World Journal of Education and Humanities
ISSN 2687-6760 (Print) ISSN 2687-6779 (Online)
Vol. 2, No. 3, 2020
www.scholink.org/ojs/index.php/wjeh

Original Paper

Algebraic Pure Tone Compositions Constructed via Similarity

Will Turner[1*]

[1] University of Aberdeen, Aberdeen, Scotland

[*] Will Turner, University of Aberdeen, Aberdeen, Scotland

Received: January 6, 2020 Accepted: January 20, 2020 Online Published: January 23, 2020

doi:10.22158/wjeh.v2n3p21 URL: http://dx.doi.org/10.22158/wjeh.v2n3p21

Abstract

We describe a family of musical compositions constructed by algebraic techniques, based on the notion of similarity between musical passages.

Keywords

algebra, composition, similarity

1. Introduction

If we hear a piece of music played twice we are in a position to recognize the similarity of the second musical experience to the first. Some pieces of music operate on memory by introducing similar passages within a single piece. Similarity operates on the mind more generally: people form sentimental attachments to things, and language is built from words, which are representations of similarity-ties. Musical traditions can carry the use of similar techniques over centuries. An excess of similarity can befelt to be bad: monotony and clichéarenegative words associated with such an excess. The response of a listener to hearing similarity in a given piece of music will vary, depending on previous experience.

In this article, we introduce a class of musical compositions built around the notion of similarity. As well as carrying internal similarities, our pieces are designed to be similar to music previously written, and we record the way in which they are so. We insist on a certain sort of progression within our pieces, in an attempt to avoid monotony. Likewise, the introduction of new techniques implies our pieces are a little different from previous musical compositions.

Algebra carries a formal notion representing a self-transformation: an end omor-phism. We use certain end zoomorphism to generate our compositions. Roughly our idea is the following: Suppose we have asset of musical phrases that is closed under concatenation, and we have set of transformations of this set that carries phrases to similar phrases. Take an initial note, apply as equence of transformation stoob-tain a sequence of notes, and concatenate to obtain a phrase, each of whose notes is similar to the previous one. Apply a sequence of transformations to this phrase to obtain as equence of phrases, and concatenate to obtain a longer phrase. Iterate this process a few times. The longest resulting phrase obtained is a musical com-position, constructed by similarity. To avoid direct monotony in the composition we use nonidentity transformations. In actuality our compositions will involve an accompaniment, constructed via a slight variation on this strategy. Figure 1 is a diagramoftransformationsinvolvedinsuchacomposition.Thetransformationof the initial note corresponding to a given leaf of the tree is obtained by tracing the unique path from the root to the leaf, multiplying all the transformations along the way.

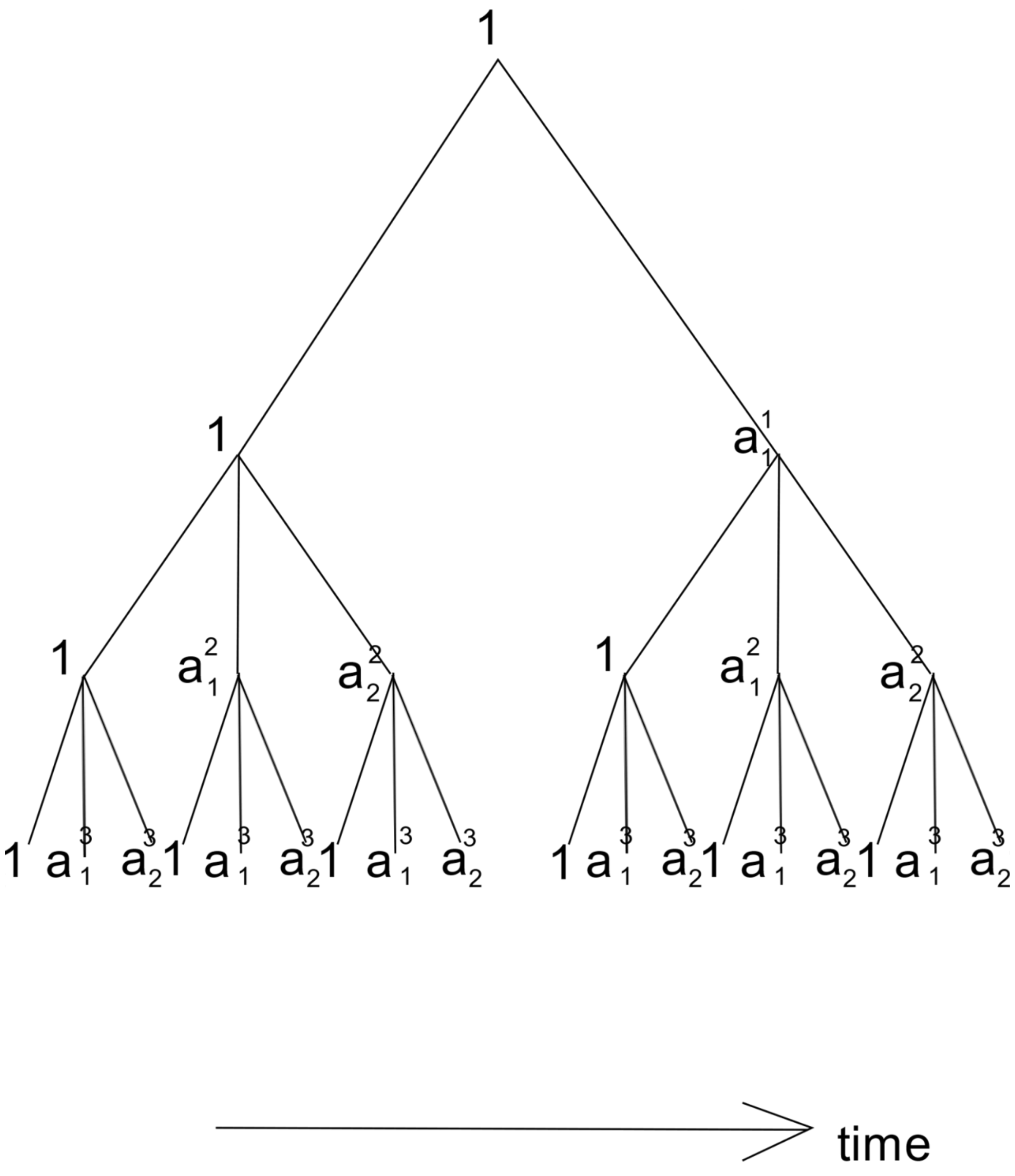

Figure 1. Transformations

Sharing music is a form of communication. This communication may occur between people in the same location at the same time, but also between people in different locations at different times. The music may have human players, in which case an additional sensual and intellectual dimension is added to the communication for those players. The music given here is to be played by a computer. However, by recording the associations we use to construct our compositions, we potentially add an intellectual dimension to the communication given by sharing the music.

Musical pieces commonly have passages designed to encourage associations in the mind of listeners. However there is potentially a great difference between the as-sociations to a piece of music experienced by a listener, and those written into the piece. For example, the associations written into the music may be of a formal nature, and difficult to detect with the ear. Furthermore, the listener may for mall kinds of additional associations that are not part of the formal design of the piece. Passage C may be designed to be similar to passage B, and passage B designed to be similar to passage A, but there may be an additional resemblance between pas-sages C and A that was not part of the formal design. Passage E may be obtained from passage D by a similarity transformation, by design, but some of the notes of passage E may coincide with notes in passage D giving additional associations between passages E and D. Rhythmic or melodic motifs may remind a listener of some other music, unknown to the composer of the piece. Via memory, a listener may form associations between the piece and a place where they hear it. Sharing the experience of a piece of music is a form of communication, and may have social associations. Further examples of musical associations beyond sounds are found when listeners take an in terestin musicians' personal lives, and beyond: in Austria Mozart's name is used to sell spherical chocolates (Mozartkugeln).

Let us comment on the relation of this work to the wider literature. Morphisms, and endomorphism's, have been extensively studied in the mathematical literature, in an abstract setting. Two classic examples are in Galois' theory of polynomial equations and in the theory of categories. Concatenation has also been studied in an abstract setting in the algebra literature, for example via quiver representations (see e.g., [5]).

A body of music to which our compositions are similar is that composed with sine waves. For example, there is the music of the Cologne school, I.3. Our so- called quasi-notes differ from the note mixtures of the Cologne School, because they have harmonics that are written onstage. The technique of combining sounds with shared harmonics (consonance), and of using similar passages one after another is standard in music for instruments with strings. Examples are to be found in Bach's Inventions.

Our pieces naturally fall into hierarchies of sequences of similar sections. The grouping of sounds into hierarchies of sections of increasing size is analyzed in some generality in A generative theory of tonal music. The psychology of the experience of similarity in music is explored in Sweet Anticipation. More ornate applications of the techniques introduced here are given elsewhere.

2. Method

In a piece of piano music, we have asset of notes on the stave. This setcan naturally be ordered by pitch, and its elements are therefore indexed by elements of Z, with middle C corresponding to zero. We call the integer corresponding to a given note on the stave its stave point. A piano note has a fundamental frequency *f*, and a set of overtones, whose frequencies are 2*f*, 3*f*, 4*f*, etc. The first five of these over tones are approximately 12, 19, 24, 28, 31 semitones above the original frequency *f*. If *S* is a subset of Z, and *m* is an element of Z then we can transform one phrase written on the stave with notes in *S* to a second by writing the elements of *S* in ascending order, and forming the second phrase by substituting the $i+m^{th}$ element of *S* for the i^{th} element of *S* wherever it appears in the phrase. We denote this transformation t_m. In case $S = Z$ such a transformation is called transposition by *m* semitones. A phrase and its transposition share common over tones of low degree in the case of transposition by 5, 7, and 12 semitones since $5 = 24-19$, $7 = 19-12$, $12 = 12 - 0 = 24 - 12$.

Formally at least, transpositions by these numbers carry special significance. Trans-position is commonly used in music written on the stave, but them ore general type of transformation we alluded to before is also used to generate similarity between phrases. For example, take *S* to correspond to the notes of the C major scale, and consider the treble part of Invention No. 1 in two parts by J. S. Bach [1]. Excepting the first note, the second bar is obtained from the first by applyingt_4. Since up to octave equivalence the notes of the major scale are related by a sequence of perfect fifths, the second bar is also close to being a transposition of the first by 7 semitones. However, bars 3 and 4 contain similar phrases that are not related by transpositions. Indeed bar 3 contains a pair of similar two crotchet motifs that are related by the transformationt_2, and the two crotchet motif at the beginning of bar 4 is related to the second of these two by t_2 as well.

The building blocks of the compositions introduced here are sounds we call *quasi-notes*, which are chords of pure tones that we treat like notes. A note on a stringed instrument has a fundamental frequency *f*, and a set of overtones, whose frequencies are 2*f*, 3*f*, 4*f*, etc. The first five of these overtones are approximately 12, 19, 24, 28, 31 semitones above the original frequency *f*. In a previous set of com-positions, we introduced set of positive integers less than orequal to 32 called *quasi-note generators*. For every point on the stave we took a quasi-note consisting of a *fundamental*, sounding together with a number of *overtones*. The fundamental is a pure tone whose frequency is given by the relevant point on the stave, whilst each over toneisa pure tone whose frequency is given by the fundamental frequency, raised by a quasi-note generator number of semitones. The fundamental, and the over tones are called the *partials* of the quasi-note. As similarity transformations in our compositions we used transposition by intervals given by the difference between a pair of quasi-note generators.

Here we use a more general sort of quasi-note, and a more general sort of similarity transformation. The motivation for generalizing is the following: since quasi-notes are synthesised, we do not need to use a fixed set of quasi-note generators as we move up the stave. We expect to be able to allow the intervals between partials to vary, and obtain a notion of quasi-note we can use. Take a set of strictly increasing maps $s_1, s_2, ..., s_p: Z \rightarrow Z$.

Our i^{th} quasi-note consists of the collection of p pure tones given by the p stave points $s_1(i), s_2(i)..., s_p(i)$, for i in Z. We call the pure tone given by $s_1(i)$ the fundamental of the i^{th} quasi-note, and the pure tone given by $s_j(i)$ the j^{th} partial of the i^{th} quasi-note. We call the j^{th} partial, for $j > 1$, an overtone of the j^{th} partial of the corresponding quasi-note. We thus have a set Q of quasi-notes, indexed by elements of Z.

Our collection of quasi-notes is given equivalently by a set of subsets $\Omega_1, \Omega_2, ..., \Omega_p \subseteq Z$ that are unbounded from above and below, with elements $\omega_1 \in \Omega_1, ..., \omega_p \in \Omega_p$.

Indeed, such data emerges when we write $\Omega_i = s_i(Z)$ and $\omega_i = s_i(0)$, for i = 1,2,...,p.

Suppose $\Omega_u \supseteq \Omega_v$.

Then thevthpartialofaquasi-noteqisequaltotheuthpartialofasecondquasi-noteqj.We write tu, v(q)=qj, and thus define a transformation tu, vof Q. We denotebyΦthecollectionofsuchtransformations.Thesetransformationstu,varethetransformationsweuseass imilaritytransformationstogenerateourcompositions.Inourexamples,wecontrolourchoicesofs1,...,spando urchoicesof the transformations tu, v we use, so that these transformations do indeed transform phrases to phrases that sound similar, beyond having common partials.

Our quasi-notes have partials corresponding to stave points, to create a similarity between our music and the large body of music that can be notated on a stave

We now record how we form a consonant accompaniment avoiding parallel motion.

Here we work under the assumption that

$\Omega_1 \supseteq \Omega_2, ..., \Omega_p$.

We discuss the construction of a minimising consonant accompaniment to a se-quence of quasi-notes, avoiding parallel motion.

Suppose we are given a sequence *s* of quasi-notes, and a subsequence *t* of *s* that dictates which elements of this sequence are to be accompanied.

Suppose we have a *consonant accompaniment* to *s*, which is a sequence of quasi- notes indexed by the elements of *t* whose fundamentals are given by overtones of the corresponding elements of *t*. We call an overtone of an element of *t* that forms a fundamental in the accompaniment a *harmonized overtone*. We say our accompaniment *avoids parallel motion* if the harmonized overtones of consecutive quasi-notes in *t* are indexed by distinct elements of {2, ..., p} (cf. Fux's rules concerning contrapuntal motion)

Suppose we fix a harmonized overtone for the first element of *t*. Suppose the i^{th} note of a consonant accompaniment of *s* is taken to avoid parallel motion, and is taken to minimize the number of semitones separating its fundamental from the fundamental of the i-1^{th}note of our accompaniment for each *i*, then we say our accompaniment is a *minimizing consonant accompaniment avoiding parallel motion*.

3. Result: The Compositions

Our compositions have a *principal part*, and an accompaniment. For n a natural number, let $\underline{n}= \{1, 2, ..., n\}$. Let r be a natural number, and let $c_1, ..., c_r$ be natural numbers. For $i = 1, ..., r$ we take maps $\lambda_i: \underline{c_i} \rightarrow$ R and $t_i: \underline{c_i} \rightarrow \Phi$, and for $i = 1, ..., r - 1$ we take maps $f_i: \underline{c_i} \rightarrow (F_2)^{c_r}$ We insist that $\lambda_i(1)$ and $t_i(1)$ are all the identity, for $i = 1, ..., r$, but that $\lambda_i(x)$ and $t_i(x)$ are different from the identity, for $x = 2, ..., c_i$ and $i = 1, ..., r$. We insist that $f_i(1) = 0$ for $i = 1, ..., r - 1$.

For $i = 1, ..., r$ we define maps $\mu_i: \underline{c_i} \rightarrow$ R and $u_i: \underline{c_i} \rightarrow End(Q)$, by

$\mu_i(x) = \lambda_i(x)\lambda_i(x - 1)...\lambda_i(1)$,

$u_i(x) = t_i(x)t_i(x - 1)...t_i(1)$, and for $i = 1, ..., r - 1$ we define maps

$g_i: \underline{c_i} \rightarrow (F_2)^{c_r}$

by

$g_i(x) = f_i(x) + f_i(x - 1) + ... + f_i(1)$.

Let us fix duration $d \in$ R, a quasi-note q with fundamental in Ω_1, and an element $b \in F^{c_r}$.

The quasi-notes of the principal part of our composition correspond to elements $x = (x_1, ..., x_r)$ of $\underline{c_1} \times \underline{c_2} \times ... \times \underline{c_r}$, ordered lexicographically. They are given by $u_1(x_1)u_2(x_2)...u_r(x_r)q$. They have duration $\mu_1(x_1)\mu_2(x_2)...\mu_r(x_r)d$.

To form the accompaniment, consider the x_r^{th} coordinate of $g_1(x_1) + g_2(x_2) + ... + g_{r-1}(x_{r-1}) + b$.

If this coordinate is 1 then we accompany our quasi-note, if it is 0 then we do not accompany our quasi-note. We fix an overtone in the first quasi-note that has an accompaniment. We then take a minimum consonantal accompaniment avoiding parallel motion.

4. Discussion: Examples

Here we record the data sets for two examples for which recordings are available.

Example 1 Let $p = 4$. Let

$\Omega_1 =$ Z, $\Omega_2 = \{0, 1, 2, 3, 4, 6, 7, 8, 9, 10\} + 12$Z,

$\Omega_3 = \{0, 1, 3, 4, 6, 7, 9, 10\} + 12$Z, $\Omega_4 = \{0, 1, 3, 4, 5, 6, 7, 9, 10, 11\} + 12$Z.

Thus

$\Omega_1 \square \Omega_2$, $\Omega_4 \square \Omega_3$.

To specify a complete set of quasi-notes it is enough to specify one:

$\omega_1 = -12$, $\omega_2 = -9$, $\omega_3 = -8$, $\omega_4 = 1$.

We take $r = 4$, $c_1 = c_2 = c_3 = 3$, and $c_4 = 7$. We take $\lambda_i(2) = \lambda_i(3)$ to be the 6^{th} root of 2, for $1 \leq i \leq 3$, and take $\lambda_4(i)$ to be 1, 1, 1 ,2, 0.5, 1 for i=2,3,4,5,6,7 respectively. We take $t_i(2) = t_{2,3}$ and $t_i(3) = t_{1,2}$ for $1 \leq i \leq 3$ and take $t_4(i)$ to be $t_{1,2}, t_{2,3}, t_{2,3}, t_{4,3}, t_{2,3}, t_{2,3}$ for i=2,3,4,5,6,7 respectively.

We choose an empty accompaniment in this case, thus $f_i = g_i = 0$ for all i and $b = 0$. We take $d = 0.5$ and q

to be the quasi-note whose fundamental is the *C* an octave below middle C.

Example 2 Let$p$ = 4. Let

$\Omega_1=\{7i|-1\leq i\leq 5\}+12Z$,$\Omega_2=\{7i|-1\leq i\leq 4\}+12Z$, $\Omega_3=\{7i|0\leq i\leq 4\}+12Z$,$\Omega_4=\{7i|0\leq i\leq 5\}+12Z$,

Thus Ω_1□ Ω_2, Ω_4□ Ω_3. To specify a complete set of quasi-notes it is enough to specify one: $\omega_1 = 17$, $\omega_2 = 9$, $\omega_3 = 0$, $\omega_4 = 4$. The partials of our quasi-notes lie in a major scale, creating a similarity between our piece and many others.

We take $r = 6$, $c_1= c_2= c_3= c_4= c_5= 2$, and $c_6= 7$. We take $\lambda_i(2)$ to be the 5^{th} root of 0.5, for $1\leq i\leq 5$, and take $\lambda_6(i)$ to be 2, 2, 0.5, 2, 0.5, 0.5 for i = 2, 3, 4, 5, 6, 7 respectively. We take $t_i(2) = t_{1,2}$ for $1 \leq i \leq 5$, and take $t_6(i)$ to be$t_{1,2}$, $t_{1,2}$, $t_{1,2}$, $t_{4,3}$, $t_{4,3}$, $t_{1,2}$ for i = 2, 3, 4, 5, 6, 7 respectively. For i = 1, ..., 5 we take $f_i(2)$ to have 0s in all seven coordinates excepting the 7-ith which is 1.

We take d = 0.125 and q to be the quasi-note whose fundamental is the *F* four octaves and a perfect fourth above middle C. We take b = (1, 0, 0, 0, 0, 0, 0). We use the second partial of q to generate our minimum consonant accompaniment avoiding parallel motion.

References

Bach, J. S. (1723). *Two Part Inventions*. BWV772-786.

Benson, D. J. (November, 2006). *Music: A Mathematical Offering*. Cambridge University Press.

Eilenberg, S., & MacLane, S. (1945). General theory of natural quivalences. *Transactions of the American Mathematical Society*, *58*, 231-294.

Fux, J. J. (1725). *Gradus ad Parnassum*.

Gabriel, P. (1972). Unzerlegbare Darstellungen. *I. Manuscripta Math*, *6*, 71-103.

Galois, E´. (1846). Oeuvres Math´ematiques. *Journalde Liouville*.

Huron, D. (2008). *Sweet Anticipation*. Music and the Psychology of Expectation. Bradford.

Lerdahl, F., & Jackendoff, R. (1983). *A generative theory of tonal music*. The MITPress.

Manning, P. (2013). *Electronic and Computer Music*. Oxford UniversityPress.

Orpen, K. S., & Huron, D. (Fall1, 1992). Measurement of Similarity in Music: A Quantitative Approach for Non-parametric Representations. *Computers in Music Research*, *4*, 1-44.

Stockhausen, K. (1954). *StudieII*.

Turner, W. (2018). Eight short mathematical compositions constructed by similarity. *Far East Journal of Mathematical Sciences*, *107*(2), 433-447.

Turner, W. (2020). *Algebraic pure tone compositions constructed via similarity, Examples 1 and 2*. Retrieved from http://homepages.abdn.ac.uk/w.turner/pages/

Turner, W. (2020). *Some scales that are similar to the chromatic scale*. Retrieved from http://homepages.abdn.ac.uk/w.turner/pages/