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Singular Value Decomposition

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Abstract

A well-known theorem is Diagonalization, where one of the factors is a diagonal matrix. In this paper we will be describing a similar way to factor/decompose a non-square matrix. The key to both of these ways to factor is eigenvalues and eigenvectors.

1 Introduction

The theory of singular value decomposition was established by mathematicians named Eugenio Beltrami, Camille Jordan, James Joseph Sylvester, Erhard Schmidt, and Hermann Weyl. Beltrami, Jordan and Sylvester came to the singular value decomposition through linear algebra in the late 1800s. Then in the early 1900s Schmidt and Weyl came to singular value decomposition by integral equations. Singular value decomposition is a generalization of the spectral decomposition, also known as factor analysis.

2 Eigenvectors and Eigenvalues

Singular Value Decomposition is related to eigenvectors and eigenvalues.

Definition: An eigenvector of an n x n matrix A is a nonzero vector $\vec{x} \in R^n$

such that $A\vec{x} = \lambda\vec{x}$ for some scalar $\lambda \in R$.

Definition: A scalar λ is called an eigenvalue of A if there is a nontrivial solution, $\vec{x}$ of $A\vec{x} = \lambda\vec{x}$.

Example:

Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Are $\vec{u}$ and $\vec{v}$ eigenvectors of A?

$$A\vec{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\vec{u}$$

$$A\vec{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Therefore, $\vec{u}$ is an eigenvector with the eigenvalue of -4, but $\vec{v}$ is not a eigenvector of A because $A\vec{v}$ is not a multiple of $\vec{v}$.

3 Diagonalization

The Diagonalization Theorem says that an $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

Example: Find a diagonal matrix D and a non-singular matrix P such that $A = PDP^{-1}$, if possible.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Step 1: Find the eigenvalues

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ -1 & \lambda - 2 & -1 \\ 1 & 0 & \lambda - 1 \end{vmatrix} = (\lambda - 2)^2(\lambda - 1) = 0$$

Eigenvalues: $\lambda = 1$ and $\lambda = 2$.

Step 2: Find linearly independent eigenvectors.

$$\lambda = 1 : \vec{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda = 2 : \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Step 3: Construct P and D using the eigenvectors and their corresponding eigenvalues.

$$P = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

4 The Spectral Theorem

The Spectral Theorem for Symmetric Matrices states that an n x n symmetric matrix A has the following properties:

1. A has n real eigenvalues, counting multiplicity.

2. The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.

3. The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.

4. A is orthogonally diagonalizable.

Example: Let

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$$

The characteristic polynomial of A is determined by $\det(\lambda I A) = 0)$ giving us,

$$(\lambda + 2)^2(\lambda - 4)$$

so the eigenvalues are

$$\lambda_1 = -2, \lambda_2 = -2, \text{ and } \lambda_3 = 4.$$

Next, to find the eigenvectors corresponding to λ_1 and λ_2, we solve the linear system $(-2I - A)x = 0$

$$\begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The eigenvectors will be:

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{and}\, \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

We can see that $\vec{v}_1$ and $\vec{v}_2$ are not orthogonal, since $\vec{v}_1 \cdot \vec{v}_2 \neq 0$. We can use the Gram-Schmidt process. We will let

$$\vec{y}_1 = \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\vec{y}_2 = \vec{v}_2 = \vec{v}_2 - \frac{\vec{x}_2 \cdot \vec{y}_1}{\vec{y}_1 \cdot \vec{y}_1} \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let

$$\vec{y*}_2 = 2\vec{y}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Now the set $\vec{y}_1$ and $\vec{y*}_2$ is an orthogonal set of eigenvectors. After normalizing there vectors, we obtain

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

Now we will find the eigenvector that corresponds to the eigenvalue 4, using the equation $(4I - A)x = 0$

$$\begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The eigenvector will be

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

After normalizing this vector we will get,

$$\vec{v}_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Now we will create P, which is made up of the columns being the eigenvectors we found.

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Now we can also find D, by using the eigenvalues that correspond to the eigenvectors we have in P.

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

We will also have to find P^-1 which is the same as P^T.

$$P^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Putting them all together in the equation $A = PDP^T$. We will get

$$A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

5 The Singular Value Decomposition

The decomposition of an $m \times n$ matrix A includes an $m \times n$ "diagonal" matrix Σ of the form

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

where D is an $r \times r$ diagonal matrix.

Definition: The number of linearly independent rows or columns is called the rank.

Definition: A diagonal matrix is a matrix where the entries $a_{ij} = 0$ if $i \neq j$.

Definition: An orthogonal matrix means $Q^-1 = Q^T$.

Theorem: Let A be an $m \times n$ matrix with rank r. Then there exists an $m \times n$ matrix Σ where the diagonal entries in D are the first r singular values of A, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, where σ_i is the square root of an eigenvalue of $A^T A$, and there exists an $m \times m$ orthogonal matrix U and an $n \times n$ orthogonal matrix V such that

$$A = U\Sigma V^T.$$

Proof: To determine V and Σ we consider the matrix $A^T A$.

(1) $A^T A$ is a positive semi-definite symmetric $n \times n$ matrix. Therefore, $A^T A = Q\Lambda Q^T$, where Q is an orthogonal matrix of eigenvectors and Λ is a diagonal matrix of non-negative eigenvalues of $A^T A$.

(2) If $A = U\Sigma V^T$ then

$$A^T A = V\Sigma^T U^T U \Sigma V^T = V(\Sigma^T \Sigma)V^T$$

Therefore we take $V = Q$ and Σ to have diagonal component D consisting of the square roots of the diagonal entries of Λ.

To determine U note that $AA^T = P\Lambda P^T$, with the same Λ since if $A^T A\vec{v} = \lambda \vec{v}$, then $AA^T A\vec{v} = \lambda A\vec{v}$.

$$AA^T = U\Sigma V^T V \Sigma^T U^T = U(\Sigma\Sigma^T)U^T$$

Therefore, we take $U = P$, the eigenvectors of AA^T.

More precisely, we can take the columns of U to be $\vec{u}_i = \dfrac{A\vec{v}_i}{\sigma_i}$.

These are orthogonal since

$$\left(\frac{A\vec{v}_i}{\sigma_i}\right)^T \left(\frac{A\vec{v}_j}{\sigma_j}\right) = \frac{\vec{v}_i^T A^T A\vec{v}_j}{\sigma_i \sigma_j} = \frac{\vec{v}_i^T \sigma_j^2 \vec{v}_j}{\sigma_i \sigma_j} = \frac{\sigma_j}{\sigma_i}\vec{v}_i^T \vec{v}_j = 0$$

since v_i and v_j are orthogonal.

Thus when A is a non-square matrix, $A = U\Sigma V^T$, where U consists of eigenvectors from AA^T, V consists of eigenvectors from $A^T A$ and Σ is an non-square matrix with a diagonal component. Moreover, we have the eigen-esque equation:

$$A\vec{v}_i = \sigma_i \vec{u}_i, \ i = 1, ..., r$$

Proposition: The symmetric matrices $A^T A$ and AA^T have the same eigenvalues.

Suppose

$$A^T A\vec{v} = \lambda\vec{v}$$

Applying A to both sides gives,

$$AA^T A\vec{v} = A\lambda\vec{v}$$

which is equivalent to

$$AA^T(A\vec{v}) = \lambda(A\vec{v})$$

A similar argument shows that if λ is an eigenvalue of AA^T with eigenvector $\vec{u}$, then λ is an eigenvalue of $A^T A$ with eigenvector $A^T\vec{u}$. Note this proof also shows a relationship between the eigenvectors of $A^T A$ and AA^T. In fact, if $\vec{v}$ is an eigenvector of $A^T A$ then $A\vec{v}$ is an eigenvector of AA^T.

Definition: A symmetric matrix A is called positive semidefinite if all of its eigenvalues are non-negative.

Proposition: The matrix $A^T A$ is a positive semidefinite matrix.

Suppose

$$A^T A\vec{v} = \lambda\vec{v}$$

Take the magnitude of $A\vec{v}$ and square it. This means the vector will be dotted with itself.

$$||A\vec{v}||^2 = A\vec{v} \cdot A\vec{v}$$

When taking a dot product it's the transpose of the vector times the vector.

$$= (A\vec{v})^T(A\vec{v})$$

The transpose of a product is the product of the transposes in the reverse order, gives

$$= (\vec{v}^T A^T)(A\vec{v})$$

Next, using associativity to regroup and using the fact λ is a eigenvalue of $A^T A$ with eigenvector $\vec{v}$, i.e. $A^T A\vec{v} = \lambda\vec{v}$, gives

$$(\vec{v}^T A^T)(A\vec{v}) = \vec{v}^T(A^T A\vec{v}) = \vec{v}^T(\lambda\vec{v})$$

Since scalars commute with matrices, we can rewrite the equation so that λ is in the front:

$$= \lambda\vec{v}^T\vec{v}$$

The dot product of a vector is the transpose times itself, which is what we have above, so this equals

$$= \lambda\vec{v} \cdot \vec{v}$$

This is the magnitude of the vector:

$$= \lambda||\vec{v}||^2.$$

Thus we have

$$||A\vec{v}||^2 = \lambda||\vec{v}||^2.$$

Since the squares are non-negative, we can conclude that

$$\lambda \geq 0,$$

and hence $A^T A$ is positive semidefinite.

Example: Construct a singular value decomposition of $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$

Step 1: Compute $A^T A$

$$A^TA = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

Step 2: Use the equation $\det(\lambda I - A^TA) = 0$ to find the eigenvalues of A^TA.

Consider the matrix $\frac{1}{10}(A^TA)$

$$\det(\lambda I - A^TA) = \det \begin{bmatrix} \lambda - 8 & -10 & -4 \\ -10 & \lambda - 17 & -14 \\ -4 & -14 & \lambda - 20 \end{bmatrix}$$

$$= \lambda - 8 \begin{vmatrix} \lambda - 17 & -14 \\ -14 & \lambda - 20 \end{vmatrix} + 10 \begin{vmatrix} -10 & -14 \\ -4 & \lambda - 20 \end{vmatrix} - 4 \begin{vmatrix} -10 & \lambda - 17 \\ -4 & -14 \end{vmatrix}$$

$= (\lambda - 8)(\lambda - 17)(\lambda - 20) - 196 + 10((-10)(\lambda - 20) - 56) - 4(140 + 4(\lambda - 17))$

$= \lambda^3 - 45\lambda^2 + 324\lambda = \lambda(\lambda^2 - 45\lambda + 324)$

The eigenvalues would equal 36, 9 and 0, but now multiply them by 10 to get eigenvalues for A^TA.

$$\lambda_1 = \sigma_1^2 = 360, \lambda_2 = \sigma_2^2 = 90 \text{ and } \lambda_3 = 0.$$

Step 3: Use the equation $(\lambda I - A^TA) = 0$ to find the eigenvectors of the matrix above.

Consider the the equation $\frac{1}{10}(\lambda I - A^TA) = 0$. When distributing the $\frac{1}{10}$ into the parentheses we would get the equation $(\frac{\lambda I}{10} - \frac{A^TA}{10})$.

$$(36I - \frac{A^TA}{10}) = \begin{bmatrix} 28 & -10 & -4 \\ -10 & 19 & -14 \\ -4 & -14 & 16 \end{bmatrix} = 0$$

When $(36I - \frac{A^TA}{10})$ is put into row reduced echelon form, it gives us:

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Where $x_1 = \frac{\text{t}}{2}$ $x_2 =$ t and $x_3 =$ t. If we let t = 2. Therefore $x_1 =$ 1, $x_2 =$ 2 and $x_3 =$ 2. We will have to divide by 3 to normalize (length = 1) the vector.

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

To find the next eigenvector we use

$$(9I - \frac{A^TA}{10}) = \begin{bmatrix} 1 & -10 & -4 \\ -10 & -8 & -14 \\ -4 & -14 & -11 \end{bmatrix} = 0$$

When $(9I - \frac{A^TA}{10})$ is put into row reduced echelon form, it gives us:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Where $x_1 =$ -t $x_2 = \frac{\text{-t}}{2}$ and $x_3 =$ t. If we let t = 2. Therefore $x_1 =$ -2, $x_2 =$ -1 and $x_3 =$ 2. We will have to divide by 3 to normalize (length = 1) the vector.

$$\vec{v}_2 = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

To find the next eigenvector we use

$$(0I - \frac{A^TA}{10}) = \begin{bmatrix} -8 & -10 & -4 \\ -10 & -17 & -14 \\ -4 & -14 & -20 \end{bmatrix} = 0$$

When $(0I - \frac{A^TA}{10})$ is put into row reduced echelon form, it gives us:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Where x_1 = 2t x_2 = -2t and x_3 = t. If we let t = 1. Therefore x_1 = 2, x_2 = -2 and x_3 = 1. We will have to divide by 3 to normalize (length = 1) the vector.

$$\vec{v}_3 = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

Step 4: Create the matrix V, whose columns are the eigenvectors of A^TA.

$$V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Step 5: To find the singular values of A, take the square roots of the eigenvalues of A^TA.

$$\sigma_1 = 6\sqrt{10}, \sigma_2 = 3\sqrt{10}, \sigma_3 = 0$$

Step 6: Make a matrix D with the non-zero singular values in the diagonal and then create the pseudo-diagonal matrix Σ containing D and block matrices of zeros.

$$D = \begin{bmatrix} 6\sqrt{10} & 0 \\ 0 & 3\sqrt{10} \end{bmatrix}, \Sigma = \begin{bmatrix} D & 0 \end{bmatrix} = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$$

Step 7: Create the matrix U, whose columns are the eigenvectors of AA^T. Find $\vec{u}_1$ and $\vec{u}_2$ by putting the eigenvector that corresponds with it over the singular value to normalize (length = 1) that vector.

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{6\sqrt{10}} \begin{bmatrix} 18 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \frac{1}{3\sqrt{10}} \begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix}$$

Step 8: Put all of the pieces into the equation $A = U\Sigma V^T$. PUT the $\vec{u}_1$ and $\vec{u}_2$ together to get the matrix U, put Σ next to it and finally transpose V.

$$A = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{2} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

6 Applications

Singular value decomposition is a very useful decomposition. One of the applications is singular value decomposition is called data compression. This is where you consider some matrix A with rank five hundred. If we had to encode this matrix on a computer it would take a lot of memory. We might want to approximate this matrix with rank one hundred. We can prove this by taking the n largest singular values of A, replacing the rest with zero and recomputing $U\Sigma V^T$ gives you the best n rank approximation to the matrix. Also the total of the first n singular values divided by the sum of all the singular values is the percentage of information that those singular values contain. This yields a weird and quick algorithm for matrices and we can use SVD. We will drop all byt a few singular values and then recompute the approximated matrix. Since we only need to store the columns of U and V that get used, we greatly reduce the memory usage. We can use this by converting a picture of a tiger to black and white and then treating this tiger as a matrix, where each element is the pixel intensity at the relevant location. It is easier to graph the singular values of the tiger picture. When we graph the singular values you can see that just fifty of the singular values already make up over 70 percent of the information contained in the tiger picture. The last thing that will have to be done is that we will have to take some approximations and plot them to see what is the best one to use to form the tiger picture clear. By using SVD, we are able to compress a 500x800 pixel image into a 50x500 matrix (for U), 50 singular values, and a 80x50 matrix (for V).

7 Conclusion

The singular value decomposition is used to factor a rectangular matrix. The key to the SVD is to apply the spectral theorem to the symmetric matrices A^TA and AA^T. SVD are used in statistics, machine learning, and computer science.

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