themes that could be used to summarize the aspects.

Findings

We first present an overview of the definitions across the 11 textbooks. Then we describe the measurement process and area definitions.

The table below shows the number of definitions, organized by type, included in each textbook.

Table 1. Number of spatial measurement definitions by textbook and definition type.

	Measure	Length	Area	Volume	Capacity
1 Bassarear (2012)		1	2	4	
2 Beckmann (2011)		1	2	2	1
3 Bennett, Burton, & Nelson (2012)	1	1	2	1	
4 Billstein, Libeskind, & Lott (2010)	1	1	2	1	
5 Sowder, Sowder, & Nickerson (2010)		2	2	2	2
6 Long & DeTemple (2012)	1		1	1	1
7 Musser, Burger, & Peterson (2011)	1	1	1	1	1
8 Parker & Baldridge (2004)	1	2	3	3	1
9 Sonnabend (2010)	1		2	1	

10 Van de Walle, Karp, & Bay-Williams (2013)		1	1	1	1
11 Wu (2011)			2		
	6	10	20	16	7

Area and volume made up 36 of the 59 definitions, which indicates more attention was paid to these concepts than to length or the measurement process.

Measurement

There were six definitions describing measurement or the measurement process. The definitions varied. For example, Long and DeTemple (2012) wrote the following definition of measurement:

> The process starts by asking what attribute of the geometric figure we want to measure. Next, we begin to discuss the general process of measurement and the concept of a unit of measurement. (i) Choose the property, or attribute (such as length, area, volume, capacity, temperature, time or weight), of an object or event that is to be measured. (ii) Select an appropriate unit of measurement. (iii) Use a measurement device to 'cover,' 'fill,' 'time,' or otherwise provide a comparison of the object with the unit. (iv) Express the measurement as the number of units used. (pp. 526-527)

In contrast, Billstein et al. (2010) quoted the National Council of Teachers of Mathematics' (2000) *Principles and Standards for School Mathematics*, "A measurable attribute is a characteristic of an object that can be quantified. Line segments have length, plane regions have area, and physical objects have mass." Both definitions include important aspects of measurement, while excluding other important aspects. For example, Long and DeTemple included a detailed description of the process of measurement while Billstein et al. did not. Both definitions include examples of measurable attributes, but Billstein et al. actually defined the term as "a characteristic of an object that can be quantified" while Long and DeTemple did not explain beyond providing examples.

After carefully reading each definition, themes began to emerge. Based on these definitions and themes, a complete definition of measurement would include a definition or description of a *measurable attribute* and a *unit of measure*, followed by the measurement process: (1) selection of object and attribute, (2) selection of unit of measure, (3) comparison of attribute and unit, and (4) expression of measure. In the table below, I show the aspects of each step in the measurement process and how they appeared in the textbook definitions.

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Table 2. Number of definitions per textbook mentioning aspects of the measurement process

	3	4	6	7	8	9
Select attribute	1		1	1		1
Select object	1			1		
Mention measurable attribute	1	1	1	1		1
Define measurable attribute		1				
Examples of measurable attribute	1	1	1	1	1	
Select unit of measure	1		1	1	1	1
Mention unit of measure	1		1	1	1	1
Define of unit of measure	1					
Mention standardized units				1	1	
Compare attribute & unit	1		1			1
Mention measuring device			1			
Mention measuring actions			1			
Mention determine number	1		1			
Express measure			1	1	1	
as multiples of a unit			1	1	1	
as number of units used			1			

Based on the textbook definitions, we describe a potential *complete* definition, using sections corresponding to the steps of the measurement process. We cite the textbooks when possible by referring to them by the number assigned to each in Table 1 (e.g., “Bennett et al. (2010)” would be referred to as “3”).

Selection of object and attribute. Select an object, or event (6), and an attribute (a property (6) or physical property (3)) to be measured (3, 7, 6, 9). A measurable attribute is a characteristic of an object that can be quantified (4). Any object or event has many attributes that can be measured (9). When you measure something, you focus on one attribute and ignore all other attributes and qualities (9). Examples of measurable attribute include: length (3, 4, 6, 7, 8), area (4, 6, 7, 8), volume (6, 7, 8), capacity (6, 8), mass (4), weight (3, 6, 7, 8), angle (8), time (6, 8), and temperature (3, 6, 7).

Selection of unit of measure. Select an appropriate (6, 7) unit (3, 6, 7, 8, 9), which may be a standard (8) or nonstandard unit, with which to measure the attribute (7). A unit of measure is any reproducible unit that can be used to measure a physical property (3) by covering, filling, timing, or otherwise (6) providing a comparison of the object's attribute with the unit (3, 6, 9).

Comparison of an object's attribute with unit of measure. Compare the attribute of the object, or event, to the unit of measure (3, 6, 9) to determine the number of units (3, 7) that will cover, fill, time, or otherwise complete the comparison of attribute and unit. This may require a measurement device (6, 7).

Expression of measure. Express final measures as multiples of the standard (8) or nonstandard unit; that is, as the number of units used (6).

Area

There were 20 area definitions, compared to only six measurement definitions. Each of the 11 textbooks included a definition, and several included two or three. Often the additional definition referred to surface area, and was still included in the analysis. Using the measurement process as a framework, we focused on the steps: (1) select an object to measure, (2) area as the selected attribute, (3) select unit of measure, (4) compare attribute of object to unit of measure, and (5) express the area measurement.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
Select object to measure											
Mention region		1		2	2	1	1	3	1	1	1
Other: space, shape, rectangle, surface, object, face	1		2		2				1		1
Define region								1			
Mention bounded / closed						1			1		

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Mention planar or two-dimensional		1		1	1	1	1	1	1	1	1
Provide examples	1		1		1			1	1		
Area as the selected attribute											
Mention as a attribute / characteristic				1	1						

Mention size, "how big", amount of space	1	1	2					1	1	1	
Mention space inside boundary								1		1	
Mention area is a measure or number	1	2	1	1		1		3	2		2
Mention attribute can be quantified				1	1			1			
Provide examples	1		1		1				1		
Select unit of measure											
Mention unit squares			1					2			2
Mention square units		1		2	1	1			1		
Mention other units	1	1		1		1	1	1			
Define unit				1	1		1	1			2
Rationale for "why squares"			1			1					
Provide examples		1				1					
Compare attribute & unit											
Mention tile, pave, cover	2	2	2	2	1		1	1	1		1
Define tile, pave, cover											1
Mention "no gaps or overlaps"		2		2							1
Mention squares or parts of squares		2									1

Express measure											
Mention number of units	1	2	2	2	1		1	3	1		1
Mention multiples of unit								1			
Describe functional relationship								1			
Describe sum of areas									1		1
Describe equal areas											1

We describe a potential *complete* definition of area and the measure of area, based on the textbook definitions. This definition is framed by the steps of the measurement process described above. We cite the textbooks when possible by referring to them by the number assigned to each in Table 1 (e.g., "Bennett et al. (2010)" would be referred to as "3").

Selection of object. We can measure the area of a region (2, 4, 5, 7, 8, 9, 10, 11). A portion of the plane is called a region (8). A region formed from several pieces is a composite region (8). The region is two-dimensional (2, 4, 5, 6, 7, 8, 9, 10) or planar (11) and is bounded (6, 9) by a closed plane curve (6). Roughly speaking, an object is two-dimensional if, at each location, there are two independent directions along which to move within the object (2). A region could refer to all the faces (1) or surfaces of a 3D figure (3, 5, 9), a rectangle (11), a triangle together with its interior (8), a polygon together with its interior (8), or a disk (8). Examples of surfaces or regions include: the area of the interior of a field (3, 5, 9), a lake (5), a country (5), a geometric shape (5), panes of glass (3), a wall (3, 5), or your body (5).

Area as the selected attribute. Area is a characteristic (or attribute) of surfaces or regions (4, 5). Area is the space within a region (8, 10). As with length, the term area is used to refer both to the attribute ("The wolf wandered over a wide area") and to the measurement ("The area of the field is 15 acres"). So area is a measure (1, 2, 3, 4, 6, 8, 9, 11). Said another way, we can quantify area by measuring it (4, 5, 8) and, when speaking of the quantification, then area is always a number (11). Thinking of area as a functional relationship, are is a way of associating to each region R a quantity Area(R) that reflects our intuitive sense of "how big" the region is, or the size of the region (1, 2, 3, 9, 10), without reference to the shape of the region (8). For example, how much it takes to cover an object like how much fertilizer to cover a lawn or how much material to cover a bed (1). You could use area to compare prices based on the area of a package of gift wrap (9).

Selection of unit of measure. To measure the sizes of surfaces, we need a new type of unit, one that can be used to cover a surface (3). To measure the area of a

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region informally, we select a an appropriate unit (1). We can choose any convenient two-dimensional shape as our unit (7). Any shape could be chosen as a unit (6), as long as they are same-sized units of area (4). The area of a "unit region" is declared to be 1 unit of area (8). The square is the most common (3, 6); we call it a square unit (2, 4, 5, 6, 9) or unit square (3, 8, 11). The size of the square is arbitrary
(6). It is natural to choose the length of a side to correspond to a unit measure of length (2, 6). That is, a square unit typically measures 1 length unit on a side (1, 4, 11). The area of a unit square is by definition the number 1 (11). Areas are usually measured in square centimeters (2), square inches (2, 6), square feet (6), and so on.

Comparison of an object's attribute with unit of measure. We cover (1, 3, 4, 5, 7, 8, 9) or tile (8, 11) or pave (11) the region with units (4, 5, 7). The units should be nonoverlapping (4); that is, there should be no gaps or overlaps between the units (2, 4, 11). The squares may be cut apart if necessary so that we may use parts of a unit (2, 11). For example, we say a collection of unit rectangles tile or pave a given rectangle R, if, by combining the collection together we get the whole rectangle R (11). By "no gaps or overlaps," we mean that the collection of unit rectangles intersect at most along their boundaries (11).

Expression of measure. We determine the number of same-sized units that are needed (1, 2, 3, 4, 5, 7, 8, 9, 11), or we can say that we express area as a multiple of the unit area (8). If two regions are congruent, then their areas are equal (11). If two regions have at most (part of) their boundaries in common, then the area of the region obtained by combining the two is the sum of their individual areas (9, 11).

Conclusion

The concepts of measurement and area seem to be fairly straightforward, especially to a layperson. The creation of definitions is not a straightforward process, however. In the textbooks examined here, no one textbook included a "perfect" definition. Even the definitions that I have constructed based on the ideas and descriptions of these 11 authors are incomplete. Future steps of this research should be to bring such a detailed definition to preservice elementary teachers to see if it comprehensible and useful to their development of a more robust understanding of measurement, area, and the process of measurement.

References

Bassarear, T. (2012). *Mathematics for elementary school teachers* (5th ed). Belmont, CA: Brooks/Cole-

Cengage.

Beckmann, S. (2011). *Mathematics for elementary school teachers* (3rd ed). Boston, MA: Pearson/Addison

Wesley.

Bennett, A.B., Burton, L. J. & Nelson, L. T. (2012). *Math for elementary teachers: A conceptual approach* (9th ed). New York, NY: McGraw-Hill Science/Engineering/Math.

Billstein, R., Libeskind, S., & Lott, J. W. (2010). *A problem solving approach to mathematics for elementary school teachers* (10th ed). Boston, MA: Pearson/Addison-Wesley.

Kamii, C., & Kysh, J. (2006). The difficulty of "length 9 width": Is a square the unit of measurement? *Journal of Mathematical Behavior*, *25(*2), 105–115.

Kloosterman, P., Rutledge, Z., & Kenney, P. A. (2009). Exploring Results of the NAEP: 1980s to the Present.

Mathematics Teaching in the Middle School, *14*(6), 357.
Long, C. T., & DeTemple, D. W. (2012). *Mathematical reasoning for elementary teachers* (6th ed.). Boston,

MA: Pearson/Addison Wesley.

Machi, L. A., & McEvoy, B. T. (2016). *The literature review: Six steps to success*. Corwin Press.

McCrory, Raven. (2006). Mathematicians and mathematics textbooks for prospective elementary teachers.

*Notices of the AMS, 51(*1), 20-29.

Musser, G. L., Burger, W. F., & Peterson, B. E. (2011). *Mathematics for elementary school teachers: A contemporary approach* (9th ed.). New York: John Wiley & Sons, Inc.

National Center for Education Statistics [NCES]. (2017). NAEP data explorer: Online questions tool. Retrieved from: http://nces.ed.gov/nationsreportcard/itmrlsx

DOI: 10.20429/stem.2019.030102

Parker, T. H., & Baldridge, S. J. (2008). *Elementary geometry for teachers* (Volume 2).

Okemos, MI: SeftonAsh Publishing.

Sonnabend, T. (2010). *Mathematics for Elementary Teachers: An Interactive Approach for*

Grades K-8 (4th ed.). Belmont, CA: Brooks/Cole-Cengage.

Sowder, J., Sowder, L., & Nickerson, S. (2010). *Reconceptualizing Mathematics for Elementary School*

Teachers: Instructors Edition. New York: W.H. Freeman and Company.

Van de Walle, John A., Karp, Karen S. and Bay-Williams, Jennifer M. (2013). Elementary and Middle School

Mathematics: Teaching Developmentally (8th ed.). New Jersey: Pearson Education, Inc.

Vinner, S. (1991). The role of definitions in learning and teaching mathematics. *Advanced*

Mathematical Thinking, 11(II), 65-81

Woodward, E. & Byrd, F. (1983). Area: Included topic, neglected concept. *School Science and Mathematics,*

*83(*4), 343-347.

Wu, H. H. (2011). Understanding Numbers in Elementary School Mathematics. AMS.