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Convergence Rates for the Central Limit Theorem for Random Sums

Christopher E. Olson

CONVERGENCE RATES FOR THE CENTRAL LIMIT THEOREM - OLSON

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CONVERGENCE RATES FOR THE
CENTRAL LIMIT THEOREM FOR RANDOM SUMS

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CONVERGENCE RATES FOR THE
CENTRAL LIMIT THEOREM FOR RANDOM SUMS

BY

CHRISTOPHER E. OLSON

DISSERTATION

Submitted in Partial Fulfillment of the
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in the Graduate School of
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CONVERGENCE RATES FOR THE
CENTRAL LIMIT THEOREM FOR RANDOM SUMS

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ABSTRACT OF DISSERTATION

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Abstract

Let $\{X_i\}$ be a sequence of independent, identically-distributed random variables with $EX_i^2 < \infty$ and $E(X_i - EX_i)^2 = 1$. Let $\overline{S}(n) = \left(\sum_{j=1}^{n} X_j - nEX_i\right)/\sqrt{n}$. If τ_n is a sequence of positive integer-valued random variables such that $\tau_n/n \xrightarrow{p} \tau$, a positive (possibly degenerate) random variable, and $\overline{G}_n(x) \equiv P[\overline{S}(\tau_n) \leq x]$, then by a theorem of Blum, Hanson, and Rosenblatt $\sup_x |\overline{G}_n(x) - \Phi(x)| \to 0$ as $n \to \infty$. This dissertation establishes sufficient conditions for series convergence rates of the form

$$\sum_n n^{-\alpha} \sup_x |\overline{G}_n(x) - \Phi(x)| < \infty. \qquad (1)$$

The exponent α in (1) depends on the number of finite absolute moments of X_i. If $E|X_i|^{2+\delta} < \infty$, $0 < \delta < 1$, then $\alpha = 1 - \delta/2$. If $EX_i^2 < \infty$ but $E|X_i|^{2+\delta} = \infty$, for all $\delta > 0$, then $\alpha = 1$. In this case, unless $EX_i^2 \log|X_i| < \infty$, the term $\Phi(x)$ in (1) is replaced by $E\Phi(x/\sigma_{\tau_n})$, where σ_n is the variance of X_i truncated at $\sqrt{n}$.

The problem is most naturally treated in two cases: In the first case, the sequences $\{X_i\}$ and $\{\tau_n\}$ are taken to be independent, and in the second, they are assumed to be dependent. In the independent case, a basic sufficient condition for (1) to hold is

$$\sum_n n^{-\alpha} \cdot P[\tau_n = k] = O(k^{-\alpha}). \qquad (2)$$

It is shown that, if τ_n is the $n\underline{th}$ epoch of a positive, integer-valued renewal process, then it satisfies (2). Another theorem gives a condition on the rate with which $\tau_n/n \xrightarrow{p} c_0$, a positive constant, which is sufficient for (2) to hold, and thus for (1).

In the dependent case, sufficient conditions are given in terms of the rate of decrease of

$$|P([\overline{S}(k) \leq x] \cap [\tau_n=k]) - P[\overline{S}(k) \leq x] \cdot P[\tau_n=k]|$$

and of

$$|P([\overline{S}(k) \leq x] \cap [\tau_n=k]) - P([\overline{S}(n) \leq x] \cap [\tau_n=k])| .$$

The sharpness of these sufficient conditions is indicated through simple examples whereas the conditions are violated and the convergence in (1) fails. A further example satisfying both of these conditions is then given.

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I. Introduction

Throughout this dissertation, let $X_1, X_2, \ldots$ be a sequence of independent, identically-distributed random variables on a probability space $(\Omega, \mathcal{A}, P)$. The distribution function of X_i is $F(x) = P(\{\omega | X_i(\omega) \leq x\})$, or more simply $F(x) = P[X_i \leq x]$.

The following conditions are assumed:

(1) $EX_i^2 < \infty$

(2) $E(X_i - \mu)^2 = 1$,

where E denotes expectation and $\mu \equiv EX_i$. Let $S(n) = \sum_{i=1}^{n} X_i$, $\overline{S}(n) = (S(n) - n\mu)/\sqrt{n}$. The distribution function of $\overline{S}(n)$ is denoted by

$$\overline{F}_n(x) = P[\overline{S}(n) \leq x] .$$

Let $\Phi(x)$ and $\varphi(x)$ be the distribution function and density function, respectively, of the normal probability distribution with zero mean and unit variance, i.e.,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt ,$$

$$\varphi(x) \equiv \Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} .$$

A truncated variance is defined for X_i by

$$\sigma_n^2 = \int_{|x| \leq \sqrt{n}} x^2 dF - \left(\int_{|x| \leq \sqrt{n}} x dF \right)^2 .$$

Note that $\lim_{n\to\infty} \sigma_n = 1$, the variance of X_i.

The central limit problem has been a principal line of research in probability theory since the inception of the theory. In its simplest form, the central limit problem consists of finding conditions under which

$$\Delta_n \equiv \sup_x |\overline{F}_n(x)-\Phi(x)| \to 0 \quad \text{as} \quad n \to \infty .$$

The history of the problem begins with DeMoivre and Laplace, who proved that $\Delta_n \to 0$ for random variables X_i with a binomial distribution. Lindeberg[17] and Feller[8] proved the necessity and sufficiency of the finiteness of $E|X_i|^2$ for $\Delta_n \to 0$, for independent, identically distributed X_i. Further work in the central limit problem has extended these classic results to include sequences of non-identically distributed random variables and dependent random variables of certain types.

In many practical applications, the limiting distribution is used to approximate the distribution of the sum $\overline{S}(n)$ for finite values of n. Such usage must be based on rigorous estimates of the remainder term or the speed of convergence. Liapounov[15,16] showed that, if $E|X_i|^{2+\delta} < \infty$, $0 < \delta < 1$, then $\Delta_n \leq cn^{-\delta/2} E|X_i|^{2+\delta}$, where the constant c may depend on the distribution of the X_i. When $E|X_i|^3 < \infty$, Berry[3] and Esseen[7] proved that

$$\Delta_n \leq c_0 n^{-1/2} E|X_i|^3 ,$$

where c_0 is a constant which does not depend on the distribution of X_i. This result was refined by Nagaev[20], who showed that

$$|\overline{F}_n(x)-\Phi(x)| \leq \frac{c_0 E|X_i|^3}{n^{1/2}(1+|x|^3)} ,$$

where c_0 is the Berry-Esseen constant. Another direction was taken by M. Katz[12], who proved that, if $g(x)$ is a real, non-negative, even function, non-decreasing on $[0,\infty)$, then $EX_i^2 g(X_i) < \infty$ is sufficient for

$$\Delta_n \leq \frac{c_0 EX_i^2 g(X_i)}{g(n^{1/2})} .$$

The results of Berry and Esseen, and of Nagaev, are the best possible in the sense that the bound is achieved by some distribution satisfying the hypotheses.

In 1966, N. Friedman, M. Katz, and L. Koopmans[9] proved a series convergence rate theorem requiring only two moments. C.C. Heyde[10] and J.A. Davis[6] each noted improvements and simplifications which led to the following result:

Theorem: If $E|X_i|^2 < \infty$, say $E|X_i|^2 = 1$ and $EX_i = 0$, then $\sum_{n=1}^{\infty} n^{-1} \sup_x |\overline{F}_n(x)-\Phi(x/\sigma_n)| < \infty$. If, in addition, $EX_i^2 \log|X_i| < \infty$, then $\sum_{n=1}^{\infty} n^{-1}\Delta_n < \infty$. Further, if $EX_i^{2+\delta} < \infty$, $0 < \delta < 1$, then

$$\sum_{n=1}^{\infty} n^{-1+\delta/2}\Delta_n < \infty.$$

Suppose now that $\{\tau_n\}$ is a sequence of positive, integer-valued random variables on $(\Omega, \mathcal{A}, P)$. Let $\overline{G}_n(x) \equiv P(\{\omega/\overline{S}(\tau_n(\omega)) \leq x\})$. Then

$$\overline{G}_n(x) = E(I_{(-\infty,x]}(\overline{S}(\tau_n))) ,$$

where I_A is the set indicator function of the set A.

Proceeding in the same line,

$$\begin{aligned}\overline{G}_n(x) &= E(I_{(-\infty,x]}(\overline{S}(\tau_n))) \\ &= E_{\tau_n}[E(I_{(-\infty,x]}(\overline{S}(\tau_n))|\tau_n)] \\ &= \sum_{k=1}^{\infty} E(I_{(-\infty,x]}(\overline{S}(k)|\tau_n=k)) \cdot P[\tau_n=k] \\ &= \sum_{k=1}^{\infty} P[\overline{S}(k)\leq x|\tau_n=k] \cdot P[\tau_n=k] .\end{aligned}$$

Note that in case $\{\tau_n\}$ is independent of $\{X_i\}$,

$$\overline{G}_n(x) = \sum_{k=1}^{\infty} \overline{F}_k(x) \cdot P[\tau_n=k] .$$

The question of when $\sup_x|\overline{G}_n(x)-\Phi(x)|$ goes to zero is a natural extension of the classical central limit problem. For example, one might wish to approximate $\overline{G}_n(x)$ when τ_n is the first passage of the sequence $S(n)$ past n, or when τ_n is an integer-multiple of n determined by the value of $S(n)$. Such an approximation would be useful in sequential analysis for tests based on $S(\tau_n)$ (see Wald[11]).

The case where the τ_n's are independent of the summands was dealt with by Robbins[12] in 1948. Anscombe[1] showed that, if $\tau_n/n \xrightarrow{p} c$, where c is a positive constant, then $\Delta_{\tau_n} \equiv \sup_x |\overline{G}_n(x) - \Phi(x)| \to 0$, regardless of the fact that the τ_n's might depend on the X_i's. This result was extended in 1960 by Renyi[23] to the case where $\tau_n/n \xrightarrow{p} \tau$, a positive discrete random variable; and by Blum, Hanson, and Rosenblatt[4], and, independently, Mogyorodi[19], in 1963, to the case where τ is any positive random variable.

In 1964, Wittenberg[27] investigated, more generally, the conditions under which the Kolmogorov-Smirnov (K-S) distance between $S(\tau_n)$ and $S(n)$ tends to zero. If $S(n)$, when properly normed, converges in distribution to a random variable with non-degenerate law, and if the K-S distance between $S(n)$ and $S(\tau_n)$ tends to zero, then $S(\tau_n)$ converges to a random variable with the same law. The theorem of Blum, Hanson and Rosenblatt becomes a particular case of Wittenberg's results.

This variety of results concerning the convergence of $\overline{G}_n(x)$ to $\Phi(x)$ will be referred to as the random central limit theorem (RCLT). In particular, a pair of sequences $\{X_i\}$, $\{\tau_n\}$ will be said to satisfy the conditions of the RCLT if $\{X_i\}$ is a sequence of independent, identically-distributed random variables with at least two finite moments, and if $\{\tau_n\}$ is a sequence of positive, integer-valued random variables such that $\tau_n/n \xrightarrow{p} \tau$, a positive (possibly degenerate) random variable.

It is the purpose of this dissertation to investigate the rate of convergence of $\bar{G}_n(x)$ to $\Phi(x)$. In order to derive series convergence rates similar to those of the Friedman-Katz-Koopmans Theorem (F-K-K), one must treat functions similar to $\Phi(x/\sigma_n)$ where the normalizing factor is defined in terms of both $\{X_i\}$ and $\{\tau_n\}$. To this end, consider

$$E\Phi(x/\sigma_{\tau_n}) = \sum_k \Phi(x/\sigma_k) \cdot P[\tau_n=k] .$$

Since $\sigma_k \to 1$, in order to avoid trivial details, it will be assumed that $\sigma_k > 0$ for all k. The series whose convergence properties are examined are the following:

$$A_1(x,\alpha) \equiv \sum_n n^{-\alpha}|\bar{G}_n(x)-E\Phi(x/\sigma_{\tau_n})|$$

$$A_2(\alpha) \equiv \sum_n n^{-\alpha} \sup_x |\bar{G}_n(x)-E\Phi(x/\sigma_{\tau_n})|$$

$$A_3(x,\alpha) \equiv \sum_n n^{-\alpha}|\bar{G}_n(x)-\Phi(x)|$$

$$A_4(\alpha) = \sum_n n^{-\alpha} \sup_x |\bar{G}_n(x)-\Phi(x)| .$$

The original statement of the F-K-K Theorem allowed for dropping the normalizing factor σ_n^{-1} of the argument of Φ provided that $\sum_{n=1}^{\infty} n^{-1}(1-\sigma_n^2) < \infty$. J.A. Davis[6] later pointed out that $\sum_n n^{-1}(1-\sigma_n^2) < \infty$ if and only if $EX_i^2 \log|X_i| < \infty$. It will be valuable to have a sufficient condition for dropping the normalizing factor $\sigma_{\tau_n}^{-1}$ from $A_1(x,\alpha)$ or $A_2(\alpha)$. The following lemma presents such a condition:

Lemma: If $A_2(\alpha)$ (or $A_1(x,\alpha)$) is finite and if

$$(1) \quad \sum_{n=1}^{\infty} n^{-\alpha} \sum_k P[\tau_n = k] \cdot |1-\sigma_k^2| < \infty ,$$

then $A_4(\alpha)$ (or $A_3(x,\alpha)$) is finite.

Proof: Throughout this and following proofs, $c, c', c'', \ldots$ will denote positive constants that arise in the arguments, the values of which are unimportant to the discussion. For a sequence $a_k \to 1$, a straightforward computation yields

$$|\Phi(a_k x)-\Phi(x)| \leq \Phi'(x)|(a_k-1)x|(1+cx^2|a_k-1|\exp(-\tfrac{1}{2}\beta_k x^2)) ,$$

where $\beta_k = 2 - a_k > 0$ as soon as $a_k < 2$. Thus, for k sufficiently large,

$$\sup_x |\Phi(a_k x)-\Phi(x)| \leq c' \sup_x \Phi'(x)|(a_k-1)x| .$$

By taking c' sufficiently large, this inequality can be made valid for all k.

Let $a_k = \sigma_k^{-1}$. Then

$$\sum_n n^{-\alpha} \sup_x |E\Phi(x/\sigma_{\tau_n}) - \Phi(x)| \le \sum_n n^{-\alpha} E(\sup_x |\Phi(x/\sigma_{\tau_n}) - \Phi(x)|)$$

$$= \sum_n n^{-\alpha} \sum_n P[\tau_n = k] \sup_x |\Phi(x/\sigma_k) - \Phi(x)|$$

$$\le c \cdot \sum_n n^{-\alpha} \sum_k P[\tau_n = k] \cdot \sup_x \{\Phi'(x) \cdot |x/\sigma_k - x|\}$$

$$= c \cdot \sum_n n^{-\alpha} \sum_k P[\tau_n = k] \cdot \sup_x \{|x\Phi'(x)| \cdot | \frac{1-\sigma_k}{\sigma_k} |\} .$$

Now, $|x\Phi'(x)|$ is bounded, so

$$\sum_n n^{-\alpha} \sup_x |E\Phi(x/\sigma_{\tau_n}) - \Phi(x)| \le c' \sum_n n^{-\alpha} \sum_k P[\tau_n = k] \cdot | \frac{1-\sigma_k}{\sigma_k} |$$

$$= c' \sum_n n^{-\alpha} \sum_k P[\tau_n = k] \cdot \frac{|1-\sigma_k|^2}{\sigma_k(1+\sigma_k)}$$

$$\le c'' \sum_n n^{-\alpha} \sum_n P[\tau_n = k] \cdot |1-\sigma_k^2| ,$$

since σ_k and $(1+\sigma_k)$ are bounded away from zero. But then

$$A_4(\alpha) \le A_2(\alpha) + c'' \sum_n n^{-\alpha} \sum_k P[\tau_n = k] \cdot |1-\sigma_k^2| , \qquad (2)$$

and the finiteness of A_4 follows from the finiteness of the right-hand side of (2). Similar reasoning yields the result for $A_3(x,\alpha)$. ∎

Theorems giving sufficient conditions for the finiteness of A_1 or A_2 thus become theorems for A_3 or A_4 with the addition of condition 1 of the lemma. This fact will not usually be repeated in the following.

II. Convergence of the A_j's when the Sequences $\{\tau_n\}$ and $\{X_i\}$ are Independent.

In this section, we assume that $\{\tau_n\}$ is an independent sequence and is independent of $\{X_n\}$. Our basic theorem states, that in order for A_2 (or A_4) to be finite, it is sufficient that the probability that a given value k recurs in a sequence of τ_n's diminishes rapidly as n increases:

<u>Theorem II A.</u> Let $E|X_i|^{2+\delta} < \infty$, $0 \leq \delta < 1$. If

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} P[\tau_n = k] = O(k^{-1+\delta/2}) , \qquad (1)$$

then $A_2(1-\delta/2) < \infty$. If $\delta > 0$ or $\delta = 0$ but $EX_i^2 \log|X_i| < \infty$, then $A_4(1) < \infty$.

<u>Proof</u>:

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \sup_x |\bar{G}_n(x) - E\Phi(x/\sigma_{\tau_n})|$$

$$\leq \sum_{n=1}^{\infty} n^{-1+\delta/2} \sum_{k=1}^{\infty} P[\tau_n = k] \sup_x |\bar{F}_k(x) - \Phi(x/\sigma_k)|$$

$$= \sum_{k=1}^{\infty} \sup_x |\bar{F}_k(x) - \Phi(x/\sigma_k)| \cdot \sum_{n=1}^{\infty} n^{-1+\delta/2} P[\tau_n = k]$$

$$\leq c \cdot \sum_{k=1}^{\infty} k^{-1+\delta/2} \sup_x |\bar{F}_k(x) - \Phi(x/\sigma_k)| .$$

This final sum is finite by the F-K-K Theorem. If $EX_i^2 \log|X_i| < \infty$, then $A_4(1) < \infty$ by the second part of the F-K-K Theorem. ■

The condition $\sum_n n^{-1+\delta/2} P[\tau_n = k] = O(k^{-1+\delta/2})$ is a basic one. We first give an example in which it does not hold and the conclusion of Theorem A is false. This example provides information concerning the sharpness of Theorem A. Then we give examples of distributions for which the condition does hold.

Example II.1. Let $\{X_i\}$ be any sequence of non-normal, independent, identically-distributed random variables satisfying $E|X_i|^{2+\delta} < \infty$, $0 < \delta < 1$. Let

$$P[\tau_n = k] = \begin{cases} (\log n)^{-1} & \text{for } k = 1 \\ 1 - (\log n)^{-1} & \text{for } k = n , \end{cases}$$

and let x_0 be chosen so that $|\overline{F}_1(x_0) - \Phi(x_0)| > 0$. Then

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \sup_x |\overline{G}_n(x) - \Phi(x)| \geq \left| \sum_{n=1}^{\infty} n^{-1+\delta/2} \sum_{k=1}^{\infty} (\overline{F}_k(x_0) - \Phi(x_0)) \cdot P[\tau_n = k] \right| .$$

But
$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \sum_{k=1}^{\infty} (\overline{F}_k(x_0) - \Phi(x_0)) \cdot P[\tau_n = k]$$
$$= \sum_{n=1}^{\infty} \left\{ n^{-1+\delta/2} (\log n)^{-1} \cdot (\overline{F}_1(x_0) - \Phi(x_0)) + n^{-1+\delta/2} \left(\frac{\log n - 1}{\log n} \right) (\overline{F}_n(x_0) - \Phi(x_0)) \right\} .$$

By Liapounov's theorem, $|\overline{F}_n(x_0) - \Phi(x_0)| \leq c \cdot n^{-\delta/2}$. Thus, for large n, say $n > N_0$,

$$\left|\sum_{n=N_0}^{\infty} n^{-1+\delta/2} \sum_{k=1}^{\infty} (\overline{F}_k(x_0)-\Phi(x_0)) \cdot P[\tau_n = k]\right|$$

$$\geq \left|\sum_{n=N_0}^{\infty} \left\{n^{-1+\delta/2}(\log n)^{-1}(\overline{F}_1(x_0)-\Phi(x_0))-c\left(\frac{\log n - 1}{\log n}\right) n^{-1}\right\}\right|$$

$$= \left|\sum_{n=N_0}^{\infty} n^{-1}\left\{c' \frac{n^{\delta/2}}{\log n} - c''\right\}\right| ,$$

which diverges faster than $\sum_n n^{-1+\gamma}$ for some $\gamma > 0$.

Ibragimov[11] has shown that, for random variables with characteristic function $f(t)$ such that $\overline{\lim}_{t\to\infty} |f(t)| < 1$, the condition

$$\sup_x |\overline{F}_n(x)-\Phi(x)| = O(n^{-(k+\delta/2)}) , \quad k = 1,2,\ldots,$$

$0 < \delta < 1$, is equivalent to the joint conditions

(i) $\int_{|x|>z} |x|^{k+2} dF = O(z^{-\delta})$, $z \to \infty$; and

(ii) the moments μ_s of X_i up to order $s = k+2$, inclusive, coincide with the corresponding moments of the normal distribution.

Thus, no condition of the form $\sum_n n^{-1+\delta/2} P[\tau_n=k] = O(k^\alpha)$, $\alpha > 0$, is necessary for $A_4 < \infty$; while the sufficiency of conditions weaker than (1) of Theorem A depends on the specific distribution of X_i.

Some familiar distributions for $\{\tau_n\}$ which satisfy the hypotheses of Theorem A are the following:

<u>Example II.2.</u> Let τ_n be Poisson, with parameter n. Here

$$P[\tau_n = k] = e^{-n}\frac{n^k}{k!} .$$

Then

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} P[\tau_n = k] = \frac{1}{k!}\sum_{n=1}^{\infty} n^{-1+\delta/2} n^k e^{-n}$$

$$= \frac{1}{k!}\sum_{n=1}^{\infty} n^{k-1+\delta/2} e^{-n}$$

$$= \frac{1}{k!e}\sum_{n=1}^{\infty} n^{k-1+\delta/2} e^{-n} .$$

The summand $n^{k-1+\delta/2}e^{-n}$ is everywhere positive and monotone decreasing for $n > 1$, so

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} P[\tau_n = k] \le \frac{1}{k!e} + \frac{1}{k!}\int_1^{\infty} t^{k-1+\delta/2} e^{-t} dt$$

$$\le \frac{1}{k!e} + \frac{1}{k!}\int_0^{\infty} t^{k-1+\delta/2} e^{-t} dt$$

$$= \frac{1}{k!e} + \frac{1}{k!}\Gamma(k+\delta/2)$$

$$= \frac{1}{k!e} + \frac{\Gamma(k+\delta/2)}{\Gamma(k+1)}$$

$$= O(k^{-1+\delta/2}) .$$ (See Ref. 19 for last equality.)

Example II.3. Let τ_n be binomial with parameters n and p, $0 < p < 1$. Then

$$P[\tau_n = k] = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}, \quad \text{for } 0 \leq k < n .$$

We have

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} P[\tau_n = k] = \sum_{n=k}^{\infty} n^{-1+\delta/2} P[\tau_n = k]$$

$$= \frac{p^k}{k!} \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} n^{-1+\delta/2} (1-p)^{n-k}$$

$$\leq k^{-1+\delta/2} \cdot (p^k/k!) \cdot \sum_{n=0}^{\infty} (n+k)(n+k-1) \cdots (n+1) \cdot (1-p)^n$$

$$= k^{-1+\delta/2} \cdot (p^k/k!) \cdot (k!/p^{k+1}) \quad \text{(Ref. 20)}$$

$$= O(k^{-1+\delta/2}) .$$

Let Θ be the class of real, non-increasing functions $\theta(t)$ such that for all $\varepsilon > 0$, $\sum_n n^{-1+\varepsilon}\theta(n) = \infty$, and let Ψ be the class of real, non-increasing functions $\psi(t)$ such that $\sum_{n=1}^{\infty} n^{-1}\psi(n) < \infty$.

Theorem II.B. If the RCLT conditions hold for $\{X_i\}$ and $\{\tau_n\}$, if $\sum_{n=1}^{\infty} n^{-1+\delta/2} E(\tau_n^{-\alpha}\psi(\tau_n)) < \infty$ for some $\psi \in \Psi \cap \Theta$, and an arbitrary α such that $\delta/2 \geq \alpha > 0$; and if $a_k = \sum_n n^{-1+\delta/2} P[\tau_n = k]$ is monotone decreasing for large k; then $A_2(1-\delta/2) < \infty$. If $\delta > 0$ or $\delta = 0$ and $EX_i^2 \log|X_i| < \infty$, then $A_4(1) < \infty$.

Proof:

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} E(\tau_n^{-\alpha} \psi(\tau_n)) = \sum_{n=1}^{\infty} n^{-1+\delta/2} \sum_{k=1}^{\infty} k^{-\alpha} \psi(k) \cdot P[\tau_n = k]$$

$$= \sum_{k=1}^{\infty} k^{-\alpha} \psi(k) \sum_{n=1}^{\infty} n^{-1+\delta/2} P[\tau_n = k] < \infty .$$

(The final sum is finite by hypothesis.)

Now, $\sum_{k=1}^{\infty} k^{-\alpha} \psi(k) a_k < \infty$ and a_k monotone imply[21] that

$k^{-\alpha} a_k = o(k^{-1})$, or

$$a_k = \sum_n n^{-1+\delta/2} P[\tau_n = k] = o(k^{-1+\alpha}) = o(k^{-1+\delta/2}) .$$

The result now follows from Theorem A. ∎

Another approach is the following:

Theorem II.C. Suppose the RCLT conditions hold for $\{X_i\}$ and $\{\tau_n\}$ with $\frac{\tau_n}{n} \xrightarrow{p} c_0$, a positive constant. Suppose further that there is a positive monotone sequence $\{\rho_n\}$, $\rho_n = o(\frac{1}{n})$, such that for all n and for some function $\psi \in \Psi$,

$$P[|\tau_n - nc_0| > n\rho_n] = O(n^{-1}\psi(n)) . \qquad (1)$$

Then $A_2(1-\delta/2) < \infty$. If, in addition, $EX_i^2 \log|X_i| < \infty$, then $A_4(1) < \infty$.

Proof: Fix k sufficiently large so that $\rho_k < c_0$ and let

$$K_1 = \left[\frac{k}{c_0+\rho_k}\right], \quad K_2 = \left[\frac{k}{c_0-\rho_k}\right],$$

where $\rho_k \leq \rho < c_0$. Here the symbol [a] denotes the largest integer not exceeding a. Then

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} P[\tau_n=k] = \left\{\sum_{n=1}^{K_1} + \sum_{n=K_1+1}^{K_2-1} + \sum_{n=K_2}^{\infty}\right\} (n^{-1+\delta/2}P[\tau_n=k])$$

$$\equiv I_1(k) + I_2(k) + I_3(k) .$$

We consider $I_2(k)$ first:

$$I_2(k) = \sum_{n=K_1+1}^{K_2-1} n^{-1+\delta/2}P[\tau_n=k] \leq (K_1+1)^{-1+\delta/2}) \cdot \left(2+ \frac{2k\rho_k}{c_0^2-\rho_k^2}\right)$$

$$= O(k^{-1+\delta/2}) , \quad \text{since} \quad \rho_k = O(\tfrac{1}{k}) .$$

$$I_3(k) = \sum_{n=K_2}^{\infty} n^{-1+\delta/2} P[\tau_n=k] \leq K_2^{-1+\delta/2} \sum_{n=K_2}^{\infty} P[\tau_n=k]$$

$$\leq K_2^{-1+\delta/2} \sum_{n=K_2}^{\infty} n^{-1}\psi(n)$$

$$= O(k^{-1+\delta/2}) , \quad \text{since} \quad \sum_{n=1}^{\infty} n^{-1}\psi(n) < \infty .$$

We next consider the term involving $I_1(k) = \sum_{n=1}^{K_1} n^{-1+\delta/2}P[\tau_n=k]$:

Let

$$J = \sum_{k=1}^{\infty} \sup_x |\overline{F}_k(x) - \Phi(x/\sigma_k)| \cdot I_1(k)$$

$$= \sum_{k=1}^{\infty} \sup_x |\overline{F}_k(x) - \Phi(x/\sigma_k)| \sum_{n=1}^{[k/(c_0+\rho_k)]} n^{-1+\delta/2} P(\tau_n = k] \ . \ .$$

For fixed n, let $\hat{n} = \min\{j / j \geq n(c_0+\rho_j)\}$, and note that $\hat{n}$ is well-defined because of the monotonicity of ρ_j. Then

$$J \leq \sum_{n=1}^{\infty} n^{-1+\delta/2} \sum_{k=\hat{n}}^{\infty} P[\tau_n = k] \sup_x |\overline{F}_k(x) - \Phi(x/\sigma_k)|$$

$$\leq \sum_{n=1}^{\infty} n^{-1+\delta/2} \sum_{k=\hat{n}}^{\infty} n^{-1} \psi(n) \sup_x |\overline{F}_k(x) - \Phi(x/\sigma_k)|$$

$$\leq \sum_{n=1}^{\infty} n^{-1+\delta/2} \sum_{k=\hat{n}}^{\infty} n^{-\delta/2} \psi(n) \left(\frac{k}{c_0+\rho_n}\right)^{-1+\delta/2} \sup_x |\overline{F}_k(x) - \Phi(x/\sigma_k)|$$

$$\leq c \cdot \sum_{k=1}^{\infty} k^{-1+\delta/2} \sup_x |\overline{F}_k(x) - \Phi(x/\sigma_k)| \cdot \sum_{n=1}^{\infty} n^{-1} \psi(n) < \infty \ ,$$

since both $\sum_n n^{-1} \psi(n)$ and $\sum_{k=1}^{\infty} k^{-1+\delta/2} \sup_x |\overline{F}_k(x) - \Phi(x/\sigma_k)|$ are finite. Thus,

$$A_2(1-\delta/2) = \sum_{k=1}^{\infty} \sup_x |\overline{F}_k(x) - \Phi(x/\sigma_k)| \cdot \{I_1(k) + I_2(k) + I_3(k)\}$$

$$\leq c' \cdot \sum_{k=1}^{\infty} k^{-1+\delta/2} \sup_x |\overline{F}_k(x) - \Phi(x/\sigma_k)| < \infty \ . \ \blacksquare$$

It may not always be obvious that a given sequence of stopping times $\{\tau_n\}$ satisfies the hypotheses of Theorems II.A, II.B, or II.C. In Theorem II.D below, a more easily recognizable class of random variables is treated.

Let $\{T_n\}$ be a sequence of non-negative, independent, identically distributed random variables, and let $\tau_n = \sum_1^n T_i$. Consider a population of individuals such that when any individual "dies" or "fails", it is replaced by a new individual; and successive individuals live and die independently of one another. If the lifetime of the <u>ith</u> individual is T_i, then τ_n represents the time of the <u>nth</u> replacement of a sequence of individuals. In such a situation, $\{T_i\}$ is called an ordinary renewal process[5], and τ_n is called the <u>nth</u> renewal epoch.

<u>Theorem II.D.</u> Suppose $\{X_i\}$ is a sequence of independent, identically-distributed random variables, $E|X_i|^{2+\delta} < \infty$ for some δ, $0 \leq \delta < 1$. Let $\{T_i\}$ be an integer-valued ordinary renewal process independent of the $\{X_i\}$, such that $P[T_i > 0] = 1$, $E|T_i|^{2+\delta} < \infty$ and $ET_i = m > 0$. Let τ_n be the <u>nth</u> renewal epoch. Then

(i) $\sup_x |\bar{G}_n(x) - E\Phi(x/\sigma_{\tau_n})| \to 0$ as $n \to \infty$.

(ii) $A_2(1-\delta/2) < \infty$.

If, in addition, $EX_i^2 \log|X_i| < \infty$, then $\sup_x |\bar{G}_n(x) - \Phi(x)| \to 0$ and $A_4(1) < \infty$.

Proof: Part (i) follows immediately from the strong law of large numbers and Anscombe's theorem, since the first implies $\frac{\tau_n}{n} \xrightarrow{p} m$.

(ii) Let $\eta(k) = \min\{n/\tau_n \geq k\}$. Then $P[\eta(k)>n-1] = P[\tau_n \leq k]$, since $P[T_i > 0] = 1$ implies that $P[\tau_n \leq k] = P[\tau_{n-1} < k]$.

Let $M = [\frac{\lambda k}{m}]$, where λ is chosen arbitrarily between zero and one. Note that

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} P[\tau_n=k] \leq \sum_{n=1}^{M-1} n^{-1+\delta/2} P[\tau_n \geq k] + \sum_{n=M}^{\infty} n^{-1+\delta/2} P[\tau_n \leq k]$$

$$\leq \sum_{n=1}^{M-1} n^{-1+\delta/2} P[\tau_n > k-1]$$

$$+ \sum_{n=M}^{\infty} n^{-1+\delta/2} P[\eta(k) > n-1]$$

$$\equiv I_1(k) + I_2(k) .$$

Chebysheff's inequality is applied to $I_1(k)$ to obtain the following inequality:

$$I_1(k) = \sum_{n=1}^{M-1} n^{-1+\delta/2} P[\tau_n - nm > k-1-nm]$$

$$\leq \sum_{n=1}^{M-1} n^{-1+\delta/2} \frac{nET_1^{2+\delta}}{(k-1-nm)^{2+\delta}} .$$

Write $\alpha = \delta/2$ and note that $M = \frac{\lambda k}{m^{\nu}}$, $\nu \geq 1$, and that for

$x > \frac{k-1}{2+\delta+m}$, $x^{\alpha}(k-1-mx)^{-2-\delta}$ is monotone decreasing. Thus,

$$I_1(k) \leq ET_1^{2+\delta} \sum_{n=1}^{M-1} n^{\alpha}(k-1-nm)^{-2-\delta} = ET_1^{2+\delta} \sum_{n=1}^{\ell_1} n^{\alpha}(k-1-nm)^{-2-\delta}$$

$$+ ET_1^{2+\delta} \sum_{n=\ell_1}^{M-1} n^{\alpha}(k-1-nm)^{-2-\delta} ,$$

where $\ell_1 = [\frac{k-1}{2+\delta+m}]$. It follows that

$$I_1(k) \leq ET_1^{2+\delta} \left(\frac{k-1}{2+\delta+m}\right)^{\alpha} \cdot \frac{k-1}{2+\delta+m} \{(k-1) \cdot (1- \frac{m}{2+\delta+m})\}^{-2-\delta}$$

$$+ ET_1^{2+\delta} \int_0^{\lambda k/m+1} x^{\alpha}(k-1-mx)^{-2-\delta}dx .$$

The first term in the above inequality is of order $O(k^{-1+\delta/2})$. Thus,

$$I_1(k) \leq O(k^{-1+\delta/2}) + ET_1^{2+\delta} \cdot \frac{k^{\alpha}}{m^{\alpha}k^{2+\delta}} \int_0^{\frac{\lambda k}{m}+1} \frac{m^{\alpha}(x/k)^{\alpha}}{(1-1/k-m(x/k))^{2+\delta}} dx$$

$$= O(k^{-1+\delta/2}) + ET_1^{2+\delta} m^{-\alpha}k^{-1-\alpha} \int_0^{\frac{\lambda k+m}{k}} y^{\alpha}(1-\frac{1}{k}-y)^{-2-\delta}dy ,$$

by the substitution of $y = \frac{mx}{k}$. It follows that

$$I_1(k) \leq O(k^{-1+\delta/2}) + ET_1^{2+\delta}\, m^{-\alpha} k^{-1-\alpha} \cdot c'(\frac{\lambda k+m}{k})^{\alpha+1} \int_0^{\frac{\lambda k+m}{k}} \frac{dy}{(1-\frac{1}{k}-y)^{2+\delta}}$$

$$= O(k^{-1+\delta/2}) + c'' k^{-1-\alpha} \{(1-\frac{1}{k}-y)^{-1-\delta}\}_{y=0}^{y=\frac{\lambda k+m}{k}}$$

$$= O(k^{-1+\delta/2}) + c'' k^{-1-\alpha} \{(\frac{k(1-\lambda)+m-1}{k})^{-1-\delta} - (\frac{k-1}{k})^{-1-\delta}\}$$

$$= O(k^{-1+\delta/2}) .$$

By Stein's theorem[25], there exists p, $0 < p < 1$, such that $P[\eta(k) \geq n] \leq p^n$. Thus

$$I_2(k) = \sum_{n=M}^{\infty} n^{-1+\delta/2}\, P[\eta(k) \geq n-1]$$

$$\leq (\frac{\lambda k}{m})^{-1-\delta/2} \sum_{n=M}^{\infty} p^{n-1} \leq c \cdot (\frac{\lambda}{m})^{-1+\delta/2} k^{-1+\delta/2}$$

$$= O(k^{-1+\delta/2}) .$$

Therefore,

$$\sum_{n=1}^{\infty} n^{-1+\delta/2}\, P[\tau_n = k] \leq I_1(k) + I_2(k) = O(k^{-1+\delta/2}) ,$$

and the result follows from Theorem II.A. ∎

We note that in the preceding proof, the positivity of $\{T_i\}$ was required in order to use the event $[\eta(k) \geq n-1]$ and its known properties.

Theorems II.C and II.D are related to the following theorem of Baum and Katz[2]:

<u>Theorem</u>: For $t > 1$, $r > 1$, $1/2 < r/t \leq 1$, the following statements are equivalent:

(a) $E|X_1|^t < \infty$ and $EX_1 = \mu$.

(b) $\sum_{n=1}^{\infty} n^{r-2} P[|S(n)-n\mu| > n^{r/t}\epsilon] < \infty$ for all $\epsilon > 0$.

For $r = t = 2$, statement (b) of Katz's theorem is similar to condition (1) of II.C, with τ_n replaced by a sum (see II.D), and with the monotone sequence p_n replaced by an arbitrary $\epsilon > 0$. A condition on $\{\tau_n\}$ which required that it "resemble", in a suitable way, a sum of random variables might allow the sequence $\{p_n\}$ in II.C to be replaced by any $\epsilon > 0$.

We close this section by considering analogues for random sums to the theorems of Berry and Esseen, and Nagaev. For an independent sequence of non-negative integer-valued random variables $\{\tau_n\}$ with finite variances ν_n and means a_n, Orazov and Hudai-Verenov[21] have shown that, if $E|X_1|^3 < \infty$ and $\frac{\tau_n}{n} \xrightarrow{P} c > 0$, then

$$\sup_x |\bar{G}_n(x)-\Phi(x)| \leq B_0 E|X_1|^3 \max(a_n^{-1/2}, a_n^{-2\Delta_n}),$$

where $\Delta_n = 1 - \frac{\log \nu_n}{\log a_n}$, and B_0 is the Berry Esseen constant. A similar result is the following:

Theorem II.E. If $E|X_1|^3 < \infty$ and the RCLT conditions hold for $\{\tau_n\}$ and $\{X_i\}$, then

$$|\bar{G}_n(x)-\Phi(x)| \leq \frac{B_0 E|X_1|^3}{1+|x|^3} \cdot E\left(\frac{1}{\sqrt{\tau_n}}\right).$$

Also,

$$\sup_x |\bar{G}_n(x)-\Phi(x)| \leq B_0 E|X_1|^3 E\left(\frac{1}{\sqrt{\tau_n}}\right).$$

Proof:

$$|\bar{G}_n(x)-\Phi(x)| \leq \sum_{k=1}^{\infty} |\bar{F}_k(x)-\Phi(x)| \cdot P[\tau_n=k]$$

$$\leq (1+|x|^3)^{-1} B_0 E|X_1|^3 \cdot \sum_{k=1}^{\infty} k^{-1/2} P[\tau_n=k]$$

(by Nagaev's theorem)

$$= B_0 E|X_1|^3 (1+|x|^3)^{-1} \cdot E\left(\frac{1}{\sqrt{\tau_n}}\right). \quad \blacksquare$$

Note that the theorem of Orazov and Hudai-Verenov deals with moments of τ_n, while Theorem II.E is expressed in terms of moments of $\frac{1}{\tau_n}$.

III. Convergence of the A_j's when the Sequences $\{\tau_n\}$ and $\{X_i\}$ are Dependent.

The case where $\{\tau_n\}$ is no longer required to be independent of $\{X_i\}$ can involve deeper complications. It will no longer suffice to give conditions based solely on $\{\tau_n\}$. Rather, the conditions must be stated in terms of the interaction of $\{\tau_n\}$ and $\{X_i\}$.

Suppose τ_n depends only on $X_1,\ldots,X_n$. Such a random variable is called a stopping time for the sequence $\{S(n)\}$. It follows from a Theorem of Renyi[22] on mixing sequences of random variable that for each x and every $\epsilon > 0$, there is a K such that for $k > K$,

$$|P([\overline{S}(k) \leq x] \cap [\tau_n = k]) - P[\overline{S}(k) \leq x]\cdot P[\tau_n = k]| \leq \epsilon \cdot P[\tau_n = k]. \quad (1)$$

For future reference, let

$$D_{n,k}(x) = |P([\overline{S}(k) \leq x] \cap [\tau_n = k]) - P[\overline{S}(k) \leq x]\cdot P[\tau_n = k]|.$$

The phenomenon exhibited in (1), the asymptotic independence of $\overline{S}(k)$ from each partial summand, suggests splitting the sums whose convergence we wish to investigate into an "independent" part and an "asymptotically independent" part. Sufficient conditions will be stated in terms of the rate with which $D_{n,k}(x)$ goes to zero.

<u>Theorem III.A.</u> Let $\{X_i\}$, $\{\tau_n\}$ satisfy the RCLT conditions. If

$$\text{(i)} \qquad \sum_{n=1}^{\infty} n^{-1+\delta/2} P[\tau_n = k] = O(k^{-1+\delta/2}) ,$$

and

$$\text{(ii)} \qquad \sum_{\{k|P[\tau_n=k]>0\}} D_{n,k}(x) = O(n^{-\delta/2}\psi(n))$$

for some $\psi \in \Psi$, then $A_1(x,1-\delta/2) < \infty$. If, in addition, $EX_1^2 \log|X_1| < \infty$, then $A_3(x,1) < \infty$.

Remark: Condition (ii) will certainly be true if

$$|P[\overline{S}(k) \leq x|\tau_n=k] - P[\overline{S}(k) \leq x]| = O(n^{-\delta/2}\psi(n)) ,$$

for all sufficiently large values of k for which $P[\tau_n = k] > 0$.

<u>Proof:</u> We proceed to split the sum A, as described above:

$$A_1(x,1-\delta/2) = \sum_{n=1}^{\infty} n^{-1+\delta/2} |P[\overline{S}(\tau_n) \leq x] - E\Phi(x/\sigma_{\tau_n})|$$

$$\leq \sum_{n=1}^{\infty} n^{-1+\delta/2} \sum_{\{k|P[\tau_n=k]>0\}} |P([\overline{S}(k) \leq x] \cap [\tau_n=k])$$

$$- P[\overline{S}(k) \leq x] \cdot P[\tau_n=k]|$$

$$+ \sum_{n=1}^{\infty} n^{-1+\delta/2} \sum_{k=1}^{\infty} P[\tau_n=k] \cdot |P[\overline{S}(k) \leq x] - \Phi(x/\sigma_k)|$$

$$\equiv I_1(x) + I_2(x) .$$

Now

$$I_2(x) = \sum_{k=1}^{\infty} |P[\bar{S}(k) \leq x]-\Phi(x/\sigma_k)| \cdot \sum_{n=1}^{\infty} n^{-1+\delta/2} P[\tau_n = k]$$

$$\leq c \cdot \sum_{k=1}^{\infty} k^{-1+\delta/2}|P[\bar{S}(k) \leq x]-\Phi(x/\sigma_k)| ,$$

which is finite by the F-K-K Theorem. But $I_1(x)$ is just

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \sum_{\{k|P[\tau_n=k]>0\}} D_{n,k}(x) ,$$

so

$$I_1(x) \leq c \cdot \sum_{n=1}^{\infty} n^{-1}\psi(n) < \infty .$$

Thus

$$A_1(x,1-\delta/2) \leq I_1(x)+I_2(x) < \infty .$$

The assertion about $A_3(x,1)$ follows from II.A. ∎

It is not a hypothesis of Theorem III.A. that τ_n depend only on $X_1,\ldots,X_n$. Referring to (1), one can see that the rate of convergence of $D_{n,k}$ to zero can be dominated either by rapidly decreasing values of $P[\tau_n=k]$, or by bounding $D_{n,k}$ by an expression like $\epsilon_{n,k}P[\tau_n=k]$, where $\epsilon_{n,k}\to 0$ with suitable speed. Thus, if τ_n does in fact depend only on $X_1,\ldots,X_n$, we need only a slightly more stringent condition on $\{\tau_n\}$ to satisfy (ii) of Theorem III.A. The following example shows, however, that simple dependence on $X_1,\ldots,X_n$ is not enough.

Example III.1. Let $X_i \geq 0$ be integer-valued, $EX_i \geq 1$, $EX_i^{2+\delta} < \infty$, $\delta > 0$, and let $\tau_n = S(n)$. Then

$$A_3(x,1) = \sum_{n=1}^{\infty} n^{-1} |P[\overline{S}(\tau_n) \leq x] - \Phi(x)|$$

$$= \sum_{n=1}^{\infty} n^{-1} \sum_{k=1}^{\infty} |P([\overline{S}(k) \leq x] \cap [\tau_n = k] - P[\tau_n = k] \cdot \Phi(x)|$$

$$= \sum_{n=1}^{\infty} n^{-1} \sum_{k=1}^{\infty} |P([\overline{S}(k) \leq x] \cap [S(n)=k]) - P[S(n)=k] \cdot \Phi(x)|$$

$$= \sum_{n=1}^{\infty} n^{-1} \Big\{ \sum_{k=1}^{n-1} |P([S(n-k) \leq x\sqrt{k} + k(\mu-1)] \cap [S(n)=k]) - P[S(n)=k] \cdot \Phi(x)|$$

$$+ \sum_{k=n}^{\infty} |P([S(k-n) \leq x\sqrt{k} + k(\mu-1)] \cap [S(n)=k]) - P[S(n)=k] \cdot \Phi(x) \Big\}$$

$$\geq \sum_{n=1}^{\infty} n^{-1} \sum_{k=n}^{\infty} P[S(n)=k] \cdot |P[S(k-n) \leq x\sqrt{k} + k(\mu-1)] - \Phi(x)| .$$

Now, $P[S(k-n) \leq x\sqrt{k} + k(\mu-1)] \to 0$ as $k \to \infty$ since

$$P[S(k-n) \leq x\sqrt{k} + k(\mu-1)] = P\left[\overline{S}(k-n) \leq x\sqrt{\frac{k}{k-n}} + \frac{n\mu-k}{\sqrt{k-n}}\right] .$$

Let the quantity $a(n,x)-1$ be the last integer for which $|P[S(k-n) \leq x\sqrt{k} + k(\mu-1)]-\Phi(x)|$ is less than some constant d where $\Phi(x) \geq d > 0$. Then there is a constant $c_1 > 0$ for which .

$$A_3(x,1) \geq c_1\Phi(x) \sum_{n=1}^{\infty} n^{-1} \sum_{k=a(n,x)}^{\infty} P[S(n)=k]$$

$$= c_1\Phi(x) \sum_{n=1}^{\infty} n^{-1} P[S(n) \geq a(n,x)] .$$

If $a(n,x) \leq \mu n + c$, $c > 0$, then $P[S(n) \geq a(n,x)] \to 1$, and $A_3(x,1) = \infty$. To be more specific, then, let

$$P[X_i=m] = \begin{cases} \frac{1}{2} & \text{for } m = 2 \\ \frac{1}{2} & \text{for } m = 0 \\ 0 & \text{otherwise .} \end{cases}$$

Here, $\mu = 1$. Consider $A_3(0,1)$ and note first that $a(n,x)-1$ is the largest integer for which $|P[S(k-n) \leq 0] - 1/2|$ is less than some constant d, $1/2 \geq d > 0$. Now $P[S(k-n) \leq 0] = (1/2)^{k-n}$, so $a(n,x) = n+2$. From the above, $A_3(0,1) = \infty$. Notice that the sequences $\{X_i\}$, $\{\tau_n\}$ satisfy the RCLT conditions and condition (i) of Theorem III.A. For let $k = 2\ell$ (otherwise $P[S(n)=k]=0$). Then

Then, $e \neq 1$. Consider [illegible]

the largest integer [illegible]

some divisors d_i [illegible]

$v(d_i) = m_i$. Then [illegible]

Lemma [illegible] (3.1) [illegible]

of Theorem III.5, [illegible]

$$\sum_{n=1}^{\infty} n^{-1} P[S(n)=k] = \sum_{n=\ell}^{\infty} n^{-1}\binom{n}{\ell}\left(\tfrac{1}{2}\right)^n \leq \left(\tfrac{k}{2}\right)^{-1} \sum_{n=\ell}^{\infty} \binom{n}{\ell}\left(\tfrac{1}{2}\right)^n$$

$$= \left(\tfrac{k}{2}\right)^{-1} \frac{1}{\ell!} \sum_{n=0}^{\infty} (n+\ell(n+\ell-1) \cdots (n-1)\left(\tfrac{1}{2}\right)^{n+\ell}$$

$$\leq \left(\tfrac{k}{2}\right)^{-1} \frac{1}{\ell!} \frac{\ell!}{\left(\tfrac{1}{2}\right)^{\ell+1}} \left(\tfrac{1}{2}\right)^{\ell} \quad \text{(Ref. 20)}$$

$$= O(k^{-1}) .$$

For future convenience, let $H_{n,k} = [\tau_n = k]$. Now consider the case where $[\overline{S}(\tau_n) \leq x]$ and $[\overline{S}(n) \leq x]$ have, in some sense, the same asymptotic degree of dependence on $H_{n,k}$. One of the problems that must be faced is the question of convergence of

$$R \equiv \sum_{n=1}^{\infty} n^{-1+\delta/2} \sup_x |\Phi(x/\sigma_n) - E\Phi(x/\sigma_{\tau_n})| . \tag{2}$$

Recall that we assume $\sigma_i \neq 0$ for all $i > 0$.

<u>Lemma:</u> Let $\{X_i\}$, $\{\tau_n\}$ satisfy the RCLT conditions. If, for some $\psi \in \Psi$,

$$\sum_{k=1}^{\infty} P(H_{n,k}) |\sigma_k^2 - \sigma_n^2| = O(n^{-\delta/2} \psi(n)) , \tag{3}$$

then R is finite.

Proof:

$$R \leq \sum_{n=1}^{\infty} n^{-1+\delta/2} \sum_{k=1}^{\infty} P(H_{n,k}) \sup_x |\Phi(x/\sigma_n)-\Phi(x/\sigma_k)|$$

$$\leq \sum_{n=1}^{\infty} n^{-1+\delta/2} \sum_{k=1}^{\infty} P(H_{n,k}) \sup_x \{\Phi'(x/\sigma_n)\cdot(1+\Theta_{n,k})\cdot \left|\frac{x}{\sigma_n} - \frac{x}{\sigma_k}\right|\} ,$$

as in the lemma of section I. Here, $\Theta_{n,k} \to 0$ as $k \to \infty$ for every n. Thus

$$R \leq c\cdot \sum_{n=1}^{\infty} n^{-1+\delta/2} \sum_{k=1}^{\infty} P(H_{n,k}) \sup_x |\Phi'(x/\sigma_n)\cdot \frac{x}{\sigma_n} \cdot \frac{\sigma_k-\sigma_n}{\sigma_k}|$$

$$\leq c'\cdot \sum_{n=1}^{\infty} n^{-1+\delta/2} \sum_{k=1}^{\infty} P(H_{n,k}) \left| \frac{\sigma_k-\sigma_n}{\sigma_k} \right| ,$$

because $t\Phi'(t)$ is bounded. Since σ_k and σ_n are bounded and nonzero,

$$R \leq c''\cdot \sum_{n=1}^{\infty} n^{-1+\delta/2} \sum_{k=1}^{\infty} P(H_{n,k}) |\sigma_k^2-\sigma_n^2| < \infty \quad \text{(by hypothesis)} . \blacksquare$$

Theorem III.B. Let $\{X_i\}$, $\{\tau_n\}$ satisfy the RCLT conditions and condition (3) of the lemma. If there is a function $f(n,k)$ and a sequence $\{a_n\}$ such that

(i) $\sum_n a_n n^{-1+\delta/2} < \infty$,

(ii) $Ef(n,\tau_n) = \sum_k f(n,k)P(H_{n,k}) \leq c_0 a_n$ for each n,

when c_0 is a fixed positive constant; and

(iii) $|P([\bar{S}(k) \leq x] \cap H_{n,k}) - P([\bar{S}(n) \leq x] \cap H_{n,k})| \leq f(n,k)\cdot P(H_{n,k})$,

then $A_2(1-\delta/2) < \infty$. If, in addition, $EX_1^2 \log|X_1| < \infty$, then $A_4(1) < \infty$.

Proof:

$$A_2(1-\delta/2) = \sum_{n=1}^{\infty} n^{-1+\delta/2} \sup_x |P[\bar{S}(\tau_n) \le x] - E\Phi(x/\sigma_{\tau_n})|$$

$$\le \sum_{n=1}^{\infty} n^{-1+\delta/2} \Big\{ \sup_x |P[\bar{S}(\tau_n) \le x] - P[\bar{S}(n) \le x]|$$

$$+ \sup_x |P[\bar{S}(n) \le x] - \Phi(x/\sigma_n)| + \sup_x |\Phi(x/\sigma_n) - E\Phi(x/\sigma_{\tau_n})| \Big\}$$

$$\equiv I_1 + I_2 + I_3 .$$

$I_3 < \infty$ by the lemma, and $I_2 < \infty$ by the F-K-K Theorem.

$$I_1 \le \sum_{n=1}^{\infty} n^{-1+\delta/2} \sum_{\{k | P(H_{n,k}) > 0\}} P(H_{n,k}) \sup_x |P[\bar{S}(k) \le x] - P[\bar{S}(n) \le x | H_{n,k}]|$$

$$\le \sum_{n=1}^{\infty} n^{-1+\delta/2} \sum_k P(H_{n,k}) f(n,k) \le c \cdot \sum_{n=1}^{\infty} n^{-1+\delta/2} a_n < \infty .$$

Thus $A_2(1-\delta/2) \le I_1 + I_2 + I_3 < \infty$. ■

One of the classical stopping times of sequential analysis is the first time the sum $S(k)$ exceeds a given value; e.g.,

$$\tau_n = \min\{k | S(k) \ge n\} .$$

To see how this sequence $\{\tau_n\}$ can lead to overdependence of $[\bar{S}(k) \le x]$ on $H_{n,k}$ and thus to divergence of A_3, consider the following example.

Example III.2. Let X_i be zero with probability q, $1 > q > 0$, and one with probability $p, q = 1-p$. Let

$$\tau_1 = \min\{i | X_i = 1\} \ , \quad \tau_n = \tau_{n-1} + \min\{i | X_{\tau_{n-1}+i} = 1\} \ .$$

It follows that

$$A_3(0,1) = \sum_{n=1}^{\infty} n^{-1} \sum_{k=1}^{\infty} |P([S(k) \leq kp] \cap [\tau_n = k]) - \Phi(x) \cdot P[\tau_n = k]|$$

$$= \sum_{n=1}^{\infty} n^{-1} \sum_{k=1}^{\infty} |P([S(k) \leq kp] \cap [S(k-1)=n-1] \cap [X_k = 1]) - \Phi(x)$$

$$\cdot P([S(k-1)=n-1] \cap [X_k = 1])| \ ,$$

since $[\tau_n = k] = [S(k-1) = n-1] \cap [X_k = 1]$. Thus

$$A_3(0,1) = \sum_{n=1}^{\infty} n^{-1} \Big\{ \sum_{k=n}^{[n/p]} p\Phi(x) \binom{k-1}{n-1} p^{n-1} q^{k-n}$$

$$+ \sum_{k=[n/p]+1}^{\infty} |p \binom{k-1}{n-1} p^{n-1} q^{k-n} - \Phi(x) p \binom{k-1}{n-1} p^{n-1} q^{k-n}| \Big\}$$

$$= \Phi(x) \cdot \sum_{n=1}^{\infty} n^{-1} \Big\{ \sum_{k=n}^{[n/p]} \binom{k-1}{n-1} p^n q^{k-n} + \frac{1-\Phi(x)}{\Phi(x)} \sum_{k=[n/p]+1}^{\infty} \binom{k-1}{n-1} p^n q^{k-n} \Big\}$$

$$\geq c' \cdot \sum_{n=1}^{\infty} n^{-1} \sum_{k=n}^{\infty} \frac{n}{k} \binom{k}{n} p^n q^{k-n} = \sum_{k=1}^{\infty} k^{-1} \sum_{n=1}^{\infty} \binom{k}{n} p^n q^{k-n}$$

$$= \sum_{k=1}^{\infty} k^{-1} (1-q^k) = \infty \ .$$

The problem we have encountered in this example is that the sum $S(\tau_n)$ is completely specified. In example III.1, a similar situation occurred--namely, the value of τ_n was too closely tied to the value of the summand $S(n)$.

In the next example, we consider a sequence $\{\tau_n\}$ which, though dependent on $X_1,\ldots,X_n$, does not lead to a completely specified value of $S(\tau_n)$ and whose dependence on $S(n)$ is such that the hypothesis of Theorem III.A are satisfied.

<u>Example III.3</u>. Let $\{X_i\}$ be an independent sequence of identically-distributed random variables with mean μ and unit variance. Let ϵ_0 be a fixed positive constant. Define a sequence $\{\tau_n\}$ as follows:

$$\tau_n = \begin{cases} n \,, & \text{if } S(n) \le n(\mu+\epsilon_0) \\ 2n \,, & \text{if } S(n) > n(\mu+\epsilon_0) \,. \end{cases}$$

Note that the event $[\tau_n=k]$ is the same as the event $[S(n) \le n(\mu+\epsilon_0)]$ for $k = n$, and the same as $[S(n) > n(\mu+\epsilon_0)]$ for $k = 2n$. The sum $\sum_{\{k|P[\tau_n=k]>0\}} D_{n,k}(x)$ becomes, for $x = 0$, simply

$$\sum_{\{k|P[\tau_n=k]>0\}} D_{n,k}(0) \le \big|P([S(n)\le n\mu] \cap [S(n)\le n(\mu+\epsilon_0)]) - P[S(n)\le n\mu]\cdot P[S(n)\le n(\mu+\epsilon_0)]\big|$$

$$+ \quad \big|P([S(2n)\le 2n\mu] \cap [S(n) > n(\mu+\epsilon_0)]) - P[S(2n)\le 2n\mu]\cdot P[S(n) > n(\mu+\epsilon_0)]\big| \,.$$

Consider the first term:

$$|P([S(n) \le n\mu] \cap [S(n) \le n(\mu+\epsilon_0)] - P[S(n) \le n\mu] \cdot P[S(n) \le n(\mu+\epsilon_0)]|$$

$$= |P[S(n) \le n\mu] - P[S(n) \le n\mu] \cdot P[S(n) \le n(\mu+\epsilon_0)]|$$

$$= P[S(n) \le n\mu] \cdot (1 - P[S(n) \le n(\mu+\epsilon_0)])$$

$$= P[S(n) \le n\mu] \cdot P[S(n) > n(\mu+\epsilon_0)] \ .$$

Let $S^*(n) = S(2n) - S(n)$. Then, for the second term,

$$|P([S(2n) \le 2n\mu] \cap [S(n) > n(\mu+\epsilon_0)]) - P[S(2n) \le 2n\mu] \cdot P[S(n) > n(\mu+\epsilon_0)]|$$

$$= |P([S^*(n) \le 2n\mu - S(n)] \cap [S(n) > n(\mu+\epsilon_0)]) - P[S(2n) \le 2n\mu] \cdot P[S(n) > n(\mu+\epsilon_0)]| \ .$$

It is certainly the case that

$$[S^*(n) \le 2n\mu - S(n)] \cap [S(n) > n(\mu+\epsilon_0)] \subset [S^*(n) \le n(\mu-\epsilon_0)] \cap [S(n) > n(\mu+\epsilon_0)] \ .$$

Thus

$$P([S^*(n) \le 2n\mu - S(n)] \cap [S(n) > n(\mu+\epsilon_0)]) \le P([S^*(n) \le n(\mu-\epsilon_0)] \cap [S(n) > n(\mu+\epsilon_0)]) \ .$$

By the strong law of large numbers and the central limit theorem, for sufficiently large n, $P[S^*(n) \leq n(\mu-\epsilon_0)] < P[S(2n) \leq 2n\mu]$, since the former goes to zero with n, and the latter to one-half. Thus,

$$P([S^*(n) \leq n(\mu-\epsilon_0)] \cap [S(n) > n(\mu+\epsilon_0)]$$

$$= P[S^*(n) \leq n(\mu-\epsilon_0)] \cdot P[S(n) > n(\mu+\epsilon_0)]$$

$$< P[S(2n) \leq 2n\mu] \cdot P[S(n) > n(\mu+\epsilon_0)] .$$

But, for sufficiently large n, it follows that

$$|P([S^*(n) \leq 2n\mu-S(n)] \cap [S(n) > n(\mu+\epsilon_0)]$$

$$- P[S(2n) \leq 2n\mu] \cdot P[S(n) > n(\mu+\epsilon_0)]|$$

$$= P[S(2n) \leq 2n\mu] \cdot P[S(n) > n(\mu+\epsilon_0)]$$

$$- P([S^*(n) \leq 2n\mu-S(n)] \cap [S(n) > n(\mu+\epsilon_0)])$$

$$\leq P[S(2n) \leq 2n\mu] \cdot P[S(n) > n(\mu+\epsilon_0)] .$$

Hence

$$\sum D_{n,k}(x) \leq 2 \cdot P[S(n) > n(\mu+\epsilon_0)] .$$

If $E|X_i|^{2+\delta} < \infty$, $\delta \geq 0$, then Chebysheff's inequality yields

$$\sum D_{n,k}(x) \leq 2 \cdot P[S(n) > n(\mu+\epsilon_0)|$$

$$\leq \frac{2 \cdot n \cdot E|X_1|^{2+\delta}}{\epsilon_0^{2+\delta}\, n^{2+\delta}} = o(n^{-\delta/2}\psi(n)) ,$$

and condition (ii) of Theorem III.A is satisfied.

Condition (i) of Theorem III.A is easily verified, since

$$\sum_{n=1}^{\infty} n^{-1+\delta/2}\, P[\tau_n = k] = p_1 k^{-1+\delta/2} + p_2 \left(\frac{k}{2}\right)^{-1+\delta/2} ,$$

where p_1 and p_2 are the probabilities that $\tau_n = n$ and $2n$, respectively. We also note that $\{X_i\}$ and $\{\tau_n\}$ satisfy the RCLT conditions, since $\frac{\tau_n}{n} \xrightarrow{p} 1$ by the weak law of large numbers. Thus, by Theorem III.A, $A_1(0,1-\delta/2) < \infty$.

This same example serves as an illustration of Theorem III.B. Referring to the theorem of Baum and Katz[2], recall that if $E|X_i|^{2+\delta} < \infty$, then

$$\sum_{n=1}^{\infty} n^{\delta} \cdot P[|S(n)-n\mu| > nc_0] < \infty \quad \text{for} \quad c_0 > 0 .$$

For condition (iii) of III.B, we have for $k = n$,

$$|P([S(n) \leq n\mu] \cap [S(n) \leq n(\mu+\epsilon_0)]) - P([S(n) \leq n\mu] \cap [S(n) \leq n(\mu+\epsilon_0)])| = 0 ,$$

and for $k = 2n$,

$$|P([S(2n) \leq 2n\mu] \cap [S(n) > n(\mu+\epsilon_0)]) - P([S(n) \leq n\mu] \cap [S(n) > n(\mu+\epsilon_0)])|$$

$$\leq P[S^{*}(n) \leq n(\mu-\epsilon_0)] \cdot P[S(n) > n(\mu+\epsilon_0)] .$$

For other values of k, $P[\tau_n=k] = 0$. Using the notation of III.B,

$$f(n,k) = 0 \quad \text{for} \quad k \neq 2n ,$$

$$f(n,k) = P[S^*(n) \leq n(\mu+\epsilon_0)] \quad \text{for} \quad k = 2n .$$

Thus

$$\sum_k f(n,k)P(H_{n,k}) \leq P[S^*(n) \leq n(\mu-\epsilon_0)] \cdot P[S(n) > n(\mu+\epsilon_0)] ,$$

and by the theorem of Baum and Katz,

$$\sum n^{-1+\delta/2} P[S(n) > n(\mu+\epsilon_0)] \cdot P[S^*(n) \leq n(\mu-\epsilon_0)] < \infty$$

with plenty to spare. The only condition of III.B not yet verified is the condition that

$$R = \sum_{k=1}^{\infty} P(H_{n,k}) \cdot |\sigma_k^2-\sigma_n^2| = o(n^{-\delta/2}\psi(n)) .$$

In this example,

$$R = |\sigma_{2n}^2-\sigma_n^2| \cdot P[S(n) > n(\mu+\epsilon_0)] .$$

Here $|\sigma_{2n}^2-\sigma_n^2|$ is bounded, and

$$P[S(n) > n(\mu+\epsilon_0)] = o(n^{-\delta/2}\psi(n))$$

by Theorem 3 and the lemma of the paper by Baum and Katz[2] . Thus the proof of Theorem III.B also yields $A_1(0,1-\delta/2) < \infty$.

We have seen examples where the sufficiency conditions of Theorems III.A and III.B do not hold, and the series in question diverge, and an example where they do hold. This indicates that these conditions are reasonably sharp. However, the search for interesting necessary conditions has been fruitless. The openness of the necessity question for independent $\{\tau_n\}$ suggests that no condition more stringent than

$$\sum_n n^{-1+\delta/2} P[\tau_n = k] < \infty$$

is likely to be, in general, necessary in the dependent case.

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