Spaces and Moduli Spaces of Flat Riemannian Metrics on Closed Manifolds

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Abstract

In the present work we study the space of isometry classes of Riemannian metrics with zero sectional curvature on a closed manifold. Specifically, in dimension 3 and for a family of manifolds in dimension 4, we give a complete algebraic description of the Teichmüller spaces and the moduli spaces of flat metrics. Furthermore, we give complete information about the topology of the moduli spaces of flat metrics of the 3-dimensional closed manifolds.

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Contents

Introduction

A flat manifold is not only locally homeomorphic to a Euclidean space but also has the same geometry, which means that near any point we can introduce coordinates so that, with respect to these coordinates, the rules of Euclidean geometry hold. More precisely, a flat manifold is a Riemannian manifold which admits a metric of zero sectional curvature, called **flat metric**.

In the present work we are interested in closed flat manifolds. The most important invariant for these spaces is the fundamental group, in the sense that if two closed flat manifolds have the same fundamental group then they are homeomorphic. This result, and the study of the types of groups that can be a fundamental group of a closed flat manifold are due to Bieberbach, who published his work about 1911 in [4].

Later, in 1973, Wolf carried out the isometric classification of closed flat manifolds in a fixed affine equivalence class, in [23]. In other words, the work of Wolf allows us to study the space of flat metrics on a closed manifold and identify the isometric ones. The space of isometry classes of flat metrics on a manifold is **the moduli space of flat metrics** of the manifold. The aim of this work is to study this moduli space.

Even though the space and the moduli space of flat metrics is perhaps the easiest curvature constraint one could ask in the space of Riemannian metrics, its understanding is not as developed as one could expect. Just recently, in 2017, Bettiol, Derdzinski and Piccione in [3] established an algebraic description of the Teichmüller space of flat metrics, which provides a straightforward method to compute it and where it is clear that it is diffeomorphic to a euclidean space. The moduli space of flat metrics will be a quotient of the Teichmüller space of flat metrics by a discrete group. If we get a better understanding of this kind of quotients then we could find out a better description for the moduli space of flat metrics, where their topology is clearer. In this work, we make a contribution to the solution of this problem, in dimension 3 and 4.

The present thesis builds upon the work of Kang, where she computed the moduli space of flat metrics for the 3-dimensional closed manifolds. Here we do some amendments and we complete her work by studying the topology of those moduli spaces of flat metrics in Theorem 3.4.3. We also compute the moduli space of flat metrics for a family of 4-dimensional closed manifolds in Theorem 4.4.2.

The approach to describe these spaces is case by case, using the description given by Wolf, where one has to compute some spaces which depend on a representation of certain class of groups, called **Bieberbach groups**. The Bieberbach groups are equivalent to closed flat manifolds. Then one has to solve some equations, where the program Mathematica helped, and in some cases, one has to study further the affine structure of the Bieberbach group.

Once one has the description of the moduli spaces of flat metrics, then one can attempt to say something about their topology. A tool we use for the study of the topology of the moduli spaces of flat metrics in dimension 3 and some cases in dimension 4 comes from number theory, because it is related to study discrete subgroups acting isometrically on the hyperbolic plane.

The study of the moduli space of flat metrics could lead to have a better understanding of the moduli space of metrics with some other curvature constraint. For instance, Tuschmann and Wiemeler in [20] worked with manifolds diffeomorphic to the product of a simply connected closed smooth manifold with a torus, and studied their moduli space of non-negative Ricci curvature Riemannian metrics, where they were able to reduce it to study the moduli space of flat metrics of the torus.

Our work is organized as follows. The first chapter contains basic notions and results. The second chapter is about the moduli space of flat metrics in dimension 2 and the tools we need later for studying the topology of the moduli spaces of flat metrics in higher dimensions. The last two chapters describe the moduli spaces of flat metrics in dimension 3 and 4.

Chapter 1

Background

1.1 Notations

Let us fix the notation we will use in our work.

- $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the natural, the integer and the real numbers, respectively.
- We let $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.
- $\mathbb{R}^+$ will denote the positive real numbers.
- Let $n \in \mathbb{N}$. We will use the following notation:
  - $\mathrm{GL}(n, \mathbb{R})$ is the group of $n \times n$ invertible matrices with real entries.
  - $\mathrm{GL}(n, \mathbb{R})^+$ is the subgroup of $\mathrm{GL}(n, \mathbb{R})$ consisting of matrices with positive determinant.
  - $\mathrm{SL}(n, \mathbb{R})$ is the subgroup of $\mathrm{GL}(n, \mathbb{R})$ of determinant one matrices.
  - $\mathrm{GL}(n, \mathbb{Z})$ is the unimodular group consisting of all integer matrices with determinant ± 1.
  - $\mathrm{SL}(n, \mathbb{Z})$ is the subgroup of $\mathrm{GL}(n, \mathbb{Z})$ of determinant one matrices.
  - $\mathrm{O}(n)$ is the orthogonal group, consisting of matrices in $\mathrm{GL}(n, \mathbb{R})$ whose inverse is equal to its transpose.
  - $\mathrm{SO}(n)$ is the subgroup of $\mathrm{O}(n)$ of determinant one matrices.
  - $\mathrm{Aff}(n)$ is the group of affine transformations of $\mathbb{R}^n$.
  - $\mathrm{Iso}(n)$ is the subgroup of $\mathrm{Aff}(n)$ consisting of isometries of $\mathbb{R}^n$ with the usual metric.

- Id will denote the identity matrix of the corresponding dimension.
- A^t is the transpose of a matrix A.
- G^+ will denote the subgroup of matrices with positive determinant in a given group G of matrices.
- $H_1 \cdot H_2 := \{h_1 \cdot h_2 \mid h_1 \in H_1,\ h_2 \in H_2\}$, where H_1 and H_2 are two subgroups of a given group.

1.2 Flat manifolds

In this section, we explain the relation between flat manifolds and Bieberbach groups, we study affine transformations and introduce an important notion: **the holonomy**. Further can be found in [22] or in [5].

By a **flat manifold** (M, g) we mean a connected n-dimensional complete Riemannian manifold with zero sectional curvature, where the Riemannian metric g will be called **flat metric**. The universal cover $\widetilde{M}$ of M is isometric to $\mathbb{R}^n$ with the usual metric denoted by σ. For a fixed manifold M and a fixed flat metric g on M, we lift the metric g to $\widetilde{M}$ and denote it by $\tilde{g}$. Then we have the next diagram

$$\begin{tikzcd}
(\widetilde{M}, \tilde{g}) \arrow[r] \arrow[d] & (\mathbb{R}^n, \sigma) \arrow[dl] \\
(M, g) &
\end{tikzcd}$$

where the horizontal arrow is an isometry. This makes $(\mathbb{R}^n, \sigma)$ also a Riemannian covering of (M, g). Let us consider its group of deck transformations, denoted by π. Then (M, g) is isometric to $(\mathbb{R}^n/\pi, \sigma)$, with σ also denoting the metric induced by the usual metric σ.

Therefore, given a manifold with a fixed flat metric, we get a group π of isometries of $(\mathbb{R}^n, \sigma)$. On the other hand, if a group of isometries π acts freely and properly discontinuously on $\mathbb{R}^n$, then $\mathbb{R}^n/\pi$ is a flat manifold. This result follows from the Killing-Hopf Theorem [22, Corollary 2.4.10]. For the case when this manifold is closed, i.e, compact and without boundary, we have the next definition.

Definition 1.2.1. A **Bieberbach group** π is a discrete subgroup of $\mathrm{Iso}(n)$ that is torsion-free and such that $\mathbb{R}^n/\pi$ is compact.

Given a Bieberbach group π, we get a closed flat manifold $\mathbb{R}^n/\pi$. Conversely, given a closed flat manifold of dimension $n \geq 2$, its fundamental group is isomorphic to a Bieberbach group.

To study the structure of π, it is important to understand the structure of the group of affine transformations of $\mathbb{R}^n$. The group of affine transformations of $\mathbb{R}^n$, $\mathrm{Aff}(n)$, has the structure of a semidirect product in the following way: any affine transformation can be expressed as $(A, v) : \mathbb{R}^n \to \mathbb{R}^n$ with $(A, v)x \mapsto Ax + v$, where $v \in \mathbb{R}^n$ and $A \in \mathrm{GL}(n, \mathbb{R})$. The product is just the composition of these transformations, as follows.
Let $(A, v), (B, u) \in \mathrm{Aff}(n)$. Then

$$(A, v)(B, u) = (AB, A(u) + v).$$

With this description, we have a semidirect product

$$\mathrm{Aff}(n) = \mathrm{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n.$$

Remark 1.2.2. Another way to think about this product is with matrices:

$$(A, v) = \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix},$$

with the product of usual matrix multiplication.

The isometries of $\mathbb{R}^n$ also have a structure of semidirect product

$$\mathrm{Iso}(n) = \mathrm{O}(n) \ltimes \mathbb{R}^n.$$

Now, we can present another important definition. Consider the projection homomorphism

$$\begin{array}{rccc} \tau: & \mathrm{Aff}(n) & \to & \mathrm{GL}(n, \mathbb{R}) \\ & (A, v) & \mapsto & A \end{array}$$

Definition 1.2.3. Let π be a Bieberbach group. The **holonomy** of π is the subgroup of $\mathrm{GL}(n, \mathbb{R})$ given by $H_\pi := \tau(\pi)$.

The kernel of τ restricted to π is denoted by L_π, the maximal normal abelian subgroup of π, which consists of all the translations (Id, v) of π. We have a short exact sequence

$$1 \to L_\pi \to \pi \to H_\pi \to 1.$$

The Bieberbach theorems provide the essential facts about the groups (the algebraic version) or, equivalently, about the closed flat manifolds (the geometric version). We present the geometric formulation:

Theorem 1.2.4 (Bieberbach theorems).

I. *Let M be a closed flat manifold. Then M is covered by a flat torus, and the covering map is a local isometry.*

II. *Let M and M' be closed flat manifolds with isomorphic fundamental groups. Then M and M' are affinely equivalent.*

III. *There are only finitely many affine equivalence classes of closed flat manifolds in any dimension.*

The list of (affine equivalent classes of) closed flat manifolds, which, by the theorem above, is in bijective correspondence with the list of (affine conjugate classes of) Bieberbach groups in $\mathrm{Iso}(n)$, is known for some values of the dimension n. We are only concerned about dimension 2, 3 and 4:

- If $n = 2$, there are 2: the torus and the Klein bottle. This is a classical fact.
- If $n = 3$, there are 10. Listed in Wolf's book [22].
- If $n = 4$, there are 74. Listed in the Ph.D. thesis of Lambert [15].

1.3 The moduli space of flat metrics

We are going to introduce the moduli space of Riemannian metrics based on [21], and then state a result of Wolf for the moduli space of flat metrics proven in [23].

Let M be a closed manifold. We denote by $\mathcal{R}(M)$ the space of all complete Riemannian metrics on M. The space $\mathcal{R}(M)$ is a subspace of $C^\infty(M, S^2T^*M)$, the real vector space of smooth symmetric $(0,2)$ tensor fields on M, where S^2T^*M denotes the second symmetric power of the cotangent bundle of M. In this setting, the space $\mathcal{R}(M)$ is usually equipped with the smooth compact-open topology or, some other times, with the Whitney topology. In here, we are only interested in the case when M is compact, where the two topologies mentioned coincide. Details about these topologies can be found in [21], [10] or [6].

We equip $\mathcal{R}(M)$ with the smooth compact-open topology. A sequence converges in this topology if and only if the k derivative converge uniformly on M, for all $k \in \mathbb{N}$.

We would like to identify the metrics that are isometric, i.e., where there exists an isometry between the given metrics. This leads us to consider the next action. Let $\operatorname{Diff}(M)$ denote the group of self-diffeomorphisms of M. Then $\operatorname{Diff}(M)$ acts on $\mathcal{R}(M)$ by pulling back metrics: $f \in \operatorname{Diff}(M)$, we let $g \cdot f := f^*(g)$ with $g \in \mathcal{R}(M)$.

Definition 1.3.1. The **moduli space** $\mathcal{M}(M)$ of complete Riemannian metrics on M is the quotient of $\mathcal{R}(M)$ by the above action of the diffeomorphism group $\operatorname{Diff}(M)$.

We can study subspaces of $\mathcal{R}(M)$ that satisfy a curvature condition and consider again the quotient by $\operatorname{Diff}(M)$. In our case, we are interested in the space of complete Riemannian metrics with zero sectional curvature, denoted by $\mathcal{R}_{flat}(M)$, leading us to the next definition:

Definition 1.3.2. The **moduli space of flat metrics** $\mathcal{M}_{flat}(M)$ is the quotient of $\mathcal{R}_{flat}(M)$ by the action of $\operatorname{Diff}(M)$.

There is a description of the moduli space $\mathcal{M}_{flat}(M)$ due to Wolf [23]. We present this description and also some properties in order to compute the moduli space $\mathcal{M}_{flat}(M)$. Some of the properties are also studied in [11], [12] or [13]. In these last three papers, they use the notation and setting of Kulkarni, Lee and Raymond [14], without using Wolf's result.

First, we have to fix a connected n-dimensional, closed flat Riemannian manifold M. From before we know that this means that we are fixing a Bieberbach group π, where $M = \mathbb{R}^n/\pi$.

A useful group is the normalizer of π in $\operatorname{Aff}(n)$, which we denote by

$$\mathrm{N}_{\operatorname{Aff}(n)}(\pi) = \{\gamma \in \operatorname{Aff}(n) \mid \gamma\pi\gamma^{-1} = \pi\}.$$

We will compute this group for the torus. For this, we need the next notion.

Definition 1.3.3. Given a basis $\{a_1, \ldots, a_n\}$ of $\mathbb{R}^n$, the **lattice** generated by that basis is $\{\sum_{i=1}^n n_i a_i \mid n_i \in \mathbb{Z}\}$. We also refer as a lattice to the group generated by the translations by a_i: $\langle(\mathrm{Id}, a_1), \ldots, (\mathrm{Id}, a_n)\rangle$.

As we know the torus can be constructed as the quotient of $\mathbb{R}^n$ and a lattice. We fix the lattice to be

$$\mathbb{Z}^n = \langle (\mathrm{Id}, e_i) \mid \{e_1, \dots, e_n\} \text{ the standard basis of} \mathbb{R}^n \rangle.$$

Then $\mathrm{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

Example 1.3.4. Let us show that $\mathrm{N}_{\mathrm{Aff}(n)}(\mathbb{Z}^n) = \mathrm{GL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n$.
We want to find $\gamma \in \mathrm{Aff}(n)$ such that $\gamma \mathbb{Z}^n \gamma^{-1} = \mathbb{Z}^n$. Let $\gamma = (X, u)$ and $(\mathrm{Id}, v) \in \mathbb{Z}^n$. Then

$$(X, u)(\mathrm{Id}, v)(X^{-1}, -X^{-1}(u)) = (\mathrm{Id}, X(v)).$$

This means that we have to find the matrices $X \in \mathrm{GL}(n, \mathbb{R})$ that preserve the lattice, i.e., send a basis that generates the whole lattice to another. Then the matrices must have integer entries and determinant ± 1. These matrices are precisely the unimodular matrices. (See, for example, [17]).

Theorem 1.3.5 ([23] **Wolf**). *The subset*

$$Iso(n) \backslash \{\gamma \in A\!f\!f(n) \mid \gamma \pi \gamma^{-1} \subset Iso(n)\} / N_{A\!f\!f(n)}(\pi)$$

of the double coset space $Iso(n) \backslash\, A\!f\!f(n)/\, N_{A\!f\!f(n)}(\pi)$, is in bijective correspondence with the set of all isometry classes of riemannian manifolds that are affinely equivalent to $M = \mathbb{R}^n/\pi$. The double coset $Iso(n)\gamma\; N_{A\!f\!f(n)}(\pi)$ corresponds to the isometry class of $\mathbb{R}^n/(\gamma \pi \gamma^{-1})$.

Every flat n-dimensional torus is affinely equivalent to $\mathbb{R}^n/\mathbb{Z}^n$. If $\gamma = (A, u) \in \mathrm{Aff}(n)$ then $\gamma \mathbb{Z}^n \gamma^{-1} = A(\mathbb{Z}^n) \subset \mathrm{Iso}(n)$. Thus the case $\pi = \mathbb{Z}^n$ of the previous theorem is as follows. (This is equivalent to [22, Lemma 3.5.11]).

Corollary 1.3.6. *The double coset space*

$$O(n) \backslash GL(n, \mathbb{R}) / GL(n, \mathbb{Z})$$

is in bijective correspondence with the set of all isometry classes of flat Riemannian n-tori. The double coset $O(n)A\; GL(n, \mathbb{Z})$ correspond to the class of $\mathbb{R}^n/A(\mathbb{Z}^n)$.

The result of Wolf gives us actually a homeomorphism. Since it seems there is no reference where the continuity of the bijection is proved, we give some details here.

As we mentioned before, the space $\mathcal{R}_{flat}(M)$ has the subspace topology from $\mathcal{R}(M)$, which is equipped with the smooth compact-open topology, and the space $\mathrm{Aff}(n)$ with the subspace topology induced from $\mathrm{GL}(n+1,\mathbb{R})$ (see Remark 1.2.2).

Fix a flat manifold $\mathbb{R}^n/\pi$. Let $g \in \mathcal{R}_{flat}(\mathbb{R}^n/\pi)$. Then we know that we get an affine transformation γ. The construction of γ is the same as in Theorem 4.1 in do Carmo's book [7], which uses the theorem of Cartan. Since we can choose the affine transformation in different ways, we will choose them as follows: first, we lift the metric g to $\mathbb{R}^n$, denoting $\mathbb{R}^n$ with the lifted metric, as before, $(\mathbb{R}^n, \tilde{g})$. Let $\{e_1, \ldots, e_n\}$ the standard basis of $\mathbb{R}^n$, $p \in \mathbb{R}^n/\pi$ and choose $p_0 \in \mathbb{R}^n$ in the fibre of p. Second, by the Gram-Schmidt process we can produce, from the standard basis, an orthonormal basis $\{u_1, \ldots, u_n\}$ with respect to the metric $\tilde{g}$. Then we consider the linear isometry between the tangent spaces $A : T_{p_0}\mathbb{R}^n \to T_0\mathbb{R}^n$, such that $u_i \mapsto e_i$ with $1 \leq i \leq n$, and 0 is the origin in $\mathbb{R}^n$. Finally, the affine transformation is the map:

$$\gamma = \exp_0^{\sigma} \circ A \circ (\exp_{p_0}^{\tilde{g}})^{-1} : (\mathbb{R}^n, \tilde{g}) \to (\mathbb{R}^n, \sigma),$$

where $\exp_{p_0}^{\tilde{g}}$ and $\exp_0^{\sigma}$ are the exponential maps define in $T_{p_0}\mathbb{R}^n$ and $T_0\mathbb{R}^n$ respectively. By Theorem 4.1 in [7] the map $\gamma : (\mathbb{R}^n, \tilde{g}) \to (\mathbb{R}^n, \sigma)$ is an isometry. Observe that $\gamma = (A, -p_0)$.
In this way, we have the map:

$$\begin{array}{ccc} \mathcal{R}_{flat}(\mathbb{R}^n/\pi) & \to & \mathrm{Aff}(n) \\ g & \mapsto & \gamma \end{array}$$

which we will see is continuous by using sequences. Let $(g_k)_{k\in\mathbb{N}}$ be a sequence of flat metrics on $\mathbb{R}^n/\pi$ that converges to a flat metric g. Then we will get a sequence of linear isometries $(A_k)_{k\in\mathbb{N}}$ converging to the linear isometry A, because the orthonormal bases we get by the Gram-Schmidt process will depend continuously on the metrics. Therefore, the affine transformations $(\gamma_k = (A_k, -p_0))_{k\in\mathbb{N}}$ will converge to $\gamma = (A, -p_0)$.

Observe that $\{\gamma \in \mathrm{Aff}(n) \mid \gamma\pi\gamma^{-1} \subset \mathrm{Iso}(n)\}$ is not a group. The important part of this space is the matrix part since for an affine transformation to be an isometry we only have to see that the matrix is orthogonal. Also, to describe the double coset in theorem 1.3.5, the important part is the matrix part of the spaces, as one can already see in the corollary. We explain this in the next remark, but first let us denote the matrix part of the spaces as follows:

$$\begin{aligned} &\mathbf{C}_\pi := \{X \in \mathrm{GL}(n,\mathbb{R}) \mid XAX^{-1} \in \mathrm{O}(n) \quad \text{for all} \quad A \in H_\pi\}, \\ &\mathcal{N}_\pi := \tau(\mathrm{N}_{\mathrm{Aff}(\mathbb{R}^n)}(\pi)). \end{aligned}$$

Then $\{\gamma \in \mathrm{Aff}(n) \mid \gamma\pi\gamma^{-1} \subset \mathrm{Iso}(n)\} = C_\pi \ltimes \mathbb{R}^n$. Sometimes we call the space C_π, the **cone space**, since it is closed under multiplication by scalars.

Remark 1.3.7. Consider the quotient

$$\mathrm{Iso}(\mathbb{R}^n)\backslash\{\gamma \in \mathrm{Aff}(\mathbb{R}^n) \mid \gamma\pi\gamma^{-1} \subset \mathrm{Iso}(\mathbb{R}^n)\} = \mathrm{O}(n) \ltimes \mathbb{R}^n \backslash C_\pi \ltimes \mathbb{R}^n.$$

The action is just by multiplication from the left and every orbit must contain the whole $\mathbb{R}^n$. Thus, the quotient is simply $\mathrm{O}(n)\backslash C_\pi$.

We have the same situation for the double quotient. Therefore the moduli space of flat metrics of $M = \mathbb{R}^n/\pi$ is

$$\mathcal{M}_{flat}(\mathbb{R}^n/\pi) = \mathrm{O}(n)\backslash C_\pi/\mathcal{N}_\pi.$$

Now, we present some properties of these matrix spaces. The next lemma is useful for computing the cone space C_π.

Lemma 1.3.8 ([12, Lemma 2.2])**.** *Let $A \in O(n)$. For any invertible matrix $X \in GL(n, \mathbb{R})$, we have $XAX^{-1} \in O(n)$ if and only if $(X^tX)A = A(X^tX)$.*

Proof. The fact that XAX^{-1} is orthogonal means that

$$(XAX^{-1})(XAX^{-1})^t = \mathrm{Id},$$

which is equivalent to $XA(X^tX)^{-1} = (X^t)^{-1}A$, and therefore equivalent to

$$(X^tX)A = A(X^tX).$$

□

We know that a Bieberbach group π contains a lattice, denoted before by L_π. Thus, the normalizer of π in $\mathrm{Aff}(n)$ is contained in the normalizer of L_π in $\mathrm{Aff}(n)$, and the matrix part of the normalizer of a lattice in $\mathrm{Aff}(n)$, $\mathcal{N}_{L_\pi}$, is a conjugate of $\mathcal{N}_{\mathbb{Z}^n}$ in $\mathrm{GL}(n, \mathbb{R})$. We state this in the following lemma.

Lemma 1.3.9. *Let π be a Bieberbach group, then*

$$N_{Aff(n)}(\pi) \subset QGL(n, \mathbb{Z})Q^{-1} \ltimes \mathbb{R}^n,$$

where $Q \in GL(n, \mathbb{R})$.

The matrix Q is the change of coordinates matrix from the basis of L_π to the standard basis.

It is convenient to study what happens to these spaces when we conjugate the Bieberbach group by an affine transformation. For the next two lemmas let π be a Bieberbach group and $\pi' = \xi\pi\xi^{-1}$, for some $\xi \in \mathrm{Aff}(n)$.

Lemma 1.3.10 ([12, Lemma 2.3])**.** *The cone spaces C_π and $C_{\pi'}$ are homeomorphic.*

Proof. Consider $\tau(\xi) = P$. If $X \in C_{\pi'}$, then $XP \in C_\pi$, giving us the homeomorphism. □

Lemma 1.3.11 ([13, page 1069])**.** *We have*

$$\xi N_{Aff(\mathbb{R}^n)}(\pi)\xi^{-1} = N_{Aff(\mathbb{R}^n)}(\xi\pi\xi^{-1}).$$

Proof. We see that $\gamma \in \mathrm{N}_{\mathrm{Aff}(\mathbb{R}^n)}(\xi\pi\xi^{-1})$ if and only if $\xi^{-1}\gamma\xi \in \mathrm{N}_{\mathrm{Aff}(\mathbb{R}^n)}(\pi)$. □

1.4 The Teichmüller space of flat metrics

We will introduce the Teichmüller space of flat metrics of a closed manifold and a straightforward method to compute it, which is proved in [3].

As before, let M be a closed manifold. We denote by $\mathrm{Diff}_0(M)$ the subgroup of $\mathrm{Diff}(M)$ of all smooth diffeomorphisms of M that are homotopic to the identity. The Teichmüller space of flat metrics of M, $\mathcal{T}_{flat}(M)$, can be defined as the quotient of $\mathcal{R}_{flat}(M)$ by restricting the action to $\mathrm{Diff}_0(M)$. This definition is equivalent to the next one, that for our purpose is the definition we use and is also used in [3].

Definition 1.4.1. The **Teichmüller space of flat metrics** $\mathcal{T}_{flat}(M)$ of $M = \mathbb{R}^n/\pi$ is the orbit space $\mathrm{O}(n)\backslash C_\pi$ of the left action of $\mathrm{O}(n)$ on C_π.

In other contexts, the definition of the Teichmüller space is slightly different: metrics that are homothetic are identified. There is a nice introduction to this notion in [9]. We will discuss the 2-dimensional case in the next remark.

Remark 1.4.2. The Teichmüller space of a closed surface S is the set of classes of hyperbolic Riemannian metrics on S given by the action of $\mathrm{Diff}_0(S)$. The Gauss-Bonnet theorem implies that any closed hyperbolic surface S has fixed area $-2\pi\chi(S)$. Then a more natural definition for the Teichmüller space of flat metrics would be to restrict to the set of unit-area flat structures and identify them by the action of $\mathrm{Diff}_0(S)$. This is the way it is done in [8]. This definition and the definition we are using differ only by an extra real line, as we are going to see in the examples of the 2-dimensional flat manifolds.

We now present the description of the Teichmüller space of flat metrics done in [3] only for manifolds, but they also prove it for orbifolds:

Theorem 1.4.3. *Let M be a closed flat manifold, and denote by W_i, $1 \leq i \leq l$, the isotopic components of the orthogonal representation of its holonomy group. Each W_i consists of m_i copies of the same irreducible representation, and we write $\mathbb{K}_i$ for $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, according to this irreducible representation being of real, complex, or quaternionic type. The Teichmüller space is diffeomorphic to*

$$\mathcal{T}_{flat}(M) \cong \prod_{i=1}^{l} \frac{GL(m_i, \mathbb{K}_i)}{O(m_i, \mathbb{K}_i)},$$

where $GL(m, \mathbb{K})$ is the group of $\mathbb{K}$-linear automorphisms of $\mathbb{K}^m$ and $O(m, \mathbb{K})$ stands for $O(m)$, $U(m)$, or $Sp(m)$, when $\mathbb{K}$ is, respectively, $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. In particular, $\mathcal{T}_{flat}(M)$ is real-analytic and diffeomorphic to $\mathbb{R}^d$.

The dimension $d = \dim \mathcal{T}_{flat}(M)$ is easily computed as the sum of the dimensions $d_i \geq 1$ of the factors $\frac{\mathrm{GL}(m_i, \mathbb{K}_i)}{\mathrm{O}(m_i, \mathbb{K}_i)} \cong \mathbb{R}^{d_i}$, $1 \leq i \leq l$, which are given by

$$d_i = \begin{cases} \frac{1}{2} m_i(m_i + 1), & \text{if} \quad \mathbb{K}_i = \mathbb{R}, \\ m_i^2, & \text{if} \quad \mathbb{K}_i = \mathbb{C}, \\ m_i(2m_i - 1), & \text{if} \quad \mathbb{K}_i = \mathbb{H}. \end{cases}$$

1.5 Further remarks

The moduli space of flat metrics can be described from the Teichmüller space of flat metrics in the following way:

$$\mathcal{M}_{flat}(M) = \mathcal{T}_{flat}(M)/\mathcal{N}_\pi,$$

where π is a Bieberbach group and $M = \mathbb{R}^n/\pi$.

The right action of $\mathcal{N}_\pi$ on C_π need not be free, so $\mathcal{M}_{flat}(M)$ may have (isolated) singularities. We have further information about the group $\mathcal{N}_\pi$. (See [3, Proposition 4.4]).

Proposition 1.5.1. *The group $\mathcal{N}_\pi$ is a discrete group.*

Proof. First we have that $\mathrm{GL}(n, \mathbb{Z})$ is discrete, since $\mathbb{Z} \subset \mathbb{R}$ is discrete. Now, for the lattice $L_\pi = \mathbb{R}^n \cap \pi$, the normalizer $\mathcal{N}_{L_\pi}$ is a conjugate of $\mathrm{GL}(n, \mathbb{Z})$ inside $\mathrm{GL}(n, \mathbb{R})$, which is also discrete. Finally, we have that $\mathcal{N}_\pi \subset \mathcal{N}_{L_\pi}$ (see Lemma 1.3.9). Therefore, $\mathcal{N}_\pi$ is discrete. □

Thus the Teichmüller space and the moduli space of flat metrics are orbifolds with the same dimension.

Corollary 1.5.2. *We have* $\dim \mathcal{T}_{flat}(M) = \dim \mathcal{M}_{flat}(M)$.

This description could be studied further since we have a diffeomorphism between $\mathcal{T}_{flat}(M)$ and $\mathbb{R}^d$. One would have to study the action of $\mathcal{N}_\pi$ via this diffeomorphism.

Another remark on these spaces of metrics is the following: the Teichmüller space $\mathcal{T}_{flat}(M)$ only depends on the holonomy of π. On the other hand, the moduli space $\mathcal{M}_{flat}(M)$ depends not only on the holonomy but also on the affine structure of the group π. This will be clarified later, in Chapter 3 and 4.

Chapter 2

2-dimensional closed flat manifolds and the action of SL(2,$\mathbb{Z}$)

In this section we recall some facts on the moduli space of flat metrics of the 2-dimensional closed manifolds and make some remarks about them. In the study of these spaces the group $\mathrm{SL}(2,\mathbb{Z})$ will appear. In addition, it will be important to study its action on the hyperbolic plane $\mathbb{H}^2$, and moreover, this action will be restricted to some subgroups of $\mathrm{SL}(2,\mathbb{Z})$ as well. This is important because in the next chapters is going to be a key point in order to study the topology of the moduli spaces of flat metrics of closed manifolds in dimension 3 and 4.

In dimension 2, there are only two closed flat manifolds: the torus and the Klein bottle.

2.1 The 2-torus

The Teichmüller space of flat metrics and the moduli space of flat metrics of the 2-torus is very well understood (see [8]). Following this book, we present some parts of this information.

Before going into details, one should notice that the definition in [8] of moduli space of flat metrics is slightly different from the one we are using. Inded, in the book [8], they are relating homothetic metrics as well, see remark 1.4.2. In our situation, we consider relations of homotheties in order to follow the same ideas, but in the final homeomorphism we add the positive reals. This is important because homothetic flat metrics are not isometric, i.e., they give us different points in the moduli space.

We proceed as in [8]. Recall from the previous chapter that:

$$\mathcal{M}_{flat}(\mathrm{T}^2) = \mathrm{O}(2)\backslash\mathrm{GL}(2,\mathbb{R})/\mathrm{GL}(2,\mathbb{Z}).$$

We start by studying the left quotient $\mathrm{O}(2)\backslash\mathrm{GL}(2,\mathbb{R})$. We use an auxiliary space, the space of ordered bases in $\mathbb{R}^2$. In this way we have two bijections. The first one is:

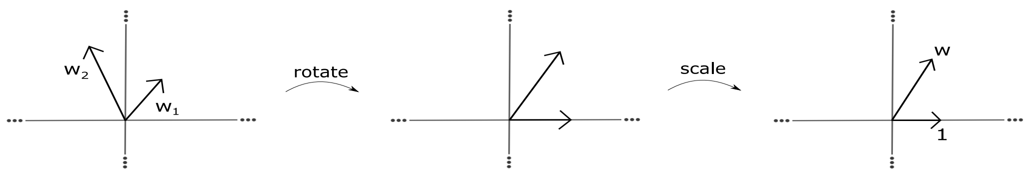
$$\frac{\mathrm{GL}(2,\mathbb{R})}{\mathrm{O}(2)\cup\{\text{homotheties}\}} \quad\leftrightarrow\quad \frac{\{\text{ordered bases in}\mathbb{R}^2\}}{\mathrm{O}(2)\cup\{\text{homotheties}\}} \tag{2.1}$$

Every regular matrix can be seen as an ordered lattice and vice versa:
$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R}) \quad\leftrightarrow\quad \{w_1 = (x_1, y_1), w_2 = (x_2, y_2)\}.$

The space $\mathrm{GL}(2,\mathbb{R})$ has a topology given by being embedded in $\mathrm{M}_{2\times 2}(\mathbb{R}) \cong \mathbb{R}^4$. Then we can give a topology on the space of ordered bases in such a way that makes the bijection in (2.1) a homeomorphism.

The second bijection is given from the ordered bases into the hyperbolic half-plane by an appropriate rotation and scaling in the following way:

$$\frac{\{\text{ordered lattices in}\mathbb{R}^2\}}{\mathrm{O}(2)\cup\{\text{homotheties}\}} \quad\leftrightarrow\quad \mathbb{H}^2 = \{x + iy \mid y > 0\} \tag{2.2}$$



First (with an appropriate rotation) we can suppose that our ordered lattice $\{w_1, w_2\}$ has the first vector w_1 in the real part of $\mathbb{C}$. From that lattice we can scale and get $\{1, w = \frac{w_2}{w_1}\}$. In this way we get two representatives $\{1, w\}$ and $\{1, -w\}$, so we choose the one that is in the upper half plane (assume it is w). Then we have a well defined $w \in \mathbb{H}^2$. This means that every $\tau \in \mathbb{H}^2$ is representing the lattice $\{1, \tau\}$.

One has the usual topology of $\mathbb{H}^2$ that is equivalent to the topology of $\mathbb{R}^2$. Then one can show that the bijection in (2.2) is also a homeomorphism.

Remark 2.1.1. One can have an easier way to see the bijection by noticing that $\mathrm{SL}(2,\mathbb{R})$ acts transitively on $\mathbb{H}^2$ with point stabilizers isomorphic to $\mathrm{SO}(2)$.

Using the two homeomorphisms in (2.1) and in (2.2), and adding the homotheties, we obtain the identification

$$\mathrm{O}(2)\backslash \mathrm{GL}(2,\mathbb{R}) \cong \mathbb{R}^+ \times \mathbb{H}^2. \tag{2.3}$$

Now we study the action of $\mathrm{GL}(2,\mathbb{Z})$ on $\mathbb{H}^2$ via the bijection resulting from the composition of the two previous bijections in (2.1) and in (2.2). This means that we have to compute the transformation we get on $\mathbb{H}^2$ from the matrix multiplication on $\mathrm{GL}(2,\mathbb{R})$.

Observe that we are quotienting out the orientation reversing matrices, since we are choosing a representative of the class of lattices. Then it only make sense to consider the ones with positive determinant, i.e., the group $\mathrm{SL}(2,\mathbb{Z})$.

Let $U \in \mathrm{SL}(2,\mathbb{Z})$ and $X \in \mathrm{GL}(2,\mathbb{R})$. We have that $\mathrm{SL}(2,\mathbb{Z})$ is acting by matrix multiplication on the right: XU. This is equivalent to the action on the left as U^tX^t. Thus

$$\begin{aligned} UX &= \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ &= \begin{pmatrix} dw_1 + cw_2 \\ bw_1 + aw_2 \end{pmatrix} = \begin{pmatrix} w_1' \\ w_2' \end{pmatrix}. \end{aligned}$$

Then the correspondent representatives in $\mathbb{H}^2$ are:

$$(w_1, w_2) \sim (1, \tfrac{w_2}{w_1}) \quad \text{and} \quad (w_1', w_2') \sim (1, \tfrac{w_2'}{w_1'}) \quad \text{with}$$

$$\frac{w_2'}{w_1'} = \frac{bw_1 + aw_2}{dw_1 + cw_2} = \frac{b + a\frac{w_2}{w_1}}{d + c\frac{w_2}{w_1}}.$$

Therefore $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathbb{H}^2$ via Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} w = \frac{aw + b}{cw + d},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ and $w \in \mathbb{H}^2$.

We can even compute the fundamental domain of the action that will give us the full information about the quotient space $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$. First, let us recall the definition of fundamental domain.

Definition 2.1.2. A **fundamental domain** for a discrete group Γ acting on $\mathbb{H}^2$, is a connected open subset D of $\mathbb{H}^2$ such that no two points of D are equivalent under Γ and $\mathbb{H}^2 = \cup_{\gamma\in\Gamma}\gamma\overline{D}$, where $\overline{D}$ is the closure of D.

To compute the fundamental domain in $\mathbb{H}^2$ given by the action of $\mathrm{SL}(2,\mathbb{Z})$, we use that $\mathrm{SL}(2,\mathbb{Z})$ has two generators $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. This means that we have two transformations:

$$T(w) = w + 1 \quad \text{and} \quad S(w) = -\frac{1}{w}.$$

The first one is just a translation and the second one is a composition of an inversion of the circle of radius one and center in the origin and a reflection. One can see that the fundamental domain looks:

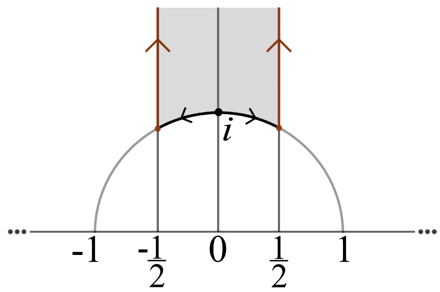


Figure 2.1: The fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}^2$.

Making the corresponding identifications we get an orbifold with two singular points that is homeomorphic to a punctured sphere which is contractible. Thus the moduli space of flat metrics on the 2-torus is the product of two contractible spaces:

$$\mathcal{M}_{flat}(\mathrm{T}^2) \cong \mathbb{R}^+ \times (\mathbb{H}^2/\mathrm{SL}(2,\mathbb{Z})) \cong \mathbb{R}^+ \times (\mathbb{S}^2 \setminus \{*\}) \cong \mathbb{R}^3.$$

2.2 The Klein bottle

Now we are going to determine the moduli space of flat metrics for the Klein bottle.

The Klein bottle K^2 can be described as a quotient of $\mathbb{R}^2$ by a discrete subgroup of $\mathrm{Iso}(2)$ as follows: $\mathrm{K}^2 = \mathbb{R}^2/\pi$, where π is a group generated by a translation and a glide reflection in independent directions. We can fix π to be generated by $(\mathrm{Id},\ e_1)$ and $(E, \frac{1}{2}e_2)$, where $E(x,y) = (-x,y)$ is the reflection with respect the y-axis and $\{e_1, e_2\}$ is the standard basis of $\mathbb{R}^2$. Observe that the holonomy of K^2 is generated by the reflection E, $\mathrm{H}_\pi = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle = \mathbb{Z}_2$.

As we have seen before, every lattice gives us a metric in the torus, but for the Klein bottle the lattice needs to behave well with respect to the glide reflection. This means that only the orthogonal lattices will induce a metric on the Klein bottle. This will be clear in the following paragraphs (see also [22, Proposition 2.5.9]).

We compute the spaces C_π and the normalizer of π as in Theorem 1.3.5, in order to compute the moduli space of flat metrics. Let us compute $\{\gamma \in \mathrm{Aff}(n) \mid \gamma\pi\gamma^{-1} \subset \mathrm{Iso}(n)\}$. For this we will use the description of Lemma 1.3.8 to compute C_π. We want to know which $X = (x_1, x_2) \in \mathrm{GL}(2,\mathbb{R})$ satisfy

$$(X^tX)E = E(X^tX), \quad \text{where} \quad E = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus we have

$$\begin{aligned} \begin{pmatrix} \langle x_1, x_1\rangle & \langle x_1, x_2\rangle \\ \langle x_2, x_1\rangle & \langle x_2, x_2\rangle \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \langle x_1, x_1\rangle & \langle x_1, x_2\rangle \\ \langle x_2, x_1\rangle & \langle x_2, x_2\rangle \end{pmatrix}, \\ \text{then} \quad \begin{pmatrix} -\langle x_1, x_1\rangle & \langle x_1, x_2\rangle \\ -\langle x_2, x_1\rangle & \langle x_2, x_2\rangle \end{pmatrix} &= \begin{pmatrix} -\langle x_1, x_1\rangle & -\langle x_1, x_2\rangle \\ \langle x_2, x_1\rangle & \langle x_2, x_2\rangle \end{pmatrix}. \end{aligned}$$

This can only happen if $x_1 \perp x_2$. This means that

$$C_\pi = \{X \in \mathrm{GL}(2,\mathbb{R}) \mid x_1 \perp x_2\} = \mathrm{O}(2) \cdot (\mathbb{R}^+)^2.$$

Now we compute the normalizer $\mathrm{N}_{\mathrm{Aff}(\mathbb{R}^2)}(\pi)$. First we find the matrix part $\mathcal{N}_\pi$, which is a matrix $X \in \mathrm{GL}(2,\mathbb{Z})$ such that $XEX^{-1} = E$, since we need to preserve the generator of the holonomy in order to generate the whole group. That is equivalent to solving the equation $XE = EX$ and adding the conditions of determinant ± 1 with integer entries. We only get

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

which can be seen as the 2-dihedral group D_2 or, equivalently, the Klein 4-group.

Second, the translation part is computed as follows. Let any translation $\gamma = (\mathrm{Id}, u) \in \mathrm{Aff}(2)$ and $\epsilon = (E, v) \in \pi$, where $u = (u_1, u_2)$ and $v = n_1 e_1 + \frac{(2n_2+1)}{2} e_2$. Then

$$\begin{aligned} \gamma\epsilon\gamma^{-1} &= (\mathrm{Id}, u)(E, v)(\mathrm{Id}, -u) \\ &= (E, -E(u_1, u_2) + v + (u_1, u_2) \\ &= (E, (2u_1, 0) + v). \end{aligned}$$

Therefore, we need that $2u_1 \in \mathbb{Z}$. This means that the translations that normalize are $\frac{1}{2}\mathbb{Z} \oplus \mathbb{R}$.

Putting together the two computations we have that

$$\mathrm{N}_{\mathrm{Aff}(\mathbb{R}^n)}(\pi) = D_2 \ltimes (\tfrac{1}{2}\mathbb{Z} \oplus \mathbb{R}).$$

With this information we can now see that the moduli space of flat metrics is

$$\begin{aligned} \mathcal{M}_{flat}(\mathrm{K}^2) &= \mathrm{O}(2) \ltimes \mathbb{R}^2 \backslash \mathrm{O}(2) \cdot (\mathbb{R}^*)^2 \ltimes \mathbb{R}^2 / D_2 \ltimes (\tfrac{1}{2}\mathbb{Z} \oplus \mathbb{R}) \\ &= \mathrm{O}(2) \backslash \mathrm{O}(2) \cdot (\mathbb{R}^*)^2 / D_2 \\ &= (\mathbb{R}^+)^2. \end{aligned}$$

Thanks to the reflection, we have more restrictions for the Klein bottle. Thus the dimension of the moduli space of flat metrics in comparison to that of the 2-torus decreases. Actually, this can be observed already in the Teichmüller space. Furthermore, for this case the moduli space of flat metrics and the Teichmüller space are homeomorphic, telling us that the normalizer is not giving us any new information about the space. This situation will change for some cases in dimension 3 and 4.

2.3 Fundamental domains for congruence subgroups

In this section we are going to explain how to compute the fundamental domain of the action of a subgroup of $\mathrm{SL}(2, \mathbb{Z})$ on the hyperbolic plane. Later, we will compute it for some specific subgroups. The information of this section is based on [1], [16] and [18].

From section 2.1 we know that $\mathrm{O}(2)\backslash\mathrm{GL}(2, \mathbb{R}) \cong \mathbb{R}^+ \times \mathbb{H}^2$, and that the group $\mathrm{SL}(2, \mathbb{Z})$ acts on the hyperbolic plane by Möbius transformations. Now we are interested in computing the fundamental domain of some subgroups of $\mathrm{SL}(2,\mathbb{Z})$ because in the next section we will find out those types of double cosets and we would like to say something about their topology.

2.3.1 Subgroups of $\mathbf{SL}(2,\mathbb{Z})$

There are a lot of discrete groups acting on $\mathbb{H}^2$, usually studied in number theory. Some of them receive a special name, for example:

- Any discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ acting on $\mathbb{H}^2$ is called **Fuchsian group**.

There are also some special names for subgroups of $\mathrm{SL}(2,\mathbb{Z})$:

- **The principal congruence subgroup** of level N is

$$\Gamma(N)^+ = \{A \in \mathrm{SL}(2,\mathbb{Z}) \mid A \equiv \mathrm{Id} \bmod N\}$$

  for any natural number N. By the congruence of matrices we mean the respective congruence of integer numbers in each entry.

- A **congruence subgroup** of $\mathrm{SL}(2,\mathbb{Z})$ is a subgroup containing $\Gamma(N)^+$ for some N. For example:

$$\Gamma_0(N)^+ = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \mid c \equiv 0 \bmod N \right\}.$$

Remark 2.3.1. The usual notation is $\Gamma(N)$ without the "+" since one always considers elements with positive determinant. In our case we consider also elements with negative determinant in order to use Wolf's notation, even though for computing the double coset it only the positive determinant elements are important.

2.3.2 Fundamental domains for Fuchsian groups

We already saw the definition of fundamental domain in 2.1.2. The next theorem will tell us how to relate the fundamental domain of a group and its subgroups:

Theorem 2.3.2. *Let Γ be a discrete subgroup of $SL(2,\mathbb{R})$, and let D be a fundamental domain for Γ. Let Γ' be a subgroup of Γ of finite index,* *and choose elements* $\gamma_1, \ldots, \gamma_m$ *in* Γ *such that we have the disjoint union* $\overline{\Gamma} = \overline{\Gamma'}\bar{\gamma}_1 \sqcup \cdots \sqcup \overline{\Gamma'}\bar{\gamma}_m$, *where a bar denotes the images in* $Aut(\mathbb{H}^2)$. *Then*

$$D' = \bigcup_{i=1}^{m} \gamma_i D$$

is a fundamental domain for Γ'.

Remark 2.3.3. It is possible to choose the γ_i so that the closure of D' is connected; the interior of the closure of D' is then a connected fundamental domain for Γ.

2.3.3 The fundamental domain for some congruence subgroups of SL$(2, \mathbb{Z})$

First consider the following subgroups of $\mathrm{GL}(2, \mathbb{Z})$ that are going to appear in the next sections:

$$\begin{aligned}
\Gamma_0(2) &= \{X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid c \equiv 0 \bmod 2\} \\
&= \left\{ \begin{pmatrix} 2a+1 & b \\ 2c & 2d+1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid a, c, d \in \mathbb{Z} \right\}, \\
\Gamma(2) &= \{X \in \mathrm{GL}(2,\mathbb{Z}) \mid X \equiv \mathrm{Id} \bmod 2\} \\
&= \left\{ \begin{pmatrix} 2a+1 & 2b \\ 2c & 2d+1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid a, b, c, d \in \mathbb{Z} \right\}.
\end{aligned}$$

For describing the next subgroup, we denote $Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then the subgroup of $\mathrm{GL}(2, \mathbb{Z})$ is given by

$$\Gamma(2)Y := \Gamma(2) \cdot \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

$$= \left\{ \begin{pmatrix} 2a+1 & 2b \\ 2c & 2d+1 \end{pmatrix}, \begin{pmatrix} 2a & 2b+1 \\ 2c+1 & 2d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid a, b, c, d \in \mathbb{Z} \right\}.$$

In this section, we consider the subgroups of the previous groups given by the matrices with positive determinant. These are the three congruence subgroups to which we will compute their fundamental domain on $\mathbb{H}^2$. The notation for these groups, as introduced in the section 1.1, is:

$$\Gamma_0(2)^+, \quad \Gamma(2)^+ \quad \text{and} \quad \Gamma(2)Y^+.$$

The algorithm to compute the fundamental domain of a subgroup Γ of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}^2$, deduced from the Theorem 2.3.2, is:

1. Compute the index of Γ in $\mathrm{SL}(2,\mathbb{Z})$.
2. Find representatives in $\mathrm{SL}(2,\mathbb{Z})$ for Γ.
3. Apply the respective representatives transformations to the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}^2$.

- We carry out these steps for $\Gamma_0(2)^+$.

1. Compute the index of $\Gamma_0(2)^+$ in $\mathrm{SL}(2,\mathbb{Z})$.

We will use that the index of $\Gamma(2)^+$ in $\mathrm{SL}(2,\mathbb{Z})$ is: $[\mathrm{SL}(2,\mathbb{Z}):\Gamma(2)^+]=6$, this is well explained in [18], page 20-22.

We have that $\Gamma(2)^+ < \Gamma_0(2)^+ < \mathrm{SL}(2,\mathbb{Z})$, then

$$[\mathrm{SL}(2,\mathbb{Z}):\Gamma(2)^+] = [\mathrm{SL}(2,\mathbb{Z}):\Gamma_0(2)^+][\Gamma_0(2)^+:\Gamma(2)^+],$$

since the index is multiplicative. This means that $[\mathrm{SL}(2,\mathbb{Z}):\Gamma_0(2)^+]\leq 3$, because $[\Gamma_0(2)^+:\Gamma(2)^+]\neq 1$.

On the other hand, we have that for any group G and subgroup $H<G$, if $g\in G$, then $g^n\in H$ with $n=[G:H]$.

Consider $B=\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\notin \Gamma_0(2)^+$. Then $B^2\notin\Gamma_0(2)^+$ but $B^3\in\Gamma_0(2)^+$. This means that $[\mathrm{SL}(2,\mathbb{Z}):\Gamma_0(2)^+]\geq 3$. Therefore, $[\mathrm{SL}(2,\mathbb{Z}):\Gamma_0(2)^+]=3$.

2. Find representatives in $\mathrm{SL}(2,\mathbb{Z})$ for $\Gamma_0(2)^+$.

Consider

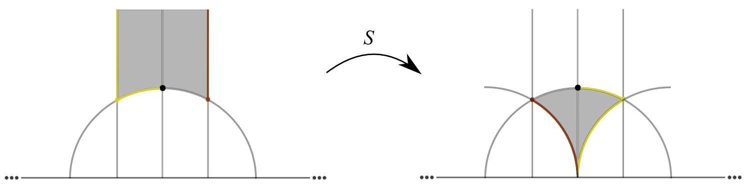
$\gamma_1=\mathrm{Id}, \qquad \gamma_2=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}=S, \qquad \gamma_3=\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}=ST.$

They are representatives of $\Gamma_0(2)^+$ since:

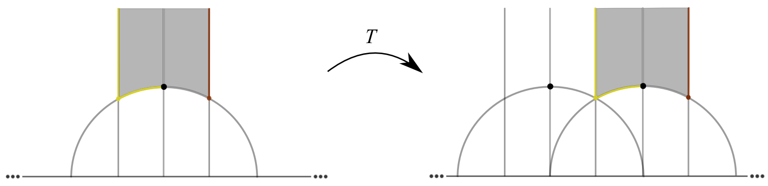
$$\begin{aligned}
\Gamma_0(2)^+ &= \left\{\begin{pmatrix} 2a+1 & b \\ 2c & 2d+1 \end{pmatrix}\in \mathrm{GL}(2,\mathbb{Z}) \mid a,c,d\in\mathbb{Z}\right\},\\
\Gamma_0(2)^+\gamma_2 &= \left\{\begin{pmatrix} a & 2b+1 \\ 2c+1 & 2d \end{pmatrix}\in \mathrm{GL}(2,\mathbb{Z}) \mid b,c,d\in\mathbb{Z}\right\},\\
\Gamma_0(2)^+\gamma_3 &= \left\{\begin{pmatrix} a & b \\ 2c+1 & 2d+1 \end{pmatrix}\in \mathrm{GL}(2,\mathbb{Z}) \mid c,d\in\mathbb{Z}\right\}.
\end{aligned}$$

3. Compute what happens to the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$. Every transformation of $\mathrm{SL}(2,\mathbb{Z})$ is generated by S and T. Then it is enough to see what these two transformations are doing to the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}^2$.

The map S is an inversion together with a reflection:



The map T is just a translation:



Since we have the representatives in terms of S and T, we make the corresponding compositions to obtain that the fundamental domain is

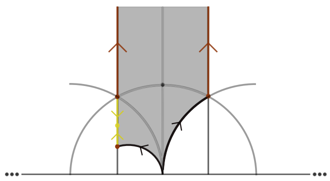


Figure 2.2: The fundamental domain of $\Gamma_0(2)^+$ on $\mathbb{H}^2$.

The borders of the fundamental domain are identified by T, $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$, and $\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \in \Gamma_0(2)^+$. Doing the border identifications we have an orbifold which is homeomorphic to a cylinder.

- We carry out the steps for $\Gamma(2)^+$.

1. Again, the index of $\Gamma(2)^+$ in $\mathrm{SL}(2,\mathbb{Z})$ is

$$[\mathrm{SL}(2,\mathbb{Z}) : \Gamma(2)^+] = 6. \quad ([18],\ \text{page } 20-22).$$

2. The representatives in $\mathrm{SL}(2,\mathbb{Z})$ for $\Gamma(2)^+$ (that we choose) are

$$\gamma_1 = \mathrm{Id},\ \gamma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = S,\ \gamma_3 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = ST,\ \gamma_4 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = T,$$
$$\gamma_5 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = TS,\ \gamma_6 = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} = TST^{-1}.$$

3. Since we already expressed the representatives in terms of the generators T and S, we can apply them easier to the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$. In this way we obtain the fundamental domain for $\Gamma(2)^+$ on $\mathbb{H}^2$:

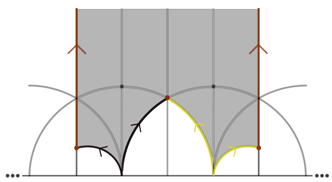


Figure 2.3: The fundamental domain of $\Gamma(2)^+$ on $\mathbb{H}^2$.

The borders of the fundamental domain are identified by T^2, $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$, and $\begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix} \in \Gamma(2)^+$. Doing the border identifications, as shown in the following picture, we have an orbifold which is homeomorphic to a 3-punctured sphere.

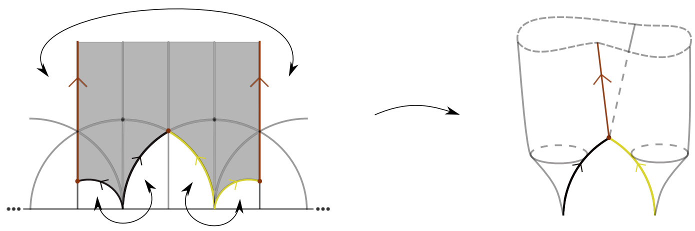


Figure 2.4: Doing the border identifications.

- The steps to compute the fundamental domain of the group $\Gamma(2)Y^+$ on $\mathbb{H}^2$:

1. With a similar procedure as the case of $\Gamma_0(2)$, we get that the index of $\Gamma(2)Y^+$ in $\mathrm{SL}(2,\mathbb{Z})$ is

$$[\mathrm{SL}(2,\mathbb{Z}) : \Gamma(2)Y^+] = 3.$$

2. The representatives in $\mathrm{SL}(2,\mathbb{Z})$ for $\Gamma(2)Y^+$ (that we choose) are

$$\gamma_1 = \mathrm{Id}, \qquad \gamma_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = T, \qquad \gamma_3 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = TS.$$

3. Since we already expressed the representatives in terms of the generators T and S, we can apply them easier to the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$. In this way we obtain the fundamental domain for $\Gamma(2)Y^+$ on $\mathbb{H}^2$:

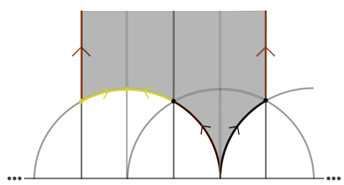


Figure 2.5: The fundamental domain of $\Gamma(2)Y^+$ on $\mathbb{H}^2$.

We can notice that the fundamental domain of $\Gamma(2)Y^+$ is quite similar to the one of $\Gamma_0(2)^+$ and it is also homeomorphic to a cylinder.

Chapter 3

3-dimensional closed flat manifolds

In this section, we present some steps and remarks necessary to compute the moduli space of flat metrics. Then, we present the lists of moduli spaces of flat metrics for 3-dimensional closed manifolds.

Recall that to describe the moduli space of flat metrics, as we have seen already in the 2-dimensional case, we have to compute two spaces:

(a) The cone space

$$C_\pi := \{X \in \mathrm{GL}(3, \mathbb{R}) \mid XH_\pi X^{-1} \subset \mathrm{O}(3)\}.$$

(b) The normalizer

$$\mathrm{N}_{\mathrm{Aff}(3)}(\pi) = \{\alpha \in \mathrm{Aff}(3) \mid \alpha\pi\alpha^{-1} = \pi\}.$$

Here, π is one of the 3-dimensional Bieberbach groups.

We will proceed as follows. First, we list the Bieberbach groups for the 3-dimensional manifolds; secondly, we determine the cone space C_π, then we determine the matrix part of the normalizer $\mathrm{N}_{\mathrm{Aff}(3)}(\pi)$ and we describe the moduli space of flat metrics. Finally, we study the topology of those moduli spaces using the tools from the previous Chapter.

3.1 List of the 3-dimensional Bieberbach groups

There are only 10 Bieberbach groups in dimension 3 up to affine change of coordinates. Out of them, six give orientable manifolds and four give non-orientable manifolds. We present a list of them. The representations can also be found in [22] or [13].

We need some notation. We shall denote by $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ the vectors of the standard basis of $\mathbb{R}^3$. The basic translations of $\mathbb{R}^3$ are denoted by $t_i = (\mathrm{Id}, e_i)$. Also, the rotation matrix by an angle $\theta \in [0, 2\pi]$ is denoted as

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

3.1.1 The orientable 3-dimensional closed flat manifolds

Here is the list of the Bieberbach groups for the orientable 3-dimensional flat manifolds.

$\mathbf{G_1} = \mathrm{T}^\mathbf{3}$: $H_\pi = \{\mathrm{Id}\}$, $\quad \pi = \langle t_1, t_2, t_3 \rangle$.

$\mathbf{G_2}$: $H_\pi = \mathbb{Z}_2$, $\quad \pi = \langle t_1, t_2, t_3, \alpha = (A, \frac{1}{2}e_1) \rangle$,

where $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & R(\pi) \end{pmatrix}.$

$\mathbf{G_3}$: $H_\pi = \mathbb{Z}_3$, $\quad \pi = \langle t_1, s_1 = (\mathrm{Id}, A(e_2)), s_2 = (\mathrm{Id}, A^2(e_2)), \alpha = (A, \frac{1}{3}e_1) \rangle$,

where $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & R(\frac{2\pi}{3}) \end{pmatrix}.$

$\mathbf{G_4}$: $H_\pi = \mathbb{Z}_4$, $\quad \pi = \langle t_1, t_2, t_3, \alpha = (A, \frac{1}{4}e_1) \rangle$,

where $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & R(\frac{\pi}{2}) \end{pmatrix}.$

$\mathbf{G_5}$: $H_\pi = \mathbb{Z}_6$, $\quad \pi = \langle t_1, s_1 = (\mathrm{Id}, A(e_2)), s_2 = (\mathrm{Id}, A^2(e_2)), \alpha = (A, \frac{1}{6}e_1) \rangle$,

where $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & R(\frac{\pi}{3}) \end{pmatrix}.$

$\mathbf{G_6}$: $H_\pi = \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \pi = \langle t_1, t_2, t_3, \alpha = (A, \frac{1}{2}e_1), \beta = (B, \frac{1}{2}(e_2 + e_3))\rangle,$

where $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$

3.1.2 The non-orientable 3-dimensional closed flat manifolds

We now list the Bieberbach groups for the non-orientable 3-dimensional flat manifolds.

$\mathbf{B_1} = \mathbb{S}^1 \times K^2$: $H_\pi = \mathbb{Z}_2, \quad \pi = \langle t_1, t_2, t_3, \epsilon = (E, \frac{1}{2}e_1)\rangle,$

where $\quad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$

$\mathbf{B_2}$: $H_\pi = \mathbb{Z}_2, \quad \pi = \langle t_1, t_2, s = (\mathrm{Id}, \frac{1}{2}(e_1 + e_2) + e_3), \epsilon = (E, \frac{1}{2}e_1)\rangle,$

where $\quad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$

$\mathbf{B_3}$: $H_\pi = \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \pi = \langle t_1, t_2, t_3, \alpha = (A, \frac{1}{2}e_1), \epsilon = (E, \frac{1}{2}e_1)\rangle,$

where $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$

$\mathbf{B_4}$: $H_\pi = \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \pi = \langle t_1, t_2, t_3, \alpha = (A, \frac{1}{2}e_1), \epsilon = (E, \frac{1}{2}(e_2 + e_3))\rangle,$

where $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$

3.2 The cone space

The cone space C_π is easy to analyze since it only depends on the holonomy. To describe the space C_π, one has to solve the equation:

$$X \in \mathrm{GL}(3, \mathbb{R}) \quad \text{such that} \quad (X^t X)A = A(X^t X) \tag{3.1}$$

for all $A \in H_\pi$ (see Lemma 1.3.8). Recall that we can consider the columns of $X = (x_1, x_2, x_3)$ as vectors in $\mathbb{R}^3$ to get the following description:

$$X^t X = \begin{pmatrix} \langle x_1, x_1 \rangle & \langle x_1, x_2 \rangle & \langle x_1, x_3 \rangle \\ \langle x_2, x_1 \rangle & \langle x_2, x_2 \rangle & \langle x_2, x_3 \rangle \\ \langle x_3, x_1 \rangle & \langle x_3, x_2 \rangle & \langle x_3, x_3 \rangle \end{pmatrix}.$$

Solving the Equation (3.1) will give some conditions for the vectors x_i.

To express these spaces, we use the notation $H_1 \cdot H_2$, given in the section 1.1. The descriptions are done in [11] and [12]. We present some details because some of the cases appear again in dimension 4.

Proposition 3.2.1. *The possible spaces C_π for the 3-dimensional closed flat manifolds are the following:*

1. *If the holonomy is trivial, then $C_\pi = GL(3, \mathbb{R})$.*
2. *If $H_\pi = \mathbb{Z}_2$, then $C_\pi = O(3) \cdot (\mathbb{R}^* \times GL(2, \mathbb{R}))$ or $O(3) \cdot (GL(2, \mathbb{R}) \times \mathbb{R}^*)$.*
3. *If $H_\pi = \mathbb{Z}_k$ with $k = 3, 4, 6$, then $C_\pi = O(3) \cdot (\mathbb{R}^* \times (\mathbb{R}^+ \times O(2)))$.*
4. *If $H_\pi = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $C_\pi = O(3) \cdot (\mathbb{R}^*)^3$.*

We explain cases 2 and 3 in the preceding proposition.

Case 2. The case of $H_\pi = \mathbb{Z}_2$ generated by $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. We solve the Equation (3.1), and we get:

$$\begin{aligned} C_\pi &= \{(x_1, x_2, x_3) \in \mathrm{GL}(3, \mathbb{R}) \mid x_1 \perp x_2 \text{ and } x_1 \perp x_3\} \\ &= \mathrm{O}(3) \cdot \left\{ \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix} \mid a \in \mathbb{R}^* \text{ and } B \in \mathrm{GL}(2, \mathbb{R}) \right\} \\ &= \mathrm{O}(3) \cdot (\mathbb{R}^* \times \mathrm{GL}(2, \mathbb{R})). \end{aligned}$$

Note that when the holonomy is generated by $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, then the second factor has different order: $\mathrm{O}(3) \cdot (\mathrm{GL}(2, \mathbb{R}) \times \mathbb{R}^*)$.

Case 3. In the cases of cyclic holonomy $\mathbb{Z}_3$, $\mathbb{Z}_4$, or $\mathbb{Z}_6$, the generator is of the form $A = \begin{pmatrix} 1 & 0 \\ 0 & R(\theta) \end{pmatrix}$ with $\theta = \frac{2\pi}{3}, \frac{\pi}{2}$ or $\frac{\pi}{3}$, respectively. Thus, when we solve the Equation (3.1), the rotation must be preserved. This gives an extra restriction on the vectors, namely:

$$\begin{aligned} C_\pi &= \{(x_1, x_2, x_3) \in \mathrm{GL}(3, \mathbb{R}) \mid x_i \perp x_j \ i \neq j \text{ and } \|x_2\| = \|x_3\|\} \\ &= \mathrm{O}(3) \cdot \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \mid a, b \in \mathbb{R}^* \right\} \\ &= \mathrm{O}(3) \cdot (\mathbb{R}^* \times (\mathbb{R}^+ \times \mathrm{O}(2)). \end{aligned}$$

Remark 3.2.2. If $A \in H_\pi$ and $X \in \mathrm{GL}(n, \mathbb{R})$ such that $XAX^{-1} \in \mathrm{O}(n)$ then $XA^rX^{-1} = (XAX^{-1})^r \in \mathrm{O}(n)$ for any $r \in \mathbb{N}$.

3.3 The normalizer

To describe the moduli space of flat metrics, it is important to compute the matrix part of the normalizer, $\mathcal{N}_\pi$, of the given Bieberbach group π in the group of affine transformations. As we saw in Corollary 1.5.2, the group $\mathcal{N}_\pi$ will not change the dimension of the moduli space of flat metrics with respect to the Teichmüller space, but $\mathcal{N}_\pi$ can make the moduli space of flat metrics have interesting topology, in the sense that it could be a non-contractible space.

We use some descriptions from [13] but we present them in a different fashion. We found out that there are some mistakes in [13]; this is going to be highlighted when is necessary. For some computations we use the program Mathematica as a tool.

3.3.1 General approach

The description of the normalizer of a Bieberbach group in the affine transformations depends not only on the holonomy but on the affine structure as well, i.e., on how the translations are acting. We have two situations:

(a) The matrix part of the normalizer $\mathcal{N}_\pi$ is not always the same as $\mathrm{N}_{\mathrm{GL}(n,\mathbb{Z})}(H_\pi)$.

(b) It is not always possible to express $\mathrm{N}_{\mathrm{Aff}(n)}(\pi)$ as a semidirect product.

This two situations lead us to the following definitions and lemmas. We are going to present an example of each definition.

Definition 3.3.1. We say that the Bieberbach group π has **trivial lattice**, when its lattice is $L_\pi = \mathbb{Z}^n \subset \pi$.

For example, G_2 has trivial lattice, but G_3 do not has trivial lattice.

Definition 3.3.2. Let π be a Bieberbach group with non trivial holonomy. We say that the group has **translation part not involved**, when for $X \in \mathrm{N}_{\mathrm{GL}(n,\mathbb{Z})}(H_\pi)$ we have that $X(v) = v = (v_1, \ldots, v_n)$ or $X(v) = (u_1, \ldots, u_n)$, with $u_i = -v_i$ for some $i \in I \subseteq \{1, \ldots, n\}$ and $u_j = v_j$ for $j \notin I$; for each generator $\alpha = (A, v)$ of π such that $A \neq \mathrm{Id}$. Otherwise, we say it has **translation part involved**.

Example 3.3.3. An example of a Bieberbach group with translation part not involved is G_2. The generator that is not a translation is $\alpha = (A, \frac{1}{2}e_1)$. Solving the equation for $X \in \mathrm{GL}(3, \mathbb{Z})$ such that $XA = AX$, where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

we conclude that the matrices X have the form

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & X_2 \end{pmatrix} \quad \text{with} \quad X_2 \in \mathrm{GL}(2, \mathbb{Z}).$$

Therefore $X(\frac{1}{2}e_1) = \pm\frac{1}{2}e_1$.

Example 3.3.4. An example of a Bieberbach group with translation part involved is B_1. The generator that is not a translation is $\epsilon = (E, \frac{1}{2}e_1)$. Solving the equation for $X \in \mathrm{GL}(3, \mathbb{Z})$ such that $XE = EX$, where

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

we conclude that the matrices X have the form

$$\begin{pmatrix} X_1 & 0 \\ 0 & \pm 1 \end{pmatrix} \quad \text{with} \quad X_1 \in \mathrm{GL}(2, \mathbb{Z}).$$

Therefore $X(\frac{1}{2}e_1)$ is not always $\pm\frac{1}{2}e_1$.

Definition 3.3.5. Let $\alpha = (A, v) \in \pi$ with $A \neq \mathrm{Id}$. Then the **lattice for** α is $\alpha L_\pi = \{(A, v)(\mathrm{Id}, u) \mid (\mathrm{Id}, u) \in L_\pi\}$, that is, all the affine transformations in π that have the matrix part equal to A.

Example 3.3.6. The lattice for ϵ in B_1 is

$$\begin{aligned} \epsilon L_\pi &= \{(E, \frac{1}{2}e_1)(\mathrm{Id}, u) \mid (\mathrm{Id}, u) \in L_\pi = \mathbb{Z}^3\} \\ &= \{(E, v) \mid v = \frac{2n_1 + 1}{2}e_1 + n_2 e_2 + n_3 e_3, \quad \text{with} \quad n_1, n_2, n_3 \in \mathbb{Z}\}. \end{aligned}$$

Using this notation we state:

Lemma 3.3.7. *Let π be a Bieberbach group with trivial lattice, translation part not involved and assume that the lattices of the generators of the holonomy are preserved by multiplying by -1, i.e., for $(A, v) \in \pi$ with $A \neq Id$, we have $(A, -v) \in \pi$. Then*

$$N_{Aff(n)}(\pi) = N_{GL(n,\mathbb{Z})}(H_\pi) \ltimes T,$$

where T are the translations of the normalizer.

Proof. Let $X \in \mathrm{GL}(n, \mathbb{Z})$ such that $XH_\pi X^{-1} = H_\pi$. For some generator (A, v) of π we have

$$(X, 0)(A, v)(X^{-1}, 0) = (XAX^{-1}, X(v)).$$

Then $XAX^{-1} \in H_\pi$ and $X(v)$ is v or $-v$ which means that the lattices with respect to each generator of H_π are preserved. $\square$

This lemma is satisfied by G_2, G_6, B_3 and B_4. Also, in general, by T^n.

Now, we can use the next property in order to see if the normalizer is a semidirect product or not.

Proposition 3.3.8. *Let $G < Aff(n)$ be a subgroup. For all $(X, u) \in G$, we have $(X, 0) \in G$ if and only if $G = M \ltimes T$, where M is the matrix part and T are the translations of G.*

We have to check this property case by case, but having it will give us the advantage to search only for the affine transformations with zero translation that normalize. All the orientable 3-dimensional closed flat manifolds have this property. In the next example we will see the case of a non-orientable one satisfying the property.

Example 3.3.9. The flat manifold B_1 satisfies the property studied in Proposition 3.3.8. Let $(X, u) \in \mathrm{N}_{\mathrm{Aff}(3)}(\pi)$ and $(E, v) \in \pi$, we have:

$$(X, u)(E, v)(X^{-1}, -X^{-1}(u)) = (XEX^{-1}, -XEX^{-1}(u) + X(v) + u).$$

Set $u = (a, b, c)$, then

$$\begin{aligned}(XEX^{-1}, -XEX^{-1}(u) + X(v) + u) &= (E, -E(u) + u + X(v)) \\ &= (E, (0, 0, 2c) + X(v)) \in \epsilon L_\pi.\end{aligned}$$

This means that the translation vector is of the form (see example 3.3.6)

$$(0, 0, 2c) + X(v) = (\frac{2n_1 + 1}{2}, n_2, 2c + n_3)$$

where the n_i are in $\mathbb{Z}$ and $2c + n_3 \in \mathbb{Z}$. This means that c can be zero, and we are still in the lattice. We have shown that $(X, 0) \in \mathrm{N}_{\mathrm{Aff}(3)}(\pi)$.

The only 3-dimensional flat manifold that does not satisfy Proposition 3.3.8 is B_2.

Example 3.3.10. The flat manifold B_2 do not satisfy the property in Proposition 3.3.8. Indeed, doing the same computation as in the previous example, we have
$(E,(0,0,2c)+X(v)) \in \epsilon L_\pi$. But now we do not have a trivial lattice. Thus the lattice of ϵ is different:

$$\epsilon L_\pi = \{(E,v) \mid v = \frac{2n_1+n_3+1}{2}e_1 + \frac{2n_2+n_3}{2}e_2 + n_3e_3, \text{ with } n_1,n_2,n_3 \in \mathbb{Z}\}.$$

As we can see, the first and second entry depend on the value of n_3. Then, the value of c could change the third entry without changing the other two. That is the reason why we can not consider the translation zero in general.

For the cases that do not have the structure of a semidirect product, it is useful to denote them as follows.

Definition 3.3.11. When Proposition 3.3.8 does not hold, we say that we have **switching cases with translations** for the elements of the normalizer that needs the translation part different from zero.

The approach to compute the matrix part of the normalizer of a Bieberbach group, π, in the group of affine transformations is:

(a) For the cases that do not have trivial lattice sometimes it is convenient to change the representation in order to have trivial lattice, and in this way, the matrix becomes an integer matrix.

(b) If the group satisfies Proposition 3.3.8, we proceed as follows:

   i. Compute the normalizer of H_π in $\mathrm{GL}(n,\mathbb{Z})$, or in a conjugation of this group for the case of non-trivial lattice:

   $$\{X \in \mathrm{GL}(n,\mathbb{Z}) \quad \text{or} \quad Q\,\mathrm{GL}(n,\mathbb{Z})Q^{-1} \mid XH_\pi = H_\pi X\},$$

   where $Q \in \mathrm{GL}(n,\mathbb{R})$.

   ii. See if the translation part is involved or not. Then see which of the matrices computed in step i preserve or interchange the lattices of the respective generators.

(c) If the group does not satisfy Proposition 3.3.8, we do the same as in step (b), but now we have to search for the switching cases with translation. To detect the possible options, one can compute the translations that normalize π.

3.3.2 The normalizer for the 3-dimensional Bieberbach groups

Having the previous notions and examples in hand, we proceed with the description of the matrix part, $\mathcal{N}_\pi$, of the normalizer of the 3-dimensional Bieberbach groups in the group of affine transformations.

First, we do a change of representation by conjugating with a suitable affine transformation as in [13, Lemma 2.2], to G_3 and G_5. Then the representations used for computing the normalizer are, using the same notation:

$$\mathbf{G_3}\text{: } H_\pi = \mathbb{Z}_3, \quad \pi = \langle t_1, t_2, t_3, \alpha = (A, \tfrac{1}{3}e_1)\rangle \quad \text{where} \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

$$\mathbf{G_5}\text{: } H_\pi = \mathbb{Z}_6, \quad \pi = \langle t_1, t_2, t_3, \alpha = (A, \tfrac{1}{6}e_1)\rangle \quad \text{where} \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Proposition 3.3.12. *Let π be one of the Bieberbach groups for the 3-dimensional orientable closed flat manifolds, then the matrix part of the normalizer of π, $\mathcal{N}_\pi$, in Aff(3) are as follows:*

1. *For T^3, $\mathcal{N}_\pi = GL(3, \mathbb{Z})$.*
2. *For G_2, $\mathcal{N}_\pi = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & B \end{pmatrix} \mid B \in GL(2, \mathbb{Z}) \right\}$.*
3. *For G_3, $\mathcal{N}_\pi = D_6 = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$.*
4. *For G_4, $\mathcal{N}_\pi = D_4 = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$.*
5. *For G_5, $\mathcal{N}_\pi = D_6 = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$.*

6. *For* G_6, $\mathcal{N}_\pi = \left\{ \begin{pmatrix} \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \end{pmatrix} \right\}$.

This is proved in [13], but we will explain some cases and make some remarks to use them later as a comparison for the cases in dimension 4.

The group G_2 has trivial lattice, and by Example 3.3.3, it has translation part not involved. When we multiply by -1 a translation vector of the lattice of the generator, it stays inside the lattice. Therefore, as we saw in Lemma 3.3.7, any matrix $X \in \mathrm{N}_{\mathrm{GL}(3,\mathbb{Z})}(H_\pi)$ will give an element $(X, 0) \in \mathrm{N}_{\mathrm{Aff}(3)}(\pi)$.

For G_3, observe that we have cyclic holonomy of order bigger than 2. In this case the next remark is useful:

Remark 3.3.13. Let π be a Bieberbach group with cyclic holonomy $H_\pi = \langle A \rangle$ of order k. To compute the normalizer one has to find $X \in \mathrm{GL}(n, \mathbb{Z})$ such that $XH_\pi X^{-1} = H_\pi$, but for this we need to satisfy $XAX^{-1} = A^r$ with $(r, k) = 1$ (relative primes) in order to always get a generator of the cyclic group. To preserve the lattices of the generators, it is enough to find the X that preserves one type of lattice for one generator.

Continuing with G_3, the preceding remark tells us that there are two cases for $X \in \mathrm{GL}(3, \mathbb{Z})$:

$$XAX^{-1} = \begin{cases} A & \text{case 1} \\ A^2 & \text{case 2.} \end{cases}$$

For both cases we need the matrix with the form $X = \begin{pmatrix} \pm1 & 0 \\ 0 & X_2 \end{pmatrix}$, where the matrix X_2 will be:

Case 1: $X_2 \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} X_2$.

Then the matrix has to be: $\quad X_2 \in \left\langle \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$.

Case 2: $X_2 \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} X_2$.

Then the matrix has to be: $\quad X_2 \in \left\{ \left\langle \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$.

As we can observe, the group G_3 has trivial lattice and translation part not involved. Then we just have to see if multiplying by -1 in the first entry affects the lattices of the generators:

$$\begin{aligned}\alpha L_\pi &= \{(A, \frac{1}{3}e_1)(\mathrm{Id}, v) \mid (\mathrm{Id}, v) \in L_\pi = \mathbb{Z}^3\}\\ &= \{(A, \hat{v}) \mid \hat{v} = \frac{3n_1+1}{3}e_1 - n_3e_2 + (n_2 - n_3)e_3, \text{ with } n_1, n_2, n_3 \in \mathbb{Z}\};\end{aligned}$$

$$\alpha^2 L_\pi = \{(A^2, \hat{v}) \mid \hat{v} = \frac{3n_1+2}{3}e_1 + (n_3 - n_2)e_2 - n_2e_3, \text{ with } n_1, n_2, n_3 \in \mathbb{Z}\}.$$

Then, the 1 in the first entry of the matrix X preserve the lattices (necessary for Case 1) and the -1 switches the lattices (necessary for Case 2). Therefore, the normalizer is

$$D_6 = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle.$$

All of the Bieberbach groups of Proposition 3.3.12 have trivial lattice, translation part not involved and the normalizer is a semidirect product, which means that to compute the normalizer of π one just has to compute the normalizer of H_π in $\mathrm{GL}(3, \mathbb{Z})$ and the lattices for αL_π to see if it is affected by multiplying by -1, as we did for the case of G_3.

Now, we present the result for the non-orientable manifolds.

Proposition 3.3.14. *Let π be one of the Bieberbach groups for the 3-dimensional non-orientable closed flat manifolds. Then the matrix part of the normalizer of π, $\mathcal{N}_\pi$, in Aff(3) is as follows:*

1. *For B_1, $\mathcal{N}_\pi = \left\{ \begin{pmatrix} \Gamma_0(2) & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}$.*
2. *For B_2, $\mathcal{N}_\pi = \left\{ \begin{pmatrix} \Gamma(2) & 0 \\ 0 & \pm 1 \end{pmatrix} \cdot \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle \right\}$.*
3. *For B_3 or B_4, $\mathcal{N}_\pi = \left\{ \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \right\}$.*

We will only prove the proposition for B_1 and B_2, which are the interesting cases.

The group B_1 has trivial lattice and the normalizer has structure of semidirect product. The difference now is that we have translation part involved as shown in example 3.3.4. This means we have to restrict to matrices in $\mathrm{N}_{\mathrm{GL}(3,\mathbb{Z})}(H_\pi)$ that preserve the corresponding lattice of the generator ϵ:

$$\epsilon L_\pi = \{(E, v) \mid v = \frac{2n_1+1}{2}e_1 + n_2e_2 + n_3e_3, \text{ with } n_1, n_2, n_3 \in \mathbb{Z}\}.$$

As we saw in Example 3.3.4, we have to search for $X_1 \in \mathrm{GL}(2,\mathbb{Z})$ such that $X_1(\frac{2n_1+1}{2}, n_2)^t = (\frac{2k_1+1}{2}, k_2)$, with n_i, $k_i \in \mathbb{Z}$ for $i = 1, 2$. This only happens for matrices in $\Gamma_0(2)$, getting the conclusion.

For B_2 we do not have trivial lattice anymore, and as we saw in Example 3.3.10, the normalizer will not have the structure of a semidirect product. We proceed as before with preserving the lattice of the generator:

$$\epsilon L_\pi = \{(E, v) \mid v = \frac{2n_1+n_3+1}{2}e_1 + \frac{2n_2+n_3}{2}e_2 + n_3e_3, \text{ with } n_1, n_2, n_3 \in \mathbb{Z}\}.$$

After considering the cases when n_3 is even and when n_3 is odd, separately, one finds that the matrices that normalize the group π without needing a translation are

$$\left\{ \begin{pmatrix} \Gamma(2) & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}.$$

We still have to consider affine transformations that normalize but with a nonzero translation (the switching case with translation). After looking at all possibilities we conclude that the switching cases with translation are

$$\left\{ \left(\begin{pmatrix} \Gamma(2) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0 \\ 0 & \pm 1 \end{pmatrix}, \tfrac{2n+1}{2}e_3 \right) \mid n \in \mathbb{Z} \right\}.$$

Thus, the whole group is:

$$\mathrm{N}_{\mathrm{Aff}(3)}(\pi) = \left(\left\{ \begin{pmatrix} \Gamma(2) & 0 \\ 0 & \pm 1 \end{pmatrix} \right\} \ltimes (\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{Z}) \right) \ltimes \langle \xi \rangle,$$

where $\xi = \left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tfrac{1}{2}e_3 \right)$.

Remark 3.3.15. There is a mistake in the computations done by Kang for the group B_1. It is stated in [13, Lemma 3.3] the group $\Gamma(2)$ instead of the group $\Gamma_0(2)$. Knowing the correct group is important because it affects the topology of the resulting spaces.

3.4 The moduli space of flat metrics

First, for completeness and because of the previous relations with the moduli space of flat metrics, we will state the Teichmüller spaces for the 3-diemnsional closed manifolds. They are computed in [11] and [12]. They are also easily computed using the description given in [3].

Theorem 3.4.1. *The Teichmüller spaces of the 3-dimensional closed flat manifolds are the following:*

1. *For T^3, $\mathcal{T}_{flat} = O(3)\backslash\, GL(3,\mathbb{R}) \cong \mathbb{R}^6$.*
2. *For G_2, B_1 and B_2, $\mathcal{T}_{flat} = O(1)\backslash\, GL(1,\mathbb{R}) \times O(2)\backslash\, GL(2,\mathbb{R})$ or $O(2)\backslash\, GL(2,\mathbb{R}) \times O(1)\backslash\, GL(1,\mathbb{R}) \cong \mathbb{R}^4$.*
3. *For G_3, G_4 and G_5, $\mathcal{T}_{flat} = O(1)\backslash\, GL(1,\mathbb{R}) \times U(1)\backslash\, GL(1,\mathbb{C}) \cong \mathbb{R}^2$.*
4. *For G_6, B_3 and B_4, $\mathcal{T}_{flat} = (O(1)\backslash\, GL(1,\mathbb{R}))^3 \cong \mathbb{R}^3$.*

To describe the moduli space of flat metrics we need to use the orthogonal representation. For the cases where we change the representation, we have to recover the orthogonal representation. This is done using Lemma 1.3.11. In this way we obtain the following groups:

$$\text{For } G_3,\, \mathcal{N}_\pi = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & R(\frac{\pi}{3}) \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle < \mathrm{O}(3).$$

$$\text{For } G_5,\, \mathcal{N}_\pi = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & R(\frac{\pi}{3}) \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle < \mathrm{O}(3).$$

We put all the information together for describing the moduli space of flat metrics and reduce the double coset whenever possible. The next result also appears in [13].

Theorem 3.4.2. *The moduli space of flat metrics of the 3-dimensional closed manifolds are:*

1. *For T^3, $\mathcal{M}_{flat} = O(3)\backslash\, GL(3,\mathbb{R})/\, GL(3,\mathbb{Z})$.*
2. *For G_2, $\mathcal{M}_{flat} = \mathbb{R}^+ \times (O(2)\backslash\, GL(2,\mathbb{R})/\, GL(2,\mathbb{Z}))$.*
3. *For G_3, G_4 and G_5, $\mathcal{M}_{flat} = (\mathbb{R}^+)^2$.*
4. *For G_6, $\mathcal{M}_{flat} = (\mathbb{R}^+)^3$.*

5. *For* B_1, $\mathcal{M}_{flat} = (O(2)\backslash\, GL(2,\mathbb{R})/\Gamma_0(2)) \times \mathbb{R}^+$.

6. *For* B_2, $\mathcal{M}_{flat} = (O(2)\backslash\, GL(2,\mathbb{R})/\Gamma(2)Y) \times \mathbb{R}^+$.

7. *For* B_3 *and* B_4, $\mathcal{M}_{flat} = (\mathbb{R}^+)^3$.

Proof. By Theorem 1.3.5 and Remark 1.3.7, the moduli space of flat metrics can be described as

$$\mathcal{M}_{flat} = \mathrm{O}(3)\backslash C_\pi/\mathcal{N}_\pi.$$

For the Bieberbach groups with cyclic holonomy with order bigger than 2, we have that when we reduce the double coset, the following factor appears:

$$\mathrm{O}(2)\backslash\mathbb{R}^* \cdot \mathrm{O}(2)/\langle R, A\rangle,$$

where R and A are generators that depend on the normalizer for each Bieberbach group with cyclic holonomy of order bigger than 2. These two generators are going to be orthogonal matrices, $R, A \in \mathrm{O}(2)$. Then

$$\mathrm{O}(2)\backslash\mathbb{R}^* \cdot \mathrm{O}(2)/\langle R, A\rangle = \mathbb{R}^+.$$

□

Now we are in the position to study the topology of these spaces. We will use the homeomorphism (2.3) and the fundamental domains we computed in Section 2.3.

Theorem 3.4.3. *There are two moduli spaces of flat metrics of 3-dimensional closed manifolds that are non-contractible, and the other moduli spaces of flat metrics are contractible.*

Proof. The proof is done case by case.

- For the 3-torus we have

  $$\mathcal{M}_{flat}(T^3) = \mathrm{O}(3)\backslash\, \mathrm{GL}(3,\mathbb{R})/\, \mathrm{GL}(3,\mathbb{Z}).$$

  This is contractible by the work of Soulé [19].

- For G_2 we have

  $$\begin{aligned}\mathcal{M}_{flat}(G_2) &= \mathbb{R}^+ \times (\mathrm{O}(2)\backslash\, \mathrm{GL}(2,\mathbb{R})/\, \mathrm{GL}(2,\mathbb{Z}))\\ &\cong \mathbb{R}^+ \times (\mathbb{R}^+ \times \mathbb{H}^2/\mathrm{SL}(2,\mathbb{Z}))\\ &\cong (\mathbb{R}^+)^2 \times \mathbb{S}^2 \setminus \{*\},\end{aligned}$$

  by the homeomorphism (2.3) given in Section 2.1, and shown in Figure 2.1.

The next cases are clearly contractible:

- For G_k, with $k = 3, 4, 5$, we have

$$\mathcal{M}_{flat}(G_k) = (\mathbb{R}^+)^2.$$

- For G_6, B_i, with $i = 3, 4$, we have

$$\mathcal{M}_{flat}(G_6) = \mathcal{M}_{flat}(B_i) = (\mathbb{R}^+)^3.$$

The non-contractible cases are the following:

- For B_1, we have

$$\begin{aligned}\mathcal{M}_{flat}(B_1) &= (\mathrm{O}(2)\backslash \mathrm{GL}(2, \mathbb{R})/\Gamma_0(2)) \times \mathbb{R}^+ \\ &\cong (\mathbb{R}^+ \times \mathbb{H}^2/\Gamma_0(2)^+) \times \mathbb{R}^+ \\ &\cong \text{cylinder} \times (\mathbb{R}^+)^2,\end{aligned}$$

  by the homeomorphism (2.3) given in Section 2.1 and the computation of the fundamental domain done in Section 2.3, and shown in Figure 2.2.

- For B_2, we have

$$\begin{aligned}\mathcal{M}_{flat}(B_2) &= (\mathrm{O}(2)\backslash \mathrm{GL}(2, \mathbb{R})/\Gamma(2)Y) \times \mathbb{R}^+ \\ &\cong (\mathbb{R}^+ \times \mathbb{H}^2/\Gamma(2)Y^+) \times \mathbb{R}^+ \\ &\cong \text{cyliner} \times (\mathbb{R}^+)^2,\end{aligned}$$

  by the homeomorphism (2.3) given in Section 2.1 and the computation of the fundamental domain done in Section 2.3, and shown in figure 2.5.

□

Remark 3.4.4. There is another erroneous claim in Kang's paper [13, Theorem 4.5]. It is stated that the moduli space of flat metrics of B_2 is

$$(\mathrm{O}(2)\backslash \mathrm{GL}(2, \mathbb{R})/\Gamma(2)) \times \mathbb{R}^+.$$

If this were true, then

$$\mathrm{O}(3)\backslash C_\pi/\mathcal{N}_\pi \cong (\mathrm{O}(2)\backslash \mathrm{GL}(2, \mathbb{R})/\Gamma(2)) \times \mathbb{R}^+,$$

but the matrix part of the normalizer is

$$\mathcal{N}_\pi = \left\{ \begin{pmatrix} \Gamma(2) & 0 \\ 0 & \pm 1 \end{pmatrix} \cdot \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle \right\}.$$

This would mean that we have the next homeomorphism:

$$(\mathrm{O}(2)\backslash \mathrm{GL}(2,\mathbb{R})/\Gamma(2)Y) \times \mathbb{R}^+ \cong (\mathrm{O}(2)\backslash \mathrm{GL}(2,\mathbb{R})/\Gamma(2)) \times \mathbb{R}^+.$$

This is a contradiction because

$$\begin{gathered} (\mathrm{O}(2)\backslash \mathrm{GL}(2,\mathbb{R})/\Gamma(2)Y) \times \mathbb{R}^+ \cong \text{cyliner} \times (\mathbb{R}^+)^2 \quad \text{and} \\ (\mathrm{O}(2)\backslash \mathrm{GL}(2,\mathbb{R})/\Gamma(2)) \times \mathbb{R}^+ \cong \text{3-punctured sphere} \times (\mathbb{R}^+)^2. \end{gathered}$$

The last homeomorphism follows from the homeomorphism (2.3) given in Section 2.1 and the fundamental domain computed in Section 2.3 and shown in Figure 2.3 and 2.4.

Chapter 4

4-dimensional closed flat manifolds

In this chapter we describe the moduli spaces of flat metrics for a family of 4-dimensional closed manifolds. We will use the same approach as in the 3-dimensional case. First we list the Bieberbach groups, then we determine the cone spaces and the normalizer groups, and finally we present the moduli spaces of flat metrics.

4.1 List of the 4-dimensional Bieberbach groups

The family of 4-dimensional closed flat manifolds that we are considering is the one whose Bieberbach groups have only one generator in their holonomy. This family consists of 16 manifolds, where 8 are orientable and 8 are non-orientable. To list their Bieberbach groups we use the classification tables done in the Ph.D. thesis of Lambert [15]. There are, in addition to this family, two further families of 2 and 3 generators in their holonomy. In total there are 74 affine equivalent classes of 4-dimensional closed flat manifolds.

Let us introduce the following notation for dimension 4, analogous to the one we used in dimension 3. We shall denote by $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0), e_3 = (0,0,1,0)$ and $e_4 = (0,0,0,1)$ the standard basis vectors of $\mathbb{R}^4$. Also, $t_i = (\mathrm{Id}, e_i)$ will denote the basic translations of $\mathbb{R}^4$.

4.1.1 The orientable 4-dimensional closed flat manifolds with a single generator in their holonomy

Here is the list of the Bieberbach groups for the orientable 4-dimensional flat manifolds with a single generator in their holonomy.

The flat torus $\mathbf{T^4} = \mathbf{O^4_1}$:$H_\pi = \mathrm{Id}$, $\quad \pi = \langle t_1, t_2, t_3, t_4 \rangle = \mathbb{Z}^4$

$\mathbf{O^4_2}$:$H_\pi = \mathbb{Z}_2$

$$\pi = \langle t_1, t_2, t_3, t_4, \alpha = (A, \tfrac{1}{2}e_4) \rangle, \qquad \text{where} \qquad A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\mathbf{O^4_3}$: $H_\pi = \mathbb{Z}_2$
$\pi = \langle t_1, t_2, t_3, s = (\mathrm{Id}, \frac{1}{2}(e_1 + e_4)), \alpha = (A, \frac{1}{2}e_2) \rangle$,

$$\text{where} \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

$\mathbf{O^4_4}$: $H_\pi = \mathbb{Z}_3$
$\pi = \langle t_1, t_2, t_3, s = (\mathrm{Id}, \frac{1}{2}e_3 + \frac{\sqrt{3}}{2}e_4)), \alpha = (A, \frac{1}{3}e_2) \rangle$,

$$\text{where} \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

$\mathbf{O^4_5}$: $H_\pi = \mathbb{Z}_3$
$\pi = \langle t_1, t_2, s_1 = (\mathrm{Id}, -\frac{1}{3}e_2 + \frac{2\sqrt{3}}{3}e_3), s_2 = (\mathrm{Id}, \frac{1}{3}e_2 + \frac{\sqrt{3}}{3}e_3 + e_4), \alpha = (A, \frac{1}{3}e_1) \rangle$,

$$\text{where} \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

$\mathbf{O^4_6}$: $H_\pi = \mathbb{Z}_4$

$$\pi = \langle t_1, t_2, t_3, t_4, \alpha = (A, \tfrac{1}{4}e_2)\rangle, \qquad \text{where} \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$\mathbf{O^4_7}$: $H_\pi = \mathbb{Z}_4$

$\pi = \langle t_1, t_2, s_1 = (\mathrm{Id}, \tfrac{1}{2}e_1 + \tfrac{1}{2}e_2 + e_3), s_2 = (\mathrm{Id}, \tfrac{1}{2}e_1 + \tfrac{1}{2}e_2 + e_4), \alpha = (A, \tfrac{1}{4}e_2)\rangle$,

$$\text{where} \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$\mathbf{O^4_8}$: $H_\pi = \mathbb{Z}_6$

$\pi = \langle t_1, t_2, t_3, s = (\mathrm{Id}, \tfrac{1}{2}e_3 + \tfrac{\sqrt{3}}{2}e_4), \alpha = (A, \tfrac{1}{6}e_2)\rangle$,

$$\text{where} \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

4.1.2 The non-orientable 4-dimensional closed flat manifolds with a single generator in their holonomy

Here is the list of the Bieberbach groups for the non-orientable 4-dimensional flat manifolds with a single generator in their holonomy.

$\mathbf{N^4_1} = K^2 \times T^2$: $H_\pi = \mathbb{Z}_2$

$$\pi = \langle t_1, t_2, t_3, t_4, \alpha = (A, \tfrac{1}{2}e_1)\rangle, \qquad \text{where} \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

$\mathbf{N^4_2}$: $H_\pi = \mathbb{Z}_2$

$\pi = \langle t_1, t_2, t_3, s = (\mathrm{Id}, \tfrac{1}{2}(e_3 + e_4)), \alpha = (A, \tfrac{1}{2}e_1)\rangle$,

$$\text{where} \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

$\mathbf{N^4_{14}}$: $H_\pi = \mathbb{Z}_2$

$$\pi = \langle t_1, t_2, t_3, t_4, \alpha = (A, \tfrac{1}{2}e_4)\rangle, \qquad \text{where} \qquad A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\mathbf{N^4_{15}}$: $H_\pi = \mathbb{Z}_4$

$$\pi = \langle t_1, t_2, t_3, t_4, \alpha = (A, \tfrac{1}{4}e_2)\rangle, \qquad \text{where} \qquad A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$\mathbf{N^4_{16}}$: $H_\pi = \mathbb{Z}_4$

$$\begin{aligned} \pi = &\langle s_1 = (\mathrm{Id}, \tfrac{1}{2}(e_1 + e_2 + e_3)), s_2 = (\mathrm{Id}, \tfrac{1}{2}(-e_1 + e_2 - e_3)), \\ &s_3 = (\mathrm{Id}, \tfrac{1}{2}(-e_1 - e_2 + e_3)), t_4, \alpha = (A, \tfrac{1}{4}e_4)\rangle, \end{aligned}$$

$$\text{where} \qquad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\mathbf{N^4_{19}}$: $H_\pi = \mathbb{Z}_6$

$$\pi = \langle t_1, t_2, t_3, s = (\mathrm{Id}, \tfrac{1}{2}e_3 + \tfrac{\sqrt{3}}{2}e_4), \alpha = (A, \tfrac{1}{6}e_2)\rangle,$$

$$\text{where} \qquad A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

$\mathbf{N^4_{20}}$: $H_\pi = \mathbb{Z}_6$

$\pi = \langle t_1, t_2, t_3, s = (\mathrm{Id}, \frac{1}{2}e_3 + \frac{\sqrt{3}}{2}e_4), \alpha = (A, \frac{1}{6}e_2)\rangle,$

$$\text{where} \qquad A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

$\mathbf{N^4_{21}}$: $H_\pi = \mathbb{Z}_6$

$$\pi = \langle t_1, t_2, t_3, t_4, \alpha = (A, \tfrac{1}{6}e_1)\rangle, \qquad \text{where} \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

4.2 The cone space

We compute the cone spaces C_π for the family of 4-dimensional closed flat manifolds with a single generator in their holonomy. As before, we continue using the same notation as in dimension 3 and Lemma 1.3.8.

Proposition 4.2.1. *The possible spaces C_π for the 4-dimensional closed flat manifolds with a single generator in their holonomy are the following:*

1. *For trivial holonomy: T^4. The space is $C_\pi = GL(4, \mathbb{R})$.*
2. *For $H_\pi = \mathbb{Z}_2$, the spaces are:*
   i. *For O^4_2 and O^4_3, $C_\pi = O(4) \cdot (GL(2, \mathbb{R}) \times GL(2, \mathbb{R}))$.*
   ii. *For N^4_1, N^4_2, and N^4_{14}, $C_\pi = O(4) \cdot (GL(3, \mathbb{R}) \times \mathbb{R}^*)$.*
3. *For cyclic holonomy of order bigger than 2, the spaces are:*
   i. *For O^4_4, O^4_5, O^4_6, O^4_7 and O^4_8, $C_\pi = O(4) \cdot (GL(2, \mathbb{R}) \times (\mathbb{R}^+ \times O(2)))$.*
   ii. *For N^4_{15}, , N^4_{19} and N^4_{20}, $C_\pi = O(4) \cdot ((\mathbb{R}^+)^2 \times O(2)) \times (\mathbb{R}^+ \times O(2)))$. For N^4_{16}, $C_\pi = O(4) \cdot ((\mathbb{R}^+ \times O(2) \times (\mathbb{R}^+)^2 \times O(2)))$.*
   iii. *For N^4_{21}, $C_\pi = O(4) \cdot (\mathbb{R}^* \times (\mathbb{R}^+ \times (0, \frac{2\pi}{3}) \times O(3)))$.*

Proof. We consider each case separately.

Case 1. When the holonomy is trivial, the result follows from Corollary 2.1.

Case 2. For $H_\pi = \mathbb{Z}_2$. The situation is similar to dimension 3 Case 2 of Proposition 3.2.1. When H_π is generated by $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ or its negative $-A$, which is the case of O_2^4 and O_3^4, we have

$$\begin{aligned} C_\pi &= \{(x_1, x_2, x_3, x_4) \in \mathrm{GL}(4, \mathbb{R}) \mid x_1 \perp x_3, x_1 \perp x_4, x_2 \perp x_3 \text{ and } x_2 \perp x_4\} \\ &= \mathrm{O}(4) \cdot \begin{pmatrix} \mathrm{GL}(2, \mathbb{R}) & 0 \\ 0 & \mathrm{GL}(2, \mathbb{R}) \end{pmatrix}. \end{aligned}$$

When the generator of H_π is $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ or its negative $-A$, which is the case of N_1^4, N_2^4, and N_{14}^4, we have

$$\begin{aligned} C_\pi &= \{(x_1, x_2, x_3, x_4) \in \mathrm{GL}(4, \mathbb{R}) \mid x_4 \perp x_i \text{ with } i = 1, 2, 3\} \\ &= \mathrm{O}(4) \cdot \left\{ \begin{pmatrix} B & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{R}^* \text{ and } B \in \mathrm{GL}(3, \mathbb{R}) \right\} \\ &= \mathrm{O}(4) \cdot (\mathrm{GL}(3, \mathbb{R}) \times \mathbb{R}^*). \end{aligned}$$

Case 3. For cyclic holonomy manifolds with order bigger than 2. The situation is similar to dimension 3, Case 3 of Proposition 3.2.1. Here we are also using remark 3.2.2. When H_π is generated by matrices of the form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R(\theta) \end{pmatrix}$, which is the case of O_4^4, O_5^4, O_6^4, O_7^4, and O_8^4, we get that

$$\begin{aligned} C_\pi = \{&(x_1, x_2, x_3, x_4) \in \mathrm{GL}(4, \mathbb{R}) \mid x_1 \perp x_3, x_1 \perp x_4, x_2 \perp x_3, x_2 \perp x_4, x_3 \perp x_4 \\ &\text{and } \|x_3\| = \|x_4\|\} \\ = \ &\mathrm{O}(4) \cdot \begin{pmatrix} \mathrm{GL}(2, \mathbb{R}) & 0 \\ 0 & \mathbb{R}^+ \times \mathrm{O}(2) \end{pmatrix} \\ = \ &\mathrm{O}(4) \cdot (\mathrm{GL}(2, \mathbb{R}) \times (\mathbb{R}^+ \times \mathrm{O}(2))). \end{aligned}$$

When the holonomy is generated by matrices of the form $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R(\theta) \end{pmatrix}$

or $\begin{pmatrix} R(\theta) & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which is the case of N^4_{15}, N^4_{16}, N^4_{19}, and N^4_{20}, we have

$$\begin{aligned} C_\pi &= \{(x_1,x_2,x_3,x_4) \in \mathrm{GL}(4,\mathbb{R}) \mid x_i \perp x_j \text{ for all } i \neq j \text{ with } i,j \in \{1,2,3,4\}, \\ &\quad \text{and } \|x_3\| = \|x_4\|\} \\ &= \mathrm{O}(4) \cdot \big(((\mathbb{R}^+)^2 \times \mathrm{O}(2)) \times (\mathbb{R}^+ \times \mathrm{O}(2))\big)\,. \end{aligned}$$

When the holonomy is generated by $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix}$, which is the case of N^4_{21}, we have

$$\begin{aligned} C_\pi = \big\{&(x_1,x_2,x_3,x_4) \in \mathrm{GL}(4,\mathbb{R}) \mid x_1 \perp x_i \text{ with } i = 2,3,4, \\ &\|x_2\| = \|x_3\| = \|x_4\| \text{ and } x_2 \cdot x_3 = x_2 \cdot x_4 = x_3 \cdot x_4\big\}\,. \end{aligned}$$

This means that the vectors x_2, x_3 and x_4 have the same length and form the same angle between them. For this situation we have that the angle is $\theta \in (0, \frac{2\pi}{3})$ since having angle $\frac{2\pi}{3}$ means that the vectors are coplanar (and not linearly independent anymore). With this information we can conclude that

$$C_\pi = \mathrm{O}(4) \cdot (\mathbb{R}^* \times (\mathbb{R}^+ \times (0, \tfrac{2\pi}{3}) \times \mathrm{O}(3)).$$

□

4.3 The normalizer

We proceed as in section 3.3.1. For the orientable manifolds we will get different situations than in dimension 3. That is why we will introduce some helpful notation.

First, we present the list of the conjugated representations of some of the Bieberbach groups. The matrix is going to be similar as in dimension 3 since we have the same holonomies. These representations are used to compute the normalizer. With an abuse of notation, we denote these groups in the same way as before.

$\mathbf{O^4_4}$: $H_\pi = \mathbb{Z}_3$

The representation is conjugated by the affine transformation $(P, 0) \in \mathrm{Aff}(4)$

with $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & \frac{1}{\sqrt{3}} \\ 0 & 0 & -1 & -\frac{1}{\sqrt{3}} \end{pmatrix}$.

The resulting integer representation is

$$\pi = \langle t_1, t_2, t_3, -t_4, \alpha = (A, \tfrac{1}{3}e_2)\rangle, \qquad \text{where} \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

$\mathbf{O^4_5}$: $H_\pi = \mathbb{Z}_3$

We do a change of representation with the same affine transformation as in O^4_4. The resulting integer representation is

$\pi = \langle t_1, t_2, s_1 = (\mathrm{Id}, -\frac{1}{3}e_2 - \frac{2\sqrt{3}}{3}e_3 - \frac{2\sqrt{3}}{3}e_4), s_2 = (\mathrm{Id}, \frac{1}{3}e_2 - \frac{2}{\sqrt{3}}e_4),$

$\alpha = (A, \frac{1}{3}e_1)\rangle,$

$$\text{where} \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

$\mathbf{O^4_8}$: $H_\pi = \mathbb{Z}_6$

The representation is conjugated by the affine transformation $(P, 0) \in \mathrm{Aff}(4)$,

where $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{\sqrt{3}} \\ 0 & 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix}$.

The resulting integer representation is

$$\pi = \langle t_1, t_2, t_3, t_4, \alpha = (A, \tfrac{1}{6}e_2)\rangle, \qquad \text{where} \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Let us introduce the following notation, which we will use in the proposition below.

$$\Gamma_0(2)^t := \left\{ \begin{pmatrix} 2a+1 & 2b \\ c & 2d+1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid a,b,c,d \in \mathbb{Z} \right\}$$
$$= \{X^t \mid X \in \Gamma_0(2)\}.$$

$$\Gamma_{0,1}(3) :=$$
$$\left\{ \begin{pmatrix} 3a+1 & 3b \\ c & 3d+1 \end{pmatrix} \text{ or } \begin{pmatrix} 3a+2 & 3b \\ c & 3d+1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid a,b,c,d \in \mathbb{Z} \right\}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid b \equiv 0 \text{ and } d \equiv 1 \bmod 3 \right\}.$$

$$\Gamma_{0,2}(3) :=$$
$$\left\{ \begin{pmatrix} 3a+1 & 3b \\ c & 3d+2 \end{pmatrix} \text{ or } \begin{pmatrix} 3a+2 & 3b \\ c & 3d+2 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid a,b,c,d \in \mathbb{Z} \right\}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid b \equiv 0 \text{ and } d \equiv 2 \bmod 3 \right\}.$$

$$\Gamma_{1,2}(3) := \left\{ \begin{pmatrix} 3a+1 & 3b \\ 3c & 3d+2 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid a,b,c,d \in \mathbb{Z} \right\}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid b,c \equiv 0,\ a \equiv 1 \text{ and } d \equiv 2 \bmod 3 \right\}.$$

$$\Gamma_{2,1}(3) := \left\{ \begin{pmatrix} 3a+2 & 3b \\ 3c & 3d+1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid a,b,c,d \in \mathbb{Z} \right\}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid b,c \equiv 0,\ a \equiv 2 \text{ and } d \equiv 1 \bmod 3 \right\}.$$

$$\Gamma_{0,1}(4) := \left\{ \begin{pmatrix} 2a+1 & 4b \\ c & 4d+1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid a,b,c,d \in \mathbb{Z} \right\}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid b \equiv 0 \text{ and } d \equiv 1 \bmod 4 \right\}.$$

$$\Gamma_{0,3}(4) := \left\{ \begin{pmatrix} 2a+1 & 4b \\ c & 4d+3 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid a,b,c,d \in \mathbb{Z} \right\}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid b \equiv 0 \text{ and } d \equiv 3 \bmod 4 \right\}.$$

$$\Gamma_{0,1}(2,4) := \left\{ \begin{pmatrix} 2a+1 & 4b \\ 2c & 4d+1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid a,b,c,d \in \mathbb{Z} \right\}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid c \equiv 0 \bmod 2,\ b \equiv 0 \text{ and } d \equiv 1 \bmod 4 \right\}.$$

$$\Gamma_{0,3}(2,4) := \left\{ \begin{pmatrix} 2a+1 & 4b \\ 2c & 4d+3 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid a,b,c,d \in \mathbb{Z} \right\}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid c \equiv 0 \bmod 2,\ b \equiv 0 \text{ and } d \equiv 3 \bmod 4 \right\}.$$

$$\Gamma_{0,1}(6) := \left\{ \begin{pmatrix} 2a+1 & 6b \\ c & 6d+1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid a,b,c,d \in \mathbb{Z} \right\}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid b \equiv 0 \text{ and } d \equiv 1 \bmod 6 \right\}.$$

$$\Gamma_{0,5}(6) := \left\{ \begin{pmatrix} 2a+1 & 6b \\ c & 6d+5 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid a,b,c,d \in \mathbb{Z} \right\}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid b \equiv 0 \text{ and } d \equiv 5 \bmod 6 \right\}.$$

Proposition 4.3.1. *The matrix part of the normalizer of π in Aff(4) for the 4-dimensional orientable closed flat manifolds with a single generator on their holonomy is as follows:*

1. *For T^4, $\mathcal{N}_\pi = GL(4,\mathbb{Z})$.*

2. *For O_2^4, $\mathcal{N}_\pi = \left\{ \begin{pmatrix} GL(2,\mathbb{Z}) & 0 \\ 0 & \Gamma_0(2) \end{pmatrix} \right\}$.*

3. *For* O^4_3, $\mathcal{N}_\pi = \left\{ \begin{pmatrix} \Gamma(2) & 0 \\ 0 & \Gamma_0(2)^t \end{pmatrix} \cdot \left\langle \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle \right\}.$

4. *For* O^4_4,
$$\mathcal{N}_\pi = \left\langle \begin{pmatrix} B & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} C & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \mid B \in \Gamma_{0,1}(3), C \in \Gamma_{0,2}(3) \right\rangle.$$

5. *For* O^4_5,
$$\mathcal{N}_\pi = \left\langle \begin{pmatrix} B & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} C & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \mid B \in \Gamma_{1,2}(3), C \in \Gamma_{2,1}(3) \right\rangle.$$

6. *For* O^4_6,
$$\mathcal{N}_\pi = \left\langle \begin{pmatrix} B & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} C & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \mid B \in \Gamma_{0,1}(4), C \in \Gamma_{0,3}(4) \right\rangle.$$

7. *For* O^4_7,
$$\mathcal{N}_\pi = \left\{ \left\langle \begin{pmatrix} B & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} C & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle \right\} \cdot \left\langle \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right\rangle,$$
where $B \in \Gamma_{0,1}(2,4), C \in \Gamma_{0,3}(2,4)$.

8. *For* O^4_8,
$$\mathcal{N}_\pi = \left\langle \begin{pmatrix} B & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} C & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \mid B \in \Gamma_{0,1}(6), C \in \Gamma_{0,5}(6) \right\rangle.$$

Proof. In the case of T^4, the result follows from example 1.3.4.

The Bieberbach groups O^4_3, O^4_5, and O^4_7 have non-trivial lattice. Then the group of matrices that normalizes the lattice is a conjugation of $\mathrm{GL}(4,\mathbb{Z})$ by a matrix $Q \in \mathrm{GL}(4,\mathbb{R})$ (Remark 1.3.9). For these cases the Q is computed, but fortunately the $X \in Q\,\mathrm{GL}(4,\mathbb{Z})Q^{-1}$ that satisfy the condition $XA = AX$ for the generator of the holonomy A are reduced to matrices in $\mathrm{GL}(4,\mathbb{Z})$. Then in all cases we can consider matrices in $\mathrm{GL}(4,\mathbb{Z})$.

In what follows we will consider all the Bieberbach groups (listed in the proposition) without taking into account the torus.

We first find all the matrices $X \in \mathrm{GL}(4,\mathbb{Z})$ that normalize the holonomy H_π. For all cases we get that the matrix must have the form

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}.$$

All of the Bieberbach groups have translation part involved. Then the lattices of the generators of the holonomy have to be computed and we have to search for matrices $X \in \mathrm{N}_{\mathrm{GL}(4,\mathbb{Z})}(H_\pi)$ that preserve or switch the lattices.

For O_2^4, the matrix $X_2 \in \mathrm{GL}(2,\mathbb{Z})$ has to preserve vectors of the form $X_2(n_3, \frac{2n_4+1}{2})^t = (k_3, \frac{2k_4+1}{2})$, with $n_i,\ k_i \in \mathbb{Z}$ for $i = 3,4$, similar to the case of B_1.

For cyclic holonomy of order k bigger than 2, we have the cases

$$XAX^{-1} = \begin{cases} A & \\ A^r & \text{with } r \in \mathbb{N}, 1 < r < k \text{ and } (r,k) = 1. \end{cases}$$

For the ones with trivial lattice, O_4^4, O_6^4 and O_8^4, we have to search for the matrices X_1 such that $X_1(n_1, \frac{kn_2+1}{k}) = (k_1, \frac{kk_2+1}{k})$ or $X_1(n_1, \frac{kn_2+1}{k}) = (k_1, \frac{kk_2+r}{k})$, with $n_i,\ k_i \in \mathbb{Z}$ for $i = 1,2$, depending on where the generator of the holonomy is sent. The matrices X_2 are going to be the same as in the cases of dimension 3 with the same respective holonomy in Proposition 3.3.12.

For the ones with non-trivial lattice, the types of vectors to preserve or switch will depend on more cases. Let us see this more closely:

For O_5^4 the lattices of the generators are:

$$\alpha L_\pi = \{(A,v) \mid v = \frac{3n_1+1}{3}e_1 + \frac{3n_2 - n_3 + n_4}{3}e_2 + (n_3+n_4)\tfrac{2}{\sqrt{3}}e_3 + n_4\tfrac{2}{\sqrt{3}}e_4 \text{ and } n_i \in \mathbb{Z}, i = 1,2,3,4\}$$

$$\alpha^2 L_\pi = \{(A,v) \mid v = \frac{3n_1+2}{3}e_1 + \frac{3n_2 - n_3 + n_4}{3}e_2 - n_4\tfrac{2}{\sqrt{3}}e_3 + n_3\tfrac{2}{\sqrt{3}}e_4 \text{ and } n_i \in \mathbb{Z}, i = 1,2,3,4\}.$$

We have the next three cases:
1. $-n_3 + n_4 \in 3\mathbb{Z}$, 2. $-n_3 + n_4 \in 3\mathbb{Z}+1$, 3. $-n_3 + n_4 \in 3\mathbb{Z}+2$.
Looking at all combinations for sending the lattices, it is concluded that not all of them are possible, leading us to get the structure of semidirect product in the normalizer.

The next two groups are the only ones that their normalizer do not accept a structure of semidirect product: O_3^4 and O_7^4. We consider each case separately.

For O^4_3, the lattice of the generator is:

$$\alpha L_\pi = \{(A,v) \mid v = \frac{2n_1+n_4}{2}e_1 + \frac{2n_2+1}{2}e_2 - n_3e_3 - \frac{n_4}{2}e_4 \text{ and } n_i \in \mathbb{Z},\\ i=1,2,3,4\}.$$

We have two cases: n_4 odd or n_4 even. Looking at all the possibilities, including the switching cases with translations, we obtain:

$$\mathrm{N}_{\mathrm{Aff}(4)}(O^4_3) = \left(\left\{\begin{pmatrix} \Gamma(2) & 0 \\ 0 & \Gamma_0(2)^t \end{pmatrix}\right\} \ltimes \mathbb{R} \oplus \mathbb{R} \oplus \tfrac{1}{2}\mathbb{Z} \oplus \tfrac{1}{2}\mathbb{Z}\right) \ltimes \langle \xi \rangle,$$

where $\xi = \left(\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \frac{1}{4}e_4\right)$.

For O^4_7, the lattices of the generators are as follows:

$$\alpha L_\pi = \{(A,v) \mid v = \tfrac{2n_1+n_3+n_4}{2}e_1 + \tfrac{4n_2+2(n_3+n_4)+1}{4}e_2 - n_4e_3 + n_3e_4 \text{ and } n_i \in \mathbb{Z},\\ i=1,2,3,4\}.$$

$$\alpha^3 L_\pi = \{(A^3,v) \mid v = \tfrac{2n_1+n_3+n_4}{2}e_1 + \tfrac{4n_2+2n_3+2n_4+3}{4}e_2 + n_4e_3 - n_3e_4 \text{ and } n_i \in \mathbb{Z},\\ i=1,2,3,4\}.$$

We will have two cases: n_3+n_4 even or n_3+n_4 odd. Looking at all possibilities, including the switching cases with translations, we obtain:

$\mathrm{N}_{\mathrm{Aff}(4)}(O^4_7) =$

$$\left(\left\langle \begin{pmatrix} \Gamma_{0,1}(2,4) & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} \Gamma_{0,3}(2,4) & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle \ltimes \mathbb{R} \oplus \mathbb{R} \oplus T \right) \ltimes \langle \xi \rangle,$$

$$\text{where} \quad \xi = \left(\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, (0,0,\frac{1}{2},-\frac{1}{2})\right), \quad \text{and}$$

$$T = \{(t, n-t) \mid t \in \tfrac{1}{2}\mathbb{Z} \text{ and } n \in \mathbb{Z}\}.$$

□

As we can see in the preceding Proposition, we have two situations similar to the situation of B_1 and B_2 in dimension 3, where the Bieberbach groups are almost the same but one of them has non-trivial lattice and consequently the normalizer $\mathrm{N}_{\mathrm{Aff}(3)}(\pi)$ does not have the structure of semidirect product. These similar situations in dimension 4 are as follows. For holonomy $\mathbb{Z}_2$ it is O_2^4 and O_3^4, and for holonomy $\mathbb{Z}_4$ it is O_6^4 and O_7^4, where the ones with normalizer $\mathrm{N}_{\mathrm{Aff}(4)}(\pi)$ without the structure of semidirect product are O_3^4 and O_7^4.

Now, we study the non-orientable manifolds. As before, we present the list of the conjugated representations in order to get a group with trivial lattice and integer matrix. This representation is the one used to compute the matrix part of the normalizer $\mathcal{N}_\pi$. With an abuse of notation, we denote these groups in the same way as before.

$\mathbf{N^4_{16}}$: $H_\pi = \mathbb{Z}_4$
The representation is conjugated by the affine transformation $(P, 0) \in \mathrm{Aff}(4)$, where $P = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.
The resulting integer representation is:

$$\pi = \langle t_1, t_2, t_3, t_4, \alpha = (A, \tfrac{1}{4}e_4)\rangle, \qquad \text{where} \qquad A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\mathbf{N^4_{19}}$: $H_\pi = \mathbb{Z}_6$
The representation is conjugated by the same affine transformation $(P, 0) \in \mathrm{Aff}(4)$ as in O_8^4. The resulting integer representation is

$$\pi = \langle t_1, t_2, t_3, t_4, \alpha = (A, \tfrac{1}{6}e_2)\rangle, \qquad \text{where} \qquad A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$\mathbf{N^4_{20}}$: $H_\pi = \mathbb{Z}_6$
The representation is conjugated by the same affine transformation $(P, 0) \in \mathrm{Aff}(4)$ as in O_8^4 and N_{19}^4. The resulting integer representation is

$$\pi = \langle t_1, t_2, t_3, t_4, \alpha = (A, \tfrac{1}{6}e_2)\rangle, \qquad \text{where} \qquad A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

As in the orientable case, let us introduce the following groups:

$$\Gamma_0(2)_3 = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \mathrm{GL}(3,\mathbb{Z}) \mid d, g \equiv 0 \bmod 2 \right\},$$

$$\Gamma_0^2(2)_3 = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \mathrm{GL}(3,\mathbb{Z}) \mid d, g, c, f \equiv 0 \bmod 2 \right\}.$$

Proposition 4.3.2. *The matrix part of the normalizer of π in Aff(4) for the 4-dimensional non-orientable closed flat manifolds with a single generator in their holonomy is as follows:*

1. *For N_1^4, $\mathcal{N}_\pi = \left\langle \begin{pmatrix} B & 0 \\ 0 & \pm 1 \end{pmatrix} \mid B \in \Gamma_0(2)_3 \right\rangle$.*

2. *For N_2^4, $\mathcal{N}_\pi = \left\{ \left\langle \begin{pmatrix} B & 0 \\ 0 & \pm 1 \end{pmatrix} \mid B \in \Gamma_0^2(2)_3 \right\rangle \right\} \cdot \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$.*

3. *For N_{14}^4, $\mathcal{N}_\pi = \left\{ \begin{pmatrix} GL(3,\mathbb{Z}) & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}$.*

4. *For N_{15}^4, $\mathcal{N}_\pi = \left\langle \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right\rangle$.*

5. *For N_{16}^4, $\mathcal{N}_\pi =$*

$$\left\langle \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right\rangle.$$

6. *For* N^4_{19}, $\mathcal{N}_\pi = \left\langle \begin{pmatrix} \pm1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} \pm1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\rangle.$

7. *For* N^4_{20}, $\mathcal{N}_\pi = \left\langle \begin{pmatrix} \pm1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} \pm1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\rangle.$

8. *For* N^4_{21}, $\mathcal{N}_\pi = \left\langle \begin{pmatrix} \pm1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \pm1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$

Proof. First we explain the case of N^4_1. The group N^4_1 has trivial lattice, translation part involved, and his normalizer has structure of semidirect product. The matrix that normalize the holonomy has to be of the form

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & \pm1 \end{pmatrix}$$

with $X_1 \in \mathrm{GL}(3,\mathbb{Z})$. In order that this matrix X is in $\mathcal{N}_\pi$, also has to preserve the lattice of the generator α

$$\alpha L_\pi = \{(A,v) \mid v = \frac{2n_1+1}{2}e_1 + n_2e_2 + n_3e_3 - n_4e_4 \text{ and } n_i \in \mathbb{Z}, \\ i = 1,2,3,4\}.$$

The case of N^4_2 is similar to N^4_1 but its group has non-trivial lattice. Then the form of the matrix X is the same but the lattice of the generator α is different:

$$\alpha L_\pi = \{(A,v) \mid v = \frac{2n_1+1}{2}e_1 + n_2e_2 + \frac{2n_3+n_4}{2}e_3 - \frac{n_4}{2}e_4 \text{ and } n_i \in \mathbb{Z}, \\ i = 1,2,3,4\},$$

with two cases: n_4 even or n_4 odd. This lead us to switching cases with translations. Therefore the whole normalizer group is

$$\mathrm{N}_{\mathrm{Aff}(4)}(N^4_2) = \\ \left(\left\langle \begin{pmatrix} 2a+1 & b & 2c & 0 \\ 2d & 2e+1 & 2f & 0 \\ 2g & h & 2i+1 & 0 \\ 0 & 0 & 0 & \pm1 \end{pmatrix} \in \mathrm{GL}(4,\mathbb{Z}) \right\rangle \ltimes T \right) \ltimes \langle \xi \rangle,$$

with $a, b, c, d, e, f, g, h, i \in \mathbb{Z}$, the translations $T = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \frac{1}{2}\mathbb{Z}$, and

$$\xi = \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, (0, 0, 0, \frac{1}{4}) \right).$$

The case of N^4_{14} is simple because the form of the matrix X is also as in N^4_1 but now the group N^4_{14} has trivial lattice and translation part not involved. Then $\mathcal{N}_\pi$ is the same as $\mathrm{N}_{\mathrm{GL}(4,\mathbb{Z})}(H_\pi)$.

The remaining groups are also simple to compute since they have trivial lattice, translation part not involved and their normalizers have structure of semidirect product. They even have the normalizer of the holonomy, $\mathrm{N}_{\mathrm{GL}(4,\mathbb{Z})}(H_\pi)$, finite, which we computed using Mathematica. We only have to be careful with multiplying by -1 in some entries of the lattice. We omit the computations. $\square$

Remark 4.3.3. The Bieberbach groups N^4_{15}, N^4_{16}, N^4_{19}, N^4_{20} and N^4_{21} have finite matrix part of the normalizer, making them similar to the case of dimension 3 for the orientable manifolds with cyclic holonomy bigger than 2.

4.4 The moduli space of flat metrics

First, for completeness and because of the previous relations with the moduli space of flat metrics, we will state the Teichmüller spaces for our manifolds which are easily computed by the description given in [3].

Theorem 4.4.1. *The Teichmüller spaces of the 4-dimensional closed flat manifolds with a single generator in their holonomy are:*

1. *For* T^4, $\mathcal{T}_{flat} = \frac{GL(4,\mathbb{R})}{O(4)} \cong \mathbb{R}^{10}$.
2. *For* O^4_2 *and* O^4_3, $\mathcal{T}_{flat} \cong \frac{GL(2,\mathbb{R})}{O(2)} \times \frac{GL(2,\mathbb{R})}{O(2)} \cong \mathbb{R}^6$.
3. *For* O^4_4, O^4_5, O^4_6, O^4_7, *and* O^4_8, $\mathcal{T}_{flat} \cong \frac{GL(2,\mathbb{R})}{O(2)} \times \frac{GL(1,\mathbb{C})}{U(1)} \cong \mathbb{R}^4$.
4. *For* N^4_1, N^4_2 *and* N^4_{14}, $\mathcal{T}_{flat} \cong \frac{GL(3,\mathbb{R})}{O(3)} \times \frac{GL(1,\mathbb{R})}{O(1)} \cong \mathbb{R}^7$.

5. *For* N^4_{15}, N^4_{16}, N^4_{19}, N^4_{20}, *and* N^4_{21}, $\mathcal{T}_{flat} \cong \frac{GL(1,\mathbb{R})}{O(1)} \times \frac{GL(1,\mathbb{R})}{O(1)} \times \frac{GL(1,\mathbb{C})}{U(1)}$ *or* $\frac{GL(1,\mathbb{C})}{U(1)} \times \frac{GL(1,\mathbb{R})}{O(1)} \times \frac{GL(1,\mathbb{R})}{O(1)} \cong \mathbb{R}^3$.

To describe the moduli spaces of flat metrics we need to use the orthogonal representation. For the cases where we changed the representation, we have to recover the orthogonal representation. This is done by conjugating with the respective transformation P^{-1} (see Lemma 1.3.11). In this way we obtain the following groups:

For O^4_4,

$$\mathcal{N}_\pi = \left\langle \begin{pmatrix} B & 0 \\ 0 & R(\frac{\pi}{3}) \end{pmatrix}, \begin{pmatrix} C & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mid B \in \Gamma_{0,1}(3), C \in \Gamma_{0,2}(3) \right\rangle.$$

For O^4_5,

$$\mathcal{N}_\pi = \left\langle \begin{pmatrix} B & 0 \\ 0 & R(\frac{\pi}{3}) \end{pmatrix}, \begin{pmatrix} C & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mid B \in \Gamma_{1,2}(3), C \in \Gamma_{2,1}(3) \right\rangle.$$

For O^4_8, $\mathcal{N}_\pi =$

$$\left\langle \begin{pmatrix} B & 0 \\ 0 & R(\frac{\pi}{3}) \end{pmatrix}, \begin{pmatrix} C & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(4,\mathbb{Z}) \mid B \in \Gamma_{0,1}(6), C \in \Gamma_{0,5}(6) \right\rangle.$$

For N^4_{16}, $\mathcal{N}_\pi =$

$$\left\langle \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right\rangle < \mathrm{O}(4).$$

For N^4_{19},

$$\mathcal{N}_\pi = \left\langle \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R(\frac{\pi}{3}) \end{pmatrix}, \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(4,\mathbb{Z}) \right\rangle < \mathrm{O}(4).$$

For N^4_{20},

$$\mathcal{N}_\pi = \left\langle \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R(\frac{5\pi}{3}) \end{pmatrix}, \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(4,\mathbb{Z}) \right\rangle < \mathrm{O}(4).$$

Let us consider the following notation:

$$\Gamma_0(3) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid b \equiv 0 \bmod 3 \right\}.$$

$$\Gamma(3) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid b, c \equiv 0 \bmod 3 \right\}.$$

$$\Gamma_0(4) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid b \equiv 0 \bmod 4 \right\}.$$

$$\Gamma_1(2)_3 := \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \mathrm{GL}(3,\mathbb{Z}) \mid d, c, f \equiv 0 \bmod 2 \right\}.$$

Theorem 4.4.2. *The moduli space of flat metrics of the 4-dimensional closed manifolds with a single generator in their holonomy are:*

1. *For* T^4, $\mathcal{M}_{flat} = O(4)\backslash\, GL(4,\mathbb{R})/\, GL(4,\mathbb{Z})$.
2. *For* O^4_2, $\mathcal{M}_{flat} = (O(2)\backslash GL(2,\mathbb{R})/GL(2,\mathbb{Z})) \times (O(2)\backslash GL(2,\mathbb{R})/\Gamma_0(2))$.
3. *For* O^4_3, $\mathcal{M}_{flat} = (O(2)\backslash\, GL(2,\mathbb{R})/\Gamma_0(2)) \times (O(2)\backslash\, GL(2,\mathbb{R})/\Gamma_0(2)^t)$.
4. *For* O^4_4, $\mathcal{M}_{flat} = (O(2)\backslash\, GL(2,\mathbb{R})/\Gamma_0(3)) \times \mathbb{R}^+$.
5. *For* O^4_5, $\mathcal{M}_{flat} = (O(2)\backslash\, GL(2,\mathbb{R})/\Gamma(3)) \times \mathbb{R}^+$.
6. *For* O^4_6, $\mathcal{M}_{flat} = (O(2)\backslash\, GL(2,\mathbb{R})/\Gamma_0(4)) \times \mathbb{R}^+$.
7. *For* O^4_7, $\mathcal{M}_{flat} = (O(2)\backslash\, GL(2,\mathbb{R})/\Gamma(2)) \times \mathbb{R}^+$.
8. *For* O^4_8, $\mathcal{M}_{flat} = (O(2)\backslash\, GL(2,\mathbb{R})/\, \langle \Gamma_{0,1}(6), \Gamma_{0,5}(6)\rangle) \times \mathbb{R}^+$.
9. *For* N^4_1, $\mathcal{M}_{flat} = (O(3)\backslash GL(3,\mathbb{R})/\Gamma_0(2)_3) \times \mathbb{R}^+$.
10. *For* N^4_2, $\mathcal{M}_{flat} = (O(3)\backslash GL(3,\mathbb{R})/\Gamma_1(2)_3) \times \mathbb{R}^+$.
11. *For* N^4_{14}, $\mathcal{M}_{flat} = (O(3)\backslash GL(3,\mathbb{R})/\, GL(3,\mathbb{Z})) \times \mathbb{R}^+$.

12. For N^4_{15}, N^4_{16}, N^4_{19}, N^4_{20}, and N^4_{21}, $\mathcal{M}_{flat} = (\mathbb{R}^+)^3$.

Proof. We use Theorem 1.3.5 and Remark 1.3.7 to obtain the moduli space of flat metrics via

$$\mathcal{M}_{flat} = \mathrm{O}(4)\backslash C_\pi/\mathcal{N}_\pi.$$

For the orientable manifolds of cyclic holonomy of order greater than 2 their double quotient is of the shape

$$\mathrm{O}(4)\backslash \mathrm{O}(4)\cdot(\mathrm{GL}(2,\mathbb{R})\times(\mathbb{R}^+\times\mathrm{O}(2)))\Big/\left\langle\left(\begin{smallmatrix}\Gamma_1 & 0\\ 0 & R\end{smallmatrix}\right),\left(\begin{smallmatrix}\Gamma_2 & 0\\ 0 & A\end{smallmatrix}\right)\right\rangle,$$

where Γ_1, $\Gamma_2 \subset \mathrm{GL}(2,\mathbb{Z})$, and R, $A \in \mathrm{O}(2)$, the respective matrices that appear in $\mathcal{N}_\pi$ for each case. Observe that $\left(\begin{smallmatrix}C & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{smallmatrix}\right) \notin \mathcal{N}_\pi$, where $C \in \Gamma_2$; this means that $\mathcal{N}_\pi$ can not be separated as the product of the groups. But we still can separate the double quotient in two factors:

$$(\mathrm{O}(2)\backslash\mathrm{GL}(2,\mathbb{R})/\langle\Gamma_1,\Gamma_2\rangle)\times\big(\mathrm{O}(2)\backslash\mathbb{R}^+\times\mathrm{O}(2)/\langle R, A\rangle\big),$$

this is because the second part of the space C_π is $\mathbb{R}^+\times\mathrm{O}(2)$ and, $\langle R, A\rangle$, the second factor of the group $\mathcal{N}_\pi$ is finite and generated by orthogonal matrices. Then we separate the double quotient in two factors and reduce the second factor as in Theorem 3.4.2.

For the non-orientable manifolds with cyclic holonomy of order greater than 2, we can reduce it because the normalizer is a subgroup of O(4) and the cone space C_π is equal to orthogonal matrices times the positive real numbers. Let us see the case of N^4_{15}:

$$\begin{aligned}
&\mathcal{M}_{flat}(N^4_{15}) = \\
&\mathrm{O}(4)\backslash\mathrm{O}(4)\cdot(((\mathbb{R}^+)^2\times\mathrm{O}(2))\times(\mathbb{R}^+\times\mathrm{O}(2)))\Big/\left\langle\begin{pmatrix}\pm1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{pmatrix},\begin{pmatrix}\pm1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\right\rangle \\
&= \mathbb{R}^+\times\mathbb{R}^+\times(\mathrm{O}(2)\backslash\mathbb{R}^+\times\mathrm{O}(2)/\langle\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right),\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)\rangle) \\
&\cong \mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+.
\end{aligned}$$

□

With the work we have done so far, we can say something about the topology of some of the moduli spaces above.

Corollary 4.4.3. *The moduli spaces of flat metrics of the 4-dimensional manifolds with Bieberbach groups O^4_2 and O^4_7 are non-contractible. On the other hand, the moduli spaces of flat metrics of the 4-dimensional manifolds with Bieberbach groups N^4_{14}, N^4_{15}, N^4_{16}, N^4_{19}, N^4_{20}, and N^4_{21} are contractible.*

Proof. The proof is done case by case.

We explain why the following moduli spaces of flat metrics are non-contractible:

- The case of O_2^4, since

$$\begin{aligned}\mathcal{M}_{flat} &= (\mathrm{O}(2)\backslash\mathrm{GL}(2,\mathbb{R})/\mathrm{GL}(2,\mathbb{Z})) \times (\mathrm{O}(2)\backslash\mathrm{GL}(2,\mathbb{R})/\Gamma_0(2)) \\ &\cong (\mathbb{R}^+ \times \mathbb{H}^2/\mathrm{SL}(2,\mathbb{Z})) \times (\mathbb{R}^+ \times \mathbb{H}^2/\Gamma_0(2)^+) \\ &\cong (\mathbb{R}^+)^2 \times \mathbb{S}^2 \setminus \{*\} \times \text{cylinder}.\end{aligned}$$

  This double cosets are studied in Theorem 3.4.3.

- The case of O_7^4, since

$$\begin{aligned}\mathcal{M}_{flat} &= (\mathrm{O}(2)\backslash \mathrm{GL}(2,\mathbb{R})/\Gamma(2)) \times \mathbb{R}^+ \\ &\cong (\mathbb{R}^+ \times \mathbb{H}^2/\Gamma(2)^+) \times \mathbb{R}^+ \\ &\cong \text{3-punctured sphere} \times (\mathbb{R}^+)^2.\end{aligned}$$

  The last homeomorphism follows from the homeomorphism (2.3) given in Section 2.1 and the fundamental domain computed in Section 2.3, and shown in figure 2.3 and 2.4.

We explain why the following moduli spaces of flat metrics are contractible:

- The case of N_{14}^4, since we have

$$\mathcal{M}_{flat}(N_{14}^4) = \mathrm{O}(3)\backslash\mathrm{GL}(3,\mathbb{R})/\,\mathrm{GL}(3,\mathbb{Z}) \times \mathbb{R}^+,$$

  and that double coset is contractible due to Soulé [19].

- The cases of N_{15}^4, N_{16}^4, N_{19}^4, N_{20}^4, and N_{21}^4, since

$$\mathcal{M}_{flat} = (\mathbb{R}^+)^3.$$

□

Remark 4.4.4. The moduli space of flat metrics of the 4-dimensional torus is non-contractible due to Tuschmann and Wiemeler in [20].

For the remaining ones we could still study their topology, and this is work in progress. For the orientable ones, we could also compute their fundamental domain as we have done so far. For the cases N_1^4 and N_2^4 potentially one could use the work of Soulé [19].

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