From the beginning of set theory to Lebesgue's measure problem

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Tiivistelmä

Cantor konstruoi reaaliluvut ensimmäistä kertaa 1870-luvun alussa ja osoitti, että ne muodostavat ylinumeroituvan joukon. Myöhemmin vuonna 1878 Cantor esitti kontinuumihypoteesin, joka johti kuvailevan joukko-opin syntyyn sekä ensimmäiseen tulokseen – Cantor–Bendixsonin lauseeseen.

Kaksi vuosikymmentä myöhemmin Cantorin työ sai huomiota Ranskassa. Ranskalaisten matematiikkojen Borelin, Bairen ja Lebesguen ideoissa mittateoriassa ja funktioiden luokittelussa hyödynnettiin laajasti Cantorin ajatuksia. Tärkeimmäksi yksittäiseksi ongelmaksi kuvailevassa joukko-opissa muodostui Lebeguen mittaongelma: mitkä reaalilukujen osajoukot ovat Lebesgue mitallisia?

Vital näytti vuonna 1904 vedoten valinta-aksioomaan, että on olemassa reaalilukujen osajoukko, joka ei ole mitallinen. Tämän tuloksen ja Zermelon vuonna 1908 esittämien joukko-opin aksioomien myötä Lebesguen mittaongelma sai matemaattisen näkökulman lisäksi myös filosofisen. Seuraavien vuosikymmenten aikana osoitettiin, että Zermelon aksioomien avulla ei voida vastata moniin keskeisiin joukko-opin kysymyksiin.

Banach ja Ulam onnistuivat kehittämään uusia aksioomia näiden ongelmien ratkaisemiseksi. Osoittautui, että heidän esittämä yleisempi mittaongelma riippuu voimakkaista suurten kardinaliteettien aksioomista. Täten Cantorin kardinaalilukujen teorialle löytyi sovelluksia jopa reaalilukujen osajoukoille.

Avainsanat: kuvaileva joukko-oppi, Lebesguen mittaongelma, suurten kardinaalilukujen aksioomat.

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Abstract

Descriptive set theory has its origins in Cantor's work on pointsets in the 1870s. Cantor's construction of real numbers and proof of the non-denumerability of real numbers were the first results towards a new theory. Later, in 1878, Cantor formulated his continuum hypothesis for the first time, which lead to the first result in descriptive set theory: the Cantor-Bendixson theorem.

Two decades later, Cantor's theory awoke interest in French analysts Borel, Baire and Lebesgue. Their work on measure theory and classification of functions rested heavily on Cantorian ideas. The most important problem for the development of descriptive set theory was Lebesgue's measure problem: which subsets of the real line are Lebesgue measurable?

After Vitali's impossibility result in 1904 and Zermelo's axiomatization of set theory ZFC in 1908 Lebesgue's measure problem gained, in addition to its mathematical framework, a philosophical one as well. This allowed for a better understanding of the underlying situation and also proof-theoretic considerations. The limit to what could be proved to be measurable in ZFC was soon achieved.

However, new ideas arose through the works of Banach and Ulam. Their more general measure problem was identified as being dependant on strong axioms of infinity, the large cardinals, which are still linked to real numbers. Cantor's theory of cardinal numbers had thus found applications even at the level of real numbers.

Keywords: descriptive set theory, Lebesgue's measure problem, ZFC axioms, large cardinals.

Contents

Chapter 1

The origins of descriptive set theory

1.1 Introduction

Descriptive set theory is the study of "well-behaved" subsets of the continuum[1]. It is said to have emerged in turn of the 20th century in works of Émile Borel, Henri Lebesgue and René-Louis Baire. However, its actual origins were in the period 1870-1885 when Georg Cantor worked on the properties of real numbers. It began with Cantor's 1872 paper on trigonometric series, where he defines real numbers as fundamental sequences of rational numbers. Then, using his concept of limit point to define the sets of the first species. After proving in 1874 that real numbers are non-denumerable, Cantor formulated his continuum hypothesis for the first time. Later, in the 1880s, Cantor's theory of pointsets advanced considerably with the publication of a series of six papers on the common title *On infinite linear point manifolds*. These papers include his famous Grundlagen, which was the first paper which concerned sets as an independent theory, and in which Cantor describes for the first time perfect sets, that later play a crucial role in the development of descriptive set theory.

[1]In other words descriptive set theory is the study of definable subsets of real numbers that are measurable, satisfy the continuum hypothesis etc.

In the following we present a chapter on Bernhard Riemann's manifolds, mention some development on pointsets before Cantor's contribution, discuss Richard Dedekind's role in Cantor's line of thought and how Riemann affected Cantor and Dedekind. Then we follow Cantor on the development of pointset theory from 1870s to 1885. There are three important things that are given special attention; Cantor's work on trigonometric series did not effect the discovery of pointset theory as much as is often written; Dedekind was more concerned with his domains than real numbers during his correspondence with Cantor; and finally that both Cantor and Dedekind worked in Riemann's tradition, albeit in different ways.

1.2 Set theory before Cantor

1.2.1 Riemann's manifolds

In 1854 Riemann gave his famous Habilitations lecture "Über die Hypothesen die der Geometrie zu Grunde liegen" ("On the hypotheses which lie at the foundation of geometry"). Because the lecture was directed to people working not only in mathematics but also in philosophy and physics it lacks a degree of mathematical detail. Still, the lecture did not go unnoticed as it affected many mathematicians and its impact, as we will discover, can be seen in the works of Cantor and Dedekind, among others. It is also remarkable, as Riemann notes in the beginning of the lecture, that there were no previous labours on the topic, with the exception of hints in the works of Carl Gauss and Johann Herbart[2]. In other words, it is safe to assume that most of the ideas originated primarily from Riemann himself.

As Riemann notes in the introduction, the goal of the lecture was to distinguish, in today's terms, the topological and metric properties of the space from each other. In the first part of the lecture that deals mostly with topological properties of the space he defines the concept of a *manifold*

[2]The influence of Herbart on Riemann is not particularly clear and those interested in the connection should see Scholz 1982a [32] for discussion. However, Riemann was probably aware of Bolzano's ideas with the paradoxes presented by infinite collections [1]

which, as we shall see later, was adopted by Cantor. For Riemann, there existed two varieties of manifolds: The *discrete*, which consists of points and the *continuous*, which consists of elements.

“According as there exists among these specializations a continuous path from one to another or not, they form a *continuous* or *discrete* manifold: the individual specializations are called in the first case points, in the second case elements, of the manifold” (Ewald [19] p.653).

It may be the case that Riemann defined the manifold to give a basis for his work on geometry and function theory (especially Riemann surfaces). This claim is supported by the fact that Riemann began using the term "continuous manifold" (although without a satisfiable definition) in his 1851 dissertation [28][3].

Riemann does not focus in discrete manifolds on his lecture, nor later on in his work, but it is clear that discrete manifolds can be seen as a concept in foundations of mathematics. Furthermore, he does not rule out the possibility that space itself might actually turn out to be a discrete manifold instead of a continuous one.

“Notions whose specializations form a discrete manifold are so common that, at least in the cultivated languages, any things being given, it is always possible to find a notion in which they are included. (Hence mathematicians might unhesitatingly found the theory of discrete magnitudes upon the postulate that certain given things are to be regarded as equivalent.)” (Ewald [19] p.653)

This conception of a discrete manifold gives undoubtedly a right to call it a "set" as it seems to agree with Cantor’s definition of a set in 1883 as a “multiplicity which can be thought of as one, i.e. every totality of determinate elements which can be united into a whole by some law” ([9] p. 587). Riemann does not define the concept more precisely, and he seems

[3] This is further supported by the publication of Riemann’s manuscripts by Scholz in 1982b [33].

to indicate that the concept is somewhat universal and that it might be dependant on the framework one is working on. Unfortunately Riemann does not go into the properties of discrete manifolds even though he remarks that at least most things might turn out to be a part of a discrete manifold.

After briefly discussing discrete manifolds, Riemann notes

"On the other hand, so few and far between are the occasions for forming notions whose specializations make up a continuous manifold, that the only simple notions whose specialization form a multiply extended manifold are the positions of perceived objects and colours. More frequent occasions for the creation and development of these notions occur first in higher mathematics." (Ewald [18] p.653)

It is pretty clear that the more interesting of the two manifolds for Riemann are the continuous ones because they arise from mathematical concepts. The lack of more discussion about the discrete manifolds, even though he explicitly remarks that the continuous manifolds are more rare, is not surprising at all from the perspective of his earlier work. And as we shall see later the continuous manifolds are the starting point of what later became known as pointset theory or topology[4].

Despite originating the concept of manifolds, Riemann did not attempt to develop an independent theory of them, instead using them as a tool to better understand geometry and space. Still, the impact on the development of pointsets and set theory is significant as we will see in the works of many mathematicians, especially Cantor and Dedekind.

1.2.2 The development of pointsets before 1870s

In the 19th century, mathematicians like Rudolf Lipschitz, Peter Dirirchlet, Hermann Hankel and Riemann were developing the theory of trigonometric series which was one of the main research subject of mathematical analysis.

[4]For the development of the concept of manifold in the direction how manifolds are nowadays defined in differential geometry see Scholz [34].

This study was started by Jean-Baptiste Fourier who noticed that seemingly arbitrary functions could be represented as trigonometric series. However, Fourier's work was not complete and many mathematicians were interested in his findings continued his work. Particular interest were sufficient and necessary conditions for a function to be represented as a trigonometric series. This problem turned out to be non-trivial and many partial results were given between years 1830 and 1870[5]. Even though many results were successfully proved, new questions arose.

Riemann was one of the first to start more rigorous research of trigonometric series already in 1850s but his approach was functional analytic and no properties of the domain of a function was considered. Still, many important tools to analyze trigonometric series appeared, certainly the most recognizable was the definition of *Riemann integral*, and thus mathematics to continue the work was developed. Riemann's work on trigonometric series really appeared already in 1854 but was left unpublished until 1868 which explains the slow development on the problem during the period 1854-1868.

One of the many papers that followed the publication of Riemann's work was Hankel's thesis in 1870. His goal was to characterize those functions that are useful and should be considered in analysis. That led Hankel to pay particular interest in discontinuous functions and it was the definition of the Riemann integral that allowed their study. In his work Hankel had to make a distinction between point-wise discontinuity and total discontinuity. To make this distinction, Hankel had to consider, in his words, how points lie on the line. This required him to focus on the domain of the function[6]. Unfortunately, Hankel's beginnings of a theory of pointsets were not developed further, due to his early death in 1873.

A small step towards the topology of the real line was also taken by Lipschitz who used concepts that are currently known as everywhere-dense

[5]See for example Dauben [13] or Grattan-Guinness [23] for early development on the problem.

[6]Hankel used expressions "fill up the line" and "lie loosely" to describe this difference. Good introduction to Hankel's ideas is found in Dauben [13] p.23-29 or in his original paper [25].

and nowhere-dense sets[7]. However, he did not really study the structure of the domain and only developed the concepts to understand the behaviour of the function in its range.

Even though the problems, especially on trigonometric series, suggested to study the structure of pointsets, no one before Cantor was able to take the step to study them in greater detail. Contemporary studies were directed towards functional analysis and the theory of pointsets in their work was never considered independently; the underlying concept of a pointset was merely used as a tool to prove theorems in functional analysis.

1.3 Cantor's early work

1.3.1 Cantor's work on trigonometric series

Riemann's work suggested[8] the so called uniqueness theorem of trigonometric series:

If a function $f(x)$ is representable as a trigonometric series:

$$f(x) = \frac{a_0}{2} + \sum(a_n \sin(nx) + b_n cos(nx)),$$

is the representation unique?

In 1870, Eduard Heine was able to prove that the representation is indeed unique, assuming uniform convergence and discontinuities only in a finite number of points. Cantor was at that time interested in number theory, which was also the topic of his doctoral dissertation but Heine encouraged Cantor to generalize his result of the uniqueness theorem, which made Cantor interested in mathematical analysis. Cantor was able to generalize Heine's result in 1870 to hold in cases where the series was convergent for all the values of x in the domain.

[7] These concepts can be found in Lipschitz [26] or in English from Dauben [13] p.20-21.

[8] In his 1854 paper [30] Riemann constructed integrable functions with diverging Fourier series and conversely functions with converging Fourier series but which were not Riemann integrable.

Cantor continued his work on trigonometric series in two subsequent papers. In the first paper, published in 1871, he suggested some improvements to the result and was able to show that the convergence for all values of x was not necessary. But the number of these exceptional points of x (where $f(x)$ did not converge) had to still be finite in finite intervals. In that paper, he further anticipated that the theorem could still be further generalized (Dauben [13] p.34-36).

1.3.2 Construction of real numbers

Some theory of pointsets was required to generalize the uniqueness theorem to hold for infinite number of exceptional points of convergence. As noted above, even the theory of irrational numbers was still in most parts undeveloped. Consequently, Cantor realized that the so-called arithemetic continuum needed to be somehow connected to the geometric continuum and that new concepts had to be created.

Cantor published the second paper "Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen" (On the generalization of a theorem from the theory of trigonometric series) in 1872 [3] to first develop the theory of real numbers and then to generalize the uniqueness theorem. In the beginning of the paper he defines a domain B which consists of symbols b which are fundamental sequences (in today's terms Cauchy sequences) of rational numbers[9]. When extending the standard arithmetic operations to hold also for the symbols b Cantor started calling them numbers[10]. Now that Cantor had defined the domain B, the same process could be done again to form domain C which consisted of symbols c that are fundamental sequences of the numbers b. Then continuing in the same fashion he also constructed domains $B, C, \ldots, L$ where L was the result of λ iterations of the process. But as Cantor himself points out all the elements of the domains $B, C, \ldots, L$

[9]Cantor seemed to assume that the limit of these sequences actually exists. Russell's critique on this can be found in I. Grattan-Guinness [23] p.300.

[10]It seems that defining the elementary operations for the symbols b made Cantor call them as numbers but Cantor did not justify this further and it seems that the symbols were called numbers just out of convenience.

could be equated and thus no new elements were formed. Cantor still needed to prove that for each number b there was a unique point in the geometric continuum (on the straight line). To this end, he recognized the need for an axiom rather than a proof[11].

Cantor was not the only one to realize the need for a more precise treatment of real numbers, as Dedekind published in the same year 1872 his own construction of real numbers[12]. In the beginning of his article, Dedekind quickly comments Cantor's paper of 1872

"It seems to me that the axiom given in Section II of that paper agrees with what I designate in Section III as the essence of continuity. But I am unable to see the use of distinguishing real numbers of a yet higher type, even if this is only done conceptually."

Thus Cantor and Dedekind were both able to recognize independently that the arithmetization of the continuum could not be done without an axiom. Also Dedekind did not see a need for the domains beyond B as no new elements appeared in the process but this exact conceptual difference led Cantor to consider the higher orders of derived sets.

1.3.3 Derived sets of real numbers

What followed the construction of real numbers in the 1872 paper was Cantor's key definition: *Derived pointsets of the first species*[13]. To define these pointsets, Cantor first defined a limit-point of a point set P as "point of the line for which in any neighbourhood (an interval that contains the point),

[11] Cantor formulates this axiom in [3] p.127. This axiom is nowadays often called Cantor-Dedekind axiom and may be formulated as "The points on a straight line can be put into a one-to-one correspondence with real numbers".

[12] Dedekind's ideas were already presented by him in 1858 as he mentions in the preface of his 1872 [14] but were not published (English translation of the paper can be found in [19] p.765-779).

[13] For Cantor pointsets P [Punktmengen] were a given finite or infinite number of points of a straight line [3] p.128 or in today's terms subsets of real numbers.

infinitely many points of P are found". Bolzano-Weierstrass theorem[14] states that every infinite bounded pointset contains at least one limit point and thus Cantor could as least "conceptually" take its set of limit points given which he called "the first derived point set of P" and denoted it by P'. And then if P' was infinite one could consider the set of its limits points and get the second derived point set P'' and so on till $P^{(v)}$. Potentially, after v iterations, the derived set $P^{(v+1)}$ may not exist, in which case Cantor described P as a being of the vth kind. All the sets P which were of the vth kind for a finite v Cantor named as the derived sets of the first species and the sets which for this did not happen he named as the *derived sets of the second species*[15].

Now that the derived sets of first species were defined, Cantor could generalize the uniqueness theorem of trigonometric series; it was enough that the series converges for all values x except for those that belong to the derived point set of the first species. Or, in other words, as Cantor wrote in the end of his 1872

"A discontinuous function $F(x)$ which is non-zero or undetermined for all values of x that correspond to points of a point set P of the vth kind in the interval $(0...(2\pi))$, cannot be represented as a trigonometric series".

At this point the derived sets do not play a big role outside of the uniqueness theorem, but their importance will be made evident later in Cantor's theory of pointsets when the basic concepts of pointsets are defined based on them. The above theorem seems to hint that Cantor was considering of sets P that had non-empty derived set $P^{(v)}$ for any finite v as early as 1872[16]. But it was not before 1880 that Cantor published

[14]Cantor does not mention in his paper where he knew the theorem but it is plausible that he heard of it from his teacher Karl Weierstrass (Moore [27] p.221-223).

[15]Cantor does not actually use the names "sets of the first species" or "sets of the second species" before his 1879 [6] p.2 but we will use them here for convenience

[16]This observation is made in Grattan-Guinness [23] p.85 and actually Cantor says in footnote in his 1880 [7] p.358 that he succeeded in finding the sequence already ten years ago. However this is criticized in a footnote in [22] p.160 where Ferreirós notes that Cantor had a habit to anticipate the dates of his findings. In either case, Cantor had found

the sequence of derived sets of the second kind which included symbols of infinity

$$P^{(\infty)}, P^{(\infty)'} = P^{(\infty+1)}, P^{(2\infty)}, P^{(\infty^n)}, P^{(\infty^\infty)}, P^{(\infty^{\infty^n})}, P^{(\infty^{\infty^\infty})}, \ldots.$$

However, there is no question that an important transition in Cantor's thinking occurred in 1872, which led Cantor to consider the infinite sets of real numbers.

It is often written that the trigonometric series and the uniqueness theorem plays a fundamental role in Cantor's findings of the beginnings of pointset theory. But as we can see from the above, the construction of real numbers and the derived sets have nothing to do with trigonometric series, apart from the fact that they offer a framework for their study. In addition, the amount of detail and discussion about these concepts is much more what is needed for the results about the trigonometric series. Further, Cantor even studied the theory of irrational numbers in his paper "Über die einfachen Zahlensysteme" (About the simple number systems) in 1869 [2], which shows his interest in them before the papers on trigonometric series. His fundamental sequences in 1872 follows Heine's work on the same year, but the connection to real line is only made by Cantor (Grattan-Guinness [23] p.82-83). It is clear that the derived sets of real numbers arose as a concept more from the construction of real numbers than from the uniqueness theorem because the idea in the definition is similar to fundamental sequences. Most importantly, we will see that in the following papers by Cantor the properties of real numbers and pointsets are considered totally independently of trigonometric series.

1.4 Cantor-Dedekind Correspondence

1.4.1 Non-denumerability of real numbers

The first written contact between Cantor and Dedekind was made in 1872 when they exchanged their papers of irrational numbers[17]. And in April

something fundamental that he was able to use later in his theory.

[17] As Ferreirós points out in [22] p.172 Cantor and Dedekind stayed in the same hotel in Switzerland in April 1872. This is often seen to be the beginning of their friendship,

they began their famous correspondence[18]. The very first question that Cantor sent during their correspondence was in November 1872:

"Take the totality of all positive whole-numbered individuals n and designate it by (n). And imagine say the totality of all positive real numerical quantities x and designate it by (x). The question is simply, Can (n) be correlated to (x) in such a way that to each individual of the one totality there corresponds one and only one of the other? At first glance one says to oneself no, it is not possible, for (n) consists of discrete parts while (x) forms a continuum. But nothing is gained by this objection, and although I incline to the view that (n) and (x) permit no one-to-one correlation, I cannot find the explanation which I seek; perhaps it is very easy."

Dedekind responded to Cantor that he could not answer the question but sent Cantor a proof that a totality of all algebraic numbers can be correlated with the totality (n) of all natural numbers. Cantor responded in turn that Dedekind's proof is approximately the same as his proof[19] on the fact that (n) can be correlated one-to-one with

$$(a_{n_1,n_2,\ldots,n_\nu}).^{20}$$

Dedekind also wrote that Cantor's question did not deserve effort because it had no practical interest. But he had to change his opinion in the 7th of December 1873 when Cantor gave his first proof that the totality (x) cannot be correlated one-to-one with the totality (n). The proof that Cantor provided was simplified after Dedekind's suggestions, but the central idea remained the same. The proof used topological properties of real

however we shall not speculate on the status of their relationship at that time, those interested in it should read Ferreirós [20].

[18]The English translation of these letters can be found in Ewald [19] p.853-878.

[19]Even if Cantor claims that their proofs are approximately the same they have some differences (see Ferreirós [22] p.179) and it is thus hard to believe that Cantor had come up with the exact content of the theorem himself.

[20]Cantor means in today's terms that (n) can be correlated one-to-one with ν-tuple $(n_1, n_2, \ldots, n_\nu)$ where each n is an integer.

numbers[21], namely Bolzano-Weierstrass theorem which already appeared in Cantor's 1872 article. Therefore, the first proof was not the famous proof by diagonalization which appeared much later in 1891. Cantor published these two theorems in 1874 with the title: On a property of the set of all real algebraic numbers[22]. The title of the publication is rather strange because an important property proven for Cantor was the difference between the continuum and totalities like algebraic or rational numbers[23]. The title and content of the paper was surely affected by Weierstrass who actually encouraged Cantor to publish the paper but only "as long as it is related to the algebraic numbers". As a result, the paper's content closely resembles another proof of the theorem of Liouville's that there exist transcendental numbers. But Cantor writes in a letter before 1874,

"there exists essential differences among the totalities and value-sets that I was until recently unable to fathom." (Ewald [19] p.846)

which is another clear implication that the most important part of the paper for Cantor was, in later terms, the non-denumerability of real numbers.

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[21] The idea of the Cantor's proof is as follows: Assume that real numbers can be given in a sequence $\omega_1, \omega_2, \ldots, \omega_v, \ldots$. Then it is enough to show that in any interval $[\alpha, \beta]$ a number η can be found such that it does not appear in the given sequence. Denote the first two numbers of the sequence that lie in the interval $[\alpha, \beta]$ by α' and β'. And then again denote the next two numbers of the sequence that lie in the interval $[\alpha', \beta']$ by α'' and β''. Continuing in the same way we get a sequence of closed nested interval $[\alpha^{(k)}, \beta^{(k)}]$. Then by the Bolzano-Weierstrass theorem we find a number η which belongs to the interval for each k. If this number η were contained in the sequence, then one would have $\eta = \omega_n$. But by construction this is not possible, so every real number cannot be a member of the sequence.

[22] Originally appeared in Cantor [4] and English translation in Ewald [19].

[23] Cantor writes only very short in the paper [4] that "I have discovered the difference between a so called continuum and any set like the totality of real algebraic numbers."

1.4.2 Dimension of a manifold

Now that Cantor knew that infinities came in different sizes it surely made him consider whether there were infinities that differ from (n) or (x). Perhaps higher dimensions would create pointsets of even higher cardinality? Or maybe there is an infinite pointset strictly between (n) and (x)? Thus it is no surprise that Cantor's next question, which he wrote to Dedekind in 1874 was:

"Can a surface be one-to-one correlated to a line so that to every point of the surface there corresponds a point of the line, and conversely to every point of the line there corresponds a point of the surface?"

At first, Cantor seemed to believe that this question should be answered negatively, as he wrote "here too one is so impelled to say no that one would like to hold the proof to be almost superfluous". But Cantor received no answer from Dedekind before 1877[24] when he had already proved his theorem to be true[25]. Cantor had also made many improvements to his notations from 1874. He now used the word *manifold* to describe surfaces, lines, etc., and thus it seems to agree mostly with Riemann's definition of continuous manifold. He also arrived at a rigorous definition of *power* which was only implicit in his 1874 article.

"If two well-defined manifolds can be correlated to one another one-to-one and completely, element for element, then I say that they have the same power." (Ewald [19] p.865)

Cantor had demonstrated more general theorem than what he sent in

[24]The reasons for a three year pause in the correspondence seems to be unknown but to those interested some speculation is found in Dauben [13] p.54 and Ferreirós [22] p.186.

[25]This is when Cantor writes the famous phrase "je le vois, mais je ne le crois pas (I see it, but I do not believe it)". This is often seen as a evidence that Cantor was surprised of his findings but closer analysis indicates that it might describe his doubts about the correctness of his proof (See Gouvêa [24]).

1874. The theorem now claimed that a manifold of n-dimensions is of equal power as a manifold of one dimension. These definitions and the proof were published in his 1878 [5] and were followed by discussion by letters between Cantor and Dedekind (Ewald [19] p.866-871) of the invariance of dimension and the definition of dimension. After Cantor's attempt in his 1879 paper on the matter the issue seemed to be settled[26] and Cantor did not consider n-dimensional manifolds in his papers to come.

In the end of his 1878 article Cantor made final important claim that "linear aggregates would consist of two classes", the denumerable ν (where ν runs through all positive numbers) and the continuous x (where x can take all real values ≥ 0 and ≤ 1). This was, of course, the first formulation of his *continuum hypothesis*. He also notes that he will study this question later, and indeed between the years 1879-1884 the focus of his research was on infinite numbers and topology of the real line.

1.4.3 Differences in the works of Cantor and Dedekind

As discussed earlier, Riemann originated the concept of continuous manifold in 1854, which Cantor by 1878 had adopted. Dedekind surely recognized the Riemannian aspect of Cantor's work and he wrote to Cantor "I should like to see the shorter and equally Riemannian word 'domain' [Gebiet] given clear preference over the clumsy word 'manifold [Mannigfaltigkeit]'" (Ewald [19] p.870). In the same letter, Dedekind suggests Cantor to develop a more rigorous foundation for the "theory of domains", and notes that he has some definitions that seem to give a solid foundation. Here Dedekind most probably refers to his manuscript written before 1872[27] where he defined basic concepts of real line topology, most notably the notion of *open set* which he

[26] Also Lüroth, Thomae, Jürgens and Netto published articles concerning this question (See Dauben [13] p.70-72). While Cantor was not happy with these considerations his proof was not also satisfactory and only in 1911 Luitzen Brouwer was able to settle the issue completely.

[27] This manuscript was probably written during 1863-1866 as Ferreirós points out in footnote in [21] p.28. The manuscript was published by Fricke, Noether and Ore in 1931 [17] p.352.

called *Körper*. This concept however was not developed further by Dedekind but it shows that he was certainly interested in advancing the subject before his 1872 article.

An important difference between Cantor's theory of irrational numbers, is that Cantor does not concern himself with the foundations at all. Dedekind's entire 1872 paper can be viewed as an attempt to codify the continuum, but Cantor constructs real numbers only because of his interest in irrational numbers. The lack of interest in foundations of mathematics actually continues in Cantor's work until 1884, and he develops new concepts only to prove theorems of the continuum. This is exactly what Riemann did after defining discrete and continuous manifolds, he did not develop an independent theory of them, but instead used them as a basis for his theorems in differential geometry. Thus it is clear that Cantor and Dedekind both worked in Riemann's tradition, but only Dedekind inherited this interest in foundations, later leading to a publication of his theory of systems[28]. One reason for the difference might be that Dedekind read Riemann much earlier, and that his field of research was mostly algebra instead of analysis.

It is often said that Dedekind was academically interested in real numbers, maybe even as much as Cantor. However, this claim is probably untrue and it is important to recognize the difference in Cantor and Dedekind when it comes to real numbers. Dedekind's 1872 article undoubtedly deals with real numbers but it does not consider any properties of them. It is also clear from his later work that he was not at any point afterwards interested in real numbers or in their structural properties. As noted before, the concepts of Dedekind's 1872 paper was known to him from the 1850s, long before his correspondence with Cantor. Already in the 1870s Dedekind seems to be into his theory of systems, which he published in 1888 on the title "Was sind und was sollen die Zahlen?" ("The nature and meaning of numbers"). In this article Dedekind does not consider directly anything about real numbers, and comments his earlier work on the theory of systems:

"Upon this unique and therefore absolutely indispensable foundation, as I

[28]The Systems seem to be really close to what Riemann meant by discrete manifolds.

have already affirmed in an announcement of this paper, 3 must, in my judgment, the whole science of numbers be established. The design of such a presentation I had formed before the publication of my paper on Continuity, but only after its appearance and with many interruptions occasioned by increased official duties and other necessary labors, was I able in the years 1872 to 1878 to commit to paper a first rough draft which several mathematicians examined and partial1y discussed with me. It bears the same title and contains, though not in the best order, all the essential fundamental ideas of my present paper, ..." (Dedekind [15] p.32)[29]

This indicates that the study of the continuum was not foremost in Dedekind's mind during the 1870s.

Cantor however was exactly the opposite and tried to find essential differences in the properties of the continuum and of the natural numbers. That is exactly what led him to the concepts like limit-point, derived sets and to look for an answer to the continuum hypothesis.

Another big difference in Dedekind's and Cantor's theories are their definition of infinity. This difference is clear from Cantor's 1878 paper, where he writes in the first page that the power of a finite manifold (a manifold that has finite number of elements) corresponds to the number of its elements and thus a proper part of a finite manifolds always has a smaller power than the manifold itself. However, he notes that this property does not hold for infinite manifolds.

"But this relation ceases to hold entirely if the manifold is infinite (manifold that consist of infinite number of elements); from the fact that an infinite manifold M is a proper part of a manifold N, or can be fully assigned to the manifold N, it can be by no means concluded that its power is smaller than the power of the manifold N." (Cantor [5] p.242)

This is in some sense a paradoxical property that a whole can have a proper part that is of the same power. In today's terms, this is known as Dedekind-infiniteness. For Cantor, this was arguably a theorem of infinite

[29]The draft that Dedekind talks about can be found in Dugac [18] p.293-309.

sets, even though he did not represent a proof[30]. Dedekind went one step further by using it as a definition of infinity[31]. And at this point a relevant question is if Dedekind-infiniteness should taken to be a theorem or a definition. Using it as a definition has a key difference to Cantor's definition of infinite[32] because it does not rely at all on the sequence of natural numbers and so one can talk about infinite sets even before defining the natural numbers.

1.5 Beginning of descriptive set theory

After his 1878 Cantor published six papers in Mathematische Annalen under the common title *Über unendliche, lineare, Punktmannigfaltigkeiten* ("On infinite linear point manifolds") that appeared during the years 1879-1884[33]. As Cantor notes in the preface of the fifth paper, the *Grundlagen*, all of the six papers are highly connected to his articles in 1872 and 1874 which means that most of the ideas were available to him before mid 1870s but now the ideas were made more rigorous.

As noted above in the first of the papers [6] 1879 Cantor studied the derived sets but this time independently of trigonometric series. Also in page 2 he says that a pointset P which is entirely or partially in closed interval $[\alpha, \beta]$ is said to be an *everywhere dense pointset* if every interval in $[\alpha, \beta]$ contains a point of P. One property of everywhere dense pointsets that Cantor proves on the next page is that they are necessarily of the second species. As he already knew that there were at least two different sizes of infinity, he remarked that the sets of the first species are all in the "class of infinitely countable" and that the sets of the second species are either

[30] The first proof that any infinite set is Dedekind-infinite is by Zermelo who proved this statement in 1904 using axiom of choice.

[31] This definition is published in his 1888 [16].

[32] Cantor defined a set to be finite if it can be one-to-one correlated with some number and infinite if not.

[33] These papers are, in chronological order, [6], [7], [8], [9], [10] and [11]. A good overview of the content of these papers are in Grattan-Guinness [23] p.92-97 or Dauben [13] p.77-119.

countable or have to same power as "continuous intervals". The classification of pointsets with certain properties continued in the fourth paper [9] with the notion of *isolated pointset*: A pointset P was called isolated if its intersection with its derived set Q' was empty. For example, Cantor noted that all isolated pointsets are countable.

After these four papers appeared the Grundlagen [10][34], in which almost all his results about pointsets appear, along with many important new ones. Certainly the most interesting concept for the descriptive set theory is the *perfect set*. Cantor called a set P to be perfect if it was equal to its derived set P'. Some properties of perfect sets were in the Grundlagen but more detailed study of them appeared a year later in the sixth paper [11] of the series. The most important result is the early version of Cantor-Bendixson[35] theorem which says that any *closed set* [36] of the continuum can be written as a disjoint union of a perfect set and a denumerable set. This result, together with the observation that every perfect set is of the same power as the interval $[0, 1]$, gives the first result towards the solution of the continuum hypothesis: at least no closed set can violate the continuum hypothesis. As Cantor notes in [11] p.488, he believed that this kind of composition theorems would ultimately solve the continuum problem.

The importance of the perfect sets can also be seen in the definition of the continuum. However, Cantor realized that the requirement that continuum is a perfect set would not be enough. Thus he introduced the term *connected set*[37] and defined the continuum as a perfect connected set.

In this way, Cantor was able to establish that closed sets satisfy the

[34]Grundlagen was the first paper which concerns sets as an independent theory and it contains the big ideas of ordinals and transfinite numbers along with a discourse about the infinity. Thus it is one of the main reasons that Cantor is known as the founder of set theory.

[35]This result was stated incorrectly in Grundlagen and together with I. Bendixson Cantor refined the result.

[36]Closed set was defined as a set that contains its own derived set.

[37]" We call T a *connected* point-set if, for any two of its point t and t' and for any arbitrarily small number ε there always exists a finite Anzahl of points $t_1, t_2, \ldots, t_v$ of T, such that the distances $\overline{tt_1}, \overline{t_1t_2}, \overline{t_2t_3}, \ldots, \overline{t_vt'}$ are less than ε. " (Ewald [19] p.906)

continuum hypothesis (to use the today's terminology). But what is the case for those sets that are not necessarily closed? This is the root of what descriptive set theory is trying to answer. However, the developments on this subject had to wait until the turn of the century and a change of scenery, first to Paris, and then later to Moscow.

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Chapter 2

Lebesgue's measure problem and its development

2.1 Introduction

Cantor's set theory and transfinite numbers did not receive a positive reception among the majority of mathematicians in his time, and as a result, many rejected his philosophy of infinity. Fortunately, after his work was translated to French, many French analysts laid interest in his findings. Three of the most notable analysts were Borel, Baire and Lebesgue. Their work, which used Cantorian ideas paired with new emerging mathematical principles, gave rise to a subject that is now called descriptive set theory. Descriptive set theory is a study of different sets of real numbers, their regularity properties and classifications. Some of the important examples of these concepts that emerged in French at the turn of the century are Borel sets, Lebesgue measurable sets and Baire classification.

Even though the theory was started by Borel, Baire and Lebesgue it is important to realize that their main research lied elsewhere. However, they were able to raise many important questions at the fundamental level of mathematics and after their work, mathematicians began a deeper study of their ideas. At the same time, interesting philosophical considerations started to emerge as new principles like the axiom of choice were introduced.

Eventually these studies by Cantor and French analysts led to the introduction of three important regularity properties of sets of real numbers: perfect set property, property of Baire and Lebesgue measurability. These properties were under research, independently of other areas of mathematics, after 1915 in Russia. Also, set theory got its first axiomatization in 1908, which led to a different program, namely, to a study of real numbers within the axioms of sets.

In this work I mostly study the origins and development of the Lebesgue's measure problem. It started with an intuitive definition of "size", that soon – after a discovery of a non-measurable set – needed clarification. I study thoroughly the early development on the two following questions:"Which sets of real numbers are measurable?" and "How the definition of size should be altered so we could attain a definition which allows us to measure more sets while keeping the definition intuitive?". We shall assert that both of these questions, surprisingly, lead to metamathematical considerations and are connected to each other and to the theory of large cardinals.

2.2 Beginning of measure theory

2.2.1 Cantor's set theory in France in 1880s

Cantor's work during the 1880s received mixed responses among the mathematicians, but one individual that expressed interest in his discoveries was Swedish mathematician Gösta Mittag-Leffler. He used Cantor's concept of derived sets in the proof of his theorem in complex analysis[1]. Mittag-Leffler had also just found a new mathematical journal *Acta Mathematica* and because of his interest in Cantor's work he began requesting new papers and translations of some older papers in French from Cantor. With the help of French mathematician Charles Hermite, seven papers were translated, but

[1]The theorem states roughly that given a set of poles (with orders) and Laurent coefficients, it is possible to find a meromorphic function with corresponding poles and Laurent coefficients. A good study of Mittag-Lefflers theorem and his usage of Cantor's theory is in [52].

in the process much of the philosophical content was removed[2]. Despite this loss of contextual richness, this made the papers more accessible to mathematicians and, more importantly, his theory was now available also to the French mathematical community.

Paul Tannery was among the first of these French mathematicians to utilize these translations. He stated a theoretical interest in them, and he claimed that he had proved the continuum hypothesis in his paper [51] (his argument of course being incorrect). Also Henri Poincaré used Cantorian ideas in his work on Fuchsian functions (automorphic forms) in [38] but the difference to Tannery was that he only used the ideas as an additional tool in his analysis instead of studying Cantor's work as an independent subject.

Another French mathematician who used Cantor's work a lot in his analysis was Camille Jordan. His famous analysis course *Cours d'Analyse* [16] contained some set theory which still did not get an independent treatment. But Jordan introduced some properties of point-sets and defined concepts like limit point, connectedness, interior, exterior and closed set. More importantly, he introduced the term *measurable set*[3]. His definition used the terms *inner content* $c_i(A)$ and *outer content* $c_e(A)$ of a set A which we understand as a least upper bound of areas formed by squares inside A and as a greatest lower bound of areas formed by squares that are allowed to intersect the boundary of A. A set is called measurable if its inner content and outer content are equal. Measurable sets were then used by Jordan in integration theory and also for the concept of additivity of measure emerged for the first time.

"Jordan showed that for any set E, if $E = E_1 \cup E_2 \cup \cdots \cup E_n$, where the

[2]It seems that Cantor's theory was too abstract to Hermite and his colleagues (Moore [33] p.95-96).

[3]Already five years before Jordan in 1887 [37] an Italian mathematician Giuseppe Peano came up with the notion of measurable set which used approximately the same ideas of inner and outer size as Jordan. Their measure is nowadays called Peano-Jordan measure.

E_p are mutually disjoint $[E_k \cap E_l = \emptyset$ for all $k \neq l]$, then

$$\sum_{p=1}^{n} c_i(E_p) \leq c_i(E) \leq c_e(E) \leq \sum_{p=1}^{n} c_e(E_p).$$

Thus, if the sets E_p are measurable, their sum, E, is likewise measurable, and $c(E) = \sum_{p=1}^{n} c(E_p)$." (Hawkins [14] s.94)

2.2.2 Lebesgue's measure problem

Jordan's work on a measure certainly affected the works of French analysts Borel and Lebesgue, who were to develop measure theory towards more preferable generality at the turn of the century. However, the initial motivation for Borel came from a different source, namely from complex analysis. This is clear from Borel's monograph *Leçons sur la théorie des fonctions* 1898 [6] in which the first section concerns measure-theoretic ideas and the second section complex analysis. Borel used the axiomatic approach[4] in his definition of measure, and thus it differs from that of Jordan's. The most important thing that Borel postulated was the countable additivity of the measure:

"When a set is formed of all the points comprised in a denumerable infinity of intervals which do not overlap and have total length s, we shall say that the set *has measure s.* When two sets do not have common points and their measures are s and s', the set obtained by uniting them, that is to say their sum, has measure $s + s'$.

More generally, if one has a denumerable infinity of sets which pairwise have no common point and having measures $s_1, s_2, \cdots, s_n, \cdots$, their sum ... has measure

$$s_1 + s_2 + \cdots + s_n + \cdots$$

[countable additivity]. All that is a consequence of the definition of measure. Now here are some new definitions: If a set E has measure s and contains all the points of a set E' of which the measure is s', *the set* $E - E'$, *formed of all the points of* E *which do not belong to* E', *will be said to have measure* $s - s'$ ....

[4]This approach comes from Drach as Borel notes "This way of proceeding has great analogies with the methods introduced by M. J. Drach, ..." [6] p.48.

The sets sets for which measure can be defined by virtue of the preceding definitions will be termed measurable sets ...." (Borel [6] p.47-48, translation by Hawkins [14] p.103)

The measurable sets that Borel introduced are what are now termed *Borel sets*. Borel did not prove many results about measurable sets, but he noted that every closed set is measurable, and that every countable set necessarily has measure zero.

Lebesgue, who attended the École Normale Supérieure, as did Borel (and also René-Louis Baire), completed his studies in 1897. Later, in his second paper [20] he began to study measures, where he represents the measure problem of the surfaces:

"The problem of measuring surfaces bounded by closed simple curves can be posed as follows: Match each surface a number called *area*, such that two equal surfaces have equal areas, and that the surface formed by the union of a finite or infinite number of surfaces that have common portions of boundary and do not overlap, has as its area the sum of the areas of the component surfaces."

Lebesgue thus accepted Borel's idea of countable additivity of measure, as opposed to just finite additivity. Still, Borel's definition of measure was not used and the theory lacked mathematical rigour.

In 1902, Lebesgue presented his thesis *Intégrale, longueur, aire* [21], where in the first chapter the measure problem was generalized and also the theory of measure received some much needed clarity. Lebesgue then formulated the measure problem in the following manner:

"We propose to attach to each bounded set a non-negative number, which we shall call its measure, that satisfy the following conditions:

(i) There exists a set whose measure is not zero,

(ii) Two equal sets have the same measure [translation invariance[5]],

[5]The translation invariance can be stated in modern notation, for a measure m and a

(iii) The measure of the union of a finite number or a countable infinity pairwise disjoint sets, is the sum of the measures of these sets." (Lebesgue [21] p.236)

Immediately after formulating this measure problem, Lebesgue notes that "We will solve this problem of measure only for sets which we will call measurable". First, Lebesgue defines the *outer measure* $m_e(A)$ of a bounded set A as the greatest lower bound of the sum of countably many intervals covering the set and then the *inner measure* $m_i(A)$ of a set A is defined as

$$m_i(A) = b - a - m_e([b, a] - A).$$

A set is then called *measurable* if $m_i(A) = m_e(A)$.

If then a measure m, that satisfies the conditions (i)-(iii) and also by convention $m([0, 1]) = 1$, is definable, the outer measure m_e must by definition be greater than or equal to m. For the inner measure, we arrive at

$$m_i(A) = b - a - m_e([b, a] - A) \leq b - a - m([b, a] - A) = m(A).$$

Thus, we have arrived at the inequality $m_i(A) \leq m(A) \leq m_e(A)$. This means that for measurable sets, $m(A)$ has to be equal to both the inner and the outer measure. As such, the measure problem has an unique solution for the measurable sets.

Lebesgue then showed that the condition (iii) holds for measurable sets, and thus the measure problem was solved at least for the measurable sets. But to show the condition (iii), he first had to demonstrate that countably many disjoint measurable sets in measurable. This result, however, requires *the axiom of choice*[6] (Moore [35] p. 137) which was not properly formulated before 1904 and, as we shall discover, will play a more crucial role in the measure problem than initially expects.

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real number x, such that

$$m(A + x) = m(A),$$

where $A+x = \{a+x \mid a \in A\}$. Also, we will later discuss about measures in n-dimensional spaces and in that context this condition should be understood as "all congruent sets have equal measure".

[6]The axiom of choice states that there exists a function f (called the *choice function*), defined on any family of non-empty sets X, such that for every set $A \in X$, $f(A) \in A$.

Next, Lebesgue proved some properties of the measure and concluded that all Jordan-measurable sets are measurable and that also every Borel set is measurable. He also argued that there are measurable sets that are not Borel sets[7]. Since there exists a perfect set that is measurable (and so every subset of it is also measurable), and because there are as many Borel sets as there are real numbers, there are more measurable sets than Borel sets.

The theory of measures that Lebesgue had developed was thus an extension of Borel's, and the definitions behind it came from very natural ideas. As we have seen, the idea of Lebesgue's measure rose from the need of a generalisation of the notion of length and area. Indeed the conditions of Lebesgue measure says, in informal terms, for example, that the measure of an interval is its length, the measure of a set is independent of the location of the set, and that a measure of a set can be calculated if the measures of its disjoint parts are known. Thus, it is of no surprise that this theory had many applications, and for Lebesgue the most important among them was the integration theory that he developed in his dissertation.

At this point it is important to reflect on the development of the theory of real numbers back to its origins. Cantor started the theory of real numbers in 1870s by first proving the non-denumerability of real numbers, then formulating the continuum hypothesis and later adding more rigour to the theory with additional definitions about pointsets. It was another twenty years before Cantor's theory finally got the attention and applications it deserved in the works of Borel and Lebesgue. However, the theory still lacked independent considerations and further development, and as a result the structure of real numbers was not still well understood. The thing that Cantor really understood, perhaps better than contemporary the French analysts, was that the theory needed the stronger foundation of set theoretic considerations, which he created precisely for this reason. Indeed, the two subjects, theory of real numbers and set theory, cannot be separated, as they are in some sense symbiotic: to understand the properties of real numbers

[7]Here again the axiom of choice is crucial, because if it were false, it could happen that there are as many measurable sets as there are Borel sets. See Moore [35] for a much deeper analysis of the role of the axiom of choice in the Lebesgue's measure problem.

completely, one has to use set theoretical ideas. The emergence of these regularity properties of real numbers in France, especially the Lebesgue measurability, whose development we will closely follow, is just further evidence of the relationship between set theory and real numbers. Thus we shall see, that all the regularity properties of real numbers, are very deeply connected to set theory, with Cantor able to foresee this as early as the 20th century.

2.2.3 Vitali's non-measurable set

Lebesgue's measure problem was settled for the measurable sets, but the question whether Lebesgue measure could measure every subset of the real line remained open[8]. Stated differently, are there sets that are not Lebesgue measurable?

Lebesgue was not the only one that developed measure theory from the works of Jordan and Borel. Italian mathematician Giuseppe Vitali, who worked in integration theory, developed a definition that is equal to Lebesgue's outer measure, and also discovered the measurable sets (Vitali 1904 [54]). However, he did not generalize the notion of the integral and after reading Lebesgue's work, Vitali published articles concerning Lebesgue's theory.

During the same period, German mathematician Ernst Zermelo published a proof [58] that every set can be well-ordered. The proof used a seemingly new, but questionable mathematical principle: the axiom of choice. However, this axiom was not actually new, because as noted above, it was used for example by Lebesgue and many other analysts[9]. The use of the axiom of choice was not made explicit before, and therefore Zermelo was the first to formulate it properly. Still, it was clear from the many implicit applications of the principle that it seemed to be a crucial part of the theory. It could even be considered as an inseparable part of it. Still, the axiom of choice provoked various reactions among mathematicians, and caused one of the most

[8]Such a measure is said to be *total*.

[9]Cantor seemed to accept axiom of choice already in 1880s as he stated in the Grundlagen [10] as "a law of thought" that every set could be well-ordered.

famous discussions among mathematicians on the validity of a mathematical principle in history (see Moore [34] for a through discussion of the axiom of choice). Even though the axiom was controversial in the mathematical community, its importance to the theory of real numbers cannot be underestimated[10]. As such, the metamathematical considerations were introduced into the theory of real numbers by adapting the axiom of choice.

The axiom of choice was involved even the Lebesgue's measure problem. Indeed, in 1905 [55], using the newly discovered principle, Vitali was able to show that the measure problem had no solution by constructing a non-measurable set, which is today known as a *Vitali set*. The proof used the following idea: Define an equivalence relation $\sim$ in the interval $[0,1]$ such that $x \sim y$ if and only if $x - y$ is a rational number. Using the axiom of choice we can select an element p from each equivalence class $A_{\sim}$. Name the set of elements p as B. Because the measure is translation invariant, the translates $B + q$ (here q is a rational number in the interval $[-1,1]$) all have the same measure. Also, the sets $B + q$ cover the interval $[0,1]$, and each has to be a subset of the interval $[-1,2]$. Therefore, the sum of their measures is between 1 and 3, and thus each translate cannot be of measure zero or have a positive measure. Vitali was well aware that the axiom of choice was not accepted by all mathematicians, and as a result, he noted in the end that "the possibility of measuring all the subsets of real line and ordering it cannot coexist".

After Vitali's impossibility result, two important questions arose that we shall consider thoroughly the rest of this work:

(I) If we accept the axiom of choice which sets of real numbers are measurable?

(II) How should we weaken the conditions on the Lebesgue's measure problem to get a measure that is total and still intuitive?

[10]Many mathematicians commented on the validity of the axioms. These include Hausdorff, Baire, Borel, Lebesgue, Poincaré, König, Peano and Brouwer among others (Moore [34]).

Both of these questions were studied by various mathematicians in the coming years. I will refer in the following to these questions simply with (I) and (II).

2.3 Measurability of sets of real numbers

2.3.1 Baire Classification

To understand the development on the question (I), we need to start from Baire's thesis [2] and his classification of functions. This classification, called the *Baire's classification*, starts with class 0 which contains all the continuous functions. The functions in the next class are those discontinuous functions that are pointwise limits of the sequences of the functions from the class 0. The class 2 is then those functions that are not in the class 0 or 1 but can be represented as a pointwise limit of the functions from those classes. In the same fashion one gets a class for every finite number n:

"A function is said to be of class n if it is the limit of a sequence of functions belonging to the classes $0, 1, 2, \ldots, n-1$ and if it does not itself belong to one of these classes." (Baire [1] p. 1622)

Baire does not stop at finitely numbered classes, but immediately continues by using transfinite numbers.

"We can go further, using the notion of transfinite number. If we have a sequence of functions, each of which belongs to one of the classes $0, 1, 2, \ldots, n, \ldots$ and if there exists a limit function which does not belong to any of these classes, we will say that it belongs to class ω. We give the possibility of the existence of functions belonging to class α, where α is any transfinite number of the second class of numbers." (Baire [1] p. 1622)

Those functions that belong to some Baire class are currently known as *Baire functions*. Some of the more obvious questions concerning the Baire functions were immediately answered by Baire: The Baire functions are closed under pointwise limits and there are functions that are not Baire. In fact, Baire showed that the set of all Baire functions has the cardinality of the continuum and thus form only a very small portion of all the real functions. One might suspect that perhaps after a particular Baire class, all the classes are empty i.e. that the Baire hierarchy of functions is not proper. However, Lebesgue managed to show in 1904 [22] that this is not the case, and therefore, all the classes are non-empty.

The following year, Lebesgue published the memoir [23] which established an important connection between Borel sets and Baire functions[11]. It was determined that the Borel sets are exactly the pre-images of Baire functions and thus the *hierarchy of Borel sets* is as follows: A set is of class n if there is an interval such that the set is a pre-image of the interval by a Baire function of a class n, and also that it is not a pre-image of a Baire function that is of class less than n. We immediately notice, that because the Baire hierarchy is proper, so is the Borel hierarchy.

Another important result that Lebesgue notes in the end of his memoir is that there exists a Lebesgue measurable set that is not a Borel set[12]. This result is a little step towards the solution of question (I) because there exists a measurable set that is not a Borel set. More general sets than Borel sets are exhausted by the measurable sets which, leads us to an error of Lebesgue's in his memoir and to the study of *Analytic sets*.

[11]Both Borel and Baire published books the same year but neither of them noticed the connection between Borel sets and Baire functions. For Lebesgue's result of this connection and for the characterization of the Baire functions see Medvedev [32] p.163 - 167.

[12]One example of such a construction is the following example which is due to Luzin [28]. Let A be a set of irrational numbers with continued factor representation

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}}$$

such that there exists an infinite subsequence a_{n_k}, where each a_{n_k} divides $a_{n_{k+1}}$. Then A is a non-Borel set that is measurable. Actually more is true: the set is also an example of an non-Borel set that is analytic (see next page for the definition of analytic set).

2.3.2 Analytic and projective sets

Lebesgue wrote in p.192 of his memoir that "The part of this set for which we have $\alpha \leq y \leq \beta$ is therefore Borel measurable, and thus is its projection on the manifold ...". By projection of a set $B \subset \mathbb{R}^{n+1}$ we mean the set

$$A = \{\langle x_1, x_2, \ldots, x_n \rangle \,|\, \exists b : \; \langle x_1, x_2, \ldots, x_n, b \rangle \in B\}.$$

This seemingly obvious claim by Lebesgue, that a projection of a Borel set is also a Borel set, is not necessarily true. And this, rather fortunate mistake, was noticed by Russian mathematician M. Suslin which led him to consider *A-sets*, now termed Analytic sets.

During 1915-1916, Nikolai Luzin and Wacław Sierpinski held seminars on many different topics in analysis. Among these was a seminar where the subject was the structure of Borel-measurable sets. Pavel Aleksandrov, a student of Luzin, who studied the Borel hierarchy was able to prove, for example, that each uncountable Borel set contains a perfect set[13] and thus has the cardinality of the continuum. The proof involved an operation called *A-operation*[14]. Mikhail Suslin, who attended the seminar noticed that the A-operation actually led to a larger class of sets than the Borel sets, namely the analytic sets, which are the projections of the countable intersections of open subsets of real numbers. (Cooke [7] p.311.) Suslin's 1917 sole publication [50] (he died in 1919) had a core assertion that a set is a Borel set if and only if, it and also its complement, are analytic. A note was added to the article by Luzin showing that every analytic set is Lebesgue measurable. Thus the class of Lebesgue measurable sets attained a new boundary and a natural

[13]In today's terms a set is said to have *a perfect set property* if it is either countable or has a non-empty perfect subset.

[14]A *defining system* is a family $\{X_s\}_s$ of sets indexed by finite sequences s of integers. The operation $A(\{X_s\}_s)$ is then defined to be:

$$x \in A(\{X_s\}_s) \text{ iff } (\exists f : \omega \to \omega)(\forall n \in \omega)(x \in X_{f|n})$$

where $f|n$ is the sequence determined by the first n values of f. Then $X \subset \mathbb{R}$ is called analytic iff $X = A(\{X_s\}_s)$ for some a defining system that consists of closed sets of reals (Kanamori [17] p.247).

question arose – can this boundary be taken a step further? Thus, a more complicated subsets of real numbers had to be considered.

There are two natural ways to attempt to move the boundary: to use the A-operation or to use projections. The latter is more interesting to the later development of the theory, but one result of the former is due to Luzin and Sierpinski. They proved in 1918 [25] that the Lebesgue measurable sets are closed under the A-operation, and thus iterating the A-operation on the analytic sets gives an extended class of Lebesgue measurable sets. Following that, their study on *co-Analytic sets* (complements of analytic sets), led them to take projection and complementation as the basic operations instead of using the A-operation. Using these operations on Borel sets Luzin and Sierpinski introduced in 1925 [26] and [41] the *projective hierarchy*. For clarity, we shall use more modern notation of projective hierarchy. A subset A of real number is projective if it belongs to Σ^1_n for some natural number n. More precisely, a subset A of real numbers is

- Σ^1_1 iff A is analytic,
- Π^1_1 iff A is co-analytic,
- Π^1_n iff the complement of A is Σ^n_1,
- Σ^1_{n+1} iff A is a projection of some set that is Π^1_n,
- Δ^1_n iff A is Σ^1_n and Π^1_n.

The basic properties were soon established and proved for the projective hierarchy. Luzin and Sierpinski used Cantor's diagonal argument in 1925 to prove that the hierarchy was proper and Sierpinski showed in 1928 [42] that the classes are closed under countable unions and intersections.

The next step in the theory of projective sets was to try to prove the basic regularity properties[15] for them. However these investigations faced heavy problems even at the bottom level of the projective hierarchy. The Σ^1_1 sets were known to satisfy all the basic regularity properties but for

[15]The term basic regularity properties means the properties that were mentioned in the introduction: The property of Baire, perfect set property and Lebesgue measurability.

the sets Π^1_1 the perfect set property could not be proved. The Lebesgue measurability of these sets did not reach much further: because a complement of a measurable set is also measurable the sets Π^1_1 were at least Lebesgue measurable, but again, the establishment of Lebesgue measurability of Σ^1_2 was problematic. Luzin notes this at the end of his [27] as follows:

"But these difficulties increase when one considers the projective sets of class two [Σ^1_2]: one does not know and one will never know if the projection of a two-dimensional co-analytic set (that is uncountable) has the power of the continuum, ... or even if it is measurable." (Luzin [27] p.1818-1819)

This forward-looking note refers to the metamathematical considerations that were starting to become increasingly prevalent as a result of axiomatization of set theory. The note by Luzin might have meant that they were potentially lacking the principles to prove the properties of projective sets. Indeed, this was verified by Kurt Gödel in the 1930s, leading us into a completely different facet of the problem: Lebesgue, Vitali, Luzin etc. were all analyzing real numbers and not the principles in the theory, where Gödel was instead interested in the metamathematics behind the theory, and analyzed the proof system.

Before going into Gödel's work, we should compare the work of the Russian school to the French analysts, and also reflect back on Cantor. The regularity properties that arose in France were studied by the Russian school in greater depth. Their point of view was completely different, as for the first time, the sets of real numbers and their properties were the main focus of study. It is reasonable to say that the subject they studied was descriptive set theory. However, the similarity of the approach to Cantor's studies was notable, as both the Russian school and Cantor had the same goal: to prove the regularity properties for as many sets of real numbers as possible. Indeed, Cantor was the one that proved that every closed set of real numbers has the perfect set property (a result that is clearly about descriptive set theory) and he thought and tried to generalize the result to hold on larger class of sets of real numbers than the closed sets. In a sense, Cantor had the correct idea for proceeding but did not have the French school's analytical tools the theory needed to be further developed. Cantor was perhaps too optimistic as he was trying to prove that the perfect set property (and so the continuum hypothesis) holds for all the sets of real numbers. This, as we shall see in the next chapter, was beyond the ability of standard principles that Cantor had available to him.

2.4 The limits of the principles

Before 1908, many "paradoxes"[16] of set theory arose and, together with a need for a theory where well-ordering theorem could be proved, served as a motivation to axiomatize the theory[17]. Indeed, in 1908 [59] Zermelo presented his first formulation of the axioms of set theory. We shall not review the axioms and their development in detail here except on one crucial point. To ease the notation, we will refer to the axioms with abbreviation ZF, the Zermelo-Fraenkel set theory without the axiom of choice, and by ZFC when the axiom of choice is included. In his axiomatization Zermelo used a term *definite proposition* that was not clearly defined before Thoralf Skolem's famous paper in 1922 [44], in which he adopts the first-order logic as the fundamental language to study set theory. Skolem defines definite proposition as "finite expression constructed from elementary propositions of the form $a \in b$ or $a = b$ by means of the five proposition mentioned [conjuction, disjunction, negation, universal and existential quantifications]". This rather simple observation was, however, crucial to the study of projective sets because it enabled Kazimierz Kuratowski and Alfred Tarski in 1931 [19] to notice the connection between descriptive set theory and logic. The basic set theoretical operations corresponded to the logical connectives and the existential

[16]Here I refer to the famous paradoxes like Russell's paradox and Burali-Forti's paradox. The use of quotation marks refers to the fact that these paradoxes were later to describe properties of sets rather than to be actual paradoxes. For example, Russell's paradox can be understood to describe that the universe of sets cannot possibly be a set itself.

[17]Needles to say, these two were not the only motivations for Zermelo to axiomatize set theory, see Moore [36] p. 149-160.

quantifier to projection. Thus, the projective sets could be studied using first order logic.

Gödel, who published his well-known incompleteness theorems in 1931 aimed to prove that generalized continuum hypothesis and the axiom of choice cannot be disproved using the axioms of ZF. This led him to define L, the *constructible universe* of sets. In the following we will discuss about Gödel's ideas only informally. The idea of the constructible universe is based on Zermelo's *cumulative hierarchy* of sets (Zermelo [60]). Zermelo's hierarchy consists of levels V_α, where α is an ordinal number, defined by transfinite induction.

- $V_0 = \emptyset$,
- $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, where $\mathcal{P}(V_\alpha)$ is the *power set* of V_α or in other words it is the set of all the possible subsets of the set V_α,
- and $V_\beta = \bigcup_{\alpha<\beta} V_\alpha$ when β is a limit ordinal[18].

Gödel defined his levels of constructible sets ([9] p.27) L_α in the same fashion differing only in the definitions of successor states: The subsets of L_α that are in $L_{\alpha+1}$ are those that are definable by a first-order formula of set theory, having parameters from L_α and the quantifiers are interpreted as ranging over L_α. A set is called *constructible* if it belongs to some L_α, and finally the totality of all the constructible sets i.e. the constructible universe, is called L. Thus, Zermelo's idea of cumulative hierarchy and Skolem's idea of using the language of first-order logic were combined. Indeed, Gödel showed that all the axioms of ZF, the generalized continuum hypothesis, and the axiom of choice hold in L. He showed that the claim that every set is constructible (often referred as $V = L$) holds in L and also proves that under these assumptions the above mentioned axioms hold (Solovay [48] p.10). This was Gödel's main goal, but as he notes, he actually proved a result about the measurability of Σ^1_2 sets as the following assertion also cannot be disproved within ZFC.

[18]Limit ordinal is an ordinal number that is not a successor of any smaller ordinal number.

"The existence of linear non-measurable sets such that both they and their complements are one-to-one projections of two-dimensional complements of analytic sets." (Gödel [9] p.26)

It is important to make a difference in the approaches by Gödel and the French and Russian analysts. Gödel proved that the standard principles, ZFC, cannot prove that the Δ_2^1 sets are measurable. Or in a more convenient terms, that there exists a *model* of ZFC where non-measurable Δ_1^2 sets exist. Thus, in terms of the projective hierarchy, the boundary of Lebesgue measurability had hit its limit at the bottom of the hierarchy, at least under the axioms of ZFC.

Nevertheless, for the first time, mathematicians were able to base their theory on the axioms of ZFC. Gödel had proven that these axioms cannot answer many of the questions pertaining to simple sets of real numbers, or settle questions on the very low level of Cantor's hierarchy of infinite cardinals. Thus one is led to think that set theory is more than the axioms of ZFC, as it seems, that ZFC does not contain the entire truth of sets. We will discuss this point in more detail below, where we mention multiple ways to strengthen the theory[19] of ZFC by introducing a notion of large cardinal which naturally arises from the measure problem.

A good way to understand why the introduction of ZFC was necessary is to refer back again to the ideas of Cantor. As noted before, it was Cantor who realised that to understand the structure of the real line, set theory must be involved. That said, the set theory that Cantor studied was much more informal than ZFC. In this case, the informality might not have been a negative. This is because mathematicians studying set theory after Cantor were limited by the axioms of ZFC. Thus, Cantor's informal definitions about sets allowed him to study the problems about real numbers much more freely. It is also important to realise that because set theory was not properly

[19] The phrase "strengthen our theory" is used here informally. It means to add new axioms to ZFC so that the theory $ZFC+''\text{new axiom}''$ is consistent (under the assumption that ZFC is itself consistent), and that this new theory can prove more than ZFC alone.

formalized, Gödel's independence results would not have been possible for Cantor to study. One may say that Cantor created set theory and realised its importance to real numbers, Zermelo axiomatized it and made it more formal and Gödel brought it back to the subsets of the real line as a different discipline than it was originally. This does not mean that Cantor was working on something more modest than Zermelo or Gödel. Rather, it is actually more reasonable to think that he had much more under investigation, as he was not limited by any particular set of axioms.

Considering the question (I) of the measure problem, we cannot consider it as answered, because as noted in the last section, the projective sets are simple subsets of real numbers and we still do not know if they are Lebesgue measurable or not. There are two rather obvious ways to approach the question. The first is to weaken the conditions of the measure that brings us to the question (II). The second is to modify our underlying principles[20]. We will first discuss the development of the former, but we will notice that both approaches are connected.

2.5 Modified measure problem

2.5.1 Finitely additive measures

There are a few ways to weaken the requirements of the measure problem. We shall consider two cases, weakening the countable additivity to finite additivity[21] or removing the translation invariance. The earliest considerations on this problem go back to 1914 to Germany when Felix Hausdorff started to analyze Lebesgue's measure problem in n-dimensional space. In his [11] Hausdorff weakens the requirement of countably additivity and considers measures that are only finitely additive. Intuitively replacing the countably additivity by finite additivity does not change the problem much because it

[20]Taking this path takes us quickly to the large cardinals, and as we will see, while we seek an answer to question (II), the large cardinals will still play a big part.

[21]Finite additivity simple means that the measure of the union of *finitely many* disjoint sets is the sum of the measure of these sets.

preserves mostly the natural geometric form of the measure and it is still applicable to at least some parts of analysis. However, resting heavily on the axiom of choice, Hausdorff managed to prove that no such measure is possible for $n \geq 3$. The proof is based on a decomposition of a sphere, or more precisely, on the contradiction that “a half of a sphere and a third of a sphere can be congruent” (Hausdorff [11] p.469). This paradoxical decomposition is now termed the *Banach-Tarski paradox*[22]. In this case, if the dimension of the underlying space is 3 or more, no progress is possible, but for the cases of line or the plane this limitation may give us new results. However, Hausdorff was not able to settle the problem for these cases $n = 1$ or $n = 2$.

Hausdorff’s ideas turned out the be fruitful, as Polish mathematician Stefan Banach continued his work on the measure problem in 1923 [3]. Banach managed to show that the answer for the two remaining cases is positive and so such a measure existed for a line and plane, with this measure extending the Lebesgue measure[23]. In the following years, the difference between the cases, when the dimension is 1 or 2 and when it is greater or equal to 3, was studied, and it turned out that the reason for the difference lies in the group of motions that is present in the higher dimensions[24].

2.5.2 Measurable cardinals

Later, in 1929, Banach was studied another generalization of the measure problem. He wanted to keep the countable additivity and instead gave up the translation invariance requirement, but he still worked within the interval $[0, 1]$. In a joint work with Kuratowski, Banach published that the measure problem did not have a solution even on this weaker form, at least

[22]The word paradox is often used to describe the discovery of Banach and Tarski. However the paradoxical nature of the statement disappears when one realizes that the decomposition of the sphere includes non-measurable sets.

[23]By saying that a measure extends Lebesgue measure we mean that if a set is Lebesgue measurable it is also measurable in the sense of this new measure, and also that the new measure and Lebesgue measure agree for Lebesgue measurable sets.

[24]See Wagon [56] for a proof of Banach’s theorem on the Hausdorff’s measure problem, and a detailed treatment of the Banach-Tarski paradox and the underlying motions.

if the continuum hypothesis is true:

"...we shall prove, assuming the continuum hypothesis, that the more general measure problem which is obtained by omitting condition I [translation invariance] and adding the condition that if X consists of a single point $m(X) = 0$ (a condition which obviously results from I and II [countable additivity]) - has no solution."[25] (Banach and Kuratowski [4] p.1)

Hausdorff's work in 1914 motivated Banach further generalizations, but at this point a core concept was still missing. In 1930, Banach came to realize that without the translation invariance, some of the geometric notion of the measure is lost, and it is natural to consider the problem in a different domain than the interval $[0, 1]$. This was a key observation, as indeed the main property one measures without the translation invariance is the cardinality of the set because in this case it is impossible to distinguish the difference between the sets of the same cardinality. The next generalization of the problem is the case where sets of arbitrary cardinality are considered, rather than just the unit interval $[0, 1]$. On the other hand, as the considered set now has arbitrary cardinality, the countable additivity seems too restrictive and it becomes natural to consider more general forms of additivity. The most straightforward generalization of countable additivity is the notion of κ-additivity: a measure is said to be κ-additive (where κ is a cardinal number) if for any pairwise disjoint sequence A_α $(\alpha < \lambda)$ of subsets of κ of length $\lambda < \kappa$, the measure of the union of these subsets is the sum of the measures. Indeed, under this terminology, $\aleph_1$-additivity is equivalent to countable additivity so the definition seems reasonable[26].

Now that the measure problem had achieved a more general framework by introducing arbitrary sets, much richer theory was possible. The first

[25]The condition that a singleton set has measure zero is to ensure that a trivial solution to the problem is excluded.

[26]Much more general forms of additivity cannot be required because if the set of cardinality κ is represented as a sum of singletons, and the measure is assumed to be more than κ additive, then the measure of the set is 0.

result in this setting was proved by Banach: without the restriction to the unit interval and with the generalized notion of additivity he proved in 1930, under the assumption of generalized continuum hypothesis[27], that

"If an $\aleph_\xi$-additive measure can be defined in a set E [the cardinality of E is assumed to be $\aleph_\xi$], then $\aleph_\xi$ is an weakly inaccessible cardinal number[28]." (Banach [5] s.98)

This result was achieved by some generalizations of the Lebesgue's measure problem: firstly by leaving out a part of the geometrical form of the measure when reducing the translation invariance[29], secondly by allowing more general sets than the subsets of real numbers and, thirdly, by using a more general notion of additivity. Thus, it may seem that the original goal of the measure problem has somehow been changed, as the question is no longer solely about the subsets of real numbers – or at least not directly. Indeed, the problem is now not just about the measure on the subsets of real numbers but rather about the measures on the cardinal of real numbers. Thus, instead of analysing the unit interval $[0, 1]$, we are analyzing the cardinal of real numbers $2^{\aleph_0}$, in the hopes of getting an answer to the original measure problem.

As is often the case with set theoretic principles, the generalized continuum hypothesis was not accepted by every mathematician, and it was a key factor in Banach's argument. Fortunately, soon it turned out that the assumption of the generalized continuum hypothesis was not in fact needed, as Polish mathematician Stanisław Ulam continued the work of Banach, and

[27] The generalized continuum hypothesis states that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for any ordinal number α.

[28] And uncountable cardinal number is said to be weakly inaccessible if it is not a successor of any other cardinal number and not a sum of smaller number of smaller cardinal numbers. So in a sense it cannot be broken into a smaller collection of smaller parts.

[29] Indeed, not all of the natural geometrical interpretation of the measure is gone. On one hand, the translation invariance is a geometric notion, but on the other hand, as the reader will notice, the generalisation of the measure still implies many geometric properties and is related to the standard Lebesgue measure.

improved the result (See Ulam [53]). Needless to say, this again opened up many avenues of further research. Ulam's and Banach's key result can be stated as follows: the cardinal of the set A that satisfies the measure problem, as defined by Banach, must be weakly inaccessible. Ulam was also able to show another result that is interesting for the measure problem, as he proved that in order to have a solution to it, a *weakly inaccessible cardinal* must exist that is less or equal to $2^{\aleph_0}$. In other words,

$$\aleph_0 < \kappa \leq 2^{\aleph_0},$$

where κ must be weakly inaccessible[30]. Hence, if the measure problem has a solution the continuum hypothesis must be violated. We return to this issue later.

To further the discussion, we adopt the modern terminology for Ulam's results. The cardinals that Banach and Ulam represented (the cardinals that satisfy the measure problem) are now identified as measurable cardinals, or more precisely:

A cardinal κ is called *measurable cardinal* if and only if there exists a κ-additive, non-trivial, $0-1$-valued measure on κ.

A cardinal κ is called *real valued measurable cardinal* if and only if there exists a κ-additive, non-trivial measure on κ.

These definitions and Ulam's results imply that a real valued measurable cardinal must be weakly inaccessible, all measurable cardinals are real valued measurable and a measurable cardinal is strongly inaccessible[31].

Cardinal notions, like measurable cardinal and strongly inaccessible cardinal, are currently called *large cardinals*. What are large cardinals? Informally, we mean a cardinal number that is large in the sense that it cannot

[30]Solovay improved Ulam's result in [47] so that κ is the κth weakly inaccessible cardinal and that κ is also weakly Mahlo.

[31]Strongly inaccessible cardinal is a cardinal that is weakly inaccessible and not a product of smaller amount of smaller cardinals.

be constructed in ZFC, i.e., that it cannot be reached by building up with the the standard operations defined in ZFC. In a sense, large cardinals are thus "above" ZFC. It was noted above that ZFC can be strengthened and, indeed, the theory formed by adding a large cardinal axiom (postulating existence of a large cardinal) adds consistency strength to ZFC. The theory ZFC + "a large cardinal $\aleph$ exists" proves the consistency of ZFC. This was discovered first by Zermelo in 1930 [60], as he showed that ZFC+"there exists an inaccessible cardinal" proves the consistency of ZFC.

Banach and Ulam discovered a framework for a far-reaching theory by introducing large cardinal notions. By analysing the measurable cardinals, we may find an answer to simple questions of real numbers: which subsets of real numbers are measurable? With this question on mind, we now turn ourselves to a new aspect of the problem, to measures on cardinals and to large cardinals. It is also important to recall that we are going to work with axioms that are "much above" the axioms of ZFC, and continue to attempt to answer questions that are seemingly at the low level of the set theoretic universe, i.e. the questions about real numbers[32]. This important matter shall be discussed in more detail below.

2.5.3 The consistency results in the measure problem

Turning back to the measure problem we can ask, using the new notions of large cardinals, if the cardinal of real numbers $2^{\aleph_0}$ is a measurable cardinal. Surely not, as $2^{\aleph_0}$ is not strongly inaccessible because it can be reached from below (from natural numbers) using the power set operation. This is because if there is a measurable cardinal κ, it is actually much "bigger" than inaccessible cardinals by Robert Solovay's result: he showed that there must exists κ strongly inaccessible cardinals less than κ[33]. This shows that

[32] When stating that large cardinals are "above" ZFC or that real numbers are at low level of the set theoretic universe, we need to do so with care, because large cardinals are connected to real numbers and to their properties. Thus I use quotation marks when I informally refer to the "size" of the large cardinals. To avoid confusion, we encourage the reader to think large cardinals only as a way to strengthen the theory of ZFC.

[33] See footnote 30.

there indeed is a "cap" between the cardinal of real numbers and the first measurable cardinal. Thus we can state informally that $2^{\aleph_0}$ is not even close of being measurable.

The cardinal of real numbers $2^{\aleph_0}$ cannot be a measurable cardinal, but can it still be a real valued measurable cardinal? In a sense this is true, or at least in some of the cases it may be true. We shall consider two such cases which were both solved by Solovay in late 1960s.

The first case is to work without the axiom of choice. In this case Solovay managed to build a model of

$ZF + DC +$ "*All sets of real numbers are Lebesgue measurable*" (Solovay [46]),

where DC denotes the axiom of dependent choice[34]. Thus it might be the case that $2^{\aleph_0}$ is a real valued measurable cardinal. To prove the result, Solovay had to postulate the existence of a strongly inaccessible cardinal[35], which is still a fairly mild assumption (with respect to consistency strength) at least in terms of large cardinal axioms. However, it is still much more than was available or agreeable at the beginning of the century, and so the leap to this result is huge for the first mathematicians attacking the measure problem, even if they would have had the tools available. The conclusion of this result would be intriguing to them, as the result actually implies that the axiom of choice is needed for the existence of a non-measurable set[36]. Therefore, the axiom of choice is no longer sufficient but also necessary.

But if we do not give up the axiom of choice, and hence start from ZFC, can we arrive at an answer? The answer is affirmative, as Solovay constructed a model for

[34]Dependent choice is a weaker form of the axiom of choice which is sufficient to develop most of the real analysis. A discussion about different choice principles can be found in [34]

[35]The existence of a strongly inaccessible cardinal is necessary for the Solovay's model, see Shelah [40].

[36]This result is of course assuming that the measure is still translation invariant.

ZFC + "Lebesgue measure has a countable additive extension μ defined on every set of reals".

To achieve these results, Solovay had to assume more than the consistency of ZFC and an existence of a inaccessible cardinal, as he postulated the existence of a measurable cardinal. As the measurable cardinals are very large, compared to the inaccessible cardinals, the leap becomes even bigger. We should not blindly accept the result from the philosophical point of view, but should instead consider whether the assumption of the existence of measurable cardinal could be weakened. Unfortunately, this is not possible, because Solovay's result was in fact an *equiconsistency* result, i.e., two theories were shown to have equal consistency strength. More precisely, two theories $ZFC+$ "A" and $ZFC+$ "B" are said to be equiconsistent if the consistency of $ZFC+$ "A" implies the consistency of $ZFC+$ "B" and vice versa. Solovay's results thus imply that the following theories are equiconsistent.

(I) $ZFC+$*"there exists a measurable cardinal"*

(II) $ZFC+$*"Lebesgue measure has a countable additive extension μ defined on every set of reals"*

(III) $ZFC+$*"there exists a real valued measurable cardinal"*

As a result, the existence of a measurable cardinal has to be postulated if we hope to extend the Lebesgue measure to every subset of real numbers. And hence, the axiom of the measurable cardinal is essentially connected to the measure problem. This might not be surprising, as the reader may recall, that measurable cardinals emerged from the measure problem in a natural way. Still, the result might seem a bit unintuitive, as measurable cardinals are "much larger" than the cardinal $2^{\aleph_0}$ of real numbers. The key observation to make, is that the cardinalities of the sets are not the most important factor, and instead it is their measures which expose the resemblance. This similarity can be seen already in Solovay's work, as he showed how to transform the two-valued measure on the cardinal κ to the Lebesgue measure on real numbers and the other way around. Stated differently, he showed that the two notions of measure are inseparable (N. Goldring [10] p.182.)

2.6 Large cardinals and the measure problem

The results so far have been merely consistency results, i.e., results in properties which can be proved or cannot be proved with certain selection of axioms. Hence, these results do not imply the existence of such elements in the universe of sets. On the other hand, we can still learn a lot from consistency results, with Gödel's result ending the search of more measurable sets than the analytic sets in the projective hierarchy (in terms of the axioms of the ZFC alone). As with the large cardinals, we were able to proceed in the measure problem, at least when considering the consistency results. Large cardinals may also help us to proof that more sets than the analytic sets are measurable in the universe of sets. However, before taking this step it is reasonable to consider if postulating an existence of a large cardinal is too strong an axiom, and perhaps one could improve the results without any large cardinal notions.

Using large cardinal axioms is not tempting at first, and the hope is that all the sets in the projective hierarchy can be proved to be measurable without relying on such axioms. We could try to achieve this by adding axioms to ZFC in such a way that our new theory is equiconsistent with it. This new theory would be of the same consistency strength as ZFC, and so the added axioms are not large cardinal axioms because a commonly agreed property for all large cardinal axioms is that they add consistency strength to ZFC.

As we noted earlier, more than analytic sets could be proven to be measurable in ZFC iterating Suslin's A-operation, but this does not help prove that more sets in the projective hierarchy are measurable. Still, using only the axioms of ZFC one can proceed forwards somewhat: R. Solovay proved that the sets that are provably in Δ^1_2 (so not all Δ^1_2 sets, only those that are provably in it) are Lebesgue measurable[37]. Nonetheless, by Gödel's result the provability requirement cannot be removed, and therefore this is the best result we can hope for.

In terms of the ZFC axioms, the measure problem is solved. This is still not enough, as we know nothing about the measurability of the sets in higher levels of the projective hierarchy. The information we have gathered along the way is also useful – we know we must seek new axioms to answer the questions about the measure problem. Of course, these new axioms do not have to be axioms that add consistency strength to ZFC. However, if they do not, we cannot advance much further. An example of an axiom, that does not add consistency strength to ZFC (and thus is not a large cardinal axiom), and proves that more set than analytic sets are measurable, is Martin's Axiom[38]. From this and from the negation of the continuum hypothesis, it follows by Donald Martin's and Solovay's result [30], that every Σ^1_2 set is Lebesgue measurable. The climb in the projective hierarchy still stops quickly as Saharon Shelah showed that the measurability of Σ^1_3 set proves the consistency of ZFC. As a result, we must add axioms that increase the consistency strength to ZFC to prove more sets to be measurable, and large cardinal axioms seem to be a natural way to accomplish this.

A result that supports the use of large cardinal axioms further is that assuming the measurability of Σ^1_3 sets implies the existence of inaccessible cardinal in L. This result, stemming from Shelah, tells us the following: it is not only that we need to add more consistency strength to the theory to prove more sets to be measurable, but also that assuming more sets to be measurable gives rise to a theory where large cardinals may indeed exist.

However, it is not immediately apparent that the large cardinal axioms

[37]A set E is provably in Δ^1_2 if there is a real number a, Σ^1_2 formula ϕ and Π^1_2 formula Φ such that $A = \{x \mid \phi(x,a)\} = \{x \mid \Phi(x,a)\}$. And in addition ZFC proves that the two formulas are equivalent.

[38]The exact statement of Martin's axiom can be found in Jech [15] p. 230. Informally it says that cardinals that are below $2^{\aleph_0}$ behave in some sense like $\aleph_0$. Martin's axiom is also implied by continuum hypothesis and consistent with ZFC+"the negation of continuum hypothesis". Thus many results that are implied by continuum hypothesis follow also from Martin's axiom, see Fremlin [8].

offer any solution to the measure problem, but the above discussions certainly hints to that direction. A logical place to start is from measurable cardinals, as they arose from the measure problem. This was also the starting point for Solovay who was the first to attain a result towards the solution of measure problem using large cardinals:

Assuming the existence of a measurable cardinal all Σ^1_2 and Π^1_2 sets are Lebesgue measurable.[39]

This represents not only a consistency result, but also a result concerning the truth in the set theoretic universe where measurable cardinals exists. This means that we can finally declare that we have progressed the limit of what can be proved: more than analytic sets can now be measurable. At this point, one should recall how large the gap between Luzin's result of measurability of analytic sets and the Solovay's measurability result of Σ^1_2 sets is, necessitating the emergence of an entirely new theory.

At first blush, Solovay's result might not seem remarkable, because as discussed earlier, the same result followed from a theory that is equiconsistent with ZFC. In fact, the true importance of Solovay's result is that large cardinals offer us a way to proceed the climb in the projective hierarchy. And, as proved by Jack Silver [43], the result of Solovay's is optimal, as the existence of a measurable cardinal is not enough to prove the Lebesgue measurability of Δ^1_3 sets. As such, we cannot hope for a generalization of Solovay's result in terms of measurable cardinals, and new large cardinal notions should instead be considered.

Indeed, by introducing the notion of the *Woodin cardinal*, one is able to proceed as Shelah and Hugh Woodin were able to prove that

Assuming the existence of infinitely many Woodin cardinals, all the sets in projective hierarchy are Lebesgue measurable. (Shelah and Woodin

[39]For Solovay's arguments on establishing the perfect set property for the Σ^1_2 sets see his 1969 [45]. The same arguments also establish the measurability, more details can be found in Kanamori [18] p.178.

[39])

This assertion finally brings us to the end of the climbing in the projective hierarchy, and also clearly shows that the large cardinals provide us the tools to achieve this.

What exactly are the Woodin cardinals, and are they essentially connected to the measure problem, as measurable cardinals were? This indeed is the case, but it might be hard to extract this from the definition of a Woodin cardinal, as it is technically challenging. The standard definition includes mappings that are called *elementary embeddings*[40]. These mappings are functions between two structures that are both total and truth-preserving. In set theory, these elementary embeddings primarily occur between the universe of all sets V and some transitive class M[41,42]. Although it may seem that embeddings are entirely separate from measures, there is in fact a way to obtain measures from elementary embeddings, and vice versa. For example, the definition of a measurable cardinal could also be presented in terms of elementary embeddings: a cardinal κ is measurable if and only if there exists a transitive set M and a non-trivial (not identity) embedding $j : V \to M$ with κ as its *critical point* (a critical point of an embedding j is an ordinal κ such that $j(\alpha) = \alpha$ for all $\alpha < \kappa$ and $j(\kappa) \neq \kappa$)[43].

These two notions, the measurable cardinal and Woodin cardinal, are indeed related, as Woodin cardinals are generalizations of measurable cardinals. There are at least two ways to investigate this generalization: the

[40]See Kanamori [18] p. 360 for the definition of Woodin cardinal.

[41]A class or a set M is said to be transitive if and only if every element of M is also a subset of M.

[42]To be more precise let V be the universe of all sets and M some transitive class. Then $j : V \to M$ is an elementary embedding iff

$$V \models \phi(a_1, \dots, a_n) \Rightarrow M \models \phi(j(a_1), \dots, j(a_n)),$$

where $\phi(a_1, \dots, a_n)$ is a formula of language of set theory.

[43]To get a measure from elementary embedding set $\mu(A) = 1 \Leftrightarrow k \in j(A)\ \forall A \subset \kappa$. And conversely, to get embeddings from measures, one uses the ultrapower construction (J. Steel [49]).

first is to discern that Woodin cardinals are limits of measurable cardinals (however, they are not measurable themselves), and the second is to consider that we are aware of multiple measures that measure smaller cardinals[44].

It is also known, by result of Martin and John Steel [31], that assuming only finitely many Woodin cardinals exists, is not enough to prove that all the projective sets are Lebesgue measurable. More precisely, they proved that the existence of finitely many Woodin cardinals is consistent with the existence of a non-measurable Σ^1_{n+2} set. This means, at least to some extent, that the assumption of infinitely many Woodin cardinals is optimal. Also, some results considering more complicated sets of real numbers than projective sets are known. Under even stronger large cardinal hypothesis, the existence of a supercompact cardinal, all the sets in $L(\mathbb{R})$ are measurable (Woodin [57]). In other words, the existence of a supercompact cardinal guarantees that all the "naturally" definable sets of real numbers are Lebesgue measurable.

These results might make one believe that large cardinals eventually provide us with a framework to prove the existence of a total (non-translation invariant) measure. This however is not true, as we have already discussed that the existence of such a measure violates the continuum hypothesis, and by Azriel Levy's and Solovay's result, large cardinals do not affect the status of continuum hypothesis (Levy and Solovay [24]).

Now that we have concluded our search for truth concerning the Lebesgue measurability of simple definable subsets of real numbers, we can confidently state, that we have found one of the most remarkable connections between the "small" sets (subsets of real numbers) and very "large" sets (large cardinals).

2.7 Conclusion

Lebesgue's natural question about the measurability of the subsets of real numbers awoke questions following Vitali's result of existence of non-measurable sets. Two decades later, the problem of measuring sets was re-

[44] In the definition of the Woodin cardinal we have many different elementary embeddings with critical points that are strictly less than the Woodin cardinal. Thus we have multiple measures that measure smaller cardinals.

vealed to be present in a very simple subsets of real numbers, namely, in the projections of analytic sets. This, together with the axiomatization of set theory, allowed mathematicians to obtain a clearer view into the underlying situation. It was revealed that some key ideas about the universe of sets were missing, which ultimately led to the considerations of large cardinal axioms.

The theory of large cardinals emerged in a natural way to the measure problem, and the analysis of the measure problem (questions I and II) led to an area of study that was entirely different than what originally expected[45]. The analysis on the question (I) culminated in the Solovay's result: assuming existence of a measurable cardinal, the sets Σ^1_2 are Lebesgue measurable. And the analysis on the question (II) in Solovay's result on the consistency strength of the theory, with extensions of Lebesgue measure.

The two questions were thus seen to be connected, on a very deep level of the theory: in the large cardinal axioms. The set theoretic universe, as the French analysts understood it, was upturned. We should not restrict ourselves to the standard axioms of ZFC, as it clearly does not capture the whole truth about sets, and instead one should search within the large cardinal axioms to obtain more answers to simple questions concerning real numbers[46]. However, this does not mean that the two are separate, or that they must be far apart, because as we have seen, real numbers and the large cardinals are certainly connected.

[45] I have not touched in this thesis the questions of the existence of measurable cardinals or even any large cardinal. For a standard discussions about this reader should see for example Kanamori [18] or Penelope [29]. For another perspective in terms of Cantor's Absolute the reader should see Hauser [13]

[46] If one tends to think large cardinals as something "huge" the last statement could be informally stated as: The answers we are looking cannot be found when looking at the things in ZFC, which we can understand as looking from "below", but rather we should take a new viewpoint on the sets of real numbers and seek an answer from "above".

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