A complete dichotomy for complex-valued holantc

Backens, Miriam

DOI:
10.4230/LIPIcs.ICALP.2018.12

A Complete Dichotomy for Complex-Valued Holant^c

Miriam Backens[1]
Department of Computer Science, University of Oxford, UK
miriam.backens@cs.ox.ac.uk

Abstract

Holant problems are a family of counting problems on graphs, parametrised by sets of complex-valued functions of Boolean inputs. HOLANT^c denotes a subfamily of those problems, where any function set considered must contain the two unary functions pinning inputs to values 0 or 1. The complexity classification of Holant problems usually takes the form of dichotomy theorems, showing that for any set of functions in the family, the problem is either #P-hard or it can be solved in polynomial time. Previous such results include a dichotomy for real-valued HOLANT^c and one for HOLANT^c with complex symmetric functions, i.e. functions which only depend on the Hamming weight of the input.

Here, we derive a dichotomy theorem for HOLANT^c with complex-valued, not necessarily symmetric functions. The tractable cases are the complex-valued generalisations of the tractable cases of the real-valued HOLANT^c dichotomy. The proof uses results from quantum information theory, particularly about entanglement. This full dichotomy for HOLANT^c answers a question that has been open for almost a decade.

2012 ACM Subject Classification Theory of computation → Complexity classes, Theory of computation → Problems, reductions and completeness

Keywords and phrases computational complexity, counting complexity, Holant problems, dichotomy, entanglement

Digital Object Identifier 10.4230/LIPIcs.ICALP.2018.12

Related Version A full version of the paper is available at [1], arXiv:1704.05798.

Acknowledgements I would like to thank Pinyan Lu for pointing out a flaw in the original statement of the main theorem. Many thanks also to Ashley Montanaro, William Whistler and Leslie Ann Goldberg for helpful comments on earlier versions of this paper.

1 Introduction

Holant problems are a framework for the analysis of counting problems defined on graphs. They encompass and generalise other counting complexity frameworks like counting constraint satisfaction problems (#CSP) [9, 10] and counting graph homomorphisms [10, 4].

A Holant instance is defined by assigning a function from a specified set to each vertex of a graph, with the edges incident on that vertex corresponding to inputs of the function. The counting problem is a sum-of-products computation: multiplying all the function values

[1] The research leading to these results has received funding from EPSRC via grant EP/L021005/1 and from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) ERC grant agreement no. 334828. The paper reflects only the author's views and not the views of the ERC or the European Commission. The European Union is not liable for any use that may be made of the information contained therein. No new data were created during this study. The majority of this research was done at the School of Mathematics, University of Bristol, UK.

45th International Colloquium on Automata, Languages, and Programming (ICALP 2018).
Editors: Ioannis Chatzigiannakis, Christos Kaklamanis, Dániel Marx, and Donald Sannella;
Article No. 12; pp. 12:1–12:14
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

and then summing over the different assignments of input values to the edges [10]. A more rigorous definition can be found in Section 2. In this work, we consider only complex-valued functions of Boolean inputs. Throughout, all numbers are assumed to be algebraic [4].

Problems expressible in the Holant framework include counting matchings or counting perfect matchings, counting vertex covers [10], and counting Eulerian orientations [14]. A Holant problem can also be thought of as the problem of contracting a tensor network; from that perspective, each function corresponds to a tensor with one index for each input [7].

The main goal in the analysis of Holant problems is the derivation of dichotomy theorems, showing that all problems in a certain family are either polynomial time solvable or #P-hard. Families of Holant problems are often defined by assuming that the function sets contain specific functions, which are said to be 'freely available'. As an example, the problem #CSP($\mathcal{F}$) for a function set $\mathcal{F}$ effectively corresponds to the Holant problem $\textsc{Holant}\left(\mathcal{F} \cup \{=_n \mid n \in \mathbb{N}_{\geq 1}\}\right)$, where $(=_1) : \{0,1\} \to \mathbb{C}$ is the function that is 1 on both inputs, and, for $n \geq 2$, $(=_n) : \{0,1\}^n \to \mathbb{C}$ is the function satisfying:

$$(=_n)(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if } x_1 = x_2 = \ldots = x_n \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The problem $\textsc{Holant}^c(\mathcal{F})$ is the Holant problem where the unary functions pinning edges to values 0 or 1 are available in addition to the elements of $\mathcal{F}$:

$$\textsc{Holant}^c(\mathcal{F}) = \textsc{Holant}\left(\mathcal{F} \cup \{\delta_0, \delta_1\}\right), \tag{2}$$

with $\delta_0(0) = 1$, $\delta_0(1) = 0$ and $\delta_1(0) = 0$, $\delta_1(1) = 1$ [9]. Another important family is $\textsc{Holant}^*$, in which all unary functions are freely available [6].

Known Holant dichotomies include a full dichotomy for $\textsc{Holant}^*$ [6], dichotomies for $\textsc{Holant}^c$ with symmetric functions, i.e. where all functions in the sets considered depend only on the Hamming weight of the input [9], and a dichotomy for real-valued $\textsc{Holant}^c$, where functions need not be symmetric but must take values in $\mathbb{R}$ instead of $\mathbb{C}$ [11]. There is also a dichotomy for symmetric $\textsc{Holant}$ [5] and a dichotomy for non-negative real-valued $\textsc{Holant}$ [18]. Both existing results about $\textsc{Holant}^c$ are proved via dichotomies for #CSP problems with complex-valued, not necessarily symmetric functions: in the first case, a dichotomy for general #CSP problems, and in the second case, a dichotomy for $\#\mathrm{CSP}_2^c$, a subfamily of #CSP in which each variable must appear an even number of times and variables can be pinned to 0 or 1.

While many dichotomies have been derived for functions taking values in some smaller set, we consider complex-valued functions to be the natural setting for Holant problems. This is motivated in part by connecting Holant problems to quantum computation, where complex numbers naturally arise: the problem of strongly classically simulating a quantum circuit with fixed input and output states can immediately be expressed as a Holant problem. The second justification for considering complex numbers is that many tractable sets find a more natural expression over $\mathbb{C}$. An example of this are the 'affine functions' (see Section 3.1): they were originally discovered as several distinct tractable sets for a smaller codomain, but their definition is vastly more straightforward when expressed in terms of complex values [10]. Thirdly, some problems parametrised in terms of real values are naturally connected by *complex-valued* holographic transformations (defined in Section 2.1): for example, the problem of counting Eulerian orientations on 4-regular graphs is expressed in the Holant framework in this way [5, 14].

We therefore build on the existing work to derive a $\textsc{Holant}^c$ dichotomy for complex-valued, not necessarily symmetric functions. In the process, we employ notation and results from quantum information theory. This approach was first used in a recent paper [2] to derive a dichotomy for Holant$^+$, in which four unary functions are freely available, including the ones available in Holant^c: $\text{Holant}^+(\mathcal{F}) = \text{Holant}(\mathcal{F} \cup \{\delta_0, \delta_1, \delta_+, \delta_-\})$, where $\delta_+(x) = 1$ for both inputs (i.e. it is the same as the unary equality function[2]) and $\delta_-(x) = (-1)^x$.

A core part of quantum theory, and also of the quantum approach to Holant problems, is the notion of entanglement. A pure[3] quantum state of n qubits, the quantum equivalents of bits, is represented by a vector in the space $(\mathbb{C}^2)^{\otimes n}$, which consists of n tensor copies of $\mathbb{C}^2$. Such a vector is called *entangled* if it cannot be written as a tensor product of vectors from each copy of $\mathbb{C}^2$.

An n-ary function $f : \{0,1\}^n \to \mathbb{C}$ can be considered as a vector in $\mathbb{C}^{2^n}$ by treating each input as an element of an orthonormal basis for that space and using the function values as coefficients in a linear combination of those basis vectors (cf. Section 2). This vector space $\mathbb{C}^{2^n}$ is isomorphic to $(\mathbb{C}^2)^{\otimes n}$, allowing functions to be brought into correspondence with quantum states. We thus call a function entangled if the associated vector is entangled. Identifying this property in Holant problems lets us apply some of the large body of work on quantum entanglement [19, 13, 12, 17] to Holant problems. The resulting complexity classification of Holant problems remains non-quantum, we simply employ a different set of mathematical tools in their analysis.

In the Holant$^+$ dichotomy, it was shown how to construct a gadget for an entangled ternary function, given an n-ary entangled function with $n \geq 3$ and using the freely-available unary functions. Furthermore, in most cases it was shown to be possible to realise a ternary symmetric function from this [2]. We show how to adapt those constructions to the Holant^c framework, where only two unary functions are freely available. This does not always work, yet if the construction fails, it is always the case that either the problem is tractable by the Holant* dichotomy or it is equivalent to $\#\text{CSP}_2^c$ using techniques from [11]. With these adaptations, we therefore extend the dichotomy theorem for real-valued Holant^c to arbitrary complex-valued functions.

In the following, Section 2 contains the formal definition of Holant problems and an overview over common strategies used in classifying their complexity. We recap existing results in Section 3. The new dichotomy and its constituent lemmas are proved in Section 4. Section 5 contains the conclusions and outlook.

2 Holant problems

Holant problems are a framework for counting complexity problems defined on graphs, first introduced in the conference version of [10]. Let $G = (V, E)$ be a graph with vertices V and edges E, which may contain self-loops and multiple edges between the same pair of vertices, and let $\mathcal{F}$ be a set of complex-valued functions of Boolean inputs. Throughout, when we refer to complex numbers we mean algebraic complex numbers. Let π be a function that assigns to each degree-n vertex v in the graph an n-ary function $f_v \in \mathcal{F}$ and also assigns one edge incident on the vertex to each input of the function. This determines a complex value

[2] The availability of the function δ_+ is indeed important to the Holant$^+$ dichotomy proof. This is a difference between the Holant framework and #CSP, where a constraint equal to δ_+ would have no effect.

[3] There are also *mixed* quantum states, which have a different mathematical representation, and which are not considered here.

associated with the tuple $(\mathcal{F}, G, \pi)$, called the *Holant* and defined as follows:

$$\mathrm{Holant}_{(\mathcal{F},G,\pi)} = \sum_{\sigma:E\to\{0,1\}} \prod_{v\in V} f\left(\sigma|_{E(v)}\right). \tag{3}$$

Here, σ is an assignment of a Boolean value to each edge in the graph and $\sigma|_{E(v)}$ is the restriction of σ to the edges incident on vertex v. The tuple $(\mathcal{F}, G, \pi)$ is called a *signature grid*.

The associated counting problem is $\textsc{Holant}\,(\mathcal{F})$: given a signature grid $(\mathcal{F}, G, \pi)$ for the fixed set of functions $\mathcal{F}$, find $\mathrm{Holant}_{(\mathcal{F},G,\pi)}$.

It is often useful to think of the functions, also called signatures, as vectors or tensors [7]. The n-ary functions can be put in one-to-one correspondence with vectors in $\mathbb{C}^{2^n}$ as follows: pick an orthonormal basis for $\mathbb{C}^{2^n}$ and label its elements $\{|x\rangle\}_{x\in\{0,1\}^n}$, i.e. each basis vector is labelled by one of the 2^n n-bit strings.[4] Then assign to each $f : \{0,1\}^n \to \mathbb{C}$ the vector $|f\rangle = \sum_{x\in\{0,1\}^n} f(x)\,|x\rangle$. Conversely, any vector $|\psi\rangle \in \mathbb{C}^{2^n}$ corresponds to an n-ary function $\psi : \{0,1\}^n \to \mathbb{C} :: x \mapsto \langle x|\psi\rangle$, where $\langle\cdot|\cdot\rangle$ denotes the inner product of two vectors.[5] The product of two functions of disjoint sets of variables corresponds to the tensor product of the associated vectors, i.e. if $h(x_1,\ldots,x_n,y_1,\ldots,y_m) = f(x_1,\ldots,x_n)g(y_1,\ldots,y_m)$ then $|h\rangle = |f\rangle \otimes |g\rangle$. Where no confusion arises, we drop the tensor product symbol and sometimes even combine labels into a single 'ket' $|\cdot\rangle$: for example, instead of writing $|0\rangle \otimes |0\rangle$, we may write $|0\rangle\,|0\rangle$ or $|00\rangle$. If g is a unary signature, we sometimes write $\langle g|_l\,|f\rangle$ to indicate that the l-th input of f is connected to a vertex with signature g.

The vector perspective is particularly useful for bipartite Holant problems, which arise on bipartite graphs if we assign functions from two different signature sets to the vertices in the two different partitions. Then the Holant becomes the inner product between two vectors corresponding to the two partitions. Formally: let $G = (V, W, E)$ be a bipartite graph with vertex partitions V and W, and let $\mathcal{F}, \mathcal{G}$ be two sets of signatures. Suppose π is a function that assigns elements of $\mathcal{F}$ to vertices from V and elements of $\mathcal{G}$ to vertices from W and otherwise acts as described above. Then:

$$\mathrm{Holant}_{(\mathcal{F}|\mathcal{G},G,\pi)} = \left(\bigotimes_{v\in V}(|f_v\rangle)^T\right)\left(\bigotimes_{w\in W}|g_w\rangle\right), \tag{4}$$

where we assume the two tensor products are arranged so that the appropriate components of the two vectors meet. The bipartite Holant problem over signature sets $\mathcal{F}$ and $\mathcal{G}$ is denoted by $\textsc{Holant}\,(\mathcal{F} \mid \mathcal{G})$.

Any Holant instance can be made bipartite without changing the value of the Holant by inserting an additional vertex in the middle of each edge and assigning it the binary equality signature $=_2$. Thus, $\textsc{Holant}\,(\mathcal{F}) \equiv_T \textsc{Holant}\,(\mathcal{F} \mid \{=_2\})$, i.e. the two problems have the same complexity.

In the following, we use the function and vector perspectives on signatures interchangeably.

2.1 Complexity classification

Most complexity results about the Holant problem take the form of *dichotomies*, showing that for all signature sets in a specific family, the problem is either #P-hard or in FP. Such a dichotomy is not expected to be true for all counting complexity problems: if $\mathsf{FP} \neq \#\mathsf{P}$, then there are problems in $\#\mathsf{P} \setminus \mathsf{FP}$ which are not #P-hard [6].

---

[4] The $|\cdot\rangle$ notation for vectors is the Dirac or bra-ket notation commonly used in quantum theory.

[5] Strictly speaking, this notation refers to the complex inner product, i.e. $\langle x|$ is the conjugate transpose of $|x\rangle$, but the distinction is irrelevant if all coefficients of $|x\rangle$ are real.

We write $A \leq_T B$ if there exists a polynomial-time reduction from problem B to problem A and $A \equiv_T B$ if $(A \leq_T B) \wedge (B \leq_T A)$. A number of polynomial-time reduction techniques are commonly used in Holant problems.

The technique of *holographic reductions* is the origin of the name Holant. Let M be a 2 by 2 invertible complex matrix and define $M \circ f = M^{\otimes \operatorname{arity}(f)} |f\rangle$, where $M^{\otimes 1} = M$ and $M^{\otimes n+1} = M \otimes M^{\otimes n}$. Furthermore, let $M \circ \mathcal{F} = \{M \circ f \mid f \in \mathcal{F}\}$. This is called a *holographic transformation*. Let $\mathcal{F}$ and $\mathcal{G}$ be two signature sets. Then:

$$\textsc{Holant}\,(\mathcal{F} \mid \mathcal{G}) \equiv_T \textsc{Holant}\,\left(M \circ \mathcal{F} \mid (M^{-1})^T \circ \mathcal{G}\right) \tag{5}$$

and, in fact, $\text{Holant}_{(\mathcal{F}|\mathcal{G},G,\pi)} = \text{Holant}_{(M\circ\mathcal{F}|(M^{-1})^T\circ\mathcal{G},G,\pi')}$; this is Valiant's Holant Theorem [20].

A second technique is that of *gadgets*. Consider a subgraph of some signature grid, which is connected to the larger graph by n edges. This subgraph can be replaced by a single degree-n vertex with an appropriate signature without changing the value of the overall Holant. Thus, if there exists some subgraph with signatures taken from $\mathcal{F}$ such that the effective signature for that subgraph is g, then [6]:

$$\textsc{Holant}\,(\mathcal{F} \cup \{g\}) \leq_T \textsc{Holant}\,(\mathcal{F})\,. \tag{6}$$

We say g is *realisable* over $\mathcal{F}$. As multiplying a signature by a non-zero constant does not change the complexity of a Holant problem, we also consider g realisable if we can construct a gadget with effective signature cg for some $c \in \mathbb{C} \setminus \{0\}$. In bipartite signature grids, we may distinguish between left-hand side (LHS) gadgets and right-hand side (RHS) gadgets, which can be used as if they are signatures for the left and right partitions, respectively.

Finally, there is the technique of *polynomial interpolation*. Let $\mathcal{F}$ be a set of signatures and suppose g is a signature that cannot be realised over $\mathcal{F}$. If, given any signature grid over $\mathcal{F} \cup \{g\}$, it is possible to set up a family of signature grids over $\mathcal{F}$ such that the Holant for the original problem instance can be determined efficiently from the Holant values of the family by solving a linear system, then g is said to be *interpolatable* over $\mathcal{F}$. We do not directly use polynomial interpolation here, though the technique is employed by many of the results we build upon. A rigorous definition of polynomial interpolation can be found in [10].

2.2 Properties of signatures

A signature is called *symmetric* if its value as a function depends only on the Hamming weight of the inputs – in other words, it is invariant under any permutation of the inputs. Symmetric functions are often written in the short-hand notation $f = [f_0, f_1, \ldots, f_n]$, where f_k is the value f takes on inputs of Hamming weight k.

A signature is *degenerate* if it can be written as a tensor product of unary signatures. Conversely, using language from quantum theory, a signature is *entangled* if it cannot be written as a tensor product of unary signatures. This corresponds to the notion of *non-degenerate* signatures in the Holant literature. For example, $|01\rangle + |11\rangle$ is not entangled because it can be rewritten as $(|0\rangle + |1\rangle) \otimes |1\rangle$. On the other hand, the binary equality signature $|00\rangle + |11\rangle$ is entangled. If $k \geq 2$, a k-ary signature can be partially decomposable into a tensor product, e.g. $|0\rangle \otimes (|00\rangle + |11\rangle)$. We say a signature is *genuinely entangled* if there is no way of decomposing it as a tensor product of signatures of any arity. A genuinely entangled signature of arity at least 3 is said to be *multipartite entangled* (as opposed to the bipartite entanglement in a signature of arity 2). A non-genuinely entangled signature has multipartite entanglement if it has a tensor factor corresponding to a genuinely entangled signature of arity at least 3 and a set of signatures has multipartite entanglement if it contains a signature that does.

Among genuinely entangled ternary signatures, we distinguish two types, also known as 'entanglement classes' [12]. Each entanglement class contains signatures that are related via local holographic transformations, i.e. two states $|f\rangle, |g\rangle$ are in the same entanglement class if and only if there exist some 2 by 2 invertible matrices A, B, C such that $(A \otimes B \otimes C) |f\rangle = |g\rangle$.

In quantum theory, the two entanglement classes are named after their representative states: the ternary equality signature $|000\rangle + |111\rangle$, called the GHZ-state, and the ternary perfect matching signature $|001\rangle + |010\rangle + |100\rangle$, called the W state. We say that a signature has GHZ type if it is equivalent to the GHZ state under local holographic transformations and that a signature has W type if it is equivalent to the GHZ state under local holographic transformations. In the Holant literature, GHZ-type signatures are called the *generic case* and W type signatures are called the *double-root case* [9].

The two types of ternary genuinely entangled signatures can be distinguished as follows [17]. Let f be a ternary signature and write:

$$|f\rangle = \sum_{k,\ell,m \in \{0,1\}} a_{k\ell m} |k\ell m\rangle, \tag{7}$$

where $a_{k\ell m} \in \mathbb{C}$ for all $k, \ell, m \in \{0, 1\}$. Then $|f\rangle$ has GHZ type if the following polynomial in the coefficients is non-zero:

$$(a_{000}a_{111} - a_{010}a_{101} + a_{001}a_{110} - a_{011}a_{100})^2 - 4(a_{010}a_{100} - a_{000}a_{110})(a_{011}a_{101} - a_{001}a_{111}). \tag{8}$$

The signature $|f\rangle$ has W type if the above polynomial is zero, and furthermore each of the following three expressions is satisfied:

$$(a_{000}a_{011} \neq a_{001}a_{010}) \vee (a_{101}a_{110} \neq a_{100}a_{111}) \tag{9}$$

$$(a_{001}a_{100} \neq a_{000}a_{101}) \vee (a_{011}a_{110} \neq a_{010}a_{111}) \tag{10}$$

$$(a_{011}a_{101} \neq a_{001}a_{111}) \vee (a_{010}a_{100} \neq a_{000}a_{110}). \tag{11}$$

If the polynomial (8) is zero and at least one of the above expressions evaluates to false, then the signature is not genuinely entangled.

There are many other classes of entangled signatures for higher arities [21, 15, 16, 3], but those are not directly relevant to this paper.

Given a set of signatures that contains multipartite entanglement in the Holant^c framework, we can assume without loss of generality that we have a genuinely multipartite-entangled signature. To see this, consider a non-zero signature $|\psi\rangle$ that has multipartite entanglement, and suppose $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$. Then at least one of the tensor factors must have multipartite entanglement, assume this is $|\psi_1\rangle$. Now, $|\psi\rangle$ is non-zero, so $\langle x|\psi_2\rangle$ must be non-zero for some bit string x. Thus we can realise $|\psi_1\rangle$ by connecting all inputs associated with $|\psi_2\rangle$ to $|0\rangle$ or $|1\rangle$, as appropriate.

3 Existing results

It is difficult to determine the complexity of the general Holant problem. Thus, all existing dichotomies make use of one or more simplifying assumptions: either they assume the availability of certain signatures in all signature sets considered [7, 6, 2], or they only consider signature sets containing functions taken from more restricted families, e.g. symmetric functions [9, 5] or functions taking only real [11] or even non-negative real values [18].

Among others, the following variants of the Holant problem have been considered:

- Holant$^*(\mathcal{F})$ = Holant $(\mathcal{F} \cup \mathcal{U})$, where $\mathcal{U}$ is the set of all unary signatures [6],

- $\textsc{Holant}^+(\mathcal{F}) = \textsc{Holant}(\mathcal{F} \cup \{\delta_0, \delta_1, \delta_+, \delta_-\})$, where $\delta_+(x) = 1$ and $\delta_-(x) = (-1)^x$ [2], and
- $\textsc{Holant}^c(\mathcal{F}) = \textsc{Holant}(\mathcal{F} \cup \{\delta_0, \delta_1\})$ [9, 11].

Several variants of complex-weighted Boolean counting constraint satisfaction problems (#CSP) have also been expressed in the Holant framework. These include:

- $\#\mathrm{CSP}(\mathcal{F}) = \textsc{Holant}(\mathcal{F} \mid \mathcal{G})$, where $\mathcal{G} = \{=_n \mid n \in \mathbb{N}_{\geq 1}\}$ is the set containing all equality signatures [10], and
- $\#\mathrm{CSP}_2^c(\mathcal{F}) = \textsc{Holant}(\mathcal{F} \mid \{\delta_0, \delta_1\} \cup \{=_{2n} \mid n \in \mathbb{N}_{\geq 1}\})$ [11].

The $\#\mathrm{CSP}_2^c$ problems assume availability of the signatures pinning inputs to 0 or 1, respectively, as well as equality signatures of even arity.

Existing results include full dichotomies for $\textsc{Holant}^*$ [6], $\textsc{Holant}^+$ [2], #CSP [10], and $\#\mathrm{CSP}_2^c$ [11]. There are also dichotomies for $\textsc{Holant}^c$ with symmetric complex-valued signatures [6], $\textsc{Holant}^c$ with arbitrary real-valued signatures [11], Holant with symmetric complex-valued signatures [5], and Holant with arbitrary non-negative real-valued signatures [18].

3.1 Preliminary definitions

The following definitions will be used throughout the dichotomy theorems. Write:

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad K = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \tag{12}$$

where $i^2 = -1$. Then let:

- $\mathcal{T}$ be the set of all unary and binary signatures,
- $\mathcal{E}$ be the set of all signatures that are non-zero only on two inputs x and $\bar{x}$, where $\bar{x}$ denotes the bit-wise complement of x, also called *generalised equality signatures*,
- $\mathcal{M}$ be the set of all signatures that are non-zero only on inputs of Hamming weight at most 1,
- $\mathcal{A}$ be the set of all *affine signatures*, i.e. functions of the form $f(x) = ci^{l(x)}(-1)^{q(x)}\chi$, where $c \in \mathbb{C}$, $l(x)$ is a linear Boolean function, $q(x)$ is a quadratic Boolean function, and χ is the indicator function for an affine space, and
- $\mathcal{L}$ be the set of all signatures f with the property that, for any bit string x in the support of f:

$$\left(\bigotimes_{j=1}^{\mathrm{arity}(f)} T^{x_j} \right) |f\rangle \in \mathcal{A}, \tag{13}$$

  where x_j is the j-th bit of x. Elements of $\mathcal{L}$ are called *local affine signatures.*

Denote by $\langle \mathcal{F} \rangle$ the closure of the signature set $\mathcal{F}$ under tensor products. It is straightforward to see that $\mathcal{A} = \langle \mathcal{A} \rangle$ and $\mathcal{L} = \langle \mathcal{L} \rangle$, i.e. these signature sets are already closed under tensor products. If n is a positive integer, we denote by $[n]$ the set $\{1, 2, \ldots, n\}$.

3.2 Dichotomies for Holant variants

The Holant dichotomies generally build upon each other. Dichotomies with fewer freely-available signatures refer to dichotomies for problems with more freely-available signatures, as all tractable cases of the latter must also be tractable cases of the former: removing signatures can never make the problem harder.

▶ **Theorem 1** (Theorem 2.2, [6]). *Let $\mathcal{F}$ be any set of complex valued functions in Boolean variables. The problem* $\text{Holant}^*(\mathcal{F})$ *is polynomial time computable if:*
- $\mathcal{F} \subseteq \langle \mathcal{T} \rangle$, *or*
- $\mathcal{F} \subseteq \langle O \circ \mathcal{E} \rangle$, *where O is a complex orthogonal 2 by 2 matrix, or*
- $\mathcal{F} \subseteq \langle K \circ \mathcal{E} \rangle$, *or*
- $\mathcal{F} \subseteq \langle K \circ \mathcal{M} \rangle$ *or* $\mathcal{F} \subseteq \langle KX \circ \mathcal{M} \rangle$.

In all other cases, $\text{Holant}^*(\mathcal{F})$ *is #P-hard.*

▶ **Theorem 2** (Theorem 6, [9]). *Let $\mathcal{F}$ be a set of complex symmetric signatures.* $\text{Holant}^c(\mathcal{F})$ *is #P-hard unless $\mathcal{F}$ satisfies one of the following conditions, in which case it is tractable:*
- $\text{Holant}^*(\mathcal{F})$ *is tractable, or*
- *there exists a 2 by 2 matrix $S \in \mathcal{S}$ such that $\mathcal{F} \subseteq S \circ \mathcal{A}$, where:*

$$\mathcal{S} = \left\{ S \,\middle|\, (S^T)^{\otimes 2}(=_2), S^T\delta_0, S^T\delta_1 \in \mathcal{A} \right\}. \tag{14}$$

▶ **Theorem 3** (Theorem 4.1, [11]). *A* $\#\text{CSP}_2^c(\mathcal{F})$ *problem has a polynomial time algorithm if one of the following holds: $\mathcal{F} \subseteq \langle \mathcal{E} \rangle$, $\mathcal{F} \subseteq \mathcal{A}$, $\mathcal{F} \subseteq T \circ \mathcal{A}$, or $\mathcal{F} \subseteq \mathcal{L}$. Otherwise, it is #P-hard.*

The preceding results all apply to complex-valued signatures, but the following theorem is restricted to real-valued ones.

▶ **Theorem 4** (Theorem 5.1, [11]). *Let $\mathcal{F}$ be a set of real-valued signatures. Then* $\text{Holant}^c(\mathcal{F})$ *is #P-hard unless $\mathcal{F}$ is a tractable family for* Holant^* *or* $\#\text{CSP}_2^c$.

3.3 Complexity results for ternary signatures

In addition to the above-mentioned Holant dichotomies, there are also some dichotomies specific to symmetric signatures on three-regular graphs. For signature sets containing a ternary GHZ-type signature, there is furthermore a direct relationship to #CSP, which allows a more general complexity classification. When deriving the Holant^c dichotomy, our general approach will be to attempt to construct a gadget for a genuinely entangled ternary signature and then use the following results.

▶ **Theorem 5** (Theorem 3.4, [9]). $\text{Holant}([y_0, y_1, y_2] | [x_0, x_1, x_2, x_3])$ *is #P-hard unless the signatures $[x_0, x_1, x_2, x_3]$ and $[y_0, y_1, y_2]$ satisfy one of the following conditions, in which case the problem is in FP:*
- $[x_0, x_1, x_2, x_3]$ *is degenerate, or*
- *there is a 2 by 2 matrix M such that:*
  - $[x_0, x_1, x_2, x_3] = M \circ [1, 0, 0, 1]$ *and* $(M^T)^{-1} \circ [y_0, y_1, y_2]$ *is in* $\mathcal{A} \cup \langle \mathcal{E} \rangle$,
  - $[x_0, x_1, x_2, x_3] = M \circ [1, 1, 0, 0]$ *and* $(M^T)^{-1} \circ [y_0, y_1, y_2]$ *is of the form* $[0, *, *]$,
  - $[x_0, x_1, x_2, x_3] = M \circ [0, 0, 1, 1]$ *and* $(M^T)^{-1} \circ [y_0, y_1, y_2]$ *is of the form* $[*, *, 0]$,

with $*$ denoting an arbitrary complex number.

The signature $|000\rangle + |111\rangle$ is invariant under holographic transformations of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \omega \end{smallmatrix}\right)$, where $\omega^3 = 1$. Therefore, a binary signature is considered to be ω-normalised if $y_0 = 0$, or there does not exist a primitive $(3t)$-th root of unity λ, where $gcd(t, 3) = 1$, such that $y_2 = \lambda y_0$. Similarly, a unary signature $[a, b]$ is ω-normalised if $a = 0$, or there does not exist a primitive $(3t)$-th root of unity λ, where $gcd(t, 3) = 1$, such that $b = \lambda a$ [9].

▶ **Theorem 6** (Theorem 4.1, [9]). *Let $\mathcal{G}_1, \mathcal{G}_2$ be two sets of signatures and let $[y_0, y_1, y_2]$ be a ω-normalised and non-degenerate signature. In the case of $y_0 = y_2 = 0$, further assume that $\mathcal{G}_1$ contains a unary signature $[a, b]$ which is ω-normalised and satisfies $ab \neq 0$. Then:*

$$\text{HOLANT}\left(\{[y_0, y_1, y_2]\} \cup \mathcal{G}_1 \mid \{[1, 0, 0, 1]\} \cup \mathcal{G}_2\right) \equiv_T \#\text{CSP}(\{[y_0, y_1, y_2]\} \cup \mathcal{G}_1 \cup \mathcal{G}_2). \quad (15)$$

More specifically, HOLANT $(\{[y_0, y_1, y_2]\} \cup \mathcal{G}_1 \mid \{[1, 0, 0, 1]\} \cup \mathcal{G}_2)$ *is #P-hard unless*

$$\{[y_0, y_1, y_2]\} \cup \mathcal{G}_1 \cup \mathcal{G}_2 \subseteq \langle\mathcal{E}\rangle \quad \text{or} \quad \{[y_0, y_1, y_2]\} \cup \mathcal{G}_1 \cup \mathcal{G}_2 \subseteq \mathcal{A}, \quad (16)$$

in which cases the problem is in FP.

The following lemmas show how to realise symmetric genuinely entangled ternary signatures from non-symmetric ones. They do not rely on any unary signatures.

▶ **Lemma 7** (Lemma 18, [2]). *Let $|\psi\rangle$ be a ternary GHZ-type signature, i.e. $|\psi\rangle = (A \otimes B \otimes C) |\mathrm{GHZ}\rangle$ for some invertible 2 by 2 matrices A, B, C. Then at least one of the three possible symmetric triangle gadgets constructed from three copies of $|\psi\rangle$ is non-degenerate, unless $|\psi\rangle \in K \circ \mathcal{E}$ and is furthermore already symmetric.*

▶ **Lemma 8** (Lemma 19, [2]). *Let $|\psi\rangle$ be a ternary W-type signature, i.e. $|\psi\rangle = (A \otimes B \otimes C) |W\rangle$ for some invertible 2 by 2 matrices A, B, C. If $|\psi\rangle \in K \circ \mathcal{M}$ (or $|\psi\rangle \in KX \circ \mathcal{M}$), assume that we also have a binary entangled signature $|\phi\rangle$ that is not in $K \circ \mathcal{M}$ (or $KX \circ \mathcal{M}$, respectively). Then we can realise a symmetric genuinely entangled ternary signature.*

3.4 Results about 4-ary signatures

Besides the above results about ternary signatures, we will also make use of the following result about realising or interpolating the 4-ary equality signature from a more general 4-ary signature.

▶ **Lemma 9** (Lemma 2.38, [8]). *Suppose $\mathcal{F}$ contains a signature f of arity 4 with:*

$$|f\rangle = a\,|0000\rangle + b\,|0011\rangle + c\,|1100\rangle + d\,|1111\rangle\,, \quad (17)$$

where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ has full rank. Then PL-HOLANT$(\{=_4\} \cup \mathcal{F}) \leq_T$ PL-HOLANT$(\mathcal{F})$.

Here, PL-HOLANT refers to the Holant problem for planar graphs; the lemma can also be used in the non-planar setting.

▶ **Lemma 10** (Lemma 5.2, [11]). *Suppose $\mathcal{F}$ contains a 4-ary generalised equality signature f, i.e. $f \in \mathcal{F} \cap \mathcal{E}$ and arity$(f) = 4$. Then* HOLANT $(\mathcal{F}) \equiv_T \#\text{CSP}_2(\mathcal{F})$, *the counting constraint satisfaction problem in which each variable appears an even number of times.*

4 The dichotomy

Our dichotomy proof uses techniques from the HOLANT$^+$ dichotomy [2] and from the real-valued HOLANT^c dichotomy [11], as well as some new results.

The core strategy in the hardness part of the HOLANT$^+$ dichotomy proof is to realise a symmetric genuinely entangled ternary signature f and a symmetric entangled binary signature g for which HOLANT $(\{f\} \mid \{g\})$ is known to be #P-hard. The techniques for realising low-arity signatures utilise knowledge from quantum information theory. They rely crucially on having access to the four unary signatures $\delta_0, \delta_1, \delta_+$ and δ_-, and do not seem directly adaptable to the HOLANT^c setting.

The dichotomy proof for real-valued Holant^c contains arity-reduction techniques that require only δ_0, δ_1, and self-loops. Yet there are two main barriers to extending this result to complex-valued signatures: firstly, some of the hardness results for genuinely entangled ternary signatures in [11] only apply to real values. Secondly, some cases of the dichotomy proof rely on being able to interpolate all unary signatures using techniques that have only been shown to work for real-valued signatures.

In the present work, our strategy is similar to that in the Holant$^+$ dichotomy: we attempt to realise symmetric genuinely entangled ternary signatures. There are now two cases in which this is not possible: either there is no multipartite entanglement, in which case the problem is in FP, or all genuinely entangled signatures in the closure of $\mathcal{F} \cup \{\delta_0, \delta_1\}$ under gadgets have even arity, in which case the smallest signature that can give hardness has arity 4. For arity reduction, we adapt techniques from [11], modifying them to work for complex values. If all genuinely entangled signatures have even arity, we show analogously to [11] that is is always possible to realise a 4-ary signature of a specific form, which can then be used to realise or interpolate $=_4$. Furthermore, we adapt symmetrisation techniques for genuinely entangled ternary signatures from [2] to work in the Holant^c setting. Thus, we never need to interpolate arbitrary unary signatures. In one subcase, we do require unary signatures other than δ_0 and δ_1, but we give a new construction for realising sufficiently many such signatures by gadgets.

In this extended abstract, we give sketch proofs of the new results; full proofs may be found in the full version of the paper [1].

4.1 Hardness proofs involving a genuinely entangled ternary signature

First, we prove two lemmas that give a complexity classification for Holant^c problems in the presence of any genuinely entangled ternary signature with complex coefficients.

▶ **Lemma 11.** *Let $f \in \mathcal{F}$ be a genuinely entangled ternary signature. Then* Holant^c $(\mathcal{F})$ *is #P-hard unless:*

- Holant* $(\mathcal{F})$ *is tractable, or*
- *$\mathcal{F} \subseteq S \circ \mathcal{A}$ for some $S \in \mathcal{S}$, as defined in* (14).

In both of those cases, the problem Holant^c $(\mathcal{F})$ *is tractable.*

Proof (sketch). Let $\mathcal{F}' = \mathcal{F} \cup \{\delta_0, \delta_1\}$. We distinguish cases according to whether f is symmetric or not, and according to its entanglement class.

If f is symmetric and has GHZ type, there exists an invertible holographic transformation M that maps f to $=_3$. Transform the Holant problem to bipartite form by adding an extra vertex carrying the signature $=_2$ in the middle of each edge. It is straightforward to see that allowing the signatures δ_0 and δ_1 on both partitions does not affect the complexity, i.e.:

$$\text{Holant}^c(\mathcal{F}) \equiv_T \text{Holant}(\{=_2\} \mid \mathcal{F}') \equiv_T \text{Holant}(\{=_2, \delta_0, \delta_1\} \mid \mathcal{F}'). \tag{18}$$

Apply Valiant's Holant theorem with the holographic transformation M identified above:

$$\text{Holant}(\{=_2, \delta_0, \delta_1\} \mid \mathcal{F}') \equiv_T \text{Holant}\left((M^{-1})^T \circ \{=_2, \delta_0, \delta_1\} \mid M \circ \mathcal{F}'\right). \tag{19}$$

As $(=_3) \in M \circ \mathcal{F}'$ by construction, the problem now has the same form as the LHS of (15), with $[y_0, y_1, y_2] = (M^{-1})^T \circ (=_2)$, $\mathcal{G}_1 = (M^{-1})^T \circ \{\delta_0, \delta_1\}$, and $\mathcal{G}_2 = M \circ \mathcal{F}'$. With a bit of effort, it can be shown that the conditions of Theorem 6 regarding ω-normalisation and unary signatures are always satisfiable by choosing M appropriately; hence:

$$\text{Holant}^c(\mathcal{F}) \equiv_T \#\text{CSP}\left((M^{-1})^T \circ \{=_2, \delta_0, \delta_1\} \cup M \circ \mathcal{F}'\right). \tag{20}$$

Now, as stated in Theorem 6, this problem is #P-hard unless $(M^{-1})^T \circ \{=_2, \delta_0, \delta_1\} \cup M \circ \mathcal{F}'$ is a subset of $\langle \mathcal{E} \rangle$ or a subset of $\mathcal{A}$. With some additional work, these conditions can be shown to correspond to $\textsc{Holant}^c(\mathcal{F})$ being tractable if $\mathcal{F} \subseteq \langle O \circ \mathcal{E} \rangle$ for some orthogonal 2×2 matrix O, if $\mathcal{F} \subseteq \langle K \circ \mathcal{E} \rangle$, or if $\mathcal{F} \subseteq S \circ \mathcal{A}$ for some $S \in \mathcal{S}$. For all other $\mathcal{F}$ containing a symmetric GHZ-type signature, $\textsc{Holant}^c(\mathcal{F})$ is #P-hard.

If f is symmetric and has W type, then:

- If $f \notin K \circ \mathcal{M} \cup KX \circ \mathcal{M}$, $\textsc{Holant}(\{=_2\} \mid \{f\})$ is #P-hard by Theorem 5.
- If $\mathcal{F} \subseteq K \circ \mathcal{M}$ or $\mathcal{F} \subseteq KX \circ \mathcal{M}$, the problem is tractable by the $\textsc{Holant}^*$ dichotomy.
- If $f \in K \circ \mathcal{M}$ but $\mathcal{F} \nsubseteq K \circ \mathcal{M}$, the problem is #P-hard by Lemma 12 below, and analogously with KX instead of K.

If f is not symmetric and $f \notin K \circ \mathcal{M} \cup KX \circ \mathcal{M}$, it is possible to realise a symmetric genuinely entangled ternary signature using Lemmas 7 and 8, so the case reduces to the above.

Finally, if f is not symmetric and $f \in K \circ \mathcal{M}$ (or $f \in KX \circ \mathcal{M}$), then either $\mathcal{F} \subseteq K \circ \mathcal{M}$ (or $\mathcal{F} \subseteq KX \circ \mathcal{M}$) and $\textsc{Holant}^c(\mathcal{F})$ is in FP, or the problem is hard by Lemma 12 below. This covers all cases. ◀

▶ **Lemma 12.** *Let $f \in \mathcal{F} \cap K \circ \mathcal{M}$ be a genuinely entangled ternary signature, and assume $\mathcal{F} \nsubseteq K \circ \mathcal{M}$. Then $\textsc{Holant}^c(\mathcal{F})$ is #P-hard. The same holds if $f \in \mathcal{F} \cap KX \circ \mathcal{M}$ and $\mathcal{F} \nsubseteq KX \circ \mathcal{M}$.*

Proof (sketch). We show hardness by either realising a symmetric genuinely entangled ternary signature that is not in $K \circ \mathcal{M} \cup KX \circ \mathcal{M}$ or by realising a symmetric binary entangled signature g such that $\textsc{Holant}(\{f\} \mid \{g\})$ is #P-hard according to Theorem 5.

The basic approach is the same as in [2], but the techniques need some modification to work in the $\textsc{Holant}^c$ setting. In particular, we show how to realise new unary signatures by gadgets using f, δ_0, δ_1, and self-loops. With these gadgets, we then realise the signatures given above. ◀

These two lemmas show that we can classify the complexity of $\textsc{Holant}^c(\mathcal{F})$ whenever $\mathcal{F}$ contains a genuinely entangled ternary signature.

4.2 Main theorem

We now have all the components required to prove the main dichotomy for $\textsc{Holant}^c$. The proof strategy is to realise certain genuinely entangled signatures of low arity. Then:

- If $\mathcal{F} \subseteq \langle \mathcal{T} \rangle$, the problem is known to be tractable.
- If $\mathcal{F}$ contains a genuinely entangled ternary signature, its complexity can be determined by Lemmas 11 and 12.
- If $=_4$ can be realised or interpolated over $\mathcal{F}$, then $\textsc{Holant}^c(\mathcal{F}) \equiv_T \#\mathrm{CSP}_2^c(\mathcal{F})$ by Lemma 10, so its complexity is determined by Theorem 3.

The arity reduction technique is adapted from that used in the real-valued $\textsc{Holant}^c$ dichotomy [11], with modifications that ensure it works for all complex-valued signatures.

▶ **Theorem 13.** *Let $\mathcal{F}$ be a set of complex-valued signatures. Then $\textsc{Holant}^c(\mathcal{F})$ is #P-hard unless:*

- *$\mathcal{F}$ is a tractable family for $\textsc{Holant}^*$,*
- *there exists $S \in \mathcal{S}$ such that $\mathcal{F} \subseteq S \circ \mathcal{A}$, or*
- *$\mathcal{F} \subseteq \mathcal{L}$.*

In all of the exceptional cases, $\textsc{Holant}^c(\mathcal{F})$ is tractable.

Proof (sketch). If $\mathcal{F}$ is one of the tractable families for Holant*, $\mathcal{F} \subseteq S \circ \mathcal{A}$ for some $S \in \mathcal{S}$, or $\mathcal{F} \subseteq \mathcal{L}$, tractability of Holant$^c(\mathcal{F})$ follows using the same algorithms as employed in the dichotomy proofs for Holant* [6], #CSP [10] (possibly after a holographic transformation), or $\#\mathrm{CSP}_2^c$ [11]. So assume otherwise. In particular, this implies that $\mathcal{F} \not\subseteq \langle\mathcal{T}\rangle$, i.e. $\mathcal{F}$ has multipartite entanglement.

Without loss of generality, we may focus on genuinely entangled signatures (cf. Section 2.2). So assume that there is some genuinely entangled signature $f \in \mathcal{F}$ of arity $n \geq 3$. If the signature has arity 3, we are done by Lemma 11. Hence assume $n \geq 4$.

As in [11], we now determine the minimum Hamming distance between any pair of bit strings in the support of f, and distinguish cases according to this value. We show that, using f, δ_0, δ_1, and self-loops, it is always possible to realise either a genuinely entangled ternary signature or a 4-ary signature of the form $a\,|0000\rangle + b\,|1100\rangle + c\,|0011\rangle + d\,|1111\rangle$ where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. In the former case, we can determine the complexity by Lemmas 11 and 12. In the latter case, we can realise or interpolate the 4-ary equality signature by Lemma 9; then, by Lemma 10, Holant$^c(\mathcal{F}) \equiv_T \#\mathrm{CSP}_2^c(\mathcal{F})$.

Thus, whenever $\mathcal{F}$ is not one of the tractable families listed in the theorem statement, the problem is #P-hard. ◀

5 Conclusions

Building on the existing dichotomies for real-valued Holant^c and for complex-valued Holant$^+$, we have derived a dichotomy for complex-valued Holant^c. The tractable cases are the complex generalisations of the tractable cases of the real-valued Holant^c dichotomy. The question of a dichotomy for complex-valued, not necessarily symmetric Holant^c had been open since the definition of the family Holant^c in 2009. Several steps in the dichotomy proof use knowledge from quantum information theory, particularly about entanglement. We expect this approach of bringing together Holant problems and quantum information theory to yield further insights into both areas of research in the future. The ultimate goals include a dichotomy for general Holant problems on the one hand, building up on existing results for symmetric functions [5] and non-negative real-valued, not necessarily symmetric functions [18]. On the other hand, we hope to gain further understanding of the complexity of classically simulating quantum circuits.

References

1 Miriam Backens. A complete dichotomy for complex-valued Holant^c. *arXiv:1704.05798 [quant-ph]*, 2017.

2 Miriam Backens. A New Holant Dichotomy Inspired by Quantum Computation. In Ioannis Chatzigiannakis, Piotr Indyk, Fabian Kuhn, and Anca Muscholl, editors, *44th International Colloquium on Automata, Languages, and Programming (ICALP 2017)*, volume 80 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 16:1–16:14, Dagstuhl, Germany, 2017. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. `doi:10.4230/LIPIcs.ICALP.2017.16`.

3 Miriam Backens. Number of superclasses of four-qubit entangled states under the inductive entanglement classification. *Physical Review A*, 95(2):022329, 2017. `doi:10.1103/PhysRevA.95.022329`.

4 J. Cai, X. Chen, and P. Lu. Graph Homomorphisms with Complex Values: A Dichotomy Theorem. *SIAM Journal on Computing*, 42(3):924–1029, jan 2013. `doi:10.1137/110840194`.

5 J. Cai, H. Guo, and T. Williams. A Complete Dichotomy Rises from the Capture of Vanishing Signatures. *SIAM Journal on Computing*, 45(5):1671–1728, 2016. doi:10.1137/15M1049798.

6 J. Cai, P. Lu, and M. Xia. Dichotomy for Holant* Problems of Boolean Domain. In *Proceedings of the Twenty-Second Annual ACM-SIAM Symposium on Discrete Algorithms*, Proceedings, pages 1714–1728. Society for Industrial and Applied Mathematics, jan 2011. doi:10.1137/1.9781611973082.132.

7 Jin-Yi Cai and Vinay Choudhary. Valiant's Holant Theorem and matchgate tensors. *Theoretical Computer Science*, 384(1):22–32, 2007. doi:10.1016/j.tcs.2007.05.015.

8 Jin-Yi Cai and Zhiguo Fu. Holographic Algorithm with Matchgates is Universal for Planar #CSP over Boolean Domain. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2017, pages 842–855, New York, NY, USA, 2017. ACM. doi:10.1145/3055399.3055405.

9 Jin-Yi Cai, Sangxia Huang, and Pinyan Lu. From Holant to #CSP and Back: Dichotomy for Holant^c Problems. *Algorithmica*, 64(3):511–533, mar 2012. doi:10.1007/s00453-012-9626-6.

10 Jin-Yi Cai, Pinyan Lu, and Mingji Xia. The complexity of complex weighted Boolean #CSP. *Journal of Computer and System Sciences*, 80(1):217–236, feb 2014. First appeared as "Holant Problems and Counting CSP" in STOC '09. doi:10.1016/j.jcss.2013.07.003.

11 Jin-Yi Cai, Pinyan Lu, and Mingji Xia. Dichotomy for Real Holant^c Problems. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1802–1821. Society for Industrial and Applied Mathematics, 2018. doi:10.1137/1.9781611975031.118.

12 W. Dür, G. Vidal, and J. I. Cirac. Three qubits can be entangled in two inequivalent ways. *Physical Review A*, 62(6):062314, 2000. doi:10.1103/PhysRevA.62.062314.

13 Mariami Gachechiladze and Otfried Gühne. Completing the proof of "Generic quantum nonlocality". *Physics Letters A*, 381(15):1281–1285, apr 2017. doi:10.1016/j.physleta.2016.10.001.

14 Sangxia Huang and Pinyan Lu. A Dichotomy for Real Weighted Holant Problems. *computational complexity*, 25(1):255–304, mar 2016. doi:10.1007/s00037-015-0118-3.

15 L. Lamata, J. León, D. Salgado, and E. Solano. Inductive classification of multipartite entanglement under stochastic local operations and classical communication. *Physical Review A*, 74(5):052336, nov 2006. doi:10.1103/PhysRevA.74.052336.

16 L. Lamata, J. León, D. Salgado, and E. Solano. Inductive entanglement classification of four qubits under stochastic local operations and classical communication. *Physical Review A*, 75(2):022318, feb 2007. doi:10.1103/PhysRevA.75.022318.

17 Dafa Li, Xiangrong Li, Hongtao Huang, and Xinxin Li. Simple criteria for the SLOCC classification. *Physics Letters A*, 359(5):428–437, dec 2006. doi:10.1016/j.physleta.2006.07.004.

18 Jiabao Lin and Hanpin Wang. The Complexity of Holant Problems over Boolean Domain with Non-Negative Weights. In Ioannis Chatzigiannakis, Piotr Indyk, Fabian Kuhn, and Anca Muscholl, editors, *44th International Colloquium on Automata, Languages, and Programming (ICALP 2017)*, volume 80 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 29:1–29:14, Dagstuhl, Germany, 2017. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. doi:10.4230/LIPIcs.ICALP.2017.29.

19 Sandu Popescu and Daniel Rohrlich. Generic quantum nonlocality. *Physics Letters A*, 166(5–6):293–297, 1992. doi:10.1016/0375-9601(92)90711-T.

20 L. Valiant. Holographic Algorithms. *SIAM Journal on Computing*, 37(5):1565–1594, jan 2008. doi:10.1137/070682575.

21 F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde. Four qubits can be entangled in nine different ways. *Physical Review A*, 65(5):052112, apr 2002. doi:10.1103/PhysRevA.65.052112.