Oct 26th, 1:30 PM - 3:00 PM

Analyzing the linearity of some operators

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Abstract. Linear operators occupy an important place in functional analysis and linear algebra, which are among the most important and substantive disciplines of mathematics, whose methods and results have created an indispensable apparatus for the development of numerical mathematics, theory of approximations, equations differential and especially mathematical physics and applied mathematics. Also, linear operators are a central object of study in vector space theory. A linear operator is a function which satisfies the conditions of additivity and homogeneity. Not every function is linear operators. We will try to explore some functions which are also linear operators.

Keywords: Function, linear operators, vectors, vector spaces

Vector spaces

Often in analytical geometry, mechanics, physics, etc., we come across object oriented objects called vectors. Those objects define linear actions, vector collection, and scalar vector multiplication. [1,4]

Thus, the set of free vectors in a straight line, plane, or ordinary three-dimensional space, with respect to the said actions, form special algebraic structures, which enjoy certain properties. These structures are called vector spaces. In the general case, by taking abstract objects, actions are defined and the conditions are formulated which must satisfy them [3].

A vector space X is an aggregate of elements, called vectors, u, v,... for which linear operations (addition u + v of two vectors u, v and multiplication α u of a vector u by a scalar α are defined and obey the usual rules of such operations. The scalars are assumed to be complex numbers unless otherwise stated (complex vector space). α u is also written as u α whenever convenient, and α^{-1}u is often written as u α . The zero vector is denoted by 0 and will not be distinguished in symbol from the scalar zero. Vectors $u_1,..., u_n$ are said to be linearly independent if their linear combination $\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n = 0$ only if $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$; otherwise they are linearly dependent [2].

The dimension of X , denoted by dim X , is the largest number of linearly independent vectors that exist in X . If there is no such finite number, we set dim X = ∞ . A subset M of X is a linear manifold or a subspace if M is itself a vector space under the same linear operations as in X .

The dimension of M does not exceed that of X . For any subset S of X , the set M of all possible linear combinations constructed from the vectors of S is a linear manifold; M is called the linear manifold determined or spanned by S or simply the (linear) span of S . According to a basic theorem on vector spaces, the span M of a set of n vectors u1,..., un is at most n - dimensional; it is exactly n -dimensional if and only if u1,..., un are linearly independent. There is only one 0-dimensional linear manifold of X , which consists of the vector 0 alone and which we shall denote simply by 0 [1,2].

Example 1.1. The set of all complex-valued continuous functions u : x → u(x) defined on an interval I of a real variable x is an infinite-dimensional vector space, with the obvious definitions of the basic operations α u + β v . The same is true when, for example, the u are restricted to be functions with continuous derivatives up to a fixed order n . Also the interval R may be replaced by a region' in the m dimensional real euclidean space R m [6].
Example 1.2. The set of all solutions of a linear homogeneous differential equation

$$u^{(n)} + a_1(x)u^{(n-1)} + \ldots + a_n(x)u = 0$$

with continuous coefficients $a_j(x)$ is an n -dimensional vector space, for any solution of this equation is expressed as a linear combination of n fundamental solutions, which are linearly independent [3].
Definition 1.1. Let X be an N -dimensional vector space and let $x_1,..., x_N$ be a family of N linearly independent vectors. Then their span coincides with X , and each u ∈ X can be expanded in the form

$$u = \sum_{j=1}^{N} \xi_j x_j$$

in a unique way. In this sense the family {x j}is called a basis of X , and the scalars j are called the coefficients (or coordinates) of u with respect to this basis.
The correspondence u (ξ_j) is an isomorphism of X onto C N (the set of numerical vectors) in the sense that it is one to one and preserves the linear operations, that is, u → (ξ_j) and v → (η_j) imply $\alpha u + \beta v \to (\alpha\xi_j + \beta\eta_j)$ As is well known, any family $x_1,..., x_p$ of linearly independent vectors can be enlarged to a basis $x_1,..., x_p, x_{p+1},..., x_N$ by adding suitable vectors $x_{p+1},..., x_N$.
Definition 1.2. For any subset S and S ' of X , the symbol S + S' is used to denote the (linear) sum of S and S ' , that is, the set of all vectors of the form u + u' with u ∈ S and u'∈ S' . If S consists of a single vector u , S + S' is simply written u + S' . If M is a linear manifold, u + M is called the inhomogeneous linear manifold (or linear variety) through u parallel to M .
The totality of the inhomogeneous linear manifolds u + M with a fixed M becomes a vector space under the linear operation $\alpha(u+M)+\beta(v+M)=(\alpha u+\beta v)+M$. This vector space is called the quotient space of X by M and is denoted by X / M . The elements of X / M are also called the cosets of M . The zero vector of X / M is the set M , and we have u + M = v + M if and only if u − v ∈ M . The dimension of X / M is called the codimension or deficiency of M (with respect to X) and is denoted by codim M . We have dim M + co dim M = dim X [4,6].

Linear operators. Matrix representations

Definition 2.1. Let X be a vector space. A complex-valued function f u defined for u X is called a linear form or a linear functional if $f[\alpha u+\beta v]=\alpha f[u]+\beta f[v]$ for all u, v of X and all scalars α, β.
Example 2.1. If X = C N (the space of N -dimensional numerical vectors), a linear form on X can be expressed in the form: $f[u]=\sum_{j=1}^{N}\alpha_j\xi_j$ for $u=(\xi_j)$. It is usual to represent f as a row vector with the components α_j , when u is represented as a column vector with the components ξ_j .
The $f[u]=\sum_{j=1}^{N}\alpha_j\xi_j$ is the matrix product of these two vectors.
Example 2.2. Consider whether linear operations from $\mathbb{R}^2$ $\mathbb{R}^2$ are linear operators:

a) $a)\ A((a, b)) = (\sin a, b)$
b) $b)\ A((a, b)) = (a - b, 0)$

Solution: We try additives and homogeneity:

Let it be $U_1 = (a_1, b_1)$ and $U_2 = (a_2, b_2)$ where $U_1, U_2 \in \mathbb{R}^2$

$$a)\ A(U_1 + U_2) = A((a_1, b_1) + (a_2, b_2))$$
$$= A(a_1 + a_2, b_1 + b_2)$$
$$= \sin(a_1 + a_2), (b_1 + b_2)$$
$$\neq (\sin a, b)\text{Type equation here.}$$

Let $U_1 = \left(\frac{\pi}{2}, 0\right), U_2 = (\pi, 1)$

$$A(U_1 + U_2) = A\left(\left(\frac{\pi}{2}, 0\right) + (\pi, 1)\right)$$
$$= A\left(\frac{3\pi}{2}, 1\right)$$
$$= \left(\sin\frac{3\pi}{2}, 1\right) = (-1,1)$$
$$A(U_1) + A(U_2) = A\left(\frac{\pi}{2}, 0\right) + A(\pi, 1)$$
$$= \left(\sin\frac{\pi}{2}, 0\right) + (\sin\pi, 1)$$
$$= (1,0) + (0,1) = (1,1)$$

Although $(-1,1) \neq (1,1)$, thus $A(U_1 + U_2) \neq A(U_1) + A(U_2)$
then the given operation is not linear operators.

$$b)\ A(U_1 + U_2) = A((a_1, b_1) + (a_2, b_2))$$
$$= A(a_1 + a_2, b_1 + b_2)$$
$$= (a_1 + a_2 - (b_1 + b_2), 0)$$
$$= ((a_1 - b_1) + (a_2 - b_2), 0)$$
$$= (a_1 - b_1, 0) + (a_2 - b_2, 0)$$
$$= A(U_1) + A(U_2)$$

Thus, $A(U_1 + U_2) = A(U_1) + A(U_2)$

$$A(\lambda U) = A(\lambda(a_1, b_1))$$
$$= A(\lambda a_1, \lambda b_1)$$
$$= (\lambda a_1 - \lambda b_1, 0)$$
$$= \lambda(a_1 - b_1, 0)$$
$$= \lambda(AU)$$

Thus, $(\lambda U) = \lambda(AU)$

Since additive and homogeneity are met then we say that it is linear operators [3].

Definition 2.2. Let X , Y be two vector spaces. A function T that sends every vector u of X into a vector v = Tu of Y is called a linear transformation or a linear operator on X to Y if T preserves linear relations, that is, if $T(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 T u_1 + \alpha_2 T u_2$ for all u1, u2 of X and all scalars α_1, α_2 . X is the domain space and Y is the range space of T .

If Y = X we say simply that T is a linear operator in X . In this book an operator means a linear operator unless otherwise stated. For any subset S of X , the set of all vectors of the form Tu with u ∈ S is called the image under T of S and is denoted by TS ; it is a subset of Y . If M is a linear manifold of X , TM is a linear manifold of Y . In particular, the linear manifold TX of Y is called the range of T and is denoted by R(T) . The dimension of R(T) is called the rank of T ; we denote it by rank T . The deficiency (codimension) of R(T) with respect to Y is called the deficiency of T and is denoted by defT .

Thus rank T + def T = dim Y . For any subset S ' of Y , the set of all vectors u ∈ X such that Tu ∈ S' is called the inverse image of S ' and is denoted by T –1S' . The inverse image of 0 ⊂ Y is a linear manifold of X ; it is called the kernel or null space T of N (T). The dimension of N (T) is called the nullity of T , which we shall denote by nul T . We have rank T + nul T = dim X . To see this it suffices to note that T maps the quotient space X / N (T) (which has dimension dim X - nul T) onto R(T) in a one-to-one fashion. If both nul T and def T are zero, then T maps X onto Y one to one. In this case the inverse operator T –1 is defined; T –1 is the operator on Y to X that sends Tu into u . Obviously we have (T) = T. T is said to be nonsingular if T –1 exists and singular otherwise. For T to be nonsingular it is necessary that dim X = dim Y . If dim X = dim Y , each of nulT = 0 and def T = 0 implies the other and therefore the nonsingularity of T [2,6].

Linear operations on operators

Definition 3.1. If T and S are two linear operators on X to Y , their linear combination $\alpha S + \beta T$ is defined by $(\alpha S + \beta T)u = \alpha(Su) + \beta(Tu)$ for all u ∈ X , and is again a linear operator on X to Y . Let us denote by ℜ(X ,Y) the set of all operators on X to Y Y; ℜ(X ,Y) is a vector space with the linear operations defined as above. The zero vector of this vector space is the zero operator 0 defined by 0u = 0 for all u ∈ X .

Problem 3.1. rank(S + T) = rankS + rankT .

The dimension of the vector space ℜ(X ,Y) is equal to NM , where N = dim X and M = dimY .

To see this, let $\{x_k\}$ and $\{y_j\}$be bases of X and Y , respectively, and let Pjk be the operator on X to Y such that

$$P_{jk}x_h = \delta_{kh}y_j, \;\; k,h = 1,...,N; j = 1,...,M \,.$$

These MN operators Pjk , are linearly independent elements of ℜ(X ,Y), and we have from $Tx_k = \sum_{j=1}^{M}\tau_{jk}y_j$, M = dim Y, yelds $T = \sum \tau_{jk}P_{jk}$. Thus $\{P_{jk}\}$ is a basis of ℜ(X ,Y), which proves the assertion. $\{P_{jk}\}$ will be called the basis of ℜ(X ,Y) associated with the bases $\{x_k\}$ and $\{y_j\}$of X and Y , respectively. The last result shows that the matrix elements τ jk are the coefficients of the "vector" T with respect to the basis $\{P_{jk}\}$.

The product TS of two linear operators T , S is defined by (TS)u = T (Su) for all u ∈ X , where X is the domain space of S , provided the domain space of T is identical with the range space Y of S [4,5]. The following relations hold for these operations on linear operators :

(TS)R = T (SR), which is denoted by TSR

$$(\alpha T)S = T(\alpha S) = \alpha(TS), \text{ denoted by } \alpha TS$$

$$(T_1 + T_2)S = T_1 S + T_2 S$$

$$T(S_1 + S_2) = TS_1 + TS_2$$

$$\text{rank}(TS) \leq \max(\text{rank}T,\ \text{rank}S)$$

The algebra of linear operators

If S and T are operators on X to itself, their product TS is defined and is again an operator on X to itself. Thus the set $\Re(X) = \Re(X, X)$ of all linear operators in X is not only a vector space but an algebra. $\Re(X)$ is not commutative for dim $X \geq 2$ since TS = ST is in general not true. When TS = ST , T and S are said to commute (with each other). We have T 0 = 0T = 0 and T1 = 1T = T for every $T \in \Re(X)$, where 1 denotes the identity operator (defined by 1u = u for every $u \in X$). Thus 1 is the unit element of $\Re(X)$. The operators of the form $\alpha 1$ are called scalar operators and in symbol will not be distinguished from the scalars α . A scalar operator commutes with every operator of $\Re(X)$.

We write TT = T 2 , TTT = T 3 and so on, and set T 0 = 1 by definition. We have

$$T^m T^n = T^{m+n},\ \left(T^m\right)^n = T^{mn},\quad m, n = 0,1,2,...$$

For any polynomial $p(z) = \alpha_0 + \alpha_1 z + ... + \alpha_n z^n$ in the indeterminate z , we define the operator p T T nT= α0 + α1 + ... + α . The mapping $p(z) \to p(T)$ is a homomorphism of the algebra of polynomials to $\Re(X)$; this means that p(z) + q(z) = r(z) or p(z)q(z) = r(z) implies p(T) + q(T) = r(T) or p(T)q(T) = r(T) respectively. In particular, it follows that p(T) and q(T) commute.

If $T \in \Re(X)$ is nonsingular, the inverse T −1 exists and belongs to $\Re(X)$; we have T −1T = TT −1 = 1. If T has a left inverse T ' (that is, a $T' \in \Re(X)$ such that T 'T = 1), T has nullity zero, for Tu = 0 implies u = T 'Tu = 0 . If T has a right inverse T ' ' (that is, TT " = 1), T has deficiency zero because every $u \in X$ lies in R(T) by u = TT "u . If dim X is finite, either of these facts implies that T is nonsingular and that T ' = T −1 or T " = T −1 , respectively [4]. If S and T are nonsingular, so is TS and $(TS)^{-1} = S^{-1}T^{-1}$ For a nonsingular T , the negative powers T −n , n = 1,2,... can be defined by $T^{-n} = \left(T^{-1}\right)^n$. In this case

$$T^m T^n = T^{m+n},\ \left(T^m\right)^n = T^{mn},\quad m, n = 0,1,2,...$$

is true for any integers m, n .

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