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AUXILIARY INFORMATION AND A PRIORI VALUES IN CONSTRUCTION OF IMPROVED ESTIMATORS

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AUXILIARY INFORMATION AND *A PRIORI* VALUES IN CONSTRUCTION OF IMPROVED ESTIMATORS

Estimators	PRE $(., S_y^2)$	
	Population I	*Population II*
s_y^2	*100*	*100*
t_1	*223.14*	*228.70*
t_2	*235.19*	*228.76*
t_r *(optimum)*	*305.66*	*232.90*
t_p *(optimum)*	*305.66*	*232.90*

PRE of different estimators of S_y^2 with respect to s_y^2

2007

Auxiliary Information and a priori Values in Construction of Improved Estimators

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In the front cover table the percent relative efficiency (PRE) of S_y^2, t_1, t_2, t_r (in optimum case) and t_p (in optimum case) are computed with respect to S_y^2.

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Contents

Preface

This volume is a collection of six papers on the use of auxiliary information and *a priori* values in construction of improved estimators. The work included here will be of immense application for researchers and students who employ auxiliary information in any form.

Below we discuss each paper:

1. **Ratio estimators in simple random sampling using information on auxiliary attribute.**

    Prior knowledge about population mean along with coefficient of variation of the population of an auxiliary variable is known to be very useful particularly when the ratio, product and regression estimators are used for estimation of population mean of a variable of interest. However, the fact that the known population proportion of an attribute also provides similar type of information has not drawn as much attention. In fact, such prior knowledge can also be very useful when a relation between the presence (or absence) of an attribute and the value of a variable, known as point biserial correlation, is observed. Taking into consideration the point biserial correlation between a variable and an attribute, Naik and Gupta (1996) defined ratio, product and regression estimators of population mean when the prior information of population proportion of units, possessing the same attribute is available. In the present paper, some ratio estimators for estimating the population mean of the variable under study, which make use of information regarding the population proportion possessing certain attribute are proposed. The expressions of bias and mean squared error (MSE) have been obtained. The results obtained have been illustrated numerically by taking some empirical populations considered in the literature.

2. **Ratio-Product type exponential estimator for estimating finite population mean using information on auxiliary attribute.**

It is common practice to use arithmetic mean while constructing estimators for estimating population mean. Mohanty and Pattanaik (1984) used geometric mean and harmonic mean, while constructing estimators for estimating population mean, using multi-auxiliary variables. They have shown that in case of multi-auxiliary variables, estimates based on geometric mean and harmonic means are less biased than Olkin's (1958) estimate based on arithmetic mean under certain conditions usually satisfied in practice. For improving the precision in estimating the unknown mean $\overline{Y}$ of a finite population by using the auxiliary variable x, which may be positively or negatively correlated with y with known $\overline{X}$; the single supplementary variable is used by Bahl and Tuteja (1991) for the exponential ratio and product type estimators. For estimating the population mean $\overline{Y}$ of the study variable y, following Bahl and Tuteja (1991), a ratio-product type exponential estimator has been proposed by using the known information of population proportion possessing an attribute (highly correlated with y) in simple random sampling. The proposed estimator has an improvement over mean per unit estimator, ratio and product type exponential estimators as well as Naik and Gupta (1996) estimators. The results have also been extended to the case of two-phase sampling.

3. **Improvement in estimating the population mean using exponential estimator in simple random sampling.**

Using known values of certain population parameter(s) including coefficient of variation, coefficient of variation, coefficient of kurtosis, correlation coefficient, several authors have suggested modified ratio estimators for estimating population mean $\overline{Y}$. In this paper, under simple random sampling without replacement (SRSWOR), authors have suggested improved exponential ratio-type estimator for estimating population mean using some known values of population parameter(s). An empirical study is carried out to show the properties of the proposed estimator.

4. **Almost unbiased exponential estimator for the finite population mean.**

Usual ratio and product estimators and also exponential ratio and product type estimators suggested by Bahl and Tuteja (1991) are biased. Biasedness of an estimator is disadvantageous in some applications. This encouraged many researchers including

Hartley and Ross (1954) and Singh and Singh (1992) to construct either estimator with reduced bias known as almost unbiased estimator or completely unbiased estimator. In this paper we have proposed an almost unbiased ratio and product type exponential estimator for the finite population mean $\overline{Y}$. It has been shown that Bahl and Tuteja (1991) ratio and product type exponential estimators are particular members of the proposed estimator. Empirical study is carried to demonstrate the superiority of the proposed estimator.

5. Almost unbiased ratio and product type estimator of finite population variance using the knowledge of kurtosis of an auxiliary variable in sample surveys.

In manufacturing industries and pharmaceutical laboratories sometimes researchers are interested in the variation of their product or yields. Using the knowledge of kurtosis of auxiliary variable Upadhyaya and Singh (1999) have suggested an estimator for population variance. In this paper following the approach of Singh and Singh (1993), we have suggested almost unbiased ratio and product type estimator for population variance.

6. A general family of estimators for estimating population variance using known value of some population parameter(s).

In this paper, a general family of estimators for estimating the population variance of the variable under study; using known values of certain population parameter(s) is proposed. It has been shown that some existing estimators in literature are particular member of the proposed class. An empirical study is caring out to illustrate the performance of the constructed estimator over other.

<div align="right">The Authors</div>

Ratio Estimators in Simple Random Sampling Using Information on Auxiliary Attribute

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Abstract

Some ratio estimators for estimating the population mean of the variable under study, which make use of information regarding the population proportion possessing certain attribute, are proposed. Under simple random sampling without replacement (SRSWOR) scheme, the expressions of bias and mean-squared error (MSE) up to the first order of approximation are derived. The results obtained have been illustrated numerically by taking some empirical population considered in the literature.

AMS Classification: **62D05.**

Key words: **Proportion, bias, MSE, ratio estimator.**

1. Introduction

The use of auxiliary information can increase the precision of an estimator when study variable y is highly correlated with auxiliary variable x. There exist situations when information is available in the form of attribute ϕ, which is highly correlated with y. For example

    a) Sex and height of the persons,

    b) Amount of milk produced and a particular breed of the cow,

    c) Amount of yield of wheat crop and a particular variety of wheat etc. (see Jhajj et al., [1]).

Consider a sample of size n drawn by SRSWOR from a population of size N. Let y_i and ϕ_i denote the observations on variable y and ϕ respectively for i^{th} unit $(i = 1,2,....N)$. Suppose there is a complete dichotomy in the population with respect to the presence or absence of an attribute, say ϕ, and it is assumed that attribute ϕ takes only the two values 0 and 1 according as

    ϕ_i = 1, if ith unit of the population possesses attribute ϕ

    = 0, otherwise.

Let $A = \sum_{i=1}^{N} \phi_i$ and $a = \sum_{i=1}^{n} \phi_i$ denote the total number of units in the population and sample respectively possessing attribute ϕ. Let $P = \dfrac{A}{N}$ and $p = \dfrac{a}{n}$ denote the proportion of units in the population and sample respectively possessing attribute ϕ.

Taking into consideration the point biserial correlation between a variable and an attribute, Naik and Gupta (1996) defined ratio estimator of population mean when the

prior information of population proportion of units, possessing the same attribute is available, as follows:

$$t_{NG} = \bar{y}\left(\frac{P}{p}\right) \qquad (1.1)$$

here $\bar{y}$ is the sample mean of variable of interest. The MSE of t_{NG} up to the first order of approximation is

$$MSE(t_{NG}) = \left(\frac{1-f}{n}\right)\left[S_y^2 + R_1^2 S_\phi^2 - 2R_1 S_{y\phi}\right] \qquad (1.2)$$

where $\quad f = \dfrac{n}{N}, \quad R_1 = \dfrac{\bar{Y}}{P}, \quad S_y^2 = \dfrac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2, \quad S_\phi^2 = \dfrac{1}{N-1}\sum_{i=1}^{N}(\phi_i - P)^2,$

$S_{y\phi} = \dfrac{1}{N-1}\sum_{i=1}^{N}(\phi_i - P)(y_i - \bar{Y}).$

In the present paper, some ratio estimators for estimating the population mean of the variable under study, which make use of information regarding the population proportion possessing certain attribute, are proposed. The expressions of bias and MSE have been obtained. The numerical illustrations have also been done by taking some empirical populations considered in the literature.

2. The suggested estimator

Following Ray and Singh (1981), we propose the following estimator

$$t_1 = \frac{\bar{y} + b_\phi(P-p)}{p}P = R^*P \qquad (2.1)$$

where $b_\phi = \dfrac{s_{y\phi}}{s_\phi^2}$, $R^* = \dfrac{\bar{y} + b_\phi(P-p)}{p}$, $s_\phi^2 = \left(\dfrac{1}{n-1}\right)\sum_{i=1}^{n}(\phi_i - p)^2$ and

$s_{y\phi} = \left(\dfrac{1}{n-1}\right)\sum_{i=1}^{n}(\phi_i - p)(y_i - \bar{Y}).$

Remark 1: When we put $b_\phi = 0$ in (2.1), the proposed estimator turns to the Naik and

Gupta (1996) ratio estimator t_{NG} given in (1.1).

MSE of this estimator can be found by using Taylor series expansion given by

$$f(p,\overline{y}) \cong f(P,\overline{y}) + \frac{\partial f(c,d)}{\partial c}\bigg|_{P,\overline{Y}}(p-P) + \frac{\partial f(c,d)}{\partial c}\bigg|_{P,\overline{Y}}(\overline{y}-\overline{Y}) \qquad (2.2)$$

where $f(p,\overline{y}) = R^*$ and $f(P,\overline{Y}) = R_1$.

Expression (2.2) can be applied to the proposed estimator in order to obtain MSE

equation as follows:

$$R^* - R_1 \cong \frac{\partial\left(\left(\overline{y}+b_\phi(P-p)\right)\right)/p}{\partial p}\bigg|_{P,\overline{Y}}(p-P) + \frac{\partial\left(\left(\overline{y}+b_\phi(P-p)\right)\right)/p}{\partial \overline{y}}\bigg|_{P,\overline{Y}}(\overline{y}-\overline{Y})$$

$$\cong -\left(\frac{\overline{y}}{p^2} + \frac{b_\phi P}{p^2}\right)\bigg|_{P,\overline{Y}}(p-P) + \frac{1}{p}\bigg|_{P,\overline{Y}}(\overline{y}-\overline{Y})$$

$$E(R^* - R_1)^2 \cong \frac{(\overline{Y}+B_\phi P)^2}{P^4}V(p) - \frac{2(\overline{Y}+B_\phi P)}{P^3}Cov(p,\overline{y}) + \frac{1}{P^2}V(\overline{y})$$

$$\cong \frac{1}{P^2}\left\{\frac{(\overline{Y}+B_\phi P)^2}{P^2}V(p) - \frac{2(\overline{Y}+B_\phi P)}{P}Cov(p,\overline{y}) + V(\overline{y})\right\} \qquad (2.3)$$

where $B_\phi = \dfrac{S_{\phi y}}{S_\phi^2} = \dfrac{\rho_{pb}S_y}{S_\phi}$.

$\rho_{pb} = \dfrac{S_{y\phi}}{S_y S_\phi}$, is the point biserial correlation coefficient.

Now,

$$MSE(t_1) = P^2 E(R_1 - R_\phi)^2$$

$$\cong \frac{(\overline{Y}+B_\phi P)^2}{P^2}V(p) - \frac{2(\overline{Y}+B_\phi P)}{P}Cov(p,\overline{y}) + V(\overline{y}) \qquad (2.4)$$

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Simplifying (2.4), we get MSE of t_1 as

$$MSE(t_1) \cong \left(\frac{1-f}{n}\right)\left[R_1^2 S_\phi^2 + S_y^2\left(1 - \rho_{pb}^2\right)\right] \tag{2.5}$$

Several authors have used prior value of certain population parameters (s) to find more precise estimates. Searls (1964) used Coefficient of Variation (CV) of study character at estimation stage. In practice this CV is seldom known. Motivated by Searls (1964) work, Sen (1978), Sisodiya and Dwivedi (1981), and Upadhyaya and Singh (1984) used the known CV of the auxiliary character for estimating population mean of a study character in ratio method of estimation. The use of prior value of Coefficient of Kurtosis in estimating the population variance of study character y was first made by Singh et al. (1973). Later, used by and Searls and Intarapanich (1990), Upadhyaya and Singh (1999), Singh (2003) and Singh et al. (2004) in the estimation of population mean of study character. Recently Singh and Tailor (2003) proposed a modified ratio estimator by using the known value of correlation coefficient.

In next section, we propose some ratio estimators for estimating the population mean of the variable under study using known parameters of the attribute ϕ such as coefficient of variation C_p, Kurtosis $(\beta_2(\phi))$ and point biserial correlation coefficient ρ_{pb}.

3. Suggested Estimators

We suggest following estimator

$$t = \frac{\bar{y} + b_\phi(P - p)}{(m_1 p + m_2)}(m_1 P + m_2) \tag{3.1}$$

where $m_1(\neq 0)$, m_2 are either real number or the functions of the known parameters of the attribute such as C_p, $(\beta_2(\phi))$ and ρ_{pb}.

The following scheme presents some of the important estimators of the population mean, which can be obtained by suitable choice of constants m_1 and m_2:

Estimator	Values of	
	m_1	m_2
$t_1 = \dfrac{\bar{y} + b_\phi(P-p)}{p}P$	1	1
$t_2 = \dfrac{\bar{y} + b_\phi(P-p)}{(p+\beta_2(\phi))}[P+\beta_2(\phi)]$	1	$\beta_2(\phi)$
$t_3 = \dfrac{\bar{y} + b_\phi(P-p)}{(p+C_p)}[P+C_p]$	1	C_p
$t_4 = \dfrac{\bar{y} + b_\phi(P-p)}{(p+\rho_{pb})}[P+\rho_{pb}]$	1	ρ_{pb}
$t_5 = \dfrac{\bar{y} + b_\phi(P-p)}{(p\beta_2(\phi)+C_p)}[P\beta_2(\phi)+C_p]$	$\beta_2(\phi)$	C_p
$t_6 = \dfrac{\bar{y} + b_\phi(P-p)}{(pC_p+\beta_2(\phi))}[PC_p+\beta_2(\phi)]$	C_p	$\beta_2(\phi)$
$t_7 = \dfrac{\bar{y} + b_\phi(P-p)}{(pC_p+\rho_{pb})}[PC_p+\rho_{pb}]$	C_p	ρ_{pb}
$t_8 = \dfrac{\bar{y} + b_\phi(P-p)}{(p\rho_{pb}+C_p)}[P\rho_{pb}+C_p]$	ρ_{pb}	C_p
$t_9 = \dfrac{\bar{y} + b_\phi(P-p)}{(p\beta_2(\phi)+\rho_{pb})}[P\beta_2(\phi)+\rho_{pb}]$	$\beta_2(\phi)$	ρ_{pb}
$t_{10} = \dfrac{\bar{y} + b_\phi(P-p)}{(p\rho_{pb}+\beta_2(\phi))}[P\rho_{pb}+\beta_2(\phi)]$	ρ_{pb}	$\beta_2(\phi)$

Following the approach of section 2, we obtain the MSE expression for these proposed estimators as –

$$MSE(t_i) \cong \left(\frac{1-f}{n}\right)\left[R_i S_\phi^2 + S_y^2(1-\rho_{pb}^2)\right], \qquad (i = 1,2,3,....,10)$$ (3.2)

where $R_1 = \dfrac{\overline{Y}}{P}$, $R_2 = \dfrac{\overline{Y}}{P + \beta_2(\phi)}$, $R_3 = \dfrac{\overline{Y}}{P + C_p}$, $R_4 = \dfrac{\overline{Y}}{P + \rho_{pb}}$,

$R_5 = \dfrac{\overline{Y}\beta_2(\phi)}{P\beta_2(\phi) + C_p}$, $R_6 = \dfrac{\overline{Y}C_p}{PC_p + \beta_2(\phi)}$, $R_7 = \dfrac{\overline{Y}C_p}{PC_p + \rho_{pb}}$, $R_8 = \dfrac{\overline{Y}\rho_{pb}}{P\rho_{pb} + C_p}$,

$R_9 = \dfrac{\overline{Y}\beta_2(\phi)}{P\beta_2(\phi) + \rho_{pb}}$, $R_{10} = \dfrac{\overline{Y}\rho_{pb}}{P\rho_{pb} + \beta_2(\phi)}$.

4. Efficiency comparisons

It is well known that under simple random sampling without replacement (SRSWOR) the variance of the sample mean is

$$V(\overline{y}) = \left(\frac{1-f}{n}\right)S_y^2$$

(4.1)

From (4.1) and (3.2), we have

$$V(\overline{y}) - MSE(t_i) \geq 0, \qquad i = 1,2,.....,10$$

$$\Rightarrow \rho_{pb}^2 > \frac{S_\phi^2}{S_y^2}R_i^2$$

(4.2)

When this condition is satisfied, proposed estimators are more efficient than the sample mean.

Now, we compare the MSE of the proposed estimators with the MSE of Naik and Gupta [2] estimator t_{NG}. From (3.2) and (1.1) we have

$$MSE(t_{NG}) - MSE(t_i) \geq 0, \quad (i = 1,2,.....,10)$$

$$\Rightarrow \rho_{pb}^2 \geq \frac{S_\phi^2}{S_y^2} \left[R_i^2 - R_\phi^2 + 2R_\phi K_{yp} \right] \tag{4.3}$$

where $K_{yp} = \rho_{yp} \dfrac{C_y}{C_p}$.

5. Empirical Study

The data for the empirical study is taken from natural population data set considered by Sukhatme and Sukhatme [12]:

$\quad$ y $\quad$ = Number of villages in the circles and

$\quad$ ϕ $\quad$ = A circle consisting more than five villages

$N = 89$, $\overline{Y} = 3.36$, $P = 0.1236$, $\rho_{pb} = 0.766$, $C_y = 0.604$, $C_p = 2.19$, $\beta_2(\phi) = 6.23181$.

In the below table 5.1 percent relative efficiencies (PRE) of various estimators are computed with respect to $\overline{y}$.

Table 5.1: PRE of different estimators of $\overline{Y}$ with respect to $\overline{y}$.

Estimator	PRE $(., \overline{y})$
$\overline{y}$	100
t_{NG}	11.61
t_1	7.36
t_2	236.55
t_3	227.69
t_4	208.09
t_5	185.42
t_6	230.72
t_7	185.27
t_8	230.77
t_9	152.37
t_{10}	237.81

From table 5.1, we observe that the proposed estimators $t_i (i = 2,.....,10)$ which uses some known values of population proportion performs better than the usual sample mean $\overline{y}$ and Naik and Gupta [2] estimator t_{NG}.

Conclusion:

We have suggested some ratio estimators for estimating $\overline{Y}$ which uses some known value of population proportion. For practical purposes the choice of the estimator depends upon the availability of the population parameters.

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Ratio-Product Type Exponential Estimator for Estimating Finite Population Mean Using Information on Auxiliary Attribute

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Abstract

In practice, the information regarding the population proportion possessing certain attribute is easily available, see Jhajj et.al. (2006). For estimating the population mean $\overline{Y}$ of the study variable y, following Bahl and Tuteja (1991), a ratio-product type exponential estimator has been proposed by using the known information of population proportion possessing an attribute (highly correlated with y) in simple random sampling. The expressions for the bias and the mean-squared error (MSE) of the estimator and its minimum value have been obtained. The proposed estimator has an improvement over mean per unit estimator, ratio and product type exponential estimators as well as Naik and Gupta (1996) estimators. The results have also been extended to the case of two phase sampling. The results obtained have been illustrated numerically by taking some empirical populations considered in the literature.

Keywords: Proportion, bias, mean-squared error, two phase sampling.

1. Introduction

In survey sampling, the use of auxiliary information can increase the precision of an estimator when study variable y is highly correlated with the auxiliary variable x. but in several practical situations, instead of existence of auxiliary variables there exists some auxiliary attributes, which are highly correlated with study variable y, such as

(i) Amount of milk produced and a particular breed of cow. (ii) Yield of wheat crop and a particular variety of wheat etc. (see Shabbir and Gupta (2006)).

In such situations, taking the advantage of point biserial correlation between the study variable and the auxiliary attribute, the estimators of parameters of interest can be constructed by using prior knowledge of the parameters of auxiliary attribute.

Consider a sample of size n drawn by simple random sampling without replacement (SRSWOR) from a population of size N. let y_i and ϕ_i denote the observations on variable y and ϕ respectively for the ith unit ($i = 1,2,...,N$). We note that $\phi_i = 1$, if i^{th} unit of population possesses attribute ϕ and $\phi_i = 0$, otherwise. Let $A = \sum_{i=1}^{N} \phi_i$ and $a = \sum_{i=1}^{n} \phi_i$ denote the total number of units in the population and sample respectively possessing attribute ϕ. Let $P = \dfrac{A}{N}$ and $p = \dfrac{a}{n}$ denote the proportion of units in the population and sample respectively possessing attribute ϕ.

In order to have an estimate of the population mean $\overline{Y}$ of the study variable y, assuming the knowledge of the population proportion P, Naik and Gupta (1996) defined ratio and product estimators of population when the prior information of population proportion of units, possessing the same attribute is available. Naik and Gupta (1996) proposed following estimators:

$$t_1 = \overline{y}\left(\frac{P}{p}\right) \tag{1.1}$$

$$t_2 = \overline{y}\left(\frac{p}{P}\right) \tag{1.2}$$

The MSE of t_1 and t_2 up to the first order of approximation are

$$MSE(t_1) = f_1 \overline{Y}^2\left[C_y^2 + C_p^2(1 - 2K_p)\right] \tag{1.3}$$

$$MSE(t_2) = f_1 \overline{Y}^2\left[C_y^2 + C_p^2(1 + 2K_p)\right] \tag{1.4}$$

where $C_y^2 = \dfrac{S_y^2}{\overline{Y}^2}, C_p^2 = \dfrac{S_\phi^2}{P^2}, f_1 = \dfrac{1}{n} - \dfrac{1}{N}, K_p = \rho_{pb}\dfrac{C_y}{C_p}, S_y^2 = \dfrac{1}{N-1}\sum_{i=1}^{N}(y_i - \overline{Y})^2,$

$S_\phi^2 = \dfrac{1}{N-1}\sum_{i=1}^{N}(\phi_i - P)^2$, $S_{y\phi} = \dfrac{1}{N-1}\left(\sum_{i=1}^{N}y_i\phi_i - NP\overline{Y}\right)$ and

$\rho_{pb} = \dfrac{S_{y\phi}}{S_y S\phi}$ is the point biserial correlation coefficient.

Following Bahl and Tuteja (1991), we propose the following ratio and product exponential estimators

$$t_3 = \overline{y}\exp\left(\frac{P - p}{P + p}\right) \tag{1.5}$$

$$t_4 = \overline{y}\exp\left(\frac{p - P}{p + P}\right) \tag{1.6}$$

2. Bias and MSE of t_3 and t_4

To obtain the bias and MSE of t_3 to the first degree of approximation, we define

$e_y = \dfrac{(\overline{y} - \overline{Y})}{\overline{Y}}, e_\phi = \dfrac{(p - P)}{P}$, therefore E $(e_i) = 0$. $i = (y, \phi)$,

$E(e_y^2) = f_1 C_y^2$, $E(e_\phi^2) = f_1 C_p^2$, $E(e_y e_\phi) = f_1\rho_{pb}C_y C_p$.

Expressing (1.5) in terms of e's, we have

$$t_3 = \overline{Y}(1 + e_y) \exp\left[\frac{P - P(1 + e_\phi)}{P + P(1 + e_\phi)}\right]$$

$$= \overline{Y}(1 + e_y) \exp\left[\frac{-e_\phi}{(2 + e_\phi)}\right] \tag{2.1}$$

Expanding the right hand side of (2.1) and retaining terms up to second powers of e's, we have

$$t_3 = \overline{Y}\left[1 + e_y - \frac{e_\phi}{2} + \frac{e_\phi^2}{8} - \frac{e_y e_\phi}{2}\right] \tag{2.2}$$

Taking expectations of both sides of (2.2) and then subtracting $\overline{Y}$ from both sides, we get the bias of the estimator t_3 up to the first order of approximation, as

$$B(t_3) = f_1 \overline{Y} \frac{C_p^2}{2}\left(\frac{1}{4} - K_p\right) \tag{2.3}$$

From (2.2), we have

$$\left(t_3 - \overline{Y}\right) \cong \overline{Y}\left[e_y - \frac{e_\phi}{2}\right] \tag{2.4}$$

Squaring both sides of (2.4) and then taking expectations we get MSE of the estimator t_3, up to the first order of approximation as

$$MSE(t_3) = f_1 \overline{Y}^2\left[C_y^2 + C_p^2(\frac{1}{4} - K_p)\right] \tag{2.5}$$

To obtain the bias and MSE of t_4 to the first degree of approximation, we express (1.6) in terms of e's

$$t_4 = \overline{Y}(1 + e_y) \exp\left[\frac{P(1 + e_\phi) - P}{P(1 + e_\phi) + P}\right] \quad (2.6)$$

and following the above procedure, we get the bias and MSE of t_4 as follows

$$B(t_4) = f_1 \overline{Y} \frac{C_p^2}{2}\left(\frac{1}{4} + K_p\right) \quad (2.7)$$

$$MSE(t_4) = f_1 \overline{Y}^2\left[C_y^2 + C_p^2(\frac{1}{4} + K_p)\right] \quad (2.8)$$

3. Proposed class of estimators

It has been theoretically established that, in general, the linear regression estimator is more efficient than the ratio (product) estimator except when the regression line of y on x passes through the neighborhood of the origin, in which case the efficiencies of these estimators are almost equal. Also in many practical situations the regression line does not pass through the neighborhood of the origin. In these situations, the ratio estimator does not perform as good as the linear regression estimator. The ratio estimator does not perform well as the linear regression estimator does.

Following Singh and Espejo (2003), we propose following class of ratio-product type exponential estimators:

$$t_5 = \overline{y}\left[\alpha \exp\left(\frac{P - p}{P + p}\right) + (1 - \alpha)\exp\left(\frac{p - P}{p + P}\right)\right] \quad (3.1)$$

where α is a real constant to be determined such that the MSE of t_5 is minimum.

For $\alpha = 1$, t_5 reduces to the estimator $t_3 = \overline{y}\exp\left(\frac{P - p}{P + p}\right)$ and for $\alpha = 0$, it reduces to

$$t_4 = \overline{y}\exp\left(\frac{p - P}{p + P}\right).$$

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Bias and MSE of t_5:

Expressing (3.1) in terms of e's, we have

$$t_5 = \overline{Y}(1+e_y)\left[\alpha \exp\left\{\frac{P-P(1+e_\phi)}{P+P(1+e_\phi)}\right\} + (1-\alpha)\exp\left\{\frac{P(1+e_\phi)-P}{P(1+e_\phi)+P}\right\}\right]$$

$$= \overline{Y}(1+e_y)\left[\alpha \exp\left\{\frac{-e_\phi}{2}\right\} + (1-\alpha)\exp\left\{\frac{e_\phi}{2}\right\}\right] \tag{3.2}$$

Expanding the right hand side of (3.2) and retaining terms up to second powers of e's, we have

$$t_5 = \overline{Y}\left[1+e_y+\frac{e_\phi}{2}-\alpha e_\phi+\frac{e_\phi^2}{8}+e_y e_\phi-\alpha e_y e_\phi\right] \tag{3.3}$$

Taking expectations of both sides of (3.3) and then subtracting $\overline{Y}$ from both sides, we get the bias of the estimator t_5 up to the first order of approximation, as

$$B(t_5) = f_1\overline{Y}\left[\frac{C_p^2}{8}+\rho_{pb}C_yC_p\left(\frac{1}{2}-\alpha\right)\right] \tag{3.4}$$

From (3.3), we have

$$(t_5-\overline{Y}) \cong \overline{Y}\left[e_y+e_\phi\left(\frac{1}{2}-\alpha\right)\right] \tag{3.5}$$

Squaring both sides of (3.5) and then taking expectations we get MSE of the estimator t_5, up to the first order of approximation as

$$MSE(t_5) = f_1\overline{Y}^2\left[C_y^2+C_p^2\left(\frac{1}{4}+\alpha^2-\alpha\right)+2\rho_{pb}C_yC_p\left(\frac{1}{2}-\alpha\right)\right] \tag{3.6}$$

Minimization of (3.6) with respect to α yields optimum value of as

$$\alpha = \frac{2K_p+1}{2} = \alpha_0 \text{ (Say)} \tag{3.7}$$

Substitution of (3.7) in (3.1) yields the optimum estimator for t_5 as $(t_5)_{opt}$ (say) with minimum MSE as

$$\min.MSE(t_5) = f_1 \overline{Y}^2 C_y^2 \left(1 - \rho_{pb}^2\right) = M(t_5)_{opt} \tag{3.8}$$

which is same as that of traditional linear regression estimator.

4. Efficiency comparisons

In this section, the conditions for which the proposed estimator t_5 is better than $\overline{y}$, t_1, t_2, t_3, and t_4 have been obtained. The variance of $\overline{y}$ is given by

$$var(\overline{y}) = f_1 \overline{Y}^2 C_y^2 \tag{4.1}$$

To compare the efficiency of the proposed estimator t_5 with the existing estimator, from (4.1) and (1.3), (1.4), (2.5), (2.8) and (3.8), we have

$$var(\overline{y}) - M(t_5)_0 = \rho_{pb}^2 \geq 0. \tag{4.2}$$

$$MSE(t_1) - M(t_5)_0 = \left(C_p - \rho_{pb} C_y\right)^2 \geq 0. \tag{4.3}$$

$$MSE(t_2) - M(t_5)_0 = \left(C_p + \rho_{pb} C_y\right)^2 \geq 0. \tag{4.4}$$

$$MSE(t_3) - M(t_5)_0 = \left(\frac{C_p^2}{2} - \rho_{pb} C_y\right)^2 \geq 0. \tag{4.5}$$

$$MSE(t_4) - M(t_5)_0 = \left(\frac{C_p^2}{2} + \rho_{pb} C_y\right)^2 \geq 0. \tag{4.6}$$

Using (4.2)-(4.6), we conclude that the proposed estimator t_5 outperforms $\overline{y}$, t_1, t_2, t_3, and t_4.

5. Empirical study

We now compare the performance of various estimators considered here using the following data sets:

Population 1. [Source: Sukhatme and Sukhatme (1970), p. 256]

y = number of villages in the circles and

ϕ = A circle consisting more than five villages.

N = 89, $\overline{Y}$ =3.360, P = 0.1236, ρ_{pb} = 0.766, C_y = 0.60400, C_p = 2.19012.

Population 2. [Source: Mukhopadhyaya, (2000), p. 44]

Y= Household size and

ϕ = A household that availed an agricultural loan from a bank.

N = 25, $\overline{Y}$ =9.44, P = 0.400, ρ_{pb} = -0.387, C_y = 0.17028, C_p = 1.27478.

The percent relative efficiency (PRE's) of the estimators $\overline{y}$, t_1-t_4 and $(t_5)_{opt}$ with respect to unusual unbiased estimator $\overline{y}$ have been computed and compiled in table 5.1.

Table 5.1: PRE of various estimators with respect to $\overline{y}$.

Estimator	PRE's $(., \overline{y})$	
	Population	
	I	**II**
$\overline{y}$	100	100
t_1	11.63	1.59
t_2	5.07	1.94
t_3	66.24	5.57
t_4	14.15	8.24
$(t_5)_0$	241.98	117.61

Table 5.1 shows that the proposed estimator t_5 under optimum condition performs better than the usual sample mean $\bar{y}$, Naik and Gupta (1996) estimators (t_1 and t_2) and the ratio and product type exponential estimators (t_3 and t_4).

6. Double sampling

In some practical situations when P is not known a priori, the technique of two-phase sampling is used. Let p' denote the proportion of units possessing attribute ϕ in the first phase sample of size n'; p denote the proportion of units possessing attribute ϕ in the second phase sample of size n < n' and $\bar{y}$ denote the mean of the study variable y in the second phase sample.

When P is not known, two-phase ratio and product type exponential estimator are given by

$$t_6 = \bar{y}\exp\left(\frac{p'-p}{p'+p}\right) \tag{6.1}$$

$$t_7 = \bar{y}\exp\left(\frac{p-p'}{p+p'}\right) \tag{6.2}$$

To obtain the bias and MSE of t_6 and t_7, we write

$$\bar{y} = \bar{Y}(1+e_y), \ p = P(1+e_\phi), \ p' = P\left(1+e'_\phi\right)$$

such that

$$E\,(e_y) = E\left(e_\phi\right) = E\left(e'_\phi\right) = 0.$$

and

$$E\left(e_y^2\right) = f_1 C_y^2, \ E\left(e_\phi^2\right) = f_1 C_p^2, \ E\left(e'_\phi\right)^2 = f_2 C_p^2, \ E\left(e_\phi e'_\phi\right) = f_2 \rho_{pb} C_y C_p.$$

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where $f_2 = \dfrac{1}{n'} - \dfrac{1}{N}$.

Expressing (6.1) in terms of e's, we have

$$t_6 = \overline{Y}(1 + e_y) \exp\left[\frac{P(1 + e'_\phi) - P(1 + e_\phi)}{P(1 + e'_\phi) + P(1 + e_\phi)}\right]$$

$$= \overline{Y}(1 + e_y) \exp\left[\frac{e'_\phi - e_\phi}{2}\right] \tag{6.3}$$

Expanding the right hand side of (6.3) and retaining terms up to second powers of e's, we have

$$t_6 = \overline{Y}\left[1 + e_y + \frac{e'_\phi}{2} - \frac{e_\phi}{2} + \frac{e'^2_\phi}{8} + \frac{e^2_\phi}{8} - \frac{e_\phi e'_\phi}{4} + \frac{e_y e'_\phi}{2} - \frac{e_y e_\phi}{2}\right] \tag{6.4}$$

Taking expectations of both sides of (6.4) and then subtracting $\overline{Y}$ from both sides, we get the bias of the estimator t_6 up to the first order of approximation, as

$$B(t_6) = f_3 \overline{Y} \frac{C_p^2}{4}(1 - 2K_p) \tag{6.5}$$

From (6.4), we have

$$(t_6 - \overline{Y}) \cong \overline{Y}\left[e_y + \frac{(e'_\phi - e_\phi)}{2}\right] \tag{6.6}$$

Squaring both sides of (6.6) and then taking expectations we get MSE of the estimator t_6, up to the first order of approximation as

$$MSE(t_6) = \overline{Y}^2\left[f_1 C_y^2 + f_3 \frac{C_p^2}{4}(1 - 4K_p)\right] \tag{6.7}$$

To obtain the bias and MSE of t_7 to the first degree of approximation, we express (6.2) in terms of e's as

$$t_7 = \overline{Y}(1+e_y)\exp\left[\frac{P(1+e_\phi)-P(1+e'_\phi)}{P(1+e_\phi)+P(1+e'_\phi)}\right]$$

$$= \overline{Y}(1+e_y)\exp\left[\frac{e_\phi - e'_\phi}{2}\right] \tag{6.8}$$

Expanding the right hand side of (6.8) and retaining terms up to second powers of e's, we have

$$t_7 = \overline{Y}\left[1 + e_y + \frac{e_\phi}{2} - \frac{e'_\phi}{2} + \frac{e'^2_\phi}{8} + \frac{e^2_\phi}{8} - \frac{e_\phi e'_\phi}{4} + \frac{e_y e_\phi}{2} - \frac{e_y e'_\phi}{2}\right] \tag{6.9}$$

Taking expectations of both sides of (6.9) and then subtracting $\overline{Y}$ from both sides, we get the bias of the estimator t_7 up to the first order of approximation, as

$$B(t_7) = f_3\overline{Y}\frac{C^2_p}{4}\left(1 + 2K_p\right) \tag{6.10}$$

From (6.9), we have

$$\left(t_7 - \overline{Y}\right) \cong \overline{Y}\left[e_y + \frac{(e_\phi - e'_\phi)}{2}\right] \tag{6.11}$$

Squaring both sides of (6.11) and then taking expectations we get MSE of the estimator t_7, up to the first order of approximation as

$$MSE(t_7) = \overline{Y}^2\left[f_1 C^2_y + f_3\frac{C^2_p}{4}\left(1 + 4K_p\right)\right] \tag{6.12}$$

7. Proposed class of estimators in double sampling

We propose the following class of estimators in double sampling

$$t_8 = \overline{y}\left[\alpha_1 \exp\left(\frac{p'-p}{p'+p}\right) + (1-\alpha_1)\exp\left(\frac{p-p'}{p+p'}\right)\right] \tag{7.1}$$

where α_1 is a real constant to be determined such that the MSE of t_8 is minimum.

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For $\alpha_1 = 1$, t_8 reduces to the estimator $t_6 = \overline{y}\exp\left(\dfrac{p'-p}{p'+p}\right)$ and for $\alpha_1 = 0$, it reduces to

$$t_7 = \overline{y}\exp\left(\dfrac{p-p'}{p+p'}\right).$$

Bias and MSE of t_8:

Expressing (7.1) in terms of e's, we have

$$t_8 = \overline{Y}(1+e_y)\left[\alpha_1 \exp\left\{\frac{P(1+e'_\phi)-P(1+e\phi)}{P(1+e'_\phi)+P(1+e\phi)}\right\}+(1-\alpha_1)\exp\left\{\frac{P(1+e_\phi)-P(1+e'_\phi)}{P(1+e_\phi)+P(1+e'_\phi)}\right\}\right]$$

$$= \overline{Y}(1+e_y)\left[\alpha_1 \exp\left\{\frac{e'_\phi-e_\phi}{2}\right\}+(1-\alpha_1)\exp\left\{\frac{e_\phi-e'_\phi}{2}\right\}\right] \tag{7.2}$$

Expanding the right hand side of (7.2) and retaining terms up to second powers of e's, we have

$$t_8 = \overline{Y}[1+e_y+\frac{e_\phi}{2}-\frac{e'_\phi}{2}-\alpha_1 e_\phi+\alpha_1 e'_\phi+\frac{e_\phi^2}{8}+\frac{{e'}_\phi^2}{8}+\frac{e_y e_\phi}{2}-\frac{e_y e'_\phi}{2}$$

$$-\frac{e_\phi e'_\phi}{4}+\alpha_1 e_y e'_\phi-\alpha_1 e_y e_\phi] \tag{7.3}$$

Taking expectations of both sides of (7.3) and then subtracting $\overline{Y}$ from both sides, we get the bias of the estimator t_8 up to the first order of approximation, as

$$B(t_8) = f_3\overline{Y}\frac{C_p^2}{8}\left[1-8K_p\left(\alpha_1-\frac{1}{2}\right)\right] \tag{7.4}$$

From (7.3), we have

$$(t_8-\overline{Y}) \cong \overline{Y}\left[e_y-\left(\alpha_1-\frac{1}{2}\right)e_\phi+\left(\alpha_1-\frac{1}{2}\right)e'_\phi\right] \tag{7.5}$$

Squaring both sides of (7.5) and then taking expectations we get MSE of the estimator t_8, up to the first order of approximation as

$$MSE(t_8) = \overline{Y}^2 \left[f_1 C_y^2 + f_3 C_p^2 \left(\alpha_1 - \frac{1}{2} \right) \left\{ \left(\alpha_1 - \frac{1}{2} \right) - 2K_p \right\} \right] \qquad (7.6)$$

Minimization of (7.6) with respect to α_1 yields optimum value of as

$$\alpha_1 = \frac{2K_p + 1}{2} = \alpha_{10} \, (\text{Say}) \qquad (7.7)$$

Substitution of (7.7) in (7.1) yields the optimum estimator for t_8 as $(t_8)_{\text{opt}}$ (say) with minimum MSE as

$$\min.MSE(t_8) = \overline{Y}^2 C_y^2 \left(f_1 - f_3 \rho_{pb}^2 \right) = M(t_8)_o, \text{ (say)} \qquad (7.8)$$

which is same as that of traditional linear regression estimator.

8. Efficiency comparisons

The MSE of usual two-phase ratio and product estimator is given by

$$MSE(t_9) = \overline{Y}^2 \left[f_1 C_y^2 + f_3 C_p^2 (1 - 2K_p) \right] \qquad (8.1)$$

$$MSE(t_{10}) = \overline{Y}^2 \left[f_1 C_y^2 + f_3 C_p^2 (1 + 2K_p) \right] \qquad (8.2)$$

From (4.1), (6.7), (6.12), (8.1), (8.2) and (7.8) we have

$$\text{var}(\overline{y}) - M(t_8)_0 = f_3 \rho_{pb}^2 \geq 0. \qquad (8.3)$$

$$MSE(t_6) - M(t_8)_0 = f_3 \left(\frac{C_p}{2} - \rho_{pb} C_y \right)^2 \geq 0. \qquad (8.4)$$

$$MSE(t_7) - M(t_8)_0 = f_3 \left(\frac{C_p}{2} + \rho_{pb} C_y \right)^2 \geq 0. \qquad (8.5)$$

$$MSE(t_9) - M(t_8)_0 = f_3 \left(C_p - \rho_{pb} C_y \right)^2 \geq 0. \qquad (8.6)$$

$$MSE(t_{10}) - M(t_8)_0 = f_3\left(C_p + \rho_{pb}C_y\right)^2 \geq 0.$$ (8.7)

From (8.3)-(8.7), we conclude that our proposed estimator t_8 is better than $\bar{y}$, t_6, t_7, t_9, and t_{10}.

9. Empirical study

The various results obtained in the previous section are now examined with the help of following data:

Population 1. [Source: Sukhatme and Sukhatme(1970), p. 256]

$N = 89$, $n' = 45$, $n = 23$, $\bar{y} = 1322$, $p = 0.1304$, $p' = 0.1333$, $\rho_{pb} = 0.408$, $C_y = 0.69144$, $C_p = 2.7005$.

Population 2. [Source: Mukhopadhyaya(2000), p. 44]

$N = 25$, $n' = 13$, $n = 7$, $\bar{y} = 7.143$, $p = 0.294$, $p' = 0.308$, $\rho_{pb} = -0.314$, $C_y = 0.36442$, $Cp = 1.34701$.

Table 9.1: PRE of various estimators (double sampling) with respect to $\bar{y}$.

Estimator	PRE's (., $\bar{y}$)	
	Population	
	I	II
$\bar{y}$	100	100
t_6	40.59	25.42
t_7	21.90	40.89
t_9	11.16	8.89
t_{10}	7.60	12.09
$(t_8)_0$	112.32	106.74

Table 9.1 shows that the proposed estimator t_8 under optimum condition performs better than the usual sample mean $\overline{y}$, t_6, t_7, t_9, and t_{10}.

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Improvement in Estimating the Population Mean Using Exponential Estimator in Simple Random Sampling

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Abstract

This study proposes some exponential ratio-type estimators for estimating the population mean of the variable under study, using known values of certain population parameter(s). Under simple random sampling without replacement (SRSWOR) scheme, mean square error (MSE) equations of all proposed estimators are obtained and compared with each other. The theoretical results are supported by a numerical illustration.

Keywords: Exponential estimator, auxiliary variable, simple random sampling, efficiency.

1. Introduction

Consider a finite population $U = U_1, U_2, \ldots, U_N$ of N unites. Let y and x stand for the variable under study and auxiliary variable respectively. Let (y_i, x_i), $i = 1, 2, \ldots, n$ denote the values of the units included in a sample s_n of size n drawn by simple random sampling without replacement (SRSWOR). The auxiliary information has been used in improving the precision of the estimate of a parameter (see Cochran (1977), Sukhatme

and Sukhatme (1970) and the reference cited there in). Out of many methods, ratio and product methods of estimation are good illustrations in this context.

In order to have a survey estimate of the population mean $\overline{Y}$ of the study character y, assuming the knowledge of the population mean $\overline{X}$ of the auxiliary character x, the well-known ratio estimator is –

$$t_r = \overline{y}\left(\frac{\overline{X}}{\overline{x}}\right) \tag{1.1}$$

Bahl and Tuteja (1991) suggested an exponential ratio type estimator as –

$$t_1 = \overline{y}\exp\left[\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}}\right] \tag{1.2}$$

Several authors have used prior value of certain population parameter(s) to find more precise estimates. Sisodiya and Dwivedi (1981), Sen (1978) and Upadhyaya and Singh (1984) used the known coefficient of variation (CV) of the auxiliary character for estimating population mean of a study character in ratio method of estimation. The use of prior value of coefficient of kurtosis in estimating the population variance of study character y was first made by Singh et.al.(1973). Later used by Singh and Kakaran (1993) in the estimation of population mean of study character. Singh and Tailor (2003) proposed a modified ratio estimator by using the known value of correlation coefficient. Kadilar and Cingi (2006(a)) and Khoshnevisan et.al.(2007) have suggested modified ratio estimators by using different pairs of known value of population parameter(s).

In this paper, under SRSWOR, we have suggested improved exponential ratio-type estimators for estimating population mean using some known value of population parameter(s).

2. The suggested estimator

Following Kadilar and Cingi (2006(a)) and Khoshnevisan (2007), we define modified exponential estimator for estimating $\overline{Y}$ as –

$$t = \overline{y}\exp\left[\frac{(a\overline{X}+b)-(a\overline{x}+b)}{(a\overline{X}+b)+(a\overline{x}+b)}\right] \tag{2.1}$$

where $a(\neq 0)$, b are either real numbers or the functions of the known parameters of the auxiliary variable x such as coefficient of variation (C_x), coefficient of kurtosis $(\beta_2(x))$ and correlation coefficient (ρ).

To, obtain the bias and MSE of t, we write

$$\bar{y} = \bar{Y}(1 + e_0), \quad \bar{x} = \bar{X}(1 + e_1)$$

such that

$$E(e_0) = E(e_1) = 0,$$

and $\quad E(e_0^2) = f_1 C_y^2, \quad E(e_1^2) = f_1 C_x^2, \quad E(e_0 e_1) = f_1 \rho C_y C_x,$

where

$$f_1 = \frac{N-n}{nN}, \quad C_y^2 = \frac{S_y^2}{\bar{Y}^2}, \quad C_x^2 = \frac{S_x^2}{\bar{X}^2}.$$

Expressing t in terms of e's, we have

$$t = \bar{Y}(1 + e_0) \exp\left[\frac{a\bar{X} - a\bar{X}(1 + e_1)}{a\bar{X} + 2b + a\bar{X}(1 + e_1)}\right]$$

$$t = \bar{Y}(1 + e_0) \exp\left[-\theta e_1 (1 + \theta e_1)^{-1}\right] \tag{2.2}$$

where $\theta = \dfrac{a\bar{X}}{2(a\bar{X} + b)}$.

Expanding the right hand side of (2.2) and retaining terms up to second power of e's, we have

$$t = \bar{Y}(1 + e_0 - \theta e_1 + \theta^2 e_1^2 + \theta e_0 e_1) \tag{2.3}$$

Taking expectations of both sides of (2.3) and then subtracting $\bar{Y}$ from both sides, we get the bias of the estimator t, up to the first order of approximation, as

$$B(t) = f_1 \bar{Y}(\theta^2 C_y^2 + \theta \rho C_y C_x) \tag{2.4}$$

From (2.3), we have

$$(t - \bar{Y}) \cong \bar{Y}(e_0 - \theta e_1) \tag{2.5}$$

Squaring both sides of (2.5) and then taking expectation, we get MSE of the estimator t, up to the first order of approximation, as

$$MSE(t) = f_1 \bar{Y}^2 (C_y^2 + \theta^2 C_x^2 - 2\theta \rho C_y C_c) \tag{2.6}$$

3. Some members of the suggested estimator 't'

The following scheme presents some estimators of the population mean which can be obtained by suitable choice of constants a and b.

Estimator	Values of	
	a	b
$t_0 = \bar{y}$ The mean per unit estimator	0	0
$t_1 = \bar{y}\exp\left[\dfrac{\overline{X}-\bar{x}}{\overline{X}+\bar{x}}\right]$ Bahl and Tuteja (1991) estimator	1	1
$t_2 = \bar{y}\exp\left[\dfrac{\overline{X}-\bar{x}}{\overline{X}+\bar{x}+2\beta_2(x)}\right]$	1	$\beta_2(x)$
$t_3 = \bar{y}\exp\left[\dfrac{\overline{X}-\bar{x}}{\overline{X}+\bar{x}+2C_c}\right]$	1	C_x
$t_4 = \bar{y}\exp\left[\dfrac{\overline{X}-\bar{x}}{\overline{X}+\bar{x}+2\rho}\right]$	1	ρ
$t_5 = \bar{y}\exp\left[\dfrac{\beta_2(x)(\overline{X}-\bar{x})}{\beta_2(x)(\overline{X}+\bar{x})+2C_x}\right]$	$\beta_2(x)$	C_x
$t_6 = \bar{y}\exp\left[\dfrac{C_x(\overline{X}-\bar{x})}{C_x(\overline{X}+\bar{x})+2\beta_2(x)}\right]$	C_x	$\beta_2(x)$
$t_7 = \bar{y}\exp\left[\dfrac{C_x(\overline{X}-\bar{x})}{C_x(\overline{X}+\bar{x})+2\rho}\right]$	C_x	ρ
$t_8 = \bar{y}\exp\left[\dfrac{\rho(\overline{X}-\bar{x})}{\rho(\overline{X}+\bar{x})+2C_x}\right]$	ρ	C_x
$t_9 = \bar{y}\exp\left[\dfrac{\beta_2(x)(\overline{X}-\bar{x})}{\beta_2(x)(\overline{X}+\bar{x})+2\rho}\right]$	$\beta_2(x)$	ρ
$t_{10} = \bar{y}\exp\left[\dfrac{\rho(\overline{X}-\bar{x})}{\rho(\overline{X}+\bar{x})+2\beta_2(x)}\right]$	ρ	$\beta_2(x)$

In addition to above estimators a large number of estimators can also be generated from the proposed estimator t at (2.1) just by putting different values of a and b.

It is observed that the expressions of the first order of approximation of bias and MSE of the given member of the family can be obtained by mere substituting the values of a and b in (2.4) and (2.6) respectively.

4. Modified estimators

Following Kadilar and Cingi (2006(b)), we propose modified estimator combining estimator t_1 and t_i $(i = 2,3,....,10)$ as follows

$$t_i^* = \alpha t_1 + (1-\alpha)t_i, \qquad (i = 2,3,....,10) \tag{4.1}$$

where α is a real constant to be determined such that the MSE of t_i^* is minimum and t_i $(i = 2,3,....,10)$ are estimators listed in section 3.

Following the procedure of section (2), we get the MSE of t_i^* to the first order of approximation as –

$$MSE(t_i^*) = f_1 \overline{Y}^2 \left[C_y^2 + C_x^2 \left(\frac{\alpha}{2} + \theta_i - \alpha\theta_i \right)^2 - 2\rho C_y C_x \left(\frac{\alpha}{2} + \theta_i - \alpha\theta_i \right) \right] \tag{4.2}$$

where

$$\theta_2 = \frac{\overline{X}}{2(\overline{X} + \beta_2(x))}, \qquad\qquad \theta_3 = \frac{\overline{X}}{2(\overline{X} + C_x)},$$

$$\theta_4 = \frac{\overline{X}}{2(\overline{X} + \rho))}, \qquad\qquad \theta_5 = \frac{\beta_2(x)\overline{X}}{2(\beta_2(x)\overline{X} + C_x))},$$

$$\theta_6 = \frac{C_x \overline{X}}{2(C_x \overline{X} + \beta_2(x))}, \qquad\qquad \theta_7 = \frac{C_x \overline{X}}{2(C_x \overline{X} + \rho)},$$

$$\theta_8 = \frac{\rho \overline{X}}{2(\rho \overline{X} + C_x)}, \qquad\qquad \theta_9 = \frac{\beta_2(x)\overline{X}}{2(\beta_2(x) + \rho)},$$

$$\theta_{10} = \frac{\rho \overline{X}}{2(\rho + \beta_2(x))}.$$

Minimization of (4.2) with respect to α yields its optimum value as

$$\alpha = \frac{2(K - \theta)}{(1 - 2\theta)} = \alpha_{opt}(say) \tag{4.3}$$

where $K = \rho \dfrac{C_y}{C_x}$.

Substitution of (4.3) in (4.10) gives optimum estimator $t_o^*(say)$, with minimum MSE as

$$\min \mathrm{MSE}(t_i^*) = f_1 \overline{Y}^2 C_y^2 (1 - \rho^2) = \mathrm{MSE}(t^*)_o \qquad (4.4)$$

The $\min \mathrm{MSE}(t_i^*)$ at (4.4) is same as that of the approximate variance of the usual linear regression estimator.

5. Efficiency comparison

It is well known that under SRSWOR the variance of the sample mean is

$$\mathrm{Var}(\overline{y}) = f_1 \overline{Y}^2 C_y^2 \qquad (5.1)$$

we first compare the MSE of the proposed estimators, given in (2.6) with the variance of the sample mean, we have the following condition:

$$K \leq \frac{\theta_i}{2}, \qquad i = 2,3,\dots,10 \qquad (5.2)$$

When this condition is satisfied, proposed estimators are more efficient than the sample mean.

Next we compare the MSE of proposed estimators t_i^* ($i = 2,3,\dots,10$) in (4.4) with the MSE of estimators listed in section 3. We obtain the following condition

$$(\theta C_x - \rho C_y)^2 \geq 0, \qquad i = 2,3,\dots,10. \qquad (5.3)$$

We can infer that all proposed estimators t_i^*, ($i = 2,3,\dots,10$) are more efficient than estimators proposed in section 3 in all conditions, because the condition given in (5.1) is always satisfied.

6. Numerical illustration

To illustrate the performance of various estimators of $\overline{Y}$, we consider the data given in Murthy (1967 pg-226). The variates are:

y : Output,   x: number of workers

$\overline{X} = 283.875$, $\overline{Y} = 5182.638$, $C_y = 0.3520$, $C_x = 0.9430$, $\rho = 0.9136$, $\beta_2(x) = 3.65$.

We have computed the percent relative efficiency (PRE) of different estimators of $\overline{Y}$ with respect to usual estimator $\overline{y}$ and complied in table 6.1:

Table 6.1: PRE of different estimators of $\overline{Y}$ with respect to $\overline{y}$

Estimator	PRE
$\overline{y}$	100
t_1	366.96
t_2	385.72
t_3	368.27
t_4	371.74
t_5	386.87
t_6	368.27
t_7	372.03
t_8	372.05
t_9	368.27
t_{10}	386.91
t_o^*	877.54

7. Conclusion

We have developed some exponential ratio type estimators using some known value of the population parameter(s), listed in section 3. We have also suggested modified estimators t_i^* ($i = 2,3,.....,10$). From table 6.1 we conclude that the proposed estimators are better than Bahl and Tuteja (1991) estimator t_1. Also, the modified estimator t_i^* ($i = 2,3,.....,10$) under optimum condition performs better than the estimators proposed and listed in section 3 and than the Bahl and Tuteja (1991) estimator t_1. The choice of the estimator mainly depends upon the availability of information about known values of the parameter(s) (C_x, ρ, $\beta_2(x)$, etc.).

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Almost Unbiased Exponential Estimator for the Finite Population Mean

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Abstract

In this paper we have proposed an almost unbiased ratio and product type exponential estimator for the finite population mean $\overline{Y}$. It has been shown that Bahl and Tuteja (1991) ratio and product type exponential estimators are particular members of the proposed estimator. Empirical study is carried to demonstrate the superiority of the proposed estimator.

Keywords: Auxiliary information, bias, mean-squared error, exponential estimator.

1. Introduction

It is well known that the use of auxiliary information in sample surveys results in substantial improvement in the precision of the estimators of the population mean. Ratio, product and difference methods of estimation are good examples in this context. Ratio

method of estimation is quite effective when there is a high positive correlation between study and auxiliary variables. On other hand, if this correlation is negative (high), the product method of estimation can be employed effectively.

Consider a finite population with N units $(U_1, U_2,...., U_N)$ for each of which the information is available on auxiliary variable x. Let a sample of size n be drawn with simple random sampling without replacement (SRSWOR) to estimate the population mean of character y under study. Let $(\bar{y}, \bar{x})$ be the sample mean estimator of $(\bar{Y}, \bar{X})$ the population means of y and x respectively.

In order to have a survey estimate of the population mean $\bar{Y}$ of the study character y, assuming the knowledge of the population mean $\bar{X}$ of the auxiliary character x, Bahl and Tuteja (1991) suggested ratio and product type exponential estimator

$$t_1 = \bar{y}\exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right) \qquad (1.1)$$

$$t_2 = \bar{y}\exp\left(\frac{\bar{x} - \bar{X}}{\bar{x} + \bar{X}}\right) \qquad (1.2)$$

Up to the first order of approximation, the bias and mean-squared error (MSE) of t_1 and t_2 are respectively given by

$$B(t_1) = \left(\frac{N-n}{nN}\right)\bar{Y}\frac{C_x^2}{2}\left(\frac{1}{2} - K\right) \qquad (1.3)$$

$$MSE(t_1) = \left(\frac{N-n}{nN}\right)\bar{Y}^2\left[C_y^2 + C_x^2\left(\frac{1}{4} - K\right)\right] \qquad (1.4)$$

$$B(t_2) = \left(\frac{N-n}{nN}\right)\bar{Y}\frac{C_x^2}{2}\left(\frac{1}{2} + K\right) \qquad (1.5)$$

$$\text{MSE}(t_2) = \left(\frac{N-n}{nN}\right)\overline{Y}^2\left[C_y^2 + C_x^2\left(\frac{1}{4} + K\right)\right] \tag{1.6}$$

where $\quad S_y^2 = \dfrac{1}{(N-1)}\sum_{i=1}^{N}(y_i - \overline{Y})^2, \quad S_x^2 = \dfrac{1}{(N-1)}\sum_{i=1}^{N}(x_i - \overline{X})^2, \quad C_y = \dfrac{S_y}{\overline{Y}}, \quad C_x = \dfrac{S_x}{\overline{X}},$

$K = \rho\left(\dfrac{C_y}{C_x}\right), \quad \rho = \dfrac{S_{yx}}{(S_y S_x)}, \quad S_{yx} = \dfrac{1}{(N-1)}\sum_{i=1}^{N}(y_i - \overline{Y})(x_i - \overline{X}).$

From (1.3) and (1.5), we see that the estimators t_1 and t_2 suggested by Bahl and Tuteja (1991) are biased estimator. In some applications bias is disadvantageous. Following Singh and Singh (1993) and Singh and Singh (2006) we have proposed almost unbiased estimators of $\overline{Y}$.

2. Almost unbiased estimator

Suppose $t_0 = \overline{y}$, $t_1 = \overline{y}\exp\left(\dfrac{\overline{X} - \overline{x}}{\overline{X} + \overline{x}}\right)$, $t_2 = \overline{y}\exp\left(\dfrac{\overline{x} - \overline{X}}{\overline{x} + \overline{X}}\right)$

such that t_0, t_1, $t_2 \in H$, where H denotes the set of all possible estimators for estimating the population mean $\overline{Y}$. By definition, the set H is a linear variety if

$$t_h = \sum_{i=0}^{2} h_i t_i \in H \tag{2.1}$$

for $\displaystyle\sum_{i=0}^{2} h_i = 1$, $h_i \in R$ \qquad (2.2)

where $h_i(i = 0,1,2)$ denotes the statistical constants and R denotes the set of real numbers.

To obtain the bias and MSE of t_h, we write

$\overline{y} = \overline{Y}(1 + e_0)$, $\overline{x} = \overline{X}(1 + e_1)$.

such that

E (e_0)=E (e_1)=0.

$$E(e_0^2) = \left(\frac{N-n}{Nn}\right)C_y^2, \ E(e_1^2) = \left(\frac{N-n}{Nn}\right)C_x^2, \ E(e_0e_1) = \left(\frac{N-n}{Nn}\right)\rho C_y C_x.$$

Expressing t_h in terms of e's, we have

$$t_h = \overline{Y}(1+e_0)\left[h_0 + h_1 \exp\left(\frac{-e_1}{2+e_1}\right) + h_2 \exp\left(\frac{e_1}{2+e_1}\right)\right] \tag{2.3}$$

Expanding the right hand side of (2.3) and retaining terms up to second powers of e's, we have

$$t_h = \overline{Y}\left[1 + e_0 - \frac{e_1}{2}(h_1 - h_2) + h_1\frac{e_1^2}{8} + h_2\frac{e_1^2}{8} - h_1\frac{e_0e_1}{2} + h_2\frac{e_0e_1}{2}\right] \tag{2.4}$$

Taking expectations of both sides of (2.4) and then subtracting $\overline{Y}$ from both sides, we get the bias of the estimator t_h, up to the first order of approximation as

$$B(t_h) = \left(\frac{N-n}{Nn}\right)\overline{Y}\frac{C_x^2}{2}\left[\frac{1}{4}(h_1 + h_2) - K(h_1 - h_2)\right] \tag{2.5}$$

From (2.4), we have

$$(t_h - \overline{Y}) \cong \overline{Y}\left[e_0 - h\frac{e_1}{2}\right] \tag{2.6}$$

where $h = h_1 - h_2$. \hfill (2.7)

Squaring both the sides of (2.7) and then taking expectations, we get MSE of the estimator t_h, up to the first order of approximation, as

$$MSE(t_h) = \left(\frac{N-n}{Nn}\right)\overline{Y}^2\left[C_y^2 + C_x^2 h\left(\frac{h}{4} - K\right)\right] \tag{2.8}$$

which is minimum when

$$h = 2K. \tag{2.9}$$

Putting this value of $h = 2K$ in (2.1) we have optimum value of estimator as t_h (optimum).

Thus the minimum MSE of t_h is given by

$$\min.MSE(t_h) = \left(\frac{N-n}{Nn}\right)\overline{Y}^2 C_y^2 \left(1-\rho^2\right)$$ (2.10)

which is same as that of traditional linear regression estimator.

From (2.7) and (2.9), we have

$h_1-h_2 = h = 2K$. (2.11)

From (2.2) and (2.11), we have only two equations in three unknowns. It is not possible to find the unique values for h_i's, i=0,1,2. In order to get unique values of h_i's, we shall impose the linear restriction

$$\sum_{i=0}^{2} h_i B(t_i) = 0.$$ (2.12)

where $B(t_i)$ denotes the bias in the ith estimator.

Equations (2.2), (2.11) and (2.12) can be written in the matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & B(t_1) & B(t_2) \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2K \\ 0 \end{bmatrix}$$ (2.13)

Using (2.13), we get the unique values of h_i's(i=0,1,2) as

$$\begin{aligned} h_0 &= 1 - 4K^2 \\ h_1 &= K + 2K^2 \\ h_2 &= -K + 2K^2 \end{aligned} \Bigg\}$$ (2.14)

Use of these h_i's (i=0,1,2) remove the bias up to terms of order $o(n^{-1})$ at (2.1).

3. Two phase sampling

When the population mean $\overline{X}$ of x is not known, it is often estimated from a preliminary large sample on which only the auxiliary characteristic is observed. The

value of population mean $\overline{X}$ of the auxiliary character x is then replaced by this estimate. This technique is known as the double sampling or two-phase sampling.

The two-phase sampling happens to be a powerful and cost effective (economical) procedure for finding the reliable estimate in first phase sample for the unknown parameters of the auxiliary variable x and hence has eminent role to play in survey sampling, for instance, see; Hidiroglou and Sarndal (1998).

When $\overline{X}$ is unknown, it is sometimes estimated from a preliminary large sample of size n' on which only the characteristic x is measured. Then a second phase sample of size $n(n < n')$ is drawn on which both y and x characteristics are measured. Let

$$\overline{x} = \frac{1}{n'}\sum_{i=1}^{n'} x_i$$ denote the sample mean of x based on first phase sample of size n';

$$\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i \text{ and } \overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$ be the sample means of y and x respectively based on second phase of size n.

In double (or two-phase) sampling, we suggest the following modified exponential ratio and product estimators for $\overline{Y}$, respectively, as

$$t_{1d} = \overline{y}\exp\left(\frac{\overline{x}' - \overline{x}}{\overline{x}' + \overline{x}}\right) \tag{3.1}$$

$$t_{2d} = \overline{y}\exp\left(\frac{\overline{x} - \overline{x}'}{\overline{x} + \overline{x}'}\right) \tag{3.2}$$

To obtain the bias and MSE of t_{1d} and t_{2d}, we write

$$\overline{y} = \overline{Y}(1 + e_0), \ \overline{x} = \overline{X}(1 + e_1), \ \overline{x}' = \overline{X}(1 + e_1')$$

such that

$$E(e_0) = E(e_1) = E(e_1') = 0$$

and

$$E(e_0^2) = f_1 C_y^2, \qquad E(e_1^2) = f_1 C_x^2, \qquad E(e_1'^2) = f_2 C_x^2,$$

$$E(e_0 e_1) = f_1 \rho C_y C_x,$$

$$E(e_0 e_1') = f_2 \rho C_y C_x,$$

$$E(e_1 e_1') = f_2 C_x^2.$$

where $f_1 = \left(\dfrac{1}{n} - \dfrac{1}{N} \right)$, $f_2 = \left(\dfrac{1}{n'} - \dfrac{1}{N} \right)$.

Following standard procedure we obtain

$$B(t_{1d}) = \overline{Y} f_3 \left[\frac{C_x^2}{8} - \frac{1}{2} \rho C_y C_x \right] \tag{3.3}$$

$$B(t_{2d}) = \overline{Y} f_3 \left[\frac{C_x^2}{8} + \frac{1}{2} \rho C_y C_x \right] \tag{3.4}$$

$$MSE(t_{1d}) = \overline{Y}^2 \left[f_1 C_y^2 + f_3 \left(\frac{C_x^2}{4} - \rho C_x C_y \right) \right] \tag{3.5}$$

$$MSE(t_{2d}) = \overline{Y}^2 \left[f_1 C_y^2 + f_3 \left(\frac{C_x^2}{4} + \rho C_x C_y \right) \right] \tag{3.6}$$

where $f_3 = \left(\dfrac{1}{n} - \dfrac{1}{n'} \right)$.

From (3.3) and (3.4) we observe that the proposed estimators t_{1d} and t_{2d} are biased, which is a drawback of an estimator is some applications.

4. Almost unbiased two-phase estimator

Suppose $t_0 = \bar{y}$, t_{1d} and t_{2d} as defined in (3.1) and (3.2) such that $t_0, t_{1d}, t_{2d} \in W$, where W denotes the set of all possible estimators for estimating the population mean $\bar{Y}$. By definition, the set W is a linear variety if

$$t_W = \sum_{i=0}^{2} w_i t_i \in W. \tag{4.1}$$

for $\sum_{i=1}^{2} w_i = 1$, $w_i \in R$. \tag{4.2}

where $w_i (i = 0,1,2)$ denotes thee statistical constants and R denotes the set of real numbers.

To obtain the bias and MSE of t_w, using notations of section 3 and expressing t_w in terms of e's, we have

$$t_w = \bar{Y}(1 + e_0)\left[w_0 + w_1 \exp\left(\frac{e_1' - e_1}{2}\right) + w_2 \exp\left(\frac{e_1 - e_1'}{2}\right)\right] \tag{4.3}$$

$$t_w = \bar{Y}[1 + e_0 - \frac{w}{2}(e_1 - e_1') + \frac{w_1}{8}(e_1^2 + e_1'^2) + \frac{w_2}{8}(e_1^2 + e_1'^2) - \left(\frac{w_1}{4} + \frac{w_2}{4}\right)e_1 e_1'$$

$$+ \frac{w}{2}(e_0 e_1' - e_0 e_1)] \tag{4.4}$$

where $w = w_1 - w_2$. \tag{4.5}

Taking expectations of both sides of (4.4) and then subtracting $\bar{Y}$ from both sides, we get the bias of the estimator t_w, up to the first order f approximation as

$$\text{Bias}(t_w) = \bar{Y}f_3\left[\left(\frac{w_1 + w_2}{8}\right)C_x^2 - \frac{w}{2}\rho C_y C_x\right] \tag{4.6}$$

From (4.4), we have

$$t_w \cong \overline{Y}\left[e_0 - \frac{w}{2}(e_1 - e_1')\right] \tag{4.7}$$

Squaring both sides of (4.7) and then taking expectation, we get MSE of the estimator t_w, up to the first order of approximation, as

$$MSE(t_w) = \overline{Y}^2\left[f_1C_y^2 + f_3wC_x^2\left(\frac{w}{4}\right) - K\right] \tag{4.8}$$

which is minimum when

$$w = 2K. \tag{4.9}$$

Thus the minimum MSE of t_w is given by –

$$\text{min}.MSE(t_w) = \overline{Y}^2C_y^2\left[f_1 - f_3\rho^2\right] \tag{4.10}$$

which is same as that of two-phase linear regression estimator. From (4.5) and (4.9), we have

$$w_1 - w_2 = w = 2K \tag{4.11}$$

From (4.2) and (4.11), we have only two equations in three unknowns. It is not possible to find the unique values for w_i's $(i = 0,1,2)$. In order to get unique values of h_i's, we shall impose the linear restriction

$$\sum_{i=0}^{2} w_i B(t_{id}) = 0 \tag{4.12}$$

where $B(t_{id})$ denotes the bias in the ith estimator.

Equations (4.2), (4.11) and (4.12) can be written in the matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & B(t_{1d}) & B(t_{2d}) \end{bmatrix}\begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2K \\ 0 \end{bmatrix} \tag{4.13}$$

Solving (4.13), we get the unique values of w_i's $(i = 0,1,2)$ as –

$$
\left.
\begin{aligned}
w_0 &= 1 - 8K^2 \\
w_1 &= K + 4K^2 \\
w_2 &= -K + 4K^2
\end{aligned}
\right\}
\qquad (4.14)
$$

Use of these w_i's $(i = 0,1,2)$ removes the bias up to terms of order $o(n^{-1})$ at (4.1).

5. Empirical study

The data for the empirical study are taken from two natural population data sets considered by Cochran (1977) and Rao (1983).

Population I: Cochran (1977)

$C_y = 1.4177$, $C_x = 1.4045$, $\rho = 0.887$.

Population II: Rao (1983)

$C_y = 0.426$, $C_x = 0.128$, $\rho = -0.7036$.

In table (5.1), the values of scalar h_i's $(i = 0,1,2)$ are listed.

Table (5.1): Values of h_i's $(i = 0,1,2)$

Scalars	Population	
	I	II
h_0	-2.2065	-20.93
h_1	2.4985	8.62
h_2	0.7079	13.30

Using these values of h_i's $(i = 0,1,2)$ given in the table 5.1, one can reduce the bias to the order $o(n^{-1})$ in the estimator t_h at (2.1).

In table 5.2, Percent relative efficiency (PRE) of $\bar{y}$, t_1, t_2 and t_h (in optimum case) are computed with respect to $\bar{y}$.

Table 5.2: PRE of different estimators of $\overline{Y}$ with respect to $\bar{y}$.

Estimators	PRE $(., \bar{y})$	
	Population I	**Population II**
$\bar{y}$	100	100
t_1	272.75	32.55
t_2	47.07	126.81
t_h (optimum)	468.97	198.04

Table 5.2 clearly shows that the suggested estimator t_h in its optimum condition is better than usual unbiased estimator $\bar{y}$, Bahl and Tuteja (1991) estimators t_1 and t_2.

For the purpose of illustration for two-phase sampling, we consider following populations:

Population III: Murthy (1967)

    y : Output
    x : Number of workers
$C_y = 0.3542$, $C_x = 0.9484$, $\rho = 0.9150$, $N = 80$, $n' = 20$, $n = 8$.

Population IV: Steel and Torrie(1960)
$C_y = 0.4803$, $C_x = 0.7493$, $\rho = -0.4996$, $N = 30$, $n' = 12$, $n = 4$.

In table 5.3 the values of scalars w_i's $(i = 0,1,2)$ are listed.

Table 5.3: Values of $w_i\text{'s}(i = 0,1,2)$

Scalars	Population I	Population II
w_0	0.659	0.2415
w_1	0.808	0.0713
w_2	0.125	0.6871

Using these values of $w_i\text{'s}(i = 0,1,2)$ given in table 5.3 one can reduce the bias to the order $o(n^{-1})$ in the estimator t_w at 5.3.

In table 5.4 percent relative efficiency (PRE) of $\bar{y}$, t_{1d}, t_{2d} and t_w (in optimum case) are computed with respect to $\bar{y}$.

Table 5.4: PRE of different estimators of $\overline{\overline{Y}}$ **with respect to** $\bar{y}$**.**

Estimators	PRE $(., \bar{y})$	
	Population I	**Population II**
$\bar{y}$	100	100
t_{1d}	128.07	74.68
t_{2d}	41.42	103.64
t_w	138.71	106.11

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Almost Unbiased Ratio and Product Type Estimator of Finite Population Variance Using the Knowledge of Kurtosis of an Auxiliary Variable in Sample Surveys

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Abstract

It is well recognized that the use of auxiliary information in sample survey design results in efficient estimators of population parameters under some realistic conditions. Out of many ratio, product and regression methods of estimation are good examples in this context. Using the knowledge of kurtosis of an auxiliary variable Upadhyaya and Singh (1999) have suggested an estimator for population variance. In this paper, following the approach of Singh and Singh (1993), we have suggested almost unbiased ratio and product-type estimators for population variance.

1. Introduction

Let $U = (U_1, U_2, \ldots, U_N)$ denote a population of N units from which a simple random sample without replacement (SRSWOR) of size n is to be drawn. Further let y and x denote the study and the auxiliary variables respectively. The problem is to estimate the parameter

$$S_y^2 = \frac{N}{N-1} \sigma_y^2 \qquad (1.1)$$

with $\sigma_y^2 = \frac{1}{N} \sum_{i=1}^{N} (y_i - \overline{Y})^2$ of the study variate y when the parameter

$$S_x^2 = \frac{N}{N-1} \sigma_x^2 \qquad (1.2)$$

with $\sigma_x^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \overline{X})^2$ of the auxiliary variate x is known,

where $\overline{Y} = \sum_{i=1}^{N} \frac{y_i}{N}$ and $\overline{X} = \sum_{i=1}^{N} \frac{x_i}{N}$; are the population means of y and x respectively.

The conventional unbiased estimator of S_y^2 is defined by

$$s_y^2 = \frac{\sum_{i=1}^{n} (y_i - \overline{Y})^2}{(n-1)} \qquad (1.3)$$

where $\overline{y} = \sum_{i=1}^{n} \frac{y_i}{n}$ is the sample mean of y.

Using information on S_x^2, Isaki (1983) proposed a ratio estimator for S_y^2 as

$$t_1 = s_y^2 \frac{S_x^2}{s_x^2} \qquad (1.4)$$

where $s_x^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (x_i - \overline{x})^2$ is unbiased estimator of S_x^2.

In many survey situations the values of the auxiliary variable x may be available for each unit in the population. Thus the value of the kurtosis $\beta_2(x)$ of the auxiliary variable x is known. Using information on both S_x^2 and $\beta_2(x)$ Upadhyay and Singh (1999) suggested a ratio type estimator for S_y^2 as

$$t_2 = s_y^2 \left[\frac{S_x^2 + \beta_2(x)}{s_x^2 + \beta_2(x)} \right] \tag{1.5}$$

For simplicity suppose that the population size N is large enough relative to the sample size n and assume that the finite population correction (fpc) term can be ignored. Up to the first order of approximation, the variance of s_y^2, and t_1 and bias and variances of t_2 (ignoring fpc term) are respectively given by

$$\text{var}(s_y^2) = \frac{S_y^4}{n} \{\beta_2(y) - 1\} \tag{1.6}$$

$$\text{var}(t_1) = \frac{S_y^4}{n} \left[\{\beta_2(y) - 1\} + \{\beta_2(x) - 1\}(1 - 2C) \right] \tag{1.7}$$

$$B(t_2) = \frac{S_y^2}{n} \left[\{\beta_2(x) - 1\}\theta(\theta - C) \right] \tag{1.8}$$

$$\text{var}(t_2) = \frac{S_y^4}{n} \left[\{\beta_2(y) - 1\} + \theta\{\beta_2(x) - 1\}(\theta - 2C) \right] \tag{1.9}$$

where $\theta = \dfrac{S_x^2}{S_x^2 + \beta_2(x)}$; $\beta_2(y) = \dfrac{\mu_{40}}{\mu_{20}^2}$; $\beta_2(x) = \dfrac{\mu_{04}}{\mu_{02}^2}$; $h = \dfrac{\mu_{22}}{(\mu_{20} \cdot \mu_{02})}$; $C = \dfrac{(h-1)}{\beta_2(x) - 1}$ and

$$\mu_{rs} = \frac{1}{N} \sum_{i=1}^{N} (y_i - \bar{Y})^r (x_i - \bar{X})^s.$$

From (1.8), we see that the estimator t_2 suggested by Upadhyay and Singh (1999) is a biased estimator. In some application bias is disadvantageous. This led authors to suggest almost unbiased estimators of S_y^2.

2. A class of ratio-type estimators

Consider $\quad t_{Ri} = s_y^2 \left(\dfrac{S_x^2 + \beta_2(x)}{s_x^2 + \beta_2(x)} \right)^i \quad$ such that $\ t_{Ri} \in R$, for $\ i = 1,2,3$; where R

denotes the set of all possible ratio-type estimators for estimating the population variance S_y^2. We define a class of ratio-type estimators for S_y^2 as –

$$t_r = \sum_{i=1}^{3} w_i t_{Ri} \in R, \tag{2.1}$$

where $\displaystyle\sum_{i=1}^{3} w_i = 1$ and w_i are real numbers. $\tag{2.2}$

For simplicity we assume that the population size N is large enough so that the fpc terms are ignored. We write

$$s_y^2 = S_y^2(1 + e_0), s_x^2 = S_x^2(1 + e_1)$$

such that $E(e_0) = E(e_1) = 0$.

Noting that for large N, $\dfrac{1}{N} \cong 0$ and $\dfrac{n}{N} \cong 0$, and thus to the first degree of approximation,

$$E(e_0^2) = \frac{\beta_2(y) - 1}{n}, \ E(e_1^2) = \frac{\beta_2(x) - 1}{n}, \ E(e_0 e_1) = \frac{(h - 1)}{n} = \frac{[\beta_2(x) - 1]C}{n}.$$

Expressing (2.1) in terms of e's we have

$$t_r = S_y^2(1 + e_0) \sum_{i=1}^{3} a_i (1 + \theta e_1)^{-i} \tag{2.3}$$

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Assume that $\left|\theta e_1\right| < 1$ so that $\left(1 + \theta e_1\right)^i$ is expandable. Thus expanding the right hand side of the above expression (2.3) and retaining terms up to second power of e's , we have

$$t_r = S_y^2 \left[1 + e_0 - \sum_{i=1}^{3} a_i i \left(\theta e_1 + \theta e_0 e_1 - \left(\frac{i+1}{2} \right) \theta^2 e_1^2 \right) \right]$$

or

$$t_r - S_y^2 = S_y^2 \left[e_0 - \sum_{i=1}^{3} a_i i \left(\theta e_1 + \theta e_0 e_1 - \left(\frac{i+1}{2} \right) \theta^2 e_1^2 \right) \right] \tag{2.4}$$

Taking expectation of both sides of (2.3) we get the bias of t_r , to the first degree of approximation, as

$$B(t_r) = \frac{S_y^2}{2n} \left[\{\beta_2(x) - 1\} \sum_{i=1}^{3} i a_i \theta(\theta i - 2C + \theta) \right] \tag{2.5}$$

Squaring both sides of (2.4), neglecting terms involving power of e's greater than two and then taking expectation of both sides, we get the mean-squared error of t_r to the first degree of approximation, as

$$MSE(t_r) = \frac{S_y^4}{n} \left[\{\beta_2(y) - 1\} + R_1 \{\theta\beta_2(x) - 1\}\{\theta R_1 - 2C\} \right] \tag{2.6}$$

where $R_1 = \sum_{i=1}^{3} i.w_i$ (2.7)

Minimizing the MSE of t_r in (2.7) with respect to R_1 we get the optimum value of R_1 as

$$R_1 = \frac{C}{\theta} \tag{2.8}$$

Thus the minimum MSE of t_r is given by

$$\text{min.MSE}(t_r) = \frac{S_y^4}{n} \left[\{\beta_2(y) - 1\} - \{\beta_2(x) - 1\}C^2 \right]$$

$$= \frac{S_y^4}{n} \left[\{\beta_2(y) - 1\}(1 - \rho_1^2) \right] \tag{2.9}$$

where $\rho_1 = \dfrac{(h-1)}{\sqrt{\{\beta_2(x)-1\}\{\beta_2(y)-1\}}}$ is the correlation coefficient between $(y - \bar{Y})^2$

and $(x - \bar{X})^2$.

From (2.2), (2.7) and (2.8) we have

$$\sum_{i=1}^{3} w_i = 1 \tag{2.10}$$

and $\displaystyle\sum_{i=1}^{3} i w_i = \frac{C}{\theta} = \frac{\rho_1}{\theta} \left\{ \frac{\beta_2(y)-1}{\beta_2(x)-1} \right\}^{\frac{1}{2}}$ (2.11)

From (2.10) and (2.11) we have three unknown to be determined from two equations

only. It is therefore, not possible to find a unique value of the constants $w_i's(i=1,2,3)$.

Thus in order to get the unique values of the constants $w_i's(i=1,2,3)$, we shall impose a

linear constraint as

$$B(t_r) = 0 \tag{2.12}$$

which follows from (2.5) that

$$(\theta - C)a_1 + (3\theta - 2C)a_2 + (6\theta - 3C)a_3 = 0 \tag{2.13}$$

Equation (2.10), (2.11) and (2.13) can be written in   the matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ (\theta - C) & (3\theta - 2C) & (6\theta - 3C) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ C / \theta \\ 0 \end{bmatrix} \tag{2.14}$$

Using (2.14) we get the unique values of $w_i's(i=1,2,3)$ as

$$w_1 = \frac{1}{\theta^2}\left[3\theta^2 - 3\theta C + C^2\right]$$
$$w_2 = \frac{1}{\theta^2}\left[-3\theta^2 + 5\theta C - 2C^2\right] \Bigg\} \qquad (2.15)$$
$$w_3 = \frac{1}{\theta^2}\left[\theta^2 - 2\theta C + C^2\right]$$

Use of these $w_i's(i=1,2,3)$ remove the bias up to terms of order $o(n^{-1})$ at (2.1).

Substitution of (2.14) in (2.1) yields the almost unbiased optimum ratio-type estimator of

the population variance S_y^2.

3. A class of product-type estimators

Consider $t_{Pi} = s_y^2 \left[\dfrac{s_x^2 + \beta_2(x)}{S_x^2 + \beta_2(x)}\right]^i$ such that $t_{Pi} \in P$, for $i = 1,2,3$; where P denotes

the set of all possible product-type estimators for estimating the population variance S_y^2.

We define a class of product-type estimators for S_y^2 as –

$$t_P = \sum_{i=1}^{3} k_i t_{Pi} \in P, \qquad (3.1)$$

where $k_i's$ $(i=1,2,3)$ are suitably chosen scalars such that

$$\sum_{i=1}^{3} k_i = 1 \text{ and } k_i \text{ are real numbers.}$$

Proceeding as in previous section, we get

$$B(t_P) = \frac{S_y^2}{2n}\left[\{\beta_2(x) - 1\}\sum_{i=1}^{3} i a_i \theta(\theta i + 2C - \theta)\right] \qquad (3.2)$$

$$MSE(t_P) = \frac{S_y^4}{n}\left[\{\beta_2(y) - 1\} + R_2\theta\{(\beta_2(x) - 1)\}(\theta R_2 + 2C)\right] \qquad (3.3)$$

where, $R_2 = \sum_{i=1}^{3} ik_i$ \qquad (3.4)

Minimizing the MSE of t_P in (3.4) with respect to R_2, we get the optimum value of R_2 as

$$R_2 = -\frac{C}{\theta} \qquad (3.5)$$

Thus the minimum MSE of t_P is given by

$$\min.MSE(t_P) = \frac{S_y^4}{n}\{\beta_2(y) - 1\}(1 - \rho_1^2) \qquad (3.7)$$

which is same as that of minimum MSE of t_r at (2.9).

Following the approach of previous section, we get

$$\left.\begin{array}{l} k_1 = \frac{1}{\theta^2}\left[3\theta^2 + 2\theta C + C^2\right] \\[2mm] k_2 = -\frac{1}{\theta^2}\left[3\theta^2 + 3\theta C + 2C^2\right] \\[2mm] k_3 = \frac{1}{\theta^2}\left[\theta^2 + \theta C + C^2\right] \end{array}\right\} \qquad (3.8)$$

Use of these k_i's (i=1,2,3) removes the bias up to terms of order O (n[-1]) at (3.1).

4. Empirical Study

The data for the empirical study are taken from two natural population data sets

considered by Das (1988) and Ahmed *et. al.* (2003).

Population I – Das (1988)

The variables and the required parameters are:

X: number of agricultural laborers for 1961.

Y: number of agricultural laborers for 1971.

$\beta_2(x) = 38.8898, \beta_2(y) = 25.8969, h=26.8142, S_x^2 = 1654.44$.

Population II – Ahmed et. al. (2003)

The variables and the required parameters are:

X: number of households

Y: number of literate persons

$\beta_2(x) = 8.05448, \beta_2(y) = 10.90334, S_x^2 = 11838.85, h=7.31399$.

In table 4.1 the values of scalars w_i's (i=1,2,3) and k_i's (i=1,2,3) are listed.

Table 4.1: Values of scalars w_i's and k_i's (i=1,2,3)

Scalars	Population		Scalars	Population	
	I	**II**		**I**	**II**
w_1	1.3942	1.1154	k_1	4.8811	5.5933
w_2	-0.4858	-0.1261	k_2	-6.0647	-7.2910
w_3	0.0916	0.0109	k_3	2.1837	2.6978

Using these values of w_i's and k_i's (i=1,2,3) given in table 4.1,one can reduce the bias to the order $O(n^{-1})$ respectively, in the estimators t_r and t_p at (2.1) and (3.1).

In table 4.2 percent relative efficiency (PRE) of s_y^2, t_1, t_2, t_r (in optimum case) and t_p (in optimum case) are computed with respect to s_y^2.

Table 4.2: PRE of different estimators of S_y^2 with respect to s_y^2

Estimators	PRE $(., S_y^2)$	
	Population I	**Population II**
S_y^2	100	100
t_1	223.14	228.70
t_2	235.19	228.76
t_r (optimum)	305.66	232.90
t_p (optimum)	305.66	232.90

Table 4.2 clearly shows that the suggested estimators t_r and t_p in their optimum case are better than the usual unbiased estimator s_y^2, Isaki's (1983) estimator t_1 and Upadhayaya and Singh (1999) estimator t_2.

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A General Family of Estimators for Estimating Population Variance Using Known Value of Some Population Parameter(s)

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Abstract

A general family of estimators for estimating the population variance of the variable under study, which make use of known value of certain population parameter(s), is proposed. Some well known estimators have been shown as particular member of this family. It has been shown that the suggested estimator is better than the usual unbiased estimator, Isaki's (1983) ratio estimator, Upadhyaya and Singh's (1999) estimator and Kadilar and Cingi (2006). An empirical study is carried out to illustrate the performance of the constructed estimator over others.

Keywords: Auxiliary information, variance estimator, bias, mean squared error.

1. Introduction

In manufacturing industries and pharmaceutical laboratories sometimes researchers are interested in the variation of their produce or yields (Ahmed et.al. (2003)).

Let $(U = U_1, U_2, \ldots, U_N)$ denote a population of N units from which a simple random sample without replacement (SRSWOR) of size n is drawn. Further let y and x denote the study and the auxiliary variables respectively.

Let $\overline{Y} = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} y_i$ and $S_y^2 = \dfrac{1}{N-1}\displaystyle\sum_{i=1}^{n}(y_i - \overline{Y})^2$ denotes respectively the unknown population mean and population variance of the study character y. Assume that population size N is very large so that the finite population correction term is ignored. It is established fact that in most of the survey situations, auxiliary information is available (or may be made to be available diverting some of the resources) in one form or the other. If used intelligibly, this information may yield estimators better than those in which no auxiliary information is used.

Assume that a simple random sample of size n is drawn without replacement. The usual unbiased estimator of S_y^2 is

$$s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \overline{y})^2 \tag{1.1}$$

where $\overline{y} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} y_i$ is the sample mean of y.

When the population mean square $S_x^2 = \dfrac{1}{N-1}\displaystyle\sum_{i=1}^{N}(x_i - \overline{X})^2$ is known, Isaki (1983) proposed a ratio estimator for S_y^2 as

$$t_1 = \left(\frac{s_y^2}{s_x^2}\right) S_x^2 \tag{1.2}$$

where $s_x^2 = \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n}(x_i - \overline{x})^2$ is an unbiased estimator of S_x^2.

Several authors have used prior value of certain population parameter(s) to find more precise estimates. The use of prior value of coefficient of kurtosis in estimating the population variance of study character y was first made by Singh et. al. (1973). Kadilar and Cingi (2006) proposed modified ratio estimators for the population variance using different combinations of known values of coefficient of skewness and coefficient of variation.

In this paper, under SRSWOR, we have suggested a general family of estimators for estimating the population variance S_y^2. The expressions of bias and mean-squared error (MSE), up to the first order of approximation, have been obtained. Some well known estimators have been shown as particular member of this family.

2. The suggested family of estimators

Motivated by Khoshnevisan et. al. (2007), we propose following ratio-type estimators for the population variance as

$$t = s_y^2 \frac{\left(aS_x^2 - b\right)}{[\alpha(as_x^2 - b) + (1 - \alpha)(aS_x^2 - b)]} \tag{2.1}$$

where $(a \neq 0)$, b are either real numbers or the function of the known parameters of the auxiliary variable x such as coefficient of variation $C(x)$ and coefficient of kurtosis $(\beta_2(x))$.

The following scheme presents some of the important known estimators of the population variance, which can be obtained by suitable choice of constants α, a and b:

Table 2.1 : Some members of the proposed family of the estimators 't'

Estimator	Values of		
	α	**a**	**b**
$t_0 = s_y^2$	0	0	0
$t_1 = \dfrac{s_y^2}{s_x^2} S_x^2$ Isaki (1983) estimator	1	1	0
$t_2 = \dfrac{s_y^2}{s_x^2 - C_x}[S_x^2 - C_x]$ Kadilar and Cingi (2006) estimator	1	1	C_x
$t_3 = \dfrac{s_y^2}{s_x^2 - \beta_2(x)}[S_x^2 - \beta_2(x)]$	1	1	$\beta_2(x)$
$t_4 = \dfrac{s_y^2}{s_x^2 \beta_2(x) - C_x}[S_x^2 \beta_2(x) - C_x]$	1	$\beta_2(x)$	C_x
$t_5 = \dfrac{s_y^2}{s_x^2 C_x - \beta_2(x)}[S_x^2 C_x - \beta_2(x)]$	1	C_x	$\beta_2(x)$
$t_6 = \dfrac{s_y^2}{s_x^2 + \beta_2(x)}[S_x^2 + \beta_2(x)]$ Upadhyaya and Singh (1999)	1	1	$-\beta_2(x)$

The MSE of proposed estimator 't' can be found by using the firs degree approximation in the Taylor series method defined by

$$MSE(t) \cong d \sum d'$$ (2.2)

where

$$h = \left[\left. \frac{\partial h(a,b)}{\partial a} \right|_{s_y^2, s_x^2} \qquad \left. \frac{\partial h(a,b)}{\partial b} \right|_{s_y^2, s_x^2} \right]$$

$$\Sigma = \begin{bmatrix} V(s_y^2) & Cov(s_y^2, s_x^2) \\ Cov(s_x^2, s_y^2) & V(s_x^2) \end{bmatrix}.$$

Here $h(a,b) = h(s_y^2, s_x^2) = t$. According to this definition, we obtain 'd' for the proposed estimator, t, as follows:

$$d = \left[1 \qquad -\frac{\alpha a S_y^2}{a S_x^2 + b} \right]$$

MSE of the proposed estimator t using (2.2) is given by

$$MSE(t) \cong V(s_y^2) - 2\alpha \left(\frac{a S_y^2}{a S_x^2 - b} \right) Cov(s_y^2, s_x^2) + \left(\frac{\alpha a S_y^2}{a S_x^2 - b} \right) V(s_x^2) \qquad (2.3)$$

where

$$\left. \begin{aligned} V(s_y^2) &= \lambda S_y^4 [\beta_2(y) - 1] \\ V(s_x^2) &= \lambda S_y^4 [\beta_2(x) - 1] \\ Cov(s_y^2, s_x^2) &= \lambda S_y^2 S_x^2 (h - 1) \end{aligned} \right\} \qquad (2.4)$$

where $\lambda = \dfrac{1}{n}$, $\beta_2(y) = \dfrac{\mu_{40}}{\mu_{20}^2}$, $\beta_2(x) = \dfrac{\mu_{04}}{\mu_{02}^2}$, $h = \dfrac{\mu_{22}}{\mu_{20}\mu_{02}}$,

$$\mu_{rs} = \frac{1}{N} \sum_{i=1}^{N} (y_i - \overline{Y})^r (x_i - \overline{X})^s \text{ , (r, s) being non negative integers.}$$

Using (2.4), MSE of t can be written as

$$MSE(t) \cong \lambda S_y^4 \left\{ \beta_2(y) - 1 - 2\alpha\theta(h - 1) + \alpha^2\theta^2 (\beta_2(x) - 1) \right\} \qquad (2.5)$$

where $\theta = \dfrac{a S_x^2}{a S_x^2 - b}$.

The MSE equation of estimators listed in Table2.1 can be written as-

$$\text{MSE}(t_i) \cong \lambda S_y^4 \{\beta_2(y) - 1 - 2\alpha\theta_i(h-1) + \alpha^2\theta_i^2(\beta_2(x) - 1)\}, \quad i = 2,3,4,5,6 \qquad (2.6)$$

where

$$\theta_2 = \frac{S_x^2}{S_x^2 - C_x}, \qquad\qquad \theta_3 = \frac{S_x^2}{S_x^2 - \beta_2(x)},$$

$$\theta_4 = \frac{S_x^2\beta_2(x)}{S_x^2\beta_2(x) - C_x}, \quad \theta_5 = \frac{S_x^2 C_x}{S_x^2 C_x - \beta_2(x)}, \qquad \theta_6 = \frac{S_x^2}{S_x^2 + \beta_2(x)}.$$

Minimization of (2.5) with respect to α yields its optimum value as

$$\alpha = \frac{C}{\theta} = \alpha_{opt} \qquad (2.7)$$

where $C = \dfrac{(h-1)}{\{\beta_2(x) - 1\}}$.

By substituting α_{opt} in place of α in (2.5) we get the resulting minimum variance of t as

$$\text{min.MSE}(t) = \lambda S_y^4 [\beta_2(y) - 1 - \{\beta_2(x) - 1\}] \qquad (2.8)$$

3. Efficiency comparisons

Up to the first order of approximation, variance (ignoring finite population correction) of $t_o = s_y^2$ and t_1 is given by –

$$\text{Var}(s_y^2) = \lambda S_y^4 [\beta_2(y) - 1] \qquad (3.1)$$

$$\text{MSE}(t_1) = \lambda S_y^4 [\{\beta_2(y) - 1\} + \{\beta_2(x) - 1\}(1 - 2C)] \qquad (3.2)$$

From (2.6), (2.8), (3.1), and (3.2), we have

$$\text{Var}(s_y^2) - \text{min.MSE}(t) = \lambda S_y^4 \{\beta_2(x) - 1\}C^2 > 0 \qquad (3.3)$$

$$\text{MSE}(t_i) - \text{min.MSE}(t) = \lambda S_y^4 \{\beta_2(x) - 1\}(\theta_i - C^2) > 0, \quad i = 1,2,3,4,5,6 \qquad (3.4)$$

provided $C \neq \theta_i$.

Thus it follows from (3.3) and (3.4) that the suggested estimator under 'optimum' condition is (i) always better then s_y^2, (ii) better than Isaki's (1983) estimator t_1 except when C = 1 in which both are equally efficient, and (iii) Kadilar and Cingi (2006) estimators $t_i (i = 2,3,4,5)$ except when $C = \theta_i (i = 2,3,4,5)$ in which t and $t_i (i = 2,3,4,5)$ are equally efficient.

4. Empirical study

We use data in Kadilar and Cingi (2004) to compare efficiencies between the traditional and proposed estimators in the simple random sampling.

In Table 4.1, we observe the statistics about the population.

Table 4.1: Data statistics of the population for the simple random sampling

$$N = 106,\ n = 20,\ \rho = 0.82,\ C_y = 4.18,\ C_x = 2.02,\ \overline{Y} = 15.37,\ \overline{X} = 243.76,$$

$$S_y = 64.25,\ S_x = 491.89,\ \beta_2(x) = 25.71,\ \beta_2(y) = 80.13,\ \lambda = 0.05,\ \theta = 33.30.$$

The percent relative efficiencies of the estimators s_y^2, $t_i (i = 2,3,4,5,6)$ and min.MSE(t) with respect to s_y^2 have been computed and presented in Table 4.2 below.

Table 4.2: Relative efficiencies (%) of s_y^2, $t_i (i = 2,3,4,5,6)$ and $\min.\mathrm{MSE}(t)$ with respect to s_y^2.

Estimator	PRE $(., s_y^2)$
$t_0 = s_y^2$	100
t_1	201.6564
t_2	201.6582
t_3	201.6782
t_4	201.6565
t_5	201.6672
t_6	201.6347
$\min.\mathrm{MSE}(t)$	214.3942

5. Conclusion

From theoretical discussion in section 3 and results of the numerical example, we infer that the proposed estimator 't' under optimum condition performs better than usual estimator s_y^2, Isaki's (1983) estimator t_1, Kadilar and Cingi's (2006) estimators (t_2, t_3, t_4, t_5) and Upadhyaya and Singh's (1999) estimator t_6.

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This volume is a collection of six papers on the use of auxiliary information and *a priori* values in construction of improved estimators. The work included here will be of immense application for researchers and students who employ auxiliary information in any form.