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MOD Pseudo Linear Algebras

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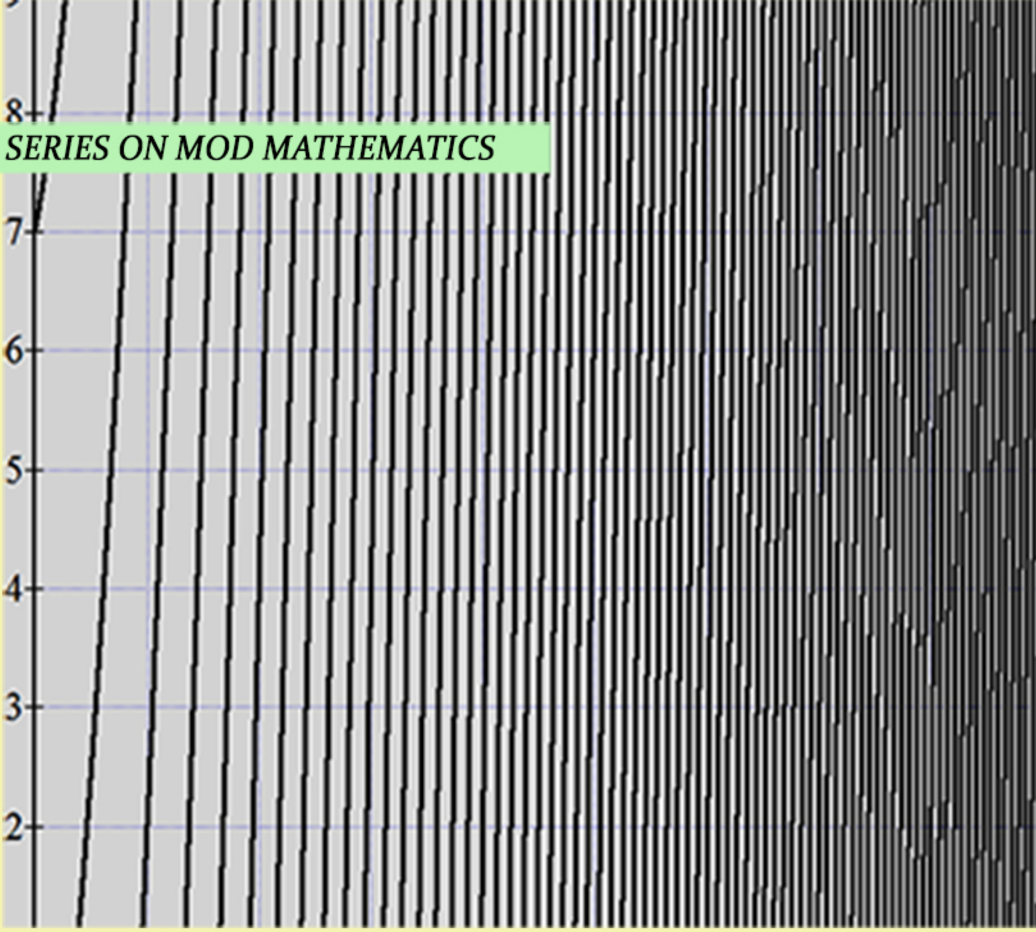

SERIES ON MOD MATHEMATICS

MOD PSEUDO LINEAR ALGEBRAS

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PREFACE

In this book authors for the first time elaborately study the notion of MOD vector spaces and MOD pseudo linear algebras. This study is new, innovative and leaves several open conjectures. In the first place as distributive law is not true we can define only MOD pseudo linear algebras.

Secondly most of the classical theorems true in case of linear algebras are not true in case of MOD pseudo linear algebras. Finding even eigen values and eigen vectors happens to be a challenging problem. Further the notion of multidimensional MOD pseudo linear algebras are defined using the notion of MOD mixed matrices.

These function only under the natural product $\times_n$ as the usual product $\times$ cannot be even defined on these mixed MOD matrices.

Several innovative and interesting results are given in this book. Many open problems are proposed.

We wish to acknowledge Dr. K Kandasamy for his sustained support and encouragement in the writing of this book.

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Chapter One

INTRODUCTION

In this chapter we introduce the notion of MOD interval, MOD plane and MOD transformation. For more about these notions [26-30].

Throughout this book $(-\infty, \infty)$ is the real line, Z_m the ring of modulo integers m and R the real plane. C the complex plane. $(-\infty i, \infty i)$ the imaginary line, R(g) ($g^2 = 0$) the dual number plane, R(h) ($h^2 = h$) the special dual like number plane. R(k); ($k^2 = -k$) the special quasi dual number plane, $C(Z_m)$ the finite complex modulo integer $i_F^2 = m - 1$.

$(-\infty g, \infty g)$, ($g^2 = 0$) the dual number line. $(-\infty h, \infty h)$ ($h^2 = h$); the special dual like number line. $(-\infty k, \infty k)$ the special quasi dual number line. $k^2 = -k$. $\langle R \cup I\rangle$ is the real neutrosophic plane. $(-\infty I, \infty I)$ is the neutrosophic number line.

$$\langle R \cup I\rangle = \{a + bI \mid a, b \in R, I^2 = I\},$$

$$R(g) = \{a + bg \mid g^2 = 0, a, b \in R\},$$

$$R(h) = \{a + bh \mid h^2 = h, a, b \in R\},$$

$$C = \{a + bi \mid a, b \in R, i^2 = -1\} \text{ and}$$

$$R(k) = \{a + bk \mid a, b \in R, k^2 = -k\}$$

denotes the neutrosophic plane, dual number plane, special dual like number plane, complex plane and special quasi dual number plane respectively.

$(-\infty, \infty)$ is the real line,

$$(-\infty g, \infty g) = \{mg \mid m \in R, g^2 = 0\},$$

$$(-\infty I, \infty I) = \{mI \mid m \in R, I^2 = 0\},$$

$$(-\infty i, \infty i) = \{mi \mid m \in R, i^2 = -1\},$$

$$(-\infty h, \infty h) = \{mh \mid m \in R, h^2 = h\} \text{ and}$$

$$(-\infty k, \infty k) = \{mk \mid m \in R, k^2 = -k^2\}$$

are the real line, dual line, neutrosophic line, complex number line, special dual like number line and special quasi dual number line respectively.

We can now proceed onto define the new notion of MOD transformation. For this to occur we recall the definition of MOD intervals of 6 types and the MOD planes of 6 types [26-30].

Let $[0, m)$; $2 \leq m < \infty$ be the interval. We define this interval as MOD real interval.

$[0, 7)$, $[0, 12)$, $[0, 148)$, $[0, 19)$ and so on $[0, m)$; $2 \leq m < \infty$ are defined as MOD real intervals.

$[0, 1)$ is defined as the fuzzy MOD interval or MOD fuzzy interval [26-30].

We have infinite number of real MOD intervals but only one MOD fuzzy interval.

[0, 43) is the MOD real interval.

[0, m) = {a | a ∈ [0, m), that is a cannot take the value m}.

[0, 5) = {a | 0 ≤ a < 5; that is a cannot take the value 5}.

[0, 24) = {a | 0 ≤ a < 24; that is a can take 23.99…9 but cannot take a = 24}.

The advantage of using MOD real intervals is there exists infinitely many MOD real intervals but however one and only one real interval (–∞, ∞).

Next we define the complex interval (–∞i, ∞i). We see we have one and only one complex interval (–∞i, ∞i).

However $[0, m)i_F$; $2 \leq m < \infty$ is the MOD complex interval where $i_F^2 = m - 1$.

We have infinite number of MOD complex intervals for each of this $i_F^2 = m - 1$, $2 \leq m < \infty$.

$[0, 3)i_F$ is the MOD complex interval; $i_F^2 = 2$ and so on.

$[0, 42)i_F$ is the MOD complex interval; $i_F^2 = 41$ and so on.

$[0, 14)i_F = \{ai_F \mid i_F^2 = 13; 0 \leq ai_F < 14i_F\}$.

$[0, 215)i_F = \{ai_F \mid 0 \leq ai_F < 215i_F, i_F^2 = 24\}$

are the MOD complex interval.

Next we proceed onto describe MOD neutrosophic intervals (–∞I, ∞I); $I^2 = I$ is defined as the neutrosophic interval.

$[0, m)I = \{aI \mid 0 \leq aI < mI\}$ is defined as the neutrosophic interval. Clearly as $2 \leq m < \infty$ we have infinitely many neutrosophic intervals.

$$[0, 5)I = \{aI \mid 0 \leq aI < 5I\},$$

$$[0, 24)I = \{aI \mid 0 \leq aI < 24I\} \text{ and so on?}$$

We now proceed onto describe and recall the MOD dual number intervals.

The dual number interval is $(-\infty g, \infty g)$; $g^2 = 0$. The MOD dual number intervals are $[0, m)g$; $2 \leq m < \infty$, $g^2 = 0$.

$$[0, 12)g = \{ag \mid 0 \leq ag < 12g\}$$

$$[0, 19)g = \{ag \mid 0 \leq ag < 19g\}$$

$$[0, 48)g = \{ag \mid 0 \leq ag < 48g\} \text{ and so on.}$$

We have infinite number of MOD dual number intervals however we have only one infinite dual number interval $(-\infty g, \infty g)$.

Next the notion of special dual like number interval

$$(-\infty h, \infty h);\ h^2 = h \text{ is described,}$$
$$(-\infty h, \infty h) = \{ah \mid -\infty h \leq ah \leq \infty h\}.$$

The corresponding MOD special dual like number interval is $[0, m)h = \{ah;\ h^2 = h,\ 0 \leq ah < mh\}$; $2 \leq m < \infty$. In fact we have infinite number of MOD special dual like number intervals.

$$[0, 9)h = \{ah \mid h^2 = h;\ 0 \leq ah < 9h\},$$

$$[0, 16)h = \{ah \mid h^2 = h;\ 0 \leq ah < 16h\}$$

$$[0, 19)h = \{ah \mid h^2 = h;\ 0 \leq ah < 19h\}$$

and so on are all MOD special dual like number intervals.

The MOD special quasi dual number interval is [0, mk) where $k^2 = (m - 1)k$. This is got from the real special quasi dual number interval $(-\infty k, \infty k)$, $k^2 = -k$.

$[0, 12)k = \{ak \mid k^2 = 11k, a \in [0, 12)\}$ is the MOD special quasi dual number interval.

$[0, 43)k = \{bk \mid b \in [0, 43), k^2 = 42k\}$ is also a MOD special quasi dual number interval.

$[0, 27)k = \{ak \mid k^2 = 26k, a \in [0, 27)\}$ is also a MOD special quasi dual number interval.

In fact we have only one real special quasi dual number interval viz $(-\infty k, \infty k)$ but however we have infinite number of MOD special quasi dual number intervals got as [0, m)k where $k^2 = (m - 1)k$; $m \in Z^+ \setminus \{0, 1\}$.

This is the main advantage in using these MOD intervals. Such multichoice of values that too using small (MOD) intervals happens to be an interesting research.

Thus there are also MOD dual number interval sets generated by

$$\{\langle [0, m) \cup g\rangle\} = \{a + bg \mid a, b \in [0, m); g^2 = 0\}.$$

As $m \in Z^+ \setminus \{1\}$ we have infinitely many such MOD dual number interval sets which are also MOD dual number planes.

For instance

$$P_1 = \{\langle [0, 20) \cup g\rangle \mid g^2 = 0, a + bg, a, b \in [0, 20)\},$$

$$P_2 = \{\langle [0, 29) \cup g\rangle \mid g^2 = 0, a + bg, a, b \in [0, 29)\},$$

$$P_3 = \{\langle [0, 5) \cup g\rangle \mid g^2 = 0, a, b \in [0, 5); a + bg\}$$

and so on are all MOD dual number interval sets.

Clearly $[0, m)g \subseteq \{\langle[0, m) \cup g\rangle\}$ is a proper subset of $\{\langle[0, m) \cup g\rangle \mid a + bg, a, b \in [0, m); g^2 = 0\}$.

We see $\langle R \cup g\rangle = \{a + bg \mid a, b \in R; g^2 = 0$, R-reals$\}$ is the infinite real dual number plane.

Likewise $\langle R \cup i\rangle = \{a + bi \mid a, b \in R; i^2 = -1$, R reals$\} = C$; is the complex plane.

Thus with default of notations we represent $\langle(-\infty, \infty) \cup g\rangle = \langle R \cup g\rangle$, $\langle(-\infty, \infty) \cup i\rangle = \langle R \cup i\rangle = C$ and $\langle(-\infty, \infty) \cup I\rangle = \langle R \cup I\rangle = R(I)$ the real dual number plane or interval, the infinite complex plane or interval and the infinite neutrosophic interval or plane.

Further $\langle(-\infty, \infty) \cup h\rangle = \langle R \cup h\rangle = R(h)$ where $h^2 = h$ is the real special dual like number plane or interval and $\langle(-\infty, \infty) \cup k\rangle = R(k)$ ($k^2 = -k$) is the real special quasi dual number plane. So we may call it as plane or interval by default of notions.

The same is true in case of MOD real neutrosophic interval or plane $\langle[0, m) \cup I\rangle = R_n^I(m)$, the MOD real dual number plane or interval. $\langle[0, m) \cup g\rangle = R_n^g(m)$ $g^2 = 0$, $\langle[0, m) \cup i_F\rangle = C_n(m)$ ($i_F^2 = m - 1$) the infinite complex modulo integer interval or plane.

$\langle[0, m) \cup h\rangle = R_n^h(m)$; $h^2 = h$ be the infinite special dual like number interval or plane.

$\langle[0, m) \cup k\rangle = R_n^k(m)$; $k^2 = (m - 1)k$ be the infinite special quasi dual number interval or plane or space.

We will represent this by some examples.

Example 1.1: Let $S = \{\langle[0, 12) \cup g\rangle; g^2 = 0\} = R_n^g(12) = \{a + bg \mid a, b \in [0, 12); g^2 = 0\}$ be the MOD dual number plane.

We can define + and × on S. (S, +) is an infinite abelian group and (S, ×) is an infinite abelian semigroup. However $a \times (b + c) \neq a \times b + a \times c$ in general for a, b, c ∈ S. {S, +, ×} is defined [30] as the MOD infinite dual number pseudo ring.

Example 1.2: Let N = {⟨[0, 11) ∪ I⟩} = { $R_n^I(11) = a + bI$, a, b ∈ [0, 11); $I^2 = I$} be the MOD infinite neutrosophic set. {N, ×} is the MOD infinite neutrosophic semigroup.

Clearly {N, ×} a commutative infinite semigroup. [0, 11)I is an ideal of infinite order. {N, +} is the MOD infinite neutrosophic group. {N, +, ×} be the MOD neutrosophic interval pseudo ring.

Example 1.3: Let S = {⟨[0, 24) ∪ I⟩; $I^2 = I$, +, ×} be the pseudo MOD neutrosophic interval ring. This ring has zero divisors, units and idempotents.

Next examples of MOD special dual like number groups, semigroups and pseudo rings are defined.

Example 1.4: Let S = {⟨[0, 16) ∪ h⟩, $h^2 = h$, +} be the MOD special dual like number group of infinite order. This has subgroups of finite and infinite order.

Example 1.5: Let M = {⟨[0, 23) ∪ h⟩ $h^2 = h$, ×} be the MOD special dual like number semigroup. M is of infinite order. M has ideals and subsemigroups. Finite order subsemigroups of M are not ideals of M. M has idempotents, zero divisors and units. M is a S-semigroup as B = {Z_{23} \ {0}, ×} ⊆ M is a subgroup.

D = {$Z_{23}h$, ×} ⊆ M is a subsemigroup of finite order.

E = {0, 0.5, 1, 1.5, 2, …, 21, 21.5, 22, 22.5} ⊆ M is only a subset and is not a subsemigroup under ×.

However E generates under product an infinite subsemigroup of M which is not an ideal of M.

Thus P = $\{\langle 0.1\rangle, \times\}$ can generate a subsemigroup of infinite order which is not an ideal of M.

In fact M has several such infinite order subsemigroups.

Finding ideals in M is a difficult task.

However L = $\{[0, 23)h, \times\}$ is an ideal of infinite order in L.

Finding other ideals is a challenging task.

Now we can build MOD special dual like number pseudo rings.

This is illustrated by the following example.

Example 1.6: Let M = $\{\langle [0, 48) \cup h\rangle, +, \times; h^2 = h\}$ be the MOD special dual like number pseudo ring.

Clearly M has units, zero divisors, pseudo zero divisors and idempotents. In fact M has finite subrings which are not pseudo.

For $\{Z_{48}, +, \times\} = P_1$ is a subring of order 48.

$P_2 = \{2Z_{48}, +, \times\} \subseteq M$ is a subring of order 24.

$P_3 = \{3Z_{48}, +, \times\}$ is a subring of order 16.

$P_4 = \{24, 0, +, \times\}$ is a subring of order two.

$P_5 = \{Z_{48}h, +, \times\}$ is a subring of order 48 and so on.

$P_6 = \{[0, 48), +, \times\}$ is a MOD pseudo special dual like number subring of infinite order which is not an ideal.

$P_7 = \{[0, 48)h, +, \times\}$ is a pseudo subring of special dual like numbers of infinite order which is an ideal of M.

In fact M has all ideals to be of infinite order. Finding ideals other than P_7 happens to be a very difficult task.

$P_8 = \{\langle Z_{48} \cup h\rangle, +, \times\}$ is again a subring of finite order which is not pseudo.

Example 1.7: Let $S = \{\langle [0, 13) \cup h\rangle, h^2 = h, +, \times\}$ be the MOD special dual like number pseudo ring. S has a few subrings of finite order. S has pseudo zero divisors as well as some zero divisors. S has units and idempotents.

Working with these MOD special dual like pseudo rings when n used in the interval [0, n) is a prime number is yet a difficult task for finding finite order subrings other than

$$P_1 = \{Z_{13}, +, \times\},$$

$$P_2 = \{Z_{13}h, +, \times\} \text{ and}$$

$$P_3 = \{\langle Z_{13} \cup h\rangle, +, \times\}$$

happens to be a very difficult task.

Next we proceed onto describe MOD special quasi dual number set and the algebraic structures on them using +, × and both + and ×.

Example 1.8: Let $M = \{\langle [0, 8) \cup k\rangle, k^2 = 7k\}$ be the MOD special quasi dual number set.

This is also the MOD special quasi dual number plane for $M = \{a + bk \mid k^2 = 7k, a, b \in [0, 8)\}$.

Here $N = \{[0, 8)k; k^2 = 7k\}$ is only a MOD special quasi dual number interval.

Example 1.9: Let $M = \{\langle [0, 19) \cup k\rangle\ k^2 = 18k\}$ be the MOD special quasi dual number interval plane.

$$P_1 = \{[0, 19)\} \text{ and}$$

$$P_2 = \{[0, 19)k\}$$

are MOD special quasi dual number interval subsets. Both are intervals. In fact P_1 is only a MOD real interval.

Example 1.10: Let M = $\{\langle[0, 12) \cup k\rangle, k^2 = 11k, +\}$ be the MOD special quasi dual number group under +. M has subgroups of both finite and infinite order.

$G_1 = \{Z_{12}, +\} \subseteq M$ is a subgroup of finite order.

$G_2 = \{Z_{12}k, +\} \subseteq M$ is also a subgroup of finite order.

$G_3 = \{\langle Z_{12} \cup k\rangle, k^2 = 11k, +\} \subseteq M$ is also a subgroup of finite order.

$G_4 = \{[0, 12); +\} \subseteq M$ is a subgroup of infinite order.

$G_5 = \{[0, 12)k, k^2 = 11k, +\}$ is also a subgroup of infinite order.

Example 1.11: Let M = $\{\langle[0, 187) \cup k\rangle, k^2 = 186k, +\}$ be the MOD special quasi dual number group. This has subgroups of both finite and infinite order.

Next a few examples of MOD special quasi dual number semigroup.

Example 1.12: Let N = $\{\langle[0, 24) \cup k\rangle, k^2 = 23k, \times\}$ be the MOD special quasi dual number semigroup of infinite order.

This has the subsemigroups of finite and infinite order. This also has zero divisors, units and idempotents.

Example 1.13: Let S = $\{\langle[0, 17) \cup k\rangle, k^2 = 16k, \times\}$ be the special quasi dual number semigroup under product. S has finite and infinite order subsemigroups.

S has pseudo zero divisors and units. However all ideals of S are of infinite order.

$K = \{[0, 17)k, k^2 = 16k, \times\}$ is an ideal of infinite order.

$P = \{[0, 17), \times\} \subseteq S$ is again a subsemigroup of infinite order which is not an ideal of S.

$T = \{Z_{17}, \times\}$ is a subsemigroup of finite order.

$W = \{Z_{17}k, \times\}$ is a subsemigroup of finite order.

$V = \{\langle Z_{17} \cup k\rangle, \times\}$ is also a subsemigroup of finite order.

Example 1.14: Let $M = \{\langle [0, 45) \cup k\rangle, k^2 = 44k, \times\}$ be a MOD special quasi dual number semigroup of infinite order.

$P = \{[0, 45)k, k^2 = 44k, \times\}$ be the subsemigroup which is also an ideal of M. P is of infinite order.

Example 1.15: Let $Q = \{\langle [0, 143) \cup k\rangle, k^2 = 142k, \times\}$ be the MOD special quasi dual number semigroup. Q has subsemigroups of finite and infinite order.

Q has ideals of infinite order. Q has finite order subsemigroups as well as infinite order subsemigroups. Q has ideals of infinite order.

Example 1.16: Let $S = \{\langle\langle [0, 10) \cup k\rangle, +\rangle, \times; k^2 = 9k\}$ be the MOD special quasi dual number semigroup. S is of infinite order. S has subsemigroups of infinite and finite order.

$P_1 = \{Z_{10}, \times\}$,

$P_2 = \{Z_{10}k, \times\}$,

$P_3 = \{\langle Z_{10} \cup k\rangle, \times\}$ are finite order subsemigroups.

$$P_4 = \{[0, 10)k \mid k^2 = 9k, \times\}$$

is a subsemigroup which is also an ideal of S.

$$P_5 = \{[0, 10), \times\}$$

is an infinite order subsemigroup which is not an ideal of S.

Example 1.17: Let $S = \{\langle\langle[0, 3) \cup k\rangle, +\rangle, k^2 = 2k, \times\}$ be the MOD special quasi dual number semigroup. S is of infinite order. S has ideals and subsemigroups of infinite order. S has pseudo zero divisors, units and idempotents. S has finite order subsemigroups.

Next we proceed onto describe pseudo special MOD quasi dual number rings.

Example 1.18: Let $V = \{\langle[0, 12) \cup k\rangle, k^2 = 11k, +, \times\}$ be the MOD special quasi dual number pseudo ring. V has pseudo ideals and pseudo subrings of infinite order.

$T_1 = \{[0, 12)k \mid k^2 = 11k\} \subseteq V$ is pseudo subring of V which is a pseudo ideal.

$T_2 = \{[0, 12)\} \subseteq V$ is only a pseudo subring and is not an ideal.

$\{Z_{12}\} = T_3$ is a subring of order 12.

$T_4 = \{Z_{12}k \mid k^2 = 11k\}$ is a subring of order 12.

$T_5 = \{\langle Z_{12} \cup k\rangle, k^2 = 11k\}$ is again a subring of finite order.

Thus all finite order subrings of V are not pseudo that is distributive law is true.

Finally there is only one unique MOD fuzzy interval given by $F = \{[0, 1)\}$.

We can define all the 3 operations on F also.

Example 1.19: Let F = {[0, 1), +} be the MOD fuzzy interval group under +. If x = 0.3389 ∈ F then y = 0.6611 ∈ F is such that x + y = 0 = y + x. F is an infinite group.

This has subgroups of finite order. P_1 = {0, 0.1, 0.2, …, 0.9} ⊆ F is a subgroups of F of finite order.

Example 1.20: Let F = {[0, 1), ×} be the MOD fuzzy semigroup. Finding subsemigroups of finite order is a very difficult task.

In the opinion of authors there does not exist a finite subsemigroup for the MOD fuzzy semigroup.

Study in this direction is interesting.

Next the MOD fuzzy ring of infinite order. Finding substructures is an open problem.

We can also take subsets of these intervals MOD interval [0, m). S = ([0, m)) = {subsets of [0, m)}.

First we will illustrate this situation by an example.

Example 1.21: Let S([0, 4)) = {Collection of all subsets from [0, 4)} = {{0}, {0, 0.31}, {0.521}, {2.72405} … {0.3035, 2.57063}, {0.1, 2.75, 3.16, 2, 1} and so on} be the real interval on [0, 4).

Example 1.22: Let M = S([0, 7)) = {Collection of all subsets from the MOD interval [0, 7)} = {{5, 2, 1}, {6, 0.312, 2.15}, {5.5, 3, 4.7521}, {0.0021, 0.73, 1.1103} and so on} be the real interval using [0, 7).

Clearly S([0, m)) is all infinite collection.

We will also denote S([0, m)) by P([0, m)) that is power set of the set [0, m).

On P([0, m)) or equivalently on S([0, m)) we can build algebraic structures like min, max or both min and max or $\cup$ or $\cap$ or both $\cup$ and $\cap$ or + or $\times$ or both + and $\times$.

This same type of subsets can be carried out on all the seven MOD planes $R_n([0, m))$, $R_n^I([0, m))$, $C_n([0, m))$, $R_n^g([0, m))$, $R_n^h([0, m))$, $R_n^k([0, m))$ and $R_n^F([0, 1))$.

Such study is interesting.

We can call them as power sets of the planes or MOD power set planes of various types.

For instance $R_n([0, 9))$ = {Collection of all subsets of the form {(0, 7), (0.332, 1), (0, 0), (0, 0.1107)} and so on.

$P(R_n^I([0, 11)))$ = {Collection of all subsets from the plane $R_n^I([0, 11))$} = {{9 + 2I, 0.33I, 6.702 + 4.02I}, {0, 0.72I + 6.1, 0.77702 + 5.2I, 6.112I + 0.73}, {0}, {1 + I}, {I}, {9} and so on}.

Note {9} = {9 + 0I} and {I} = {0 + I}, it is by default of notations.

$P(R_n^g([0, 12)))$ = {Collection of all subsets from the MOD dual number plane} = {{5g + 3.21}, {0}, {0.8g + 0.115}, {0.7 + 8.73g, 0 + 2g, 0.3g + 4} and so on} is the power set of MOD plane of dual numbers.

Let A = {6.8 + 2g, 0.7 + 4.2g, 10.5 + 5.2g} and

B = {5 + 5g, 2 + 2g, 10 + 10g} $\in P(R_n^g([0, 12)))$.

$A \cap B = \{\phi\}$,

$A \cup B$ = {6.8 + 2g, 0.7 + 4.2g, 10.5 + 5.2g, 5 + 5g, 10 + 10g, 2 + 2g},

$A + B = \{11.8 + 7g, 5.7 + 9.2g, 3.5 + 10.2g, 8.8 + 4g, 2.7 + 6.2g, 0.5 + 7.2g, 4.8 + 0g, 10.7 + 2.2g, 8.5 + 3.2g\}$,

$A \times B = \{8 + 4g, 7 + g, 9 + g, 10 + 8g, 2 + 2g, 6 + 2g, 1.6 + 5.6g, 1.4 + 9.8g, 9 + 8g\}$

This is the way operations are performed on subsets of MOD dual number plane.

Next we see the structure of the MOD special dual like number interval subsets by some examples.

Example 1.23: Let P([0, 10)) = {Collection of all subsets from the interval [0, 10)h} = {{0, 5h, 6.332h}, {4h, 0.5h, 0.678h}, {9.28h, 0.73h}, {0.0432h, 6.0013h} and so on}.

We can perform the $\cup$ operation, $\cap$ operation, + operation, $\times$ operation and $\{+, \times\}$ and $\{\cup, \cap\}$ operation on P([0, 10)h).

This is realized as a matter of routine so left as an exercise to the reader.

$A = \{0.8h, 5h, 4h\}$ and

$B = \{6h, 4h, 0.5h, 0.01h\} \in P([0, 10h))$

$A \cap B = 4h$,

$A \cup B = \{0.8h, 5h, 4h, 6h, 0.5h, 0.01h\}$

$A + B = \{6.8h, h, 0, 4.8h, 9h, 8h, 1.3h, 5.5h, 4.5h, 0.81h, 5.01h, 4.01h\}$,

$A \times B = \{4.8h, 0, 4h, 3.2h, 6h, 0.4h, 2.5h, 2h, 0.008h, 0.05h, 0.04h\}$

P([0, 10)h) has zero divisors.

$x = \{6h\}, y = \{5h\} \in P([0, 10)h); x \times y = \{0\}$,

$x_1 = \{2.5h\}$ and $y_1 = \{4h\} \in P([0, 10)h)$ $x_1 \times y_1 = \{0\}$,

$x_2 = \{5h\}$ and $y_2 = \{8h\} \in P([0, 10)h)$; $x_2 \times y_2 = \{0\}$.

Thus P([0, 10)h) has several zero divisors.

Example 1.24: Let S = P([0, 7)g); $g^2 = 0$. S be the collection of all subsets of the interval [0, 7)g.

Let A, B $\in$ S; A $\times$ B = {0}, A $\times$ A = {0}, B $\times$ B = {0}.

P([0, 9)h) is the collection of all MOD subsets special quasi dual numbers of the interval [0, 9)k. $k^2 = 8k$.

S = P([0, 9)k) = {{0}, {2k + 3}, {7 + 0.5k}, {1 + k, 2 + 0.5k, 0.332 + 4k, 0.0006k, 6.332001k, 0.01115} and so on}.

On S we can define $\cup$, $\cap$, + and $\times$ and they are only semigroups. Even under '+'; S is only a semigroup.

For if A $\in$ S we cannot find a B in general such that A + B = {0}.

Let A = {0.3k} then there exist a unique B = {8.7k} such that A + B = {0}. However if the number of elements in A is greater than one we cannot find a B such that A + B = {0}.

Thus P([0, m)k) can only be a MOD semigroup under + and never a group. Likewise P([0, m)k) can only be a MOD semigroup under $\times$ and never a group.

But R = {P([0, m)k), +, $\times$} be a MOD pseudo semiring as + and $\times$ do not satisfy distributive law. However R is an infinite commutative MOD pseudo semiring.

This semiring has subsemirings may or may not contain finite order ideals. P has $Z_m k$ to be a finite ring.

So R is a SS-semiring. If m is a prime number or Z_m is a S-ring then R is a SSS-semiring. There may be MOD special quasi dual number subset interval pseudo semirings R which are not SSS-semirings however all semirings R are SS-semirings.

Let x = {8.037k} ∈ R (m = 20) = {P([0, 20)k | k^2 = 19k};

$$x \times x = 8.037k \times 8.037k$$
$$= \{7.274011k\}.$$

$$\text{Let } y = \{19k\} \in R;$$

$$y \times y = \{19k\} \times \{19k\} = \{19 \times 19k^2\}$$
$$= \{19 \times 19 \times 19k\} \text{ (as } k^2 = 19k)$$
$$= 19k = y \text{ is the idempotent.}$$

The only open conjecture is;

Can P([0, m)k); k^2 = (m – 1)k have idempotent sets other than {0}, {(m – 1)k} and {0, (m – 1)k} in R?

Next we consider the notion of MOD real planes, MOD complex modulo integer planes or intervals

$C_n(m)$ = {C([0, m))} = {a + bi_F | a, b ∈ [0, m); i_F^2 = (m – 1)}.

{⟨[0, m) ∪ I⟩} = $R_n^I(m)$ = {a + bI | a, b ∈ [0, m); I^2 = I} is the MOD neutrosophic interval or plane.

$R_n^g(m)$ = {a + bg | a, b ∈ [0, m), g^2 = 0} = {⟨[0, m) ∪ g⟩ | g^2 = 0} is the MOD dual number interval or plane.

$R_n^h(m)$ = {a + bh | a, b ∈ [0, m), h^2 = h} = {⟨[0, m) ∪ h⟩} be the MOD special dual like number plane or interval.

$R_n^k(m) = \{a + bk \mid a, b \in [0, m), k^2 = (m-1)k\} = \{\langle [0, m) \cup k \rangle\}$ be the MOD special quasi dual number plane or interval. Thus we have given six new types of MOD planes.

Now we can include the MOD fuzzy plane
$R_n(1) = \{(a, b) \mid a, b \in [0, 1)\}$.

Let x = (0.31, 0.002), y = (0.71, 0.5) and z = (0.003, 0.71) $\in R_n(1) = [0, 1) \times [0, 1)$.

Consider $(x \times y) \times z$

$= [(0.31, 0.002) \times (0.71, 0.5)] \times [(0.003, 0.71)]$

$= (0.2201, 0.001) \times (0.003, 0.71)$

$= (0.0006603, 0.00071)$ … I

$x \times (y \times z) = x \times [(0.71, 0.5) \times (0.003, 0.71)]$

$= x \times [(0.00213, 0.355)]$

$= (0.31, 0.002) \times (0.00213, 0.355)$

$= (0.0006603, 0.00071)$ … II

I and II are identical.

It is observed $R = [0, 1) \times [0, 1)$ is a ring as distributive law is true in the fuzzy MOD plane. R is a group of infinite order under +. R is a group under $\times$ which is commutative and is of infinite order.

Let $R_n^I(m) = \{a + bI \mid a, b \in [0, m), I^2 = I\}$ be the MOD neutrosophic plane

$\{R_n^I(m), +\}$ is a group under addition modulo m.

$\{R_n^I(m), \times\}$ is a semigroup under product.

Clearly $a \times (b + c) \neq a \times b + a \times c$ and
$(b + c) \times a \neq b \times a + c \times a$ in general $a, b, c \in R_n^I(m)$.

Hence $\{R_n^I(m), +, \times\}$ is only a MOD neutrosophic pseudo ring. For more refer [30].

We authors choose to recall these structures mainly to make this study popular.

Likewise we can find the pseudo ring of MOD dual number plane.

$R_n^g(m) = \{a + bg \mid a, b \in [0, m), g^2 = 0\}$ is the MOD dual number plane.

Clearly $R_n^g(m)$ is a MOD dual number pseudo ring. For $[0, m) \subseteq R_n^g(m)$ is only a MOD dual number pseudo subring.

However $[0, m)g \subseteq R_n^g(m)$ is a MOD dual number subring which is not pseudo as $[0, m)g$ is a zero square subring.

It is difficult to find subrings otherwise which are not pseudo.

Next we consider the MOD special dual like number plane

$$R_n^h(m) = \{a + bh \mid a, b \in [0, m), h^2 = h\}.$$

Clearly $\{R_n^h(m), +\}$ is group of infinite order and $\{R_n^h(m), \times\}$ is only a commutative semigroup of infinite order.

In spite of this $\{R_n^h(m), +, \times\}$ is a MOD special dual like number pseudo ring only as $a \times (b + c) \neq a \times b + a \times c$ for all a, b, c in $R_n^h(m)$.

Example 1.25: Let $R_n^h(27) = \{a + bh \mid h^2 = h, +, \times\}$ be the MOD special dual like number pseudo ring.

$$x = 0.32 + 0.4h,$$

$$y = 0.8 + 0.05h,$$

$$z = 1.2 + 2.5h \in R_n^h(27).$$

Consider $x \times (y + z)$

$$= (0.32 + 0.4h) \times [0.8 + 0.05h + 1.2 + 2.5h]$$

$$= (0.32 + 0.4h) \times (2 + 2.55h)$$

$$= 0.64 + 0.8h + 0.816h + 1.02h$$

$$= 0.64 + 2.636h \quad \ldots \quad \text{I}$$

$$x \times y + x \times z = 0.32 + 0.4h \times 0.8 + 0.05h + 0.32 + 0.4h \times 1.2 + 2.5h$$

$$= 0.256 + 0.32h + 0.0160h + 0.02h + 0.384 + 0.48h + 0.8h + h$$

$$= 0.64 + 2.556h \quad \ldots \quad \text{II}$$

Clearly I and II are different. Hence distributive law is not true in general. For more refer [30].

Next we proceed onto define the notion of MOD special quasi dual number planes.

$R_n^k(m) = \{a + bk \mid k^2 = (m - 1)k, a, b \in [0, m)\}$ is the MOD special quasi dual number plane. In fact we have infinite number of such planes; $m = 2, \ldots, \infty$. $\{R_n^k(m), +\}$ is a MOD special quasi dual number plane group of infinite order which is commutative. This has also subgroups of finite order.

$S = \{R_n^k(m), \times\}$ is a (MOD special quasi dual number) semigroup of infinite order. Clearly S has zero divisors, units and idempotents; the latter two depends on the m. S is a S-semigroup if Z_m is a S-semigroup.

But working with S and finding ideals happens to be a very difficult task.

Further S has subsemigroups of finite order also.

If $W = \{R_n^k(m), +, \times; k^2 = (m-1)k\}$ be the MOD special quasi dual number plane pseudo ring. W has units, zero divisors, idempotents and nilpotents mostly depending on m.

We see W has ideals which are only of infinite order, but has subrings of finite order which are not ideals. In fact these finite order subrings satisfy the distributive law.

Hence study in the direction of finding subrings of infinite order which are not ideals is interesting.

Clearly $P = \{[0, m), +, \times\}$ is a subring of W of infinite order which is pseudo and P is not an ideal only a subring of W.

Now we supply one or two examples of this situation.

Example 1.26: Let $S = \{R_n^k(24), +, \times, k^2 = 23k\}$ be the MOD special quasi dual number pseudo ring which is of infinite order and is non commutative. Z_{24} is a subring of order 24. $Z_{24}k$ is a subring of order 24.

$P = \{[0, 24), +, \times\}$ be the pseudo subring of infinite order.

$M = \{[0, 24)k, +, \times, k^2 = 23k\}$ is also a pseudo subring of infinite order which is an ideal. P is not an ideal only pseudo subring.

This has several zero divisors x = 12 + 8k, y = 6 + 12k ∈ S is such that x × y = 0 + 0k.

Example 1.27: Let M = { $R_n^k(13)$; k^2 = 12k, +, ×} be the MOD special quasi dual number pseudo ring.

Let x = 6.5 + 6.5k and y = 4 + 2k ∈ M,

we see x × y = 0 + 0k.

Thus even if m is a prime we have M to have zero divisors.

B = {[0, 13)k | k^2 = 12k, +, ×} is a pseudo subring.

D = {[0, 13); k^2 = 12k, +, ×} is also a pseudo subring.

Both B and D are not ideals of M. Z_{13} is a subring which is a field. Thus M is a S–pseudo ring.

Example 1.28: Let D = { $R_n^k(64)$, k^2 = 63k, +, ×} be the MOD special quasi dual number pseudo ring.

M = Z_{64} is a subring of order 64. N = Z_{64}k is also a subring of order 64.

Both M and N are ideals of D.

P_1 = {[0, 64); +, ×} is a pseudo subring of infinite order which is not an ideal.

P_2 = {[0, 64)k, +, ×} be the pseudo subring which is also an ideal of D.

This pseudo ring D has several zero divisors.

We suggest the following problems.

Problems

1. Obtain any special feature about the MOD real plane $R_n(m)$.

2. What are the advantages associated with the planes real MOD planes $R_n(m)$; $1 < m < \infty$?

3. Obtain the important properties associated with $\eta : R \to [0, m)$.

4. Find all important features enjoyed by η_r: $[0, m) \to R$; $1 < m < \infty$.

5. Compare η and η_r in problems (3) and (4).

6. Study the MOD neutrosophic intervals $[0, m)I$; $I^2 = I$, $1 < m < \infty$.

7. Can $R = \{[0, m)I, +, \times\}$ be a ring or a pseudo ring?

8. Find the special features enjoyed by $[0, m)g$, the MOD dual numbers $g^2 = 0$; $1 < m < \infty$.

9. Find the special properties associated with $[0, m)h$, MOD special dual like number interval $h^2 = h$; $1 < m < \infty$.

10. Study all the special properties enjoyed by the MOD special quasi dual number interval $[0, m)k$, $k^2 = (m - 1)k$; $1 < m < \infty$.

11. Let $R_n(m) = \{(a, b) \mid a, b \in [0, m); 1 < m < \infty\}$ be the MOD real plane.

    Let η_r: $R_n(m) \to R \times R$ be the map.

    i. Find the special features enjoyed by η_r.

12. Can $\{R_n(m), +, \times\}$ the pseudo ring have subrings which are not pseudo?

13. Can $\{R_n(m), +, \times\}$ have pseudo ideals of finite order?

14. What are the advantages of using the pseudo ring $\{R_n(m), +, \times\}$?

15. Can there be an infinite order S-subring of $\{R_n(m), +, \times\}$ which is not pseudo?

16. Prove only finite order subrings are not pseudo using MOD planes.

17. Let η: $R \times R \to R_n(m)$; $1 < m < \infty$ be the map.

    i. Is η a function?
    ii. Is η well defined?
    iii. Is η periodic?
    iv. Can η be a special type of function?
    v. Can we have any special property associated with η?

18. Let $S = \{R_n(16), +, \times\}$ be the MOD real plane pseudo ring.

    Study questions (12) to (16) for this S.

19. Let $W = \{R_n(23), +, \times\}$ be the MOD real plane pseudo ring.

    Study questions (12) to (16) for this W.

20. Let $\{R_n^I(m)\}$ be the MOD neutrosophic plane.

    i. What are special features enjoyed by $\{R_n^I(m)\}$?
    ii. Can $\{R_n^I(m)\}$ have distinct properties from $R_n(m)$?
    iii. Prove $\{R_n^I(m), +\}$ is an infinite group under +.

iv. Show $\{R_n^I(m), \times\}$ is a semigroup.
v. Prove $\{R_n^I(m), \times\}$ has pseudo ideals.
vi. Prove $\{R_n^I(m), \times\}$ is always a semigroup for any m having pseudo ideals.
vii. Prove $\{R_n^I(m), \times\}$ is a S-semigroup if Z_m is a S-semigroup.
viii. Can $\{R_n^I(m), \times\}$ be a MOD neutrosophic plane S-semigroup even if Z_m is not a S-semigroup?

21. Let $S = \{R_n^I(47), \times\}$ be the MOD neutrosophic plane semigroup.

Study questions (i) to (viii) of problem (20) for this S.

22. Let $M = \{R_n^I(56), \times\}$ be the MOD neutrosophic plane semigroup.

Study questions (i) to (viii) of problem (20) for this M.

23. Let $P = \{R_n^I(24), \times\}$ be the MOD plane neutrosophic semigroup.

Study questions (i) to (viii) of problem (20) for this P.

24. Let $S = \{R_n^I(m), +, \times\}$ be the MOD neutrosophic plane pseudo ring.

i. Prove S is a commutative pseudo ring.
ii. Prove all ideals are pseudo and are of infinite order.
iii. Show there exists subrings of finite order which are not pseudo ideals.
iv. Prove there exists infinite order subrings which are not pseudo ideals.
v. Prove S has infinite number of zero divisors.
vi. Can S have idempotents?
vii. Is it possible for S to have nilpotents?

viii. When is S a S-ring?
ix. Obtain any other special feature enjoyed by this MOD neutrosophic plane pseudo ring.

25. Let S = $\{R_n^I(40), +, \times\}$ be the MOD neutrosophic plane pseudo ring.

    Study questions (i) to (ix) of problem (24) for this S.

26. Let S_1 = $\{R_n^I(53), +, \times\}$ be the MOD neutrosophic plane pseudo ring.

    Study questions (i) to (ix) of problem (24) for this S_1.

27. Let R = $\{R_n^I(64), +, \times\}$ be the MOD neutrosophic plane pseudo ring.

    Study questions (i) to (ix) of problem (24) for this R.

28. Let M = $\{R_n^g(m) \mid g^2 = 0\}$ be the MOD dual number plane.

    i. Obtain all the special features associated with M.
    ii. Prove $\{M, +\}$ is a group.
    iii. Can $\{M, +\}$ have subgroups of finite order?
    iv. Prove $\{M, +\}$ can have subgroups of infinite order.
    v. Prove $\{M, \times\}$ is a MOD dual number plane semigroup.
    vi. Find ideals in $\{M, \times\}$.
    vii. Can ideals in $\{M, \times\}$ be of finite order?
    viii. Can $\{M, \times\}$ have subsemigroups none of which are ideals?
    ix. Prove $\{M, \times\}$ can have subsemigroups which are

infinite order which are not ideals.

x. Prove {M, ×} have infinite number of zero divisors.

xi. Can {M, ×} have idempotents?

xii. Prove {M, ×} can have zero square subsemigroups of both finite and infinite order.

xiii. Obtain any other special feature enjoyed by {M, ×} the MOD dual number plane semigroup.

xiv. Study L = {M, +, ×} is only a pseudo ring of MOD dual number plane.

xv. Prove L has both finite and infinite order subrings which are not ideals.

xvi. Is every ideal in L is of infinite order?

xvii. Prove L has zero square subrings of both finite and infinite order.

xviii. Can L be a S-ring?

xix. Does there exist MOD dual number plane pseudo rings which are not S-ring?

xx. Obtain any other special property associated with MOD dual number plane pseudo rings.

29. Let $\{R_n^g(48), g^2 = 0\}$ be the MOD dual number plane.

    Study questions (i) to (xx) of problem (28) for this $\{R_n^g(48), g^2 = 0\}$.

30. Let $\{R_n^g(151), g^2 = 0\}$ be the MOD dual number plane.

    Study questions (i) to (xx) of problem (28) for this $\{R_n^g(151), g^2 = 0\}$.

31. Let $\{R_n^g(729), g^2 = 0\}$ be the MOD dual number plane.

    Study questions (i) to (xx) of problem (28) for this $\{R_n^g(729), g^2 = 0\}$.

32. Let $\{R_n^h(m), h^2 = h\}$ be the MOD special dual like number plane.
    i. Study all properties associated with $\{R_n^h(m), h^2 = h\}$.
    ii. $S = \{R_n^h(m), h^2 = h, +\}$ is an infinite abelian group which has both subgroups of finite order as well as subgroups of infinite order.
    iii. Prove $W = \{R_n^h(m), h^2 = h, \times\}$ is a commutative semigroup of MOD special dual like number plane.
    iv. Can W have zero divisors?
    v. Can W have units?
    vi. Can W have idempotents?
    vii. When will W have nilpotents?
    viii. Can W have ideals of finite order?
    ix. Prove W can have subsemigroups of both finite and infinite order.
    x. Is W a S-semigroup?
    xi. Is $V = \{R_n^h(m), h^2 = h, +, \times\}$ be the MOD special dual like number plane pseudo ring commutative?
    xii. Can V have subrings of finite order which are pseudo?
    xiii. Is all pseudo subrings of V are of infinite order?
    xiv. Can V have pseudo ideals of finite order?

xv. Prove all infinite order subrings are not pseudo ideals.

xvi. Can V have zero divisors?

xvii. Obtain all idempotents of V.

xviii. Is V a S-ring?

xix. Can V have S-zero divisors?

xx. Can V have S-nilpotents?

xxi. Can V have S-units?

xxii. Obtain any other special features associated with MOD special dual like number pseudo rings.

33. Let Z = { $R_n^h(36)$, $h^2 = h$} be the MOD special dual like number plane.

    Study questions (i) to (xxii) of problem (32) for this Z.

34. Let B = { $R_n^h(59)$, $h^2 = h$} be the MOD special dual like number plane.

    Study questions (i) to (xxii) of problem (32) for this B.

35. Let S = { $R_n^h(3125)$, $h^2 = h$} be the MOD special dual like number plane.

    Study questions (i) to (xxii) of problem (32) for this S.

36. Let X = { $R_n^h(m)$, $k^2 = (m - 1)k$} be the MOD special quasi dual number plane.

    i. Obtain all the special features associated with X.
    ii. Study the important properties associated with the group (×, +).
    iii. Prove (X, +) has MOD special quasi dual number

subgroups of both finite and infinite order.

iv. Is P = {X, ×} a special quasi dual number plane S-semigroup?

v. Prove P has both subsemigroups of finite and infinite order.

vi. Can P have ideals of finite order?

vii. Is P a S-semigroup even if Z_m is not a S-semigroup under product?

viii. Can P have S-zero divisors?

ix. Can P have S-units?

x. Can P have S-idempotents?

xi. Can Y = {X, +, ×} be the MOD special quasi dual number plane pseudo ring be a S-pseudo ring?

xii. What are the advantages of studying MOD special quasi dual number plane pseudo rings?

xiii. Can Y have zero divisors?

xiv. Can Y have S-units?

xv. Can Y have S-idempotents?

xvi. Can Y have S-ideals?

xvii. Can Y have pseudo ideals of infinite order?

xviii. Prove there exists finite order subrings of Y.

xix. Prove Y has subrings of infinite order which are not ideals.

xx. Obtain any other special feature enjoyed by MOD special quasi dual number plane pseudo rings.

37. Let E = { $R_n^k(24)$, $k^2 = 23k$} be the MOD special quasi dual number plane.

Study questions (i) to (xx) of problem (36) for this E.

38. Let F = { $R_n^k(61)$, $k^2 = 60k$} be the MOD special quasi dual number plane.
Study questions (i) to (xx) of problem (36) for this F.

39. Let D = { $R_n^k(7^9)$, $k^2 = 7^9 - 1$} be the MOD special quasi dual number plane.
Study questions (i) to (xx) of problem (36) for this D.

40. Let G = {$C_n(m)$, $i_F^2 = m - 1$} be the MOD complex modulo finite number plane.

   i. Obtain all the special features enjoyed by G.
   ii. Distinguish G from all the other 5 MOD planes.
   iii. Can {G, +} the MOD complex finite number plane group have subgroups of finite order?
   iv. Obtain all infinite order MOD complex finite number subgroups of G.
   v. If {G, ×} be the MOD complex number plane semigroup. Find all finite order MOD complex plane subsemigroups.
   vi. Is {G, ×} a S-MOD complex number plane semigroup?
   vii. Find all zero divisors of {G, ×}.
   viii. Can {G, ×} have S-units?
   ix. Find all S-idempotents of {G, ×}.
   x. Can {G, ×} have ideals of finite order?
   xi. Can {G, +, ×} have infinite order subrings which satisfy distributive laws?

xii. Show $\{G, +, \times\}$ has finite order subrings which are not pseudo.

xiii. Is every ideal of $\{G, +, \times\}$ infinite order?

xiv. Can $\{G, +, \times\}$ have infinite order subrings which are not ideals?

xv. Find S-zero divisors of $\{G, +, \times\}$.

xvi. Show all units in $\{G, +, \times\}$ in general are not S-units.

xvii. Find any other special feature enjoyed by $\{G, +, \times\}$.

41. Let $S = \{C_n(19);\ i_F^2 = 18\}$ be the MOD complex finite number plane.

    Study questions (i) to (xvii) of problem (40) for this S.

42. Let $M = \{C_n(48);\ i_F^2 = 47\}$ be the MOD finite complex number plane.

    Study questions (i) to (xvii) of problem (40) for this M.

43. Let $P = \{C_n(128);\ i_F^2 = 127\}$ be the MOD finite complex number plane.

    Study questions (i) to (xvii) of problem (40) for this P.

Chapter Two

REAL MOD MATRICES AND THEIR PROPERTIES

In this chapter we introduce the notion of real MOD matrices and develop some of their properties. Throughout this chapter by a real MOD matrix we mean only a matrix which takes its entries from [0, m); $1 < m < \infty$.

We first define the real MOD matrix in the following.

DEFINITION 2.1: *Let $A = \{(a_{ij})_{p \times n}; 1 \leq i \leq p, 1 \leq j \leq b. a_{ij} \in [0, m)\}$ be a $p \times n$ matrix with entries from the MOD interval [0, m). We define A as the MOD $p \times n$ matrix defined on the interval [0, m). If p = 1 we call A the MOD row matrix.*

If n = 1 then A is defined as the MOD column matrix.

If p = n then A is called as the MOD square matrix.

We will illustrate first these situations by some examples.

Example 2.1: Let $M = \{(a_1, a_2, a_3, a_4)$ where $a_i \in [0, 5)$; $1 \leq i \leq 4\}$ be the real MOD row matrices built on the MOD interval [0, 5).

Example 2.2: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 425); 1 \leq i \leq 9 \right\}$$

be the MOD real square matrices built using the MOD interval [0, 425).

Example 2.3: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \end{bmatrix} \text{ where } a_i \in [0, 142); 1 \leq i \leq 8 \right\}$$

be the MOD real column matrix built using the MOD interval [0, 142).

Example 2.4: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \text{ where } a_i \in [0, 49); 1 \leq i \leq 12 \right\}$$

be the MOD real 3×4 matrices with entries from [0, 49).

Example 2.5: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \text{ where } a_i \in [0, 143); 1 \leq i \leq 10 \right\}$$

be the MOD real 5×2 matrices with entries from [0, 143).

Now having seen examples of MOD real matrices we define operations on them.

Example 2.6: Let

$$P = \{(a_1, a_2, a_3, a_4, a_5, a_6) \mid a_i \in [0, 7); 1 \leq i \leq 7, +\}$$

be the collection of all MOD real row matrix.

$$\text{Let } x = (0.8, 3.1, 4.5, 1.3, 0, 6.8, 4.1),$$

$$y = (4.2, 4.8, 4.7, 0.8, 6, 0.1, 0) \in P.$$

$$x + y = (5, 0.9, 2.2, 2.1, 6, 6.9, 4.1) \in P.$$

Thus {P, +} is an abelian group of infinite order known as the MOD group of row matrices.

Example 2.7: Let

$$x = \begin{bmatrix} 0.8 \\ 1.5 \\ 8.3 \\ 1.5 \\ 6.9 \end{bmatrix} \text{ and } y = \begin{bmatrix} 6 \\ 3.2 \\ 4.5 \\ 6.3 \\ 0.7 \end{bmatrix}$$

be two MOD column matrices with elements from

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_i \in [0, 12); 1 \leq i \leq 5, + \right\}.$$

$$x + y = \begin{bmatrix} 0.8 \\ 1.5 \\ 8.3 \\ 1.5 \\ 6.9 \end{bmatrix} + \begin{bmatrix} 6 \\ 3.2 \\ 4.5 \\ 6.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 6.8 \\ 4.7 \\ 0.8 \\ 7.8 \\ 7.6 \end{bmatrix} \in M.$$

This is the way operation is performed in M.

Example 2.8: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 15); 1 \leq i \leq 9, + \right\}$$

be the MOD real square matrix.

$$\text{Let } A = \begin{bmatrix} 3.5 & 2.7 & 8.5 \\ 6.8 & 10.3 & 12.1 \\ 7.8 & 6.6 & 7.1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6.9 & 9.8 & 7.8 \\ 9.2 & 12.6 & 4.9 \\ 6.9 & 7.8 & 11.4 \end{bmatrix}$$

be in M.

$$A + B = \begin{bmatrix} 10.4 & 12.5 & 1.3 \\ 1 & 14.9 & 2 \\ 14.7 & 14.4 & 3.5 \end{bmatrix} \in M.$$

$$A - B = \begin{bmatrix} 11.6 & 7.9 & 0.7 \\ 13.6 & 13.7 & 7.2 \\ 0.9 & 13.8 & 10.7 \end{bmatrix} \in M.$$

$$\text{Consider } B - A = \begin{bmatrix} 3.4 & 7.1 & 5.7 \\ 2.4 & 2.3 & 7.8 \\ 11.1 & 1.2 & 4.3 \end{bmatrix} \in M.$$

Clearly $A - B \neq B - A$.

$$\text{Now } (0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M \text{ acts as the additive identity.}$$

$$\text{For } A \in M; -A = \begin{bmatrix} 11.5 & 12.3 & 6.5 \\ 8.2 & 4.7 & 2.9 \\ 7.2 & 8.4 & 7.9 \end{bmatrix}.$$

Now this group of 3×3 matrix will be known as the MOD real square matrix. Clearly $|M| = \infty$.

Likewise we can get MOD real plane matrices using the MOD real plane $R_n(m) = \{(a, b); a, b \in [0, m)\}$.

We give a simple illustration of matrix MOD real plane.

Example 2.9: Let

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \text{ where } a_i \in R_n(20); 1 \leq i \leq 4, +\right\}$$

be a MOD real plane matrix group.

$$\text{Let } x = \begin{bmatrix} (0, 0.7) \\ (2.1, 0.6) \\ (0, 0) \\ (1, 0.1) \end{bmatrix} \text{ and } y = \begin{bmatrix} (6.32, 2) \\ (0, 4.2) \\ (9.06, 6.08) \\ (19.2, 10) \end{bmatrix} \in P.$$

$$x + y = \begin{bmatrix} (6.32, 2.7) \\ (2.1, 4.8) \\ (9.06, 6.08) \\ (0.2, 16.1) \end{bmatrix} \in P.$$

This is the way sum operation is performed on P.

Clearly P is an abelian group.

P has both finite order subgroups as well as P has infinite order subgroups.

$$M = \left\{ \begin{bmatrix} (a_1, b_1) \\ (a_2, b_2) \\ (a_3, b_3) \\ (a_4, b_4) \end{bmatrix} \middle| a_i, b_i \in Z_{20}; 1 \leq i \leq 4, +\right\} \subseteq P$$

is a subgroup of finite order.

$$W = \left\{ \begin{bmatrix} (a_1, 0) \\ (a_2, 0) \\ (a_3, 0) \\ (a_4, 0) \end{bmatrix} \middle| a_i \in [0, 20); 1 \leq i \leq 4, +\right\} \subseteq P$$

is a subgroup of P of infinite order.

Example 2.10: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \text{where } a_i = (x_i, y_i) \in R_n(201); \right.$$

$$\left. 1 \leq i \leq 12, +\right\}$$

be a MOD real matrix plane group under +.

This V has subgroups of finite order as well as infinite order.

Next we introduce the notion of MOD complex modulo integer matrix group under + through examples.

Example 2.11: Let

$$Z = \left\{ \left. \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \right| a_i \in C_n(23);\ 1 \le i \le 10, + \right\}$$

be the MOD complex modulo integer matrix group of infinite order under +.

This Z has subgroups of both finite and infinite order.

Example 2.12: Let

$$S = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right| a_i \in C_n(12);\ 1 \le i \le 9, + \right\}$$

be the MOD complex modulo integer matrix group.

$$P_1 = \left\{ \left. \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right| a_1 \in Z_{12}, + \right\} \subseteq S,$$

$$P_2 = \left\{ \left. \begin{bmatrix} 0 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right| a_2, a_3 \in Z_{12}, + \right\} \subseteq S \text{ and}$$

$$P_3 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \end{bmatrix}\right| a_1, a_2, a_3 \in Z_{12}, +\right\} \subseteq S$$

are subgroup of finite order in S.

Example 2.13: Let

$$R = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix}\right| a_i \in C_n(5);\ 1 \leq i \leq 15, +\right\}$$

be a MOD matrix complex modulo integer matrix group of infinite order.

Example 2.14: Let M = $\{(a_1, a_2, a_3) \mid a_i \in C_n(14);\ 1 \leq i \leq 3, +\}$ be the MOD complex modulo integer matrix group.

Let $x = (9.3 + 4.5i_F, 8.7 + 6.3i_F, 12.8 + 10.3i_F)$ and

$y = (0.8 + 7.2i_F, 9.2 + 2.6i_F, 2.2 + 1.4i_F) \in M$.

$$\begin{aligned} x + y &= (9.3 + 4.5i_F, 8.7 + 6.3i_F, 12.8 + 10.3i_F) \\ &\quad + (0.8 + 7.2i_F, 9.2 + 2.6i_F, 2.2 + 1.4i_F) \\ &= (10.1 + 11.7i_F, 3.9 + 8.9i_F, 1 + 11.7i_F) \in M. \end{aligned}$$

Thus we see M is a group. M has subgroups of both finite order and subgroups of infinite order.

These concepts are used in chapter III of this book in the construction of MOD vector spaces.

Next just matrices constructed using the MOD neutrosophic plane $R_n^I(m)$ is described and developed by examples.

Example 2.15: Let

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \; a_i \in R_n^I(10); 1 \leq i \leq 4 \right\}$$

be the MOD neutrosophic 1×4 column matrices.

$\{P, \times_n\}$ is a MOD neutrosophic semigroup.

$$M_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \; a_i \in Z_{10} \times Z_{10}, 1 \leq i \leq 4, \times_n \right\} \subseteq P$$

is a finite order MOD neutrosophic subsemigroup of P.

$$M_2 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1 \in R_n^I(10), \times_n \right\} \subseteq P$$

is an infinite order MOD neutrosophic subsemigroup of P which is also an ideal of P.

$$M_3 = \left\{\left.\begin{bmatrix} a_1 + b_1I \\ a + bI \\ 0 \\ 0 \end{bmatrix}\right| a_1 + b_1I \in R_n^I(10),\right.$$

$$\left. a + bI \in \langle Z_{10} \cup I\rangle, \times_n\right\} \subseteq P$$

is a subsemigroup of P which is not an ideal of P.

In fact we can define on P the notion of group under addition modulo 10. Thus {P, +} is an abelian MOD neutrosophic matrix group of infinite order.

{P, +} has both subgroups of finite and infinite order.

Clearly {P, +, $\times_n$} is known or defined as the MOD neutrosophic matrix pseudo ring of infinite order.

This ring has both finite order subrings as well as infinite order subrings.

However all ideals of {P, +, $\times_n$} are only of infinite order.

Example 2.16: Let

$$W = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right| a_i \in R_n^I(43); 1 \leq i \leq 9\right\}$$

be the MOD neutrosophic square matrix.

{W, +} is a MOD neutrosophic matrix group of infinite order which is abelian.

{W, $\times_n$} is a MOD neutrosophic matrix semigroup of infinite order which is commutative.

$\{W, \times_n\}$ has subsemigroups of both finite and infinite order.

All ideals of $\{W, \times_n\}$ are of infinite order. In fact $\{W, \times_n\}$ has many ideals.

Now $\{W, +, \times_n\}$ is the MOD neutrosophic matrix pseudo ring only as the distributive law is not true in general.

W has subrings of both finite and infinite order.

However all ideals of W are of infinite order.

Now $\{W, \times\}$ is also a MOD neutrosophic semigroup where $\times$ is the usual product of matrices and not the natural product $\times_n$. $\{W, \times\}$ is only a non commutative semigroup.

This has ideals all of which are of infinite order $\{W, \times\}$ has both right and left ideals.

Now $\{W, +, \times\}$ is only a pseudo MOD neutrosophic matrix ring which is non commutative and is of infinite order.

We see $\{W, +, \times\}$ has subrings of finite order as well as subrings of infinite order.

All ideals of W are only of infinite order.

Example 2.17: Let

$$S = \left\{ \left. \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \right| a_i \in R_n^g(20);\ 1 \leq i \leq 8 \right\}$$

be the MOD dual number plane of matrices. S is of infinite order. $\{S, \times_n\}$ is a semigroup of infinite order and $\{S, +\}$ is an abelian group of infinite order.

$\{S, +\}$ has both finite and infinite order subgroups. $\{S, \times_n\}$ has both finite and infinite order subsemigroups.

However all ideals of $\{S, \times_n\}$ is only a infinite order. Further $\{S, \times_n\}$ has subsemigroups which are zero square subsemigroups.

Clearly $\{S, +, \times_n\}$ is only a pseudo ring. This has ideals all of them are of infinite order.

$\{S, +, \times_n\}$ has subrings which are zero square subrings which are both of finite and infinite order.

For more about these concepts refer [21].

Example 2.18: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| \, a_i \in R_n^g(31);\ 1 \leq i \leq 15 \right\}$$

be the MOD dual number matrix collection. Any $a_i \in R_n^g(31)$ is of the form a + bg where a, b $\in$ [0, 31).

$\{V, +\}$ is defined as the MOD dual number matrix group and this is an infinite order group which is commutative.

$\{V, \times_n\}$ is the MOD dual number matrix semigroup of infinite order which is commutative. $\{V, \times_n\}$ has subsemigroups of both finite and infinite order.

But all ideals of $\{V, \times_n\}$ is of infinite order.

$\{V, \times_n\}$ has infinite number zero divisors and in fact has subsemigroups which are zero square subsemigroups.

Now $\{V, +, \times_n\}$ is the MOD dual number pseudo matrix ring. This ring too has subrings of both finite and infinite order.

Further this ring has zero square subrings which are not pseudo of infinite as well as finite order.

However all ideals of $\{V, +, \times_n\}$ are only of infinite order.

Next we proceed onto describe MOD special dual like number matrices by some examples.

For more about these concepts refer [18, 21].

Example 2.19: Let

$$T = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| a_i \in R_n^h(24);\, h^2 = h,\, 1 \leq i \leq 6 \right\}$$

be the MOD special dual like number matrix collection.

$\{T, +\}$ is an abelian group of infinite order. $\{T, +\}$ has subgroups of both finite and infinite order.

$\{T, \times_n\}$ is a commutative semigroup of infinite order under the natural product $\times_n$. $\{T, \times_n\}$ has zero divisors which are infinite in number.

$\{T, \times_n\}$ has idempotents and units and both of them are only finite in number.

$\{T, \times_n\}$ has subsemigroups of both finite and infinite order.

All ideals are of infinite order in $\{T, \times_n\}$.

$\{T, +, \times_n\}$ is only a MOD special dual like number matrix pseudo ring, as the distributive laws are not true in general in $\{T, +, \times_n\}$.

$\{T, +, \times_n\}$ has subrings of both finite and infinite order; but all ideals of $\{T, +, \times_n\}$ are of infinite order.

Example 2.20: Let

$$X = \{(a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mid a_i \in R_n^h(43); h^2 = h; 1 \leq i \leq 7\}$$

be the MOD special dual like number matrices.

$\{X, +\}$ is an infinite abelian group.

$\{X, \times_n\}$ is an infinite semigroup and $\{X, +, \times_n\}$ is an infinite MOD special dual like number pseudo ring of infinite order.

All properties associated with these structures can be studied as it is considered as a matter of routine so left as an exercise to the reader.

Next we proceed onto study MOD special quasi dual number matrices by some examples.

Example 2.21: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \end{bmatrix} \middle| \; a_i \in R_n^k(25); k^2 = 24k; 1 \leq i \leq 21 \right\}$$

be the MOD special quasi dual number 7×3 matrix collection.

{M, +} is a special quasi dual number matrix abelian group of infinite order.

$$P_1 = \left\{ \begin{bmatrix} a + bk & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a + bk \in Z_{25}(k); k^2 = 24k, + \right\} \subseteq M$$

is a subgroup of finite order.

Thus M has several subgroups of finite order.

M has also subgroups of infinite order.

$$R_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a_1 + b_1k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1 + b_1k \in R_n^k(25); + \right\} \subseteq M$$

is an infinite order subgroup of M.

Further $\{M, \times_n\}$ is an abelian MOD special quasi dual number semigroup of matrices under natural product.

$\{M, \times_n\}$ is a commutative semigroup.

$$L_1 = \left\{ \begin{bmatrix} a_1 + b_1k & a_2 + b_2k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1 + b_1k, a_2 + b_2k \in Z_{25}(k); k^2 = 24k, \times_n \right\} \subseteq M$$

is a subsemigroup of finite order which is not an ideal of M.

Thus M has several such finite order subsemigroups.

$$L_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a+bk & c+dk \end{bmatrix} \middle| \; a + bk, c + dk \in R_n^k(25), \times_n \right\} \subseteq M$$

is a subsemigroup of infinite order which is also an ideal of M.

All infinite order subsemigroups in general are not ideals, however all ideals of M are of infinite order.

$$N_1 = \left\{ \begin{bmatrix} a+bk & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c+dk \end{bmatrix} \middle| \; a + bk \in R_n^k(25), c + dk \in Z_{25}(k), k^2 = 24k, \times_n \right\} \subseteq M$$

is a subsemigroup of infinite order.

But N_1 is not an ideal of M.

Next we define MOD special quasi dual number pseudo ring.

$\{M, +, \times_n\}$ is the MOD special quasi dual number pseudo ring.

This pseudo ring is of infinite order and is commutative. M has subrings of both finite and infinite order which are not ideals.

Let

$$B_1 = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right| a_1, a_2, a_3 \in Z_{251}, +, \times_n \right\}$$

be a subring of M which is not pseudo and is of finite order.

Clearly B_1 is not an ideal of M.

$$B_2 = \left\{ \left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_1 & d_2 & d_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right| d_i \in Z_{25}(k), 1 \leq i \leq 3, \right.$$

$$k^2 = 24k, +, \times_n\} \subseteq M$$

is a subring which is a special quasi dual number subring which is not pseudo and is of finite order.

$$B_3 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, a_i \in [0, 25); 1 \leq i \leq 3, +, \times_n \right\} \subseteq M$$

is a subring of infinite order which is pseudo.

Clearly B_3 is not an ideal of M.

$$B_4 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, a_i \in [0, 25)k, 1 \leq i \leq 6, +, \times_n \right\}$$

is a MOD special quasi dual number pseudo subring of M which is also an ideal of M.

Clearly B_4 is of infinite order.

Let

$$B_5 = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_4 & a_5 & a_6 \end{bmatrix} \right| a_1, a_2, a_3 \in Z_{25}, a_4, a_5, \right.$$

$$\left. a_6 \in R_n^k(25), k^2 = 24k, +, \times_n \right\} \subseteq M$$

be a MOD special quasi dual number pseudo subring of M of infinite order which is not an ideal of M.

Likewise the interested reader can study the MOD special quasi dual number matrices given in the following example.

Example 2.22: Let

$$P = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \end{bmatrix} \right| a_i \in R_n^k(43); \right.$$

$$\left. k^2 = 42k; 1 \leq i \leq 20, +, \times_n \right\}$$

be the MOD special quasi dual number pseudo ring.

As in case of example 2.21 study all properties of this P.

For more about these concepts refer [19, 21, 30].

However the notion of algebraic structures on subsets of the MOD real plane, MOD neutrosophic plane, MOD complex finite modulo integer plane, MOD dual number plane, MOD special dual like number plane and MOD special quasi dual number plane is not within the purview of this book.

But a new type of matrices called mixed matrices and strongly mixed matrices will be illustrated by examples; for these concepts find their place in chapter IV of this book.

Example 2.23: Let V = $\{(a_1, a_2, a_3, a_4, a_5, a_6) \mid a_1 \in R_n(10), a_2, a_3 \in R_n(5), a_4, a_5 \in R_n(23)$ and $a_6 \in R_n(12)\}$; V is defined as the mixed MOD row real matrices.

We see V under + defined component wise is a group.

$\{V, \times\}$, $\times$ defined component wise is a semigroup of infinite order.

$\{V, +, \times\}$ is in fact an infinite pseudo ring.

Example 2.24: Let

$$W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle| \; a_1 \in R_n(9), a_2 \in R_n(11) \text{ and } a_3 \in R_n(4) \right\}$$

be the mixed MOD real column matrix collection.

$\{W, +\}$ is a group and $\{W, \times_n\}$ is a semigroup.

$$\text{Let } A = \begin{bmatrix} (3, 7) \\ (0.5, 10) \\ (3.3, 2.1) \end{bmatrix} \text{ and } B = \begin{bmatrix} (8.2, 3.5) \\ (10.8, 3.3) \\ (2.3, 1.5) \end{bmatrix} \in W;$$

$$A + B = \begin{bmatrix} (3, 7) \\ (0.5, 10) \\ (3.3, 2.1) \end{bmatrix} + \begin{bmatrix} (8.2, 3.5) \\ (10.8, 3.3) \\ (2.3, 1.5) \end{bmatrix}$$

$$= \begin{bmatrix} (3, 7) + (8.2, 3.5) \\ (0.5, 10) + (10.8, 3.3) \\ (3.3, 2.1) + (2.3, 1.5) \end{bmatrix}$$

$$= \begin{bmatrix} (3 + 8.2, 7 + 3.5) \ (\text{mod } 9) \\ (0.5 + 10.8, 10 + 3.3) \ (\text{mod } 11) \\ (3.3 + 2.3, 2.1 + 1.5) \ (\text{mod } 4) \end{bmatrix}$$

$$= \begin{bmatrix} (2.2, 1.5) \\ (10.5, 3.1) \\ (1.4, 3.8) \end{bmatrix} \in W.$$

This is the way + operation is performed on W.

Clearly $\begin{bmatrix} (0, 0) \\ (0, 0) \\ (0, 0) \end{bmatrix}$ acts as the additive identity of W.

For $A = \begin{bmatrix} (3, 7) \\ (0.5, 10) \\ (3.3, 2.1) \end{bmatrix}$ in W we have a unique

$$-A = \begin{bmatrix} (6, 2) \\ (10.5, 1) \\ (0.7, 1.9) \end{bmatrix} \in W$$

such that

$$A + (-A) = \begin{bmatrix} (0, 0) \\ (0, 0) \\ (0, 0) \end{bmatrix}.$$

–A is called the inverse of A and vice versa.

Thus {W, +} is the MOD real mixed matrix group of infinite order.

Now on W we can define $\times_n$ the natural product operation {W, $\times_n$} is the MOD real mixed matrix semigroup or in fact a monoid.

Just we show how the natural product operation $\times_n$ is performed in W.

$$A \times_n B = \begin{bmatrix} (3, 7) \\ (0.5, 10) \\ (3.3, 2.1) \end{bmatrix} \times_n \begin{bmatrix} (8.2, 3.5) \\ (10.8, 3.3) \\ (2.3, 1.5) \end{bmatrix}$$

$$= \begin{bmatrix} (3, 7) \times (8.2, 3.5) \text{ (mod 9)} \\ (0.5, 10) \times (10.8, 3.3) \text{ (mod 11)} \\ (3.3, 2.1) \times (2.3, 1.5) \text{ (mod 4)} \end{bmatrix}$$

$$= \begin{bmatrix} (24.6, 24.5) \text{ (mod 9)} \\ (54, 33) \text{ (mod 11)} \\ (7.59, 3.15) \text{ (mod 4)} \end{bmatrix}$$

$$= \begin{bmatrix} (6.6, 6.5) \\ (10, 0) \\ (3.59, 3.15) \end{bmatrix} \in W.$$

This is the way $\times_n$ operation is performed on W.

Now

$$I = \begin{bmatrix} (1, 1) \\ (1, 1) \\ (1, 1) \end{bmatrix} \in W \text{ is such that } I \times A = A \times I = A \text{ for all } A \in W.$$

$\{W, +, \times_n\}$ can be easily realized as the MOD real mixed matrix pseudo ring.

Study of ideals, zero divisors, units, subrings and idempotents are realized as a matter of routine so left as an exercise to the reader.

Example 2.25: Let

$$Z = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right| a_1, a_4 \in R_n(40), a_2, a_3 \in R_n(19),$$

$$a_5 \in R_n(10), a_6, a_{12} \in R_n(90), a_7, a_{11} \in R_n(24), a_8, a_{10} \in R_n(12), a_9 \in R_n(120)\}$$

be the MOD mixed real matrix collection.

$\{Z, +\}$ is a MOD real mixed matrix under component addition.

$\{Z, \times_n\}$ is the MOD real mixed semigroup under natural product $\times_n$.

$\{Z, +, \times_n\}$ be the MOD real mixed matrix pseudo ring of infinite order.

Example 2.26: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \middle| \; a_1, a_{10} \in R_n(43), a_2, a_4 \in R_n(2), \right.$$

$$\left. a_5, a_3, a_{11} \in R_n(12), a_6, a_7 \in R_n(3), a_8, a_9, a_{12} \in R_n(5) \right\}$$

be the MOD real mixed matrix collection.

$\{M, +, \times_n\}$ is a MOD real mixed matrix pseudo ring of infinite order which has subrings of finite order.

Example 2.27: Let

$$V = \{(a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mid a_1 \in R_n^I(10),$$
$$a_2, a_3 \in R_n^I(40), a_4, a_7 \in R_n^I(23), a_5, a_6 \in R_n^I(41)\}$$

be the MOD neutrosophic mixed matrix collection.

$\{V, +\}$ is a group under component wise addition.

Let

$$x = (0.3 + 7I, 9.3 + 2I, 25 + 36I, 22, 12.5I, 40.5, 35 + 28.5I) \text{ and}$$

$$y = (0, 10 + 5.8I, 15 + 14I, 3.8I, 10 + 7I, 31 + 40I, 0) \in V.$$

$$x + y = (0.3 + 7I, 19.3 + 7.8I, 10I, 22 + 38I, 10 + 19.5I, 30.5 + 40I, 35 + 28.5I) \in V.$$

$\{V, \times_n\}$ is the semigroup of MOD neutrosophic mixed matrix semigroup.

Thus $\{V, +, \times_n\}$ be the MOD neutrosophic mixed matrix pseudo ring.

Example 2.28: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \middle| a_1 \in R_n^I(2), a_2 \in R_n^I(4), a_3 \in R_n^I(3), \right.$$

$$\left. a_4 \in R_n^I(5), a_5 \in R_n^I(6) \text{ and } a_6 \in R_n^I(7) \right\}$$

be the MOD neutrosophic mixed matrix.

We see $\{P, +\}$ is the MOD neutrosophic mixed matrix group.

$$\text{Let } x = \begin{bmatrix} 0.1 & 2 + 3I \\ 0.2I & 0.3 + 4I \\ 4I + 2 & 3.2I \end{bmatrix} \text{ and}$$

$$y = \begin{bmatrix} 0.2I & 1 + 2I \\ 0.4 + 0.7I & 2I \\ 3I & 4 + 2I \end{bmatrix} \in P$$

$$x + y = \begin{bmatrix} 0.1 & 2+3I \\ 0.2I & 0.3+4I \\ 4I+2 & 3.2I \end{bmatrix} + \begin{bmatrix} 0.2I & 1+2I \\ 0.4+0.7I & 2I \\ 3I & 4+2I \end{bmatrix}$$

$$= \begin{bmatrix} 1+0.2I & 3+I \\ 0.4+0.9I & 0.3+I \\ I+2 & 4+5.2I \end{bmatrix} \in P.$$

This is the way '+' operation is performed.

{P, +} is the MOD neutrosophic mixed matrix group.

Now $\{P, \times_n\}$ is the MOD neutrosophic mixed matrix semigroup.

For the same x and y in P

$$x \times_n y = \begin{bmatrix} 0.1 & 2+3I \\ 0.2I & 0.3+4I \\ 4I+2 & 3.2I \end{bmatrix} \times_n \begin{bmatrix} 0.2I & 1+2I \\ 0.4+0.7I & 2I \\ 3I & 4+2I \end{bmatrix}$$

$$= \begin{bmatrix} 0.02I & 2+I \\ 0.22I & 3.6I \\ 6I & 5.2I \end{bmatrix} \in P.$$

Clearly $\{P, \times_n\}$ has zero divisors and ideals.

Further $\{P, +, \times_n\}$ is the MOD neutrosophic mixed matrix pseudo ring of infinite order.

Next we proceed onto describe MOD complex Modulo integer mixed matrix collection.

Example 2.29: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \middle| \; a_1, a_2 \in C_n(8), a_3, a_4 \in C_n(24), \right.$$

$$\left. a_5, a_6 \in C_n(3), a_7, a_8 \in C_n(43) \text{ and } a_9 \in C_n(14) \right\}$$

be the MOD complex modulo integer mixed matrix collection. {M, +} is defined as the MOD complex modulo integer mixed group.

{M, +} is of infinite order and has subgroups of both finite and infinite order.

$\{M, \times_n\}$ is the MOD complex modulo integer mixed matrix semigroup.

This semigroup $\{M, \times_n\}$ has subsemigroups of both finite and infinite order.

However all ideals of $\{M, \times_n\}$ are of infinite order.

M has zero divisors, units and idempotents

$$I = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in M$$

is the identity elements of M with respect to the natural product $\times_n$.

Further $\{M, +, \times_n\}$ is the MOD complex modulo integer mixed matrix pseudo ring. M has ideals of infinite order only but has subrings of both finite and infinite order.

$\{M, +, \times_n\}$ has units, zero divisors and idempotents.

Example 2.30: Let

$$B = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \middle| \; a_1 \in C_n(3), a_2 \in C_n(7), \right.$$

$$\left. a_3 \in C_n(5), a_4, a_5 \in C_n(6) \text{ and } a_6 \in C_n(10) \right\}$$

be the MOD complex modulo integer mixed matrices collection.

$\{B, +\}$ is a MOD mixed matrix complex modulo integer group of infinite order.

$$\text{Let } x = \begin{pmatrix} 0.2 + i_F & 4 + 2.1i_F & 0.7 + 0.6i_F \\ 4 + 2i_F & 0.3i_F & 7.1 \end{pmatrix} \text{ and}$$

$$y = \begin{pmatrix} 0.4i_F & 0.3i_F & 2 + i_F \\ 0.5 + 0.2i_F & 1 + 4i_F & 2i_F \end{pmatrix} \text{ belong to B.}$$

$$x + y = \begin{pmatrix} 0.2 + i_F & 4 + 2.1i_F & 0.7 + 0.6i_F \\ 4 + 2i_F & 0.3i_F & 7.1 \end{pmatrix} +$$

$$\begin{pmatrix} 0.4i_F & 0.3i_F & 2 + i_F \\ 0.5 + 0.2i_F & 1 + 4i_F & 2i_F \end{pmatrix}$$

$$= \begin{pmatrix} 0.2 + 1.4i_F & 4 + 2.4i_F & 2.7 + 1.6i_F \\ 4.5 + 2.2i_F & 1 + 4.3i_F & 7.1 + 2i_F \end{pmatrix} \in B.$$

This is the way + operation is performed on B.

Next $\{B, \times_n\}$ is the MOD mixed matrix complex modulo integer semigroup of infinite order.

$$x \times_n y = \begin{pmatrix} 0.2 + i_F & 4 + 2.1i_F & 0.7 + 0.6i_F \\ 4 + 2i_F & 0.3i_F & 7.1 \end{pmatrix} \times_n$$

$$\begin{pmatrix} 0.4i_F & 0.3i_F & 2 + i_F \\ 0.5 + 0.2i_F & 1 + 4i_F & 2i_F \end{pmatrix}$$

$$= \begin{pmatrix} 0.08i_F + 1.2 & 1.2i_F + 3.78 & 0 + 1.9i_F \\ 4 + 1.8i_F & 0.3i_F & 4.2i_F \end{pmatrix} \in B.$$

Thus $\{B, \times_n\}$ is a semigroup of infinite order. B has zero divisors, units and idempotents.

Now $\{B, +, \times_n\}$ is the MOD mixed matrix complex modulo integer pseudo ring. This has zero divisors, units and idempotents.

This pseudo ring has subrings of finite order and subrings of infinite order. All ideals of B are of infinite order.

Example 2.31: Let

$$A = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} \middle| \; a_4, a_1 \in R_n^g(3), \right.$$

$$\left. a_2, a_3 \in R_n^g(10), a_5, a_6, a_7 \in R_n^g(8), a_8 \in R_n^g(5) \right\}$$

be the MOD dual number mixed matrix collection. $\{A, +\}$ is the MOD dual number mixed matrix group. A is of infinite order.

$\{A, +\}$ has subgroups of finite order as well as of infinite order. $\{A, \times_n\}$ be the MOD dual number mixed matrix semigroup.

$\{A, \times_n\}$ has subsemigroups of infinite order as well as finite order. $\{A, \times_n\}$ is an ideal and is of infinite order.

$\{A, +, \times_n\}$ is the MOD dual number mixed matrix pseudo ring. $\{A, +, \times_n\}$ has subrings of finite order and $\{A, +, \times_n\}$ has subrings of infinite order which are not ideals.

However $\{A, +, \times_n\}$ has ideals all of which are of infinite order.

Example 2.32: Let

$$C = \{(a_1, a_2, a_3, a_4) \mid a_1 \in R_n^g(3), a_2 \in R_n^g(10), a_3 \in R_n^g(5), a_4 \in R_n^g(6)\}$$

be the MOD dual number mixed matrix collection. $\{C, +\}$ is the MOD dual number mixed matrix group.

Clearly $\{C, +\}$ is a group of infinite order.

Let

$$x = (2 + 1.1g, 4 + 5g, 3 + 2g, 4 + 5g)$$

and

$$y = (0.5 + 0.6g, 4g, 0.5g + 2, g + 2) \in C.$$

$$\begin{aligned} x + y &= (2 + 1.1g, 4 + 5g, 3 + 2g, 4 + 5g) + (0.5 + 0.6g, \\ &\quad 4g, 0.5g + 2, g + 2) \\ &= (2.5 + 1.7g, 4 + 9g, 2.6g, 0) \in C. \end{aligned}$$

This is the way operation of + is performed on C.

$$\begin{aligned} x \times y &= (2 + 1.1g, 4 + 5g, 3 + 2g, 4 + 5g) \times (0.5 + 0.6g, \\ &\quad 4g, 0.5g + 2, g + 2) \\ &= (1 + 1.75g, 6g, 1 + 0.5g, 2 + 2g). \end{aligned}$$

Thus $\{C, \times\}$ is the MOD dual number mixed matrix semigroup.

$\{C, +, \times\}$ is the MOD dual number mixed matrix pseudo ring of infinite order.

Example 2.33: Let

$$W = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle|\; a_1, a_2 \in R_n^h(10), a_3, a_4 \in R_n^h(8),\right.$$

$$\left. a_5, a_6, a_7 \in R_n^h(5), a_8 \in R_n^h(2), h^2 = h\right\}$$

be the MOD special dual like number mixed matrix collection.

{W, +} is the MOD special dual like number mixed matrix group of infinite order.

{W, +} has subgroups of both finite and infinite order.

$\{W, \times_n\}$ is the MOD special dual like number mixed matrix semigroup of infinite order. $\{W, \times_n\}$ has subsemigroups of finite order as well as subsemigroups of infinite order.

$\{W, \times_n\}$ has zero divisors, units and idempotents.

$\{W, +, \times_n\}$ be the MOD special dual like number mixed matrix pseudo ring. This has subrings of both finite and infinite order.

Example 2.34: Let

$$M = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle|\; a_1 \in R_n^h(10), a_2 \in R_n^h(5),\right.$$

$$\left. a_3 \in R_n^h(2), a_4 \in R_n^h(3), h^2 = h\right\}$$

be the MOD special dual like number mixed matrix collection.

{M, +} is the MOD special dual like number mixed matrix group of infinite order.

$$\text{Let } A = \begin{bmatrix} 5+0.3h \\ 4.2+4h \\ 0.7+h \\ 2+0.4h \end{bmatrix} \text{ and } B = \begin{bmatrix} 8+4h \\ 3+0.5h \\ 1+0.2h \\ 1.4h \end{bmatrix} \in M.$$

$$A + B = \begin{bmatrix} 5+0.3h \\ 4.2+4h \\ 0.7+h \\ 2+0.4h \end{bmatrix} + \begin{bmatrix} 8+4h \\ 3+0.5h \\ 1+0.2h \\ 1.4h \end{bmatrix}$$

$$= \begin{bmatrix} 3+4.3h \\ 2.2+4.5h \\ 1.7+1.2h \\ 2+1.8h \end{bmatrix} \in M.$$

This is the way + operation is performed on M.

Let $\{M, \times_n\}$ be the MOD special dual like number mixed matrix semigroup.

For A, B $\in$ M we define

$$A \times_n B = \begin{bmatrix} 5+0.3h \\ 4.2+4h \\ 0.7+h \\ 2+0.4h \end{bmatrix} \times_n \begin{bmatrix} 8+4h \\ 3+0.5h \\ 1+0.2h \\ 1.4h \end{bmatrix}$$

$$= \begin{bmatrix} 40 + 2.4h + 20h + 1.2h \\ 12.6 + 12h + 2.1h + 2h \\ 0.7 + h + 0.14h + 0.2h \\ 2.8h + 0.56h \end{bmatrix}$$

$$= \begin{bmatrix} 3.6h \\ 2.6 + 6.1h \\ 0.7 + 1.34h \\ 0.36h \end{bmatrix} \in M.$$

Thus $\{M, \times_n\}$ is the semigroup of infinite order. $\{M, +, \times_n\}$ be the MOD special dual like number mixed matrix pseudo ring.

$\{M, +, \times_n\}$ has both finite and infinite order MOD special dual like number mixed matrix pseudo subrings.

$\{M, +, \times_n\}$ has zero divisors, units and idempotents.

All ideals of $\{M, +, \times_n\}$ is of infinite order.

Next we describe the MOD special quasi dual number mixed matrix collection by the following example.

Example 2.35: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle| \; a_3, a_4 \in R_n^k(5), k^2 = 4k, \right.$$

$$\left. a_5, a_6 \in R_n^k(12), k^2 = 11k, a_7, a_8 \in R_n^k(4), k^2 = 3k \right\}.$$

$\{P, +\}$ is the MOD special quasi dual number mixed matrix group. $\{P, +\}$ is a group of infinite order known as the MOD special quasi dual number mixed matrix group.

$\{P, \times_n\}$ is the MOD special quasi dual number mixed matrix semigroup. This has subsemigroups of both finite and infinite order. $\{P, \times_n\}$ has zero divisors, units and idempotents.

$\{P, +, \times_n\}$ be the MOD special quasi dual number mixed matrix pseudo ring.

$\{P, +, \times_n\}$ has subrings of both finite and infinite order.

All ideals of $\{P, +, \times_n\}$ are of infinite order. As in case of $\{P, \times_n\}$, $\{P, +, \times_n\}$ has zero divisors, units and idempotents.

Example 2.36: Let

$$N = \{(a_1, a_2, a_3, a_4) \mid a_1 \in R_n^k(5), k^2 = 4k, a_2 \in R_n^k(6), k^2 = 5k, a_3 \in R_n^k(10), k^2 = 9k \text{ and } a_4 \in R_n^k(8), k^2 = 7k\}$$

be the MOD special quasi dual number mixed matrix collection.

$\{N, +\}$ is the MOD special quasi dual number mixed matrix group of infinite order which is commutative.

Let $x = (3 + 2.5k, 4.5 + 2k, 6 + 5k, 7 + 5k)$ and

$$y = (2.5 + 2k, 3k + 2.5, 0.4 + 2.5k, 0.8 + k) \in N.$$

$$\begin{aligned} x + y &= (3 + 2.5k, 4.5 + 2k, 6 + 5k, 7 + 5k) + (2.5 + 2k, 3k + 2.5, 0.4 + 2.5k, 0.8 + k) \\ &= (0.5 + 4.5k, 1 + 5k, 6.4 + 7.5k, 7.8 + 6k) \in N. \end{aligned}$$

This is the way the operation of addition is performed on N.

Now $(N, \times)$ is a MOD special quasi dual number mixed matrix semigroup.

$$\begin{aligned} x \times y &= (3 + 2.5k, 4.5 + 2k, 6 + 5k, 7 + 5k) \times (2.5 + 2k, 3k + 2.5, 0.4 + 2.5k, 0.8 + k) \\ &= (7.5 + 6.25k + 6k + 5 \times 4k \pmod 5, 13.5k + 30k + 5k + 11.25 \pmod 6, 2.4 + 2k + 15k + 112.5k \pmod{10}, 5.6 + 7k + 4k + 5 \times 7k \pmod 8) \\ &= (2.5 + 2.25k, 5.25 + 0.5k, 2.4 + 9.5k, 5.6 + 6k) \in N. \end{aligned}$$

This is the way $\times$ operation is performed on N.

$(N, \times)$ has zero divisors, units and ideals.

All ideals of $\{N, \times\}$ are of infinite order.

However $\{N, \times\}$ has subsemigroups of finite order which are not pseudo.

$\{N, \times\}$ has subsemigroups of infinite order which are subsemigroups which are not ideals of $\{N, \times\}$.

$\{N, \times, +\}$ is the MOD special quasi dual number mixed matrix pseudo ring.

Clearly $\{N, +, \times\}$ is commutative and is of infinite order. This has subrings of finite order which are not pseudo.

$\{N, +, \times\}$ has subrings of infinite order which are pseudo and are not ideals. All ideals of $\{N, +, \times\}$ are pseudo subrings and are of infinite order.

Next we proceed onto describe and develop the notion of MOD special multi mixed matrices using $R_n(m)$, $C_n(m)$, $R_n^I(m)$, $R_n^g(m)$, $R_n^h(m)$ and $R_n^k(m)$ by appropriate examples.

Example 2.37: Let

$$M = \left\{\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \middle| \; a_1 \in R_n(12), a_2 \in R_n^I(12), \right.$$
$$\left. a_3, a_4 \in R_n^g(12), a_5, a_6 \in C_n(12)\right\}.$$

M is defined as the special multi MOD mixed matrices collection.

Example 2.38: Let

$$W = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \middle| \; \text{where } a_1 \in R_n(3), a_2, a_3 \in C_n(7), \right.$$

$$a_4, a_5 \in R_n^I(14), a_6, a_7, a_8 \in R_n^g(17), a_9, a_{10}, a_{11} \in R_n^k(19),$$
$$a_{12} \in C_n(12), a_{13}, a_{14} \in R_n^I(27), a_{15}, a_{16}, a_{17} \in R_n^h(2) \text{ and}$$
$$\left. a_{18} \in R_n(48)\right\}$$

be the MOD special multi mixed matrix collection we see the matrix collection given in examples 2.37 and 2.38 are different.

For in 2.37 all the MOD planes are built using the MOD interval [0, 12) whereas in example 2.38 the MOD planes are built using very may distinct/different MOD intervals like [0, 3), [0, 14), [0, 7), [0, 19) and so on. So we call those special multi mixed matrix collection which uses the same base interval [0, m) as the same base interval special multi mixed collection.

However both types of special multi mixed matrix collections are well defined, however for us to define the notion of MOD vector spaces or pseudo MOD linear algebras we can use only the MOD same base interval special multi mixed matrix collection.

Example 2.39: Let

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \right| a_1 \in R_n(4),\ a_2, a_3, a_4 \in C_n(5),$$

$$a_5, a_6, a_7 \in R_n^I(10) \text{ and } a_8, a_9 \in R_n^g(6) \text{ and } a_{10} \in R_n^I(2)\}$$

be the MOD special multi mixed matrices.

We can define + operation component wise on P.

Thus {P, +} will be an abelian group of infinite order.

$\{P, \times_n\}$ will be a defined as the MOD multi special mixed matrix semigroup of infinite order.

$\{P, \times_n\}$ is a monoid which is commutative.

In fact $\{P, +, \times_n\}$ is defined as the MOD special multi mixed matrix pseudo ring of infinite order.

We will see how operations are performed on these newly defined structures by an example.

Example 2.40: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \middle| \, a_1 \in R_n(5),\ a_2 \in R_n^I(3),\ a_3 \in C_n(6), \right.$$

$$\left. a_4 \in R_n^g(2),\ a_5, a_6 \in R_n^k(7);\ k^2 = 6k,\ g^2 = g,\ I^2 = I,\ i_F^2 = 5 \right\}$$

be the MOD special multi mixed matrix collection.

Let {M, +} be the MOD special multi mixed matrix group of infinite order.

$$\text{Let } A = \begin{bmatrix} (3, 2.6) & 1 + 2.1I \\ 0.3 + 0.6i_F & 0.9 + 1.2g \\ 5 + 0.6k & 0.7 + 4k \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} (0.7, 0.5) & 0.2 + 0.5I \\ 1 + 0.5i_F & 1 + 0.5g \\ 2.3 + 5k & 4.5 + 3.7k \end{bmatrix} \in M.$$

$$A + B = \begin{bmatrix} (3, 2.6) & 1 + 2.1I \\ 0.3 + 0.6i_F & 0.9 + 1.2g \\ 5 + 0.6k & 0.7 + 4k \end{bmatrix} +$$

$$\begin{bmatrix} (0.7, 0.5) & 0.2 + 0.5I \\ 1 + 0.5i_F & 1 + 0.5g \\ 2.3 + 5k & 4.5 + 3.7k \end{bmatrix}$$

$$= \begin{bmatrix} (3.7, 3.1) & 1.2 + 2.6I \\ 1.3 + 1.1i_F & 9.9 + 1.7g \\ 0.3 + 5.6k & 5.2 + 0.7k \end{bmatrix} \in M$$

This is the way + operation is performed on M. Clearly {M, +} is abelian; for it is easily verified A + B = B + A for all A, B ∈ M.

Further $0 = \begin{bmatrix} (0, 0) & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ acts as the additive identity of M.

For $A = \begin{bmatrix} (3, 2.6) & 1 + 2.1I \\ 0.3 + 0.6i_F & 0.9 + 1.2g \\ 5 + 0.6k & 0.7 + 4k \end{bmatrix}$ the inverse

$-A = \begin{bmatrix} (2, 3.4) & 2 + 0.9I \\ 5.7 + 5.4i_F & 1.1 + 0.8g \\ 2 + 6.4k & 6.3 + 3k \end{bmatrix} \in M$ is such that

$A + (-A) = \begin{bmatrix} (0, 0) & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and –A is unique for the given A.

We see {M, $\times_n$} is the MOD special multi mixed matrix semigroup. We show how the natural product operation $\times_n$ is performed on M.

$$A \times_n B = \begin{bmatrix} (3, 2.6) & 1 + 2.1I \\ 0.3 + 0.6i_F & 0.9 + 1.2g \\ 5 + 0.6k & 0.7 + 4k \end{bmatrix} \times_n \begin{bmatrix} (0.7, 0.5) & 0.2 + 0.5I \\ 1 + 0.5i_F & 1 + 0.5g \\ 2.3 + 5k & 4.5 + 3.7k \end{bmatrix}$$

$$= \begin{bmatrix} (3, 2.6) \times (0.7, 0.5) & (1 + 2.1I) \times (0.2 + 0.5I) \\ (0.3 + 0.6i_F) \times (1 + 0.5i_F) & (0.9 + 1.2g) \times (1 + 0.5g) \\ (5 + 0.6k) \times (2.3 + 5k) & (0.7 + 4k) \times (4.5 + 3.7k) \end{bmatrix}$$

$$= \begin{bmatrix} (2.1, 3) & 0.2 + 1.97I \\ 1.8 + 0.75i_F & 0.9 + 1.65g \\ 4.5 + 0.8k & 3.5 + 2.7k \end{bmatrix} \in M.$$

This is the way $\times_n$ operation is performed on M.

Clearly A $\times_n$ B = B $\times_n$ A. We see $\{M, +, \times_n\}$ is the MOD special multi mixed matrix pseudo ring. This pseudo ring has ideals all of which are of infinite order subrings of both finite and infinite order.

As it is a matter of routine this work is left as an exercise to the reader. $\{M, +, \times_n\}$ has idempotents, units and zero divisors.

The only difference is the sum and product are carried out component wise. If M is a square matrix certainly the usual product $\times$ is not defined for lack of component wise compatibility.

Example 2.41: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_1 \in R_n(5), a_2 \in C_n(5), a_3 \in R_n^I(5) \text{ and } a_4 \in R_n^g(5),\ i_F^2 = 4, I^2 = I \text{ and } g^2 = 0 \right\}$$

be the MOD special same base multi mixed matrix collection. We show how the operations + and $\times_n$ is defined.

We also show how the usual matrix product $\times$ is not defined on S.

Let {S, +} be the abelian group where $(0) = \begin{bmatrix} (0, 0) & (0, 0) \\ (0, 0) & (0, 0) \end{bmatrix}$ is the identity of S with respect to +.

For every $A = \begin{bmatrix} (4.3, 0.7) & 4 + 0.2i_F \\ 4.2 + 0.2I & 0.4 + 3g \end{bmatrix} \in S$ we have

A + (0) = (0) + A = A. Now for every A ∈ M there is unique –A ∈ M such that

$$A + (-A) = \begin{bmatrix} (0, 0) & (0, 0) \\ (0, 0) & (0, 0) \end{bmatrix}.$$

For this A; $-A = \begin{bmatrix} (0.7, 4.3) & 1 + 4.8i_F \\ 0.8 + 4.8I & 4.6 + 2g \end{bmatrix} \in M$.

We see

$$A \times_n A = \begin{bmatrix} (4.3, 0.7) & 4 + 0.2i_F \\ 4.2 + 0.2I & 0.4 + 3g \end{bmatrix} \times_n \begin{bmatrix} (4.3, 0.7) & 4 + 0.2i_F \\ 4.2 + 0.2I & 0.4 + 3g \end{bmatrix}$$

$$= \begin{bmatrix} (3.49, 0.49) & 1 + 1.76i_F \\ 2.64 + 1.72I & 0.16 + 2.4g \end{bmatrix} \in M.$$

We show the usual matrix product is not defined.

For

$$A \times A = \begin{bmatrix} (4.3, 0.7) & 4 + 0.2i_F \\ 4.2 + 0.2I & 0.4 + 3g \end{bmatrix} \times \begin{bmatrix} (4.3, 0.7) & 4 + 0.2i_F \\ 4.2 + 0.2I & 0.4 + 3g \end{bmatrix}$$

is not defined as $4 + 0.2i_F \times 4.2 + 0.2I$ is not defined.

Hence A × A is not defined in case of both MOD special same base mixed multi square matrices as well as in case of MOD special mixed multi dimensional square matrices.

We suggest some problems some of which are open conjectures.

Problems

1. Let M = {all p × q matrices with entries from $R_n(m)$ the real MOD planes}.

   i. What is the highest algebraic structure enjoyed by M?
   ii. Prove {M, +} is an abelian group of infinite order.
   iii. Prove {M, +} has subgroups of both finite and infinite order.
   iv. Show {M, $\times_n$} is a commutative semigroup of infinite order.
   v. Can {M, $\times_n$} have ideals of finite order? Justify your claim.
   vi. Show {M, $\times_n$} has both finite and infinite order subsemigroups.
   vii. Can {M, $\times_n$} have zero divisors which are not S-zero divisors?

viii. Is $\{M, \times_n\}$ a S-semigroup?
ix. Can $\{M, \times_n\}$ have units?
x. Can $\{M, \times\}$ have S-idempotents?
xi. Prove $\{M, +, \times_n\}$ is only a pseudo ring of MOD $p \times q$ real matrices.
xii. Can $\{M, +, \times_n\}$ have pseudo ideals of finite order?
xiii. Prove $\{M, +, \times_n\}$ can have subrings which satisfy the distributive laws.
xiv. Prove $\{M, +, \times_n\}$ has subrings of infinite order which are pseudo.
xv. Characterize all S-zero divisors and zero divisors of $\{M, +, \times_n\}$.
xvi. Can we say all units of $\{M, +, \times_n\}$ is the same set of units of $\{M, \times_n\}$?
xvii. Characterize the S-idempotents and idempotents of $\{M, +, \times_n\}$.
xviii. Obtain any other special feature enjoyed by $\{M, +, \times_n\}$.

2. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_7 \\ a_8 & a_9 & a_{10} & \ldots & a_{14} \\ a_{15} & a_{16} & a_{17} & \ldots & a_{21} \end{bmatrix} \middle| \; a_i \in R_n(24); \right.$

$1 \leq i \leq 21\}$ be the MOD 3×14 real matrices.

Study questions (i) to (xviii) of problem (1) for this M.

3. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \middle| a_i \in R_n(19), 1 \leq i \leq 10 \right\}$ be the MOD 5×2 real matrices.

   Study questions (i) to (xviii) or problem (1) for this M.

4. Let $P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \middle| a_i \in C_n(29), 1 \leq i \leq 10 \right\}$ be the collection of all 10×1 MOD complex matrices.

   Study questions (i) to (xviii) or problem (1) for this P.

5. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i \in C_n(12), 1 \leq i \leq 9 \right\}$ be the collection of 3×3 square MOD complex Modulo integer matrices.

   Study questions (i) to (xviii) or problem (1) for this M.

6. Let $S = \{(a_1, a_2, \ldots, a_9) \mid a_i \in R_n^I(48); 1 \leq i \leq 9\}$ be the collection of all 1×9 MOD neutrosophic row matrices.

   Study questions (i) to (xviii) or problem (1) for this S.

7. Let $P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in R_n^I(53), \right.$

   $1 \leq i \leq 12\}$ be the collection of all 3×4 MOD neutrosophic matrices.

   Study questions (i) to (xviii) or problem (1) for this P.

8. Compare the real MOD 1×5 row matrix collection with the MOD complex modulo 1×5 row matrix collection and 1×5 MOD neutrosophic row matrix collection.

9. Let $B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix} \middle| \; a_i \in \right.$

   $R_n^g(14)$, $g^2 = 0$, $1 \leq i \leq 14\}$ be the collection of all MOD special dual number 2×7 matrices.

   i. Study questions (i) to (xviii) of problem (1) for this B.
   ii. Prove this B has zero square subsemigroups of finite order.
   iii. Show B has subrings which are zero square rings of both finite and infinite order.

10. Let $S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_i \in R_n^g(23); g^2 = 0, \right.$

    $1 \leq i \leq 16\}$ be the collection of all MOD special dual number square matrix.

i. Study questions (i) to (xviii) of problem (1) for this S.
ii. Prove if on S '×' the usual product is defined instead of the natural product $\times_n$ then {S, ×} is a non commutative semigroup and derive the special features enjoyed by them.
iii. Compare {S, ×} and {S, $\times_n$} as MOD matrix semigroups.
iv. Study the MOD matrix pseudo ring. {S, +, ×} and develop the special features associated with it.
v. Compare {S, +, ×} and {S, +, $\times_n$} as MOD matrix pseudo rings.

11. Let $Z = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \right| a_i \in R_n^h(12), 1 \le i \le 18,$

$h^2 = h\}$ be the 6 × 3 matrix collection of MOD special dual like number matrices.

i. Study question (i) to (xviii) of problem (1) for this Z.
ii. Enumerate all the special features enjoyed by Z.

12. Let $F = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix} \right| a_i \in R_n^h(43);$

$h^2 = h, 1 \le i \le 25\}$ be the MOD special dual like number square matrix collection.

i. Study questions (i) to (xviii) of problem (1) for this F.
ii. Study questions (ii) to (v) of problem (10) for this F.
iii. If $R_n^h(43)$ is replaced by $R_n^h(48)$.
What are the differences enjoyed by that collection?

13. Let R = $\left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}\right| a_i \in R_n^k(31);\right.$

$k^2 = 30k$, $1 \leq i \leq 24\}$ be the collection of all MOD special quasi dual number 4×6 matrix collection.

i. Study questions (i) to (xviii) of problem (1) for this R.
ii. If $R_n^k(31)$ is replaced by $R_n^k(24)$.

Compare the properties enjoyed by them.

14. Let S = $\left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{25} & a_{26} & a_{27} & a_{28} & a_{29} & a_{30} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \end{bmatrix}\right| a_i \in R_n^k(20);\right.$

$1 \leq i \leq 36$, $k^2 = 19k\}$ be the MOD special quasi dual number square matrix collection.

i. Study questions (i) to (xviii) of problem (1) for this S.
ii. Study questions (ii) to (v) of problem (10) for this S.

15. Let W = $\{(a_1, a_2, a_3, a_4, a_5) \mid a_1 \in R_n(7)$, $a_2 \in R_n^I(9)$, $a_3 \in R_n^g(12)$, $a_4 \in R_n^h(15)$, $a_5 \in C_n(13)$; $I^2 = I$, $g^2 = 0$, $h^2 = h$ and

$i_F^2 = 12\}$ be the collection of all MOD special multi mixed row matrix collection.

i. Prove {W, +} is a group of infinite order.
ii. {W, +} has subgroups of finite order prove.
iii. Can {W, +} have subgroups of infinite order?
iv. Prove {W, ×} is a commutative semigroup.
v. Find ideals of {W, ×}.
vi. Can ideals of {W, ×} be of finite order?
vii. Find all subsemigroups of finite order.
viii. Prove {W, ×} has subsemigroups of infinite order which are not ideals.
ix. Can {W, ×} have zero divisors which are not S-zero divisors?
x. Can {W, ×} have S-idempotents?
xi. Find all special features enjoyed by {W, ×}.
xii. Prove {W, +, ×} is only a pseudo ring.
xiii. Can {W, +, ×} have ideals of finite order?
xiv. Can {W, +, ×} be a S-ring?
xv. Can {W, +, ×} have S-ideals?
xvi. Can {W, +, ×} have S-subrings which are not pseudo?
xvii. Prove or disprove all zero divisors of {W, +, ×} is the same as zero divisors of {W, ×}.
xviii. Prove ideals of {W, ×} are not ideals of {W, +, ×}.
xix. Can {W, +, ×} have S-units?
xx. Discuss any other special feature enjoyed by {W, +, ×}.

16. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \end{bmatrix} \right| a_{11}, a_1 \in R_n(5),$

$a_2, a_4 \in R_n^I(4)$, $a_3 \in C_n(7)$, $a_5, a_6, a_{19} \in R_n(12)$, $a_7, a_8, a_{20} \in R_n^g(20)$, $a_9, a_{10} \in C_n(53)$, $a_{12}, a_{18} \in R_n^k(2)$, $a_{13}, a_{16} \in$

$R_n^I(43)$, a_{14}, $a_{15} \in R_n^k(4)$, $a_{16} \in R_n^g(13)\}$ be the MOD special multi mixed matrix collection.

Study questions (i) to (xx) of problem (15) for this M. Here $\times$ is replaced by $\times_n$.

17. Let S = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \right|$ $a_1, a_{16}, a_7 \in C_n(20)$, a_2, a_3,

$a_{12} \in R_n(14)$, $a_4, a_{15} \in R_n^g(5)$, $a_6, a_9 \in R_n^I(4)$, a_5, a_8 $a_{10} \in R_n^k(8)$, $a_{11}, a_{13} \in C_n(5)$, $a_{14} \in R_n^I(7)\}$ be the MOD special multi mixed 4×4 square matrix collection.

i. Study questions (i) to (xx) of problem (15) for this S.
ii. Prove on S only the natural product $\times_n$ can be defined.
iii. Prove on S one cannot define the usual product $\times$.

18. Let R = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \right|$ $a_1 \in C_n(4)$, $a_4, a_2 \in R_n(4)$, $a_3 \in R_n^I(4)$,

$a_5 \in R_n^g(4)$, $a_6 \in R_n^h(4)$; $I^2 = I$, $i_F^2 = 3$, $g^2 = 0$, $h^2 = h\}$ be the MOD special multi mixed same base matrix.

Study questions (i) to (xx) of problem (15) for this R.

Compare R with MOD special multi mixed 6×1 column matrix collection which does not enjoy the same base.

19. Let B = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \right|$ $a_1, a_4,$

    $a_{24} \in R_n^I(17)$, $a_2, a_9, a_{10} \in C_n(17)$, $a_3, a_{23}, a_{22}, a_{16} \in R_n(17)$, $a_5, a_{14}, a_{20} \in R_n^k(17)$, $a_6, a_7, a_8, a_{21}, a_{19}, a_{17}, a_{15}, a_{18} \in R_n^g(17)$, $a_{11}, a_{12}, a_{13} \in R_n^h(17)$, $h^2 = h$, $k^2 = 16k$, $I^2 = I$, $g^2 = 0$, $i_F^2 = 16\}$ be the MOD special multi mixed same base matrix collection.

    i. Study questions (i) to (xx) of problem (15) for this B.
    ii. Can B have infinite number zero divisors?
    iii. Can B have infinite number of idempotents?
    iv. Is it possible for B to have infinite number of subsemigroups of finite order? Justify.

20. What are the advantages of using same base MOD special multi mixed matrices?

21. Why even for square matrices in case of MOD special multi mixed matrices we cannot define usual product $\times$?

22. Explain the situation in problem 21 by a 3×3 square MOD special multi mixed matrices.

23. Prove problem 21 is not true even if the MOD special multi mixed square matrix collection is replaced by MOD special same base multi mixed square matrix collection.

24. What are the striking properties associated with this new structure?

25. What will be the probable applications of this new structure?

26. Do you think using these MOD matrices is advantageous over using real or complex or dual number planes of infinite order?

27. Compare MOD special multi mixed matrix collection with MOD multi matrix collection.

28. Prove algebraically as group, semigroup or as pseudo ring all these three MOD multi mixed collections behave in the same way.

Chapter Three

ALGEBRAIC STRUCTURES ON MOD SUBSETS OF MOD PLANES

In this chapter authors for the first time introduce the notion of MOD vector spaces using real MOD intervals [0, m), MOD real planes $R_n(m)$, MOD complex intervals, MOD complex planes $C_n(m)$, MOD neutrosophic intervals and MOD neutrosophic planes $R_n^I(m)$, MOD dual number intervals and MOD dual number planes and so on.

We will define, develop and describe them.

Example 3.1: Let V = {[0, 5), +} be the MOD real interval group under +. [0, 5) is a vector space over Z_5. In fact V is a vector space of infinite order. We define V as a MOD interval vector space. This has MOD subspace of finite as well as infinite dimension; $Z_5 \subseteq V$ is a MOD subspace of V of one dimension.

W = {0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5} $\subseteq$ V is again a MOD subspace of finite dimension.

Example 3.2: Let V = {[0, 13), +} be a vector space over the field Z_{13}. V is the MOD interval vector space over Z_{13}.

We proceed to define the new notion of MOD interval vector space.

DEFINITION 3.1: *Let V = {[0, n), +; where n is a prime} be the MOD interval abelian group. Clearly V is a vector space over the field F = Z_n. We define V as the MOD interval vector space over F = Z_n. Clearly V is infinite dimensional MOD interval vector space over F = Z_n.*

V has subspaces of finite dimension over F. Finding a basis for V is a very difficult problem in fact it is left as an open conjecture at this stage.

We will provide more examples before we proceed on to develop the associated properties of these MOD interval vector spaces.

Example 3.3: Let V = {[0, 43), +} be the MOD interval vector space over the field F = Z_{43}.

Z_{43} is a vector subspace over Z_{43} of dimension 1.

Clearly P_1 = {0, 0.5, 1, 1.5, 2, …, 41.5, 42, 42.5} $\subseteq$ V is a vector subspace of finite dimension over Z_{43}.

Similarly P_2 = {0, 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2, …, 41, 41.25, 41.5, 41.75, 42, 42.25, 42.5, 42.75} $\subseteq$ V is also a vector subspace of finite dimension over Z_{43}.

In fact V has many subspaces of finite dimension over F = Z_{43}.

Example 3.4: Let V = {[0, 53), +} be the MOD interval vector space over the field Z_{53}. V has several vector subspaces of finite order.

$W = \{0, 0.1, 0.2, \ldots, 0.9, 1, 1.1, 1.2, \ldots, 43, 43.1, 43.2, \ldots, 44, 52, 52.1, 52.2, \ldots, 52.9\} \subseteq V$ is a subspace of V over Z_{53}. Like this V has several subspaces.

In this book we for the first time we propose several open problems (conjectures).

Next we give examples of MOD vector space using MOD intervals [0, n).

Example 3.5: $V = \{([0, 7) \times [0, 7) \times [0, 7)), +\}$ be the MOD interval matrix vector space over the field Z_7.

Clearly this has subspaces of both finite and infinite dimension. Further the dimension of V over Z_7 is infinite.

$W_1 = \{([0, 7) \times \{0\} \times \{0\})\} \subseteq V$,

$W_2 = \{(\{0\} \times [0, 7) \times \{0\})\} \subseteq V$ and

$W_3 = \{(\{0\} \times \{0\} \times [0, 7))\} \subseteq V$ are all subspaces of V over Z_7.

We see $V = W_1 \oplus W_2 \oplus W_3$ is the direct sum as $W_i \cap W_j = \{\{0\}, \{0\}, \{0\}\}$. $i \neq j$; $1 \leq i, j \leq 3$.

Each subspace W_i is of infinite dimension over Z_7.

Consider $P_i = \{(Z_7 \times Z_7 \times \{0\})\}$ and
$P_j = \{(\{0\} \times \{0\} \times Z_7)\}$ subspace of V of finite dimension over Z_7.

$P_i \cap P_j = \{(\{0\}, \{0\}, \{0\})\}$.

But $P_i + P_j \neq V$ so cannot be a direct sum and in fact cannot be completed to get the direct sum.

There are several finite dimensional subspaces which cannot be completed for direct sum.

Now $M_1 = \{Z_7 \times [0, 7) \times \{0\}\}$ and

$M_2 = \{\{0\} \times \{0\} \times Z_7\}$ are subspaces of V.

M_1 is of infinite dimensional subspace of V and M_2 is a finite dimensional subspace of V such that

$M_1 \cap M_2 = (\{0\} \times \{0\} \times \{0\})$ but $M_1 + M_2 \neq V$.

Example 3.6: Let

$$S = \left\{\begin{bmatrix} [0, 23) \\ [0, 23) \\ [0, 23) \\ [0, 23) \\ [0, 23) \\ [0, 23) \end{bmatrix}, +\right\}$$

be the MOD interval vector space over the field Z_{23}.

$$V_1 = \left\{\begin{bmatrix} [0, 23) \\ Z_{23} \\ \{0\} \\ \{0\} \\ \{0\} \\ Z_{23} \end{bmatrix}\right\},\ V_2 = \left\{\begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ [0, 23) \end{bmatrix}\right\},\ V_3 = \left\{\begin{bmatrix} Z_{23} \\ [0, 23) \\ [0, 23) \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix}\right\},$$

$$V_4 = \left\{\begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ [0, 23) \\ \{0\} \end{bmatrix}\right\} \text{ and } V_5 = \left\{\begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ [0, 23) \\ \{0\} \\ \{0\} \end{bmatrix}\right\}$$

be five MOD vector subspaces of S of infinite dimension over the field Z_{23}.

Clearly $S = V_1 \oplus V_2 \oplus \ldots \oplus V_5$ and

$$V_i \cap V_j \neq \left\{\begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix}\right\} \text{ in general for all i and j. } i \neq j.$$

However if

$$W_1 = \left\{\begin{bmatrix} [0, 23) \\ \{0\} \\ \vdots \\ \{0\} \end{bmatrix}\right\},\ W_2 = \left\{\begin{bmatrix} \{0\} \\ [0, 23) \\ \{0\} \\ \vdots \\ \{0\} \end{bmatrix}\right\},\ W_3 = \left\{\begin{bmatrix} \{0\} \\ \{0\} \\ [0, 23) \\ \{0\} \\ \vdots \\ \{0\} \end{bmatrix}\right\},$$

$$W_4 = \left\{ \begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ [0, 23) \\ \{0\} \\ \{0\} \end{bmatrix} \right\},$$

$$W_5 = \left\{ \begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ [0, 23) \\ \{0\} \\ \{0\} \end{bmatrix} \right\}$$

$$\text{and } W_6 = \left\{ \begin{bmatrix} \{0\} \\ \vdots \\ \{0\} \\ [0, 23) \\ \{0\} \end{bmatrix} \right\}$$

are six MOD interval subspaces of S such that

$$W_i \cap W_j = \left\{ \begin{bmatrix} \{0\} \\ \{0\} \\ \vdots \\ \{0\} \end{bmatrix} \right\}. \; i \neq j;\ 1 \leq i, j \leq 6 \text{ and}$$

$S = W_1 \oplus W_2 \oplus \ldots \oplus W_6$.

Thus S is the direct sum of these MOD interval vector subspaces of S over Z_{23}.

$$L_1 = \left\{\begin{bmatrix} [0, 23) \\ \{0\} \\ \{0\} \\ \{0\} \\ Z_{23} \\ \{0\} \end{bmatrix}\right\}, L_2 = \left\{\begin{bmatrix} \{0\} \\ Z_{23} \\ \{0\} \\ \{0\} \\ \{0\} \\ [0, 23) \end{bmatrix}\right\},$$

$$L_3 = \left\{\begin{bmatrix} \{0\} \\ \{0\} \\ Z_{23} \\ [0, 23) \\ \{0\} \\ \{0\} \end{bmatrix}\right\} \text{ and } L_4 = \left\{\begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ [0, 23) \\ Z_{23} \end{bmatrix}\right\}$$

are MOD interval vector subspaces of S and we see $L_1 + L_2 + L_3 + L_4$ is not a direct sum.

In fact all the subspaces L_i are infinite dimensional over S.

Example 3.7: Let

$$S = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \, a_i \in [0, 11); 1 \leq i \leq 12, +\right\}$$

be the MOD interval matrix vector space over the field Z_{11}. S has MOD interval subspaces of infinite dimension over Z_{11}.

S has also finite dimensional subspaces over Z_{11}.

$$P_1 = \left\{\left.\begin{bmatrix} a_1 & a_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right| a_1, a_2 \in [0, 11), +\right\} \subseteq S,$$

$$P_2 = \left\{\left.\begin{bmatrix} 0 & 0 & a \\ b & d & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right| a, b, d \in [0, 11), +\right\} \subseteq S,$$

$$P_3 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ y & z & 0 \\ 0 & 0 & 0 \end{bmatrix}\right| x, y, z \in [0, 11), +\right\} \subseteq S \text{ and}$$

$$P_4 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t \\ s & v & u \end{bmatrix}\right| t, s, u, v \in [0, 11), +\right\} \subseteq S$$

are all infinite dimensional MOD vector subspaces of S such that
$P_1 + P_2 + P_3 + P_4 = S$ and

$$P_i \cap P_j = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}; \text{ if } i \neq j;\ 1 \leq i, j \leq 4$$

is a direct sum. We can write S as a direct sum in many different ways.

$$\text{For } T_1 = \left\{\left.\begin{bmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ a_5 & a_6 & 0 \\ a_7 & a_8 & 0 \end{bmatrix}\right| a_i \in [0, 11); 1 \leq i \leq 8, +\right\} \subseteq S$$

$$\text{and } T_2 = \left\{\left.\begin{bmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & a_4 \end{bmatrix}\right| a_i \in [0, 11), +\right\} \subseteq S$$

are both MOD interval subspaces of S infinite dimension over Z_{11}.

Clearly $S = T_1 + T_2$ and

$$T_i \cap T_j = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}; i \neq j; 1 \leq i, j \leq 2.$$

$$\text{Let } B_1 = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \end{bmatrix}\right| a_i \in [0, 11), 1 \leq i \leq 6, +\right\} \subseteq S$$

$$\text{and } B_2 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 0 \\ b_4 & b_5 & b_6 \end{bmatrix} \middle| \, b_j \in [0, 11), 1 \leq j \leq 6, +\right\} \subseteq S$$

be two MOD interval vector subspaces of S over Z_{11}.

We see $B_1 + B_2 = S$ and

$$B_1 \cap B_2 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}.$$

Thus S is a direct sum of B_1 and B_2.

Consider the subspaces

$$D_1 = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, a_i \in Z_{11}; 1 \leq i \leq 3\right\} \subseteq S,$$

$$D_2 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, a_i \in Z_{11}; 1 \leq i \leq 6\right\} \text{ and}$$

$$D_3 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_1 & c_2 & c_3 \end{bmatrix}\right| c_i \in Z_{11};\ 1 \leq i \leq 3\right\} \subseteq S$$

be three vector subspaces of S over Z_{11}. Clearly all the 3 subspaces are of finite dimension over Z_{11}.

But $D_1 + D_2 + D_3 = W \subseteq S$ is only a subspace of finite dimension over Z_{11} and hence sum is not a direct sum.

Example 3.8: Let

$$B = \left\{\left.\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix}\right| \text{where } a_i \in [0, 29);\right.$$

$$\left. 1 \leq i \leq 12, +\right\}$$

be the MOD interval matrix vector space over the field Z_{29}.

Clearly B is infinite dimensional MOD space over Z_{29}.

B has both finite and infinite dimensional vector subspaces.

Only infinite dimensional subspaces under special conditions contribute to direct sums.

Let

$$C_1 = \left\{\left.\begin{pmatrix} a_1 & a_2 & \dots & a_6 \\ b_1 & b_2 & \dots & b_6 \end{pmatrix}\right| a_i \in [0, 29) \text{ and } b_i \in Z_9,\right.$$

$$\left. 1 \leq i \leq 6; +\right\} \subseteq B$$

is an infinite dimensional subspace of B but C_1 cannot contribute to any direct sum of B.

In fact C_1 cannot be completed to get the direct sum of subspaces

$$W_1 = \left\{\begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix} \middle| a_1 \in Z_{29}\right\} \subseteq B$$

is a one dimensional subspace of B.

$$W_2 = \left\{\begin{pmatrix} a_1 & 0 & \dots & 0 \\ a_2 & 0 & \dots & 0 \end{pmatrix} \middle| a_1, a_2 \in Z_{29}\right\} \subseteq B$$

is a two dimensional vector subspace of B and so on.

Example 3.9: Let

$$Z = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| a_i \in [0, 61); 1 \leq i \leq 15, +\right\}$$

be the MOD interval matrix vector space over the field Z_{61}. Clearly Z is of infinite dimension over Z_{61}.

Z has both subspaces of finite and infinite dimension over Z_{61}. Z has one dimensional, two dimensional etc., finite dimensional vector subspaces over Z_{61}.

Next we proceed onto develop the notion of MOD linear transformation and MOD linear operator of MOD interval vector spaces defined over the field Z_p.

In fact the definition is a matter of routine so we proceed onto describe this situation by some examples.

Example 3.10: Let

$$V = \{(a_1, a_2, a_3) \mid a_i \in [0, 13); 1 \le i \le 3; +\}$$

and

$$W = \left\{ \left. \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \right| a_i \in [0, 13), 1 \le i \le 6; +\right\}$$

be any two MOD interval matrix vector spaces over the field Z_{13}.

Define $\eta: V \to W$ by

$$\eta\{(a_1, a_2, a_3)\} = \left\{ \begin{bmatrix} a_1 & 0 \\ a_2 & 0 \\ a_3 & 0 \end{bmatrix} \right\} \text{ for every } (a_1, a_2, a_3) \in V.$$

Clearly it is verified η is a MOD interval linear transformation of V to W.

Can we have ker η to be different from the zero space?

Let $\eta : W \to V$ be defined as

$$\eta \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \right\} = (a_1, a_3, a_5)$$

$$\text{for every } \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \in W.$$

$$\text{Clearly ker } \eta \neq \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right..$$

$$\text{Thus we have } \eta\text{: } W \to V \text{ with ker } \eta \neq \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}.$$

Let $\mu : V \to W$ defined by

$$\mu((a_1, a_2, a_3)) = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

be a MOD linear transformation of V to W. Clearly ker $\mu \neq \{(0, 0, 0)\}$.

We give one or two examples before we proceed onto describe MOD linear operations.

Example 3.11: Let

$$V = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \text{ where } a_i \in [0, 23); 1 \leq i \leq 9, +\right\}$$

and

$$W = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in [0, 23); 1 \leq i \leq 5, +\}$$

be two MOD interval matrix vector spaces over the field Z_{23}.

Define $\eta: V \to W$ by

$$\eta\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right) = (a_1, a_2, a_3, a_4, a_5)$$

be the MOD interval matrix linear transformation from V to W.

$$\text{Clearly ker } \eta \neq \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\} \text{ for}$$

$$\eta\left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right\} = (0\ 0\ 0\ 0\ 0),$$

$$\eta\left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & 0 \end{bmatrix}\right\} = (0\ 0\ 0\ 0\ 0),$$

$$\eta\left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_1 \end{bmatrix}\right\} = (0\ 0\ 0\ 0\ 0),$$

$$\eta\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\} = (a_1, a_2, a_3, 0, 0)$$

and so on hence the claim.

Define $\delta : W \to V$ by

$$\delta\{(a_1, a_2, a_3, a_4, a_5)\} = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for all $(a_1, a_2, a_3, a_4, a_5) \in W$.

$\ker \delta \neq \{(0\ 0\ 0\ 0\ 0)\}$ for

$$\delta\{(0\ 0\ 0\ a_4\ a_5)\} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\delta\{(0\ 0\ 0\ 0\ a_5)\} = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\},$$

$$\delta\{(0\ 0\ 0\ a_4\ 0)\} = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}.$$

Hence $\ker \delta = \{(0\ 0\ 0\ a_4,\ a_5) \mid a_4, a_5 \in [0, 23)\} \subseteq W$ is a subspace of W which is certainly different from the zero space of W.

Define ϕ: V $\subseteq$ W by

$$\phi\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right\} = (a_1\ a_2\ a_0\ a_2\ a_1)$$

$$\text{for all } \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \in V.$$

$$\ker\phi \neq \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}$$

in fact

$$\ker\phi = \left\{\begin{bmatrix} 0 & 0 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 23);\ 3 \leq i \leq 9\right\} \subseteq V$$

is a subspace of V of infinite dimension over Z_{23}.

Example 3.12: Let

$$V = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_i \in [0, 43);\ 1 \leq i \leq 5, +\right\}$$

and

$$W = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in [0, 43);\ 1 \leq i \leq 5, +\}$$

be two MOD interval vector spaces over the field Z_{43}.

Define η: V $\rightarrow$ W by

$$\eta\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}\right) = (a_1, a_2, a_3, a_4, a_5)$$

$$\text{for every } \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_5 \end{bmatrix} \in V.$$

It is easily verified η is a one to one MOD linear transformation for

$$\ker \eta \neq \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

Example 3.13: Let

$$V = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \; a_i \in [0, 5); 1 \leq i \leq 4, +\right\}$$

and

$$W = \{(a_1, a_2, a_3, a_4, a_5, a_6) \mid a_i \in [0, 5); 1 \leq i \leq 6, +\}$$

be two MOD interval matrix vector spaces over the field Z_5.

Define η: V $\to$ W by

$$\eta\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}\right) = (a_1, a_2, a_3, a_4, 0, 0)$$

$$\text{for every } \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \in V.$$

$$\text{Clearly ker } \eta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Next we proceed onto describe MOD linear operators on MOD interval matrix vector spaces by the following examples.

Example 3.14: Let

$$V = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \text{ where } a_i \in [0, 17); 1 \leq i \leq 4, +\right\}$$

be the MOD interval matrix vector space over the field Z_{17}.

Let $\phi : V \to V$ be a MOD linear operator defined by

$$\phi\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right) = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

$$\text{for every } \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in V.$$

$$\ker \phi = \left\{\begin{bmatrix} 0 & a_2 \\ a_3 & 0 \end{bmatrix} \middle|\ a_3, a_2 \in [0, 17)\right\}.$$

Clearly ker ϕ is also a MOD interval vector subspace of V.

Let $\eta : V \to V$ defined by

$$\eta\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right) = \begin{bmatrix} a_4 & a_3 \\ a_2 & a_1 \end{bmatrix}$$

$$\text{for every } \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in V.$$

$$\ker \eta = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\} \text{ is a null subspace of V.}$$

Define δ: $V \to V$ by

$$\delta\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right) = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{for every } \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in V.$$

$$\ker \delta = \left\{ \left. \begin{bmatrix} 0 & a_2 \\ a_3 & a_4 \end{bmatrix} \right| a_i \in [0, 17); 2 \leq i \leq 4 \right\} \subseteq V$$

is a MOD interval subspace of V over Z_{17} of infinite dimension.

Let Hom(V, W) denote the collection of all MOD linear transformations from V to W of two MOD interval vector spaces over the field Z_p.

The following questions are suggested as problems.

i. Will Hom(V, W) be a MOD vector space of infinite dimension over the field Z_p?

ii. Will Hom (V, W) $\cong$ Hom(W, V)?

iii. Show these MOD interval vector spaces built using [0, p) over the field Z_p cannot satisfy any of the properties of finite dimensional vector space.

iv. Show a MOD vector space cannot be in any way related to usual vector space of infinite dimension.

v. Find the dimension of Hom(V, V) where V is a MOD interval vector space built on [0, n) over the field Z_n.

vi. Can there be any vector space which is isomorphic to the MOD interval vector space over the field Z_n?

Next we see the notion of linear functional is not an easy work.

For all MOD interval vector spaces are built always on a finite field Z_n and all these MOD interval vector spaces are of infinite dimension over Z_n.

Next we proceed onto define the notion of S-MOD interval vector space or MOD interval S-vector spaces over S-rings.

DEFINITION 3.2: *Let V = {[0, n), +} be the MOD interval group. Clearly V is a MOD vector space over Z_n where Z_n is S-ring then we define V to be the S-MOD interval vector space over Z_n or MOD interval S-vector space over Z_n.*

We will illustrate this situation by some examples.

Example 3.15: Let V = {[0, 12), +} be the MOD interval S-vector space over the S-ring Z_{12}. This has S-vector subspaces as well V can also be realized as a MOD interval vector space over the field. {0, 4, 8} $\cong Z_3$.

Example 3.16: Let V = {[0, 14), +} be a MOD interval vector space over the field F = {0, 2, 4, 6, 8, 10, 12} where 8 is the multiplicative identity. V is also a S-MOD interval vector space over the S-ring Z_{14}.

Thus in case of MOD interval [0, n) we can define two types of MOD interval vector spaces one usual MOD interval vector space over the field $P \subseteq Z_n$ another S-MOD interval vector space over the S-ring Z_n.

We will illustrate a few more examples.

Example 3.17: Let V = {[0, 15), +} be a MOD interval vector space over the field F_1 = {0, 5, 10} $\cong Z_3$ (or F_2 = {0, 3, 6, 9, 12} $\cong Z_5$). V is also a MOD interval S-vector space over the S-ring Z_{15}.

Example 3.18: Let V = {[0, 24), +} be the MOD interval vector space over the field F = {0, 8, 16}. V is also a S-MOD interval vector space over Z_{24} the S-ring.

In view of all this we have the following theorem.

THEOREM 3.1: *Let V = {[0, n), +} be the group under + (n a composite number). V is a MOD interval vector space over a* *field $F \subseteq Z_n$ if and only if V is a S-MOD interval vector space over the S-ring Z_n.*

Proof: Let V be a MOD interval vector space over the field $F \subseteq Z_n$. Then this implies Z_n is a S-ring; hence V is a MOD interval S-vector space over the S-ring Z_n.

Conversely if V is a S-MOD interval vector space over the S-ring Z_n; then from the fact Z_n is a S-ring; Z_n contains a non empty subset F such that F under the operations of Z_n is a field. Thus V is a MOD interval vector space over the field F. Hence the claim of the theorem.

We have also seen several such examples to this effect.

Now we can have MOD linear transformations related to the S-MOD interval vector spaces as well as MOD interval vector spaces.

All these will be illustrated by the following examples.

Example 3.19: Let

$V = \{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 14), 1 \leq i \leq 4, +\}$

and

$$W = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \ a_i \in [0, 14); 1 \leq i \leq 4, +\right\}$$

be any two S-MOD interval vector space over the field;

$$F = \{0, 7\} \subseteq Z_{14}.$$

η_1: $V \to W$ is a S-MOD linear transformation from V to W defined by

$$\eta_1(a_1, a_2, a_3, a_4) = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}$$

for every $(a_1, a_2, a_3, a_4) \in V$.

Clearly

$$\ker \eta_1 = \{(0, 0, a_1, a_2) \mid a_1, a_2 \in [0, 14), +\} \subseteq V$$

is subspace of V of infinite dimension over F.

Define $\eta : V \to W$ by

$$\eta_2\{(a_1, a_2, a_3, a_4)\} = \begin{bmatrix} 0 & a_2 \\ 0 & a_4 \end{bmatrix}.$$

η_2 is also a S-MOD linear transformation from V to W.

$\ker \eta_2 = \{(a_1\ 0\ a_3\ 0) \mid a_1, a_3 \in [0, 14), +\} \subseteq V$ is a S-MOD interval vector subspace of V of infinite dimension.

Let

$$\eta_3\{(a_1, a_2, a_3, a_4)\} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

for every $(a_1, a_2, a_3, a_4) \in V$. $\ker \eta_3 = \{(0, 0, 0, 0)\}$.

Thus S-MOD interval linear transformations on S-MOD interval vector spaces V remain the same and become just MOD interval linear transformation of V if V is considered as the MOD interval vector space over a field $F \subseteq Z_{14}$.

This is true universally.

In the view of this we have the following theorem.

THEOREM 3.2: *Let V and W be any two MOD interval vector spaces over the field $F \subsetneq Z_n$. If η: $V \to W$ is the MOD interval linear transformation of V to W then η is also the S-MOD interval linear transformation of the same MOD interval S-vector spaces over the S-ring Z_n.*

Part of the proof follows from the earlier result. The later part can be derived from the definition so can be easily proved by the reader, hence left as an exercise to the reader.

Example 3.20: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \middle| \; a_i \in [0, 20); 1 \leq i \leq 10, +\right\}$$

be the MOD interval vector space over the field F = {0, 4, 8, 12, 16} $\subseteq Z_{20}$.

Let η: V $\to$ V be the MOD real interval vector space operator defined by

$$\eta\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix}\right) = \begin{bmatrix} a_1 & 0 \\ a_2 & 0 \\ a_3 & 0 \\ a_4 & 0 \\ a_5 & 0 \end{bmatrix}.$$

η is also the MOD real interval S-linear operator of V to V; where V can also be realized as a S-MOD interval vector space over Z_{20}.

Example 3.21: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{15} \end{bmatrix} \middle| \; a_i \in [0, 35); 1 \leq i \leq 15, + \right\}$$

be the MOD interval vector space over the field F = {0, 7, 14, 21, 28} where 21 serves as the multiplicative identity of F.

Define $\eta : V \to V$ by

$$\eta\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{15} \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ 0 \\ a_3 \\ 0 \\ a_5 \\ 0 \\ a_7 \\ \vdots \\ 0 \\ a_{15} \end{bmatrix};$$

η is a MOD interval linear operator on V.

The same η is also a S-MOD interval linear operator or MOD interval linear S-operator on V where V is realized as the S MOD interval vector space over Z_{35}.

In view of this we have the following theorem.

THEOREM 3.3: *Let V be a MOD interval vector space over the field $F \subseteq Z_n$. If η: $V \to V$ is a MOD interval linear operator on V then the same map η is also a S-MOD linear operator on V realized as a S-MOD interval vector space over the S-ring Z_n.*

Proof follows from the definitions.

Next we proceed onto discuss about pseudo linear algebras.

In the first place none of these MOD interval vector spaces can be made into a MOD interval linear algebra.

Of course we can define product in all cases but never it is possible to see that + and × are distributive over each other.

The lack of distributivity forces us to discover only pseudo MOD linear algebras structure using the MOD intervals [0, n).

We will describe them by the following examples.

Example 3.22: Let V = {[0, 7), +, ×} be the MOD interval pseudo linear algebra over the field Z_7. Since [0, 7) under + and × does not form a ring only a pseudo ring as + and × do not distribute over each other.

For if x = 3.5, y = 2.5 and z = 4.5 ∈ [0, 7);

$$\begin{aligned} \text{now } x \times (y + z) &= 3.5 \times (2.5 + 4.5) \\ &= 3.5 \times 0 \pmod 7 \\ &= 0 \qquad \ldots \qquad \text{I} \end{aligned}$$

$$\begin{aligned} \text{Consider } x \times y + x \times z &= 3.5 \times 2.5 + 3.5 \times 4.5 \\ &= 3.5 \qquad \ldots \qquad \text{II} \end{aligned}$$

As I and II are distinct we see $x \times (y + z) \neq x \times y + x \times z$. Thus V is only a MOD interval pseudo linear algebra over the field Z_7.

Thus no MOD interval vector space can be a MOD interval linear algebra.

Maximum it can be only a MOD pseudo linear algebra of infinite dimension over the field Z_p.

We will illustrate this situation by an example or two.

Example 3.23: Let $V = \{[0, 23), +, \times\}$ be the MOD interval pseudo linear algebra over the field Z_{23}.

Example 3.24: Let
$M = \{([0, 11) \times [0, 11) \times [0, 11) \times [0, 11)), +, \times\}$ be the MOD interval pseudo linear algebra over the field Z_{11}.

Example 3.25: Let

$$M = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \right| a_i \in [0, 43); 1 \leq i \leq 9, +, \times \right\}$$

be the MOD interval pseudo linear algebra over the field Z_{43}.

$$P_1 = \left\{ \left. \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right| a_1 \in [0, 43), +, \times \right\}$$

is a MOD interval pseudo linear subalgebra of dimension over Z_{43}.

$$P_2 = \left\{ \left. \begin{bmatrix} 0 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right| a_2 \in [0, 43), +, \times \right\}$$

be the MOD interval pseudo linear subalgebra of M and so on has pseudo MOD interval linear algebras as of infinite order.

Let

$$T_1 = \left\{ \left. \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right| a_1 \in Z_{43}, +, \times \right\} \subseteq M$$

be a MOD interval linear subalgebra which is not pseudo but T_1 is finite dimensional linear subalgebra.

Can one say all finite dimensional MOD interval linear subalgebras would be non pseudo? The answer is no in general.

For consider

$$V_1 = \left\{ \left. \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right| a_1 \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 42, 42.5\}, +, \times \right\} \subseteq M$$

is the MOD interval sublinear algebra which is finite dimensional but is pseudo.

$$\text{For } x = \begin{bmatrix} 0.5 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, y = \begin{bmatrix} 40.5 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ and } z = \begin{bmatrix} 2.5 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

are in V_1.

$$x \times (y + z) = \begin{bmatrix} 0.5 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \times \left(\begin{bmatrix} 40.5 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 2.5 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0.5 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \ldots \qquad \text{I}$$

Consider

$$x \times y + x \times z = \begin{bmatrix} 0.5 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \times \begin{bmatrix} 40.5 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \times \begin{bmatrix} 2.5 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 20.25 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 1.25 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 21.5 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \ldots \text{ II}$$

As I and II are different we see though V_1 is finite dimensional hence V_1 is only a MOD pseudo linear subalgebra of finite dimension over Z_{43}.

Thus M has several finite dimensional MOD interval linear subalgebras which are pseudo.

In view of all these we have the following theorem.

THEOREM 3.4: *Let V be a MOD interval pseudo linear algebra over the field Z_p.*

i. V has finite dimensional MOD interval linear subalgebra which is not pseudo.

ii. V has finite dimensional MOD interval linear subalgebra which are also pseudo.

Proof is direct hence left as an exercise to the reader.

Next we proceed onto give example of MOD pseudo interval linear algebras.

Example 3.26: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_i \in [0, 19), 1 \leq i \leq 16, +, \times \right\}$$

be the MOD interval pseudo linear algebra over the field Z_{19}.

Clearly V is a non commutative MOD interval pseudo linear algebra.

$$P_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_4 & 0 \\ a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| \; a_i \in Z_{19}, 1 \leq i \leq 4, +, \times \right\} \subseteq V$$

is a MOD interval linear algebra of finite dimension over Z_{19} which is not pseudo.

However

$$P_2 = \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| \; a_1 \in \{0, 0.5, 1, 1.5, 2, \ldots, 17.5, 18, 18.5\}, +, \times \right\} \subseteq V$$

is MOD interval pseudo linear subalgebra of finite dimension over Z_{19}.

Clearly the distributive law is not true.

$$P_3 = \left\{\left.\begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right| a_1 \in [0, 19); +, \times\right\} \subseteq V$$

is a MOD interval pseudo linear subalgebra of infinite dimension over Z_{19}.

Example 3.27: Let

$$W = \left\{\left.\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{pmatrix}\right| a_i \in [0, 23), 1 \leq i \leq 15, +, \times_n\right\}$$

be the MOD interval pseudo linear algebra over the field Z_{23}.

W has MOD interval pseudo linear subalgebra of finite dimension as well as infinite dimension.

However W has also MOD interval linear subalgebras of finite dimension which are not pseudo linear subalgebras.

$$\text{Let } P_1 = \left\{\left.\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right| a_i \in Z_{23}, 1 \leq i \leq 5, +, \times_n\right\}$$

be the MOD interval linear subalgebra over Z_{23} of finite dimension over Z_{23} which is not pseudo.

Let

$$P_2 = \left\{\left.\begin{pmatrix} a_1 & a_4 & 0 & 0 & 0 \\ a_2 & a_5 & 0 & 0 & 0 \\ a_3 & a_6 & 0 & 0 & 0 \end{pmatrix}\right| a_i \in \{0, 0.1, 0.2, \ldots, 0.9, 1, 1.1,\right.$$

$$\ldots, 21.1, 21.2, \ldots, 22, 22.1, \ldots, 22.9\}, 1 \leq i \leq 6; +, \times_n\} \subseteq W$$

is a finite dimensional MOD interval pseudo linear subalgebra over Z_{23}; this is pseudo only.

Example 3.28: Let

$$M = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix}\right| a_i \in [0, 43); 1 \leq i \leq 15, +, \times_n\right\}$$

be the MOD interval pseudo linear algebra of infinite dimension over the field Z_{43}.

M has both finite and infinite dimensional MOD interval pseudo linear subalgebra over Z_{43}.

M has also finite dimensional MOD interval linear subalgebra which is not pseudo.

Next we see we can define MOD interval pseudo linear transformation as in case of vector spaces.

Example 3.29: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{10} \end{bmatrix} \middle| \, a_i \in [0, 13); 1 \leq i \leq 10, +, \times_n \right\}$$

be the MOD interval pseudo linear algebra over Z_{13}.

$$N = \left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \end{pmatrix} \middle| \, a_i \in [0, 13); 1 \leq i \leq 10, +, \times_n \right\}$$

be the MOD interval pseudo linear algebra over Z_{13}.

Define η: M $\rightarrow$ N by

$$\eta \left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \right) = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

be the MOD interval pseudo linear transformation.

$$\ker \eta \neq \left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$$

in fact

$$\ker \eta = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_6 \\ a_7 \\ \vdots \\ a_{10} \end{bmatrix} \middle| \; a_i \in [0, 13); 6 \leq i \leq 10 \right\} \subseteq M$$

is a subspace of M.

It is left as an open conjecture to find the algebraic structure enjoyed by $\mathrm{Hom}_{Z_n}(V, W)$ where V and W are MOD interval pseudo linear algebras built using [0, n) over the field Z_n.

Next we proceed onto describe the notion of MOD interval pseudo linear operator on the MOD interval pseudo linear algebra over field Z_n, n a prime.

We will describe this by the following example.

Example 3.30: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \middle| \; a_i \in [0, 5); 1 \leq i \leq 10, +, \times_n \right\}$$

be the MOD interval pseudo linear algebra over the field Z_5.

Define η: M → M be the MOD interval pseudo linear operator defined by

$$\eta\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix}\right) = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \\ a_5 & a_6 \\ 0 & 0 \\ a_9 & a_{10} \end{bmatrix}.$$

$$\text{Clearly ker } \eta \neq \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}.$$

$$\ker \eta = \left\{\begin{bmatrix} 0 & 0 \\ a_1 & a_2 \\ 0 & 0 \\ a_4 & a_5 \\ 0 & 0 \end{bmatrix} \middle| \; a_i \in [0, 5); i = 1, 2, 4, 5; +, \times_n\right\}$$

is the MOD interval linear operator where kernel η is different from

$$\left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}.$$

Example 3.31: Let

$$V = \{(a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mid a_i \in [0, 11); 1 \leq i \leq 7, +, \times\}$$

be the MOD interval pseudo linear algebra over the field Z_{11}.

Let $\eta: V \to V$ be defined by

$$\eta(a_1, a_2, \ldots, a_7) = (0, a_2, 0, 0, 0, 0, 0)$$

$$\text{for every } (a_1, a_2, \ldots, a_7) \in V.$$

η is the MOD interval pseudo linear operator of V with

$$\ker \eta = \{(a_1\ 0\ a_3 \ldots a_7) \mid a_1, a_3, a_4, \ldots, a_7 \in [0, 7), +, \times\}$$

is a pseudo sublinear algebra of V.

Next we proceed onto describe and develop the notion of MOD interval pseudo S-linear algebra over the S-ring.

Example 3.32: Let

$$P = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in [0, 21); 1 \leq i \leq 12, +, \times_n\right\}$$

be the MOD interval pseudo S-linear algebra over the S-ring.

Example 3.33: Let

$$M = \left\{\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{pmatrix} \middle| \; a_i \in [0, 24); 1 \leq i \leq 8, +, \times_n\right\}$$

be the MOD interval pseudo S-linear algebra over the S-ring.

Example 3.34: Let

$$M = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} \end{bmatrix} \right| a_i \in [0, 18); 1 \leq i \leq 24, +, \times_n\right\}$$

be the MOD interval pseudo S-linear algebra over the S-ring Z_{18}.

We see M has S-pseudo MOD interval sublinear algebras of both infinite and finite dimension.

However S-MOD interval linear algebras which are non pseudo are of finite dimension

$$P_1 = \left\{\left.\begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right| a_1 \in Z_{18}; +, \times_n\right\} \subseteq M$$

is a MOD interval S-linear algebra which is not pseudo but P_1 is of finite dimension.

$$P_2 = \left\{\left.\begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right| a_1 \in [0, 18), +, \times_n\right\} \subseteq M$$

is a MOD interval pseudo S-linear subalgebra of infinite order.

$$P_3 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1 \in \{0, 0.5, 1, 1.5, 2, \ldots, 17, 17.5\} \subseteq [0, 18), +, \times_n \right\} \subseteq M$$

is a MOD interval pseudo S-linear subalgebra of finite dimension over S-ring Z_{18}.

Next we consider the MOD interval pseudo linear algebra built using [0, n) over a subset which is a field $\subseteq Z_n$ by some examples.

Example 3.35: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \middle| \; a_i \in [0, 46); 1 \leq i \leq 10, +, \times_n \right\}$$

be the MOD interval pseudo linear algebra over the field $F = \{0, 23\} \subseteq [0, 46)$.

W has MOD interval pseudo linear subalgebras of both finite and infinite dimension.

Further W has all its MOD interval linear subalgebra which is not pseudo is finite dimensional.

However we cannot say by this all finite dimension pseudo linear subalgebras are not pseudo for there exists MOD interval pseudo linear subalgebras of finite dimension over F.

For $V = \left\{\left.\begin{bmatrix} a & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right| a \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots,\right.$

$44, 44.5, 45, 45.5\} \subseteq [0, 46), +, \times_n\} \subseteq W$

is a MOD interval pseudo linear subalgebra of finite dimension over F = {0, 23}.

Hence the claim.

$$T = \left\{\left.\begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right| a, b \in Z_{46}, +, \times_n\right\}$$

is a MOD interval linear subalgebra of finite dimension over F = {0, 23} which is not a pseudo linear subalgebra of W.

Thus MOD interval pseudo linear algebras can have MOD interval linear algebras which are not pseudo.

Further these MOD interval pseudo linear algebras can have MOD interval vector subspaces which are not linear subalgebras.

For

$$S = \left\{\begin{bmatrix} a_1 & a_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 44, 44.5, 45, 45.5\} \subseteq [0, 46); 1 \leq i \leq 4\right\} \subseteq W$$

is only a MOD interval pseudo vector subspace of W and is not a pseudo linear subalgebra for if

$$x = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0.5 \end{bmatrix} \text{ and } y = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.5 & 0 \end{bmatrix} \in S$$

then

$$x \times_n y = \begin{bmatrix} 0.25 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \notin S.$$

Thus W has proper subsets which are MOD interval vector subspaces but they are not MOD interval linear subalgebras as product operation $\times_n$ is not closed on S.

Thus W has MOD interval vector subspaces which are not MOD interval pseudo linear subalgebras.

However all MOD interval pseudo linear subalgebras are MOD interval vector subspaces.

Finally the following result is true in case of MOD interval pseudo linear algebras.

THEOREM 3.5: *Let S be a MOD interval pseudo linear algebra related to the interval [0, n) over the field Z_n.*

S is a MOD interval vector space over Z_n.

Proof is direct and follows from definitions.

We have MOD interval vector subspaces of S(mentioned in the above theorem) which are not MOD interval pseudo linear algebras.

This situation will be represented by examples.

Example 3.36: Let $S = \{([0, 7) \times [0, 7) \times [0, 7) \times [0, 7)) \mid +, \times\}$ $= \{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 7), 1 \leq i \leq 4, +, \times\}$ be the MOD interval pseudo linear algebra over the field Z_7.

Consider
$M_1 = \{(a_1, a_2, a_3, a_4) \mid a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 6, 6.5\};$ $1 \leq i \leq 4, +\} \subseteq S$ is a MOD interval vector subspace of S but is not a linear subalgebra of S.

Hence the claim.

Let
$M_2 = \{(a_1, a_2, 0, 0) \mid a_1, a_2 \in \{0, 0.1, 0.2, \ldots, 6, 6.1, \ldots, 6.9\}, +\}$ is a MOD interval vector subspace of S and M_2 is not a MOD interval pseudo linear subalgebra of S.

Several interesting properties can be derived.

Example 3.37: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in [0, 5); 1 \leq i \leq 12, +, \times_n \right\}$$

be the MOD interval pseudo linear algebra over the field Z_5.

$$P_1 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2 \in \{0, 0.5, 1.0, 1.5, \ldots, 4, 4.5\} \subseteq [0, 5), + \right\}$$

is a MOD interval vector subspace of V and is not a MOD interval pseudo linear subalgebra of V for if

$$x = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in P_1.$$

$$x \times_n y = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.75 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \notin P_1.$$

So P_1 is not a MOD interval pseudo linear subalgebra of V.

Let

$$P_2 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2 \in \{0, 0.1, 0.2, \ldots, 4, 4.1, 4.2, \ldots, 4.9\} \subseteq [0, 5); + \right\} \subseteq V$$

be the MOD interval vector subspace of V.

Clearly P_2 is not a MOD interval pseudo linear subalgebra of V. For if

$$x = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.7 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$y = \begin{bmatrix} 0.9 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ are in } P_2,$$

$$\text{then } x \times_n y = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.7 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 0.9 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.09 & 0 & 0 \\ 0.35 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \notin P_2.$$

So P_2 is not closed under $\times_n$ hence P_2 is not a MOD interval pseudo linear subalgebra of V.

Next we proceed onto describe S-MOD interval pseudo linear algebras or MOD interval pseudo S-linear algebras over the S-ring Z_n.

Example 3.38: Let V = $\{(a_1, a_2) \mid a_1, a_2 \in [0, 12); +, \times\}$ be the MOD interval pseudo linear algebra over the S-ring Z_{12}.

$P_1 = \{(a_1, 0) \mid a_1 \in [0, 12), +, \times\} \subseteq V$ is a MOD interval pseudo S-linear subalgebra of V over Z_{12}.

$P_2 = \{(0, a_2) \mid a_2 \in [0, 12), +, \times\}$ is a S-MOD interval pseudo linear subalgebra of V over Z_{12}.

$P_1 \cap P_2 = \{(0, 0)\}$ and $V = P_1 \cup P_2$. Thus V is a direct sum of P_1 and P_2.

Further $M = \{(a_1, a_2) \mid a_1, a_2 \in Z_{12}, +, \times\}$ is a MOD interval S-linear subalgebra of V which is not pseudo.

$L = \{(a_1, a_2) \mid a_1\ a_2 \in \{0, 0.1, 0.2, \ldots, 1, 1.1, \ldots, 10, 10.1, \ldots, 10.9, 11, \ldots, 11.9\} \subseteq [0, 12), +\}$ is only a MOD interval S-vector subspace of V over Z_{12}.

Clearly L is not a S-MOD interval pseudo linear subalgebra of V as if

x = (0.1, 0.7) and y = (0.2, 0.4) are in L then

x × y = (0.1, 0.7) × (0.2, 0.4) = (0.02, 0.28) $\notin$ L hence L is not a S-MOD pseudo interval linear subalgebra of V over Z_{12}.

Example 3.39: Let

$$W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \; a_i \in [0, 55); 1 \leq i \leq 6, +, \times_n \right\}$$

be the S-MOD interval pseudo linear algebra over the S-ring Z_{55}.

W has both S-MOD interval vector spaces which are not S-MOD interval linear subalgebras as well as S-MOD interval vector subspaces which are S-MOD interval linear subalgebras.

Take

$$V_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_1 \in [0, 55), +, \times_n \right\} \subseteq W$$

to be a S-MOD interval vector subspace of W which is also a S-MOD interval linear subalgebra of W.

Consider

$$V_2 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_1 \in \{0, 0.5, 1, 1.5, \ldots, 53, 53.5, 54, 54.5\}, +\right\} \subseteq W$$

is only a S-MOD interval subvector space of W and is not a MOD interval S-linear subalgebra of V.

$$\text{That is if } x = \begin{bmatrix} 1.5 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in V_2 \text{ then}$$

$$x \times_n x = \begin{bmatrix} 1.5 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \times \begin{bmatrix} 1.5 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 2.25 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \notin V_2.$$

Hence the claim.

Finding MOD linear transformations of MOD interval linear algebras over a field Z_p or a field $F \subseteq Z_n$ are considered as a matter of routine and hence left as an exercise to the reader.

Similarly finding S-MOD linear transformation of S-MOD interval linear algebras over S-ring is also left for the reader as exercise. Clearly we face with the problem when we have to define MOD interval linear functional.

To overcome this we develop and define a special type of MOD interval linear algebras and vector spaces.

Example 3.40: Let V = $\{(x_1, x_2, x_3) \mid x_i \in [0, 17); 1 \leq i \leq 3, +\}$ be a special strong MOD interval S-vector space (SS-MOD interval S-vector space) over the pseudo S-ring [0, 17).

We see V is a SS-MOD interval S-vector space over the pseudo S-ring [0, 17) is a finite dimensional.

Dimension of V over the S-ring [0, 17) is three.

Example 3.41: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \end{bmatrix} \middle| \text{ where } a_i \in [0, 5); 1 \leq i \leq 8, +\right\}$$

be the SS-MOD interval S-vector space over the pseudo S-ring [0, 5).

Clearly the dimension of M over the pseudo S-ring [0, 5) is eight.

We see

$$P_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| a_1 \in [0, 5), +\right\}$$

be the SS-MOD interval S-vector subspace of M over the pseudo S-ring [0, 5).

Clearly dimension of P_1 is one.

Let

$$P_2 = \left\{ \begin{bmatrix} 0 \\ a_2 \\ a_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_2, a_3 \in [0, 5), + \right\} \subseteq M$$

be a SS-MOD interval S-vector subspace over the pseudo S-ring [0, 5).

Clearly dimension of P_2 is two over the pseudo S-ring [0, 5).

We can write M as a direct sum of SS-MOD interval S- vector subspaces of M over the pseudo S-ring.

Let

$$P_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_1, a_2 \in [0, 5), + \right\}$$

be the SS-MOD interval S-vector subspace over the S-pseudo ring [0, 5).

Let

$$P_4 = \left\{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle|\, a_1, a_2, a_3 \in [0, 5), +\right\} \subseteq M$$

be the SS-MOD interval S-vector subspace over the S-pseudo ring [0, 5).

Clearly $M = P_1 + P_2 + P_3 + P_4$ such that

$$P_i \cap P_j = \left\{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\}; i \neq j; 1 \leq i, j \leq 4$$

is the direct sum of SS-MOD interval S-vector space M over the pseudo S-ring [0, 5).

Example 3.42: Let

$$B = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \middle|\, a_i \in [0, 43); 1 \leq i \leq 15, +\right\}$$

be the SS-MOD interval S-vector space over the pseudo S-ring [0, 43).

Clearly B is a finite dimensional SS-MOD interval S-vector space over the pseudo S-ring T = [0, 43).

Dimension of B over T is 15.

In fact we can write B as a direct sum of two SS-MOD interval S-subspaces in many ways. For take

$$M_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_i \in [0, 43); 1 \leq i \leq 6, + \right\} \subseteq B$$

is a SS-MOD interval S-vector subspace of B over T.

$$M_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 43); 1 \leq i \leq 9, + \right\} \subseteq B$$

is a SS-MOD interval S-vector subspace of B over T is such that

$$M_1 + M_2 = B. \; M_1 \cap M_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}.$$

Let

$$N_1 = \left\{\left.\begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 0 & 0 \\ a_4 & 0 & 0 \\ a_5 & 0 & 0 \end{bmatrix}\right| a_i \in [0, 43), 1 \leq i \leq 5, +\right\} \subseteq B$$

and

$$N_2 = \left\{\left.\begin{bmatrix} 0 & a_1 & a_6 \\ 0 & a_2 & a_7 \\ 0 & a_3 & a_8 \\ 0 & a_4 & a_9 \\ 0 & a_5 & a_{10} \end{bmatrix}\right| a_i \in [0, 43); 1 \leq i \leq 10, +\right\} \subseteq B$$

are SS-MOD interval S-vector subspace of B over T.

$$N_1 \cap N_2 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}$$

and $B = N_1 \oplus N_2$ is a direct sum.

Let

$$L_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_4 & a_5 & a_6 \end{bmatrix} \middle| a_i \in [0, 43), 1 \leq i \leq 6, + \right\} \subseteq B$$

and

$$L_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ 0 & 0 & 0 \end{bmatrix} \middle| a_i \in [0, 43); 1 \leq i \leq 9, + \right\} \subseteq B$$

be two SS-MOD interval S-vector subspace of B over the pseudo S-ring [0, 43).

Clearly

$$L_1 \cap L_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \text{ and } B = L_1 \oplus L_2 \text{ is a direct sum.}$$

Let

$$T_1 = \left\{\left.\begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right| a_1 \in [0, 43), +\right\} \subseteq B$$

and

$$T_2 = \left\{\left.\begin{bmatrix} 0 & a_1 & a_2 \\ a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} \\ a_{12} & a_{13} & a_{14} \end{bmatrix}\right| a_i \in [0, 43); 1 \leq i \leq 14, +\right\} \subseteq B$$

be two SS-MOD interval S-vector subspace of B over the pseudo S-ring [0, 43).

$$T_1 \cap T_2 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}, B = T_1 \oplus T_2$$

is a direct sum.

Let

$$S_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & 0 \end{bmatrix} \middle| \; a_i \in [0, 43); 1 \leq i \leq 14, +\right\} \subseteq B$$

and

$$S_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_1 \end{bmatrix} \middle| \; a_1 \in [0, 43), +\right\} \subseteq B$$

be two SS-MOD interval S-vector subspaces of B over the pseudo S-ring [0, 43).

We see

$$S_1 \cap S_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

and $B = S_1 \oplus S_2$ so B is a direct sum.

Let

$$D_1 = \left\{ \begin{bmatrix} a_1 & a_3 & 0 \\ a_2 & a_4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_i \in [0, 43), 1 \leq i \leq 4, + \right\} \subseteq B$$

and

$$D_2 = \left\{ \begin{bmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} \end{bmatrix} \middle| \; a_i \in [0, 43), 1 \leq i \leq 11, + \right\} \subseteq B$$

are SS-MOD interval S-vector subspaces of B over T; such that

$$D_1 \cap D_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \text{ and } B = D_1 \oplus D_2$$

is the direct sum of subspaces. Consider

$$E_1 = \left\{ \begin{bmatrix} 0 & a_1 & 0 \\ 0 & a_2 & 0 \\ 0 & a_3 & 0 \\ 0 & a_4 & 0 \\ 0 & a_5 & 0 \end{bmatrix} \middle| \; a_i \in [0, 43), 1 \leq i \leq 5, + \right\} \subseteq B$$

and

$$E_2 = \left\{ \begin{bmatrix} a_1 & 0 & a_6 \\ a_2 & 0 & a_7 \\ a_3 & 0 & a_8 \\ a_4 & 0 & a_9 \\ a_5 & 0 & a_{10} \end{bmatrix} \middle| \; a_i \in [0, 43), 1 \leq i \leq 10, + \right\} \subseteq B$$

be two SS-MOD interval S-vector subspaces of B.

$$E_1 \cap E_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

and $B = E_1 \oplus E_2$ is a direct sum.

We can also represent B as a direct sum of three subspaces in many ways.

Let

$$A_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 0 & 0 \\ a_4 & 0 & 0 \\ a_5 & 0 & 0 \end{bmatrix} \middle| \; a_i \in [0, 43), 1 \leq i \leq 5, + \right\},$$

$$A_2 = \left\{ \begin{bmatrix} 0 & a_1 & a_2 \\ 0 & a_3 & a_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_i \in [0, 43), 1 \le i \le 4, + \right\} \text{ and}$$

$$A_3 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_1 & a_2 \\ 0 & a_3 & a_4 \\ 0 & a_5 & a_6 \end{bmatrix} \middle| \; a_i \in [0, 43), 1 \le i \le 6, + \right\}.$$

$$A_i \cap A_j = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}, \; i \ne j; \; 1 \le i, j \le 3$$

and $B = A_1 \oplus A_2 \oplus A_3$ be the direct sum of SS-MOD interval S-vector subspaces of B.

Let

$$F_1 = \left\{ \begin{bmatrix} 0 & 0 & a_1 \\ a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} \end{bmatrix} \middle| \; a_i \in [0, 43), 1 \le i \le 13, + \right\} \subseteq B,$$

$$F_2 = \left\{\left.\begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right| a_1 \in [0, 43), +\right\} \subseteq B \text{ and}$$

$$F_3 = \left\{\left.\begin{bmatrix} 0 & a_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right| a_1 \in [0, 43), +\right\} \subseteq B$$

are three SS-MOD interval vector S-subspaces of B such that

$$F_i \cap F_j = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}; 1 \leq i, j \leq 3, i \neq j;$$

$F_1 \oplus F_2 \oplus F_3 = B$ is the direct sum.

Let

$$R_1 = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right| a_i \in [0, 43), 1 \leq i \leq 3, +\right\} \subseteq B,$$

$$R_2 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ a_1 & a_5 & 0 \\ a_2 & a_6 & 0 \\ a_3 & 0 & 0 \\ a_4 & 0 & 0 \end{bmatrix} \middle| \; a_i \in [0, 43); 1 \leq i \leq 6, +\right\} \text{ and}$$

$$R_3 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ 0 & a_3 & a_4 \\ 0 & a_5 & a_6 \end{bmatrix} \middle| \; a_i \in [0, 43); 1 \leq i \leq 6, +\right\}$$

be there SS-MOD interval S-vector subspaces of B over the pseudo S-ring T.

$$\text{We see } R_i \cap R_j = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}, i \neq j, 1 \leq i, j \leq 3.$$

$B = R_1 \oplus R_2 \oplus R_3$ is a direct sum of SS-MOD interval S-vector subspaces of B over T.

Let

$$Q_1 = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_i \in [0, 43), 1 \leq i \leq 3, +\right\} \subseteq B,$$

$$Q_2 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 0 & 0 \\ a_4 & 0 & 0 \end{bmatrix}\right|\, a_i \in [0, 43);\ 1 \leq i \leq 4, +\right\} \subseteq B,$$

$$Q_3 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & a_2 & 0 \\ 0 & a_3 & 0 \\ 0 & a_4 & 0 \end{bmatrix}\right|\, a_i \in [0, 43),\ 1 \leq i \leq 4\right\} \subseteq B \text{ and}$$

$$Q_4 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & a_4 \end{bmatrix}\right|\, a_i \in [0, 43);\ 1 \leq i \leq 4\right\} \subseteq B$$

be four SS-MOD interval S-vector subspaces of B.

$$Q_i \cap Q_j = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\};\ i \neq j,\ 1 \leq i, j \leq 4\}.$$

$B = Q_1 \oplus Q_2 \oplus Q_3 \oplus Q_4$ is a direct sum of SS-MOD interval S-vector subspaces of B.

Likewise the reader can write B as a direct sum.

The maximum number of subspace a direct sum can have is only 15 and it cannot exceed 15.

Example 3.43: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix} \middle| \; a_i \in [0, 53), +\right\}$$

be a SS-MOD interval S-vector space over the pseudo S-ring [0, 53).

$$\text{Let } W_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_i \in [0, 53), +, 1 \leq i \leq 2\right\} \subseteq V,$$

$$W_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_3 \\ a_4 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_3, a_4 \in [0, 53); +\right\} \subseteq V,$$

$$W_3 = \left\{ \left. \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_5 \\ a_6 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right| a_5, a_6 \in [0, 53), + \right\} \subseteq V,$$

$$W_4 = \left\{ \left. \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_7 \\ a_8 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right| a_7, a_8 \in [0, 53); + \right\} \subseteq V,$$

$$W_5 = \left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ a_9 \\ a_{10} \\ 0 \\ 0 \end{bmatrix} \right| a_9, a_{10} \in [0, 53), +\} \subseteq V$$

and

$$W_6 = \left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ a_{11} \\ a_{12} \end{bmatrix} \right| a_{11}, a_{12} \in [0, 53), +\} \subseteq V$$

be six SS-MOD interval S-vector subspaces over the pseudo S-ring [0, 53).

$$\text{Clearly } W_i \cap W_j = \left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \right\} \; i \neq j;\ 1 \leq i, j \leq 6.$$

Thus $V = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5 \oplus W_6$ is a direct sum.

In fact we can write V as a direct sum using any number of SS-MOD interval S-vector subspaces of V over the pseudo S-ring [0, 53).

Example 3.44: Let

$$T = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \text{ where } a_i \in [0, 3); 1 \leq i \leq 16, +\right\}$$

be the SS-MOD interval S-vector space over the pseudo S-ring S = [0, 3).

We can write T as a direct sum of subspaces in many ways.

In fact T has several SS-MOD interval S-vector subspaces each of finite dimension over S.

Further as T is finite dimensional SS-MOD interval vector S-space over S every SS-MOD interval S-subspace of T will be finite dimensional.

Example 3.45: Let

$$W = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \middle| \text{ where } a_i \in [0, 29); 1 \leq i \leq 12; +\right\}$$

be the SS-MOD interval S-vector space over the pseudo S-ring X = [0, 29).

W has several SS-MOD interval S-vector subspaces. All SS-MOD interval S-vector subspaces are finite dimension over X as W itself is a finite dimensional SS-MOD interval S-vector subspaces.

Next we proceed onto describe SS-MOD transformation of the S-vector spaces over X.

Example 3.46: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 11); 1 \leq i \leq 9, +\right\}$$

and

$$W = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix} \middle| \; a_i \in [0, 11); 1 \leq i \leq 12, +\right\}$$

be two SS-MOD interval S-vector spaces over the pseudo S-ring.

Define $\eta : V \to W$ by

$$\eta_1 \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \right\} = \begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ a_4 & a_5 & a_6 & 0 \\ a_7 & a_8 & a_9 & 0 \end{pmatrix}.$$

η_1 is a SS-MOD interval linear transformation from V to W.

Clearly

$$\ker \eta_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Next η_2: V → W;

$$\eta_2\left(\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}\right) = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

η_2 is also a SS-MOD interval linear transformation of V and W.

$$\ker \eta_2 = \left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & a_1 & a_2 \\ a_3 & a_4 & a_5 \end{pmatrix} \middle|\ a_i \in [0, 11); 1 \leq i \leq 5\right\} \subseteq V$$

is a nontrivial SS-MOD interval S-vector subspace of V of dimension 5.

Thus the SS-MOD interval linear transformation can given non empty kernel as well as only null space.

Example 3.47: Let

$$V = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \\ a_{13} & a_{14} \end{bmatrix} \middle|\ a_i \in [0, 41); 1 \leq i \leq 14, +\right\}$$

be the SS-MOD interval pseudo S-vector space over the pseudo S-ring P = [0, 41).

Let

$$W = \left\{\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \end{pmatrix} \middle| \; a_i \in [0, 41); \; 1 \leq i \leq 14, +\right\}$$

be the SS-MOD interval pseudo S-vector space over the pseudo S-ring P.

Let η_1: V $\to$ W be defined by

$$\eta_1\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{13} & a_{14} \end{bmatrix}\right) = \begin{pmatrix} a_1 & a_2 & \dots & a_7 \\ a_8 & a_9 & \dots & a_{14} \end{pmatrix}$$

$$\text{for all } \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{13} & a_{14} \end{bmatrix} \in V. \ \ker \eta_1 = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}\right\}.$$

Define η_2 : V $\to$ W by

$$\eta_2\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{13} & a_{14} \end{bmatrix}\right) = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_7 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

be the SS-MOD interval linear transformation of V to W.

Clearly

$$\ker \eta_2 = \left\{ \left. \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \\ a_{13} & a_{14} \end{bmatrix} \right| a_i \in [0, 41);\ 8 \leq i \leq 14 \right\} \subseteq V$$

is a SS-MOD interval pseudo S-vector subspace of V.

Define η_3: V $\to$ W by

$$\eta_3\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{13} & a_{14} \end{bmatrix}\right) = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \\ a_3 & a_4 \\ 0 & 0 \\ a_5 & a_6 \\ 0 & 0 \\ a_7 & a_8 \end{bmatrix}$$

$$\text{for all } \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{13} & a_{14} \end{bmatrix} \in V.$$

Clearly η_3 is a SS-MOD interval linear transformation of the S-pseudo vector space.

$$\ker \eta_3 = \left\{ \begin{bmatrix} 0 & 0 \\ a_1 & a_2 \\ 0 & 0 \\ a_3 & a_4 \\ 0 & 0 \\ a_5 & a_6 \end{bmatrix} \middle| \; a_i \in [0, 41); 1 \leq i \leq 6, +\right\} \subseteq V$$

is a SS-MOD interval pseudo S-vector subspace of V.

There are several such SS-MOD interval linear transformation.

Study of the structure of $H_{[0, 41)}(V, W)$ happens to be a very challenging problem.

Next we can describe MOD interval linear operator of S-vector spaces V over the pseudo S-ring.

Example 3.48: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in [0, 17); 1 \leq i \leq 12, +\right\}$$

be the SS-MOD interval S-vector space over the pseudo S-ring [0, 17).

Let η_1: V $\rightarrow$ V defined by

$$\eta_1\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix}\right) = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_4 & a_5 & a_6 \end{bmatrix}$$

be the SS-MOD interval linear operator on V.

$$\text{Clearly ker } \eta_1 \neq \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}$$

$$\text{for ker } \eta_1 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, a_i \in [0, 17); 1 \leq i \leq 6, +\right\}$$

is a SS-MOD interval linear operator on V.

$\eta_2 : V \to V$ be defined as follows:

$$\text{Let } \eta_2 = \left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix}\right) = \begin{bmatrix} a_1 & 0 & a_2 \\ 0 & a_3 & 0 \\ a_4 & 0 & a_5 \\ 0 & a_6 & 0 \end{bmatrix}$$

be the SS-MOD interval linear operator on V.

$$\ker \eta_2 = \left\{\left.\begin{bmatrix} 0 & a_1 & 0 \\ a_2 & 0 & a_3 \\ 0 & a_4 & 0 \\ a_5 & 0 & a_6 \end{bmatrix}\right| a_i \in [0, 17); 1 \leq i \leq 6, +\right\} \subseteq V$$

is a SS-MOD interval S-vector subspace of V.

Let $\eta_3 : V \to V$

$$\eta_3\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix}\right) = \begin{bmatrix} a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_1 & a_2 & a_3 \end{bmatrix}$$

be a SS-MOD interval pseudo linear operator so that

$$\ker \eta_3 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}.$$

Example 3.49: Let

$$V = \left\{\left.\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}\right| a_i \in [0, 29); 1 \leq i \leq 6, +\right\}$$

be a SS-MOD interval S-vector space.

$\eta_1: V \to V$; Let

$$\eta_1\left\{\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}\right\} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 0 & a_2 \end{pmatrix}$$

be the SS-MOD interval operator on S-vector space with

$$\ker \eta_1 = \left\{\begin{pmatrix} 0 & a_1 & a_2 \\ a_3 & a_4 & 0 \end{pmatrix} \middle| \, a_i \in [0, 29); 1 \leq i \leq 4; +\right\} \subseteq V;$$

is a SS-MOD interval S-vector subspace of V.

Let $\eta_2 : V \to V$ be defined by

$$\eta_2 \left\{\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}\right\} = \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \end{pmatrix}$$

be the SS-MOD interval operator on S-vector space V.

$$\ker \eta_2 = \left\{\begin{pmatrix} 0 & 0 & 0 \\ a_1 & a_2 & a_3 \end{pmatrix} \middle| \, a_i \in [0, 29); 1 \leq i \leq 3, +\right\} \subseteq V$$

is a SS-MOD interval S-vector subspace of V.

Finding the algebraic structure enjoyed by $P = \text{Hon}_{[0,\, n)}(V, V)$ is a different task.

Is $P \cong V$?

Next we proceed onto give one or two examples of SS-MOD interval pseudo S-linear algebras.

Example 3.50: Let

$$W = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \, a_i \in [0, 43); 1 \leq i \leq 9; +, \times_n\right\}$$

be the SS-MOD interval pseudo S-linear algebras over the pseudo S-interval ring [0, 43).

W cannot have subspaces of finite order.

Every subspace of W is finite dimensional as W is a SS-MOD interval pseudo S-linear algebra over [0, 43) the pseudo S-interval ring.

Example 3.51: Let

$$M = \left\{ \left. \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{17} & a_{18} \end{bmatrix} \right| a_i \in [0, 15); 1 \leq i \leq 18, +, \times_n \right\}$$

be the SS-MOD interval pseudo S-linear algebra over the pseudo S-interval ring P = [0, 15). M is a finite dimensional SS-MOD interval pseudo S-linear algebra over P.

Clearly dim M = 15. Hence all SS-MOD interval pseudo S-linear subalgebras of M are finite dimensional.

$$P_1 = \left\{ \left. \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \right| a_1, a_2 \in [0, 15), +, \times_n \right\} \subseteq M,$$

$$P_2 = \left\{ \left. \begin{bmatrix} 0 & 0 \\ a_3 & a_4 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \right| a_3, a_4 \in [0, 15), +, \times_n \right\} \subseteq M \text{ and}$$

$$P_3 = \left\{\left.\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a_1 & a_2 \\ \vdots & \vdots \\ a_{13} & a_{14} \end{bmatrix} \right| a_i \in [0, 15), 1 \leq i \leq 14; +, \times_n\right\} \subseteq M$$

are SS-MOD interval pseudo S-linear subalgebras of M over P.

Further $M = P_1 \oplus P_2 \oplus P_3$ is a direct sum and as

$$P_i \cap P_j = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}\right\} i \neq j \text{ and } 1 \leq i, j \leq 3.$$

We can write M as a direct sum and of SS-MOD interval S-pseudo linear subalgebras in many ways.

Example 3.52: Let

$$V = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right| a_i \in [0, 33); 1 \leq i \leq 9, +, \times\right\}$$

be the SS-MOD interval S-pseudo linear algebra over the S-interval pseudo ring P = [0,33).

Clearly V is non commutative as $\times$ is a non commutative operation on V.

Clearly dimension of V over P is 9 and all SS-MOD interval pseudo S-linear subalgebras of V are also finite dimensional.

V can be written as a direct sum of SS-MOD interval S-linear subalgebras.

Example 3.53: Let

$$S = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{array} \right] \middle| \; a_i \in [0, 10); 1 \leq i \leq 10, +, \times_n \right\}$$

be the SS-MOD interval pseudo S-linear algebra over the pseudo S-ring R = [0, 10).

Dimension of S over R is 10.

All S-sublinear subalgebras of S over R is also finite and is of dimension less than 10.

Now we proceed onto define the notion of SS-MOD interval transformation, SS-MOD interval linear operator and SS-MOD interval linear functional.

Only in case SS-MOD interval S-pseudo vector spaces as well as SS-MOD interval S-pseudo interval linear algebras we can define the notion of SS-MOD interval linear functional.

All these we only illustrate by examples.

Example 3.54: Let

$$S = \left\{ \left[\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{array} \right] \middle| \; a_i \in [0, 43); 1 \leq i \leq 12, + \right\}$$

and

$$R = \left\{\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \end{pmatrix} \middle| \; a_i \in [0, 43); \right.$$

$$\left. 1 \leq i \leq 14, + \right\}$$

be any two SS-MOD interval S-pseudo vector space over the S-pseudo ring X = [0, 43).

$\eta_1 : S \to R$ be the SS-MOD interval linear transformation defined by

$$\eta_1 \left\{\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix}\right\} = \begin{pmatrix} 0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & 0 \end{pmatrix}.$$

$$\ker \eta_1 = \left\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right\}.$$

Define $\eta_2 : S \to R$ by

$$\eta_2 \left\{\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix}\right\} = \begin{pmatrix} a_1 & a_3 & a_5 & 0 & 0 & 0 & 0 \\ a_2 & a_4 & a_6 & 0 & 0 & 0 & 0 \end{pmatrix}$$

be the SS-MOD interval linear transformation with

$$\ker \eta_2 = \left\{\left.\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a_1 & a_2 \\ a_3 & a_4 & a_5 & a_6 \end{pmatrix}\right| a_i \in [0, 43);\ 1 \leq i \leq 6; +\right\}$$

is a SS-MOD interval linear transformation with nontrivial kernel.

Example 3.55: Let

$$V = \left\{\left.\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix}\right| \text{where } a_i \in [0, 24);\ 1 \leq i \leq 16, +\right\}$$

be the SS-MOD interval S-pseudo vector space over the S-pseudo ring B = [0, 24).

Let $\eta_1 : V \to V$ defined by

$$\eta_1\left\{\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix}\right)\right\} = \begin{bmatrix} a_1 & 0 \\ a_3 & 0 \\ \vdots & \vdots \\ a_{15} & 0 \end{bmatrix}$$

be the SS-MOD interval linear operator of the S-pseudo vector space.

Clearly

$$\ker \eta_1 = \left\{\left.\begin{bmatrix} 0 & a_1 \\ 0 & a_2 \\ \vdots & \vdots \\ 0 & a_8 \end{bmatrix}\right| a_i \in [0, 24); +\right\} \subseteq V$$

is a SS-MOD interval S-pseudo vector subspace of V over the pseudo S-ring [0, 24).

Define

$$\eta_2 : V \to V; \eta_2\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix}\right) = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ a_5 & a_6 \end{bmatrix}.$$

η_2 is a SS-MOD interval linear operator of the S-pseudo vector space V to V.

$$\ker \eta_2 = \left\{\left.\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ 0 & 0 \end{bmatrix}\right| a_i \in [0, 24); 1 \le i \le 10, +\right\} \subseteq V$$

is a SS-MOD interval S-vector subspace of V over the pseudo S-ring [0, 24).

Example 3.56: Let

$$V = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right| \text{where } a_i \in [0, 48); 1 \leq i \leq 9, + \right\}$$

be the SS-MOD interval S-pseudo vector space over the pseudo S-ring [0, 48).

Find T = $\mathrm{Hom}_{[0, 48)}(V, V)$.

Prove all ker η_i; $\eta_i \in$ T are SS-MOD interval S-pseudo vector subspaces of V over the S-pseudo ring [0, 48).

This simple task is left as an exercise to the reader.

Next we proceed onto describe the notion of SS-MOD interval S-pseudo linear algebra over the S-pseudo ring [0, n).

Example 3.57: Let

$$V = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \right| \text{where } a_i \in [0, 29); 1 \leq i \leq 12, +, \times_n \right\}$$

be the SS-MOD interval S-pseudo linear algebra over the S-pseudo ring P = [0, 29).

V has SS-MOD interval S-pseudo linear subalgebras over the S-pseudo ring [0, 29).

This V has SS-MOD interval S-pseudo linear subalgebras.

Since V is of finite order over P every S-pseudo linear subalgebra is also finite dimensional over P.

V can also be written as a direct sum of SS-MOD interval S-pseudo linear subalgebra over the S-pseudo ring [0, 29).

Example 3.58: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_9 \\ a_{10} & a_{11} & \dots & a_{18} \\ a_{19} & a_{20} & \dots & a_{27} \end{bmatrix} \middle| \text{ where } a_i \in [0, 65); \right.$$

$$\left. 1 \leq i \leq 27, +, \times_n \right\}$$

be the SS-MOD interval S-pseudo linear algebra over the S-pseudo ring R = [0, 65). W is of dimension 27 over R.

W has sublinear algebras all of which are finite dimensional over R.

W can be written as a direct sum of SS-MOD interval pseudo S-linear subalgebras.

Let

$$V_1 = \left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_9 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix} \middle| a_i \in [0, 65);\ 1 \leq i \leq 9, +, \times_n \right\} \subseteq W.$$

$$V_2 = \left\{\left.\begin{pmatrix} 0 & 0 & \dots & 0 \\ a_1 & a_2 & \dots & a_9 \\ 0 & 0 & \dots & 0 \end{pmatrix}\right| a_i \in [0, 65); 1 \leq i \leq 9, +, \times_n\right\} \subseteq W$$

and

$$V_3 = \left\{\left.\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ a_1 & a_2 & \dots & a_9 \end{pmatrix}\right| a_i \in [0, 65); 1 \leq i \leq 9, +, \times_n\right\} \subseteq W$$

be SS-MOD interval S-pseudo linear subalgebras of W.

Clearly $W = V_1 \oplus V_2 \oplus V_3$ and

$$V_i \cap V_j = \left\{\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}\right\} i \neq j; 1 \leq i, j \leq 3.$$

Hence W is a direct sum of SS MOD interval S-pseudo linear subalgebra each of V_i is of dimension 9 over R; $i = 1, 2, 3$.

We can also have

$$P_1 = \left\{\left.\begin{pmatrix} a_1 & 0 & \dots & 0 \\ a_2 & 0 & \dots & 0 \\ a_3 & 0 & \dots & 0 \end{pmatrix}\right| a_i \in [0, 65); 1 \leq i \leq 3; +, \times_n\right\} \subseteq W,$$

$$P_2 = \left\{ \left. \begin{pmatrix} 0 & a_1 & 0 & \ldots & 0 \\ 0 & a_2 & 0 & \ldots & 0 \\ 0 & a_3 & 0 & \ldots & 0 \end{pmatrix} \right| a_i \in [0, 65); 1 \leq i \leq 3, +, \times_n \right\} \subseteq W,$$

$$P_3 = \left\{ \left. \begin{pmatrix} 0 & 0 & a_1 & 0 & \ldots & 0 \\ 0 & 0 & a_2 & 0 & \ldots & 0 \\ 0 & 0 & a_3 & 0 & \ldots & 0 \end{pmatrix} \right| a_i \in [0, 65); \right.$$

$$\left. 1 \leq i \leq 3, +, \times_n \right\} \subseteq W,$$

$$P_4 = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & a_1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & a_2 & 0 & \ldots & 0 \\ 0 & 0 & 0 & a_3 & 0 & \ldots & 0 \end{pmatrix} \right| a_i \in [0, 65); \right.$$

$$\left. 1 \leq i \leq 3, +, \times_n \right\} \subseteq W,$$

$$P_5 = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & 0 & a_1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & a_2 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & a_3 & 0 & \ldots & 0 \end{pmatrix} \right| a_i \in [0, 65); \right.$$

$$\left. 1 \leq i \leq 3, +, \times_n \right\} \subseteq W,$$

$$P_6 = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 \end{pmatrix} \right| a_i \in [0, 65); \right.$$

$$\left. 1 \leq i \leq 3, +, \times_n \right\} \subseteq W,$$

$$P_7 = \left\{\left.\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 \end{pmatrix}\right| a_i \in [0, 65);\right.$$

$$\left. 1 \leq i \leq 3, +, \times_n\right\} \subseteq W,$$

$$P_8 = \left\{\left.\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 \end{pmatrix}\right| a_i \in [0, 65);\right.$$

$$\left. 1 \leq i \leq 3, +, \times_n\right\} \subseteq W \text{ and}$$

$$P_9 = \left\{\left.\begin{pmatrix} 0 & 0 & \dots & 0 & a_1 \\ 0 & 0 & \dots & 0 & a_2 \\ 0 & 0 & \dots & 0 & a_3 \end{pmatrix}\right| a_i \in [0, 65); +, \times_n\right\} \subseteq W$$

be nine SS-MOD interval S-pseudo linear algebras of W of each dimension three over R.

Clearly $W = P_1 + P_2 + \dots + P_9$,

$$P_i \cap P_j = \left\{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right\} i \neq j; 1 \leq i, j \leq 9$$

is a direct sum of W. One can also define SS-MOD interval projection of W to P_i say p_1: $W \to P_1$ is a projection defined by

$$p_1\left\{\begin{pmatrix} a_1 & a_4 & \dots & a_{25} \\ a_2 & a_5 & \dots & a_{26} \\ a_3 & a_6 & \dots & a_{27} \end{pmatrix}\right\} = \begin{pmatrix} a_1 & 0 & 0 & 0 & \dots & 0 \\ a_2 & 0 & 0 & 0 & \dots & 0 \\ a_3 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

It is easily verified p_1 is a projection of V on to P_1.

Likewise p_2: V → V can also be defined as the SS-MOD real linear operator of the pseudo S-linear algebras defined by

$$p_2\left\{\begin{pmatrix} a_1 & a_2 & \dots & a_9 \\ a_{10} & a_{11} & \dots & a_{18} \\ a_{19} & a_{20} & \dots & a_{27} \end{pmatrix}\right\} = \begin{pmatrix} 0 & a_2 & 0 & 0 & \dots & 0 \\ 0 & a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{20} & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Clearly p_2 is a SS-MOD interval projection of V onto P_2.

Likewise p_3 : V → V defined by

$$p_3\left\{\begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_9 \\ a_{10} & a_{11} & a_{12} & \dots & a_{18} \\ a_{19} & a_{20} & a_{21} & \dots & a_{27} \end{pmatrix}\right\} = \begin{pmatrix} 0 & 0 & a_3 & 0 & \dots & a_9 \\ 0 & 0 & a_{12} & 0 & \dots & a_{18} \\ 0 & 0 & a_{21} & 0 & \dots & a_{27} \end{pmatrix}.$$

Thus p_3 is the SS-MOD interval linear operator of the pseudo S-linear algebra. p_3 is a SS-MOD interval linear projection of V to P_3.

Likewise p_4, p_5, …, p_9 can be defined as SS-MOD interval linear operator of the pseudo S-linear algebras which can also be realized as projections of the spaces P_4, P_5, …, P_9 respectively.

Now having seen projections, it is left as an exercise to the reader to prove the primary decomposition theorem for finite dimensional SS-MOD interval pseudo S-linear algebra (or S-vector space)V over the S-pseudo ring [0, n).

But the most relevant question at this stage is that the very MOD interval rings are pseudo so MOD interval polynomials more so are pseudo as the distributive law is not true in case of polynomial rings.

So we mention at this stage it is difficult to get the primary decomposition theorem for SS-MOD interval S-vector spaces.

However projections exist by getting the primary decomposition is little difficult. One step backward to this situation is the SS-MOD characteristic values of the linear operator of these SS-MOD interval S-pseudo vector spaces.

So we proceed onto discuss and describe essential properties need for all these from the SS-MOD interval S-pseudo vector space over the S-pseudo ring; [0, n).

To this end we start defining the MOD characteristic values of a MOD interval linear operator of the finite dimensional vector space V over the S-pseudo ring [0, n).

Let us define the $n \times n$ MOD interval matrix A over the S-pseudo ring F = [0, n), a MOD characteristic value of A in F is a scalar c in F such that (A – cI) is singular (not invertible).

Since c is a MOD interval characteristic value of A if and only if det(A – cI) = 0, where I is the $n \times n$ identity matrix; or equivalently if and only if det(cI – A) = 0 we form the matrix (xI – A) with MOD polynomial entries and MOD polynomial f = det(xI – A).

Clearly MOD characteristics value of A in F cannot be just the elements c of [0, n) such that f(c) = 0 for one side and f(c) ≠ 0 as the distributive law is not true.

We will first show this problem by the following example.

Example 3.59: Let

$$A = \begin{bmatrix} 0.3 & 0 & 0 \\ 1 & 0.7 & 0 \\ 0 & 0 & 2.3 \end{bmatrix}$$

be the MOD interval matrix we find the MOD characteristic value of A if and only if det(xI – A)

$$= \det \begin{bmatrix} x + 2.7 & 0 & 0 \\ 0 & x + 2.3 & 0 \\ 0 & 0 & x + 0.7 \end{bmatrix}$$

$$= (x + 2.7) \times (x + 2.3) \times (x + 0.7)$$

$$= (x^2 + 2.7x + 2.3x + 2.7 \times 2.3) \times (x + 0.7)$$

$$= (x^2 + 2x + 0.21) \times (x + 0.7)$$

$$= x^3 + 2x^2 + 0.21x + 0.7x^2 + 1.4x + 0.147$$

$$= x^3 + 2.7x^2 + 1.61x + 0.147.$$

Let us denote by

$p(x) = (x + 2.7) \times (x + 2.3) \times (x + 0.7)$ and

$q(x) = x^3 + 2.7x^2 + 1.61x + 0.147.$

Put x = 0.3 then p(0.3) = 0 but

$$q(0.3) = 0.081 + 0.243 + 0.483 + 0.147$$

$$= 0.954 \neq 0.$$

So $p(x) \neq q(x)$ for x = 0.3.

We now find p(0.7) = 0 but

$$q(0.7) = (0.7)^3 + 2.7(0\ )^2 + 1.61 \times 0.7 + 0.147$$

$$= 0.343 + 1.323 + 1.127 + 0.147$$

$= \quad 2.94 \neq 0.$

So $p(0.7) \neq q(0.7)$ so we are in trouble.

Let $p(2.3) = 0$ but

$$q(2.3) = (2.3)^3 + 2.7 \times (2.3)^2 + 1.61 \times 2.3 + 0.147$$

$$= 0.167 + 2.283 + 0.703 + 0.147$$

$$= 0.3 \neq 0.$$

So $p(2.3) = 0$ and $q(2.3) \neq 0$.

Thus all zeros of p(x) are not equal to zero q(x).

Thus it is left as an open conjecture to find means and methods to over this problem.

For in general if $p(x) = |\det(xI - A)|$ for any $m \times m$ MOD interval matrix with entries from [0, m) then as the distributive laws are not true in pseudo polynomial rings we see $p(x) = |\det(xI - A)|$ and q(x) is the expanded form of p(x).

Then $p(x) \neq q(x)$ on the roots.

Thus we face a very odd situation which has been explained by the example.

Example 3.60: Let

$$A = \begin{bmatrix} 2.5 & 0 & 0 & 0 \\ 0 & 4.2 & 0 & 0 \\ 0 & 0 & 3.5 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

be the MOD interval matrix with entries from [0, 5).

$$\det|xI - A| = \begin{bmatrix} 2.5 + x & 0 & 0 & 0 \\ 0 & x + 0.8 & 0 & 0 \\ 0 & 0 & x + 1.5 & 0 \\ 4 & 0 & 0 & x + 4 \end{bmatrix} = p(x)$$

$= \quad (x + 2.5)\,(x + 0.8)\,(x + 1.5) \times (x + 4)$

$= \quad (x^2 + 2.5x + 0.8x + 2.5 \times 0.8) \times (x^2 + 1.5x +$
$4x + 1.5 \times 4)$

$= \quad (x^2 + 3.3x + 2) \times (x^2 + 0.5x + 1)$

$= \quad x^4 + 3.3x^3 + 2x^2 + 0.5x^3 + 3.3\times 0.5x^2 + x$
$+ x^2 + 3.3x + 2$

$= \quad x^4 + 3.8x^3 + 4.65x^2 + 4.3x + 2$

$= \quad q(x).$

We see $p(x) \neq q(x)$ for when $x = 2.5$ then $p(x) = p(2.5) = 0$ but

$q(2.5) \quad = \quad (2.5)^4 + 3.8 \times (2.5)^3 + 4.65(2.5)^2 + 4.3 \times 2.5 + 2$

$= \quad 4.0625 + 4.375 + 4.0625 + 0.75 + 2$

$= \quad 0.25 \neq 0.$

Thus $p(2.5) \neq q(2.5)$

$p(4.2) = 0$ but

$q(4.2) \quad = \quad (4.2)^4 + 3.8 \times (4.2)^3 + 4.65 \times (4.2)^2 +$
$4.3 \times 4.2 + 2$

$= \quad 1.1696 + 1.5344 + 2.026 + 3.06 + 2$

$= \quad 4.7900 \neq 0.$

So $p(4.2) \neq q(4.2)$ but 4.2 is a root of $p(x)$ and is not a root of $q(x)$.

This is because the distributive law is not true.

Likewise the roots of $p(x) = 3.5$ and 1 are not roots of $q(x)$. This is left as an exercise for the reader.

So for the introduction of the notion of MOD characteristic values and MOD characteristic vectors we will have several values and one has to choose the approximately close one for even the simple function $y = x^3 + 1$ has several zeros depending on the MOD interval $[0, n)$. [26-30].

So the problem of diagonalization of SS-MOD interval linear operator on a SS-MOD interval finite dimensional S-pseudo vector space is a open conjecture.

So the concept of spectral theorem, minimal polynomial, characteristic polynomial or for that matter MOD roots of the MOD polynomials is left as a open conjecture for the following two reasons:

i. The fundamental theorems of algebra is not true in case of MOD polynomials for a n^{th} degree polynomial can have more than n roots or less than n roots.

ii. If $p(x) = (x - \alpha_1) \ldots (x - \alpha_n)$ is a MOD polynomial then $p(x) \neq x^n - (\alpha_1 + \ldots + \alpha_n)x^{n-1} + \sum_{i \neq j} \alpha_i\alpha_j \; x^{n-2} - \ldots \pm \alpha_1 \ldots \alpha_n$.

iii. We face the above problem as distributive laws are not true in general.

Further we see SS-MOD interval S-pseudo vector spaces are finite dimensional spaces over the S-pseudo ring $[0, n)$.

Next we proceed onto describe the notion of SS-MOD interval linear functional on of S-pseudo finite dimensional vector spaces over the S-pseudo ring [0, 1).

Example 3.61: Let V = $\{(a_1, a_2, a_3) \mid a_i \in [0, 7); 1 \leq i \leq 3, +\}$ be the SS-MOD interval pseudo S-vector space S-linear algebra over the S-pseudo ring S = [0, 7).

Clearly V is finite dimensional over S = [0, 7).

Clearly [0, 7) is a SS-MOD interval pseudo S-vector space (S-linear algebra) over S = [0, 7) of dimension one.

Define a function $f_n : V \to S$

$f_n(a_1, a_2, a_3) = a_1t_1 + a_2t_2 + a_3t_3$ where $t_i \in S$; $1 \leq i \leq 3$.

f_n is defined as the SS-MOD interval linear functional from V to S.

$$f_n(3.03, 6.1117, 4.052) = 3.03 + 6.1117 + 4.052$$

$$= 6.1937 \pmod 7$$

is a MOD interval linear functional of V to S.

Can the collection of all MOD interval linear functional of V to S. $L_{[0, n)}(V, S)$ be the dual MOD space of V?

This is also left as an open conjecture for the reader.

Equivalently we propose the conjecture as follows:

Is $V^* \cong L_{[0, n)}(V, S)$?

Is V^* finite dimensional over [0, n)?

We give some more MOD interval linear functional of V in the following examples.

Example 3.62: Let

$$V = \left\{\begin{bmatrix} a_1 & a_6 \\ a_2 & a_7 \\ a_3 & a_8 \\ a_4 & a_9 \\ a_5 & a_{10} \end{bmatrix} \middle| \; a_i \in [0, 12); 1 \leq i \leq 10, +\right\}$$

be the MOD interval pseudo vector space over the pseudo MOD interval S-ring, S = [0, 12).

Define f_n: V → S be the MOD interval linear functional given by

$$f_n\left(\begin{bmatrix} a_1 & a_6 \\ a_2 & a_7 \\ a_3 & a_8 \\ a_4 & a_9 \\ a_5 & a_{10} \end{bmatrix}\right) = a_1 + a_6 + a_5 + a_{10} \ (\text{mod } 12)$$

$$\text{that is } f_n\left(\begin{bmatrix} 3.12 & 6.76 \\ 1.006 & 7.8801 \\ 5.0613 & 2.11731 \\ 0.1138 & 10.1107 \\ 5.00087 & 11.60012 \end{bmatrix}\right)$$

$$= \quad 3.12 + 6.76 + 5.00087 + 11.60012$$

$$= \quad 2.48099 \pmod{12}.$$

This is the way the MOD interval linear functional functions behave is

$$\ker f_n \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}; \text{ for if } A = \begin{bmatrix} 0 & 0 \\ 3.115 & 8.872 \\ 1.168 & 2.881 \\ 9.201 & 11.211 \\ 0 & 0 \end{bmatrix}$$

then $\quad f_n(A) \quad = \quad f_n\left(\begin{bmatrix} 0 & 0 \\ 3.115 & 8.872 \\ 1.168 & 2.881 \\ 9.201 & 11.211 \\ 0 & 0 \end{bmatrix}\right)$

$$= \quad a_1 + a_6 + a_5 + a_{10}$$

$$= 0?$$

Can $\ker f_n$ be again a MOD interval pseudo vector subspace of dimension 6 over the S-pseudo interval MOD ring, $S = [0,12)$?

We can have several such MOD interval linear functionals η_I from $V \to S$ such that the $\ker\eta_I$ is a proper MOD interval pseudo vector subspace of V.

The following problem is also left as an open conjecture.

Can we define the concept of MOD interval dual basis in case of the MOD interval pseudo vector space W, provided $V \cong W = L_{[0,\, n)}(V, S)$?

Of course we have the notion of MOD interval pseudo subspace of V provided dimV = rank f_n + nullity f_n!

Clearly [0, n) = S the pseudo MOD interval vector space of dimension 1 over the pseudo MOD interval ring S = [0, n).

So all these notions can be settled provided for any f_n: V $\to$ S rank f_n + nullity f_n = dim V where V is finite dimensional over the pseudo MOD interval S-ring S = [0, n).

Hence we encounter at each stage with several problems when we try to do small (MOD) interval pseudo vector spaces more so with all MOD interval vector spaces (or pseudo linear algebras).

So the notion of MOD interval pseudo hyperspace of a MOD interval finite dimensional pseudo vector space V will have meaning provided only proves dim V = rank f_n + nullity f_n.

$$f_n: V \to S = [0, n).$$

All relation results are at stake and several of them can be considered as open conjecture.

Next we can study the concept of MOD interval pseudo vector space of MOD polynomials.

Let us recall

$$[0, n)[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in [0, n) \right\}; \text{ clearly } [0, n)[x]$$

is only a MOD interval pseudo-ring of polynomials as the distributive law is not true in general.

Now we can realize S[x] = [0, n)[x] (where S = [0, n) as the MOD interval pseudo vector space of MOD polynomials over the MOD interval pseudo S-ring S = [0, n).

We will illustrate this by the following examples.

Example 3.63: Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in [0, 7); + \right\}$$

be the MOD interval pseudo vector space of MOD interval polynomials over the pseudo MOD interval S-ring S = [0, 7).

Clearly $\{1, x, x^2, x^3, \ldots, x^n, \ldots\}$ = B form the basis of S[x] or equivalently the set B spans S[x].

Thus S[x] is an infinite dimensional MOD interval pseudo vector space over the pseudo MOD ring S = [0, 7).

In fact by varying S = [0, n) for all finite $n \in Z^+ \setminus \{1\}$ we get infinite number of MOD interval pseudo vector space of polynomials over each S = [0, n); $n \in Z^+ \setminus \{1\}$. In case of reals or rationals we get only one vector space of polynomials over reals or rationals.

It is important to keep on record that solving polynomial equations over R[x] or Q[x] happens to be a difficult task no solving equations of polynomials in spaces S[x]; S = [0, n); $n \in Z^+ \setminus \{1\}$ happens to be a very difficult one for even the simple equation $x^3 = 1$ has many zeros [Books 1 and 2].

So at this stage one cannot blindly say all properties of polynomial vector spaces can be derived in case of MOD interval pseudo vector space of polynomials S[x] over S = [0, n).

In fact the fundamental theorem of algebra about roots of a n^{th} degree MOD polynomials in S[x] is not in general true.

If $p(x) \in S[x]$ and deg $p(x) = n$; $p(x)$ may have n roots or more than n roots or less than n roots in S.

Further if $\alpha_1, \ldots, \alpha_n$ are roots of $p(x)$.

$p(x) = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$ then $p(x) \neq x^n - (\alpha_1 + \ldots + \alpha_n)x^{n-1} + \ldots \pm \alpha_1 \ldots \alpha_n = q(x)$ then $p(x) \neq q(x)$, which is clearly attributed to the simple fact in $S = [0, n)$ or in S[x] in general the distributive laws are not true.

That is why S and S[x] are only pseudo MOD interval rings.

Further if p(x) and $q(x) \in S[x]$ and p(x) is of degree m and q(x) is of degree t then

i. p(x) q(x) need not in general be a non zero polynomial and

ii. deg p(x) + deg q(x) $\neq$ deg (p(x)q(x)).

So the study of MOD interval polynomials S[x] as pseudo vector spaces over S in general be have in a very chaotic way.

Secondly we can as a matter of routine define the notion of MOD interval pseudo linear algebra of polynomials over the S-MOD interval pseudo ring $S = [0, n)$.

We leave this to the reader but provide examples of them.

Example 3.64: Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in [0, 15) = S, +, \times \right\}$$

be the MOD interval pseudo linear algebra of polynomials over the S-MOD interval pseudo ring $S = [0, 15)$.

Dimension of S[x] over S is two. A basis of S[x] is {1, x}.

Now there are R[x] and Q[x] linear algebras over R in case of R[x] and over Q in case of R[x] and Q[x]. But in case S[x] we have infinite number such S as S = [0, n) and n $\in Z^+ \setminus \{1\}$.

So we have infinite number of MOD interval pseudo linear algebras of polynomials S[x] over the S-MOD pseudo ring S = [0, n); n $\in Z^+ \setminus \{1\}$.

All the spaces are of dimension two over S = [0, n); n $\in Z^+ \setminus \{1\}$.

Now we can have infinite dimensional MOD interval pseudo linear algebras (or vector spaces) over the S-ring Z_n.

We will give one or two example of them.

Example 3.65: Let

$$S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 43), +, \times\right\}$$

be a MOD interval pseudo linear algebra of infinite dimension over the field Z_{43}.

Clearly S[x] is infinite dimensional.

Example 3.66: Let

$$S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 17), +\right\}$$

be a MOD interval vector space of polynomials over the field Z_{17}.

S[x] is of infinite dimension over Z_{17}.

Example 3.67: Let

$$V = S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 12), +\right\}$$

be the MOD interval S-vector space of polynomials over the S-ring Z_{12}.

V is an infinite dimensional S-vector space over Z_{12}.

Example 3.68: Let

$$W = S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 15); +, \times\right\}$$

be the MOD interval S-pseudo linear algebra of MOD polynomials over the S-ring Z_{15}.

Example 3.69: Let

$$M = S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 33); +\right\}$$

be the MOD interval vector space of polynomials of infinite dimension over the field $F = \{0, 11, 22\} \subseteq Z_{33}$.

Example 3.70: Let

$$V = S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 12); +, \times\right\}$$

be the MOD interval pseudo linear algebra of MOD polynomials over the field $F = \{0, 4, 8\}$.

Now working with these spaces of MOD polynomials happens to be a difficult task for many of the results true in general are not true in case of small (MOD) spaces.

So all these study is both innovative and interesting hence left as open problems for the reader.

Finally now one can easily follow why we encountered with so much of problems while trying to solve MOD characteristic equations and in finding MOD characteristic values in case of MOD interval linear operators.

Thus while trying to study small mathematics or small (MOD) vector spaces we face with lots and lots of interesting and new things.

This has left any mathematician with lots of open problems as a new opening for the study of small or MOD mathematics in particular.

We propose the following problems.

Problems

1. Let M = {[0, 23), +} be the MOD real interval group under + be the vector space over Z_{23}.

   i. Prove M is an infinite dimensional vector space over Z_{23}.

   ii. Can M have finite dimensional subspaces?

   iii. Find all finite dimensional vector subspaces of M.

iv. Find $\text{Hom}_{Z_{23}}$ (M M) = N.

v. Is $N \cong M$?

2. Let W = {[0, 43), +} be the MOD real interval vector space over Z_{43}.

   Study questions (i) to (v) of problem 1 for this W.

3. Let V = {[0, 7), +} be the MOD real interval vector space over Z_7.

   Study questions (i) to (v) of problem 1 for this V.

4. Let S = $\left\{\left.\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{pmatrix} \right| a_i \in [0, 11); 1 \leq i \leq 10,\right.$

   +} be the MOD real interval vector space over Z_{11}.

   Study questions (i) to (v) of problem 1 for this S.

5. Let M = $\left\{\left.\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \right| a_i \in [0, 47); +\right\}$ be the MOD interval

   vector space over Z_{47}.

Study questions (i) to (v) of problem 1 for this M.

6. Let N = $\left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix}\right| a_i \in [0, 29);\ 1 \leq i \leq 12, +\right\}$

   be the MOD interval vector space over Z_{29}.

   Study questions (i) to (v) of problem 1 for this N.

7. Let T = $\left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix}\right| a_i \in [0, 48);\ 1 \leq i \leq 12, +\right\}$

   be the S-MOD interval vector space over a S-ring Z_{48}.

   i. What is the dimension of T over Z_{48}?
   ii. Can T have subspaces of finite dimension?
   iii. Final $\text{Hom}_{Z_{48}}(T,T) = M$.
   iv. Find η_1 and η_2 in M so that ker η_1 and kerη_2 are subspaces different from the zero space.
   v. Is $M \cong T$?
   vi. In how many ways can T be written as a direct sum of subspaces?
   vii. Can we define MOD projection from T to a subspace of T?
   viii. Is it possible to define projection from T to

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in Z_{48}, 1 \leq i \leq 12, + \right\} \subseteq T$$

as a subspace of T?

8. Let $S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| \; a_i \in [0, 12); \right.$

$1 \leq i \leq 15, +\}$ be the MOD interval S-vector space over Z_{12}.

Study questions (i) to (viii) of problem 7 for this S.

9. Let $W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \middle| \; a_i \in [0, 14); \right.$

$1 \leq i \leq 24, +\}$ be the MOD interval S-vector space over Z_{14}.

Study questions (i) to (viii) of problem 7 for this W.

10. Let V = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{20} \end{bmatrix} \middle| a_i \in [0, 7); 1 \leq i \leq 20, \times_n, +\right\}$ be the MOD interval pseudo linear algebra over the field Z_7.

    i. Find dim V over Z_7.
    ii. Find a basis of V over Z_7.
    iii. Can V have several basis or only one basis?
    iv. Write V as a direct sum.

11. Find the algebraic structure enjoyed by $\text{Hom}_{Z_n}(V, V)$ where V is the MOD interval vector space over the field Z_n.

12. Let V = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \middle| a_i \in [0, 43); 1 \leq i \leq 10; +\right\}$ be the MOD interval vector space over the field Z_{43}.

    i. Find S = $\text{Hom}_{Z_{43}}(V, V)$.
    ii. Is S a MOD interval vector space over Z_{43}?
    iii. What is dimension of S over Z_{43}?
    iv. Is V $\cong$ S?

13. Let M = $\left\{\left.\begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \end{bmatrix}\right| a_i \in [0, 17); 1 \leq i \leq 30,\right.$

    $+\}$ be the MOD interval vector space over Z_{17}.

    Study questions (i) to (iv) of problem 12 for this M.

14. Let R = $\left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right| a_i \in [0, 29); 1 \leq i \leq 9, +, \times_n\right\}$ be

    the MOD interval pseudo linear algebra over Z_{29}.

    Study questions (i) to (iv) of problem 12 for this R.

15. Will $\mathrm{Hom}_{Z_{29}}$ (R, R) be a MOD interval pseudo linear algebra over Z_{29}?

16. Let W = $\left\{\left.\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix}\right| a_i \in [0, 13); 1 \leq i \leq 10, +, \times_n\right\}$ be the

    MOD interval pseudo linear algebra over the field Z_{13}.

    Study questions (i) to (iv) of problem 12 for this W.

Can $Hom_{Z_{13}}$ (W, W) be defined as the dual MOD interval pseudo linear algebra of W?

17. Let V = $\left\{\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{pmatrix} \middle| a_i \in [0, 42);\right.$ $1 \leq i \leq 10, +\}$. S-MOD interval vector space over the S-ring Z_{42}.

i. Prove V is infinite dimensional over Z_{42}.
ii. How many S-MOD vector subspaces of V are finite dimensional over the S-ring Z_{42}?
iii. Can V have infinite dimensional S-MOD vector subspaces?
iv. Find $Hom_{Z_{42}}$ (V, V) = S.
v. Is S ≅ V?
vi. Find in S at least two S-MOD linear transformation which have kernel to be the zero vector.
vii. Find at least two S-MOD linear transformations of V which has nontrivial kernel.

18. Let W = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{26} & a_{27} & a_{28} & a_{29} & a_{30} \end{bmatrix} \middle| a_i \in [0, 24);\right.$

$1 \leq i \leq 10, +\}$ be the S-MOD interval vector space over the S-ring Z_{24}.

Study questions (i) to (vii) of problem 17 for this W.

19. Let $W = \left\{\begin{pmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \end{pmatrix} \middle| a_i \in [0, 19); 1 \leq i \leq 20, +, \right.$

$\times_n\}$ be the MOD interval pseudo linear algebra over the field

Z_{19} and

$$P = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \middle| a_i \in [0, 19); 1 \leq i \leq 10, +, \times_n\right\}$$ be a

MOD interval pseudo linear algebra over the same field Z_{19}.

i. Find $S = \text{Hom}_{Z_{19}}(P, W)$ and $R = \text{Hom}_{Z_{19}}(W, P)$.
ii. Is R and S in (i) related?
iii. Find those MOD linear transformation in S such that $\ker \eta = \{\text{zero space}\}$, $\eta \in \text{Hom}_{Z_{19}}(P, W)$.
iv. What is the special algebraic structure enjoyed by R and S?

20. Let $V = \{(a_1, a_2, \dots, a_8) \mid a_i \in [0, 29); 1 \leq i \leq 8, +, \times_n\}$ and

$$W = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i \in [0, 29); 1 \leq i \leq 9, +, \times_n\right\}$$ be any

two MOD interval pseudo linear algebras over the field Z_{29}.

Study questions (i) to (iv) of problem 19 for this V and W.

21. Let X = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle|\, a_i \in [0, 43); 1 \leq i \leq 6, +, \times_n\right\}$ and

$Y = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle|\, a_i \in [0, 43), 1 \leq i \leq 8, +, \times_n\right\}$ be any two

MOD interval pseudo linear algebras defined over the field $F = Z_{43}$.

Study questions (i) to (iv) of problem 19 for this X and Y.

22. What are the other special features associated with this $\text{Hom}_{Z_p}(V,W)$, p a prime V and W MOD interval pseudo linear algebras built over Z_p?

23. What is the algebraic structure enjoyed by $\text{Hom}_{Z_p}(V,V)$ where V is a MOD interval pseudo linear algebra over Z_p?

24. Can these infinite structures be useful in applications where distributive laws in general are not true?

25. Let X = $\{(a_1, a_2, \ldots, a_7) \mid a_i \in [0, 42); 1 \leq i \leq 7, +\}$ be the MOD interval S-vector space over the S-ring Z_{42}=S.

    i. What is the dim of X over s-ring Z_{42}?
    ii. Find a basis of X over S.
    iii. Can X have more number of basis?
    iv. Can we write X as a direct sum?
    v. Find $\text{Hom}_{Z_p}(X, X) = P$.
    vi. What is dim P?

26. Let Y = $\left\{\left.\begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \\ a_{31} & a_{32} & \ldots & a_{40} \\ a_{41} & a_{42} & \ldots & a_{50} \\ a_{51} & a_{52} & \ldots & a_{60} \end{bmatrix}\right| \text{ where } a_i \in [0, 15), +, \times_n\right\}$

    be the MOD interval S-linear algebra over the S-ring $B = Z_{15}$.

    Study questions (i) to (vi) of problem 25 for this Y.

27. Let

    $$M = \left\{\left.\begin{pmatrix} a_1 & a_2 & \ldots & a_7 \\ a_5 & a_9 & \ldots & a_{14} \\ a_{15} & a_{16} & \ldots & a_{21} \end{pmatrix}\right| a_i \in [0, 15); 1 \leq i \leq 21;\right.$$

    $+, \times_n\}$ be the MOD interval S-vector space over the S-ring $S = Z_{15}$.

    Study questions (i) to (vi) of problem 25 for this M.

28. Let T = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle| \; a_i \in [0, 55); 1 \leq i \leq 8, +, \times_n\right\}$

    be the MOD interval S-linear algebra over the S-ring Z_{55}.

    Study questions (i) to (vi) of problem 25 for this T.

29. Let V be a SS-MOD interval S-vector space over the MOD interval [0, n).

    i. Find the algebraic structure enjoyed by $\text{Hom}_{[0,n)}$ (V,V) = P.
    ii. Find whether V ≅ P?
    iii. Is P a SS-MOD interval S-vector space of infinite dimensions over [0, n).

30. Let V = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_{10} \\ a_{11} & a_{12} & a_{13} & \dots & a_{20} \\ a_{21} & a_{22} & a_{23} & \dots & a_{30} \end{bmatrix} \middle| \; a_i \in [0, 47),\right.$

    $1 \leq i \leq 30, +\}$ be a SS-MOD interval S-vector space over the S-pseudo ring S = [0,47).

    i. Find SS-MOD interval S-vector subspaces of V.
    ii. Can V have SS-MOD interval S-vector subspace of finite dimension?
    iii. Find P = $\text{Hom}_{[0,47)}$ (V,V).
    iv. Find at least two SS-MOD interval linear operators in P such that kernel of them is the zero subspace.
    v. Find $\eta_1, \eta_2 \in$ P so that kerη_1 and kerη_2 are non zero subspaces.

31. Let $W = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \end{bmatrix} \middle| \, a_i \in [0, 53); 1 \leq i \leq 20, +\right\}$

    be the SS-MOD interval S-vector space over the pseudo S-ring.

    Study questions (i) to (v) of problem 30 for this W.

32. Let $B = \left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_8 \\ a_9 & a_{10} & \cdots & a_{16} \\ a_{17} & a_{18} & \ldots & a_{24} \end{bmatrix} \middle| \, a_i \in [0,13); 1 \leq i \leq 24, +\right\}$

    be the SS-MOD interval S-vector space over the pseudo S-ring.

    Study questions (i) to (v) 30 of problem for this B.

33. Let $D = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \, a_i \in [0, 2)\ 1 \leq i \leq 9, +\right\}$ be the

    SS-MOD interval S-vector space over the S-pseudo ring [0,2).

    Study questions (i) to (v) of problem 30 for this D.

34. Let $N = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix}\right| a_i \in m[0, 47), 1 \leq i \leq 15, +\right\}$

    be the SS-MOD interval S-vector space over the S-pseudo ring.

    Study questions (i) to (v) of problem 30 for this N.

35. Let

    $M = \left\{\left.\begin{bmatrix} a_1 & a_2 & \ldots & a_9 \\ a_{10} & a_{11} & \ldots & a_{18} \end{bmatrix}\right| a_i \in [0,53); 1 \leq i \leq 18, +, \times_n\right\}$

    be the SS-MOD interval S-pseudo linear algebra over the S pseudo ring [0, 53).

    Study questions (i) to (v) of problem 30 for this M.

36. Let $P = [0, 23)[x] = \left\{\left.\sum_{i=0}^{\infty} a_i x^i \right| a_i \in [0, 23), +\right\}$ be the MOD interval polynomial vector space over the field $F = Z_{23}$.

    i. What is dimension of P over F?
    ii. Can P have more than one basis?
    iii. Can P be written as a direct sum?
    iv. Find all subspaces of P.
    v. Can P have finite dimensional vector subspaces?
    vi. Find any other special feature enjoyed by P.

37. Let $W = [0, 29)[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 29), +\right\}$ be the MOD interval S-pseudo vector space over the pseudo S-ring S = [0, 29).

    i. Find dim of W over S.
    ii. Find a basis of W over S.
    iii. Can W have more than one basis set?
    iv. Can W be written as direct sum of subspaces?
    v. Is it possible to have a subspace of finite dimension over S?

38. Let $R = [0, 42)[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 42), +\right\}$ be the MOD interval S-vector space of polynomial over the pseudo S-ring.

    Study questions (i) to (v) of problem 37 for this R.

39. Let $T = [0, 19)[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 19), \times, +\right\}$ be the MOD interval S-pseudo linear algebra over the S-pseudo interval ring [0, 19) = S.

    i. Find dimension of T over the S-pseudo interval ring S.
    ii. Find a basis of T over S.
    iii. Can T have more than one basis over S?
    iv. Can T have a finite S-pseudo MOD interval linear subalgebra over S?
    v. Obtain any other interesting and special features enjoyed by T over S.

40. Let $V = [0, 22)[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 22), +, \times \right\}$ be the S-MOD interval pseudo linear algebra over the S-pseudo interval MOD ring S = [0, 22).

    i. Find $\text{Hom}_{[0, 22)}(V, V) = P$.
    iii. What is dimension of P over [0, 22)?
    iii. For T: V → V a MOD S-linear operator find
        1. Characteristic MOD equation.
        2. Characteristic MOD values.
        3. Characteristic MOD vectors.
    iv. Define $f_n: V \to S = [0, 22)$.
    v. Can $\text{Hom}_s(V, S) \cong V$?
    vi. Can the concept of hyper space possible in V?

41. Can $V = \{(a_1\ a_2\ a_3\ a_4\ a_5) \mid a_i \in [0, 13); 1 \leq i \leq 5, +\}$ be the MOD interval finite dimensional vector space over the MOD pseudo interval S-ring; S = [0, 13) satisfy the primary decomposition theorem?

42. Obtain any MOD interval finite dimensional vector space over a S-pseudo MOD ring [0, n) which satisfies primary decomposition theorem for vector spaces.

43. Can the notion of diagonalization ever possible in finite dimension MOD vector spaces over S = [0, n), a MOD interval pseudo S-ring?

44. Can any of the special classical theorems be true in case of MOD interval vector spaces over the S-MOD pseudo interval ring [0, n)?

45. Will for any linear operator T : V → V where V is a MOD interval finite dimensional vector space over the S-MOD interval pseudo ring S = [0, n) satisfy

    nullity T + rank T = dim V? Justify your claim.

46. Study problem 45 if V is a S-MOD pseudo linear algebra over S = [0, n).

47. Give an example of a MOD polynomial of degree $p(x) \in [0, 29)[x]$ which has only 9 real MOD roots.

48. Give a MOD polynomial of degree 5 which has more than 25 roots in [0, 625)[x].

49. Obtain any special or interesting features enjoyed by S-MOD interval finite dimensional polynomial pseudo linear algebras over S = [0, n) a S-pseudo interval MOD ring.

50. Is there a MOD interval polynomial of degree greater than or equal to five which has only five roots in the interval [0, 10)?

51. Can we ever have the concept of MOD interval dual space?

52. What are the advantages of using small interval polynomials?

53. Find some nice applications of MOD interval linear operators of MOD interval finite dimensional S-vector spaces over the MOD S-pseudo ring [0, n).

54. Prove all MOD interval vector spaces / pseudo linear algebras defined over Z_n (n a prime) are always of infinite dimension hence all properties associated with finite dimension can never be true in these cases.

55. Let $V = \{(a_1\ a_2\ \ldots\ a_9) \mid a_i \in [0, 29);\ 1 \leq i \leq 9, +, \times\}$ be a SS-MOD interval S-pseudo linear algebra over the pseudo S-ring; S = [0, 29).

    i. What is dimension of V over S?
    ii. How many basis exists for V over S?
    iii. Can rank T + nullity T = dim V true for any $T: V \to V$?

iv. Can hyperspace be defined for f_n: V $\rightarrow$ S?
v. Is spectral theorem true in case of V?
vi. Does there exist a T: V $\rightarrow$ V so that diagonalization is possible?
vii. Can we have any subspace of V so that V cannot be completed as a direct sum of subspaces?
viii. Can primary decomposition theorem be true in case of V?
ix. Find the algebraic structure enjoyed by $\text{Hom}_{[0, 29)}(V, V)$.
x. Find the algebraic structure enjoyed by L(V, S) = P.
xi. Is P $\cong$ V?
xii. Obtain any other special property enjoyed by V.

56. Study questions (i) to (xii) when V in problem 55 is replaced by $W = \{[0, 43)\ [x]^{12} = \sum_{i=0}^{12} a_i x^i, +, \times\}$ over the pseudo MOD interval S-ring S = [0, 43).

    (W contains all polynomials of degree less than or equal 12 so that $x^{13} = 1$ and $x^{14} = x$ and so on).

57. Study questions (i) to (xii) when V in problem 55 is replaced by $R = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \ldots & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} & a_{13} & \ldots & a_{17} & a_{18} \end{bmatrix} \right|$ $a_i \in [0, 24); 1 \leq i \leq 18, +, \times_n\}$ over the S-pseudo MOD interval ring S = [0, 24).

Chapter Four

MULTIDIMENSIONAL MOD PSEUDO LINEAR ALGEBRAS

In this chapter for the first time authors define the new notion of multidimensional MOD vector spaces and MOD pseudo linear algebras defined over finite fields or finite S-rings Z_m.

The basic concept of MOD multidimensional group and pseudo rings are recalled in chapter II of this book.

First we give some examples before we make a formal definition of it.

Example 4.1: Let V = $\{(R_n(7) \times R_n^I(7) \times [0, 7)k \times C_n(7)) = (a, b, c, d) \mid a \in R_n(7), b \in R_n^I(7), c \in [0, 7)$ and $d \in C_n(7); I^2 = I, k^2 = 6k, i_F^2 = 6, +\}$ be a mod multi dimensional vector space over the field $Z_7 = F$.

Clearly if x = $((3, 2.5), 6 + 2.1I, 6k, 0.5 + 1.5i_F)$ and y = $((6, 5), 3 + 1.05I, 3k, 1 + 3i_F) \in V$ then x and y are linearly independent.

Further V has MOD subspaces of both finite and infinite dimension.

For $V_1 = \{(Z_7 \times Z_7 \times Z_7k \times Z_7) \mid k^2 = 6k\} \subseteq V$ is a MOD subspace of finite dimension over $F = Z_7$.

$V_2 = \{(Z_7 \times \{0\} \times \{0\} \times \{0\})\} \subseteq V$; is a subspace of dimension one over Z_7.

$V_3 = \{(Z_7 \times Z_7 \times \{0\} \times \{0\})\} \subseteq V$ is a subspace of dimension tow over Z_7.

$V_4 = \{([0, 7) \times \{0\} \times \{0\} \times \{0\})\} \subseteq V$ is a subspace of infinite dimension over Z_7.

We see none of these subspaces can be completed with other subspace of V to get the direct sum of V.

Consider

$P_1 = \{(R_n(7) \times \{0\} \times \{0\} \times \{0\})\} \subseteq V$ is a subspace of V of infinite dimension over $F = Z_7$.

$P_2 = \{(\{0\} \times R_n^I(7) \times \{0\} \times \{0\})\} \subseteq V$ is a subspace of V of infinite dimension over $F = Z_7$.

$P_3 = \{(\{0\} \times \{0\} \times [0, 7)k \times \{0\})\} \subseteq V$ is a subspace of V of infinite dimension over the field $F = Z_7$.

$P_4 = \{(\{0\} \times \{0\} \times \{0\} \times C_n(7))\} \subseteq V$ is a subspace of V of infinite dimension over Z_7.

Clearly $P_i \cap P_j = (\{0\} \times \{0\} \times \{0\} \times \{0\})$; $i \neq j$, $1 \leq i, j \leq 4$.

Further $V = P_1 \oplus P_2 \oplus P_3 \oplus P_4$. Hence the sum is a direct sum of subspaces.

We have V to be a direct sum in other ways also having different set of subspaces.

Working with this is a matter of routine and left as an exercise to the reader.

Example 4.2: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_5 \\ a_2 & a_6 \\ a_3 & a_7 \\ a_4 & a_8 \end{bmatrix} \right| \text{ where } a_1, a_2 \in C_n(13), a_3, a_4 \in R_n(13),$$

$$a_5, a_6 \in R_n^I(13) \text{ and } a_7, a_8 \in R_n^g(13); g^2 = 0, I^2 = I, i_F^2 = 12, +\}$$

be a MOD multi dimensional vector space over the field Z_{13}.

Clearly W is an infinite dimensional MOD multi dimensional vector space over the field Z_{13}.

Finding a basis is a challenging problem.

$$M_1 = \left\{ \begin{bmatrix} a_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right| a_1 \in C_n(13)\} \subseteq W,$$

$$M_2 = \left\{ \begin{bmatrix} 0 & a_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right| a_2 \in R_n^I(13)\} \subseteq W,$$

$$M_3 = \left\{\begin{bmatrix} 0 & 0 \\ a_1 & a_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \middle| \; a_1 \in C_n(13),\, a_2 \in R_n^I(13)\right\} \subseteq W,$$

$$M_4 = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a_3 & 0 \\ a_4 & 0 \end{bmatrix} \middle| \; a_3, a_4 \in R_n(13)\right\} \subseteq W \text{ and}$$

$$M_5 = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & a_1 \\ 0 & a_2 \end{bmatrix} \middle| \; a_1, a_2 \in R_n^g(13)\right\} \subseteq W$$

are all subspaces of W of infinite dimension over $F = Z_{13}$.

Clearly $M_i \cap M_j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$; $i \neq j$; $1 \leq i, j \leq 4$.

Further it is easily verified. $W = M_1 \oplus M_2 \oplus M_3 \oplus M_4$. There are several subspaces of finite dimension over Z_{13}. It is at this juncture left as an open conjecture to find how many basis can W have over the field $F = Z_{13}$. Even finding the basis of the subspaces of M_i; i = 1, 2, 3, 4 happens to be a challenging open conjecture.

Example 4.3: Let

$$S = \left\{\left[\begin{matrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{matrix}\right] \right| \text{ where } a_1, a_2, a_3 \in C_n(43),$$

$$i_F^2 = 42, a_4, a_5, a_6 \in R_n(43), a_7, a_8, a_9 \in R_n^I(43)$$
$$\text{and } a_{10}, a_{11}, a_{12} \in R_n^g(43); I^2 = I, g^2 = 0, +\}$$

be the MOD multi dimensional matrix vector space over the field Z_{43}. This S has subspaces of finite dimension over Z_{43}.

S has also subspaces of infinite dimension over Z_{43}. Since all these MOD spaces are of infinite dimension over Z_{43} the properties pertaining to finite dimensional spaces has no relevance.

Next we study the properties of MOD transformation and MOD linear operator of a MOD vector space.

At the outset one will be in a position to define MOD linear transformation of vector space only if both the spaces are defined over the same field.

We will illustrate this situation by some examples.

Example 4.4: Let

$$M = \left\{\left[\begin{matrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{matrix}\right] \right| a_1, a_2, a_3 \in R_n(23), a_4,$$

$$a_5, a_6 \in C_n(23), a_7, a_8, a_9 \in R_n^g(23);\ i_F^2 = 22 \text{ and } g^2 = 0, +\}$$

be the MOD multi matrix vector space defined over the field $F = Z_{23}$.

Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \middle| \; a_1, a_2 \in R_n^g(23), a_3, a_4 \in R_n(23) \right.$$

$$\left. \text{and } a_5, a_6 \in C_n(23), g^2 = 0, i_F^2 = 22, + \right\}$$

be the MOD interval multi matrix vector space over the field $F = Z_{23}$.

Define $\eta : V \to W$ by

$$\eta\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right) = \begin{bmatrix} a_7 & a_8 \\ a_1 & a_2 \\ a_4 & a_5 \end{bmatrix}.$$

It is easily verified η is a MOD multi linear transformation of V to W.

$$\text{Clearly } \ker \eta \neq \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}.$$

We can define several such MOD multi linear transformations. It is left as an open conjecture.

i) What the algebraic structure enjoyed by $\text{Hom}_{Z_{23}}(V, W)$?

ii) Is $\text{Hom}_{Z_{23}}(V, W)$ an infinite dimensional space over Z_{23}?

If the entries of one MOD multi matrix has its entries from MOD fields say $R_n^g(m)$, $R_n^I(m)$, $C_n(m)$ and $R_n^k(m)$ and another from $R_n^h(m)$, $R_n(m)$ and $C_n(m)$; m a prime number and they are MOD multi matrix vector spaces over Z_m.

Then certainly one cannot define MOD multi linear transformation as the first space has elements from different MOD planes and another from other planes has its entries from different MOD planes.

If still one wants to define we can map the elements to zero if they are not the same MOD planes.

If there is no MOD plane common in between the two MOD multi matrix spaces V and W then the MOD multi matrix transformation is only a zero transformation. So under special conditions only we can define MOD multi matrix transformations.

Next we proceed onto explain MOD multi matrix operator from V to V where V is the MOD multi matrix operator by some examples.

Example 4.5: Let

$$V = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right| a_1, a_2 \in R_n(19), a_3, a_4, \right.$$

$$\left. a_5, a_6 \in R_n^I(19), a_7, a_8 \in R_n^g(19), a_9, a_{10}, a_{11}, a_{12} \in C_n(19), + \right\}$$

be the MOD multi matrix vector space over the field $F = Z_{19}$.

Define $\pi : V \to V$ the MOD linear operator in the following way.

$$\pi\left(\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix}\right) = \begin{bmatrix} a_1 & 0 & a_3 & 0 \\ a_5 & 0 & a_7 & 0 \\ a_9 & 0 & a_{11} & 0 \end{bmatrix}.$$

Clearly π is a linear operator and ker $\pi \neq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

There exists several such MOD multi linear operators.

Example 4.6: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \middle| \; a_1, a_2, a_3 \in R_n(29), a_4, a_7, a_{10}, \right.$$

$$a_{13}, a_{16} \in R_n^I(29), a_5, a_6, a_8, a_9 \in C_n(29), a_{11}, a_{12}, a_{14}, a_{15}, a_{17}, a_{18} \in R_n^g(29), +\}$$

be the multi dimension matrix vector space over the field Z_{29}. We have several MOD linear operators.

Further V has several MOD multi matrix vector subspaces of finite dimension over Z_{29}. V also has infinite dimensional MOD multi matrix vector subspaces over Z_{29}.

Next we proceed onto discuss about S-MOD multi matrix dimensional vector spaces over the S-ring Z_m. We will describe this by some examples.

Example 4.7: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_1, a_2 \in R_n(14), a_3, a_6, a_9 \in C_n(14), \right.$$

$$\left. a_4, a_7, a_8 \in R_n^I(14), a_5 \in R_n^g(14), i_F^2 = 13, I^2 = I, g^2 = 0, + \right\}$$

be the S-MOD multi matrix vector space over the S-ring Z_{14}.

Clearly V has MOD multi matrix vector subspaces over the S-ring Z_{14}. Further V has S-MOD multi matrix vector subspaces over the S-ring Z_{14}.

$$M_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2 \in R_n(14), a_3 \in C_n(14), + \right\} \subseteq V$$

is a S-MOD vector subspace of V of infinite dimension over Z_{14}.

$$P_1 = \left\{ \begin{bmatrix} a_1 & 0 & a_4 \\ a_2 & 0 & a_5 \\ a_3 & 0 & a_6 \end{bmatrix} \middle| \; a_1, a_2, a_3, a_4, a_5, a_6 \in Z_{14}, + \right\} \subseteq V$$

be the S-MOD multi matrix finite dimensional vector subspace over the S-ring Z_{14}.

$$R_1 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \middle| \; a_1, a_2 \in R_n(14), a_3 \in C_n(14), + \right\} \subseteq V$$

is a S-MOD multi matrix infinite dimensional vector subspace defined over Z_{14}.

We see V has both S-MOD finite and infinite dimensional vector subspace over the S-ring Z_{14}.

Example 4.8: Let

$$P = \left\{ \left[\begin{array}{cc} a_1 & a_6 \\ a_2 & a_7 \\ a_3 & a_8 \\ a_4 & a_9 \\ a_5 & a_{10} \end{array}\right] \middle| \; a_1, a_6 \in C_n(15), a_2, a_7 \in R_n(15), \right.$$

$$a_3, a_8 \in R_n^I(15), a_4, a_9 \in R_n^g(15), a_5, a_{10} \in R_n^k(15),$$
$$\left. k^2 = 14k, \; i_F^2 = 14, I^2 = I, g^2 = 0, + \right\}$$

be the S-MOD multi dimensional matrix vector space over the S-ring Z_{15}.

We see

$$B_1 = \left\{ \left[\begin{array}{cc} a_1 & a_6 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}\right] \middle| \; a_1, a_6 \in C_n(15) \right\} \subseteq P$$

is a S-MOD multi dimensional matrix vector subspace over the S-ring Z_{15} of infinite dimension.

$$B_2 = \left\{ \begin{bmatrix} 0 & 0 \\ a_2 & a_7 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \middle| \; a_2, a_7 \in Z_{15}, +\right\} \subseteq P$$

is a S-MOD multi dimensional matrix vector subspace of finite dimension over the S-ring Z_{15}.

$$B_3 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a_3 & a_8 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \middle| \; a_3, a_8 \in \langle Z_{15} \cup I\rangle, I^2 = I\right\} \subseteq P$$

be the S-MOD multi dimensional matrix vector subspace of finite dimension over Z_{15} the S-ring.

$$B_4 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a_3 & a_8 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \middle| \; a_3, a_8 \in [0, 15)I, I^2 = I\right\} \subseteq P$$

is again a S-MOD multi dimensional matrix vector subspace of infinite dimension over Z_{15} the S-ring.

$$\text{We see } B_4 \cap B_3 \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

We proceed to define given V the S-MOD multi dimension matrix vector space or a S-MOD multi dimension matrix vector space over a S-ring or a field respectively the notion of orthogonality.

At the outset we acknowledge that the notion of orthogonality is impossible in MOD vector space as product cannot be defined. So if we have a product to be defined on V we need V to be a MOD multi dimensional matrix linear algebra.

But we see $\{R_n(m), +, \times\}$ is only a pseudo ring so any MOD vector space comprises of $R_n^I(m)$, $C_n(m)$, $R_n^g(m)$ and so on which means it should be only a pseudo ring so only a pseudo linear algebra or S-pseudo linear algebra.

Now we proceed onto describe MOD multi dimensional matrix pseudo linear algebra by some examples.

Example 4.9: Let

$$V = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \middle| \; a_1, a_4 \in R_n(7), a_2, a_5 \in C_n(7) \text{ and } a_3, a_6 \in R_n^g(7), g^2 = 0, i_F^2 = 6, +, \times_n \right\}$$

be the MOD multi dimensional matrix pseudo linear algebra over the field Z_7.

V has MOD multi dimensional matrix pseudo sublinear algebras of both finite and infinite dimension over Z_7.

However all finite dimensional linear subalgebras are not pseudo. But all infinite dimensional linear subalgebras are pseudo.

$$V_1 = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \middle| \, a_i \in Z_7; 1 \leq i \leq 6 \right\} \subseteq V$$

is a finite dimensional linear subalgebra of V which is not pseudo.

However

$$V_2 = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \end{pmatrix} \middle| \, a_2, a_1 \in R_n(7), +, \times_n \right\}$$

is a MOD multi dimensional matrix pseudo linear algebra of infinite dimension over Z_7.

Clearly in V the distributive law is not true.

Example 4.10: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \end{bmatrix} \middle| \text{ where } a_1, a_2, a_3, a_4, \right.$$

$$a_5 \in C_n(17), a_6, a_7, a_8, a_9, a_{10} \in R_n(17), a_{11}, a_{12}, a_{13},$$
$$a_{14}, a_{15} \in R_n^I(17) \text{ and } a_{16}, a_{17}, a_{18}, a_{19}, a_{20} \in R_n^g(17),$$
$$\left. I^2 = I, g^2 = 0, i_F^2 = 16, +, \times_n \right\}$$

be a MOD multi dimensional matrix pseudo linear algebra over the field Z_{17}.

Clearly dimension of S over Z_{17} is infinite.

S has both finite and infinite dimensional sublinear algebras over Z_{17} which are not pseudo and pseudo respectively.

$$W_1 = \left\{\left.\begin{bmatrix} a_1 & 0 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 & 0 \\ a_3 & 0 & 0 & 0 & 0 \\ a_4 & 0 & 0 & 0 & 0 \end{bmatrix}\right| a_1 \in C_n(17),\ a_2 \in R_n(17),\right.$$

$$a_3 \in R_n^I(17) \text{ and } a_4 \in R_n^g(17),\ g^2 = 0,$$
$$I^2 = I,\ i_F^2 = 16, +, \times_n\} \subseteq S$$

is a MOD multi dimensional matrix pseudo linear subalgebra of infinite dimension over Z_{17}.

$$W_2 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{bmatrix}\right| a_i \in Z_{17};\right.$$

$$1 \leq i \leq 10, +, \times_n\} \subseteq S$$

is a finite dimensional linear subalgebra of S which is not pseudo over Z_{17}.

We can as in case of MOD multi dimensional matrix vector spaces define the notion of linear transformation of MOD multi dimensional matrix linear algebras provided they are defined over the same field Z_m.

We will describe this by some examples.

Example 4.11: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_1, a_2 \in R_n(29),\, a_3, a_4 \in C_n(29), \right.$$

$$\left. a_5 \in R_n^g(29);\; i_F^2 = 28,\, g^2 = 0, +, \times_n \right\}$$

be the MOD multi dimensional matrix pseudo linear algebra over the field Z_{29}.

$$\text{Let } W = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{pmatrix} \middle| \; a_1, a_6, a_2, \right.$$

$$a_7 \in R_n(29),\, a_3, a_8 \in C_n(29),\, a_4, a_9, a_5, a_{10} \in R_n^g(29),$$
$$\left. i_F^2 = 28,\, g^2 = 0, +, \times_n \right\}$$

be the MOD multi dimensional matrix pseudo linear algebra over the field Z_{29}.

$T : V \to W$ is defined as

$$T\left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \right\} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is a MOD linear multi dimensional transformation of the two MOD pseudo linear algebra.

It is in fact a difficult problem to find the structure of $\text{Hom}_{Z_{29}}(V, W)$.

Example 4.12: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \middle| \text{ where } a_1, a_2 \in R_n(46), a_3, a_4 \in C_n(46), \right.$$

$$a_5, a_6 \in R_n^I(46), a_7, a_8 \in R_n^g(46) \text{ and } a_9, a_{10} \in R_n^h(46); g^2 = 0,$$
$$\left. I^2 = I, h^2 = h \text{ and } i_F^2 = 45, +, \times_n \right\}$$

be the S-MOD multi mixed dimensional pseudo linear algebra over the S-ring Z_{46}.

$$S = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \middle| \; a_1, a_2 \in R_n(46), a_5, a_6 \in R_n^I(46), \right.$$

$$\left. a_{10}, a_9 \in R_n^h(46), a_8, a_7 \in R_n^g(46), a_3, a_4 \in C_n(46), +, \times_n \right\}$$

be S-MOD mixed multi dimensional pseudo linear algebra over the S-ring Z_{46}.

Define T : W → S by

$$T\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix}\right\} = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$$

is a S-MOD pseudo linear transformation of the S-MOD linear algebras.

$$\text{Clearly kernel } T \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ in fact kernel}$$

$$T = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \middle|\ a \in R_n^h(46)\right\}$$

is a S-MOD multi dimensional matrix pseudo linear subalgebra of W.

Thus we can have S-linear MOD transformations of S-pseudo linear algebras which have non zero kernel.

Example 4.13: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \right| a_1, a_2 \in R_n(10), a_3, a_4 \in C_n(10),$$

$$a_5, a_6, a_7, a_8, a_9, a_{10} \in R_n^I(10), a_{11}, a_{12}, a_{13}, a_{14}, a_{15} \in R_n^g(10);$$
$$g^2 = 0, I^2 = I, i_F^2 = 9, +, \times_n\}$$

be the S-MOD multi mixed dimensional matrix pseudo linear algebra over the S-ring Z_{10}.

M has S-linear subalgebras which are not pseudo or mixed dimensional.

For take

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \right| a_i \in Z_{10}, 1 \leq i \leq 15, +, \times_n\} \subseteq M$$

is a S-linear subalgebra of M which is not pseudo.

In fact M has several such S-linear subalgebras of finite dimension over the S-ring Z_{10} which are not pseudo or multi mixed dimensional.

Consider

$$R = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| \; a_1, a_2 \in Z_{10}, a_3, a_4 \in C(Z_{10}), a_5, a_6,\right.$$

$$a_7, a_8, a_9, a_{10} \in \langle Z_{10} \cup I\rangle, a_{11}, a_{12}, a_{13},$$

$$\left. a_{14}, a_{15} \in \langle Z_{10} \cup g\rangle, +, \times_n \right\} \subseteq M$$

is a S-MOD linear subalgebra of M which is not pseudo but is multi dimensional and mixed.

We can also say when are two elements in M orthogonal.

$$\text{Let } A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_2 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & b_1 & b_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_3 \end{bmatrix} \in M.$$

$$A \times_n B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so in the language of linear algebra we call this as orthogonal elements. (We can call A as the dual of B and vice versa.)

We can develop entirely the existing classical theorems in case of dual elements or orthogonal complements.

We see if

$$A = \left\{\begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_2 & 0 & 0 \end{bmatrix} \middle| \; a_1 \in R_n(10),\, a_2 \in R_n^g(10)\right\}$$

be the S-pseudo linear subalgebra of M then

$$A^{\perp} = \left\{\begin{bmatrix} 0 & a_1 & a_2 \\ a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} \\ 0 & a_{12} & a_{13} \end{bmatrix} \middle| \; a_1 \in R_n(10),\, a_2, a_3 \in C_n(10),\right.$$

$$i_F^2 = 9,\, a_4, a_5, a_6, a_7, a_8, a_9 \in R_n^I(10),\, I^2 = I,\, a_{10}, a_{11},$$

$$\left. a_{12}, a_{13} \in R_n^g(10),\, g^2 = 0, +, \times_n\right\} \subseteq M$$

is a S-pseudo linear subalgebra and $A \oplus A^{\perp} = M$ so $A^{\perp}$ is the orthogonal complement of A.

Suppose

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_1, a_2 \in R_n(10), a_3, a_4 \in C_n(10), \right.$$

$$a_5, a_6 \in R_n^I(10), a_{13}, a_{14}, a_{15} \in R_n^g(10), g^2 = 0, I^2 = I; \; i_F^2 = 9, +, \times_n \} \subseteq M$$

is a S-pseudo sublinear algebra of M.

$$S^\perp = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2, a_3, a_4 \in R_n^I(10), I^2 = I, \right.$$

$$a_5, a_6 \in R_n^g(10), g^2 = 0, +, \times_n \} \subseteq M$$

is a S-pseudo linear subalgebra of M over the S-ring Z_{10}.

We see $S \cap S^\perp = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $S \oplus S^\perp = M$ is the direct sum.

$$\text{Let } X = \begin{bmatrix} (3, 0.5) & 2 + 0.9i_F & 0 \\ 0 & 0.7 + 4.2I & 0 \\ 0 & 0 & 0 \\ 0 & 5 + 0.2g & 0.7 + 6.5g \\ 0.7 + 0.7g & 0 & 4 + 5.2g \end{bmatrix} \in M.$$

We find $X^{\perp}$ is also a S-pseudo linear subalgebra of M.

$$\text{Now } X^{\perp} = \left\{ \begin{bmatrix} 0 & 0 & a_1 \\ a_2 & 0 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & 0 & 0 \\ 0 & a_8 & 0 \end{bmatrix} \middle| \; a_2, a_1 \in C_n(10), a_3, a_4, a_5, \right.$$

$$a_6, a_7 \in R_n^I(10), a_8 \in R_n^g(10), i_F^2 = 9, I^2 = I,$$
$$\left. g^2 = 0, +, \times_n \right\} \subseteq M$$

is the S-sublinear algebra of M which is also pseudo.

Clearly as X is only one element its complement is a S-pseudo linear subalgebra of M.

However $X + X^{\perp} \neq M$.

In view of all these we have all the classical theorems of linear algebra is true in case of S-MOD multi dimensional matrix pseudo linear algebras in terms of orthogonality.

However we cannot have the notion of norm or inner product in the usual manner. But we can built or define these notions with appropriate changes.

Example 4.14: Let

$$S = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}\right| a_1 \in R_n(19), a_2 \in C_n(19), a_3 \in R_n^I(19),\right.$$

$$a_4 \in R_n^g(19), a_5 \in R_n^k(19) \text{ and } a_6 \in R_n^h(19);$$

$$\left. k^2 = 18k, h^2 = h, I^2 = I, g^2 = 0, i_F^2 = 18, +, \times_n\right\}$$

be the MOD multi dimensional matrix pseudo linear algebra defined over the field Z_{19}.

$$P_1 = \left\{\left.\begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right| a_1 \in R_n(19), +, \times_n\right\} \subseteq S$$

is a MOD multi dimensional matrix pseudo linear subalgebra of S.

We see

$$P_2 = \left\{\begin{bmatrix} 0 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_2 \in C_n(19), +, \times_n\right\} \subseteq S$$

is also MOD multi dimensional matrix pseudo linear subalgebra of S.

We see $P_1 \cap P_2 = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}$ and P_1 is orthogonal P_2 as MOD

pseudo linear subalgebras however $P_1 \oplus P_2 \neq S$, so P_2 cannot be the MOD orthogonal pseudo linear subalgebra which is complement of P_1 or vice versa.

In fact

$$P_3 = \left\{\begin{bmatrix} 0 \\ 0 \\ a_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_3 \in R_n^I(19), I^2 = I, +, \times_n\right\} \subseteq S$$

is also a MOD pseudo linear subalgebra of S.

We see P_1 and P_3 are perpendicular to each other however P_3 is not the orthogonal complement of P_1 as $P_1 + P_3 \neq S$ and

$$P_1 \cap P_3 = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

Consider

$$P_4 = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_4 \in R_n^g(19), g^2 = 0, +, \times_n\right\} \subseteq S$$

is also a pseudo sublinear algebra orthogonal with P_1 however it is not the orthogonal complement of P_1 as $P_1 + P_4 \neq S$ but we have

$$P_1 \cap P_4 = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

We can have several such orthogonal pseudo sublinear algebras but they are not orthogonal complements of P_1.

There is only one MOD pseudo linear algebra.

$$W = \left\{ \begin{bmatrix} 0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_1 \in C_n(19),\, a_2 \in R_n^I(19),\, a_3 \in R_n^g(19), \right.$$

$$\left. a_4 \in R_n^k(19) \text{ and } a_5 \in R_n^h(19) \text{ with } I^2 = I, \; g^2 = 0,\, k^2 = 18k,\, h^2 = h \text{ and } i_F^2 = 18, +, \times_n \right\} \subseteq S$$

is a MOD multi dimensional matrix pseudo linear algebra which is the orthogonal complement of P_1 and

$$P_1 \cap W = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ and } P_1 \oplus W = S.$$

So the orthogonal complement of any pseudo linear subalgebra is unique but one can have several pseudo linear subalgebras as well as subsets which can be orthogonal to P_1 but not the orthogonal complement of P_1.

Example 4.15: Let

$$B = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_1 \in C_n(15),\, a_2 \in R_n^I(15), \right.$$

$$\left. a_3 \in R_n^g(15) \text{ and } a_4 \in R_n^k(15),\; i_F^2 = 14,\, I^2 = I, \; g^2 = 0,\, k^2 = 14k, +, \times_n \right\}$$

be the S-MOD multi dimensional matrix pseudo linear algebra over the S-ring Z_{15}.

Clearly under the usual product the product is not defined though the collection is 2×2 square matrix collection.

$$\text{Let } L_1 = \left\{ \begin{bmatrix} 0 & 0 \\ a_1 & 0 \end{bmatrix} \middle| \; a_1 \in R_n^g(15), +, \times_n \right\}$$

be the S-MOD matrix pseudo linear subalgebra of B.

We have several S-MOD matrix pseudo linear subalgebras which are orthogonal to L_1 but there is only one S-MOD matrix pseudo linear subalgebra which is the orthogonal complement of L_1 given by

$$M_1 = \left\{ \begin{bmatrix} a_2 & a_1 \\ 0 & a_3 \end{bmatrix} \middle| \; a_2 \in C_n(15), a_1 \in R_n^I(15), \right.$$

$$\left. a_3 \in R_n^k(15), I^2 = I, k^2 = 14k, i_F^2 = 14, +, \times_n \right\} \subseteq B$$

is a S-MOD multi dimensional matrix pseudo linear subalgebra of B over Z_{15} which is such that

$$M_1 \cap L_1 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \text{ and } M_1 \oplus L_1 = B.$$

So M_1 is the unique complement S-MOD pseudo linear subalgebra of L_1.

$$\text{Now consider } P_1 = \left\{ \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix} \middle| \; a_1 \in Z_{15}, +, \times_n \right\} \subseteq B$$

is a S-linear subalgebra of B over Z_{15} which is such that it is orthogonal to L_1 and

$$L_1 \cap P_1 = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\} \text{ but } L_1 + P_1 \neq B$$

hence P_1 cannot serve as the orthogonal complement of L_1 further M_1 is of dimension one over the S-ring Z_{15}.

Thus it is important to note that we can have S-linear subalgebras which are orthogonal to L_1 can be of any finite dimension over Z_{15}.

Further even single elements like

$$M_2 = \begin{bmatrix} 4.3 + 6.75i_F & 0 \\ 0 & 0 \end{bmatrix} \in B \text{ can be orthogonal to } L_1.$$

$$M_2 = \begin{bmatrix} 3 + 0.7i_F & 0 \\ 0 & 7.25 + 10.35k \end{bmatrix} \in B$$

is such that L_1 is orthogonal to M_3.

It is very important to note that in general we may not be in a position to define inner product on these MOD multi dimensional matrix vector spaces or pseudo linear algebras; for we can have the inner product to be zero without the vector being zero.

This will be first illustrated by some examples.

Example 4.16: Let $V = \{R_n(17), +\}$ be a MOD vector space over the field Z_{17}. In fact V is a MOD pseudo linear algebra over the field Z_{17}.

We see the notion of inner product is not possible on V.

For if $\alpha = (1, 4) \in V$ then $(\alpha, \alpha) = ((1, 4), (1, 4))$

$$= 1 + 16$$

$$= 0 \pmod{17}.$$

Thus $\alpha \neq (0, 0)$ but $(\alpha, \alpha) = 0$.

So only we can define only pseudo inner product on V.

Further as the distributive laws are not true in case of MOD pseudo linear algebras even the famous classical Cauchy-Schwarz inequality in general is not true for $\alpha, \beta \in V$.

For if $\alpha = (1, 4)$ and $\beta = (4, 1)$ then

$$\alpha + \beta = (5, 5)$$

$$\text{and } \langle(\alpha + \beta), (\alpha + \beta)\rangle = 25 + 25$$

$$= 16 \qquad \ldots \text{ I}$$

$(||\alpha||^2 + ||\beta||^2) = 0$ so $16 < 0$.

Two things are to be recalled in this situation

i) No ordering is possible in $[0, n)$; $n < \infty$.

ii) Thus there sort of classical inequalities are impossible in these MOD intervals more so on MOD vector spaces and MOD pseudo linear algebras.

Keeping all this in mind we claim that as the very notion of inner product does not exist even if we say $\langle\alpha, \alpha\rangle \neq 0$ if $\alpha \neq 0$ still over coming this by pseudo is not proving to be effective.

Finally the Gram-Schmidt orthogonalization process becomes meaningless in case of MOD vector spaces and pseudo linear algebras.

So the other concepts based on inner product become meaningless.

Further as all these MOD spaces are infinite dimensional all concepts related to finite dimensional spaces become inappropriate. Thus the limitations of these spaces at present ends with orthogonality.

Even the notion of orthonormality is meaningless for here we define orthogonality of two MOD vectors by saying their product is a zero vector and nothing more.

Only this raw or crude definition of orthogonality is taken as the only definition in the case MOD vector spaces.

Still more problem is that as we are using only natural product $\times_n$ the notion of determinants even in case of square matrices is not possible.

More so in case of multi dimensional matrices the natural product alone is defined even in case of square matrices the usual product is not defined.

We give a simple illustration.

Let

$$A = \begin{bmatrix} 3 + 2i_F & 7g + 4 \\ 0.8I + 2 & 2k + 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 4 + 0.7i_F & 2 + 3.2g \\ 0.7I & 4 + 0.5k \end{bmatrix}$$

where

$$3 + 2i_F,\ 4 + 0.7i_F \in C_n(5);$$

$$i_F^2 = 4,\ 7g + 4,\ 2 + 3.2g \in R_n^g(5),\ g^2 = 0,$$

$$0.8I + 2,\ 0.7I \in R_n^I(5);$$

$I^2 = I$ and $2k + 1$ and $4 + 0.5k \in R_n^k(5)$; $k^2 = 4k$.

$$A \times_n B = \begin{bmatrix} 3 + 2i_F & 7g + 4 \\ 0.8I + 2 & 2k + 1 \end{bmatrix} \times_n \begin{bmatrix} 4 + 0.7i_F & 2 + 3.2g \\ 0.7I & 4 + 0.5k \end{bmatrix}$$

$$= \begin{bmatrix} 3.4 + 0i_F & 3 + 1.8g \\ 1.96I & 4 + 2.5k \end{bmatrix}.$$

But $A \times B$ is not defined for we cannot product $7g + 4$ with $0.7I$ and so on.

Thus the notion of determinant has no sense in case of MOD multi dimensional matrices more so on in MOD matrices as distributive laws are not true in $R_n(m)$.

So all results related to these concepts cannot be extended in case of these MOD spaces.

Further we are trying to find the feasibility of getting finite dimensional MOD multi dimensional vector spaces and MOD multi dimensional pseudo linear algebras or their Smarandache analogue in the following.

We will define and describe them.

DEFINITION 4.1: *Let $V = \{p \times q$ matrices with entries from $R_n(m)$, $R_n^I(m)$, $C_n(m)$, $R_n^g(m)$, $R_n^h(m)$, $R_n^k(m)$, $k^2 = (m - 1)k$, $g^2 = 0$, $h^2 = h$, $I^2 = I$, $i_F^2 = m - 1$ and m a prime, $+\}$.*

V is a Smarandache MOD interval MOD multi dimensional matrix vector space over the MOD interval S-ring $[0, m)$.

We will give examples of them in the following.

Example 4.17: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_1, a_2 \in R_n(7), a_3, a_4 \in C_n(7), \right.$$

$$a_5, a_6 \in R_n^I(7), a_7, a_8 \in R_n^g(7), a_9, a_{10} \in R_n^k(7),$$

$$\left. a_{11}, a_{12} \in R_n^h(7);\; i_F^2 = 6, I^2 = I, g^2 = 0, k^2 = 6k, h^2 = h, + \right\}$$

be the Smarandache MOD interval multi dimensional matrix vector space over the MOD interval S-ring R = [0, 7).

Clearly V is a finite dimensional space over the S-ring R = [0, 7).

Now in this case all the finite dimensional properties barring inner product can be derived with simple appropriate modifications.

Example 4.18: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \middle| \; a_1 \in R_n(11), a_2, a_3 \in C_n(11), a_4 \in R_n^I(11), \right.$$

$$a_5 \in R_n^g(11), a_6 \in R_n^h(11) \text{ and } a_7 \in R_n^k(11);\; i_F^2 = 10,$$

$$\left. I^2 = I, g^2 = 0, h^2 = h, k^2 = 10k, + \right\}$$

be the Smarandache MOD interval multi dimensional matrix vector space of finite dimension over the MOD interval S-ring R = [0, 11).

This has subvector spaces. The possible basis as follows:

$$B = \left\{ \begin{bmatrix} (0,1) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} (1,0) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1+0i_F \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0+i_F \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1+0i_F \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \right.$$

$$\begin{bmatrix} 0 \\ 0 \\ 0+i_F \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1+0I \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0+I \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1+0g \\ 0 \\ 0 \end{bmatrix},$$

$$\left. \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0+1g \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1+0h \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0+h \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1+0k \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0+k \end{bmatrix} \right\}.$$

This MOD interval multi dimension of M over R = (0, 11) is 14.

This is the way basis are calculated. One can find several sets of basis but all of them will have the same cardinality. However finding the basis is not a very easy job.

Example 4.19: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \end{bmatrix} \right| \text{ where } a_1, a_2, a_3, a_4 \in R_n(23),$$

$a_5, a_6, a_7, a_8 \in R_n^I(23)$, $a_9, a_{10}, a_{11}, a_{12} \in C_n(23)$, $a_{13}, a_{14}, a_{15}, a_{16} \in R_n^g(23)$ and $R_{17}, a_{18}, R_{19}, R_{20} \in R^k(23)$, $I^2 = I$, $g^2 = 0$; $k^2 = 22k$, $i_F^2 = 22$; $+\}$

be the Smarandache MOD interval MOD multi dimensional matrix vector space over the S-MOD interval ring [0, 23).

Find the basis of S. S has subspaces all of them are finite dimension over the S-MOD interval ring.

$$P_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right| a_1 \in R_n(23), +\} \subseteq S$$

is a S-MOD interval subspace of dimension two where the basis

$$B_1 = \left\{ \begin{bmatrix} (1,0) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} (0,1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}$$

of P_1 over the S-MOD interval ring [0, 23).

$$\text{Let } P_2 = \left\{ \begin{bmatrix} 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a+bk \end{bmatrix} \middle| \; a_1 \in R_n(23),\ a+bk \in R_n^k(23),\ k^2 = 22k \right\} \subseteq S$$

be the S-MOD interval multi dimensional vector space over the S-MOD interval ring [0,23).

The basis B_2 of P_2 is as follows:

$$B_2 = \left\{ \begin{bmatrix} 0 & (1,0) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & (0,1) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\left.\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+0k \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0+k \end{bmatrix}\right\}.$$

Thus dimension of P_2 is four.

$$\text{Let } P_3 = \left\{\begin{bmatrix} 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| \; a_1 \in R_n^I(23),\ a_2 \in C_n(23),\right.$$

$$\left. a_3 \in R_n^g(23)\right\} \subseteq S$$

be the S-MOD interval MOD multi dimensional matrix vector subspace over the S-MOD interval ring [0, 23).

The basis B_3 of P_3 is as follows:

$$B_3 = \left\{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1+0I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0+I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\right.$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1+0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0+i_F & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left.\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+0g \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0+g \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\}.$$

Clearly dimension of B_3 is 6 over the S-MOD interval ring.

Can we say the dimension of S is 40 over S-MOD interval ring?

Example 4.20: Let

$$D = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \middle| \; a_1, a_2, a_3 \in R_n(19), a_4, a_5,\right.$$

$$a_6 \in C_n(19), a_7, a_8, a_9 \in R_n^g(19), a_{10}, a_{11}, a_{12} \in R_n^I(19), a_{13}, a_{14},$$

$$a_{15} \in R_n^h(19), a_{16}, a_{17}, a_{18} \in R_n^k(19);\ i_F^2 = 18, g^2 = 0,$$

$$\left. I^2 = I, h^2 = h, k^2 = 18k, + \right\}$$

be the S-MOD interval MOD multi dimension matrix vector space over the S-MOD interval ring [0, 19).

We have several subspaces of dimension 1, dimension 2 and so on. The maximum dimension of D is 36.

Let

$$P_1 = \left\{ \begin{bmatrix} (a, 0) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, a \in R_n(19), + \right\} \subseteq D$$

is a S-MOD interval multi mixed matrix vector subspace of dimension one over the S-MOD interval ring [0,19).

$$P_2 = \left\{ \begin{bmatrix} (0, a) & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, a \in [0, 19), b \in R_n(19) \right\} \subseteq D$$

is a S-MOD interval multi dimensional matrix vector subspace of D over the S-MOD interval ring [0, 19).

Dimension of P_2 over [0, 19) is three.

Next we proceed onto describe some properties of these finite dimensional S-MOD interval multi dimensional matrix vector space over the S-MOD interval ring R = [0, m); m a prime.

Example 4.21: Let S = $\{(a_1, a_2, a_3) \mid a_1 \in R_n(3), a_2 \in C_n(3), a_3 \in R_n^I(3), +\}$ be the S-MOD interval multi dimensional matrix vector space over the S-interval ring [0, 3).

Dimension of S over [0, 3) is two.

So S is finite dimensional vector space over the S-MOD interval ring.

Now we can describe the S-MOD interval linear transformation in case of these S-MOD interval multi dimensional matrix vector spaces over the S-MOD interval ring [0, m) by the following examples.

Example 4.22: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \; a_1, a_2 \in R_n(7) \text{ and } a_3, a_4 \in R_n^I(7), + \right\}$$

and

$$W = \{(a_1, a_2, a_3, a_4, a_5, a_6) \mid a_1, a_2, a_3 \in R_n(7) \text{ and } a_4, a_5, a_6 \in R_n^I(7), +\}$$

be two S-MOD interval multi dimensional matrix vector space over the S-MOD interval ring [0, 7).

Let $T : V \to W$ defined by

$$T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \right) = (a_1, 0, a_2, 0, a_3, 0).$$

$$\ker T = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \end{bmatrix} \middle| \; a_4 \in R_n^I(7) \right\}$$

is a subvector space of S-MOD dimension two. Clearly the S MOD interval dimension of V is 8 and that of W is 12.

Clearly rank T is 6. Hence rank T + nullity T = dim V. Thus 6 + 2 = 8. Hence the result.

Interested reader can derive all results about these finite dimensional S-MOD interval multi dimensional matrix vector spaces; as it is considered as a matter of routine.

All results can be obtained with simple and appropriate modifications.

Next we proceed onto define the notion of S-MOD interval multi dimensional matrix pseudo linear algebras over the S-MOD interval pseudo ring [0, m).

At this juncture we are forced to observe the following. In the MOD interval [0, m); m need not be always a prime for [0, m) to be a S-MOD interval ring. [0, m) the MOD interval is a S-ring provided Z_m is a S-ring and m need not be prime for that situation.

Example 4.23: Let

$$M = \left\{ \left[\begin{matrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{matrix}\right] \middle| \; a_1, a_2 \in R_n^I(15), a_3, a_4 \in R_n(15), \right.$$

$$\left. a_5, a_6, a_7 \in C_n(15), a_8, a_9 \in R_n^g(15), + \right\}$$

be the S-MOD interval multi dimensional matrix vector space of dimension 18 one the S-MOD interval ring [0, 15).

This does not make any changes when Z_m is a S-ring or Z_m is a field.

However we will give examples of S-MOD interval multi dimensional pseudo linear algebras.

Example 4.24: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle| \; a_1, a_2 \in R_n(13), a_3, a_4 \in C_n(13), \right.$$

$$a_5, a_6 \in R_n^I(13) \text{ and } a_7, a_8 \in R_n^g(13), \left. g^2 = 0, I^2 = I, i_F^2 = 12, +, \times_n \right\}$$

be the S-MOD interval multi dimensional matrix pseudo linear algebra over the S-MOD interval ring S = [0, 13).

Clearly V is finite dimensional over S = [0, 13). All linear subalgebras of V over S = [0, 13) are pseudo and are only finite dimensional. In fact this gives an infinite collection of S-MOD interval pseudo linear algebras of finite dimension.

However still we cannot define inner product on them. That direction is left open for any serious researcher.

Further study of $\text{Hom}_{[0, m)}(V, W)$ and $\text{Hom}_{[0, m)}(V, V)$ are S-MOD vector spaces of finite dimension V and W are left as an exercise for the reader.

The only things important to mention at this juncture is that by using these S-MOD interval multi dimensional matrix vector spaces alone we are in a position to define linear functional not treating them as inner product spaces.

We realize [0, m) as a S-MOD interval vector space of dimension one over the S-MOD interval ring [0, m). So here linear functionals, S-MOD are linear transformations from

$V \to [0, m)$ where V is a S-MOD interval matrix linear pseudo algebra or vector space over the S-MOD interval ring $[0, m)$.

So if V is a S-MOD interval vector space of dimension t then the map $f : V \to [0, m)$ makes kernel f to be a $(t - 1)$ dimensional S-MOD interval vector subspace of V over $[0, m)$. Thus these subspaces are defined as S-MOD interval hyperspaces.

We can as in case of MOD multi dimensional matrix vector spaces define the notion of orthogonality. Two matrices A, B as V are orthogonal if $A \times_n B = (0)$ is the zero matrix.

All related results of orthogonality can be derived with appropriate modifications in case of these special spaces.

Once again finding eigen values of eigen vector in case of these spaces happens to be an impossibility as the operation + and $\times$ does not satisfy the distributive laws in the MOD interval $[0, m)$ or in all the six MOD planes $R_n(m)$, $C_n(m)$, $i_F^2 = m - 1$, $R_n^I(m)$, $I^2 = I$, $R_n^g(m)$, $g^2 = 0$, $R_n^k(m)$; $k^2 = (m - 1)k$ and $R_n^h(m)$, $h^2 = h$.

Hence we are not in a position to overcome this problem. So all properties classical or otherwise related with inner product cannot be studied in case of MOD vector spaces and MOD pseudo linear algebras of finite or infinite dimension as we cannot give any form of proper ordering in them.

So the classical spectral theorem cannot be even imagined in case of these pseudo linear algebras.

However it is left as an exercise to the reader to find the algebraic structure enjoyed by $\text{Hom}_{[0, m)}(V, [0, m))$.

Will $\text{Hom}_{[0, m)}(V, [0, m))$ have same dimension as a pseudo linear algebra V over the S-MOD interval ring $[0, m)$?

That is will $\text{Hom}_{[0, m)}(V, [0, m)) \cong V$?

Such problems are left open for the reader.

Next a brief information regarding the impossibility of using polynomials as pseudo linear algebras as distributive law is not true.

Further we can have MOD vector spaces as polynomials. Here we illustrate them by some examples.

Example 4.25: Let $\{R_n^{11}[x], +\} = V$ be the MOD polynomial real vector space over the field Z_{11}.

Example 4.26: Let $W = \{R_n^{I}(23)[x], +\}$ be the MOD polynomial neutrosophic vector space over the field Z_{23}.

Example 4.27: Let $M = \{R_n^{g}(15)[x], +\}$ be the S-polynomial MOD vector space over the S-pseudo ring Z_{15}.

Finding basis and working with them is a matter of routine. All the three spaces given above are of infinite dimension.

We can have also finite dimensional MOD polynomial vector spaces using the MOD interval pseudo S-ring [0, m).

This is illustrated by some examples.

Example 4.28: Let $M = \{R_n^{h}(17)[x], h^2 = h, +\}$ be the S-MOD interval polynomial MOD vector space over the S-MOD interval pseudo ring $S = [0, 17)$.

Clearly M is of infinite dimension over $S = [0, 17)$.

Example 4.29: Let $P = \{R_n^{k}(23)[x], k^2 = k, +\}$ be the S-MOD interval polynomial MOD vector space over the S-MOD interval pseudo ring [0, 23).

This has subspaces. All subspaces is only of infinite dimension over [0, 23).

Example 4.30: Let W = $\{C_n(7)[x],\ i_F^2 = 6, +\}$ be the S-MOD interval polynomial S-vector space over the S-MOD interval pseudo ring [0, 7).

Example 4.31: Let M = $\{R_n^h(14)[x],\ h^2 = h, +\}$ be the S-MOD interval polynomial S-vector space over the S-MOD interval pseudo ring [0, 14).

Finding basis is an interesting feature.
All spaces are of infinite dimension over [0, 14).

Next we proceed to get the finite dimensional S-MOD interval polynomial vector spaces over the S-MOD interval pseudo ring.

Example 4.32: Let V = $\{R_n(13)[x]_9$ | All polynomial of degree less than or equal to 9 alone is taken with coefficients from (a, b) $\in R_n(13)$, +} be the S-MOD interval polynomial S-vector space over the S-MOD interval pseudo ring [0, 13).

The basis for V is {(0, 1), (1, 0), (1, 0)x, (1, 0)x^2, (1, 0)x^3, (1, 0)x^4, …, (1, 0)x^9, (0, 1)x, (0, 1)x^2, …, (0, 1)x^9}. That is MOD dimension of V over the S-MOD interval pseudo ring [0, 13) is 20.

Likewise we can get finite dimensional S-MOD interval polynomial S-vector spaces over the S-MOD interval pseudo ring S = [0, m).

Example 4.33: Let M = $\{C_n(10)[x]_6$ = {Collection of all polynomials of degree less than or equal to six with coefficients from $C_n(10)$, $i_F^2 = 9\}$ be the S-MOD interval polynomial S-vector space over the S-MOD interval ring [0, 10).

The basis of M is B = $\{1 + 0i_F, 0 + 1i_F, x, i_Fx, x^2, i_Fx^2, x^3, i_Fx^3, x^4, i_Fx^4, x^5, i_Fx^5, x^6, i_Fx^6\}$. Clearly cardinality of B is 14 and dimension of M over S is 14.

Interested reader can find more sets of basis and prove all basis is of same cardinality.

Clearly though product is defined for polynomials yet the distributive law is not true we have problems working with them as $(x + 2) \times (x + 3) \times (x + 4) \neq x^3 + 9x^2 + 26x + 24$ we face lots of problems.

Thus we cannot define the notion of MOD pseudo linear algebras using MOD polynomials over S-MOD interval ring [0, m).

Example 4.34: Let W = { $R_n^I(28)[x]_5$ all polynomials of degree less than or equal to 5 with coefficients from $R_n^I(27)$, +, $I^2 = I$} be the S-MOD interval polynomial MOD S-vector space over the S-MOD interval pseudo ring S = [0, 28).

This space is finite dimensional over [0, 28).

A basis for W over S is
B = $\{1, I, x, Ix, x^2, Ix^2, x^3, Ix^3, x^4, Ix^4, x^5, Ix^5\}$. Thus W is a S-vector space of dimension 12 over S = [0, 28).

Any polynomial p(x) ∈ W is of the form $p(x) = (3.7 + 2.5I)x^5 + (0.72 + 4.9I)x^4 + (27 + 14I)x^3 + 6.732Ix^2 + 10.7354x + (20.331 + 1.7234I)$.

So in general product operation is not well defined as the distributive laws is not true.

It is matter of routine to find subspaces, basis direct sum, linear transformation, linear operator on any of the S-MOD interval polynomial MOD S-vector spaces over the S-MOD interval ring S = [0, m); m can be prime or Z_m should be a

S-ring which is the main criteria for [0, m) to be a S-pseudo ring.

We suggest the following problems for the reader.

Problems

1. Let V = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix} \right|$ $a_1, a_2, a_6, a_{11} \in$

   $C_n(3)$, $a_7, a_{12}, a_3 \in R_n(3)$, $a_8, a_{13}, a_4, a_9 \in R_n^I(3)$, $a_{14}, a_5, a_{10}, a_{15} \in R_n^g(3)$, +} be the MOD multi matrix vector space over the field Z_3.

   i) Find the number of vector subspaces of finite dimension over Z_3.
   ii) How many MOD vector subspaces of infinite dimension over Z_3?
   iii) Write V as the direct sum of subspaces.
   iv) Find at least 3 distinct basis of V over Z_3.
   v) Can V have more than one basis?

2. Obtain all the special features enjoyed by the MOD multi matrix dimensional vector space

   V = $\left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{24} \\ a_{25} & a_{26} & \ldots & a_{36} \end{bmatrix} \right|$ $a_1, a_2, a_3, \ldots, a_{12} \in R_n(m)$,

   $a_{13}, a_{14}, \ldots, a_{24} \in C_n(m)$ and $a_{25}, a_{26}, \ldots, a_{36} \in R_n^I(m)$; m a prime, +} over the field Z_m.

3. Let V and W be any two MOD multi matrix vector spaces over the field Z_m (m a prime).

   i) Find $\text{Hom}_{Z_m}(V, W) = S$.
   ii) What is the algebraic structure enjoyed by S?
   iii) Is S a MOD multi matrix vector space of infinite dimension over Z_m?
   iv) Find $R = \text{Hom}_{Z_m}(W, V)$.
   v) What is the algebraic structure enjoyed by R?
   vi) Will $S \cong R$?

4. Let V = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix} \right|$ $a_1, a_2, a_3 \in$

   $R_n(41)$, $a_4, a_5, a_6 \in R_n^I(41)$, $a_7, a_8, a_9 \in R_n^g(41)$, $a_{10}, a_{11}, a_{12} \in R_n^h(41)$, $a_{13}, a_{14}, a_{15} \in C_n(41)$; $i_F^2 = 40$, $h^2 = h$, $g^2 = 0$, $I^2 = I$, +} be the MOD multi dimensional matrix vector space

   over the field Z_{41} and B = $\left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \right|$ $a_1, a_2 \in R_n(41)$, a_3,

   $a_4 \in R_n^I(41)$, $a_5, a_6 \in R_n^g(41)$, $a_7, a_8 \in R_n^h(41)$, $a_9, a_{10} \in C_n(41)$, $I^2 = I$, $g^2 = 0$, $h^2 = h$, $i_F^2 = 40$, +} be the MOD multi dimensional matrix vector space over the field Z_{41}.

   Study questions (i) to (vi) of problem (3) for this V and W.

5. Let V = $\{(a_1, a_2, a_3, a_4, a_5, a_6) \mid a_1, a_2 \in R_n(26), a_3 \in R_n^I(26), a_4, a_5 \in R_n^g(26), a_6 \in C_n(26), +\}$ be the S-MOD multi dimensional vector space over the S-ring Z_{26}.

   Study questions (i) to (v) of problem (1) for this V.

6. Let S = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| \; a_1, a_2, a_3, a_4, a_5, a_6 \in R_n(17),\right.$

   $a_8, a_7, a_9 \in C_n(17)$, $a_{10}, a_{11}, a_{12} \in R_n^g(17)$, $a_{13}, a_{14}, a_{15} \in R_n^k(17)$, $i_F^2 = 16$, $g^2 = 0$, $k^2 = 16k$, $+, \times_n\}$ be the MOD pseudo multi dimension pseudo linear algebra over the field Z_{17}.

   i) Study questions (i) to (v) of problem (1) for this S.
   ii) Does S enjoy any other special feature as a pseudo linear algebra?
   iii) Can S have subspaces which are not linear subalgebras?
   iv) Obtain a basis of S over Z_{17}.

7. Let V and W be any two multi dimensional matrix pseudo linear algebras over the field Z_p (p a prime).

   i) Study the structure of $S = \text{Hom}_{Z_p}(V, W)$.
   ii) Find $\text{Hom}_{Z_p}(W, V) = R$.
   iii) Does there exist any relation between S and R?
   iv) Find $\text{Hom}_{Z_p}(V, V) = P$.

v) Find $\text{Hom}_{Z_p}(W, W) = Q$.

vi) Does there exist any relation between P and V?

vii) Does there exist any relation between Q and W?

viii) Compare Q, W, V, P, S and R as pseudo linear algebras.

8. Let $V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \right.$ $a_1, a_2, a_3 \in R_n(31)$, a_4, $a_5, a_6 \in C_n(31)$, $a_7, a_8, a_9 \in R_n^I(31)$, $a_{10}, a_{11}, a_{12} \in R_n^g(31)$, $+, \times_n\}$ and

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \end{bmatrix} \middle| \right. a_5, a_1, a_2, a_3, a_4$$

$\in R_n(31)$, $a_6, a_7, a_8, a_9, a_{10} \in R_n^I(31)$, $a_{11}, a_{12}, a_{13}, a_{14}, a_{15} \in C_n(31)$, $a_{16}, a_{17}, a_{18}, a_{19}, a_{20} \in R_n^g(31)$, $+, \times_n\}$ be the multi dimensional matrix pseudo linear algebras over Z_{31}.

Study questions (i) to (viii) for this V and W.

9. Let V = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \right.$ $a_2, a_1 \in R_n(10)$, $a_3, a_4,$

$a_5, a_6 \in R_n^I(10)$, $a_7, a_8, a_9, a_{10} \in R_n^g(10)$ and $a_{11}, a_{12} \in R_n^k(10)$, $I^2 = I$, $g^2 = 0$, $k^2 = 9k$, $+, \times_n\}$ and

W = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \middle| \right.$ $a_1, a_2 \in R_n(10)$, $a_3, a_4, a_5 \in R_n^I(10)$, a_6

$a_7, a_8 \in R_n^g(10)$ and $a_9, a_{10} \in R_n^k(10)$, $g^2 = 0$, $I^2 = I$, $k^2 = 9k$, $+, \times_n\}$ be two S-MOD multi dimensional matrix pseudo linear algebras over the S-ring Z_{10}.

Study questions (i) to (viii) of problem (7) for this V and W.

10. Let W = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \middle| \right.$ a_1 to $a_6 \in$

$R_n(39)$, a_7 to $a_{12} \in C_n(39)$, a_{13} to $a_{18} \in R_n^I(39)$, a_{19} to $a_{24} \in R_n^g(39)$; $g^2 = 0$, $I^2 = I$, $i_F^2 = 38$, $+, \times_n\}$ and

$$V = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} \\ a_{25} & a_{26} & a_{27} \\ a_{28} & a_{29} & a_{30} \end{array}\right] \right| \; a_1 \text{ to } a_9 \in R_n(39), a_{10} \text{ to } a_{15} \in$$

$C_n(39)$, a_{16} to $a_{24} \in R_n^I(39)$ and a_{25} to $a_{30} \in R_n^g(39)$; $g^2 = 0$, $I^2 = I$, $i_F^2 = 39, +, \times_n\}$ be two MOD multi mixed dimensional matrix pseudo S-linear algebras over the S-ring Z_{39}.

Study questions (i) to (viii) of problem (7) for this V and W.

11. Let $V = \left\{ \left[\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \end{array}\right] \right| \; a_1, a_2, a_3,$

$a_4 \in C_n(14)$, a_5 to $a_{12} \in R_n^I(14)$, a_{13} to $a_{16} \in R_n^g(14)$ and a_{17},

$a_{18} \in R_n^k(14)$; $g^2 = 0$, $I^2 = I$, $k^2 = 13k$ and $i_F^2 = 13, +, \times_n\}$

be the S-MOD interval multi dimensional matrix pseudo linear algebra over the S-MOD interval pseudo ring [0, 14).

$$W = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \right| a_1 \text{ to } a_4 \in C_n(14),\ a_5 \text{ to}$$

$a_8 \in R_n^I(14)$, a_9 to $a_{12} \in R_n^g(14)$ and a_{13} to $a_{16} \in R_n^k(14)$; $g^2 = 0$, $I^2 = I$, $i_F^2 = 13$ and $k^2 = 13k$, $+, \times_n\}$ be the S-MOD interval multi dimensional matrix S-pseudo linear algebra over the S-pseudo interval MOD ring [0, 14).

i) Study questions (i) to (viii) of problem (7) for this V and W.
ii) If $\times_n$ is replaced by $\times$ the usual matrix product in W prove the product is not defined.

iii) Obtain any other special features enjoyed by V and W.

12. Can there be a square multi dimensional matrix collection M so that under usual matrix product $\times$, $\{M, \times\}$ is a semigroup?

    That is for A, B $\in$ M; $A \times B \in M$. Justify your claim.

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ABOUT THE AUTHORS

Dr.W.B.Vasantha Kandasamy is a Professor in the Department of Mathematics, Indian Institute of Technology Madras, Chennai. In the past decade she has guided 13 Ph.D. scholars in the different fields of non-associative algebras, algebraic coding theory, transportation theory, fuzzy groups, and applications of fuzzy theory of the problems faced in chemical industries and cement industries. She has to her credit 694 research papers. She has guided over 100 M.Sc. and M.Tech. projects. She has worked in collaboration projects with the Indian Space Research Organization and with the Tamil Nadu State AIDS Control Society. She is presently working on a research project funded by the Board of Research in Nuclear Sciences, Government of India. This is her 110th book.

On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.

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Dr. Florentin Smarandache is a Professor of Mathematics at the University of New Mexico in USA. He published over 75 books and 200 articles and notes in mathematics, physics, philosophy, psychology, rebus, literature. In mathematics his research is in number theory, non-Euclidean geometry, synthetic geometry, algebraic structures, statistics, neutrosophic logic and set (generalizations of fuzzy logic and set respectively), neutrosophic probability (generalization of classical and imprecise probability). Also, small contributions to nuclear and particle physics, information fusion, neutrosophy (a generalization of dialectics), law of sensations and stimuli, etc. He got the 2010 Telesio-Galilei Academy of Science Gold Medal, Adjunct Professor (equivalent to Doctor Honoris Causa) of Beijing Jiaotong University in 2011, and 2011 Romanian Academy Award for Technical Science (the highest in the country). Dr. W. B. Vasantha Kandasamy and Dr. Florentin Smarandache got the 2012 New Mexico-Arizona and 2011 New Mexico Book Award for Algebraic Structures. He can be contacted at smarand@unm.edu

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