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Conditional-Mean Hedging Under Transaction Costs in Gaussian Models

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We consider so-called regular invertible Gaussian Volterra processes and derive a formula for their prediction laws. Examples of such processes include the fractional Brownian motions and the mixed fractional Brownian motions. As an application, we consider conditional-mean hedging under transaction costs in Black–Scholes type pricing models where the Brownian motion is replaced with a more general regular invertible Gaussian Volterra process.

Keywords: delta-hedging; option pricing; prediction; transaction costs.

1. Introduction

We consider discrete imperfect hedging under proportional transaction costs in Black–Scholes type pricing models where the asset price is driven by a relatively general Gaussian process; a so-called regular invertible Gaussian Volterra process. These are continuous Gaussian processes that are non-anticipative linear transformations of continuous Gaussian martingales.

For European vanilla type options we construct the so-called conditional-mean hedge. This means that at each trading time the value of the conditional mean of the discrete hedging strategy coincides with the frictionless price. By frictionless we mean the continuous-trading hedging price without transaction costs. The key ingredient in constructing the conditional mean hedging strategy is a representation for the regular conditional laws of regular invertible Gaussian Volterra processes

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*Typeset names in 8 pt roman, uppercase. Use the footnote to indicate the present or permanent address of the author.

which we provide in Section 4. Let us note that in our models there may be arbitrage strategies with continuous trading without transaction costs, but not with discrete trading strategies, even in the absence of trading costs.

For the classical Black–Scholes model driven by the Brownian motion, the study of hedging under transaction costs goes back toLeland (1985). See also Denis & Kabanov (2010) and Kabanov & Safarian (2009) for a mathematically rigorous treatment. For the fractional Black–Scholes model driven by the long-range dependent fractional Brownian motion, the study of hedging under transaction costs was studied in Azmoodeh Azmoodeh (2013). In the series of articles Shokrollahi *et al.* (2016), Wang (2010a,b), Wang *et al.* (2010a,b) the discrete hedging in the fractional Black–Scholes model was studied by using the economically dubious Wick–Itô–Skorohod interpretation of the self-financing condition. Actually, with the economically solid forward-type pathwise interpretation of the self-financing condition, these hedging strategies are valid, not for the geometric fractional Brownian motion, but for a geometric Gaussian process where the driving noise is a Gaussian martingale with the same variance function as the corresponding fractional Brownian motion would have, see Gapeev *et al.* (2011). Our approach here builds on the works Sottinen & Viitasaari (2017) and Shokrollahi & Sottinen (2017). The novelty of this note is twofold: First, we extend the results to a more general class of Gaussian processes than just the long-range dependent fractional Brownian motions, thus allowing more flexible models that can allow different features such as non-stationary increments, short range dependence, or other stylised facts one wants to include to the model. Second, we emphasize the models where there exists a non-trivial quadratic variation. This makes the formulas and the analysis very different from the long-range dependent fractional Brownian case. Indeed, in the case of the fractional Brownian motion with $H > \frac{1}{2}$, the quadratic variation vanishes and consequently, the formulas simplifies significantly. For example, in this case the hedging strategy ϕ_t depends only on the spot, i.e. $\phi_t = \phi(S_t)$, while in our case it depends also on time, i.e. $\phi_t = \phi(t, S_t)$.

The rest of the paper is organized as follows: In Section 2 we introduce our pricing model with a regular invertible Gaussian Volterra process as the driving noise, and develop a transfer principle for the noise. In Section 3 we investigate arbitrage and hedging in our pricing models. In Section 4 we provide prediction formulas for the driving noise and for Markovian functionals of the asset price. Finally, in Section 5 we provide formulas for conditional-mean hedging under transaction costs.

2. Pricing Model with Invertible Gaussian Volterra Noise

Let $T > 0$ be a fixed time of maturity of the contingent claim under consideration. We are interested in imperfect hedging in a geometric Gaussian model where the discounted risky asset follows the dynamics

$$\frac{\mathrm{d}S_t}{S_t} = \mathrm{d}\mu(t) + \mathrm{d}X_t, \quad t \in [0, T], \tag{2.1}$$

where $\mu \colon [0,T] \to \mathbb{R}$ is a known excess return of the asset and X is a driving Gaussian noise, and the stochastic differential equation is understood in a pathwise sense. We assume that μ is continuous with bounded variation. We note that here we only assume the dynamics (2.1) formally. However, it turns out that the solution S exists and is unique (cf. Section 3). For the noise X we assume that it is continuous and centered with $X_0 = 0$ and covariance function

$$R(t,s) = \mathbb{E}\left[X_t X_s\right], \quad s,t \in [0,T]. \tag{2.2}$$

To analyze the pricing model (2.1), we make the following rather technical Definition 2.1 that ensures the invertible Volterra representation and continuous quadratic variation for the noise process X. We note that Definition 2.1 is not very restrictive: many interesting Gaussian models satisfy it (see Example 2.1 below).

Recall that a kernel $K(t,u)$ is called Volterra kernel if $K(t,u) = 0$ for all $u > t$.

Definition 2.1 (Regular Invertible Gaussian Volterra Process). A centered Gaussian process over an interval $[0,T]$ with covariance function R is a *regular invertible Gaussian Volterra process* if

(1) There exists a continuous increasing function $m \colon [0,T] \to \mathbb{R}_+$ and a Volterra kernel $K \in L^2([0,T]^2, \mathrm{d}m \times \mathrm{d}m)$ such that: K is non-decreasing in the first variable, for each $s \in [0,T]$ the mapping $t \mapsto K(t,s)$ is continuously differentiable on (s,T), i.e. the derivative exists and is continuous, and the mapping $t \mapsto K(t,t)$ is continuously differentiable on $[0,T]$, and

$$R(t,s) = \int_0^{t \wedge s} K(t,u) K(s,u) \, \mathrm{d}m(u). \tag{2.3}$$

(2) For each $t \in [0,T]$, the equation

$$\mathrm{K}^*[f](s) = 1_{[0,t)}(s) \tag{2.4}$$

has a ($\mathrm{d}m$-almost everywhere) unique solution, where

$$\mathrm{K}^*[f](s) = f(s) K(T,s) + \int_s^T [f(t) - f(s)] \frac{\partial K}{\partial t}(t,s) \, \mathrm{d}t. \tag{2.5}$$

Example 2.1 (Examples and Counterexamples).

(1) Obviously, any continuous Gaussian martingale M is a regular invertible Gaussian Volterra process. Indeed, since M is continuous, so is the bracket $\langle M \rangle$. Consequently, we can take $m = \langle M \rangle$ and $K(t,s) = 1_{t>s}$.

(2) Fractional Brownian motions with Hurst index $H \in [1/2, 1)$ are regular invertible Gaussian Volterra processes. Indeed, this follows from the already well-known Volterra representation with respect to the standard Brownian motion; see e.g. Mishura (2008), Section 1.8, for details.

(3) Mixed fractional Brownian motions with Hurst index $H \in [1/2, 1)$ are regular invertible Gaussian Volterra processes. See Cai *et al.* (2016) for details.

(4) Fractional or mixed fractional Brownian motions with Hurst index $H \in (0, 1/2)$ are not regular invertible Gaussian Volterra processes, since they have infinite quadratic variation, cf. Lemma 3.1 below.

(5) The Gaussian slope $X_t = t\xi$, where ξ is a standard Gaussian random variable is an invertible Gaussian Volterra process in the sense that it is generated non-anticipatively from a Gaussian martingale. It is not regular, however, since the generating martingale cannot be continuous due to the jump in the filtration of X at zero, cf. Theorem 2.1 below.

We note that we have the following isometry for all step-functions f and g:

$$\mathbb{E}\left[\int_0^T f(t)\mathrm{d}X_t \int_0^T g(t)\mathrm{d}X_t\right] = \int_0^T \mathrm{K}^*[f](t)\mathrm{K}^*[g](t)\,\mathrm{d}m(t). \qquad (2.6)$$

By using this isometry, we can extend the Wiener-integral with respect to X to the closure of step-functions under this isometry; we refer to Sottinen & Viitasaari (2016) for details.

Write

$$K^{-1}(t, s) = (\mathrm{K}^*)^{-1}\left[1_{[0,t)}\right](s). \qquad (2.7)$$

Theorem 2.1 (Invertible Volterra Representation). *Let X be a continuous regular invertible Gaussian Volterra process. Then the process*

$$M_t = \int_0^t K^{-1}(t, s)\,\mathrm{d}X_s, \quad t \in [0, T], \qquad (2.8)$$

is a continuous Gaussian martingale with bracket m, and X can be recovered from it by

$$X_t = \int_0^t K(t, s)\,\mathrm{d}M_s, \quad t \in [0, T]. \qquad (2.9)$$

The martingale M in Theorem 2.1 is called a *fundamental martingale*.

Proof. By using the definition of K^{-1} together with the isometry (2.6) we get

$$\mathbb{E}\left[M_t M_s\right] = \mathbb{E}\left[\int_0^T (\mathrm{K}^*)^{-1}\left[1_{[0,t)}\right](u)\mathrm{d}X_u \int_0^T (\mathrm{K}^*)^{-1}\left[1_{[0,s)}\right](u)\mathrm{d}X_u\right]$$

$$= \int_0^T \mathrm{K}^*\left[(\mathrm{K}^*)^{-1}\left[1_{[0,t)}\right]\right](u)\mathrm{K}^*\left[(\mathrm{K}^*)^{-1}\left[1_{[0,s)}\right]\right](u)\,\mathrm{d}m(u)$$

$$= \int_0^T 1_{[0,t)}(u)1_{[0,s)}(u)\,\mathrm{d}m(u)$$

$$= m(t \wedge s),$$

which implies that M is a martingale. It remains to prove the continuity of M. However, this follows from the continuity of m. Indeed, by defining $\tilde{M}_t = W_{m(t)}$, where W is a Brownian motion, we observe that $\tilde{M}$ is continuous and has the same finite dimensional distributions as M. This implies that M is continuous. $\qquad \square$

Remark 2.1 (Continuity). We remark that we assumed the continuity of X *a priori*. In general, the process X is always L^2-continuous. Indeed, this follows directly from the Itô isometry. However, this does not necessarily imply almost surely continuous sample paths, as the modulus of continuity in L^2 depends on the function m which in general may behave badly. On the other hand, if m is absolutely continuous with respect to the Lebesgue measure, then X is even Hölder continuous.

3. Quadratic Variation and Robust Hedging

The form of the solution of risky-asset dynamics (2.1) depend on the quadratic variation of the noise process X. Recall that the (pathwise) quadratic variation of a process X is defined as

$$q^2(t) = \langle X \rangle_t = \lim_{n \to \infty} \sum_{t_k^n \leq t} \left(X_{t_k^n} - X_{t_{k-1}^n} \right)^2, \tag{3.1}$$

where $\{t_0^n = 0 < t_1^n < \cdots < t_n^n = T\}$ is a sequence of partitions of $[0, T]$ such that $\max_k |t_k^n - t_{k-1}^n| \to 0$. We remark that in general the existence or the value of the quadratic variation may depend on the chosen sequence of partitions. In our case however, it does not.

Lemma 3.1 (Quadratic Variation). *For a regular invertible Gaussian Volterra process the quadratic variation always exists along arbitrary sequence of partitions. Furthermore, it is deterministic, independent of the chosen sequence of partitions, and given by*

$$q^2(t) = \int_0^t K(s, s)^2 \, \mathrm{d}m(s). \tag{3.2}$$

Proof. By (Viitasaari 2015 Theorem 3.1), the convergence of quadratic variation of a Gaussian process X holds also in L^p for any $p \geq 1$. Suppose first that the quadratic variation is deterministic. Then, by using representation (2.9) we obtain that

$$\mathbb{E}\left[(X_t - X_{t-\Delta t})^2 \right]$$
$$= \int_0^T \left(K(t, u) - K(t - \Delta t, u) \right)^2 \, \mathrm{d}m(u)$$
$$= \int_{t-\Delta t}^t K(t, u)^2 \, \mathrm{d}m(u) + \int_0^{t-\Delta t} \left(K(t, u) - K(t - \Delta t, u) \right)^2 \, \mathrm{d}m(u),$$

where we have used the fact that K is a Volterra kernel. As here Δt is arbitrary, we can consider arbitrary sequence of partitions. As a result, the claim follows for deterministic quadratic variations from this by using Taylor's approximation for the kernel in the latter integral.

It remains to prove that the quadratic variation is deterministic along arbitrary sequence of partitions. By (Viitasaari 2015 Theorem 3.1), it suffices to prove that

$$\max_{1 \le j \le n} \sum_{t_k^n \le t} \mathbb{E}\left[\left(X_{t_k^n} - X_{t_{k-1}^n}\right)\left(X_{t_j^n} - X_{t_{j-1}^n}\right)\right] \to 0. \tag{3.3}$$

Let $k > j$. Representation (2.9) together with the Itô isometry and the fact that K is Volterra kernel yields

$$\mathbb{E}\left[\left(X_{t_k^n} - X_{t_{k-1}^n}\right)\left(X_{t_j^n} - X_{t_{j-1}^n}\right)\right]$$

$$= \int_{t_{j-1}^n}^{t_j^n} \left(K(t_k^n, u) - K(t_{k-1}^n, u)\right) K(t_j^n, u)\, dm(u)$$

$$+ \int_0^{t_{j-1}^n} \left(K(t_k^n, u) - K(t_{k-1}^n, u)\right)\left(K(t_j^n, u) - K(t_{j-1}^n, u)\right) dm(u).$$

For the first term we use the fact that $t \mapsto K(t, u)$ is increasing together with the bound $K(t_j^n, u) \le K(T, u)$. Hence we observe that summing with respect to either of the variables and letting $\max_k |t_k^n - t_{k-1}^n| \to 0$ yields convergence towards zero. For example, we have

$$\sum_{t_j^n \le t} \int_{t_{j-1}^n}^{t_j^n} \left(K(t_k^n, u) - K(t_{k-1}^n, u)\right) K(t_j^n, u)\, dm(u)$$

$$\le \sum_{t_j^n \le T} \int_{t_{j-1}^n}^{t_j^n} \left(K(t_k^n, u) - K(t_{k-1}^n, u)\right) K(T, u)\, dm(u)$$

$$= \int_0^T \left(K(t_k^n, u) - K(t_{k-1}^n, u)\right) K(T, u)\, dm(u) \to 0.$$

For the second term, it suffices to observe

$$\int_0^{t_{j-1}^n} \left(K(t_k^n, u) - K(t_{k-1}^n, u)\right)\left(K(t_j^n, u) - K(t_{j-1}^n, u)\right) dm(u)$$

$$\le \int_0^T \left(K(t_k^n, u) - K(t_{k-1}^n, u)\right)\left(K(t_j^n, u) - K(t_{j-1}^n, u)\right) dm(u).$$

Hence summing with respect to either of the variables and letting $\max_k |t_k^n - t_{k-1}^n| \to 0$ we get (3.3). $\qquad\square$

By Lemma 3.1 and Föllmer (1981), the solution to the stochastic differential equation (2.1) defining the discounted risky asset price is given by

$$S_t = S_0 \exp\left\{\mu(t) - \frac{1}{2}q^2(t) + X_t\right\} \tag{3.4}$$

and the quadratic variation of S is

$$\langle S \rangle_t = \int_0^t S_s^2\, dq^2(s). \tag{3.5}$$

Write

$$q^2(s,t) = q^2(t) - q^2(s). \tag{3.6}$$

Suppose $q^2(t)$ is non-vanishing on every interval. Then it follows from the robust replication theorem of Bender *et al.* (2008) that the pricing model is free of arbitrage under so-called allowed strategies and replications of vanilla claims are robust in the sense that, as replicating strategies are involved, one can replace X with a Gaussian martingale with bracket $q^2(t)$. Thus, we have the following proposition:

Proposition 3.1 (Robust Hedging). *Let $f(S_T)$ be a European claim such that $f(S_T) \in L^2(\Omega)$ and f is continuous. Then its Markovian replicating strategy is given by the delta-hedge*

$$\pi_t = \frac{\partial v}{\partial x}(t, S_t), \tag{3.7}$$

where

$$v(t, S_t) = \int_{-\infty}^{\infty} f\left(S_t e^{-\frac{1}{2}q^2(t,T) + q(t,T)z}\right) \phi(z) \mathrm{d}z \tag{3.8}$$

is the value of the replicating strategy π at time t. Here $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ is the density function of a standard normal random variable.

Proof. We make use of the robust replication theorem 5.4 of Bender *et al.* (2008). First of all, note that since S is continuous, cf. Example 5.6 in Bender *et al.* (2008) and Remark 3.1 below, it suffices to check condition (H) of Bender *et al.* (2008). However, as G can be viewed as a time-changed Brownian motion, this follows simply from a time change. Now theorem 5.4 of Bender *et al.* (2008) gives

$$V_t^\pi = \mathbb{E}_{\mathbb{Q}}\left[f(S_T) \,\middle|\, \mathcal{F}_t\right], \tag{3.9}$$

where, under $\mathbb{Q}$, the price process S is the exponential martingale driven by a Gaussian martingale G with bracket q^2.

By equality of filtrations of G and S driven by G, we have

$$V_t^\pi = \mathbb{E}\left[f\left(S_0 e^{G_T - \frac{1}{2}q^2(T)}\right) \,\middle|\, \mathcal{F}_t^G\right]$$
$$= \mathbb{E}\left[f\left(S_t e^{-\frac{1}{2}q^2(t,T) + (G_T - G_t)}\right) \,\middle|\, \mathcal{F}_t^G\right].$$

The claim follows from this, since $G_T - G_t$ is independent of $\mathcal{F}_t^G$ and is Gaussian with zero mean and variance $q^2(t,T)$. $\qquad\square$

Remark 3.1 (Black–Scholes Type BPDE). If q^2 is absolutely continuous with respect to the Lebesgue measure then the European vanilla option $f(S_T)$ can be replicated by solving its time-value from the Black–Scholes type backward partial differential equation

$$\frac{\partial v}{\partial t}(t,x) + \frac{1}{2}x^2 \frac{\mathrm{d}q^2}{\mathrm{d}t}(t) \frac{\partial^2 v}{\partial x^2}(t,x) = 0, \tag{3.10}$$

$$v(T,x) = f(x). \tag{3.11}$$

Remark 3.2 (Vanishing Quadratic Variation). Proposition 3.1 remains formally true for $q^2 \equiv 0$. However, in this case the replicating strategy is very simple:

$$\pi_t = f'(S_t). \tag{3.12}$$

Remark 3.3 (Simple Arbitrage). If the quadratic variation measure q^2 vanishes on some interval, then there are simple arbitrage opportunities. Indeed, suppose $q^2(s,t) = 0$ for some $0 \le s < t \le T$. Then, by Bender *et al.* (2007) and Bender *et al.* (2011),

$$(S_t - S_s)^+ = \int_s^t 1_{[S_s,\infty)}(S_u) \, \mathrm{d}S_u. \tag{3.13}$$

So, a buy-and-hold-when-expensive strategy would generate arbitrage.

4. Prediction

Prediction of the asset price or the noise is possible because all the filtrations $\mathcal{F}_t^S$, $\mathcal{F}_t^X$ and $\mathcal{F}_t^M$ are the same, and for regular invertible Gaussian Volterra processes we can use the theorem of Gaussian correlations in an explicit manner. Write

$$\hat{X}_t(u) = \mathbb{E}\left[X_t \,\middle|\, \mathcal{F}_u^X\right], \tag{4.1}$$

$$\hat{R}(t,s|u) = \mathbb{C}\mathrm{ov}\left[X_t, X_s \,\middle|\, \mathcal{F}_u^X\right]. \tag{4.2}$$

Theorem 4.1 (Prediction). *Let X be a regular invertible Gaussian Volterra process with fundamental martingale M. Then the conditional process $X_t(u) = X_t|\mathcal{F}_u^X$, $t \in [u,T]$, is Gaussian with $\mathcal{F}_u^X$-measurable mean*

$$\hat{X}_t(u) = X_u - \int_0^u \Psi(t,s|u) \, \mathrm{d}X_s, \tag{4.3}$$

where

$$\Psi(t,s|u) = (\mathrm{K}^*)^{-1} \left[K(t,\cdot) - K(u,\cdot)\right](s), \tag{4.4}$$

and deterministic covariance

$$\hat{R}(t,s|u) = R(t,s) - \int_0^u K(t,v)K(s,v) \, \mathrm{d}m(v). \tag{4.5}$$

Proof. Consider first the conditional mean. By Theorem 2.1 together with the fact that K is a Volterra kernel we obtain

$$\begin{aligned}
\hat{X}_t(u) &= \mathbb{E}\left[\int_0^t K(t,s) \, \mathrm{d}M_s \,\middle|\, \mathcal{F}_u^M\right] \\
&= \int_0^u K(t,s) \, \mathrm{d}M_s \\
&= \int_0^u K(u,s) \, \mathrm{d}M_s - \int_0^u \left[K(u,s) - K(t,s)\right] \mathrm{d}M_s \\
&= X_u - \int_0^u \left[K(u,s) - K(t,s)\right] \mathrm{d}M_s.
\end{aligned}$$

The conditional expectation formula follows from this by using the isometric definition of Wiener integration with respect to X.

Consider then the conditional covariance. By Theorem 2.1 and calculations above

$$
\begin{aligned}
\hat{R}(t,s|u) &= \mathbb{E}\left[\int_u^t K(t,v)\,\mathrm{d}M_v \int_u^s K(s,v)\,\mathrm{d}M_v \,\Big|\, \mathcal{F}_u^M\right] \\
&= \mathbb{E}\left[\int_u^t K(t,v)\,\mathrm{d}M_v \int_u^s K(s,v)\,\mathrm{d}M_v\right] \\
&= \int_u^{t\wedge s} K(t,v)K(s,v)\,\mathrm{d}m(v) \\
&= \int_0^{t\wedge s} K(t,v)K(s,v)\,\mathrm{d}m(v) - \int_0^u K(t,v)K(s,v)\,\mathrm{d}m(v) \\
&= R(t,s) - \int_0^u K(t,v)K(s,v)\,\mathrm{d}m(v). \qquad\qquad \square
\end{aligned}
$$

Write

$$
\hat{\rho}(t|u) = \sqrt{\hat{R}(t,t|u)}, \tag{4.6}
$$

$$
\beta(u,t) = \mu(u,t) - \frac{1}{2}q^2(u,t) \tag{4.7}
$$

Then

$$
S_t = S_u \mathrm{e}^{\beta(u,t)+(X_t-X_u)} \tag{4.8}
$$

and

$$
\mathbb{V}\mathrm{ar}[X_t - X_u \,|\, \mathcal{F}_u] = \hat{\rho}^2(t|u). \tag{4.9}
$$

The following corollary is the key result that allows us to calculate the conditional-mean hedging strategies in Section 5.

Corollary 4.1 (Prediction). *Let $0 \le u \le t \le T$. Let $f\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ be such that $f(t,S_t)$ is integrable. Let ϕ be the standard Gaussian density function. Then*

$$
\mathbb{E}\left[f(t,S_t)\,\Big|\, \mathcal{F}_u^S\right] = \int_{-\infty}^{\infty} f\left(t, S_u \mathrm{e}^{\beta(u,t)-\int_0^u \Psi(t,s|u)\mathrm{d}X_s+\hat{\rho}(t|u)z}\right)\phi(z)\mathrm{d}z. \tag{4.10}
$$

Proof. Given Theorem 4.1, the equality of filtrations and the Föllmer–Itô formula, the claim follows from straightforward calculations:

$$
\begin{aligned}
\mathbb{E}\left[f(t,S_t)\,\Big|\, \mathcal{F}_u^S\right] &= \mathbb{E}\left[f\left(t, S_u\mathrm{e}^{\beta(u,t)+(X_t-X_u)}\right)\,\Big|\, \mathcal{F}_u^X\right] \\
&= \int_{-\infty}^{\infty} f\left(t, S_u\mathrm{e}^{\beta(u,t)+\left(\hat{\rho}(t|u)z+\hat{X}_t(u)\right)-X_u}\right)\phi(z)\mathrm{d}z \\
&= \int_{-\infty}^{\infty} f\left(t, S_u\mathrm{e}^{\beta(u,t)+\hat{\rho}(t|u)z+\left(\hat{X}_t(u)-X_u\right)}\right)\phi(z)\mathrm{d}z \\
&= \int_{-\infty}^{\infty} f\left(t, S_u\mathrm{e}^{\beta(u,t)+\hat{\rho}(t|u)z-\int_0^u \Psi(t,s|u)\mathrm{d}X_s}\right)\phi(z)\mathrm{d}z,
\end{aligned}
$$

proving the claim. □

5. Conditional-Mean Hedging

We are interested in the pricing and hedging of European vanilla options $f(S_T)$ of the single discounted underlying asset $S = (S_t)_{t \in [0,T]}$, where $T > 0$ is a fixed time of maturity of the option.

We assume that the trading only takes place at fixed preset time points $0 = t_0 < t_1 < \cdots < t_N < T$. We Write by π^N the discrete trading strategy

$$\pi_t^N = \pi_0^N 1_{\{0\}}(t) + \sum_{i=1}^{N} \pi_{t_{i-1}}^N 1_{(t_{i-1}, t_i]}(t). \tag{5.1}$$

The value of the strategy π^N is given by

$$V_t^{\pi^N, k} = V_0^{\pi^N, k} + \int_0^t \pi_u^N \, \mathrm{d}S_u - \int_0^t kS_u |\mathrm{d}\pi_u^N| \tag{5.2}$$

$$= V_0^{\pi^N, k} + \sum_{t_i \le t} \pi_{t_{i-1}}^N \Delta S_{t_i} + \sum_{t_i \le t} kS_{t_{i-1}} |\Delta \pi_{t_i}^N|, \tag{5.3}$$

where $\Delta S_{t_i} = S_{t_i} - S_{t_{i-1}}$, $\Delta \pi_{t_i}^N = \pi_{t_i}^N - \pi_{t_{i-1}}^N$, and $k \in [0,1)$ is the proportional transaction cost.

Under transaction costs perfect hedging is not possible. In this case, it is natural to try to hedge on average in the sense of the following definition:

Definition 5.1 (Conditional-Mean Hedge). Let $f(S_T) \in L^2(\Omega)$ be a European vanilla type option with convex or concave payoff function f. Let π be its Markovian replicating strategy of Proposition 3.1: $\pi_t = g(t, S_t)$ for some function g. We call the discrete-time strategy π^N a *conditional-mean hedge*, if for all trading times t_i,

$$\mathbb{E}\left[V_{t_{i+1}}^{\pi^N, k} \,|\, \mathcal{F}_{t_i}\right] = \mathbb{E}\left[V_{t_{i+1}}^{\pi} \,|\, \mathcal{F}_{t_i}\right]. \tag{5.4}$$

Here $\mathcal{F}_{t_i}$ is the information generated by the asset price process S up to time t_i.

Remark 5.1 (Conditional-Mean Hedge as Tracking Condition). Criterion (5.4) is actually a tracking requirement. We do not only require that the conditional means agree on the last trading time before the maturity, but also on all trading times. In this sense the criterion has an "American" flavor in it. From a purely "European" hedging point of view, one can simply remove all but the first and the last trading times.

Remark 5.2 (Arbitrage and Uniqueness of Conditional-Mean Hedge). Note that the conditional-mean hedging strategy π^N depends on the continuous-time hedging strategy π. Since there may be strong arbitrage in the pricing model (zero can be perfectly replicated with negative initial wealth), the replicating strategy π may not be unique. However, the strong arbitrage strategies are very complicated. Indeed, it follows directly from the Föllmer–Itô change-of-variables formula

that in the class of Markovian strategies $\pi_t = g(t, S_t)$, the delta-hedge coming from the Black–Scholes type backward partial differential equation is the unique replicating strategy for the claim $f(S_T)$.

Remark 5.3 (No Martingale Measures). We stress that the expectation in (5.4) is with respect to the true probability measure; not under any equivalent martingale measure. Indeed, equivalent martingale measures may not even exist.

To find the solution to (5.4) one must be able to calculate the conditional expectations involved.

Let π be the continuous-time Markovian hedging strategy of the claim $f(S_T)$ and let V^π be its value process. Write

$$\Delta \hat{X}_{t_{i+1}}(t_i) = \hat{X}_{t_{i+1}}(t_i) - X_{t_i}, \tag{5.5}$$

$$= \mathbb{E}\left[X_{t_{i+1}} | \mathcal{F}_{t_i}\right] - X_{t_i}, \tag{5.6}$$

$$\Delta \hat{S}_{t_{i+1}}(t_i) = \hat{S}_{t_{i+1}}(t_i) - S_{t_i} \tag{5.7}$$

$$= \mathbb{E}\left[S_{t_{i+1}} | \mathcal{F}_{t_i}\right] - S_{t_i}, \tag{5.8}$$

$$\Delta \hat{V}^\pi_{t_{i+1}}(t_i) = \hat{V}^\pi_{t_{i+1}}(t_i) - V^\pi_{t_i} \tag{5.9}$$

$$= \mathbb{E}\left[V^\pi_{t_{i+1}} | \mathcal{F}_{t_i}\right] - V^\pi_{t_i}, \tag{5.10}$$

$$\Delta \hat{V}^{\pi^N, k}_{t_{i+1}}(t_i) = \hat{V}^{\pi^N, k}_{t_{i+1}}(t_i) - V^\pi_{t_i} \tag{5.11}$$

$$= \mathbb{E}\left[V^{\pi^N, k}_{t_{i+1}} | \mathcal{F}_{t_i}\right] - V^{\pi^N, k}_{t_i}. \tag{5.12}$$

Write

$$\gamma(s, t, T) = \beta(s, t) - \frac{1}{2} q^2(t, T). \tag{5.13}$$

Lemma 5.1 below states that all these conditional gains listed above can be calculated explicitly.

Lemma 5.1 (Conditional Gains). *Suppose that f is polynomially bounded. Then*

$$\Delta \hat{X}_{t_{i+1}}(t_i) = -\int_0^u \Psi(t, s | u) \, \mathrm{d}X_u, \tag{5.14}$$

$$\Delta \hat{S}_{t_{i+1}}(t_i) = S_{t_i} \left(e^{\beta(t_i, t_{i+1}) + \frac{1}{2}\hat{\rho}^2(t_{i+1} | t_i) + \Delta \hat{X}_{t_{i+1}}(t_i)} - 1 \right), \tag{5.15}$$

$$\Delta \hat{V}^\pi_{t_{i+1}}(t_i) = \int_{-\infty}^\infty \left[\int_{-\infty}^\infty f\left(S_{t_i} e^{\gamma(t_i, t_{i+1}, T) + \hat{\rho}(t_{i+1} | t_i) y + q(t_{i+1}, T) z} \right) \phi(y) \mathrm{d}y \right.$$
$$\left. - f\left(S_{t_i} e^{-\frac{1}{2}q^2(t_i, T) + q(t_i, T) z} \right) \right] \phi(z) \mathrm{d}z, \tag{5.16}$$

$$\Delta \hat{V}^{\pi^N, k}_{t_{i+1}}(t_i) = \pi^N_{t_i} \Delta \hat{S}_{t_{i+1}}(t_i) - k S_{t_i} |\Delta \pi^N_{t_i}|. \tag{5.17}$$

Proof. The formula for $\Delta \hat{X}_{t_{i+1}}(t_i)$ is given by Theorem 4.1.

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Consider $\Delta \hat{S}_{t_{i+1}}(t_i)$. By Corollary 4.1,

$$
\begin{aligned}
\hat{S}_{t_{i+1}}(t_i) &= \int_{-\infty}^{\infty} S_{t_i} \mathrm{e}^{\beta(t_i,t_{i+1})+\Delta \hat{X}_{t_{i+1}}(t_i)+\hat{\rho}(t_{i+1}|t_i)z}\,\phi(z)\mathrm{d}z \\
&= S_{t_i} \mathrm{e}^{\beta(t_i,t_{i+1})+\Delta \hat{X}_{t_{i+1}}(t_i)} \int_{-\infty}^{\infty} \mathrm{e}^{\hat{\rho}(t_{i+1}|t_i)z}\,\phi(z)\mathrm{d}z \\
&= S_{t_i} \mathrm{e}^{\beta(t_i,t_{i+1})+\frac{1}{2}\hat{\rho}^2(t_{i+1}|t_i)+\Delta \hat{X}_{t_{i+1}}(t_i)}.
\end{aligned}
$$

Consequently,

$$
\Delta \hat{S}_{t_{i+1}}(t_i) = S_{t_i}\left(\mathrm{e}^{\beta(t_i,t_{i+1})+\frac{1}{2}\hat{\rho}^2(t_{i+1}|t_i)+\Delta \hat{X}_{t_{i+1}}(t_i)} - 1 \right). \tag{5.18}
$$

Consider then $\Delta \hat{V}^{\pi}_{t_{i+1}}(t_i)$. Since f is polynomially bounded, we may use Fubini theorem for conditional expectations to obtain

$$
\hat{V}_{t_{i+1}}(t_i) = \int_{-\infty}^{\infty} \mathbb{E}\left[f\left(S_{t_{i+1}} \mathrm{e}^{-\frac{1}{2}q^2(t_{i+1},T)+q(t_{i+1},T)z} \right) \,\Big|\, \mathcal{F}_{t_i} \right] \phi(z)\mathrm{d}z. \tag{5.19}
$$

Now,

$$
\begin{aligned}
&\mathbb{E}\left[f\left(S_{t_{i+1}} \mathrm{e}^{-\frac{1}{2}q^2(t_{i+1},T)+q(t_{i+1},T)z} \right) \,\Big|\, \mathcal{F}_{t_i} \right] \\
&= \mathbb{E}\left[f\left(S_{t_i} \mathrm{e}^{\beta(t_i,t_{i+1})+\frac{1}{2}q^2(t_i,T)+(X_{t_{i+1}}-X_{t_i})+q(t_{i+1},T)z} \right) \,\Big|\, \mathcal{F}_{t_i} \right] \\
&= \int_{-\infty}^{\infty} f\left(S_{t_i} \mathrm{e}^{\beta(t_i,t_{i+1})-\frac{1}{2}q^2(t_{i+1},T)+\hat{\rho}(t_{i+1}|t_i)y+q(t_{i+1},T)z} \right) \phi(y)\mathrm{d}y.
\end{aligned}
$$

Since

$$
V^{\pi}_{t_i} = \int_{-\infty}^{\infty} f\left(S_{t_i} \mathrm{e}^{-\frac{1}{2}q^2(t_i,T)+q(t_i,T)z} \right) \phi(z)\mathrm{d}z, \tag{5.20}
$$

we obtain

$$
\begin{aligned}
&\Delta \hat{V}^{\pi}_{t_{i+1}}(t_i) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(S_{t_i} \mathrm{e}^{\gamma(t_i,t_{i+1},T)+\hat{\rho}(t_{i+1}|t_i)y+q(t_{i+1},T)z} \right) \phi(y)\mathrm{d}y\phi(z)\mathrm{d}z \\
&\quad - \int_{-\infty}^{\infty} f\left(S_{t_i} \mathrm{e}^{-\frac{1}{2}q^2(t_i,T)+q(t_i,T)z} \right) \phi(z)\mathrm{d}z \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f\left(S_{t_i} \mathrm{e}^{\gamma(t_i,t_{i+1},T)+\hat{\rho}(t_{i+1}|t_i)y+q(t_{i+1},T)z} \right) \phi(y)\mathrm{d}y \right. \\
&\quad \left. - f\left(S_{t_i} \mathrm{e}^{-\frac{1}{2}q^2(t_i,T)+q(t_i,T)z} \right) \right] \phi(z)\,\mathrm{d}z.
\end{aligned}
$$

Finally, we calculate

$$
\begin{aligned}
\hat{V}_{t_{i+1}}^{\pi^N,k}(t_i) &= \mathbb{E}\left[V_{t_{i+1}}^{\pi^N,k}\,\big|\,\mathcal{F}_{t_i}\right] \\
&= V_{t_i}^{\pi^N,k} + \mathbb{E}\left[\int_{t_i}^{t_{i+1}}\pi_u^N\,\mathrm{d}S_u - \int_{t_i}^{t_{i+1}}kS_u|\mathrm{d}\pi_u^N|\,\Big|\,\mathcal{F}_{t_i}\right] \\
&= V_{t_i}^{\pi^N,k} + \pi_{t_i}^N\left(\mathbb{E}\left[S_{t_{i+1}}\big|\mathcal{F}_{t_i}\right] - S_{t_i}\right) - kS_{t_i}|\Delta\pi_{t_i}^N| \\
&= V_{t_i}^{\pi^N,k} + \pi_{t_i}^N\Delta\hat{S}_{t_{i+1}}(t_i) - kS_{t_i}|\Delta\pi_{t_i}^N|.
\end{aligned}
$$

The formula for $\Delta\hat{V}_{t_{i+1}}^{\pi^N,k}(t_i)$ follows from this. $\qquad\square$

Now we are ready to state and prove our main result. We note that, in principle, our result is general: it is true in any pricing model where the option $f(S_T)$ can be replicated. In practice, our result is specific to the regular invertible Gaussian Volterra noise pricing model via Lemma 5.1.

Theorem 5.1 (Conditional-Mean Hedging Strategy). *The conditional mean hedge of the European vanilla type option with convex or concave positive payoff function f with proportional transaction costs k is given by the implicit equation*

$$
\pi_{t_i}^N = \frac{\Delta\hat{V}_{t_{i+1}}^{\pi}(t_i) + (V_{t_i}^{\pi} - V_{t_i}^{\pi^N,k}) + kS_{t_i}|\Delta\pi_{t_i}^N|}{\Delta\hat{S}_{t_{i+1}}(t_i)}, \tag{5.21}
$$

where $V_{t_i}^{\pi^N,k}$ is determined by (5.2).

Proof. Let us first consider the left hand side of (5.4). We have

$$
\begin{aligned}
\mathbb{E}\left[V_{t_{i+1}}^{\pi^N,k}\,\big|\,\mathcal{F}_{t_i}\right] &= \mathbb{E}\left[V_{t_i}^{\pi^N,k} + \int_{t_i}^{t_{i+1}}\pi_u^N\,\mathrm{d}S_u - k\int_{t_i}^{t_{i+1}}S_u|\mathrm{d}\pi_u^N|\,\Big|\,\mathcal{F}_{t_i}\right] \\
&= V_{t_i}^{\pi^N,k} + \pi_{t_i}^N\mathbb{E}\left[S_{t_{i+1}}(t_i) - S_{t_i}\,\big|\,\mathcal{F}_{t_i}\right] - kS_{t_i}|\Delta\pi^N t_i| \\
&= V_{t_i}^{\pi^N,k} + \pi_{t_i}^N\Delta\hat{S}_{t_{i+1}}(t_i) - kS_{t_i}|\Delta\pi_{t_i}^N|.
\end{aligned}
$$

For the right-hand-side of (5.4), we simply write

$$
\mathbb{E}\left[V_{t_{i+1}}^{\pi}\,\big|\,\mathcal{F}_{t_i}\right] = \Delta\hat{V}_{t_{i+1}}^{\pi}(t_i) + V_{t_i}^{\pi}. \tag{5.22}
$$

Equating the sides we obtain (5.21) after a little bit of simple algebra. $\qquad\square$

Remark 5.4 (Interpretation). Taking the expected gains $\Delta\hat{S}_{t_{i+1}}(t_i)$ to be the numéraire, one recognizes three parts in the hedging formula (5.21). First, one invests on the expected gains in the time-value of the option. This "conditional-mean delta-hedging" is intuitively the most obvious part. Indeed, a naïve approach to conditional-mean hedging would only give this part, forgetting to correct for the tracking-errors already made, which is the second part in (5.21). The third part in (5.21) is obviously due to the transaction costs.

Remark 5.5 (Initial Position). Note that the equation (5.21) for the strategy of the conditional-mean hedging is recursive: in addition to the filtration $\mathcal{F}_{t_i}$, the position $\pi^N_{t_{i-1}}$ is needed to determine the position $\pi^N_{t_i}$. Consequently, to determine the conditional-meand hedging strategy by using (5.21), the initial position π^N_0 must be fixed. The initial position is, however, not uniquely defined. Indeed, let β^N_0 be the position in the riskless asset. Then the conditional-mean criterion (5.4) only requires that

$$\beta^N_0 + \pi^N_0 \mathbb{E}[S_{t_1}] - kS_0|\pi^N_0| = \mathbb{E}[V^\pi_{t_1}]. \tag{5.23}$$

There are of course infinite number of pairs (β^N_0, π^N_0) solving this equation. A natural way to fix the initial position (β^N_0, π^N_0) for the investor interested in conditional-mean hedging would be the one with minimal cost. If short-selling is allowed, the investor is then faced with the minimization problem

$$\min_{\pi^N_0 \in \mathbb{R}} w(\pi^N_0), \tag{5.24}$$

where the initial wealth w is the piecewise linear function

$$w(\pi^N_0) = \beta^N_0 + \pi^N_0 S_0 \tag{5.25}$$

$$= \begin{cases} \mathbb{E}[V^\pi_{t_1}] - \left(\Delta \hat{S}_{t_1}(0) - kS_0\right)\pi^N_0, & \text{if} \quad \pi^N_0 \geq 0, \\ \mathbb{E}[V^\pi_{t_1}] - \left(\Delta \hat{S}_{t_1}(0) + kS_0\right)\pi^N_0, & \text{if} \quad \pi^N_0 < 0. \end{cases} \tag{5.26}$$

Clearly, the minimal solution π^N_0 is independent of $\mathbb{E}[V^\pi_{t_1}]$, and, consequently, of the option to be replicated. Also, the minimization problem is bounded if and only if

$$k \geq \left|\frac{\Delta \hat{S}_{t_1}(0)}{S_0}\right|, \tag{5.27}$$

i.e. the proportional transaction costs are bigger than the expected return on $[0, t_1]$ of the stock. In this case, the minimal cost conditional mean-hedging strategy starts by putting all the wealth in the riskless asset.

Acknowledgments

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