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N-dimensional Fractional Fourier Transform and its Eigenvalues and Eigenfunctions

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Abstract

In this paper, we have established the N- dimentional fractional Fourier transform and its mathematical expression in a easier manner and discus the eigenvalues and eigenfunctions of N-dimensional fractional Fourier transform.

Keywords: Fourier transform; Fractional Fourier Transform; Eigenvalues;Eigenfunctions .

1. Introduction

In 1929 Norbert Wiener introduce the idea of the fractional power of the Fourier operator[1], later in 1in 1937 and 1939 Condon and Kober extended this idea of fractional order [2,3] but it is established in more precise manner by Victor Namias in 1980 and introduced the fractional Fourier transform as a generalized form of classical Fourier transform and successfully established the mathematical properties of this very important transform, he also applied the FRFT to solve the Schrodinger wave equations [4]. Later on McBride and Kerr extended this 1idea of fractional powers of the Fourier operator in terms of mathematical formulation [5] and in 1994 Almaida used this transform in time frequency plane [6]. FRFT has been used quantum mechanics , optics and signal processing. Since 1990 the mathematician, physicist and scientist highly attracted towards this famous transforms a series of boom publication was published and it is still going on. We are in this article focus with the mathematical formulation of N-dimensional fractional Fourier transform and investigating their eigenvalues eigenfunctions in a similar manner as for classical Fourier transform.

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1.1. The N-dimensional Fourier transform (FT)

In this section we recall definition and some of the properties and eigenvalues and eigenfunction of the classical continuous Fourier transform so that it will motivate to dicuss the eigen values for N-dimensional fractional Fourier transform. The Fourier and its inverse Fourier transform is defined by the following pair of equations

$$F(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iut}\, dt \qquad (1)$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(u) e^{iut}\, dt \qquad (2)$$

The one of the main application of Fourier transform (FT) find in designing of filters real filters, reconstruction of signal from digital to analog conversion, sinusoidal modulation and pulse amplitude modulation schemes in facial recognition the predefined eigenfaces are used to construct the images from eigenvalues [7]. One of the important role of eigenfunctions as a role in linear time invariant systems where eigenfunctions represent as arbitrary signals .The eigenvalues and eigenfunctions of the ordinary Fourier transform are well known and discussed in [8]. There are Hermite-Gaussian functions $\Psi_n(u)$commonly known as the eigensolutions of the harmonic oscillator in quantum mechanics or the modes of propagation of quadratic graded-index media in optics. The eigenvalues may be expressed as $exp(-in\pi/2)$ and the eigen value equation for the ordinary Fourier transform may be written as

$$\mathcal{F}\Psi_n(u) = e^{-in\pi/2}\Psi_n(u) \qquad (3)$$

A detailed discussions is found in [9] on the analysis of eigenvalues eigenfunctions of N- dimensional Fourier transform and are given by $\left(\sqrt{2\pi}\right)^n, -\left(\sqrt{2\pi}\right)^n, i\left(\sqrt{2\pi}\right)^n, -i\left(\sqrt{2\pi}\right)^n$ There are number of applications of eigenvalues and eigenfunctions are found in various areas like of eigenfunctions is also find in image processing, voice recognition, medical imaging analysis, handwriting analysis, lip reading and sign language/hand gestures interpretation[9]. In case a signal is represented in linear combination of eigenfunctions then it will reduce the evalution of spectrum and also useful in developing . This representation is also useful in developing the new signal compression techniques.The eigenfunction which correspond the eigenvalues are also useful in study of signal spaces spanned and can revolutionize the transmission of data such as higher dimensional signals like images, video etc.

2. Results and Discussions

2.1. One dimensional Fractional Fourier transform

The one dimensional linear operator version of fractional Fourier transform is given as follows,

$$\mathcal{F}^1[f(t)] = \mathcal{F}^\alpha[f(t)] = \int_{-\infty}^{\infty} f(t)k^\alpha(t,u)dt \qquad (4)$$

The $k^\alpha(t,u)$ is the kernel of transform defined in [6]

$$k^\alpha(t,u) = \begin{cases} b(\alpha)\exp j\left(\frac{u^2}{2}+\frac{t^2}{2}\right)cot\alpha - jutcsc\alpha & if\alpha \neq n\pi \\ \delta(t-u) & if\alpha = 2n\pi \\ \delta(t+u) & if\alpha = (2n\pm 1)n\pi \end{cases} \qquad (5)$$

where $b(\alpha) = \sqrt{\frac{1-jcot\alpha}{2\pi}}$, $\alpha\epsilon\mathbb{R}$, $n \in Z$, and δ□is the Dirac delta function

The inverse-FRFT with respect to angle α is the FRFT with angle $-\alpha$ given by from eqn (4) is

$$f(t) = \int_{-\infty}^{\infty} f(t)k^{-\alpha}(t,u)dt \qquad (6)$$

Since FRFT is the generalized form of Fourier transform. In classical Fourier transform the concept of rotating angle is $\pi/2$ where in FrFT the concept of rotating angleis generalized to over an arbitrary angle

$\alpha= a\pi/2$ with $a\epsilon\mathbb{R}$. Where a is the order Fractional Fourier transform This FRFT operator shall be denoted as F_a where $F_1 = F$ corresponds to the classical FT operator over rotational angle $\pi/2$, therefore it is similar to the Fourier Transform whenever the rotation of angle is $\pi/2$, for $a = 2$ then F_2 represent the axis as the reverese time axix known as the parity operator , i.e., the time axis rotated over an angle□π, for $a = 3$ the rotation of the axis over an angle $3\pi/2$ or $-\pi/2$ which represent the inversion Fourier transform ,i.e., $F_3 = F^{-1}$. It will now be clear that by another rotation of angle $\pi/2$,i.e., $\alpha = 2\pi$ and $a = 4$, which brings us back to the original time axis, hence $F_4 = F$ [6].

2.2. Two dimensional Fractional Fourier transform

TIn two-dimensional (2D)-fractional Fourier transform that is extension of equation (4)

with two variable and with two angles it is denoted as

$$\mathcal{F}^2\{f(t_1,t_2)\} = \mathcal{F}^{\alpha_1,\alpha_2}\{f(t_1,t_2)\}(u_1,u_2) = \mathcal{F}^{\alpha_1,\alpha_2}(u_1,u_2)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t_1,t_2)K^{\alpha_1,\alpha_2}(t_1,t_2;u_1,u_2)\ dt_1dt_2 \qquad (5)$$

where $K^{\alpha_1,\alpha_2}(t_1,t_2;u_1,u_2) = K^{\alpha_1}(t_1,u_1)K^{\alpha_2}(t_2,u_2)$ and defined

$$K^{\alpha_1,\alpha_2}(x_1,x_2;u_1,u_2) =$$
$$b(\alpha)\exp\left\{j\left(\frac{{u_1}^2}{2}+\frac{{t_1}^2}{2}\right)cot\alpha_1 + j\left(\frac{{u_2}^2}{2}+\frac{{t_2}^2}{2}\right)cot\alpha_2 - ju_1t_1csc\alpha_1 - ju_2t_2csc\alpha_2\right\} \qquad (6)$$

Since two dimensional fractional Fourier transform is the generalized form of two dimensional (2-D) classical Fourier transform if two angles of rotation replaced by $\alpha_1 = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ and $\alpha_2 = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ it will reduce the(2-D)FT with same rotational axis properties . If one of these angle is zero, then (2-D) FRFT reduced to (1-D) FRFT.

2.3. N- dimensional Fractional Fourier transform

In a similar manner equation (6) can be extended to 3-D and so on up to n-dimensional fractional Fourier transform. Let us consider a function with n variables and with n rotational angles the FRFTcan be denoted and defined defined as follows

$$\mathcal{F}^N\{f(t_1, t_2, \dots t_N)\} = \mathcal{F}^{\alpha_1, \alpha_2, \dots \alpha_N}\{f(t_1, t_2, \dots t_N)\}(u_1, u_2, \dots, u_N) = \mathcal{F}^{\alpha_1, \alpha_2, \dots \alpha_N}(u_1, u_2, \dots, u_N)$$
$$= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(t_1, t_2) K^{\alpha_1, \alpha_2}(t_1, t_2; u_1, u_2)\ dt_1 \dots dt_N \qquad (5)$$

Where

$$K^{\alpha_1, \alpha_2, \dots, \alpha_N}(t_1, t_2, \dots t_N; u_1, u_2, \dots, u_N)$$
$$= b(\alpha) \exp\left\{ j\left(\frac{{u_1}^2}{2} + \frac{{t_1}^2}{2}\right) cot\alpha_1 + j\left(\frac{{u_2}^2}{2} + \frac{{t_2}^2}{2}\right) cot\alpha_2 + \dots + j\left(\frac{{u_N}^2}{2} + \frac{{t_N}^2}{2}\right) cot\alpha_N \right.$$
$$\left. - ju_1 t_1 csc\alpha_1 - ju_2 t_2 csc\alpha_2 - \dots - ju_N t_N \right\} \qquad (6)$$

The N- dimensional fractional Fourier transform is the generalized form of N- dimensional classical Fourier transform if N-angles of rotation replaced by $\alpha_1 = \frac{\pi}{2}$, $\alpha_2 = \frac{\pi}{2}, \dots, \alpha_N = \frac{\pi}{2}$ furthe if each rotational angle is rotated over an angle $\frac{\pi}{2}$ it will posses the same roatational properties of N-D Fourier transform. If one of these angle is zero, then N-D FRFT reduced to (n-1) dimensional FRFT.

2.4. Eigenvalues and Eigenfunctions

The eigenfunctions of the fractional Fourier transform are the Hermite functions which is same as for classical N-dimensional Fourier transform also hold eqn (3) and the corresponding eigenvalues are $b(\alpha)e^{in\alpha}$ If we put $\alpha = 0$ leads to identity operator and we take $\alpha = \pi$ the result leads to the parity operator. If we define the order *n* of the fractional Fourier transform by $n = \alpha/(\pi/2)$, we see that the regular Fourier transform is of order 1. Order 0 is the identity operation and order 2 is the parity transformation. Negative orders correspond to inverse transforms. The whole range of transforms is covered by the orders $-2 \leq \alpha \leq 2$. Transforms with non-integral values of *n* may be called Fourier transforms of fractional order. Thus we can define the eigen values for the N-D fractional Fourier transform as $\left(b(\alpha)e^{in\beta}\right)^a$, where $a = 1,2,3, \dots . N$ If we consider $a = 1$, then it will be eigenvalues of I-D fractional Fourier transform where $\beta = \alpha = \alpha_1$ For $a = 2$ then it will eigenvalues of 2-D fractional Fourier transform ain this case $\beta = \frac{\alpha_1 + \alpha_1}{2}$

Similarly if $a = N$ then it will eigenvalues for N-D fractional Fourier transform and $\beta = \frac{\alpha_1+\alpha_1+\cdots+\alpha_N}{N}$

For if $\alpha = \alpha_1 = \alpha_1 = \cdots = \alpha_N = \frac{\pi}{2}$ then it will easily obtain the same eigenvalues as for N-dimensional Fourier transform $\left(\sqrt{2\pi}\right)^N, -\left(\sqrt{2\pi}\right)^N, j\left(\sqrt{2\pi}\right)^N, -j\left(\sqrt{2\pi}\right)^N$

3. Conclusion

We have eatablished the mathematical formulation of N-dimesional Fractional Fourier transform in very easier manner and discuss their eigenvalues and eigen functions in a simila manner as for we have classical Fourier transform it will help to apply these concepet in more meaning full manner in various applications specially in case of face deduction from images.

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