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Adaptive Output Regulation via Nonlinear Luenberger Observers ⋆

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Abstract: In Marconi et al. 2007, it was shown that a solution to the output regulation problem for minimum-phase normal forms always exists. That approach, however, yields only an existence result, and no general analytic procedure is known to actually choose the regulator's degrees of freedom, even for simple problems. In this paper we propose an adaptive regulator that, leveraging the aforementioned existence result, self-tunes online according to an optimization policy. To this aim, the regulator may employ every system identification scheme that fulfills some given strong stability properties, and the asymptotic regulation error is proved to be directly related to the prediction capabilities of the identifier.

Keywords: Output Regulation;Adaptive Output Regulation;Identification for Control

1. INTRODUCTION

Output regulation refers to the class of control problems in which some outputs of the controlled system are required to follow some desired reference behaviors, despite the presence of external disturbances and model uncertainties. In the seminal works (Francis and Wonham, 1975, 1976), the output regulation problem for linear systems has been cast in a very elegant framework, providing a unifying treatise of tracking and disturbance rejection problems under the assumption that references and disturbances are generated by a linear external process, called the *exosystem*. The authors have shown that any regulator solving the problem in a "robust" way *necessarily* embeds, in the control loop, an *internal model* of the exosystem. Constructive designs for linear systems, based on the concept of internal model, have been given later in (Davison, 1976), thus closing de facto the problem of linear output regulation.

Output regulation for nonlinear systems is, instead, a much wider and more challenging problem, nowadays still open. The research on nonlinear output regulation has started around the 90s, in the seminal works (Isidori and Byrnes, 1990; Huang and Rugh, 1990; Huang, 1995; Byrnes et al., 1997). These first results concerned "local" designs, and they were still biased by a linear perspective. Nevertheless, they made clear how, in a nonlinear setting, the role of the *zero dynamics* of the controlled plant and of the exosystem blend together, and how an internal model of the sole exosystem is far from being sufficient for the design of a regulator. A purely nonlinear theory of output regulation has appeared only in the 2000s, starting with the works (Byrnes and Isidori, 2003; Byrnes et al., 2003; Byrnes and Isidori, 2004), and related regulator designs for single-input-single-output normal forms have been proposed, for instance, in (Byrnes and Isidori, 2004; Huang and Chen, 2004; Chen and Huang, 2005; Marconi et al., 2007). In (Marconi et al., 2007), in particular, it was shown that for the class of minimum-phase single-input-single-output normal forms a solution of the output regulation problem always exists. This result, however, is not *constructive*, in the sense that, although the existence of a regulator is guaranteed, no general procedure is given to choose all the degrees of freedom characterizing it. By leveraging this existence result, in (Marconi and Praly, 2008) some methods have been proposed to construct "approximate" regulators. Nevertheless, the construction of the regulator remains mostly impractical.

In this paper we propose an adaptive regulator that complements the result of (Marconi et al., 2007) by adding an adaptation unit in the control loop, whose role is to tune at run-time the degrees of freedom of the regulator that cannot be computed analytically in advance. Adaptation is cast as a user-defined *system identification* problem, defined on the closed-loop measurable signals. Sufficient conditions are given under which a desired identification algorithm can be used, and the main result relates the prediction capabilities of the identifier to the asymptotic regulation performance. The proposed approach shares some similarities with the "high-gain" approach of (Forte et al., 2017) and the linear design of (Bin et al., 2019), insofar as the identifier is characterized by the same stability properties. However, unlike the latter, the proposed design is purely nonlinear, and unlike the former, we do not rely on a high-gain internal models, and a regression is guaranteed to exist in steady-state between the two input signals of the identifier without any assumption on the steady-state error-zeroing input.

The paper is organized as follows. In Section 2 we describe our nonlinear framework, we recall the main result of (Marconi et al., 2007), and we highlight the contribution of the paper. The adaptive regulator is constructed in Section 3, and the main result of the paper is given in Section 4.

Notation: $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ the set of naturals and $\mathbb{R}_+ := [0, \infty)$. If S is a set, $\mathbb{H}(S)$ denotes the set of functions $S \to \mathbb{R}_+$. Norms are denoted by $|\cdot|$. If $S \subset$

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$\mathbb{R}^n$ ($n \in \mathbb{N}$) is a closed set, $|x|_S := \inf_{s\in S} |x - s|$ denotes the distance of $x \in \mathbb{R}^n$ to S. For a signal $x : \mathbb{R} \to \mathbb{R}^n$, we let $|x|_{[0,t)} := \sup_{s\in[0,t)} |x(s)|$ and $|x|_\infty = |x|_{[0,\infty)}$. We denote by $\mathcal{C}^1$ set of continuously differentiable functions. A continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is of class-K ($f \in \mathcal{K}$) if it is strictly increasing and $f(0) = 0$. It is of class-K_∞ ($f \in \mathcal{K}_\infty$) if $f \in \mathcal{K}$ and $f(s) \to \infty$ as $s \to \infty$. A continuous function $\beta : \mathbb{R}^2_+ \to \mathbb{R}_+$ is of class-KL ($\beta \in \mathcal{KL}$) if $\beta(\cdot, t) \in \mathcal{K}$ for each $t \in \mathbb{R}_+$ and $\beta(s, \cdot)$ is strictly decreasing to zero for each $s \in \mathbb{R}_+$. $\mathrm{Id} \in \mathcal{K}$ denotes the map $\mathrm{Id}(s) = s$. With $h : \mathbb{R}^n \to \mathbb{R}$ a $\mathcal{C}^1$ function in the arguments $x_1, \dots, x_n$, and $f : \mathbb{R}^n \to \mathbb{R}$, for each $i \in \{1, \dots, n\}$ we denote by $L^{(x_i)}_f h$ the map $x \mapsto L^{(x_i)}_{f(x)} h(x) := \partial h/\partial x_i(x) f(x)$. In the text "ISS" stands for Input-to-State Stability (Sontag, 1989).

2. THE FRAMEWORK

2.1 Problem Statement

We consider systems of the form

$$\begin{aligned} \dot z &= f(w, z, y) \\ \dot y &= q(w, z, y) + u, \end{aligned} \tag{1}$$

with state (z, y) taking values in $\mathbb{R}^{n_z} \times \mathbb{R}$, control input $u \in \mathbb{R}$, measured output $y \in \mathbb{R}$, and with $w \in \mathbb{R}^{n_w}$ an exogenous input that we suppose to belong to the set of solutions of an *exosystem* of the form

$$\dot w = s(w), \tag{2}$$

originating in a compact invariant subset W of $\mathbb{R}^{n_w}$. We further assume that $f : \mathbb{R}^{n_w} \times \mathbb{R}^{n_z} \times \mathbb{R} \to \mathbb{R}^{n_z}$ is locally Lipschitz and $q : \mathbb{R}^{n_w} \times \mathbb{R}^{n_z} \times \mathbb{R} \to \mathbb{R}$ is $\mathcal{C}^1$. In this paper we consider the problem of *approximate output regulation* for systems of the form (1), which reads as follows: given a compact set $\mathrm{Z}_0 \times \mathrm{Y}_0 \subset \mathbb{R}^{n_z} \times \mathbb{R}$ and a *performance measure* $\mu^\star : \mathbb{H}(\mathrm{W}) \to \mathbb{R}_+$, find a regulator of the form

$$\begin{cases} \dot x_c = f_c(x_c, y) \\ u = h_c(x_c, y), \end{cases} \tag{3}$$

with state $x_c \in \mathbb{R}^{n_c}$, and a set $\mathrm{X}_c \subset \mathbb{R}^{n_c}$, such that the solutions $\mathbf{x} := (w, z, y, x_c)$ of the closed-loop system (1),(2),(3) originating in $\mathrm{W} \times \mathrm{Z}_0 \times \mathrm{Y}_0 \times \mathrm{X}_c$ are complete, uniformly eventually equibounded [1] and satisfy

$$\limsup_{t\to\infty} |y(t)| \le \mu^\star(w).$$

For a particular solution w of (2), the positive scalar $\mu^\star(w)$ represents the desired asymptotic bound on the output y, thus capturing regulation objectives milder than the usual *asymptotic output stabilization*, obtained whenever $\mu^\star = 0$.

We consider the problem at hand under the following minimum-phase assumption.

A1) *There exists a* $\mathcal{C}^1$ *map* $\pi : \mathrm{W} \to \mathbb{R}^{n_z}$ *satisfying*

$$L^{(w)}_{s(w)} \pi(w) = f(w, \pi(w), 0)$$

in an open set including $\mathrm{W} \times \mathbb{R}^{n_z}$*, such that the system*

$$\begin{aligned} \dot w &= s(w) \\ \dot z &= f(w, z, y) \end{aligned}$$

is ISS relative to the compact set

$$\mathcal{A} = \{(w, z) \in \mathrm{W} \times \mathbb{R}^{n_z} \; : \; z = \pi(w)\}$$

and with respect to the input y*.*

We observe that, by using the same arguments of (Marconi et al., 2007), A1 could be weakened to a local asymptotic stability requirement of the set $\mathcal{A}$ for the *zero dynamics*

$$\dot w = s(w), \qquad \dot z = f(w, z, 0), \tag{4}$$

as long as the domain of attraction includes $\mathrm{W} \times \mathrm{Z}_0 \times \mathrm{Y}_0$. This, however, comes at the price of a more involved technical treatise leading to minor conceptual contribution. Assumption A1 is customary in the literature of output regulation (see e.g. (Isidori, 2017; Pavlov et al., 2006)). Necessary and sufficient conditions for the existence of a single-valued steady-state map π can be found in (Pavlov et al., 2006). By definition, continuity of π holds whenever $\mathcal{A}$ is closed, while its differentiability has to be assumed.

2.2 The Marconi-Praly-Isidori Regulator

Under A1, it is proved in (Marconi et al., 2007) that the problem of asymptotic output regulation for systems of the form (1) can be always solved by means of a controller of the form

$$\begin{aligned} \dot \eta &= F\eta + Gu \\ u &= \gamma(\eta) + \kappa(y) \end{aligned} \tag{5}$$

with state η taking values in $\mathbb{R}^{n_\eta}$, with $n_\eta = 2(n_w + n_z + 1)$, (F, G) a controllable pair with F a Hurwitz matrix, and with $\gamma : \mathbb{R}^{n_\eta} \to \mathbb{R}$ and $\kappa : \mathbb{R} \to \mathbb{R}$ suitably defined continuous functions. More precisely, in (Marconi et al., 2007) it was shown that:

1. the pair (F, G) can be chosen as a real realization of any complex pair (F_c, G_c) of dimension $n_\eta/2$, with G_c a vector with non zero entries, and $F_c = \mathrm{diag}(\lambda_1, \dots, \lambda_{n_\eta/2})$ such that the eigenvalues λ_i have sufficiently negative real part and $(\lambda_1, \dots, \lambda_{n_\eta/2})$ is outside a set of zero-Lebesgue measure.
2. the map κ must verify $\kappa(e)e < 0$ for all non zero e and, if $\mathcal{A}$ is also locally exponentially stable for (4), then it can be taken linear.

2.3 Contribution of the Paper

Although the existence of the map γ is always guaranteed, its construction is challenging and, apart from linear systems, there exists no general procedure to construct it. Some approximating algorithms have been proposed in (Marconi and Praly, 2008), yet their implementation remains tedious. In this paper we propose a regulator that employs online adaptation to find the map γ at run time, by requiring only a "qualitative" a priori knowledge on it, necessary to set up a meaningful identification problem. Leveraging the result of (Marconi et al., 2007), the proposed regulator is obtained as an extension of (5), which is equipped with an adaptive unit that produces and updates an estimate of the "right" function γ to employ. The adaptive unit, called the *identifier*, is a system that solves a user-defined *system identification* problem (Ljung, 1999), cast on the closed-loop signals. Instead of proposing a particular design of the identifier, we give sufficient stability conditions characterizing a class of algorithms that can be used, thus leaving to the designer a further

[1] That is, there exists a compact set $\mathrm{K} \subset \mathbb{R}^{n_z} \times \mathbb{R} \times \mathbb{R}^{n_c}$ and a $\tau \ge 0$ such that every solution $\mathbf{x} := (z, y, x_c)$ of (1)-(3) originating in $\mathrm{Z} \times \mathrm{Y} \times \mathrm{X}_c$ satisfies $\mathbf{x}(t) \in \mathrm{K}$ for all $t \ge \tau$.

degree of freedom. The main result of the paper consists in relating the asymptotic bound on the regulated variable y to the prediction capabilities of the identifier chosen, thus leading to asymptotic regulation whenever the "right" γ is reproducible by the identifier.

3. THE REGULATOR STRUCTURE

We consider a controller of the form [2]

$$\begin{aligned} \dot\eta &= F\eta + Gu \\ \dot\xi &= \varphi(\xi,\eta,u) \\ \dot\zeta &= \ell(\zeta,y) \\ \dot u &= \kappa(y,\zeta) + \psi(\xi,\eta,u) \end{aligned} \tag{6}$$

with state $(\eta,\xi,\zeta,u) \in \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\zeta} \times \mathbb{R}$, input y, and output u. We refer to the subsystems η, ξ, ζ and u respectively as the *internal model unit*, the *identifier*, the *derivative observer* and the *stabilizer*. These subsystems are constructed in the next subsections.

3.1 The Internal Model Unit

The dimension n_η of η and the pair (F,G) are chosen according to (Marconi et al., 2007), by following the guidelines reported in Section 2.2. This choice of (n_η,F,G) and the arguments of (Marconi et al., 2007) yield the following result.

Lemma 1. Suppose that A1 holds and pick $n_\eta = 2(n_w + n_z + 1)$. Then there exist a Hurwitz matrix $F \in \mathbb{R}^{n_\eta \times n_\eta}$, a matrix $G \in \mathbb{R}^{n_\eta \times 1}$, and continuous maps $\tau : \mathbb{R}^{n_w} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_\eta}$ and $\gamma : \mathbb{R}^{n_\eta} \to \mathbb{R}$ such that

$$\gamma \circ \tau(w,z) = -q(w,z,0) \qquad \forall (w,z) \in \mathcal{A} \tag{7}$$

and the system

$$\begin{aligned} \dot w &= s(w) \\ \dot z &= f(w,z,y) \\ \dot\eta &= F\eta - Gq(w,z,y) + \delta_0. \end{aligned}$$

is ISS relative to the set

$$\mathcal{G} := \Big\{(w,z,\eta) \in \mathcal{A} \times \mathbb{R}^{n_\eta} \,:\, \eta = \tau(w,\pi(w))\Big\},$$

and with respect to the input (y,δ_0).

Lemma 1, and in particular the equality (7), claims that, when y and δ_0 are small, the quantity $\gamma(\eta)$ gives a proxy for the ideal feedforward action

$$u^\star(w) := -q(w,\pi(w),0)$$

that holds the regulated variable y to zero in steady state (hence its use in (5)). We stress that the map γ is only used for the analysis and it is not used by the controller. The identifier subsystem described hereafter has indeed the aim of identifying it online.

3.2 The Identifier

The identifier subsystem is a system built to estimate online the unknown map γ given by Lemma 1. The estimation of γ here is cast as a system identification problem defined on generic inputs, and the data (n_ξ,φ) defining the identifier are constructed according to a "design requirement" detailed hereafter, representing sufficient conditions for its correct successive embedding in the closed-loop system. In this respect, we stress that we do not rely on a particular choice of (n_ξ,φ), and every identification algorithm that fulfills the requirement can be used.

We consider a class $\mathcal{I}$ of functions of the form $(\alpha_{\rm in},\alpha_{\rm out})$, with $\alpha_{\rm in} : \mathbb{R}_+ \to \mathbb{R}^{n_\eta}$, $\alpha_{\rm out} : \mathbb{R}_+ \to \mathbb{R}$, and compact sets $\mathrm{A}_{\rm in} \subset \mathbb{R}^{n_\eta}$, $\mathrm{A}_{\rm out} \subset \mathbb{R}$ such that for each $\alpha^\star = (\alpha^\star_{\rm in},\alpha^\star_{\rm out}) \in \mathcal{I}$, we have $\alpha^\star(\mathbb{R}_+) \subseteq \mathrm{A}_{\rm in} \times \mathrm{A}_{\rm out}$. Based on the available knowledge on $\mathcal{I}$, we fix $n_\xi, n_\theta \in \mathbb{N}$, a function $\varphi : \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta} \times \mathbb{R} \to \mathbb{R}^{n_\xi}$, a bounded function $h : \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_\theta}$, a C^1 map $\hat\gamma : \mathbb{R}^{n_\theta} \times \mathbb{R}^{n_\eta} \to \mathbb{R}$, and we consider the following *virtual system*

$$\begin{aligned} \dot\xi &= \varphi(\xi,\alpha_{\rm in},\alpha_{\rm out}) \\ \theta &= h(\xi). \end{aligned} \tag{8}$$

The design of φ is approached in such a way that the virtual system (8) asymptotically converges to a trajectory $\xi^\star : \mathbb{R}_+ \to \mathbb{R}^{n_\xi}$ whose output $\theta^\star := h(\xi^\star)$ is such that the inferred *prediction model*

$$\alpha_{\rm out} = \hat\gamma(\theta^\star,\alpha_{\rm in}) \tag{9}$$

fits "at best" the measurements available of the inputs $\alpha_{\rm in}$ and $\alpha_{\rm out}$. The meaning of "at best" is made formal throughout the definition of a cost functional associated to the elements in $\mathcal{I}$, assigning to each value of θ and each $t \in \mathbb{R}_+$ a comparable value. More precisely, to each $\alpha := (\alpha_{\rm in},\alpha_{\rm out}) \in \mathcal{I}$, we associate a function $\mathcal{J}_\alpha : \mathbb{R}_+ \to \mathbb{H}(\mathbb{R}^{n_\theta})$ of the form

$$\mathcal{J}_\alpha(t)(\theta) := \int_0^t \mathrm{c}\big(\varepsilon(\alpha(s),\theta),\, t,\, s\big)ds + \varrho(\theta), \tag{10}$$

in which

$$\varepsilon(\alpha(s),\theta) := \alpha_{\rm out}(s) - \hat\gamma(\theta,\alpha_{\rm in}(s))$$

denotes the *prediction error* of the model $\hat\gamma(\theta,\cdot)$ computed at time s and corresponding to a given choice of $\theta \in \mathbb{R}^{n_\theta}$, $\mathrm{c} : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ assigns to each prediction error and each time instant s a "cost" to be weighted in the integral, and $\varrho : \mathbb{R}^{n_\theta} \to \mathbb{R}_+$ is a *regularization* term. To the cost functional (10) we associate the set-valued map $\vartheta^\circ_\alpha : \mathbb{R}_+ \rightrightarrows \mathbb{R}^{n_\theta}$ defined as

$$\vartheta^\circ_\alpha(t) := \operatorname*{arg\,min}_{\theta \in \mathbb{R}^{n_\theta}} \mathcal{J}_\alpha(t)(\theta).$$

The functions φ and h in (8) are then designed to satisfy the conditions stated in the following requirement, consisting of a robust stability property relative to an "optimal steady state" $\xi^\star$ whose output $\theta^\star := h(\xi^\star)$ is a pointwise solution to the minimization problem associated with (10).

Requirement 1. The pair (φ,h) is said to fulfill the *identifier requirement* relative to $\mathcal{I}$ if h is bounded and there exist $\beta_\xi \in \mathcal{KL}$, $\rho_\xi \in \mathcal{K}$, and a compact set $\Xi \subset \mathbb{R}^{n_\xi}$, such that for each $\alpha = (\alpha^\star_{\rm in},\alpha^\star_{\rm out}) \in \mathcal{I}$, there exists a function $\xi^\star : \mathbb{R}_+ \to \mathbb{R}^{n_\xi}$ satisfying $\xi^\star(t) \in \Xi$ for all $t \geq 0$ and such that the following properties hold:

(1) **Optimality:** the output $\theta^\star := h(\xi^\star)$ satisfies
$$\theta^\star(t) \in \vartheta^\circ_\alpha(t)$$
for each $t \in \mathbb{R}_+$.

(2) **Stability:** for each $d = (d_{\rm in},d_{\rm out}) : \mathbb{R}_+ \to \mathbb{R}^{n_\eta} \times \mathbb{R}$ the system
$$\dot\xi = \varphi(\xi,\alpha^\star_{\rm in} + d_{\rm in},\alpha^\star_{\rm out} + d_{\rm out})$$
satisfies
$$|\xi(t) - \xi^\star(t)| \leq \max\big\{\beta_\xi(|\xi(0)-\xi^\star(0)|,t),\, \rho_\xi(|d|_{[0,t)})\big\}$$
each $t \in \mathbb{R}_+$.

[2] We notice that (6) has the form (3) with $x_c := \mathrm{col}(\eta,\xi,\zeta,u)$, $f_c(x_c,y) := \mathrm{col}(F\eta + Gu,\, \varphi(\xi,\eta,u),\, \ell(\zeta,y),\, \kappa(y,\zeta) + \psi(\xi,\eta,u))$ and $h_c(x_c) := u$.

(3) **Regularity:** the map
$$\lambda(\xi,\alpha_{\rm in},\alpha_{\rm out}) := \lim_{\epsilon\to 0}\frac{h(\xi+\epsilon\varphi(\xi,\alpha_{\rm in},\alpha_{\rm out}))-h(\xi)}{\epsilon}$$
is well-defined and continuous on $\Xi\times A_{\rm in}\times A_{\rm out}$.

Examples of identifiers that fulfill such conditions include least-square algorithms and can be found for instance in (Forte et al., 2017; Bin et al., 2019; Bin, 2018). The identifier is interconnected with the rest of the regulator by setting
$$\alpha_{\rm in}=\eta, \qquad \alpha_{\rm out}=u.$$
This choice is motivated by the fact that, close to the attractor $\mathcal{G}$ introduced in Lemma 1, η is a good "proxy variable" for the quantity $\tau(w,\pi(w))$ while, as detailed in the next section, u provides a proxy for the quantity $u^\star(w) = -q(w,\pi(w),0)$. Thus, in view of the relation (7), if η and u are close enough to the ideal quantities $\tau(w,\pi(w))$ and $u^\star(w)$, the identification problem solved by the identifier provides an estimate of the ideal unknown function γ close to the optimal one. In view of this discussion, we make the following assumption

A2) *The pair (φ,h) fulfills the identifier requirement relative to a class of functions $\mathcal{I}$ satisfying*
$$\begin{aligned}\mathcal{I}\supset\big\{(\alpha^\star_{\rm in},\alpha^\star_{\rm out}) :\mathbb{R}\to\mathbb{R}^{n_\eta}\times\mathbb{R} \,:\, \\ \alpha^\star_{\rm in}=\tau(w,\pi(w)),\ \alpha^\star_{\rm out}=-q(w,\pi(w),0)),\\ w \text{ solves (2) with } w(0)\in \mathrm{W}\big\}\end{aligned}$$
and with $A_{\rm in}\subset\mathbb{R}^{n_\eta}$ and $A_{\rm out}\subset\mathbb{R}$ compact supersets of $\tau(\mathrm{W},\pi(\mathrm{W}))$ and $-q(\mathrm{W},\pi(\mathrm{W}),0)$.

To make the dependency on w explicit, we associate with each solution w of (2) a signal $\xi^\star_w$ defined as the optimal steady state of ξ introduced in the identifier requirement corresponding to the inputs
$$\alpha^\star_{\rm in} := \tau(w,\pi(w)), \quad \alpha^\star_{\rm out} := -q(w,\pi(w),0), \tag{11}$$
and we let $\theta^\star_w(t) := h(\xi^\star_w(t))$ be the corresponding optimal parameter. According to (7), the quantities (11) satisfy $\alpha^\star_{\rm out}=\gamma(\alpha^\star_{\rm in})$, so that we can associate with each solution w of (2) the signal
$$\varepsilon^\star_w := \gamma\big(\tau(w,\pi(w))\big) - \hat\gamma\big(\theta^\star_w,\tau(w,\pi(w))\big), \tag{12}$$
which represents the *optimal prediction error*, i.e. the prediction error attained by the optimal model, along the solution w.

We remark that, while the knowledge of the maps (11) is not assumed in this paper, the choice of the identifier, and in particular of the structure and parametrization of the map $\hat\gamma$, is necessarily guided by the a priori qualitative and quantitative information that the designer has on the ideal steady-state signals $(\alpha^\star_{\rm in}(t),\alpha^\star_{\rm out}(t))$. We underline, however, that such information is only needed for the purpose of setting up the identification problem. In turn, the choice of the structure of $\hat\gamma$, and thus of the corresponding identification algorithm, depends on the amount and quality of the available information on (11), and it may range from a very specific set of functions, such as linear regressions, to *universal approximators*, such as wavelet bases or neural networks. We also stress that the inferred parametrization of $\hat\gamma$ does not hide any assumption on the structure of the map γ, and the fact that there may not exist any $\theta^\star\in\mathbb{R}^{n_\theta}$ such that $\gamma(\cdot)=\hat\gamma(\theta,\cdot)$ jut implies that the optimal prediction error (12) may not be zero at the steady state. In turn, the main result of the paper relates the asymptotic bound on the regulated variable y to the optimal prediction error (12) (i.e., to the best prediction performance of the identifier), and it thus results in an approximate regulation property, which can be strengthened to asymptotic regulation only if γ is reproducible by the identifier for some $\theta^\star$.

3.3 The Stabilizer

As q is $\mathcal{C}^1$, we can immerse (1) into the system
$$\begin{aligned}\dot z &= f(w,z,\chi_1)\\ \dot\chi_1 &= \chi_2\\ \dot\chi_2 &= q'(w,z,\chi)+\dot u,\end{aligned} \tag{13}$$
with $\chi_1=y$, $\chi_2=q(w,z,y)+u$, and where
$$q'(w,z,\chi) := \Big(L^{(w)}_{s(w)} + L^{(z)}_{f(w,z,\chi_1)} + L^{(\chi_1)}_{\chi_2}\Big)\, q(w,z,\chi_1).$$
According to Lemma 1, the internal model unit must be ideally fed by the input $u^\star(w) = -q(w,\pi(w),0)$, which is, however, not available for feedback. A natural proxy variable for $-q(w,\pi(w),0)$, is given by the quantity $-q(w,z,\chi_1)$, which, on the heels of (Freidovich and Khalil, 2008), can be expressed in terms of the new state variables and inputs of (13) as
$$-q(w,z,\chi_1) = u-\chi_2. \tag{14}$$
Clearly, as $\chi_2=\dot y$ is not measured, neither (14) is available for feedback. Nevertheless, we can obtain an estimate of χ_2 by means of a derivative observer, here provided by the subsystem ζ. We postpone the definition of ζ to Section 3.4, and here we provide the necessary background by designing the stabilizer assuming that χ_2 is available.

As a first step, with $\hat\gamma$ the prediction model of the identifier given in (9), and with λ the map introduced in the identifier requirement, we define the continuous map $\hat\gamma' : \Xi\times A_{\rm in}\times A_{\rm out}\to\mathbb{R}$ as
$$\begin{aligned}&\hat\gamma'(\xi,\alpha_{\rm in},\alpha_{\rm out})\\ &\qquad := \Big(L^{(h(\xi))}_{\lambda(\xi,\alpha_{\rm in},\alpha_{\rm out})} + L^{(\alpha_{\rm in})}_{F\alpha_{\rm in}+G\alpha_{\rm out}}\Big)\,\hat\gamma(h(\xi),\alpha_{\rm in}).\end{aligned}$$
We then let $\psi:\mathbb{R}^{n_\xi}\times\mathbb{R}^{n_\eta}\times\mathbb{R}\to\mathbb{R}$ be any bounded and uniformly continuous function that agrees with $\hat\gamma'$ on $\Xi\times A_{\rm in}\times A_{\rm out}$, and we consider the following virtual system
$$\begin{aligned}\dot w &= s(w),\\ \dot z &= f(w,z,\chi_1)\\ \dot\eta &= F\eta + G(-q(w,z,\chi_1)+\chi_2+\delta)\\ \dot\xi &= \varphi(\xi,\eta,-q(w,z,\chi_1)+\chi_2+\delta)\\ \dot\chi_1 &= \chi_2+\delta\\ \dot\chi_2 &= \Delta(w,z,\eta,\chi_1,\chi_2+\delta)+\kappa_0(\chi).\end{aligned} \tag{15}$$
with
$$\begin{aligned}\Delta(w,z,\eta,\chi) = q'(w,z,\chi) - q'(w,\pi(w),0)&\\ +\psi(\xi^\star_w,\eta,-q(w,z,\chi_1)+\chi_2)&\\ -\hat\gamma'(\xi^\star_w,\tau(w,\pi(w)),-q(w,\pi(w),0)),&\end{aligned} \tag{16}$$
with input δ, and with κ_0 a "nominal" stabilizing action to be fixed. For compactness, we define
$$\begin{aligned}\widetilde{\mathcal{G}} &:= \big\{(w,z,\eta,\xi)\in\mathcal{G}\times\mathbb{R}^2 \,:\, \xi=\xi^\star_w\big\},\\ \mathcal{B} &:= \big\{(w,z,\eta,\xi,\chi)\in\widetilde{\mathcal{G}}\times\mathbb{R}^2 \,:\, \chi=0\big\}.\end{aligned}$$
Then, the following result holds.

Lemma 2. Suppose that A1 and A2 hold. Then for each pair of compact sets $\mathrm{Z}_0 \subset \mathbb{R}^{n_z}$ and $\mathrm{Y}_0 \subset \mathbb{R}$, and for each $\bar{\delta} > 0$, there exist a $\mathcal{C}^1$ function κ_0, $\beta_0 \in \mathcal{KL}$, and $\rho_0 \in \mathcal{K}$, such that for every input $\delta : \mathbb{R} \to \mathbb{R}$ fulfilling $|\delta|_\infty \leq \bar{\delta}$, each solution of (15) originating in $\mathrm{W} \times \mathrm{Z}_0 \times \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_\xi} \times \mathrm{Y}_0 \times \mathbb{R}$ satisfies

$$|(w(t), z(t), \eta(t), \xi(t), \chi(t))|_{\mathcal{B}} \leq \max\Big\{\beta_0\big(|(w(0), z(0), \eta(0), \xi(0), \chi(0))|_{\mathcal{B}}, t\big), \rho_0(|\delta|_{[0,t)})\Big\}$$

for all $t \geq 0$.

The proof of Lemma 2 can be deduced from the "small-gain arguments" in the proofs of Theorem 2 and 3 of (Marconi et al., 2007), once noted that: i) A1 and Lemma 1 suffice to show that the subsystem (w, z, η) of (15) is ISS with respect to the set $\mathcal{G}$ and relative to the input (χ, δ); ii) from A2, with $d_{\text{in}} = \eta - \tau(w, \pi(w))$ and $d_{\text{out}} = -q(w, z, \chi_1) + \chi_2 + q(w, \pi(w), 0)$, we deduce that the cascade (w, z, η, ξ) is ISS with respect to the set $\widetilde{\mathcal{G}}$ and relative to the input (χ, δ); iii) by boundedness of ψ and continuity of q', given $\bar{\delta}$ and the sets W, Z_0 and Y_0, there exist compact sets $\mathrm{Z} \subset \mathbb{R}^{n_z}$ and $\mathrm{Y} \subset \mathbb{R}$, and a map $\underline{\kappa}_0$ such that for all κ_0 satisfying $|\kappa_0(y)| \geq |\underline{\kappa}_0(y)|$ and $\kappa_0(y)y < 0$ for all y, every solution of (15) initialized in $\mathrm{W} \times \mathrm{Z}_0 \times \mathbb{R}^{n_\eta} \times \mathrm{Y}_0 \times \mathbb{R}$ and corresponding to an input δ satisfying $|\delta|_\infty \leq \bar{\delta}$ is bounded, complete and such that $(z, y) \in \mathrm{Z} \times \mathrm{Y}$ at all times; iv) by continuity of q' and from the definition of ψ, there exists $\varpi \in \mathcal{K}$ such that $|\Delta_0(w, z, \eta, \xi, \chi)| \leq \varpi(\max\{|(w, z, \eta, \xi)|_{\widetilde{\mathcal{G}}}, |\chi|\})$ on $\mathrm{W} \times \mathrm{Z} \times \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_\xi} \times \mathrm{Y} \times \mathbb{R}$ and κ_0 can be chosen so that a small-gain condition holds in the overall interconnection between (w, z, η, ξ) and χ.

We further notice that, in view of (Marconi et al., 2007, Theorem 3), if Δ_0 is locally Lipschitz and if $\mathcal{A}$, defined in A1, is locally exponentially stable for the zero-dynamics (4), then κ_0 can be taken linear of the form $\kappa_0(\chi) = -k(\chi_2 + a\chi_1)$ with $a > 0$ and with $k > 0$ sufficiently large. The design of the stabilizer is then concluded in the following section by choosing κ in (6) on the basis of the functions κ_0 defined here and by substituting to χ_2 its estimate provided by the derivative observer.

3.4 The Derivative Observer

Since many design techniques already exist in the literature (see e.g. (Teel and Praly, 1995; Atassi and Khalil, 1999; Andrieu et al., 2008)), we do not detail here a particular choice of the derivative observer, represented in (6) by the subsystem ζ. Rather, we suppose that the designer has already available a design choice of the degrees of freedom (n_ζ, ℓ) such that, for some known map $\nu : \mathbb{R}^{n_\zeta} \to \mathbb{R}$, the unimplementable stabilizing control law claimed by Lemma 2 can be substituted by a control action which employs the estimate $\hat{\chi}_2 := \nu(\zeta)$ in place of the unmeasured derivative χ_2 of the output $y = \chi_1$. We thus rely on a separation principle for the stabilization problem (15) by assuming that the new control law makes the interconnection $(w, z, \eta, \xi, \chi, \zeta)$ ISS relative to the set

$$\mathcal{D} := \big\{(w, z, \eta, \xi, \chi, \zeta) \in \mathcal{B} \times \mathbb{R}^{n_\zeta} \,:\, \nu(\zeta) = 0\big\}$$

and with respect to the input δ, thus complementing the result of Lemma 2.

A3) *For each pair of compact subsets* $\mathrm{Z}_0 \subset \mathbb{R}^{n_z}$ *and* $\mathrm{Y}_0 \subset \mathbb{R}$*, and for each* $\bar{\delta} > 0$*, there exist a function* $\kappa : \mathbb{R}^2 \to \mathbb{R}$*,* $\beta \in \mathcal{KL}$*,* $\rho \in \mathcal{K}$ *and a non-empty subset* $\mathrm{T}_0 \subset \mathbb{R}^{n_\zeta}$ *such that, for every solution of the system*

$$\begin{aligned}
\dot{w} &= s(w), \\
\dot{z} &= f(w, z, \chi_1) \\
\dot{\eta} &= F\eta + G(-q(w, z, \chi_1) + \chi_2 + \delta) \\
\dot{\xi} &= \varphi(\xi, \eta, -q(w, z, \chi_1) + \chi_2 + \delta) \\
\dot{\chi}_1 &= \chi_2 + \delta \\
\dot{\chi}_2 &= \Delta(w, z, \eta, \chi_1, \chi_2 + \delta) + \kappa(\chi_1, \nu(\zeta)) \\
\dot{\zeta} &= \ell(\zeta, \chi_1)
\end{aligned} \tag{17}$$

originating in $\mathrm{W} \times \mathrm{Z}_0 \times \mathbb{R}^{n_\eta} \times \mathrm{Y}_0 \times \mathbb{R} \times \mathrm{T}_0$*, and corresponding to an input* $\delta : \mathbb{R} \to \mathbb{R}$ *fulfilling* $|\delta|_\infty \leq \bar{\delta}$*, the following bound holds*

$$|(w(t), z(t), \eta(t), \xi(t), \chi(t), \zeta(t))|_{\mathcal{D}} \leq \max\Big\{\beta\big(|(w(0), z(0), \eta(0), \chi(0), \zeta(0))|_{\mathcal{D}}, t\big), \rho(|\delta|_{[0,t)})\Big\}$$

for all $t \in \mathbb{R}_+$*.*

We remark that, in view of the separation principle of (Atassi and Khalil, 1999), if Δ is locally Lipschitz then (n_η, ℓ) and ν can be chosen to implement a "dirty" high-gain observer, i.e. such that

$$\dot{\zeta}_1 = \zeta_2 + L(\chi_1 - \zeta_1), \quad \dot{\zeta}_2 = L^2(\chi_1 - \zeta_1), \quad \nu(\zeta) = \zeta_2$$

with L sufficiently large, and κ can be taken as a saturated version of κ_0.

4. MAIN RESULT

We now consider the interconnection of the forced plant (1), (2) with the controller (6), which is a system with state $(w, z, y, \eta, \xi, \zeta, u) \in \mathcal{O} := \mathrm{W} \times \mathbb{R}^{n_z} \times \mathbb{R} \times \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\zeta} \times \mathbb{R}$. For the sake of readability, we define the set

$$\mathcal{M} := \Big\{(w, z, y, \eta, \xi, \zeta, u) \in \mathcal{O} \,:\, (w, z, \eta, \xi) \in \widetilde{\mathcal{G}},\, y = 0\Big\}.$$

Then the main result reads as follows.

Proposition 1. Suppose that A1, A2 and A3 hold. Then for each pair of compact subsets $\mathrm{Z}_0 \subset \mathbb{R}^{n_z}$ and $\mathrm{Y}_0 \subset \mathbb{R}$ there exist $\kappa : \mathbb{R}^2 \to \mathbb{R}$, $\beta_{\mathbf{x}} \in \mathcal{KL}$, $\rho_{\mathbf{x}} \in \mathcal{K}$, and a non-empty subset $\mathrm{T}_0 \subset \mathbb{R}^{n_\zeta}$ such that each solution $\mathbf{x} := (w, z, y, \eta, \xi, \zeta, u)$ of the closed-loop system (1), (2), (6) originating in $\mathrm{W} \times \mathrm{Z}_0 \times \mathrm{Y}_0 \times \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_\xi} \times \mathrm{T}_0 \times \mathbb{R}$ fulfills

$$|\mathbf{x}(t)|_{\mathcal{M}} \leq \max\big\{\beta_{\mathbf{x}}(|\mathbf{x}(0)|_{\mathcal{M}}, t),\, \rho_{\mathbf{x}}(|\varepsilon_w^\star|_{[0,t)})\big\},$$

for all $t \in \mathbb{R}_+$. In particular, each of those solutions satisfies

$$\limsup_{t\to\infty} |y(t)| \leq \rho_{\mathbf{x}}\Big(\limsup_{t\to\infty} |\varepsilon_w^\star(t)|\Big).$$

The claim of the proposition states that the regulation performance, expressed in terms of the asymptotic bound on y, are directly related to the prediction performance of the identifier, expressed in terms of the optimal prediction error $\varepsilon_w^\star$ defined in (12), with the latter depending on how well the estimate $\hat{\gamma}(\theta_w^\star, \cdot)$ approximates $\gamma(\cdot)$. Finally, we remark that if there exists $\theta^\star$ such that $\gamma = \hat{\gamma}(\theta^\star, \cdot)$, then $\varepsilon_w^\star = 0$ and asymptotic regulation is achieved.

Proof. Consider a solution $\mathbf{x}$ of (1), (2), (6) originating in $\mathrm{W} \times \mathrm{Z}_0 \times \mathrm{Y}_0 \times \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_\xi} \times \mathrm{T}_0 \times \mathbb{R}$, and let $\chi := (y, \dot{y})$. Then, $(w, z, \chi, \eta, \xi, \zeta, u)$ is solution to

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(w, z, \chi_1) \\ \dot{\chi}_1 &= \chi_2 \\ \dot{\chi}_2 &= q'(w, z, \chi) + \kappa(\chi_1, \nu(\zeta)) + \psi(\xi, \eta, \zeta) \\ \dot{\eta} &= F\eta + Gu \\ \dot{\xi} &= \varphi(\xi, \eta, u) \\ \dot{\zeta} &= \ell(\zeta, \chi_1) \\ \dot{u} &= \kappa(\chi_1, \nu(\zeta)) + \psi(\xi, \eta, \zeta). \end{aligned} \tag{18}$$

Consider the change of coordinates $\chi \mapsto \tilde{\chi}$, in which

$$\begin{aligned} \tilde{\chi}_1 &:= \chi_1 \\ \tilde{\chi}_2 &:= \chi_2 - q(w, \pi(w), 0) - \hat{\gamma}(\theta_w^\star, \tau(w, \pi(w))). \end{aligned}$$

Noting that $-q(w, \pi(w), 0) = \gamma(\tau(w, \pi(w)))$, (12) yields $\chi = \tilde{\chi} - E\varepsilon_w^\star$, with $E := \mathrm{col}(0, 1)$. Moreover, we have

$$\dot{\tilde{\chi}}_1 = \tilde{\chi}_2 - \varepsilon_w^\star, \quad \dot{\tilde{\chi}}_2 = \Lambda(w, z, \eta, \tilde{\chi}, \xi, u) + \kappa(\tilde{\chi}_1, \nu(\zeta))$$

where

$$\begin{aligned} \Lambda(\cdot) := q'(w, z, \tilde{\chi} - E\varepsilon_w^\star) - q'(w, \pi(w), 0) + \psi(\xi, \eta, u) \\ - \hat{\gamma}'\big(\xi_w^\star, \tau(w, \pi(w)), -q(w, \pi(w), 0)\big). \end{aligned}$$

We thus notice that, in view of (14),

$$u = -q(w, z, \tilde{\chi}_1) + \tilde{\chi}_2 - \varepsilon_w^\star$$

and, in view of (16), Λ can be written as

$$\Lambda(w, z, \eta, \tilde{\chi}, \xi, u) = \Delta(w, z, \eta, \tilde{\chi}_1, \tilde{\chi}_2 + \delta).$$

with $\delta = -\varepsilon_w^\star$. Now from (12), by boundedness of h and continuity of γ, τ, π and $\hat{\gamma}$, there exists a nonnegative scalar $\overline{\varepsilon_w^\star}$ such that $|\varepsilon_w^\star(t)| \le \overline{\varepsilon_w^\star}$ for all t and for any solution w of (2) in W. The result then follows directly from A3 with $\bar{\delta} = \overline{\varepsilon_w^\star}$. ■

5. CONCLUSION

This paper provides a constructive design of a regulator solving the (approximate) output regulation problem for a class of single-input-single-output normal forms. The proposed regulator is based on the existence result of (Marconi et al., 2007), and it employs adaptation to tune online the unknown quantities characterizing the internal model unit. Adaptation is cast as a system identification problem, and the main result relates the asymptotic regulation performances to the prediction capabilities of the chosen identification algorithm. The proposed approach thus complements the framework of (Marconi et al., 2007), providing a constructive adaptive design that yields an "optimal", and possibly asymptotic, regulation result.

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