DECOMPOSITION OF COMPLETE DESIGNS

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Dedication

This study is sincerely dedicated to my beloved parents, who have been my source of inspiration and gave me strength when I thought of giving up, who continually provide their moral, spiritual, emotional, and financial support.

Abstract

Through six chapters, the concept of decomposing the *complete design* is demonstrated. Group divisible designs, symmetric designs, strongly regular graphs, and association schemes are examples of the combinatorial objects that complete designs are decomposed into.

Reconstructing McFarland designs leads to the existence of sets of designs with disjoint incidence matrices whose sum is the complete design. The existence of infinite classes of symmetric association schemes follows from the decomposition.

Applying a similar technique on the Spence designs provides sets of designs all sharing the same complete tripartite graphs. By appropriately splitting the designs a decomposition of the complete design is obtained leading to an infinite class of non-commutative association schemes.

A final attempt is made to combine the constructed decomposition with specific classes of balanced generalized weighing matrices.

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Last but not the least, I am grateful to my parents, siblings, friends and acquaintances who remembered me in their prayers for the ultimate success and who helped me survive all the stress and not letting me give up. I consider myself nothing without them.

Contents

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List of Symbols and Abbreviations

Symbol or Abbreviation	Definition
$\mathbb{Z}$	The set of integers
q	A prime power number
d	A positive integer
I_v	The identity matrix of size ν
J_v	The $\nu \times \nu$ matrix all of whose entries are one
$\mathbb{R}_q$	The back diagonal identity matrix of size q
circ	Circulant matrix
$BGW(\nu,k,\lambda)$	Balanced generalized weighing matrix with parameters ν, k and λ over a cyclic group
$SRG(\nu,k,\lambda,\mu)$	Strongly regular graph with parameters ν, k, λ and μ
$GH(q,q^t)$	Generalized Hadamard matrix with parameters q and q^t
$DDG(\nu,k,\lambda,\mu,m,n)$	Divisible design graph with parameters ν, k, λ, μ, m, and n
$GDD(m,n,k;\lambda,\mu)$	Group divisible design with parameters m, n, ν, k, λ and μ
(ν,k,λ)-*BIBD*	Balanced incomplete block design with parameters ν, k and λ
C_p	Circulant matrix of order p with the first row $(0,1,0,\ldots,0)$
G_p	Cyclic group of order p generated by C_p
$\Omega(p,m)$	Group of order p^m whose elements are Kronecker product of the elements of G_p
ω_i	An element of $\Omega(p,m)$
$\Delta(GH(q,q^\theta),d)$	A set consisting of $q^{\theta+1}$ $(0,1)$-matrices of size q^{d+1}, where $0 \le \theta \le d-1$
$\Delta'(GH(q,q^\theta),d)$	A set made by making the elements of $\Delta(GH(q,q^\theta),d)$ symmetric
$\Delta_\varepsilon(GH(q,q^\theta),d)$	A generalization of the set $\Delta(GH(q,q^\theta),d)$
$\Delta'_\varepsilon(GH(q,q^\theta),d)$	A set made by making the elements of $\Delta_\varepsilon(GH(q,q^\theta),d)$ symmetric
$\Xi(q,d)$	$\bigcup_{\theta=0}^{d-1} \Delta(GH(q,q^\theta),d) \cup \{I_q \otimes J_{q^d}\} \cup \{0\}$

$\Xi_R(q,d)$	$\bigcup_{\theta=0}^{d-1} \Delta'(GH(q,q^\theta),d) \cup \{R_q \otimes J_{q^d}\} \cup \{0\}$
$\Xi(q,d,\varepsilon)$	$\bigcup_{\theta=0}^{d-1} \Delta_\varepsilon(GH(q,q^\theta),d) \cup \{\omega_\varepsilon \otimes J_{q^d}\} \cup \{0\}$
$\Xi_R(q,d,\varepsilon)$	$\bigcup_{\theta=0}^{d-1} \Delta'_\varepsilon(GH(q,q^\theta),d) \cup \{\omega_\varepsilon R_q \otimes J_{q^d}\} \cup \{0\}$
$\Xi'(3,d)$	$\bigcup_{\theta=0}^{d-1} \Delta(GH(3,3^\theta),d) \cup \{(J_3 - I_3) \otimes J_{3^d}\}$
$\Xi'_R(3,d)$	$\bigcup_{\theta=0}^{d-1} \Delta'(GH(3,3^\theta),d) \cup \{(J_3 - I_3) \otimes J_{3^d}\}$
$\Xi'(3,d,\varepsilon)$	$\bigcup_{\theta=0}^{d-1} \Delta_\varepsilon(GH(3,3^\theta),d) \cup \{(J_3 - I_3) \otimes J_{3^d}\}$
$\Xi'_R(3,d,\varepsilon)$	$\bigcup_{\theta=0}^{d-1} \Delta'_\varepsilon(GH(3,3^\theta),d) \cup \{(J_3 - I_3) \otimes J_{3^d}\}$
$\Lambda(q,\varepsilon)$	$\bigcup_{\theta=0}^{d-1} \Delta'_\varepsilon(GH(3,3^\theta),d) \cup \{(J_3 - \omega_\varepsilon) \otimes J_{3^d}\}$
$\Lambda_R(q,\varepsilon)$	$\bigcup_{\theta=0}^{d-1} \Delta'_\varepsilon(GH(3,3^\theta),d) \cup \{(J_3 - \omega_\varepsilon R_3) \otimes J_{3^d}\}$

Chapter 1

Introduction

1.1 Motivation and Plans

The purpose of this study is to introduce a complete design decomposition and its applications. The concepts of complete design decomposition, complete graph decompositions, and association schemes have a lot in common. A complete graph is always decomposed to combinatorial objects with symmetric adjacency matrices while symmetry is not necessary when one talks about a complete design decomposition.

A complete graph decomposition consisting of three strongly regular graphs is an amorphic association scheme [22]. Association schemes were originally introduced by Bose and Shimamoto [4], and symmetric association scheme have drawn much attention in the last few decades [1, 5, 19, 20].

Some constructions of symmetric designs lead to a complete graph decomposition. Balanced generalized weighing matrices have been used in the construction of symmetric designs since the 70's [8, 9]. In [15], Kharaghani and Torabi used particular product on balanced generalized weighing matrices to find symmetric designs with new parameters. Using their symmetric designs, Klin, Reichard, and Woldar generated a complete graph decomposition into distance regular graphs, see [18].

In [16], the authors present a decomposition of the complete graph on $2^{2m}(2^m+2)$ vertices. Then, they show how to use this decomposition to make an infinite class of symmetric association schemes.

My main motivation originates from [10], where Ionin and Kharaghani generated infinite classes of symmetric designs and strongly regular graphs with new parameters. Reconstructing their designs and graphs leads to the existence of sets of designs with disjoint incidence matrices whose sum is the complete design.

In Section 1.2, all primary and crucial definitions needed is described. In this thesis, Balanced Incomplete Block Designs, abbreviated to BIBDs, are the key combinatorial objects into which complete designs are decomposed. To achieve a greater understanding of balanced incomplete block designs, the second chapter takes a dive into this area. In Section 2.1, BIBDs are defined, and their properties are investigated. Among all kinds of BIBDs, we mostly focus on symmetric designs or symmetric BIBDs. In Section 2.2, symmetric BIBDs are studied, and Section 2.3 utilizes projective planes and more generally vector spaces on finite fields to construct symmetric designs.

In Chapter 3, we introduce a new method to reconstruct McFarland and Spence designs. McFarland and Spence designs are reviewed in Section 3.1. In Section 3.2, a new method to reconstruct McFarland and Spence designs is introduced. Section 3.2 has three subsections. First subsection is about the materials needed in the construction method. Section 3.2.2 explains how to use the new method to construct symmetric designs with the same parameters of McFarland designs while Section 3.2.3 reconstructs Spence designs.

Chapter 4 introduces and enhances the concept of decomposition of complete designs. In Section 4.1, we introduce complete design decompositions. Sections 4.2 and 4.3 are generalizations of Sections 3.2.2 and 3.2.3, respectively. In Sections 4.2 and 4.3, we introduces a new infinite class of decompositions of complete designs into McFarland and Spence Designs.

In Section 5.1 and 5.2, we show how to combine McFarland and Spence designs generated in Sections 3.2.2 and 3.2.3 with specific classes of balanced generalized weighing matrices to make symmetric designs and also strongly regular graphs. In Section 5.3, we introduce new series of infinite classes of decompositions using designs and graphs made in Sections 5.1 and 5.2.

In Chapter 6, the complete designs are decomposed into different kinds of association schemes. In Section 6.1, we introduce a new infinite class of non-commutative association schemes. For $q = 2^m$, McFarland designs always have symmetric incidence matrices. In Section 6.2, we use these designs to decompose complete designs and also complete graphs. Moreover, this decomposition leads to a new infinite class of symmetric association schemes which are studied in Section 6.3.

1.2 Preliminaries

A set with two distinct elements is called a two-set. A graph is a pair $G = (V, E)$ such that E is a set of two-sets of V. Elements of V are called *vertices* and elements of E are called *edges*. The notation xy shows the edge $\{x, y\}$ which joins x and y. Two vertices x and y are *incident* with xy, and x and y are *adjacent* to or *neighbours* of each other. The number of neighbours a vertex has is called the *degree* of the vertex.

Definition 1.1 ([3, Definition 1.2, p. 7])**.** The *adjacency matrix* of a graph on m vertices is a $(0,1)$-matrix $A = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if the vertices } \nu_i \text{ and } \nu_j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

A graph $G = (V, E)$ is called *complete* if for any two distinct vertices x and y in V, $xy \in E$. The notation K_n is shown the complete graph on n vertices. Two $(0,1)$-matrices A and B are called *disjoint* if the matrix $A + B$ is a $(0,1)$-matrix. Two graphs are called *disjoint* if their adjacency matrices are disjoint. If J_n (or simply J) is the $n \times n$ matrix of all ones, I_n (or simply I) is the identity matrix of order n, and A is the adjacency matrix of K_n, then $A = J_n - I_n$.

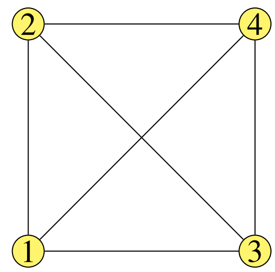

Figure 1.1: The complete graph K_4

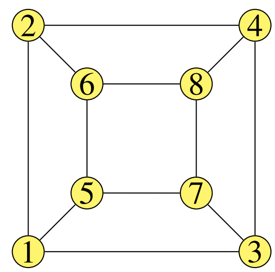

Figure 1.2: A regular graph

Example 1.2. The adjacency matrix A of the complete graph K_4 shown in Figure 1.1 is

$$A = J_4 - I_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

A graph G is a *regular graph* of order k, if degrees of all of vertices are equal to k. Figure 1.2 shows a regular graph of order 3. If A is the adjacency matrix of a regular graph of order k, then $AJ = JA = kJ$.

Example 1.3. The adjacency matrix A of the regular graph shown in Figure 1.2 is

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

and we have $AJ = JA = 3J$.

A *strongly regular graph* is a regular graph $G = (V, E)$ of order k with $|V| = \nu$ such that every two adjacent vertices have λ common neighbors, and every two non-adjacent vertices

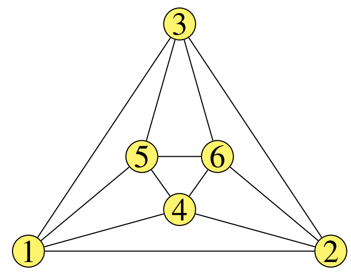

Figure 1.3: A $SRG(6,4,2,4)$

have μ common neighbors. Such graph is said to be a $SRG(\nu,k,\lambda,\mu)$.

Figure 1.3 shows a strongly regular graph with parameter $SRG(6,4,2,4)$. The regular graph shown in Figure 1.2 is not a strongly regular graph as two pairs of non-adjacent vertices $(2,3)$ and $(2,7)$ have different numbers of common neighbors; i.e. 2 and 0 respectively. For an adjacency matrix A of a $SRG(\nu,k,\lambda,\mu)$, we have $A^2=(k-\mu)I+(\lambda-\mu)A+\mu J$, or equivalently

$$A^2 = kI + \lambda A + \mu(J-I-A). \tag{1.1}$$

Example 1.4. The adjacency matrix A of the strongly regular graph shown in Figure 1.3 is:

$$A=\begin{bmatrix}0&1&1&1&1&0\\1&0&1&1&0&1\\1&1&0&0&1&1\\1&1&0&0&1&1\\1&0&1&1&0&1\\0&1&1&1&1&0\end{bmatrix}$$

For this matrix, we have

$$A^2=\begin{bmatrix}4&2&2&2&2&4\\2&4&2&2&4&2\\2&2&4&4&2&2\\2&2&4&4&2&2\\2&4&2&2&4&2\\4&2&2&2&2&4\end{bmatrix}=\begin{bmatrix}4&0&0&0&0&0\\0&4&0&0&0&0\\0&0&4&0&0&0\\0&0&0&4&0&0\\0&0&0&0&4&0\\0&0&0&0&0&4\end{bmatrix}+\begin{bmatrix}0&2&2&2&2&0\\2&0&2&2&0&2\\2&2&0&0&2&2\\2&2&0&0&2&2\\2&0&2&2&0&2\\0&2&2&2&2&0\end{bmatrix}+\begin{bmatrix}0&0&0&0&0&4\\0&0&0&0&4&0\\0&0&0&4&0&0\\0&0&4&0&0&0\\0&4&0&0&0&0\\4&0&0&0&0&0\end{bmatrix}$$

So $A^2=4I+2A+4(J-I-A)$. Hence, the matrix A satisfies (1.1).

If $A=[a_{ij}]$ is a $m\times n$ matrix and $B=[b_{ij}]$ is a $p\times q$ matrix, then the *Kronecker product* or *Hadamard product* $A\otimes B$ is the $mp\times nq$ block matrix $A\otimes B=[a_{ij}B]$.

Example 1.5. For $A=\left[\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right]$ and $B=\begin{bmatrix}a&b&c\\e&f&g\\i&j&k\end{bmatrix}$, we have

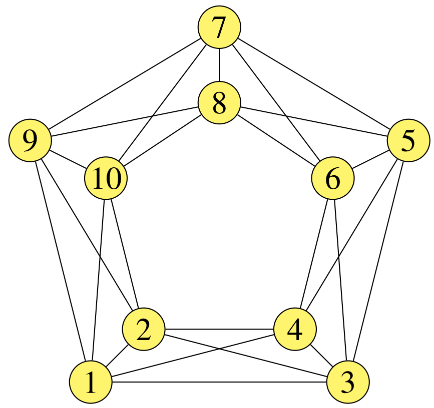

Figure 1.4: $DDG(10,5,4,2,5,2)$, adopted from [7, p. 2]

$$A \otimes B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} a & b & c \\ e & f & g \\ i & j & k \end{bmatrix} = \begin{bmatrix} a & b & c & 2a & 2b & 2c \\ e & f & g & 2e & 2f & 2g \\ i & j & k & 2i & 2j & 2k \\ 3a & 3b & 3c & 4a & 4b & 4c \\ 3e & 3f & 3g & 4e & 4f & 4g \\ 3i & 3j & 3k & 4i & 4j & 4k \end{bmatrix}.$$

Definition 1.6 ([7, Definition 1.1, p. 2])**.** A regular graph on ν vertices and k edges at each vertex is a *divisible design graph* (*DDG* for short) with parameters (ν,k,λ,μ,m,n) if the vertex set can be partitioned into m classes each of size n, such that two distinct vertices from the same class have exactly λ common neighbors, and two vertices from different classes have exactly μ common neighbors. A divisible design graph $DDG(\nu,k,\lambda,\mu,m,n)$ is called *improper* if $m=1$, $n=1$, or $\lambda=\mu$. Otherwise, it is called *proper*.

The adjacency matrix A of a divisible design graph satisfies:

$$A^2 = kI_\nu + \lambda((I_m \otimes J_n) - I_\nu) + \mu(J_\nu - (I_m \otimes J_n)). \tag{1.2}$$

Example 1.7 ([7, p. 2])**.** The graph, presented in Figure 1.4, is a $DDG(10,5,4,2,5,2)$. The adjacency matrix of this graph is:

$$A = \begin{bmatrix} 0&1&1&1&0&0&0&0&1&1 \\ 1&0&1&1&0&0&0&0&1&1 \\ 1&1&0&1&1&1&0&0&0&0 \\ 1&1&1&0&1&1&0&0&0&0 \\ 0&0&1&1&0&1&1&1&0&0 \\ 0&0&1&1&1&0&1&1&0&0 \\ 0&0&0&0&1&1&0&1&1&1 \\ 0&0&0&0&1&1&1&0&1&1 \\ 1&1&0&0&0&0&1&1&0&1 \\ 1&1&0&0&0&0&1&1&1&0 \end{bmatrix}.$$

One can check to see A satisfies (1.2):

$$A^2 = \begin{bmatrix} 5&4&2&2&2&2&2&2&2&2 \\ 4&5&2&2&2&2&2&2&2&2 \\ 2&2&5&4&2&2&2&2&2&2 \\ 2&2&4&5&2&2&2&2&2&2 \\ 2&2&2&2&5&4&2&2&2&2 \\ 2&2&2&2&4&5&2&2&2&2 \\ 2&2&2&2&2&2&5&4&2&2 \\ 2&2&2&2&2&2&4&5&2&2 \\ 2&2&2&2&2&2&2&2&5&4 \\ 2&2&2&2&2&2&2&2&4&5 \end{bmatrix} = 5I_{10} + 4((I_5 \otimes J_2) - I_{10}) + 2(J_{10} - (I_5 \otimes J_2)).$$

Definition 1.8 ([7, Definition 1.2, p. 2])**.** An incidence structure with constant block size k is a *group divisible design*, denoted by $GDD(m,n,k;\lambda,\mu)$, whenever the point set can be partitioned into m classes each of size n, such that two points from one class occur together in exactly λ blocks, and two points from different classes occur together in exactly μ blocks.

Definition 1.9 ([6, p. 17])**.** Let $r \geq 2$ be an integer. A graph $G = (V,E)$ is called *r-partite* or *multipartite* if V admits a partition into r classes such that every edge has its ends in different classes; vertices in the same partition class must not be adjacent. Instead of '2-partite' one usually says *bipartite*, and instead of '3-partite' one may say *tripartite*. An r-partite graph in which every two vertices from different partition classes are adjacent is called *complete*. The complete r-partite graphs for all r together are the *complete multipartite* graphs. A complete r-partite graph is denoted by $K_{n_1,\ldots,n_r}$, where n_i are the cardinality of the i-th class. If $n_1 = \ldots = n_r = s$, we abbreviate to K_s^r.

The adjacency matrix of a complete r-partite $K_{n_1,\ldots,n_r}$ is $J_\nu - diag(J_{n_1},\ldots,J_{n_r})$, where $\nu = n_1 + \ldots + n_\nu$ and $diag(J_{n_1},\ldots,J_{n_r})$ is a block matrix with the J_{n_i} on its diagonal and zero off diagonal.

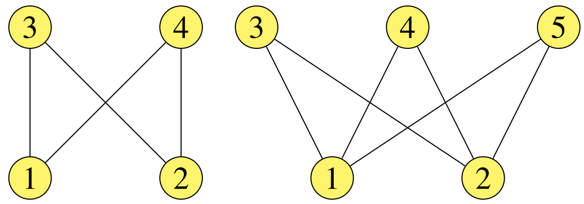

Figure 1.5: Two Complete Bipartite Graph

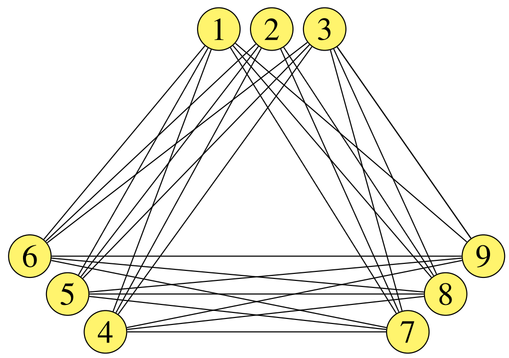

Figure 1.6: A Complete Tripartite Graphs

Example 1.10. The graphs shown in Figure 1.5 are complete bipartite graphs K_2^2 and $K_{2,3}$ with the adjacency matrices $A_1 = \begin{bmatrix} 0&0&1&1\\ 0&0&1&1\\ 1&1&0&0\\ 1&1&0&0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0&0&1&1&1\\ 0&0&1&1&1\\ 1&1&0&0&0\\ 1&1&0&0&0\\ 1&1&0&0&0 \end{bmatrix}$, respectively. The graph shown in Figure 1.6 is a complete tripartite graph K_3^3 with the adjacency matrix

$$A_3 = \begin{bmatrix} 0&0&0&1&1&1&1&1&1\\ 0&0&0&1&1&1&1&1&1\\ 0&0&0&1&1&1&1&1&1\\ 1&1&1&0&0&0&1&1&1\\ 1&1&1&0&0&0&1&1&1\\ 1&1&1&0&0&0&1&1&1\\ 1&1&1&1&1&1&0&0&0\\ 1&1&1&1&1&1&0&0&0\\ 1&1&1&1&1&1&0&0&0 \end{bmatrix}.$$

A *Latin square* is an n by n array filled with n different symbols, each occurring exactly once in each row and exactly once in each column. The matrix $\begin{bmatrix} 1&2&3\\ 3&1&2\\ 2&3&1 \end{bmatrix}$ is a Latin square of size 3 defined on three symbols 1, 2, and 3. A Latin square L is symmetric if $L = L^T$. A Latin square L is constant (back) diagonal if all of (back) diagonal entries of L are equal.

Definition 1.11 ([12, Definition 1.3, p. 227])**.** Let G be a multiplicatively written finite group. A matrix $W = [w_{ij}]$ of order ν with entries from the set $G \cup \{0\}$ is called a *balanced generalized weighing matrix* with *parameters* (ν, k, λ) over G if each row of W contains k

non-zero entries and, for all distinct $i, j \in \{1,2,\ldots,\nu\}$ the multiset

$$\{w_{ik}^{-1}w_{jk} : 1 \leq k \leq \nu, w_{ik} \neq 0, w_{jk} \neq 0\}$$

consists of exactly $\lambda/|G|$ copies of each element of G. This matrix is denoted by $BGW(\nu,k,\lambda)$. If all entries of W are nonzero, it is called *generalized Hadamard matrix* of order m over G and is denoted by $GH(n,\lambda)$, where $n = |G|$ and $\lambda = m/n$. Finally, any $BGW(n+1,n,n-1)$ is called a *generalized conference matrix*.

Example 1.12 ([12, p. 227])**.** Let $G = \{1,\omega,\omega^2\}$. Then,

$$\begin{bmatrix} 0 & 1 & \omega & \omega & 1 \\ 1 & 0 & 1 & \omega & \omega \\ \omega & 1 & 0 & 1 & \omega \\ \omega & \omega & 1 & 0 & 1 \\ 1 & \omega & \omega & 1 & 0 \end{bmatrix}$$

is a balanced generalized conference matrix $BGW(5,4,3)$ over G and

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \omega & \omega^2 & 1 & \omega^2 & 1 & \omega \\ \omega & 1 & \omega^2 & \omega^2 & \omega & 1 \\ 1 & \omega^2 & \omega^2 & 1 & \omega & \omega \\ \omega^2 & \omega^2 & 1 & \omega & \omega & 1 \\ \omega^2 & 1 & \omega^2 & \omega & 1 & \omega \end{bmatrix}$$

is a generalized Hadamard matrix $GH(3,2)$ over G.

Chapter 2

Balanced Incomplete Block Designs

Balanced Incomplete Block Designs are combinatorial objects into which complete designs are decomposed. In this Chapter, BIBDs are defined, and their properties are studied.

2.1 Introduction to Balanced Incomplete Block Design

Definition 2.1 ([21, Definition 1.1, p. 2])**.** A *design* is a pair $(X, \mathbf{A})$, where X is a set of elements called points and $\mathbf{A}$ is a multiset of nonempty subsets of X called blocks.

As is mentioned in [21, p. 2], two identical blocks in a design are said to be *repeated blocks*. A design with no repeated block is said to be a *simple design*. In [21, p. 2], the notation [ ] is used to show a multiset. Based on this notation, we have

$$[1,3,4,3] \neq \{1,3,4,3\} = \{1,3,4\}.$$

Definition 2.2 ([21, Definition 1.2, p. 2])**.** Let ν, k, and λ be positive integers such that $\nu > k$. A (ν, k, λ)-*balanced incomplete block design* (which we abbreviate to (ν, k, λ)-BIBD) is a design $(X, \mathbf{A})$ such that the following properties are satisfied:

1. $|X| = \nu$.

2. Each block contains exactly k points.

3. Every pair of distinct points is contained in exactly λ blocks.

In [21, p. 2], it is noted that the "balance" property is due to Part 3 in Definition 2.2. Since $k < v$, a BIBD is called an incomplete block design, and so all blocks are *incomplete blocks*. In [23, p. 167], when $k = v$ the design is called *complete*. We rephrase this definition in the way that the size of incidence matrix (Definition 2.11) of a complete design is flexible.

Definition 2.3. A design $(X, \mathbf{A})$ is called *complete* if

$$\mathbf{A} = [X, X, \ldots, X]$$

with $|\mathbf{A}| = n$, where $n \geq 1$.

Following [21, p. 2], blocks are written in the form of *abc* instead of $\{a, b, c\}$.

Example 2.4 ([21, Example 1.3, p. 2])**.** The design $(X, \mathbf{A})$ is a $(7,3,1)$-BIBD, where $X = \{1,2,3,4,5,6,7\}$ and $\mathbf{A} = \{123, 145, 167, 246, 257, 347, 356\}$.

Example 2.5 ([21, Example 1.7, p. 3])**.** The design $(X, \mathbf{A})$ is a $(7,3,2)$-BIBD with repeated blocks, where $X = \{1,2,3,4,5,6,7\}$ and

$$\mathbf{A} = [123, 145, 167, 246, 257, 347, 356, 123, 147, 156, 245, 267, 346, 357].$$

Example 2.6. The design $(X, \mathbf{A})$ is complete, where $X = \{1, \ldots, 5\}$ and $\mathbf{A} = [X, X, X, X, X]$.

Two basic properties of BIBD are mentioned by the following theorems.

Theorem 2.7 ([21, Theorem 1.8, p. 4])**.** *In a (v, k, λ)-BIBD, every point occurs in exactly*

$$r = \frac{\lambda(v-1)}{k-1}$$

blocks.

Theorem 2.8 ([21, Theorem 1.9, p. 5])**.** *A (v, k, λ)-BIBD has exactly*

$$b = \frac{rv}{k} = \frac{\lambda(v^2 - v)}{k^2 - k}$$

blocks.

The notation (v, b, r, k, λ)-BIBD features all five parameters. The following corollary is concluded from Theorem 2.7, Theorem 2.8, and the fact that b and r must be integers.

Corollary 2.9 ([21, Corollary 1.10, p. 5])**.** *If a* (v, k, λ)*-BIBD exists, then* $\lambda(v-1) \equiv 0 \pmod{k-1}$ *and* $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$.

For example, there is no $(9, 4, 1)$-BIBD considering $\lambda(v-1) = 8 \not\equiv 0 \pmod 3$, and there is no $(11, 7, 1)$-BIBD considering $\lambda v(v-1) = 110 \not\equiv 0 \pmod{42}$.

In [21, p. 5], it is explained that one of the principal focus of combinatorial design theory is to establish necessary and sufficient conditions for the existence of a (v, k, λ)-BIBD. One of the general applications of Corollary 2.9 is to decide necessary conditions for families of BIBDs having fixed values of k and λ.

Example 2.10 ([21, p. 5])**.** For the existence of a $(v, 3, 1)$-BIBD, Corollary 2.9 proves that the following conditions are required:

$$\lambda(v-1) = v-1 \equiv 0 \pmod 2$$

and

$$\lambda v(v-1) = v(v-1) \equiv 0 \pmod 6.$$

By these two conditions, the cases $v \equiv 0, 2, 4 \pmod 6$ are impossible as we know $v-1 \equiv 0 \pmod 2$. The case $v \equiv 1 \pmod 6$ is possible beacuse $v \equiv 1 \pmod 6$ implies $v(v-1) \equiv 0 \pmod 6$. The case $v \equiv 3 \pmod 6$ is possible because $v \equiv 3 \pmod 6$ implies $v(v-1) \equiv 0 \pmod 6$. The case $v \equiv 5 \pmod 6$ is impossible as $v \equiv 5 \pmod 6$ results in $v(v-1) \not\equiv 0 \pmod 6$. Therefore, there is a $(v, 3, 1)$-BIBD only if $v \equiv 1, 3 \pmod 6$.

Definition 2.11 ([21, Definition 1.11, p. 6]). Let $(X,\mathbf{A})$ be a design where $X=\{x_1,\ldots,x_\nu\}$ and $\mathbf{A}=[A_1,\ldots,A_b]$. The incidence matrix of $(X,\mathbf{A})$ is the $\nu\times b$, $(0,1)$-matrix $M=[m_{ij}]$ defined by the rule

$$m_{ij}=\begin{cases}1 & \text{if } x_i\in A_j\\ 0 & \text{if } x_i\notin A_j.\end{cases}$$

In [21, p. 2], three features of the incidence matrix M of a (ν,b,r,k,λ)-BIBD are listed as follow:

1. In each column, M has exactly k ones.
2. In each row, M has exactly r ones.
3. The inner product of any two distinct rows of M is equal to λ.

Note that $\mathbf{u}_n$ stands for a row vector of length n that all of its coordinates are equal to one.

Theorem 2.12 ([21, Theorem 1.13, p. 6]). *Let M be a $\nu\times b$ $(0,1)$-matrix and let $2\le k<\nu$. Then, M is the incidence matrix of a (ν,b,r,k,λ)-BIBD if and only if $MM^T=\lambda J_\nu+(r-\lambda)I_\nu$ and $\mathbf{u}_\nu M=k\mathbf{u}_b$.*

Example 2.13. In this example, we study the incidence matrices of all designs mentioned in Examples 2.4, 2.5, and 2.6. The incidence matrix N_1 of the $(7,3,1)$-BIBD of Example 2.4 is

$$N_1=\begin{bmatrix}1&1&1&0&0&0&0\\1&0&0&1&1&0&0\\1&0&0&0&0&1&1\\0&1&0&1&0&1&0\\0&1&0&0&1&0&1\\0&0&1&1&0&0&1\\0&0&1&0&1&1&0\end{bmatrix}.$$

We have

$$N_1N_1^T=\begin{bmatrix}3&1&1&1&1&1&1\\1&3&1&1&1&1&1\\1&1&3&1&1&1&1\\1&1&1&3&1&1&1\\1&1&1&1&3&1&1\\1&1&1&1&1&3&1\\1&1&1&1&1&1&3\end{bmatrix}=J_7+2I_7$$

and $\mathbf{u}_7 N_1 = 3\mathbf{u}_7$. The incidence matrix N_2 of the $(7,3,2)$-BIBD of Example 2.5 is a 7×14 $(0,1)$-matrix

$$N_2 = \begin{bmatrix} 1&1&1&0&0&0&0&1&1&1&0&0&0&0 \\ 1&0&0&1&1&0&0&1&0&0&1&1&0&0 \\ 1&0&0&0&0&1&1&1&0&0&0&0&1&1 \\ 0&1&0&1&0&1&0&0&1&0&1&0&1&0 \\ 0&1&0&0&1&0&1&0&0&1&1&0&0&1 \\ 0&0&1&1&0&0&1&0&0&1&0&1&1&0 \\ 0&0&1&0&1&1&0&0&1&0&0&1&0&1 \end{bmatrix}.$$

We have

$$N_2 N_2^T = \begin{bmatrix} 6&2&2&2&2&2&2 \\ 2&6&2&2&2&2&2 \\ 2&2&6&2&2&2&2 \\ 2&2&2&6&2&2&2 \\ 2&2&2&2&6&2&2 \\ 2&2&2&2&2&6&2 \\ 2&2&2&2&2&2&6 \end{bmatrix} = 4I_7 + 2J_7$$

and $\mathbf{u}_7 N_2 = 3\mathbf{u}_{14}$. The incidence matrix N_3 of the complete design $(X,\mathbf{A})$ of Example 2.6 is $N_3 = \begin{bmatrix} 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \end{bmatrix}$.

In [21, p. 15], the design $(X, \{X \setminus A : A \in \mathbf{A}\})$ is called the *block complement* or simply the *complement* of $(X,\mathbf{A})$.

Theorem 2.14 ([21, Theorem 1.32, p. 15])**.** *Suppose $(X,\mathbf{A})$ is a (ν,b,r,k,λ)-BIBD, where $k \leq \nu - 2$. Then, the complement of $(X,\mathbf{A})$ is a $(\nu, b, b-r, \nu-k, b-2r+\lambda)$-BIBD.*

Example 2.15 ([21, p. 16])**.** Let $(X,\mathbf{A})$ be the $(7,3,1)$-BIBD or $(7,7,3,3,1)$-BIBD mentioned in Example 2.4. The complement of $(X,\mathbf{A})$ is $(X,\mathbf{A}')$, where

$$\mathbf{A}' = \{X \setminus A : A \in \mathbf{A}\} = \{4567, 2367, 2345, 1357, 1346, 1256, 1247\}.$$

The design $(X,\mathbf{A}')$ is a $(7,4,2)$-BIBD or $(7,7,4,4,2)$-BIBD.

As is mentioned in [21, p. 16], one of the remarkable necessary condition for the existence of a (ν,k,λ)-BIBD is known as Fisher's Inequality.

Theorem 2.16 ([21, Theorem 1.33, p. 16])**.** *In any (ν,b,r,k,λ)-BIBD, $b \geq \nu$.*

2.2 Symmetric Designs

In this thesis, the focus is mainly on symmetric BIBDs. Therefore, we inspect different features of these designs.

Definition 2.17 ([21, Definition 2.1, p. 23])**.** A BIBD in which $b = \nu$ (or, equivalently, $r = k$ or $\lambda(v-1) = k^2 - k$) is called a *symmetric BIBD*.

Note that the incidence matrix of a symmetric BIBD is not necessarily symmetric. We use symmetric design and symmetric BIBD interchangeably.

Example 2.18 ([21, p. 23])**.** For a point set X with $|X| = \nu$, let $\mathbf{A}$ consists of all k-subsets of X. Then, $(X, \mathbf{A})$ is (ν, k, λ)-BIBD. If $k = \nu - 1$, $(X, \mathbf{A})$ is a symmetric $(\nu, \nu-1, \nu-2)$-BIBD or symmetric $(\nu, \nu-1, \nu-2)$-design.

The following theorem expresses one of the important properties of a symmetric design.

Theorem 2.19 ([21, Theorem 2.2, p. 23])**.** *Suppose that $(X, \mathbf{A})$ is a symmetric (ν, k, λ)-BIBD and denote $\mathbf{A} = \{A_1, \ldots, A_\nu\}$. Suppose that $1 \leq i, j \leq \nu$, $i \neq j$. Then, $|A_i \cap A_j| = \lambda$.*

By Theorems 2.19, the following corollaries are achieved.

Corollary 2.20 ([21, Corollary 2.4, p. 25])**.** *Suppose M is the incidence matrix of a symmetric (ν, k, λ)-BIBD. Then, M^T is also the incidence matrix of a symmetric (ν, k, λ)-BIBD.*

Corollary 2.21 ([21, Corollary 2.5, p. 25])**.** *Suppose that μ is a positive integer and $(X, \mathbf{A})$ is a (ν, b, r, k, λ)-BIBD such that $|A \cap A'| = \mu$ for all $A, A' \in \mathbf{A}$. Then, $(X, \mathbf{A})$ is a symmetric BIBD and $\mu = \lambda$.*

We proceed analyzing symmetric designs by studying a method of generating BIBDs from symmetric designs.

Definition 2.22 ([21, Definition 2.6, p. 26])**.** Suppose $(X, \mathbf{A})$ is a symmetric (ν, k, λ)-BIBD and let $A_0 \in \mathbf{A}$. Define

$$Der(X, \mathbf{A}, A_0) = (A_0,\ \{A \cap A_0\colon\ A \in \mathbf{A},\ A \neq A_0\})$$

and

$$Res(X,\mathbf{A},A_0) = (X \setminus A_0,\ \{A \setminus A_0\colon\ A \in \mathbf{A},\ A \neq A_0\}).$$

The design $Der(X,\mathbf{A},A_0)$ is called a *derived* BIBD, and the design $Res(X,\mathbf{A},A_0)$ is called a *residual BIBD*.

In the following theorem, the parameters of $Der(X,\mathbf{A},A_0)$ and $Res(X,\mathbf{A},A_0)$ are provided.

Theorem 2.23 ([21, Theorem 2.7, p. 26])**.** *Suppose* $(X,\mathbf{A})$ *is a symmetric* (v,k,λ)*-BIBD and let* $A_0 \in \mathbf{A}$*. Then,* $Der(X,\mathbf{A},A_0)$ *is a* $(k, v-1, k-1, \lambda, \lambda-1)$*-BIBD provided that* $\lambda \geq 2$*. Moreover,* $Res(X,\mathbf{A},A_0)$ *is a* $(v-k, v-1, k, k-\lambda, \lambda)$*-BIBD provided that* $k \geq \lambda + 2$*.*

Example 2.24 ([21, Example 2.8, p. 26])**.** The eqaulity $2(11-1) = 5(5-1)$ implies that an $(11,5,2)$-BIBD is symmetric. For an $(11,5,2)$-BIBD, we have $\lambda \geq 2$ and $k \geq \lambda + 2$. Therefore, we can use Theorem 2.23. Let $X = \{0,1,\ldots,10\}$ and $A_0 = \{1,3,4,5,9\}$. Suppose $(X,\mathbf{A})$ be the symmetric $(11,5,2)$-design whose incidence matrix is

$$\begin{bmatrix}
0&0&1&0&0&0&1&1&1&0&1\\
1&0&0&1&0&0&0&1&1&1&0\\
0&1&0&0&1&0&0&0&1&1&1\\
1&0&1&0&0&1&0&0&0&1&1\\
1&1&0&1&0&0&1&0&0&0&1\\
1&1&1&0&1&0&0&1&0&0&0\\
0&1&1&1&0&1&0&0&1&0&0\\
0&0&1&1&1&0&1&0&0&1&0\\
0&0&0&1&1&1&0&1&0&0&1\\
1&0&0&0&1&1&1&0&1&0&0\\
0&1&0&0&0&1&1&1&0&1&0
\end{bmatrix}.$$

Then, the design $Der(X,\mathbf{A},A_0)$ is a $(5,2,1)$-BIBD and its incedence matrix is

$$\begin{bmatrix}
0&0&1&0&0&0&1&1&1&0\\
0&1&0&0&1&0&0&0&1&1\\
1&0&1&0&0&1&0&0&0&1\\
1&1&0&1&0&0&1&0&0&0\\
0&0&0&1&1&1&0&1&0&0
\end{bmatrix}.$$

Similarly, the design $Res(X,\mathbf{A},A_0)$ is a $(6,3,2)$-BIBD and its incedence matrix is

$$\begin{bmatrix} 0&1&0&0&0&1&1&1&0&1 \\ 1&0&0&1&0&0&0&1&1&1 \\ 1&1&1&0&1&0&0&1&0&0 \\ 0&1&1&1&0&1&0&0&1&0 \\ 0&0&1&1&1&0&1&0&0&1 \\ 1&0&0&0&1&1&1&0&1&0 \end{bmatrix}.$$

2.3 Symmetric Designs and Projective Planes

There are various ways to define a projective plane. Following [21, p. 27], we use a BIBD with particular parameters to define a projective plane.

Definition 2.25 ([21, Definition 2.9, p. 27])**.** An $(n^2+n+1, n+1, 1)$-BIBD with $n \geq 2$ is said a *projective plane* of *order n*.

Since $n^2+n = (n+1)n$, projective planes are symmetric *BIBDs*. An example of a projective plane of order $n = 2$ is the $(7,3,1)$-BIBD given in Example 2.4. In the following example, a projective plane of order $n = 3$ is exhibited.

Example 2.26 ([21, Example 2.11, p. 28])**.** Let $X = \{0,2,\dots,12\}$ and $(X,\mathbf{A})$ be the symmetric $(13,4,1)$-design whose incidence matrix is

$$\begin{bmatrix} 1&1&0&1&0&0&0&0&0&1&0&0&0 \\ 0&1&1&0&1&0&0&0&0&0&1&0&0 \\ 0&0&1&1&0&1&0&0&0&0&0&1&0 \\ 0&0&0&1&1&0&1&0&0&0&0&0&1 \\ 1&0&0&0&1&1&0&1&0&0&0&0&0 \\ 0&1&0&0&0&1&1&0&1&0&0&0&0 \\ 0&0&1&0&0&0&1&1&0&1&0&0&0 \\ 0&0&0&1&0&0&0&1&1&0&1&0&0 \\ 0&0&0&0&1&0&0&0&1&1&0&1&0 \\ 0&0&0&0&0&1&0&0&0&1&1&0&1 \\ 1&0&0&0&0&0&1&0&0&0&1&1&0 \\ 0&1&0&0&0&0&0&1&0&0&0&1&1 \\ 1&0&1&0&0&0&0&0&1&0&0&0&1 \end{bmatrix}.$$

Then, the design $(X,\mathbf{A})$ is a projective plane of order 3.

The following theorem guarantees the existence of a projective plane of order q, for each prime power q.

Theorem 2.27 ([21, Theorem 2.10, p. 28]). *There exists a symmetric* $(q^2+q+1, q+1, 1)$-*design, for every prime power q.*

As is mentioned in [21], one of the greatly admired open problems in design theory is the question regarding the existence of a projective plane of non-prime power order. Currently, no projective plane of non-prime power order is known.

Definition 2.28 ([21, Definition 2.12, p. 29]). Let $n \geq 2$. An $(n^2, n^2+n, n+1, n, 1)$-BIBD is called an *affine plane* of *order n*.

Theorem 2.23 confirms that the residual BIBD of a projective plane of order n or a $(n^2+n+1, n+1, 1)$-BIBD is an $(n^2, n^2+n, n+1, n, 1)$-BIBD which is an affine plane of order n. Hence, Theorem 2.27 results in the following Theorem.

Theorem 2.29 ([21, Theorem 2.13, p. 29]). *For every prime power* $q \geq 2$*, there is a* $(q^2, q, 1)$-*BIBD (i.e., an affine plane of order q).*

Theorem 2.27 is the specific case ($d = 2$) of the following theorem.

Theorem 2.30 ([21, Theorem 2.14, p. 30]). *Suppose* $q \geq 2$ *is a prime power and* $d \geq 2$ *is an integer. Then there is a symmetric*

$$\left(\frac{q^{d+1}-1}{q-1}, \frac{q^d-1}{q-1}, \frac{q^{d-1}-1}{q-1}\right)\text{-}BIBD.$$

As a consequence of Theorem 2.23, the following corollary provides the parameters of derived and residual BIBDs of the BIBDs made in Theorem 2.31.

Corollary 2.31 ([21, Corollary 2.15, p. 30]). *Let* $q \geq 2$ *be a prime power and* $d \geq 2$ *be an integer. Then, there exists a*

$$\left(q^d, q^{d-1}, \frac{q^{d-1}-1}{q-1}\right)\text{-}BIBD.$$

Moreover, if $d > 2$, there is a

$$\left(\frac{q^d-1}{q-1}, \frac{q^{d-1}-1}{q-1}, \frac{q(q^{d-2}-1)}{q-1}\right)\text{-}BIBD.$$

In [21, p. 30], the following two theorems explain a couple of necessary existence conditions for symmetric designs.

Theorem 2.32 ([21, Theorem 2.16, p. 30])**.** *Suppose there is a symmetric (ν, k, λ)-BIBD with ν even. Then $k-\lambda$ is a perfect square.*

Theorem 2.33 ([21, Theorem 2.19, p. 32])**.** *Suppose there exists a symmetric (ν, k, λ)-BIBD with ν odd. Then there exist integers x, y, and z (not all 0) such that*

$$x^2 = (k-\lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2.$$

Chapter 3

A New Construction for McFarland and Spence Designs

In this Section, we explain how to reconstruct the structure of McFarland and Spence Designs introduced in [10]. In [10], Ionin and Kharaghani utilized hyperplanes of vector spaces to generate McFarland and Spence Designs. When the parameters of the designs are growing, the method becomes more complicated. Therefore, it was practically infeasible to use computer programming for doing further research on these designs.

My supervisor suggested a new method in which we use generalized Hadamard matrices. The new method is computer-friendly and it directly generates the incidence matrix of a design. Applying the new method provides us with new results on McFarland and Spence designs. We first review the original construction in Section 3.1 and then introduce the new construction in Section 3.2.

3.1 McFarland and Spence Designs

In this section, we review [10] to explain the original method. Let V be a $(d+1)$-dimensional vector space over the field $GF(q)$, where q is a prime power and d is a positive integer. A d-dimensional subspace of V is called a *hyperplane*. A coset of a hyperplane is called a *d-flat*. If H is a hyperplane and $x \neq y \in V$, then $H+x$ and $H+y$ are considered parallel. The space V has $r = (q^{(d+1)} - 1)/(q-1)$ hyperplanes, $H_1, H_2, \dots, H_r$. For each i, all d-flats parallel to H_i make a *parallel class* Π_i with $|\Pi_i| = q$, see [10, p. 213].

Example 3.1. We set $q = 3$ and $d = 1$. Let V be the 2-dimensional vector space over $GF(3) = \{-1, 0, 1\}$. The point set of this vector space is $\{1 = (-1, -1), 2 = (-1, 0), 3 = (-1, 1), 4 = (0, -1), 5 = (0, 0), 6 = (0, 1), 7 = (1, -1), 8 = (1, 0), 9 = (1, 1)\}$. This vector space has $r = (q^{d+1} - 1)/(q-1) = (3^2 - 1)/(3-1) = 4$ hyperplanes $H_1 = \{4, 5, 6\}$, $H_2 = \{2, 5, 8\}$, $H_3 = \{7, 5, 3\}$, and $H_4 = \{1, 5, 9\}$. Then, the parallel classes containing parallel 1-flats are:

$$\Pi_1 = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}\},$$

$$\Pi_2 = \{\{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 9\}\},$$

$$\Pi_3 = \{\{4, 2, 9\}, \{7, 5, 3\}, \{6, 8, 1\}\},$$

and

$$\Pi_4 = \{\{2, 6, 7\}, \{1, 5, 9\}, \{8, 4, 3\}\}.$$

Let $\mathcal{F}$ be the set consisting of the entire space V, the empty set, all d-flats, and their complements. Let m be a positive integer and let $\mathcal{F}^m$ be the set of all ordered m-tuples of elements of $\mathcal{F}$.

Definition 3.2 ([10, Definition 4.1, p. 214])**.** A *McFarland* $(r+1)$*-tuple* is an $(r+1)$-tuple $\mathbf{A} = (A_1, A_2, \ldots, A_{r+1}) \in \mathcal{F}^{r+1}$ such that one of the sets $A_1, A_2, \ldots, A_{r+1}$ is empty and the other r sets are pairwise non-parallel d-flats. A Spence r-tuple is an r-tuple $\mathbf{A} = (A_1, A_2, \ldots, A_r) \in \mathcal{F}^r$ such that one of the sets $A_1, A_2, \ldots, A_r$ is the complement of a d-flat parallel to H_1 and the other $r-1$ sets are pairwise non-parallel d-flats, which are not parallel to H_1.

Example 3.3. Consider the vector space mentioned in Example 3.1. The 5-tuple $\mathbf{A} = (A_1, A_2, \ldots, A_5) \in \mathcal{F}^5$, where A_1 is empty, $A_2 = \{1, 2, 3\}$, $A_3 = \{1, 4, 7\}$, $A_4 = \{4, 2, 9\}$, and $A_5 = \{2, 6, 7\}$, is a McFarland 5-tuple. The 4-tuple $\mathbf{A} = (A_1, A_2, A_3, A_4) \in \mathcal{F}^4$, where $A_1 = \{4, 5, 6, 7, 8, 9\}$, $A_2 = \{1, 4, 7\}$, $A_3 = \{4, 2, 9\}$, and $A_4 = \{2, 6, 7\}$, is a Spence 4-tuple. We note that $A_1 = \{4, 5, 6, 7, 8, 9\}$ is the complement of the d-flat $\{1, 2, 3\}$ which is parallel to $H_1 = \{4,5,6\}$. We also note that $A_2 = \{1,4,7\}$, $A_3 = \{4,2,9\}$, and $A_4 = \{2,6,7\}$ are pairwise non-parallel d-flats, which are not parallel to H_1.

In [10, p. 213], it is mentioned that a finite abelian group $G = \{x_1, x_2, \dots, x_n\}$ is equipped with a *symmetric order* if $x_i + x_{n+1-i}$ is the same for $i = 1, 2, \dots, n$. Now, consider a finite abelian group $G = \{x_1, x_2, \dots, x_n\}$ with a symmetric order. For any subset A of G, the $(0,1)$ matrix $M(A) = [m_{ij}(A)]$ of order n is defined by

$$m_{ij}(A) = \begin{cases} 1 & \text{if } x_{n+1-i} - x_j \in A, \\ 0 & \text{if } x_{n+1-i} - x_j \notin A. \end{cases}$$

The matrix $M(A)$ is symmetric because the group G is equipped with a symmetric order.

Example 3.4. The vector space V mentioned in Example 3.1 is a finite abelian group under addition which is equipped with a symmetric order. The sets $A_1, A_2, \dots, A_5$ of the McFarland 5-tuple in Example 3.3 are subsets of V. Now, we want to find $M(A_i)$ which is a 9×9 $(0,1)$-matrix, for each $i = 1, 2, \dots, 5$. Since A_1 is empty, all entries of the matrix $M(A_1)$ are zero. We have

$$M(A_1) = \begin{bmatrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{bmatrix} \quad M(A_2) = \begin{bmatrix} 1&1&1&0&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&1&1&1 \\ 0&0&0&1&1&1&0&0&0 \\ 0&0&0&1&1&1&0&0&0 \\ 0&0&0&1&1&1&0&0&0 \end{bmatrix} \quad M(A_3) = \begin{bmatrix} 1&0&0&1&0&0&1&0&0 \\ 0&0&1&0&0&1&0&0&1 \\ 0&1&0&0&1&0&0&1&0 \\ 1&0&0&1&0&0&1&0&0 \\ 0&0&1&0&0&1&0&0&1 \\ 0&1&0&0&1&0&0&1&0 \\ 1&0&0&1&0&0&1&0&0 \\ 0&0&1&0&0&1&0&0&1 \\ 0&1&0&0&1&0&0&1&0 \end{bmatrix}$$

$$M(A_4) = \begin{bmatrix} 0&0&1&0&1&0&1&0&0 \\ 0&1&0&1&0&0&0&0&1 \\ 1&0&0&0&0&1&0&1&0 \\ 0&1&0&1&0&0&0&0&1 \\ 1&0&0&0&0&1&0&1&0 \\ 0&0&1&0&1&0&1&0&0 \\ 1&0&0&0&0&1&0&1&0 \\ 0&0&1&0&1&0&1&0&0 \\ 0&1&0&1&0&0&0&0&1 \end{bmatrix} \quad M(A_5) = \begin{bmatrix} 0&0&1&1&0&0&0&1&0 \\ 0&1&0&0&0&1&1&0&0 \\ 1&0&0&0&1&0&0&0&1 \\ 1&0&0&0&1&0&0&0&1 \\ 0&0&1&1&0&0&0&1&0 \\ 0&1&0&0&0&1&1&0&0 \\ 0&1&0&0&0&1&1&0&0 \\ 1&0&0&0&1&0&0&0&1 \\ 0&0&1&1&0&0&0&1&0 \end{bmatrix}$$

The following theorem explains how McFarland tuples form symmetric designs.

Theorem 3.5 ([10, Theorem 4.1, p. 214])**.** *Let q be a prime power, d a positive integer, and V a $(d+1)$-dimensional vector space over the field $GF(q)$. Suppose $[A_{ij}]$, $i,j = 1,\dots,r+1$, be an array of subsets of V, all rows and all columns of which are McFarland $(r+1)$-tuples. Then the block matrix $N = [M(A_{ij})]$ is the incidence matrix of a symmetric $((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$-design.*

Example 3.6 ([21, Example 5.25, p. 114])**.** We use the matrices $M(A_1), M(A_2), \dots, M(A_5)$ found in Example 3.4 to generate a matrix N satisfying the condition of Theorem 3.5. In the Latin square $\begin{bmatrix} 1&2&3&4&5\\ 2&3&4&5&1\\ 3&4&5&1&2\\ 4&5&1&2&3\\ 5&1&2&3&4 \end{bmatrix}$, we replace $i = 1,2,\dots,5$ with $M_i = M(A_i)$. The resulted matrix is

$$N = \begin{bmatrix} M_1 & M_2 & M_3 & M_4 & M_5\\ M_2 & M_3 & M_4 & M_5 & M_1\\ M_3 & M_4 & M_5 & M_1 & M_2\\ M_4 & M_5 & M_1 & M_2 & M_3\\ M_5 & M_1 & M_2 & M_3 & M_4 \end{bmatrix}.$$

By Theorem 3.5, N is the incidence matrix of a symmetric $(45,12,3)$-design which is given in Appendix 1.

We have a similar structure using Spence tuples. Spence tuples generate a symmetric design only when $q = 3$.

Theorem 3.7 ([10, Theorem 4.2, p. 215])**.** *Let V be the $(d+1)$-dimensional vector space over $GF(3)$ and $r = (3^{d+1}-1)/2$. Let $[A_{ij}]$, $i,j = 1,2,\dots,r$, be an array of subsets of V, all rows and all columns of which are Spence r-tuples. Then the block matrix $N = [M(A_{ij})]$ is the incidence matrix of a symmetric $(3^{d+1}(3^{d+1}-1)/2, 3^d(3^{d+1}+1)/2, 3^d(3^d+1)/2)$-design.*

3.2 A Construction for Symmetric Designs

Let q be a prime power, d a positive integer, and $r = \dfrac{q^{d+1}-1}{q-1}$. In Section 3.1, McFarland and Spence tuples generate symmetric $((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$-designs and symmetric $\left(\dfrac{3^{d+1}(3^{d+1}-1)}{2}, \dfrac{3^d(3^{d+1}+1)}{2}, \dfrac{3^d(3^d+1)}{2}\right)$-designs, respectively.

In this Section, we use generalized Hadamard matrices to construct symmetric $((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$-designs and $\left(\frac{3^{d+1}(3^{d+1}-1)}{2}, \frac{3^d(3^{d+1}+1)}{2}, \frac{3^d(3^d+1)}{2}\right)$-designs.

In Section 3.2.1, we discuss the main materials of our structure. In Section 3.2.2, we reconstruct designs made by McFarland tuples while using generalized Hadamard matrices. In Section 3.2.3, we see how to reconstruct designs made by Spence tuples.

3.2.1 Introduction

We start this chapter by defining specific classes of Latin squares needed for the construction methods. A Latin square L of size n is *constant back diagonal* if there exists a number x such that for all $1 \leq k \leq n$, $L_{k,n+1-k} = x$. For any given n, any circulant Latin square of size n is a constant diagonal Latin square which is not necessarily symmetric. For example, $L = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix}$ is a circulant Latin square of size 4. Moreover, any back circulant Latin square of size n is a symmetric Latin square which is not necessarily constant diagonal but is constant back diagonal. For example, $L = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$ is a constant back diagonal Latin square of size 4.

For any even n, there is a symmetric constant diagonal Latin square of size n. In Chapter 6, we use Latin squares of even size which are symmetric, constant diagonal, and constant back diagonal. In the following lemma, we explain how to construct such Latin squares.

Lemma 3.8 ([13, Lemma 2, p. 70])**.** *For any even n, there is a symmetric Latin square L of order n with constant diagonal and constant back diagonal.*

Proof. Take addition table mod $n-1$. Move the diagonal entries to form the first row and the first column. Then, fill-in the diagonal with the remaining number. $\square$

Example 3.9. To construct a symmetric constant diagonal Latin square of size 6, we form the addition table on $\{1,\ldots,5\}$ to get a symmetric Latin square of size 5. We have $A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$. All the numbers appear on the diagonal. Now, we define the symmetric constant diagonal Latin square $L = \begin{bmatrix} 5&0&2&4&1&3 \\ 0&5&1&2&3&4 \\ 2&1&5&3&4&0 \\ 4&2&3&5&0&1 \\ 1&3&4&0&5&2 \\ 3&4&0&1&2&5 \end{bmatrix}$ which also has constant back diagonal.

Now, we describe a group $\Omega(p,m)$ which will be used in our method. The elements of this group are $(0,1)$-matrices and the operation of this group is the matrix multiplication.

Definition 3.10. Let p^m be a prime power and G_p the cyclic group generated by C_p, where $C_p = circ(0,1,0,\ldots,0)$ is a circulant matrix of order p. Then, $\Omega(p,m)$ stands for the group $\{A_1 \otimes \ldots \otimes A_m | A_1,\ldots,A_m \in G_p\}$ of $(0,1)$-matrices whose operation is the matrix multiplication.

In Definition 3.10, let $\omega = A_1 \otimes \ldots \otimes A_m$ be an arbitrary element of $\Omega(p,m)$. The inverse of ω is $A_1^{-1} \otimes \ldots \otimes A_m^{-1}$, where A_i^{-1} is the inverse of A_i in G_p, $i = 1,\ldots,m$. The identity element of $\Omega(p,m)$ is I_{p^m}, the identity matrix of order p^m.

Lemma 3.11. *Let p^m be a prime power. We have*

(i) $\sum_{A \in \Omega(p,m)} A = J_{p^m}$;

(ii) For each matrix $A \in \Omega(p,m)$, we have $A^{-1} = A^T$, where A^T is the transpose of A.

(iii) For any two elements $\omega_1 = C_p^{\alpha_1} \otimes \ldots \otimes C_p^{\alpha_m}$ and $\omega_2 = C_p^{\beta_1} \otimes \ldots \otimes C_p^{\beta_m}$ of $\Omega(p,m)$, the equality $\omega_1 = \omega_2$ implies $C_p^{\alpha_i} = C_p^{\beta_i}$, for all $i = 1,\ldots,m$.

Proof. We prove (i) by induction on m.

Base case: We need to prove $\sum_{A \in G} A = J_p$. By Definition 3.10, we know $G_p = < C_p >$, where $C_p = circ(0,1,0,\ldots,0)$ is a circulant matrix of size p. We have

$$\begin{aligned}\sum_{A \in G} A &= C_p^0 + C_p^1 + C_p^2 + \ldots + C_p^{p-1} \\ &= circ(1,0,0,\ldots,0) + circ(0,1,0,\ldots,0) + \cdots + circ(0,0,0,\ldots,1) \\ &= circ(1,1,\ldots,1) = J_p.\end{aligned}$$

Inductive step: Suppose $\sum_{A\in\Omega(p,m-1)} A = J_{p^{m-1}}$. We have

$$\begin{aligned}\sum_{A\in\Omega(p,m)} A = \sum_{A\in\Omega(p,m-1)} \sum_{B\in G_p} A\otimes B &= \sum_{B\in G_p} \Big[\sum_{A\in\Omega(p,m-1)} A\Big]\otimes B\\ &= \sum_{B\in G_p} J_{p^{m-1}}\otimes B\\ &= J_{p^{m-1}}\otimes \Big[\sum_{B\in G_p} B\Big]\\ &= J_{p^{m-1}}\otimes J_p\\ &= J_{p^m}.\end{aligned}$$

To prove (ii), we note that the rows of $C_p = circ(0,1,0,\dots,0)$ are pairwise orthogonal and so we have $C_pC_p^T = I_p$. From this, we conclude that $BB^T = I_p$ for any element $B\in G_p$. Hence, $AA^T = I_{p^m}$.

We prove (iii) by induction on m.

Base case: We set $m=2$. We need to prove $C_p^{\alpha_1}\otimes C_p^{\alpha_2} = C_p^{\beta_1}\otimes C_p^{\beta_2}$ implies $C_p^{\alpha_i} = C_p^{\beta_i}$, for $i=1,2$. Let $C_{ij}^{\alpha_1}$ and $C_{ij}^{\beta_1}$ be the (i,j)-entry of $C_p^{\alpha_1}$ and $C_p^{\beta_1}$, respectively. We have $C_p^{\alpha_1}\otimes C_p^{\alpha_2} = [C_{ij}^{\alpha_1}C_p^{\alpha_2}]$ and $C_p^{\beta_1}\otimes C_p^{\beta_2} = [C_{ij}^{\beta_1}C_p^{\beta_2}]$. Hence, the equality $C_p^{\alpha_1}\otimes C_p^{\alpha_2} = C_p^{\beta_1}\otimes C_p^{\beta_2}$ implies $C_{ij}^{\alpha_1}C_p^{\alpha_2} = C_{ij}^{\beta_1}C_p^{\beta_2}$, for all $i,j=1,\dots,p$. Since all elements of G_p are non-zero $(0,1)$- matrices, we have $C_{kl}^{\alpha_1} = 1$, for some k and l. The equality $C_{kl}^{\alpha_1}C_p^{\alpha_2} = C_{kl}^{\beta_1}C_p^{\beta_2}$ proves $C_{kl}^{\beta_1} = 1$. Hence, we have $C_p^{\alpha_2} = C_p^{\beta_2}$ and

$$\begin{aligned}0 =&C_{ij}^{\alpha_1}C_p^{\alpha_2} - C_{ij}^{\beta_1}C_p^{\beta_2}\\ =&C_{ij}^{\alpha_1}C_p^{\alpha_2} - C_{ij}^{\beta_1}C_p^{\alpha_2}\\ =&(C_{ij}^{\alpha_1} - C_{ij}^{\beta_1})C_p^{\alpha_2}.\end{aligned}$$

Since $C_p^{\alpha_2}$ is a non-zero matrix, we have $C_{ij}^{\alpha_1} - C_{ij}^{\beta_1} = 0$ or $C_{ij}^{\alpha_1} = C_{ij}^{\beta_1}$, for all $i,j = 1,\dots,p$. Therefore, we conclude $C_p^{\alpha_1} = C_p^{\beta_1}$.

Inductive step: Suppose $C_p^{\alpha_1} \otimes \ldots \otimes C_p^{\alpha_{m-1}} = C_p^{\beta_1} \otimes \ldots \otimes C_p^{\beta_{m-1}}$ implies $C_p^{\alpha_i} = C_p^{\beta_i}$, for all $i = 1, \ldots, m-1$. The matrices $C_p^{\alpha_1} \otimes \ldots \otimes C_p^{\alpha_{m-1}}$ and $C_p^{\beta_1} \otimes \ldots \otimes C_p^{\beta_{m-1}}$ are nonzero $(0,1)$-matrix. Using the same logic of the base case, the equality $C_p^{\alpha_1} \otimes \ldots \otimes C_p^{\alpha_m} = C_p^{\beta_1} \otimes \ldots \otimes C_p^{\beta_m}$ implies $C_p^{\alpha_1} \otimes \ldots \otimes C_p^{\alpha_{m-1}} = C_p^{\beta_1} \otimes \ldots \otimes C_p^{\beta_{m-1}}$ and $C_p^{\alpha_m} = C_p^{\beta_m}$. By induction hypothesis, $C_p^{\alpha_1} \otimes \ldots \otimes C_p^{\alpha_{m-1}} = C_p^{\beta_1} \otimes \ldots \otimes C_p^{\beta_{m-1}}$ implies $C_p^{\alpha_i} = C_p^{\beta_i}$, for all $i = 1, \ldots, m-1$. Therefore, the equality $C_p^{\alpha_1} \otimes \ldots \otimes C_p^{\alpha_m} = C_p^{\beta_1} \otimes \ldots \otimes C_p^{\beta_m}$ implies implies $C_p^{\alpha_i} = C_p^{\beta_i}$, for all $i = 1, \ldots, m$.

$\square$

Let p be a prime number, m a positive integer, and $\mathbb{Z}_p^m$ the group whose elements are m-tuples of vectors with entries in $\mathbb{Z}_p$ and whose operation is vector addition.

Theorem 3.12. *Any generalized Hadamard matrix over the additive group $\mathbb{Z}_p^m$ is also a generalized Hadamard matrix over the group $\Omega(p,m)$ and visa versa.*

Proof. It is enough to prove that two groups $\Omega(p,m)$ and $\mathbb{Z}_p^m$ are isomorphic. The group $G_p = < C_p >$, defined in Definition 3.10, is a cyclic group of order p. Naturally, we define an isomorphism $\varphi : \mathbb{Z}_p^m \longrightarrow \Omega(p,m)$ by $\varphi(\alpha_1, \ldots, \alpha_m) = C_p^{\alpha_1} \otimes \ldots \otimes C_p^{\alpha_m}$. We consider two arbitrary elements $\omega_1 = C_p^{\alpha_1} \otimes \ldots \otimes C_p^{\alpha_m}$ and $\omega_2 = C_p^{\beta_1} \otimes \ldots \otimes C_p^{\beta_m}$ of $\Omega(p,m)$. By Lemma 3.11, the equality $\omega_1 = \omega_2$ implies $C_p^{\alpha_i} = C_p^{\beta_i}$ and so $\alpha_i = \beta_i$, for all $i = 1, \ldots, m$. Therefore, φ is a bijection. To prove that φ preserves the given group operations, let $(\alpha_1, \ldots, \alpha_m)$ and $(\beta_1, \ldots, \beta_m)$ be two arbitrary elements of $\mathbb{Z}_p^m$. Then, we have

$$\begin{aligned}
\varphi((\alpha_1, \ldots, \alpha_m) + (\beta_1, \ldots, \beta_m)) &= \varphi(\alpha_1 + \beta_1, \ldots, \alpha_m + \beta_m) \\
&= C_p^{\alpha_1+\beta_1} \otimes \ldots \otimes C_p^{\alpha_m+\beta_m} \\
&= (C_p^{\alpha_1} \otimes \ldots \otimes C_p^{\alpha_m})(C_p^{\beta_1} \otimes \ldots \otimes C_p^{\beta_m}) \\
&= \varphi(\alpha_1, \ldots, \alpha_m)\varphi(\beta_1, \ldots, \beta_m).
\end{aligned}$$

$\square$

Example 3.13. For $q = 2^2$, we have $G_2 =< circ(0,1) >=< \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} >= \{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \}$, and

$$\Omega(2,2) = \{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \} =$$

$$\left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right\}.$$

By proof of Theorem 3.12, we have an isomorphism $\varphi : \mathbb{Z}_2^2 \longrightarrow \Omega(2,2)$ such that $\varphi(0,0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\varphi(0,1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $\varphi(1,0) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, and $\varphi(1,1) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. The following matrix is a generalized Hadamard matrix over $\mathbb{Z}_2^2$, and so over $\Omega(2,2)$ via the above isomorphism:

$$GH(4,1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & 3 & 1 & 2 \end{bmatrix},$$

where $0 = (0,0)$, $1 = (0,1)$, $2 = (1,0)$, and $3 = (1,1)$.

Lemma 3.14 ([12, Proposition 1.4, p. 234])**.** *For each prime power q, there exists a $GH(q,1)$ over the additive group of the finite field $GF(q)$.*

Proof. The multiplication table of $GF(q)$ is a $GH(q,1)$ over the group $(GF(q),+)$. □

By Theorem 3.12, the following holds.

Corollary 3.15. *For each prime power $q = p^m$, there exists a generalized Hadamard matrix $GH(q,1)$ over the group $\Omega(p,m)$.*

The following lemma shows the Kronecker product of two generalized Hadamard matrices over a group G is a generalized Hadamard matrix over G.

Theorem 3.16 ([11, Theorem 10.4.9, p. 343])**.** *For a prime power $q = p^m$, let $GH(q,q^r)$ and $GH(q,q^t)$ be generalized Hadamard matrices over the group $G = \Omega(p,m)$. Then, $T = GH(q,q^r) \otimes GH(q,q^t)$ is a generalized Hadamard matrix $GH(q,q^{r+t+1})$ over G.*

Theorem 3.17 ([12, Theorem 4.3, p. 235])**.** *For each prime power $q = p^m$ and an integer $\theta \geq 0$, there exists a generalized Hadamard matrix $GH(q,q^{\theta})$ over the group $\Omega(p,m)$.*

3.2.2 Symmetric $((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$-Design

We use generalized Hadamard matrices to generate symmetric designs and strongly regular graphs. Let r_α denote the α-th row of $GH(q,q^\theta)$. Then, r_α^* stands for the column whose j-th coordinate is the inverse of the j-th coordinate of r_α. Since the operation we use to define $r_\alpha^* r_\alpha$ is the same as the operation of the group $\Omega(p,m)$, each of the entries of the matrix $r_\alpha^* r_\alpha$ is an element of $\Omega(p,m)$. Using these matrices, the following definition is made.

Definition 3.18. Let $q = p^m$ be a prime power, d a positive integer, and $GH(q,q^\theta)$ a generalized Hadamard matrix over $\Omega(p,m) = \{\omega_0, \ldots, \omega_{p^m-1}\}$, where $0 \leq \theta \leq d-1$. The notation $\Delta(GH(q,q^\theta), d)$ is used to show the set

$$\{(r_\alpha^* r_\alpha) \otimes J_{q^{d-\theta-1}} | 1 \leq \alpha \leq q^{\theta+1},\ r_\alpha \text{ is the } \alpha\text{-th row of } GH(q,q^\theta)\}$$

of $(0,1)$-matrices of size q^{d+1}. If $d = 1$, we may use the notation $\Delta(GH(q,q^\theta))$ instead of $\Delta(GH(q,q^\theta), 1)$.

Example 3.19. In this example, we assume $q = 2$ and $d = 2$. The purpose of this example is to figure out all two elements of $\Delta(GH(2,1),2)$, where $GH(2,1) = \begin{bmatrix} \omega_0 & \omega_0 \\ \omega_0 & \omega_1 \end{bmatrix}$ is a generalized Hadamard matrix over the group $\Omega(2,1) = \{\omega_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \omega_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}$. We note that the inverse of each element of $\Omega(2,1)$ is itself. Then, we have $r_1^* r_1 = \begin{bmatrix} \omega_0 \\ \omega_0 \end{bmatrix} \begin{bmatrix} \omega_0 & \omega_0 \end{bmatrix} = \begin{bmatrix} \omega_0 & \omega_0 \\ \omega_0 & \omega_0 \end{bmatrix}$ and $r_2^* r_2 = \begin{bmatrix} \omega_0 \\ \omega_1 \end{bmatrix} \begin{bmatrix} \omega_0 & \omega_1 \end{bmatrix} = \begin{bmatrix} \omega_0 & \omega_1 \\ \omega_1 & \omega_0 \end{bmatrix}$. Hence, we have

$$(r_1^* r_1) \otimes J_2 = \begin{bmatrix} \omega_0 & \omega_0 \\ \omega_0 & \omega_0 \end{bmatrix} \otimes J_2 = \begin{bmatrix} 1&0&1&0 \\ 0&1&0&1 \\ 1&0&1&0 \\ 0&1&0&1 \end{bmatrix} \otimes J_2 = \begin{bmatrix} 1&1&0&0&1&1&0&0 \\ 1&1&0&0&1&1&0&0 \\ 0&0&1&1&0&0&1&1 \\ 0&0&1&1&0&0&1&1 \\ 1&1&0&0&1&1&0&0 \\ 1&1&0&0&1&1&0&0 \\ 0&0&1&1&0&0&1&1 \\ 0&0&1&1&0&0&1&1 \end{bmatrix} \in \Delta(GH(2,1),2)$$

and

$$(r_2^* r_2) \otimes J_2 = \begin{bmatrix} \omega_0 & \omega_1 \\ \omega_1 & \omega_0 \end{bmatrix} \otimes J_2 = \begin{bmatrix} 1&0&0&1 \\ 0&1&1&0 \\ 0&1&1&0 \\ 1&0&0&1 \end{bmatrix} \otimes J_2 = \begin{bmatrix} 1&1&0&0&0&0&1&1 \\ 1&1&0&0&0&0&1&1 \\ 0&0&1&1&1&1&0&0 \\ 0&0&1&1&1&1&0&0 \\ 0&0&1&1&1&1&0&0 \\ 0&0&1&1&1&1&0&0 \\ 1&1&0&0&0&0&1&1 \\ 1&1&0&0&0&0&1&1 \end{bmatrix} \in \Delta(GH(2,1),2).$$

Lemma 3.20. *Let $q = p^m$ be a prime power, d a positive integer, and $GH(q,q^\theta)$ a generalized Hadamard matrix over the group $\Omega(p,m)$, where $0 \leq \theta \leq d-1$. Then, the matrices in the set $\Delta(GH(q,q^\theta),d)$ are all symmetric.*

Proof. We have

$$\Delta(GH(q,q^\theta),d) = \{(r_\alpha^* r_\alpha) \otimes J_{q^{d-\theta-1}} | 1 \leq \alpha \leq q^{\theta+1},\ r_\alpha \text{ is the } \alpha\text{-th row of } GH(q,q^\theta)\}.$$

For each α, we only need to prove that $r_\alpha^* r_\alpha$ is symmetric because $J_{q^{d-\theta-1}}$ is symmetric and the Kronecker product of two symmetric matrices is a symmetric matrix. Since we have $(r_\alpha^* r_\alpha)^* = r_\alpha^* r_\alpha$, we conclude that the matrix $r_\alpha^* r_\alpha$ is symmetric. □

Definition 3.21. Let $q = p^m$ be a prime power, d a positive integer. Suppose $GH(q,1)$, $GH(q,q^1)$, …, $GH(q,q^{d-1})$ are generalized Hadamard matrices over $\Omega(p,m)$. We use the notation $\Xi(q,d)$ to denote the set

$$\Delta(GH(q,1),d) \cup \ldots \cup \Delta(GH(q,q^{d-1}),d) \cup \{O\} \cup \{I_q \otimes J_{q^d}\}.$$

Lemma 3.22. *Let $q = p^m$ be a prime power, d a positive integer, and $GH(q,q^\theta)$ a generalized Hadamard matrix over the group $\Omega(p,m)$, where $0 \leq \theta \leq d-1$. The elements of the set $\Xi(q,d)$ satisfy the following conditions:*

1. *For any two distinct elements $A \neq O$ and $B \neq O$ in the set $\Xi(q,d)$, we have $AB^T = q^{d-1} J_{q^{d+1}}$.*

2. *For any nonzero element $A \in \Xi(q,d)$, we have $AA^T = q^d A$.*

3. $\displaystyle\sum_{A\in\Delta(GH(q,q^\theta),d)} A = q^{\theta+1}(I_{q^{\theta+2}}\otimes J_{q^{d-\theta-1}}) + q^\theta((J_{q^{\theta+1}} - I_{q^{\theta+1}})\otimes J_{q^{d-\theta}})$.

Proof. 1. By Definition 3.21, we have

$$\Xi(q,d) = \Delta(GH(q,1),d)\cup\ldots\cup\Delta(GH(q,q^{d-1}),d)\cup\{O\}\cup\{I_q\otimes J_{q^d}\}.$$

We first assume $A,B\in\Delta(GH(q,q^\theta),d)$, where $0\le\theta\le d-1$. By Definition 3.18, $A = (r_\alpha^* r_\alpha)\otimes J_{q^{d-\theta-1}}$ and $B = (r_\beta^* r_\beta)\otimes J_{q^{d-\theta-1}}$ for some α and β. By Lemma 3.20, the matrices A and B are symmetric. Hence, we have

$$\begin{aligned} AB^T = AB &= [(r_\alpha^* r_\alpha)\otimes J_{q^{d-\theta-1}}][(r_\beta^* r_\beta)\otimes J_{q^{d-\theta-1}}] \\ &= [(r_\alpha^* r_\alpha)(r_\beta^* r_\beta)]\otimes[J_{q^{d-\theta-1}}J_{q^{d-\theta-1}}] \\ &= q^{d-\theta-1}[(r_\alpha^* r_\alpha)(r_\beta^* r_\beta)]\otimes[J_{q^{d-\theta-1}}]. \end{aligned}$$

The (i,j)-block of size q of the matrix $(r_\alpha^* r_\alpha)(r_\beta^* r_\beta)$ is equal to $\displaystyle\sum_{k=1}^{q^{\theta+1}}(r_{\alpha i}^{-1}r_{\alpha k})(r_{\beta k}^{-1}r_{\beta j})$. We have

$$\sum_{k=1}^{q^{\theta+1}}(r_{\alpha i}^{-1}r_{\alpha k})(r_{\beta k}^{-1}r_{\beta j}) = r_{\alpha i}^{-1}r_{\beta j}\sum_{k=1}^{q^{\theta+1}} r_{\alpha k}r_{\beta k}^{-1}. \tag{3.1}$$

By the definition of a generalized Hadamard matrix, we have

$$r_{\alpha i}^{-1}r_{\beta j}\sum_{k=1}^{q^{\theta+1}} r_{\alpha k}r_{\beta k}^{-1} = q^\theta\sum_{i=0}^{q-1}\omega_i \tag{3.2}$$

and by Lemma 3.11, we have

$$\sum_{i=0}^{q-1}\omega_i = J_q. \tag{3.3}$$

From (3.1), (3.2), and (3.3), we conclude

$$\sum_{k=1}^{q^{\theta+1}} (r_{\alpha i}^{-1} r_{\alpha k})(r_{\beta k}^{-1} r_{\beta j}) = q^{\theta} J_q. \tag{3.4}$$

Thus, (i,j)-block of size q of the matrix $(r_\alpha^* r_\alpha)(r_\beta^* r_\beta)$ is equal to $q^\theta J_q$, for all i and j. Hence, we have $(r_\alpha^* r_\alpha)(r_\beta^* r_\beta) = q^\theta J_{q^{\theta+1}}$. This implies that

$$\begin{aligned} AB^T &= q^{d-\theta-1}[(r_\alpha^* r_\alpha)(r_\beta^* r_\beta)] \otimes [J_{q^{d-\theta-1}}] \\ &= q^{d-\theta-1}[q^\theta J_{q^{\theta+1}}] \otimes [J_{q^{d-\theta-1}}] \\ &= q^{d-1} J_{q^{d+1}}. \end{aligned}$$

Now, assume $A \in \Delta(GH(q,q^s),d)$ and $B \in \Delta(GH(q,q^\theta),d)$, where $s < \theta$. From Definition 3.18, $A = (r_\alpha^* r_\alpha) \otimes J_{q^{d-s-1}}$ and $B = (r_\beta^* r_\beta) \otimes J_{q^{d-\theta-1}}$ for some α and β. We have

$$\begin{aligned} AB^T = AB &= [(r_\alpha^* r_\alpha) \otimes J_{q^{d-s-1}}][(r_\beta^* r_\beta) \otimes J_{q^{d-\theta-1}}] \\ &= [((r_\alpha^* r_\alpha) \otimes J_{\theta-s})(r_\beta^* r_\beta)] \otimes [J_{q^{d-\theta-1}} J_{q^{d-\theta-1}}] \\ &= q^{d-\theta-1}[((r_\alpha^* r_\alpha) \otimes J_{q^{\theta-s}})(r_\beta^* r_\beta)] \otimes [J_{q^{d-\theta-1}}]. \end{aligned}$$

We also have

$$[((r_\alpha^* r_\alpha) \otimes J_{q^{\theta-s}})(r_\beta^* r_\beta)] = q^\theta J_{q^{\theta+2}}. \tag{3.5}$$

Therefore, we have

$$\begin{aligned} AB^T = AB &= q^{d-\theta-1}[((r_\alpha^* r_\alpha) \otimes J_{q^{\theta-s}})(r_\beta^* r_\beta)] \otimes [J_{q^{d-\theta-1}}] \\ &= q^{d-\theta-1}[q^\theta J_{q^{\theta+2}}] \otimes [J_{q^{d-\theta-1}}] \\ &= q^{d-1} J_{q^{d+1}}. \end{aligned}$$

Now, suppose $A = I_q \otimes J_{q^d}$ and $B \in \Delta(GH(q,q^\theta),d)$, where $0 \le \theta \le d-1$. By Definition 3.18, $B = (r_\alpha^* r_\alpha) \otimes J_{q^{d-\theta-1}}$ for some α. We have

$$(I_q \otimes J_{q^{\theta+1}})(r_\alpha^* r_\alpha) = q^\theta J_{q^{\theta+2}}. \tag{3.6}$$

Therefore, we have

$$\begin{aligned} AB^T = BA^T &= (I_q \otimes J_{q^{\theta+1}})(r_\alpha^* r_\alpha \otimes J_{q^{d-\theta-1}}) \\ &= (I_q \otimes J_{q^{\theta+1}} \otimes J_{q^{d-\theta-1}})(r_\alpha^* r_\alpha \otimes J_{q^{d-\theta-1}}) \\ &= [(I_q \otimes J_{q^{\theta+1}}) r_\alpha^* r_\alpha] \otimes [J_{q^{d-\theta-1}} J_{q^{d-\theta-1}}] \\ &= [q^\theta J_{q^{\theta+2}}] \otimes [q^{d-\theta-1} J_{q^{d-\theta-1}}] \\ &= q^{d-1} J_{q^{d+1}}. \end{aligned}$$

2. By Definition 3.18, $A = (r_\alpha^* r_\alpha) \otimes J_{q^{d-\theta-1}}$ for some α. We have

$$\begin{aligned} AA^T = AA &= [(r_\alpha^* r_\alpha)(r_\alpha^* r_\alpha)] \otimes [J_{q^{d-\theta-1}}] \\ &= [r_\alpha^* (r_\alpha r_\alpha^*) r_\alpha] \otimes [J_{q^{d-\theta-1}}] \\ &= [q^q (r_\alpha^* r_\alpha)] \otimes [J_{q^{d-\theta-1}}] \\ &= q^d ((r_\alpha^* r_\alpha) \otimes J_{q^{d-\theta-1}}) \\ &= q^d A. \end{aligned}$$

Now, suppose $A = I_q \otimes J_{q^d}$. We have

$$\begin{aligned} AA^T = AA &= (I_q \otimes J_q^d)(I_q \otimes J_q^d) \\ &= (I_q I_q) \otimes (J_q^d J_q^d) \\ &= (I_q) \otimes (q^d J_{q^d}) \end{aligned}$$

$$= q^d I_q \otimes J_q^d$$
$$= q^d A.$$

3. By Definition 3.18, we have

$$\sum_{A\in\Delta(GH(q,q^\theta),d)} A = \sum_{\alpha=1}^{q^{\theta+1}} (r_\alpha^* r_\alpha) \otimes J_{q^{d-\theta-1}} = (\sum_{\alpha=1}^{q^{\theta+1}} r_\alpha^* r_\alpha) \otimes J_{q^{d-\theta-1}}. \tag{3.7}$$

If $(\sum_{\alpha=1}^{q^{\theta+1}} r_\alpha^* r_\alpha)_{ij}$ is the (i,j)-block of the matrix $\sum_{\alpha=1}^{q^{\theta+1}} r_\alpha^* r_\alpha$, we have

$$(\sum_{\alpha=1}^{q^{\theta+1}} r_\alpha^* r_\alpha)_{ii} = \sum_{\alpha=1}^{q^{\theta+1}} r_{\alpha i}^{-1} r_{\alpha i} = \sum_{\alpha=1}^{q^{\theta+1}} \omega_0 = q^{\theta+1}\omega_0 = q^{\theta+1} I_q, \tag{3.8}$$

for all $i = 0,\ldots,q^d$. For $i \neq j$, we have

$$(\sum_{\alpha=1}^{q^{\theta+1}} r_\alpha^* r_\alpha)_{ij} = \sum_{\alpha=1}^{q^{\theta+1}} r_{\alpha i}^{-1} r_{\alpha j} = q^\theta J_q. \tag{3.9}$$

From (3.8) and (3.9), we conclude

$$\sum_{\alpha=1}^{q^{\theta+1}} r_\alpha^* r_\alpha = q^{\theta+1}(I_{q^{\theta+2}}) + q^\theta((J_{q^{\theta+1}} - I_{q^{\theta+1}}) \otimes J_q). \tag{3.10}$$

From (3.7) and (3.10), we have

$$\sum_{A\in\Delta(GH(q,q^\theta),d)} A = q^{\theta+1}(I_{q^{\theta+2}} \otimes J_{q^{d-\theta-1}}) + q^\theta((J_{q^{\theta+1}} - I_{q^{\theta+1}}) \otimes J_{q^{d-\theta}}).$$

□

We know $\Xi(q,d) = \Delta(GH(q,1),d) \cup \ldots \cup \Delta(GH(q,q^{d-1}),d) \cup \{O\} \cup \{I_q \otimes J_{q^d}\}$. To generate a symmetric $((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$-design, we need to convert the set $\Xi(q,d)$ into an indexed family using the index set $\{1,\ldots,r+1\}$. The number of elements are the same in the two sets $\Xi(q,d)$ and $\{1,\ldots,r+1\}$. We define a bijective function $M:\{1,\ldots,r+1\}\rightarrow\Xi(q,d)$ such that $M(r)=I_q\otimes J_{q^d}$ and $M(r+1)$ is equal to the zero matrix. Now we let $M(1)$ to be an element in $\Xi(q,d)\setminus\{M(r),M(r+1)\}$. Then inductively we assign $M(i+1)$ to an element in $\Xi(q,d)\setminus\{M(r),M(r+1),M(k)|k=1,\ldots,i\}$.

Remark 3.23. Note that there are many possibilities in defining different values for M and this indicate that different selection may lead to unequivalent designs, but this is out of the scope of this thesis.

We now present the new method. In the following theorem, we use the set $\Xi(q,d)$ and a Latin square L of size $r+1$ to construct a symmetric $((r+1)q^{d+1},rq^d,(r-1)q^{d-1})$-design.

Theorem 3.24. *Let $q=p^m$ be a prime power, d be a positive integer, $r=(q^{d+1}-1)/(q-1)$, and L a Latin square over $\{1,\ldots,r+1\}$. Substitute i with $M(i)$ in the Latin square L to make a $(0,1)$-matrix N of size $(r+1)q^{d+1}$. Then, N is the incidence matrix of a symmetric $((r+1)q^{d+1},rq^d,(r-1)q^{d-1})$-design.*

Proof. We need to prove $NN^T=rq^dI_{q^{d+1}}+(r-1)q^{d-1}(J_{q^{d+1}}-I_{q^{d+1}})$. For any $1\leq i,j\leq r+1$, let N_{ij} be the (i,j)-block of size q^{d+1} of the matrix $N=[N_{ij}]$. Let $NN^T=[M_{ij}]$, where M_{ij} is the (i,j)-block of size q^{d+1} of NN^T. Let $B(\theta)=q^\theta(I_{q^{\theta+1}}\otimes J_{q^{d-\theta}})$. For $i=j$, Lemma 3.22 proves

$$
\begin{aligned}
M_{ii} &= \sum_{A\in\Xi(q,d)} AA^T \\
&= ((I_q\otimes J_{q^d})(I_q\otimes J_{q^d}))+(\sum_{\theta=0}^{d-1}\sum_{\{A\in\Delta(GH(q,q^\theta),d)\}} AA^T) \\
&= q^d(I_q\otimes J_{q^d})+(\sum_{\theta=0}^{d-1} q^d \sum_{\{A\in\Delta(GH(q,q^\theta),d)\}} A) \\
&= q^d(I_q\otimes J_{q^d})+q^d\sum_{\theta=0}^{d-1}\ [q^{\theta+1}(I_{q^{\theta+2}}\otimes J_{q^{d-\theta-1}})+q^\theta((J_{q^{\theta+1}}-I_{q^{\theta+1}})\otimes J_{q^{d-\theta}})] \\
&= q^d(I_q\otimes J_{q^d})+q^d\sum_{\theta=0}^{d-1}\ [q^{\theta+1}(I_{q^{\theta+2}}\otimes J_{q^{d-\theta-1}})+q^\theta(J_{q^{d+1}})-q^\theta(I_{q^{\theta+1}}\otimes J_{q^{d-\theta}})]
\end{aligned}
$$

$$
\begin{aligned}
&= q^d(I_q \otimes J_{q^d}) + q^d \sum_{\theta=0}^{d-1} [B(\theta+1) + q^\theta J_{q^{d+1}} - B(\theta)] \\
&= q^d(I_q \otimes J_{q^d}) + q^d \sum_{\theta=0}^{d-1} [B(\theta+1) - B(\theta)] + q^d J_{q^{d+1}} \sum_{\theta=0}^{d-1} q^\theta \\
&= q^d(I_q \otimes J_{q^d}) + q^d[B(d-1) - B(0)] + (q^d[\frac{q^d-1}{q-1}])J_{q^{d+1}} \\
&= q^{2d} I_{q^{d+1}} + (r-1)q^{d-1} J_{q^{d+1}} \\
&= (q^{2d} + (r-1)q^{d-1}) I_{q^{d+1}} + (r-1)q^{d-1}(J_{q^{d+1}} - I_{q^{d+1}}) \\
&= rq^d I_{q^{d+1}} + (r-1)q^{d-1}(J_{q^{d+1}} - I_{q^{d+1}}).
\end{aligned}
$$

Now, suppose $i \neq j$. By Lemma 3.22, we have

$$M_{ij} = \sum_{k=1}^{r+1} N_{ik} N_{jk} = (r-1)q^{d-1} J_{q^{d+1}}.$$

Hence, we have $NN^T = rq^d I_{q^{d+1}} + (r-1)q^{d-1}(J_{q^{d+1}} - I_{q^{d+1}})$ and so N is the incidence matrix of a symmetric $((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$-design. □

If we use a symmetric Latin square in Theorem 3.24, the incidence matrix that is constructed in Theorem 3.24 is symmetric. Since for any even integer n, there is a symmetric constant diagonal Latin square of order n, we conclude the following corollary.

Corollary 3.25. *Let $q = p^m$ be a prime power, d be a positive integer. If $r = \dfrac{q^{d+1}-1}{q-1}$ is odd, let L be a symmetric constant diagonal Latin square over $\{1, \ldots, r+1\}$ with $r+1$ on the diagonal. To make a $(0,1)$-matrix N of size $(r+1)q^{d+1}$, substitute i with $M(i)$ in the Latin square L. Then, N is the adjacency matrix of a*

$$SRG((r+1)q^{d+1}, rq^d, (r-1)q^{d-1}, (r-1)q^{d-1}).$$

Example 3.26 ([21, Example 5.25, p. 114])**.** In this example, we show how to construct a symmetric $(45,12,3)$-design. Let $q=3$ and $d=1$. Then,

$$\Omega(3,1)=G_3=<circ(0,1,0)>=\left\{\omega_0=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\ \omega_1=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix},\ \omega_2=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\right\}.$$

Since $d=1$, we only need $GH(3,1)=\begin{bmatrix}\omega_0&\omega_0&\omega_0\\\omega_0&\omega_1&\omega_2\\\omega_0&\omega_2&\omega_1\end{bmatrix}$ over $\Omega(3,1)$. We have:

$$r_1^*r_1=\begin{bmatrix}\omega_0\\\omega_0\\\omega_0\end{bmatrix}[\omega_0\ \omega_0\ \omega_0]=\begin{bmatrix}\omega_0&\omega_0&\omega_0\\\omega_0&\omega_0&\omega_0\\\omega_0&\omega_0&\omega_0\end{bmatrix}=\begin{bmatrix}1&0&0&1&0&0&1&0&0\\0&1&0&0&1&0&0&1&0\\0&0&1&0&0&1&0&0&1\\1&0&0&1&0&0&1&0&0\\0&1&0&0&1&0&0&1&0\\0&0&1&0&0&1&0&0&1\\1&0&0&1&0&0&1&0&0\\0&1&0&0&1&0&0&1&0\\0&0&1&0&0&1&0&0&1\end{bmatrix}.$$

$$r_2^*r_2=\begin{bmatrix}\omega_0\\\omega_2\\\omega_1\end{bmatrix}[\omega_0\ \omega_1\ \omega_2]=\begin{bmatrix}\omega_0&\omega_1&\omega_2\\\omega_2&\omega_0&\omega_1\\\omega_1&\omega_2&\omega_0\end{bmatrix}=\begin{bmatrix}1&0&0&0&1&0&0&0&1\\0&1&0&0&0&1&1&0&0\\0&0&1&1&0&0&0&1&0\\0&0&1&1&0&0&0&1&0\\1&0&0&0&1&0&0&0&1\\0&1&0&0&0&1&1&0&0\\0&1&0&0&0&1&1&0&0\\0&0&1&1&0&0&0&1&0\\1&0&0&0&1&0&0&0&1\end{bmatrix}.$$

$$r_3^*r_3=\begin{bmatrix}\omega_0\\\omega_1\\\omega_2\end{bmatrix}[\omega_0\ \omega_2\ \omega_1]=\begin{bmatrix}\omega_0&\omega_2&\omega_1\\\omega_1&\omega_0&\omega_2\\\omega_2&\omega_1&\omega_0\end{bmatrix}=\begin{bmatrix}1&0&0&0&0&1&0&1&0\\0&1&0&1&0&0&0&0&1\\0&0&1&0&1&0&1&0&0\\0&1&0&1&0&0&0&0&1\\0&0&1&0&1&0&1&0&0\\1&0&0&0&0&1&0&1&0\\0&0&1&0&1&0&1&0&0\\1&0&0&0&0&1&0&1&0\\0&1&0&1&0&0&0&0&1\end{bmatrix}.$$

Therefore, we have

$$\Delta(GH(3,1))=\left\{\begin{bmatrix}1&0&0&1&0&0&1&0&0\\0&1&0&0&1&0&0&1&0\\0&0&1&0&0&1&0&0&1\\1&0&0&1&0&0&1&0&0\\0&1&0&0&1&0&0&1&0\\0&0&1&0&0&1&0&0&1\\1&0&0&1&0&0&1&0&0\\0&1&0&0&1&0&0&1&0\\0&0&1&0&0&1&0&0&1\end{bmatrix},\begin{bmatrix}1&0&0&0&1&0&0&0&1\\0&1&0&0&0&1&1&0&0\\0&0&1&1&0&0&0&1&0\\0&0&1&1&0&0&0&1&0\\1&0&0&0&1&0&0&0&1\\0&1&0&0&0&1&1&0&0\\0&1&0&0&0&1&1&0&0\\0&0&1&1&0&0&0&1&0\\1&0&0&0&1&0&0&0&1\end{bmatrix},\begin{bmatrix}1&0&0&0&0&1&0&1&0\\0&1&0&1&0&0&0&0&1\\0&0&1&0&1&0&1&0&0\\0&1&0&1&0&0&0&0&1\\0&0&1&0&1&0&1&0&0\\1&0&0&0&0&1&0&1&0\\0&0&1&0&1&0&1&0&0\\1&0&0&0&0&1&0&1&0\\0&1&0&1&0&0&0&0&1\end{bmatrix}\right\}.$$

We first index the elements of $\Xi(3,1)=\Delta(GH(3,1))\cup\{O\}\cup\{I_3\otimes J_3\}$ using the bijection function $M : \{1,\ldots,r+1\} \to \Xi(3,1)$ defined by:

$$M(1) = \begin{bmatrix} 1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&0\\ 0&0&1&0&0&1&0&0&1\\ 1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&0\\ 0&0&1&0&0&1&0&0&1\\ 1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&0\\ 0&0&1&0&0&1&0&0&1 \end{bmatrix},\ M(2) = \begin{bmatrix} 1&0&0&0&1&0&0&0&1\\ 0&1&0&0&0&1&1&0&0\\ 0&0&1&1&0&0&0&1&0\\ 0&0&1&1&0&0&0&1&0\\ 1&0&0&0&1&0&0&0&1\\ 0&1&0&0&0&1&1&0&0\\ 0&1&0&0&0&1&1&0&0\\ 0&0&1&1&0&0&0&1&0\\ 1&0&0&0&1&0&0&0&1 \end{bmatrix},\ M(3) = \begin{bmatrix} 1&0&0&0&0&1&0&1&0\\ 0&1&0&1&0&0&0&0&1\\ 0&0&1&0&1&0&1&0&0\\ 0&1&0&1&0&0&0&0&1\\ 0&0&1&0&1&0&1&0&0\\ 1&0&0&0&0&1&0&1&0\\ 0&0&1&0&1&0&1&0&0\\ 1&0&0&0&0&1&0&1&0\\ 0&1&0&1&0&0&0&0&1 \end{bmatrix},$$

$$M(4) = \begin{bmatrix} 1&1&1&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0\\ 0&0&0&1&1&1&0&0&0\\ 0&0&0&1&1&1&0&0&0\\ 0&0&0&1&1&1&0&0&0\\ 0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&1&1&1 \end{bmatrix},\ M(5) = \begin{bmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix}.$$

In the Latin square $L = \begin{bmatrix} 5&4&1&2&3\\ 3&5&4&1&2\\ 2&3&5&4&1\\ 1&2&3&5&4\\ 4&1&2&3&5 \end{bmatrix}$, replace each i with $M(i)$. The resulted matrix, called N, is the incidence matrix of a symmetric $(45,12,3)$-design which appears in Appendix 2.

3.2.3 Symmetric $(3^{d+1}r, 3^d(r+1), 3^d(r-3^d+1))$-Design

Let d be a positive integer. Now, we apply the new method to make a symmetric $(3^{d+1}(3^{d+1}-1)/2, 3^d(3^{d+1}+1)/2, 3^d(3^d+1)/2)$-design which has the parameters as the parameters of Spence designs, [10].

Definition 3.27. Let d be a positive integer and $GH(3,1)$, $GH(3,3)$, …, $GH(3,3^{d-1})$ generalized Hadamard matrices over $\Omega(3,1)$. Then, we define

$$\Xi'(3,d) = \Delta(GH(3,1),d) \cup \ldots \cup \Delta(GH(3,3^{d-1}),d) \cup \{J_{3^{d+1}} - (I_3 \otimes J_{3^d})\}.$$

The number of elements are the same in the two sets $\Xi'(3,d)$ and $\{1,\ldots,r\}$. We define a bijective function $M' : \{1,\ldots,r\} \to \Xi'(3,d)$ such that $M'(r) = J_{3^{d+1}} - (I_3 \otimes J_{3^d})$. Now we let $M'(1)$ to be an element in $\Xi'(3,d) \setminus \{M'(r)\}$. Then inductively we assign $M'(i+1)$ to an element in $\Xi'(3,d) \setminus \{M'(r), M'(k) | k = 1,\ldots,i\}$.

Theorem 3.28. *Let d be a positive integer, $r = (3^{d+1}-1)/2$, and L be a Latin square over $\{1,\ldots,r\}$. Substitute i with $M'(i)$ in the Latin square L to make a $(0,1)$-matrix N of size*

$r(3^{d+1})$. *Then, N is the incidence matrix of a symmetric*

$$(3^{d+1}(3^{d+1}-1)/2, 3^d(3^{d+1}+1)/2, 3^d(3^d+1)/2)\text{-}design.$$

Proof. We need to prove

$$NN^T = 3^d[\frac{3^{d+1}+1}{2}]I_{3^{d+1}} + 3^d[\frac{3^d+1}{2}](J_{3^{d+1}} - I_{3^{d+1}}). \tag{3.11}$$

For any $1 \le i, j \le r$, let N_{ij} be the (i, j)-block of size 3^{d+1} of the matrix $N = [N_{ij}]$. Now, we want to calculate the matrix $NN^T = [M_{ij}]$. For $i = j$, we have

$$\begin{aligned}
M_{ii} &= \sum_{A\in\Xi'(3,d)} AA^T = (J_{3^{d+1}} - I_3 \otimes J_{3^d})(J_{3^{d+1}} - I_3 \otimes J_{3^d}) \sum_{\theta=0}^{d-1} \sum_{\{A\in\Delta(GH(3,3^\theta),d)\}} AA^T \\
&= 3^d(J_{3^{d+1}} + I_3 \otimes J_{3^d}) + \sum_{\theta=0}^{d-1} \sum_{\{A\in\Delta(GH(3,3^\theta),d)\}} AA^T \\
&= 3^d(J_{3^{d+1}} + I_3 \otimes J_{3^d}) + 3^d[3^d I_{3^{d+1}} - I_3 \otimes J_{3^d}] + 3^d[\frac{3^d-1}{3-1}]J_{3^{d+1}} \\
&= 3^{2d} I_{3^{d+1}} + (3^d[\frac{3^d+1}{2}])J_{3^{d+1}} \\
&= (3^d[\frac{3^{d+1}+1}{2}])I_{3^{d+1}} + (3^d[\frac{3^d+1}{2}])(J_{3^{d+1}} - I_{3^{d+1}}).
\end{aligned}$$

Therefore, diagonal blocks of NN^T satisfy (3.11). For any $A \in \Xi'(3,d) \setminus \{J_{3^{d+1}} - (I_3 \otimes J_{3^d})\}$, we have $A(J_{3^{d+1}} - I_3 \otimes J_{3^d})^T = (J_{3^{d+1}} - I_3 \otimes J_{3^d})A^T = 2 \times 3^{d-1} J_{3^{d+1}}$. For $i \neq j$, we have

$$\begin{aligned}
M_{ij} &= \sum_{k=1}^{r} N_{ik} N_{jk} = \\
&= 2(2 \times 3^{d-1} J_{3^{d+1}}) + \sum_{i=0}^{r-2} 3^{d-1} J_{3^{d+1}} \\
&= (4 \times 3^{d-1} J_{3^{d+1}}) + (r-2)3^{d-1} J_{3^{d+1}} \\
&= 3^{d-1}(r+2) J_{3^{d+1}} \\
&= 3^{d-1}(\frac{3^{d+1}-1}{2} + 2) J_{3^{d+1}}
\end{aligned}$$

$$
\begin{aligned}
&= 3^{d-1}\left(\frac{3^{d+1}+3}{2}\right)J_{3^{d+1}} \\
&= 3^{d}\left(\frac{3^{d}+1}{2}\right)J_{3^{d+1}}.
\end{aligned}
$$

Hence, we have $NN^T = 3^d[\frac{3^{d+1}+1}{2}]I_{q^{d+1}} + 3^d[\frac{3^d+1}{2}](J_{q^{d+1}} - I_{q^{d+1}})$ and so N is the incidence matrix of a symmetric $(3^{d+1}(3^{d+1}-1)/2, 3^d(3^{d+1}+1)/2, 3^d(3^d+1)/2)$-design. $\square$

Since for any even integer n, there is a symmetric constant diagonal Latin square of order n, we conclude the following corollary.

Corollary 3.29. *Let d be a positive integer. If $r = \dfrac{q^{d+1}-1}{q-1}$ is even, let L be a symmetric constant diagonal Latin square over $\{1,\ldots,r\}$ with r on the diagonal. To makes a $(0,1)$-matrix N of size $r(3^{d+1})$, Substitute i with $M'(i)$ in the Latin square L. Then, N is the adjacency matrix of a $SRG(3^{d+1}(3^{d+1}-1)/2, 3^d(3^{d+1}+1)/2, 3^d(3^d+1)/2, 3^d(3^d+1)/2)$.*

Example 3.30. In this example, we construct a symmetric $(36,15,6)$-design. Since $d=1$, we use the same set $\Delta(GH(3,1))$ made in Example 3.26, i.e.,

$$
\Delta(GH(3,1)) = \left\{
\begin{bmatrix}
1&0&0&1&0&0&1&0&0\\
0&1&0&0&1&0&0&1&0\\
0&0&1&0&0&1&0&0&1\\
1&0&0&1&0&0&1&0&0\\
0&1&0&0&1&0&0&1&0\\
0&0&1&0&0&1&0&0&1\\
1&0&0&1&0&0&1&0&0\\
0&1&0&0&1&0&0&1&0\\
0&0&1&0&0&1&0&0&1
\end{bmatrix},
\begin{bmatrix}
1&0&0&0&1&0&0&0&1\\
0&1&0&0&0&1&1&0&0\\
0&0&1&1&0&0&0&1&0\\
0&0&1&1&0&0&0&1&0\\
1&0&0&0&1&0&0&0&1\\
0&1&0&0&0&1&1&0&0\\
0&1&0&0&0&1&1&0&0\\
0&0&1&1&0&0&0&1&0\\
1&0&0&0&1&0&0&0&1
\end{bmatrix},
\begin{bmatrix}
1&0&0&0&0&1&0&1&0\\
0&1&0&1&0&0&0&0&1\\
0&0&1&0&1&0&1&0&0\\
0&1&0&1&0&0&0&0&1\\
0&0&1&0&1&0&1&0&0\\
1&0&0&0&0&1&0&1&0\\
0&0&1&0&1&0&1&0&0\\
1&0&0&0&0&1&0&1&0\\
0&1&0&1&0&0&0&0&1
\end{bmatrix}
\right\}.
$$

We first index the elements of the set $\Xi'(3,1)$:

$$
M'(1) = \begin{bmatrix}
1&0&0&0&0&1&0&1&0\\
0&1&0&1&0&0&0&0&1\\
0&0&1&0&1&0&1&0&0\\
0&1&0&1&0&0&0&0&1\\
0&0&1&0&1&0&1&0&0\\
1&0&0&0&0&1&0&1&0\\
0&0&1&0&1&0&1&0&0\\
1&0&0&0&0&1&0&1&0\\
0&1&0&1&0&0&0&0&1
\end{bmatrix},\quad
M'(2) = \begin{bmatrix}
1&0&0&1&0&0&1&0&0\\
0&1&0&0&1&0&0&1&0\\
0&0&1&0&0&1&0&0&1\\
1&0&0&1&0&0&1&0&0\\
0&1&0&0&1&0&0&1&0\\
0&0&1&0&0&1&0&0&1\\
1&0&0&1&0&0&1&0&0\\
0&1&0&0&1&0&0&1&0\\
0&0&1&0&0&1&0&0&1
\end{bmatrix},
$$

$$
M'(3) = \begin{bmatrix}
1&0&0&0&1&0&0&0&1\\
0&1&0&0&0&1&1&0&0\\
0&0&1&1&0&0&0&1&0\\
0&0&1&1&0&0&0&1&0\\
1&0&0&0&1&0&0&0&1\\
0&1&0&0&0&1&1&0&0\\
0&1&0&0&0&1&1&0&0\\
0&0&1&1&0&0&0&1&0\\
1&0&0&0&1&0&0&0&1
\end{bmatrix},\quad
M'(4) = \begin{bmatrix}
0&0&0&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1\\
1&1&1&0&0&0&1&1&1\\
1&1&1&0&0&0&1&1&1\\
1&1&1&0&0&0&1&1&1\\
1&1&1&1&1&1&0&0&0\\
1&1&1&1&1&1&0&0&0\\
1&1&1&1&1&1&0&0&0
\end{bmatrix}.
$$

In the Latin square $L = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$, replace i with $M'(i)$. The resulted matrix, called N, is the incidence matrix of a symmetric $(36, 15, 6)$-design. Since N is symmetric zero diagonal, it is also the adjacency matrix of a $SRG(36, 15, 6, 6)$.

Chapter 4

Decompositions and Symmetric Designs

The main purpose of this chapter is to introduce the concept of complete design decomposition. The two concepts complete design decompositions and complete graph decompositions have a lot in common. The concept of strongly regular decompositions of complete graphs was introduced in many papers including [15]. We start this chapter by reviewing decompositions of complete graphs.

Suppose that A is the incidence matrix of the complete design with ν points, and B is the adjacency matrix of the complete graph on ν vertices. Then, we have $A = B + I_\nu$. Suppose that the incidence matrix N of a symmetric (ν,k,λ)-design is symmetric with constant diagonal. If N has "0" on the diagonal, N is the adjacency matrix of an $SRG(\nu,k,\lambda,\lambda)$. Suppose that N has "1" on the diagonal. Since the graphs are without loops, $N-I$ is the adjacency matrix of a $SRG(\nu,k-1,\lambda-2,\lambda)$. Therefore, the concept of graphs and the concept of symmetric designs are closely related. The set $\{G_1,\ldots,G_d\}$ of graphs is a *decomposition* of a graph G if for any pair of adjacent vertices in G, the two vertices are adjacent in exactly one graph G_i, the vertices set of G_i and G are the same, for all i, and there is at least one edge in each G_i, [22, p. 182]. We can restate the definition of a decomposition of a graph using the adjacency matrix of a graph.

Definition 4.1. Let G be a graph whose adjacency matrix A_G is a $\nu \times \nu$ $(0,1)$-matrix. The set $\mathcal{G}$ of symmetric non-zero $\nu \times \nu$ $(0,1)$-matrices form a decomposition of G if $\sum\limits_{N\in\mathcal{G}} N = A_G$.

If all graphs in a complete graph decomposition are strongly regular graphs, the decomposition is said to be *strongly regular*. A graph and its complement form a trivial decomposition of a complete graph. Since the complement of a strongly regular graph is a strongly regular graph, any strongly regular graph and its complement form a strongly regular decomposition of a complete graph. If the adjacency matrices of all graphs in a complete graph decomposition commute, the decomposition is called *commutative*, [22, p. 183].

4.1 Introducing Complete Design Decomposition

The incidence matrix of a complete design with $|X| = v$ is J_v. The design $(X, \mathbf{A})$ is a complete design with 5 points, where

$$X = \{1,2,3,4,5\}, \text{ and}$$
$$\mathbf{A} = [X,X,X,X,X].$$

The incidence matrix of this design is

$$J_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Definition 4.2. A set $\mathcal{D}$ of $v \times v$ pairwise disjoint $(0,1)$-matrices is called a *symmetric design decomposition* of the complete design with v points if

1. $\sum_{N \in \mathcal{D}} N = J_v$.
2. Each matrix $N \in \mathcal{D}$ is the incidence matrix of a symmetric design.

Example 4.3. Consider the following set of three 7×7 $(0,1)$-matrices:

$$\mathcal{D} = \left\{ N_1 = \begin{bmatrix} 0&1&1&0&1&0&0 \\ 0&0&1&1&0&1&0 \\ 0&0&0&1&1&0&1 \\ 1&0&0&0&1&1&0 \\ 0&1&0&0&0&1&1 \\ 1&0&1&0&0&0&1 \\ 1&1&0&1&0&0&0 \end{bmatrix},\ N_2 = \begin{bmatrix} 0&0&0&1&0&1&1 \\ 1&0&0&0&1&0&1 \\ 1&1&0&0&0&1&0 \\ 0&1&1&0&0&0&1 \\ 1&0&1&1&0&0&0 \\ 0&1&0&1&1&0&0 \\ 0&0&1&0&1&1&0 \end{bmatrix},\ N_3 = \begin{bmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix} \right\}.$$

The set $\mathcal{D}$ is a symmetric design decomposition of the complete design with 7 points. i.e.,

$$\begin{bmatrix} 0&1&1&0&1&0&0 \\ 0&0&1&1&0&1&0 \\ 0&0&0&1&1&0&1 \\ 1&0&0&0&1&1&0 \\ 0&1&0&0&0&1&1 \\ 1&0&1&0&0&0&1 \\ 1&1&0&1&0&0&0 \end{bmatrix} + \begin{bmatrix} 0&0&0&1&0&1&1 \\ 1&0&0&0&1&0&1 \\ 1&1&0&0&0&1&0 \\ 0&1&1&0&0&0&1 \\ 1&0&1&1&0&0&0 \\ 0&1&0&1&1&0&0 \\ 0&0&1&0&1&1&0 \end{bmatrix} + \begin{bmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix} = \begin{bmatrix} 1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1 \end{bmatrix} = J_7.$$

The matrices N_1 and N_2 are the incidence matrices of symmetric $(7,3,1)$-designs and N_3 is the incidence matrix of a symmetric $(7,1,0)$-design.

The more general definition of decomposition of a complete design can be achieved by removing the second condition from Definition 4.2. In this way, the combinatorial objects, into which the complete designs are decomposed, can be different from symmetric designs. For example, a set $\mathcal{D}$ of $\nu \times \nu$ $(0,1)$-matrices decomposes the complete design with ν points into symmetric designs and DDGs if

1. $\sum_{N \in \mathcal{D}} N = J_\nu$.
2. Each matrix $N \in \mathcal{D}$ is the incidence matrix of a symmetric design or is the adjacency matrix of a DDG.

For decomposing the complete design with ν points, we need to know if there is any (non-complete and nontrivial) symmetric design with ν points. For example, the only (non-complete) symmetric designs of order 12 are symmetric $(12,11,10)$-design and its complement symmetric $(12,1,0)$-design both of which are trivial. Therefore, the complete design on 12 points can be decomposed to $(12,11,10)$-design and its complement only.

Definition 4.4. For any positive integer ν, we define a set

$$\mathcal{K}_\nu = \{k| \text{ There is a symmetric } (\nu, k, \frac{k(k-1)}{\nu-1})\text{-design.}\}$$

and we call it the *symmetric design capacity set* of ν.

Obviously, $k \in \mathcal{K}_\nu$ only if $\frac{k(k-1)}{\nu-1}$ is an integer. Moreover, $k \in \mathcal{K}_\nu$ if and only if $\nu - k \in \mathcal{K}_\nu$.

Theorem 4.5. *For a positive integer $n = mk$, let $D = (J_k - I_k) \otimes J_m$ which is the adjacency matrix of a $DDG(n, n-m, n-m, n-2m, k, m)$. If there is a decomposition of D into h symmetric (n, l, λ)-designs, then $\frac{(n-m)(n-m-h)}{\lambda(n-1)}$ is a perfect square. If n is even, then $\frac{(n-m)((h-1)n+m)}{\lambda(n-1)}$ is also a perfect square.*

Proof. The h symmetric (n, l, λ)-designs decompose D. So, we have $m + hl = n$ or $l = \dfrac{n-m}{h}$. Since there is a symmetric (n, l, λ)-design, we have $\lambda = \dfrac{l(l-1)}{n-1}$ and

$$\lambda = \frac{\frac{n-m}{h}(\frac{n-m}{h} - 1)}{n-1} = \frac{(n-m)(n-m-h)}{h^2(n-1)}.$$

Hence,

$$h^2 = \frac{(n-m)(n-m-h)}{\lambda(n-1)}.$$

Therefore, $\dfrac{(n-m)(n-m-h)}{\lambda(n-1)}$ is a perfect square.

Now, suppose n is even. Since there is a symmetric (n, l, λ)-design, Theorem 2.32 proves $l - \lambda$ is perfect square. Since l is equal to $\frac{n-m}{h}$, we have $l - \lambda = \frac{(n-m)\left((h-1)n+m\right)}{h^2(n-1)}$. Therefore, $\frac{(n-m)((h-1)n+m)}{h^2(n-1)}$ is a perfect square. Since h^2 is a perfect square, we conclude $\frac{(n-m)((h-1)n+m)}{(n-1)}$ is a perfect square. $\square$

In the next section, we first generate symmetric designs with disjoint incidence matrices and then we use them to decompose the complete designs.

4.2 Constructing Pairwise Disjoint Symmetric Designs

Let q be a prime power, d a positive integer, $r = \dfrac{q^{d+1}-1}{q-1}$, and $D = (J_{r+1} - I_{r+1}) \otimes J_{q^{d+1}}$ which is a $DDG((r+1)q^{d+1}, rq^{d+1}, rq^{d+1}, (r-1)q^{d+1}, r+1, q^{d+1})$. By Theorem 4.5, we have the necessary condition for a decomposition of D into q symmetric $((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$-designs. We note $n = (r+1)q^{d+1}$, $m = q^{d+1}$, $k = r+1$, $h = q$, $\lambda = (r-1)q^{d-1}$, and

$$\frac{(n-m)(n-m-h)}{\lambda(n-1)} = \frac{((r+1)q^{d+1} - q^{d+1})((r+1)q^{d+1} - q^{d+1} - q)}{(r-1)q^{d-1}((r+1)q^{d+1} - 1)} = q^2.$$

We first generate q disjoint symmetric $((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$-designs which decompose D. We then use more combinatorial objects to achieve a decomposition of a complete design on $(r+1)q^{d+1}$ points. We need to generalize the method introduced in Section 3.2 starting with modifying Definition 3.18.

Definition 4.6. Let $q = p^m$ be a prime power, d a positive integer, $\varepsilon \in \{0, \ldots, q-1\}$, and $GH(q, q^\theta)$ a generalized Hadamard matrix over the group $\Omega(p, m) = \{\omega_0, \ldots, \omega_{p^m-1}\}$, where $0 \le \theta \le d-1$. We use the notation $\Delta_\varepsilon(GH(q, q^\theta), d)$ to show the set

$$\{[(I_{q^{\theta+1}} \otimes \omega_\varepsilon)(r_\alpha^* r_\alpha)] \otimes J_{q^{d-\theta-1}} | 1 \le \alpha \le q^{\theta+1},\ r_\alpha \text{ is the } \alpha\text{-th row of } GH(q, q^\theta)\}$$

of $(0,1)$-matrices of size q^{d+1}. If $d = 1$, we may use the notation $\Delta_\varepsilon(GH(q, q^\theta))$ instead of $\Delta_\varepsilon(GH(q, q^\theta), 1)$. We have $\Delta_0(GH(q, q^\theta), d) = \Delta(GH(q, q^\theta), d)$.

Definition 4.7. Let $q = p^m$ be a prime power, $\varepsilon \in \{0, \ldots, q-1\}$, and d a positive integer. Suppose $GH(q, 1)$, ..., $GH(q, q^{d-1})$ are generalized Hadamard matrices over $\Omega(p, m)$. We use the notation $\Xi(q, d, \varepsilon)$ to show the set

$$\Delta_\varepsilon(GH(q, 1), d) \cup \ldots \cup \Delta_\varepsilon(GH(q, q^{d-1}), d) \cup \{O\} \cup \{\omega_\varepsilon \otimes J_{q^d}\}.$$

Example 4.8. In this example, we assume $q=2$ and $d=2$. The purpose of this example is to demonstrate $\Delta_0(GH(2,1),2)$ and $\Delta_1(GH(2,1),2)$, where $GH(2,1)=\left[\begin{smallmatrix}\omega_0 & \omega_0\\ \omega_0 & \omega_1\end{smallmatrix}\right]$ is a generalized Hadamard matrix over the group $\Omega(2,1)=\{\omega_0=\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right],\omega_1=\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]\}$. Since we know $\Delta_0(GH(2,1),2)=\Delta(GH(2,1),2)$, we have $\Delta_0(GH(2,1),2)$ from Example 3.19. i.e.,

$$\Delta_0(GH(2,1),2)=\left\{\begin{bmatrix}1&1&0&0&1&1&0&0\\1&1&0&0&1&1&0&0\\0&0&1&1&0&0&1&1\\0&0&1&1&0&0&1&1\\1&1&0&0&1&1&0&0\\1&1&0&0&1&1&0&0\\0&0&1&1&0&0&1&1\\0&0&1&1&0&0&1&1\end{bmatrix},\begin{bmatrix}1&1&0&0&0&0&1&1\\1&1&0&0&0&0&1&1\\0&0&1&1&1&1&0&0\\0&0&1&1&1&1&0&0\\0&0&1&1&1&1&0&0\\0&0&1&1&1&1&0&0\\1&1&0&0&0&0&1&1\\1&1&0&0&0&0&1&1\end{bmatrix}\right\}.$$

To examine the set $\Delta_1(GH(2,1),2)$, we note

$$[(I_2\otimes\omega_1)(r_1^*r_1)]\otimes J_2=(\begin{bmatrix}\omega_1&0\\0&\omega_1\end{bmatrix}\begin{bmatrix}\omega_0&\omega_0\\\omega_0&\omega_0\end{bmatrix})\otimes J_2=\begin{bmatrix}\omega_1&\omega_1\\\omega_1&\omega_1\end{bmatrix}\otimes J_2=\begin{bmatrix}0&0&1&1&0&0&1&1\\0&0&1&1&0&0&1&1\\1&1&0&0&1&1&0&0\\1&1&0&0&1&1&0&0\\0&0&1&1&0&0&1&1\\0&0&1&1&0&0&1&1\\1&1&0&0&1&1&0&0\\1&1&0&0&1&1&0&0\end{bmatrix}$$

and

$$[(I_2\otimes\omega_1)(r_2^*r_2)]\otimes J_2=(\begin{bmatrix}\omega_1&0\\0&\omega_1\end{bmatrix}\begin{bmatrix}\omega_0&\omega_1\\\omega_1&\omega_0\end{bmatrix})\otimes J_2=\begin{bmatrix}\omega_1&\omega_0\\\omega_0&\omega_1\end{bmatrix}\otimes J_2=\begin{bmatrix}0&0&1&1&1&1&0&0\\0&0&1&1&1&1&0&0\\1&1&0&0&0&0&1&1\\1&1&0&0&0&0&1&1\\1&1&0&0&0&0&1&1\\1&1&0&0&0&0&1&1\\0&0&1&1&1&1&0&0\\0&0&1&1&1&1&0&0\end{bmatrix}.$$

Therefore, we have

$$\Delta_1(GH(2,1),2)=\left\{\begin{bmatrix}0&0&1&1&0&0&1&1\\0&0&1&1&0&0&1&1\\1&1&0&0&1&1&0&0\\1&1&0&0&1&1&0&0\\0&0&1&1&0&0&1&1\\0&0&1&1&0&0&1&1\\1&1&0&0&1&1&0&0\\1&1&0&0&1&1&0&0\end{bmatrix},\begin{bmatrix}0&0&1&1&1&1&0&0\\0&0&1&1&1&1&0&0\\1&1&0&0&0&0&1&1\\1&1&0&0&0&0&1&1\\1&1&0&0&0&0&1&1\\1&1&0&0&0&0&1&1\\0&0&1&1&1&1&0&0\\0&0&1&1&1&1&0&0\end{bmatrix}\right\}.$$

For $\varepsilon\neq\vartheta$, there is a natural isomorphism between $\Delta_\varepsilon(GH(q,q^\theta),d)$ and $\Delta_\vartheta(GH(q,q^\theta),d)$ which corresponds each element of $\Delta_\varepsilon(GH(q,q^\theta),d)$ to an element of $\Delta_\vartheta(GH(q,q^\theta),d)$. Let $A(\varepsilon)=[(I_{q^{\theta+1}}\otimes\omega_\varepsilon)(r_\alpha^*r_\alpha)]\otimes J_{q^{d-\theta-1}}$ be an arbitrary element of the set $\Delta_\varepsilon(GH(q,q^\theta),d)$, where $\alpha\in\{1,\dots,q^{\theta+1}\}$. Then the image of $A(\varepsilon)$ under the isomorphism is equal to $A(\vartheta)=[(I_{q^{\theta+1}}\otimes\omega_\vartheta)(r_\alpha^*r_\alpha)]\otimes J_{q^{d-\theta-1}}$.

The same isomorphism is defined when $A(\varepsilon) \in \Xi(q,d,\varepsilon)$. We may correspond $A(\varepsilon)$ to an element in $A(\vartheta) \in \Xi(q,d,\vartheta)$. In this case, if $A(\varepsilon) \in \Delta_\varepsilon(GH(q,q^\theta),d)$ and $A(\vartheta) \in \Delta_\vartheta(GH(q,q^\eta),d)$, then we have $\theta = \eta$. In the following lemma, we prove $A(\varepsilon)$ and $A(\vartheta)$ are disjoint whenever $\varepsilon \neq \vartheta$.

Lemma 4.9. *Let $q = p^m$ be a prime power, d a positive integer, and $GH(q,q^\theta)$ a generalized Hadamard matrix over the group $\Omega(p,m) = \{\omega_0, \ldots, \omega_{p^m-1}\}$, where $0 \leq \theta \leq d-1$. For $\varepsilon \neq \vartheta$, suppose $A(\varepsilon) \in \Delta_\varepsilon(GH(q,q^\theta),d)$ and $A(\vartheta) \in \Delta_\vartheta(GH(q,q^\theta),d)$. Then, $A(\varepsilon)$ and $A(\vartheta)$ are disjoint.*

Proof. We have $A(\varepsilon) = [(I_{q^{\theta+1}} \otimes \omega_\varepsilon)(r_\alpha^* r_\alpha)] \otimes J_{q^{d-\theta-1}}$ and $A(\vartheta) = [(I_{q^{\theta+1}} \otimes \omega_\vartheta)(r_\alpha^* r_\alpha)] \otimes J_{q^{d-\theta-1}}$, for some $\alpha \in \{0, \ldots, q^{\theta+1}\}$. Lemma 3.11 proves that the elements of the set $\Omega(p,m)$ are pairwise disjoint. Therefore, two matrices $(I_{q^{\theta+1}} \otimes \omega_\varepsilon)(r_\alpha^* r_\alpha)$ and $(I_{q^{\theta+1}} \otimes \omega_\vartheta)(r_\alpha^* r_\alpha)$ are disjoint. Hence, $A(\varepsilon)$ and $A(\vartheta)$ are disjoint. □

We have the following lemma.

Lemma 4.10. *Let $q = p^m$ be a prime power, d a positive integer, $\varepsilon \in \{0, \ldots, q-1\}$, and $GH(q,q^\theta)$ a generalized Hadamard matrix over the group $\Omega(p,m) = \{\omega_0, \omega_1, \ldots, \omega_{p^m-1}\}$, where $0 \leq \theta \leq d-1$. The elements of the set $\Xi(q,d,\varepsilon)$ satisfy the following conditions:*

1. *For any two distinct elements $A \neq O$ and $B \neq O$ of the set $\Xi(q,d,\varepsilon)$, we have $AB^T = q^{d-1} J_{q^{d+1}}$;*
2. *For any nonzero element $A(\varepsilon) \in \Xi(q,d,\varepsilon)$, we have $A(\varepsilon)A(\varepsilon)^T = q^d A(0)$;*
3. $\displaystyle\sum_{A(\varepsilon) \in \Delta_\varepsilon(GH(q,q^\theta),d)} A(\varepsilon) = q^{\theta+1}(I_{q^{\theta+1}} \otimes \omega_\varepsilon \otimes J_{q^{d-\theta-1}}) + q^\theta((J_{q^{\theta+1}} - I_{q^{\theta+1}}) \otimes J_{q^{d-\theta}})$.

We know

$$\Xi(q,d,\varepsilon) = \Delta_\varepsilon(GH(q,1),d) \cup \ldots \cup \Delta_\varepsilon(GH(q,q^{d-1}),d) \cup \{O\} \cup \{\omega_\varepsilon \otimes J_{q^d}\}.$$

To generate a symmetric $((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$-design, we need to convert the set $\Xi(q,d,\varepsilon)$ into an indexed family using the index set $\{1,\ldots,r+1\}$. The number of elements are the same in the two sets $\Xi(q,d,\varepsilon)$ and $\{1,\ldots,r+1\}$. We define a bijective function $M_\varepsilon : \{1,\ldots,r+1\} \to \Xi(q,d,\varepsilon)$ such that $M_\varepsilon(r) = \omega_\varepsilon \otimes J_{q^d}$ and $M_\varepsilon(r+1)$ is equal to the zero matrix. Now we let $M_\varepsilon(1)$ to be an element in $\Xi(q,d) \setminus \{M_\varepsilon(r), M_\varepsilon(r+1)\}$. Then inductively we assign $M_\varepsilon(i+1)$ to an element in $\Xi(q,d,\varepsilon) \setminus \{M_\varepsilon(r), M_\varepsilon(r+1), M_\varepsilon(k) | k = 1,\ldots,i\}$.

Theorem 4.11. *Let p be prime, $q = p^m$, $\varepsilon \in \{0,\ldots,q-1\}$, d a positive integer, $r = \frac{q^{d+1}-1}{q-1}$, and L a Latin square over $\{1,\ldots,r+1\}$. Substitute i with $M_\varepsilon(i)$ in the Latin square L to construct a $(0,1)$-matrix $N(\varepsilon)$ of size $(r+1)q^{d+1}$. Then, $N(\varepsilon)$ is the incidence matrix of a symmetric $((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$-design.*

Proof. We need to prove

$$N(\varepsilon)N(\varepsilon)^T = rq^d I_{q^{d+1}} + (r-1)q^{d-1}(J_{q^{d+1}} - I_{q^{d+1}}). \tag{4.1}$$

Let M_{ij} be the (i,j)-block of size q^{d+1} of the matrix $N(\varepsilon)N(\varepsilon)^T$. For $i = j$, Lemma 4.10 and Theorem 3.24 implies

$$\begin{aligned}
M_{ii} &= \sum_{A(\varepsilon)\in\Xi(q,d,\varepsilon)} A(\varepsilon)A(\varepsilon)^T \\
&= ((\omega_\varepsilon \otimes J_{q^d})(\omega_\varepsilon^T \otimes J_{q^d})) + (\sum_{\theta=0}^{d-1} \sum_{\{A(\varepsilon)\in\Delta_\varepsilon(GH(q,q^\theta),d)\}} A(\varepsilon)A(\varepsilon)^T) \\
&= q^d(\omega_0 \otimes J_{q^d}) + (\sum_{\theta=0}^{d-1} q^d \sum_{\{A(0)\in\Delta(GH(q,q^\theta),d)\}} A(0)) \\
&= q^d(I_q \otimes J_{q^d}) + q^d \sum_{\theta=0}^{d-1} [q^{\theta+1}(I_{q^{\theta+2}} \otimes J_{q^{d-\theta-1}}) + q^\theta((J_{q^{\theta+1}} - I_{q^{\theta+1}}) \otimes J_{q^{d-\theta}})] \\
&= rq^d I_{q^{d+1}} + (r-1)q^{d-1}(J_{q^{d+1}} - I_{q^{d+1}}).
\end{aligned}$$

Therefore, diagonal blocks of $N(\varepsilon)N(\varepsilon)^T$ are as in (4.1). For any $1 \le i,j \le r+1$, let $N(\varepsilon)_{ij}$

be the (i, j)-block of size q^{d+1} of the matrix $N(\varepsilon) = [N(\varepsilon)_{ij}]$. Now, suppose $i \neq j$. By Lemma 4.10, we have

$$M_{ij} = \sum_{k=1}^{r+1} N(\varepsilon)_{ik} N(\varepsilon)_{jk} = (r-1)q^{d-1} J_{q^{d+1}}.$$

Hence, we have $N(\varepsilon)N(\varepsilon)^T = rq^d I_{q^{d+1}} + (r-1)q^{d-1}(J_{q^{d+1}} - I_{q^{d+1}})$ and so $N(\varepsilon)$ is the incidence matrix of a symmetric $((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$-design. $\square$

In the following theorem, we prove $N(\varepsilon)$ and $N(\vartheta)$ have disjoint incidence matrices whenever $\varepsilon \neq \vartheta$.

Theorem 4.12. *Suppose $q = p^m$ is prime power, d is a positive integer, $\varepsilon, \vartheta \in \{0, \ldots, q-1\}$, and $\varepsilon \neq \vartheta$. Define $r = (q^{d+1} - 1)/(q-1)$. Let L be any Latin square over $\{1, \ldots, r+1\}$. Substitute i with $M_\varepsilon(i)$ and $M_\vartheta(i)$ in the Latin square L to construct $(0,1)$-matrices $N(\varepsilon)$ and $N(\vartheta)$, respectively. Then, two matrices $N(\varepsilon)$ and $N(\vartheta)$ are disjoint.*

Proof. Since we use the same Latin square for $N(\varepsilon)$ and $N(\vartheta)$, the theorem follows from Lemma 4.9. $\square$

Example 4.13. In this example, we give three disjoint matrices. These matrices are the incidence matrices of symmetric $(45, 12, 3)$-designs. For this purpose, we set $q = 3$ and $d = 1$. Then, $\Omega(3,1) = G_3 = < circ(0,1,0) > = \{\omega_0 = \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix}, \omega_1 = \begin{bmatrix} 0&1&0\\0&0&1\\1&0&0 \end{bmatrix}, \omega_2 = \begin{bmatrix} 0&0&1\\1&0&0\\0&1&0 \end{bmatrix}\}$. Since $d = 1$, we only need the generalized Hadamard matrix $GH(3,1) = \begin{bmatrix} \omega_0&\omega_0&\omega_0\\ \omega_0&\omega_1&\omega_2\\ \omega_0&\omega_2&\omega_1 \end{bmatrix}$ over the group $\Omega(3,1)$. While using Definition 4.6, we first define three sets $\Delta_0(GH(3,1))$, $\Delta_1(GH(3,1))$, and $\Delta_2(GH(3,1))$. Since $\Delta_0(GH(3,1)) = \Delta(GH(3,1))$, from Example 3.26, we have:

$$\Delta_0(GH(3,1)) = \left\{ \begin{bmatrix} 1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&0\\ 0&0&1&0&0&1&0&0&1\\ 1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&0\\ 0&0&1&0&0&1&0&0&1\\ 1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&0\\ 0&0&1&0&0&1&0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0&0&1&0&0&0&1\\ 0&1&0&0&0&1&1&0&0\\ 0&0&1&1&0&0&0&1&0\\ 0&0&1&1&0&0&0&1&0\\ 1&0&0&0&1&0&0&0&1\\ 0&1&0&0&0&1&1&0&0\\ 0&1&0&0&0&1&1&0&0\\ 0&0&1&1&0&0&0&1&0\\ 1&0&0&0&1&0&0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0&0&0&1&0&1&0\\ 0&1&0&1&0&0&0&0&1\\ 0&0&1&0&1&0&1&0&0\\ 0&1&0&1&0&0&0&0&1\\ 0&0&1&0&1&0&1&0&0\\ 1&0&0&0&0&1&0&1&0\\ 0&0&1&0&1&0&1&0&0\\ 1&0&0&0&0&1&0&1&0\\ 0&1&0&1&0&0&0&0&1 \end{bmatrix} \right\}.$$

Now, we demonstrate $\Delta_1(GH(3,1))$:

$$(I_3 \otimes \omega_1)(r_1^* r_1) = \begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_1 & 0 \\ 0 & 0 & \omega_1 \end{bmatrix} \begin{bmatrix} \omega_0 & \omega_0 & \omega_0 \\ \omega_0 & \omega_0 & \omega_0 \\ \omega_0 & \omega_0 & \omega_0 \end{bmatrix} = \begin{bmatrix} \omega_1 & \omega_1 & \omega_1 \\ \omega_1 & \omega_1 & \omega_1 \\ \omega_1 & \omega_1 & \omega_1 \end{bmatrix} = \begin{bmatrix} 0&1&0&0&1&0&0&1&0 \\ 0&0&1&0&0&1&0&0&1 \\ 1&0&0&1&0&0&1&0&0 \\ 0&1&0&0&1&0&0&1&0 \\ 0&0&1&0&0&1&0&0&1 \\ 1&0&0&1&0&0&1&0&0 \\ 0&1&0&0&1&0&0&1&0 \\ 0&0&1&0&0&1&0&0&1 \\ 1&0&0&1&0&0&1&0&0 \end{bmatrix}.$$

$$(I_3 \otimes \omega_1)(r_2^* r_2) = \begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_1 & 0 \\ 0 & 0 & \omega_1 \end{bmatrix} \begin{bmatrix} \omega_0 & \omega_1 & \omega_2 \\ \omega_2 & \omega_0 & \omega_1 \\ \omega_1 & \omega_2 & \omega_0 \end{bmatrix} = \begin{bmatrix} \omega_1 & \omega_2 & \omega_0 \\ \omega_0 & \omega_1 & \omega_2 \\ \omega_2 & \omega_0 & \omega_1 \end{bmatrix} = \begin{bmatrix} 0&1&0&0&0&1&1&0&0 \\ 0&0&1&1&0&0&0&1&0 \\ 1&0&0&0&1&0&0&0&1 \\ 1&0&0&0&1&0&0&0&1 \\ 0&1&0&0&0&1&1&0&0 \\ 0&0&1&1&0&0&0&1&0 \\ 0&0&1&1&0&0&0&1&0 \\ 1&0&0&0&1&0&0&0&1 \\ 0&1&0&0&0&1&1&0&0 \end{bmatrix}.$$

$$(I_3 \otimes \omega_1)(r_3^* r_3) = \begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_1 & 0 \\ 0 & 0 & \omega_1 \end{bmatrix} \begin{bmatrix} \omega_0 & \omega_2 & \omega_1 \\ \omega_1 & \omega_0 & \omega_2 \\ \omega_2 & \omega_1 & \omega_0 \end{bmatrix} = \begin{bmatrix} \omega_1 & \omega_0 & \omega_2 \\ \omega_2 & \omega_1 & \omega_0 \\ \omega_0 & \omega_2 & \omega_1 \end{bmatrix} = \begin{bmatrix} 0&1&0&1&0&0&0&0&1 \\ 0&0&1&0&1&0&1&0&0 \\ 1&0&0&0&0&1&0&1&0 \\ 0&0&1&0&1&0&1&0&0 \\ 1&0&0&0&0&1&0&1&0 \\ 0&1&0&1&0&0&0&0&1 \\ 1&0&0&0&0&1&0&1&0 \\ 0&1&0&1&0&0&0&0&1 \\ 0&0&1&0&1&0&1&0&0 \end{bmatrix}.$$

Therefore, we have

$$\Delta_1(GH(3,1)) = \left\{ \begin{bmatrix} 0&1&0&0&1&0&0&1&0 \\ 0&0&1&0&0&1&0&0&1 \\ 1&0&0&1&0&0&1&0&0 \\ 0&1&0&0&1&0&0&1&0 \\ 0&0&1&0&0&1&0&0&1 \\ 1&0&0&1&0&0&1&0&0 \\ 0&1&0&0&1&0&0&1&0 \\ 0&0&1&0&0&1&0&0&1 \\ 1&0&0&1&0&0&1&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0&0&0&1&1&0&0 \\ 0&0&1&1&0&0&0&1&0 \\ 1&0&0&0&1&0&0&0&1 \\ 1&0&0&0&1&0&0&0&1 \\ 0&1&0&0&0&1&1&0&0 \\ 0&0&1&1&0&0&0&1&0 \\ 0&0&1&1&0&0&0&1&0 \\ 1&0&0&0&1&0&0&0&1 \\ 0&1&0&0&0&1&1&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0&1&0&0&0&0&1 \\ 0&0&1&0&1&0&1&0&0 \\ 1&0&0&0&0&1&0&1&0 \\ 0&0&1&0&1&0&1&0&0 \\ 1&0&0&0&0&1&0&1&0 \\ 0&1&0&1&0&0&0&0&1 \\ 1&0&0&0&0&1&0&1&0 \\ 0&1&0&1&0&0&0&0&1 \\ 0&0&1&0&1&0&1&0&0 \end{bmatrix} \right\}.$$

In a similar way, we have

$$\Delta_2(GH(3,1)) = \left\{ \begin{bmatrix} 0&0&1&0&0&1&0&0&1 \\ 1&0&0&1&0&0&1&0&0 \\ 0&1&0&0&1&0&0&1&0 \\ 0&0&1&0&0&1&0&0&1 \\ 1&0&0&1&0&0&1&0&0 \\ 0&1&0&0&1&0&0&1&0 \\ 0&0&1&0&0&1&0&0&1 \\ 1&0&0&1&0&0&1&0&0 \\ 0&1&0&0&1&0&0&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&1&1&0&0&0&1&0 \\ 1&0&0&0&1&0&0&0&1 \\ 0&1&0&0&0&1&1&0&0 \\ 0&1&0&0&0&1&1&0&0 \\ 0&0&1&1&0&0&0&1&0 \\ 1&0&0&0&1&0&0&0&1 \\ 1&0&0&0&1&0&0&0&1 \\ 0&1&0&0&0&1&1&0&0 \\ 0&0&1&1&0&0&0&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&1&0&1&0&1&0&0 \\ 1&0&0&0&0&1&0&1&0 \\ 0&1&0&1&0&0&0&0&1 \\ 1&0&0&0&0&1&0&1&0 \\ 0&1&0&1&0&0&0&0&1 \\ 0&0&1&0&1&0&1&0&0 \\ 0&1&0&1&0&0&0&0&1 \\ 0&0&1&0&1&0&1&0&0 \\ 1&0&0&0&0&1&0&1&0 \end{bmatrix} \right\}.$$

Then, we have $\Xi(3,1,\varepsilon) = \Delta_\varepsilon(GH(3,1)) \cup \{0\} \cup \{\omega_\varepsilon \otimes J_3\}$. For each $\varepsilon = 0,1,2$, replace

j with $M_\varepsilon(j)$ in the Latin square $L = \begin{bmatrix} 5&4&1&2&3\\ 3&5&4&1&2\\ 2&3&5&4&1\\ 1&2&3&5&4\\ 4&1&2&3&5 \end{bmatrix}$. The resulted matrix, called $N(\varepsilon)$, is the incidence matrix of a symmetric $(45,12,3)$-design by Theorem 4.11. Theorem 4.12 guarantees that the elements of the set $\{N(0),N(1),N(2)\}$ are pairwise disjoint. The elements of the set $\{N(0),N(1),N(2)\}$ appears in Appendix 4.

In Lemma 3.20, we see that all matrices in the set $\Delta(GH(q,q^\theta),d)$ or $\Delta_0(GH(q,q^\theta),d)$ are symmetric. However, the matrices in the set $\Delta_\varepsilon(GH(q,q^\theta),d)$ are not necessarily symmetric when $\varepsilon \neq 0$. One may be curious if there is a way to make all matrices of the set $\Delta_\varepsilon(GH(q,q^\theta),d)$ symmetric for all ε. Fortunately, our method can be modified to achieve this. We first make the matrices of the set $\Omega(p,m)$ to be all symmetric.

Lemma 4.14. *Let $q = p^m$ be a prime power and R_q be the back diagonal identity matrix of size q. For any element $\omega \in \Omega(p,m)$, we have*

1. *ωR_q is symmetric,*
2. *$\omega R_q = R_q \omega^T$, and*
3. *$(\omega R_q)(\omega R_q) = I_q$.*

Proof. 1. Let $\omega \in \Omega(p,m)$. From Definition 3.10, we have $\omega = A_1 \otimes \ldots \otimes A_m$ for some $A_1,\ldots,A_m \in G_p$, where G_p is a cyclic group of order p generated by the circulant matrix $C_p = circ(0,1,0,\ldots,0)$. Any matrix A in G_p is circulant. Therefore, we have $AR_p = R_pA^T$. On the other hand, we have $(AR_p)^T = R_p^TA^T = R_pA^T$. Hence, $AR_p = (AR_p)^T$ and we conclude AR_p is symmetric. Moreover, we have

$$\begin{aligned} \omega R_q &= (A_1 \otimes \ldots \otimes A_m)R_q \\ &= (A_1 \otimes \ldots \otimes A_m)(R_p \otimes \ldots \otimes R_p) \\ &= A_1R_p \otimes \ldots \otimes A_mR_p. \end{aligned}$$

Since $A_1R_p,\ldots,A_mR_p$ are symmetric, ωR_q is equal to Kronecker product of symmetric matrices. Thus, ωR_q is symmetric.

2. Since ωR_q and R_q are symmetric, we have

$$\omega R_q = (\omega R_q)^T = R_q^T \omega^T = R_q \omega^T.$$

3. By Lemma 3.11, we have

$$\begin{aligned}(\omega R_q)(\omega R_q) &= \omega R_q R_p \omega^T \\ &= \omega\omega^T \\ &= \omega\omega^{-1} \\ &= I_p.\end{aligned}$$

Thus, $(\omega R_q)(\omega R_q) = I_q$. □

Definition 4.15. Let $q = p^m$ be a prime power and R_q be the back diagonal identity matrix of size q. We define a set $\Omega_R(p,m)$ of $(0,1)$-matrices by $\Omega_R(p,m) = \{\omega R_q | \omega \in \Omega(p,m)\}$.

Lemma 4.14 and Lemma 3.11 prove the following lemma:

Lemma 4.16. *Let $q = p^m$ be a prime power. For any element $A \in \Omega_R(p,m)$, we have*

1. *A is symmetric,*
2. *$A = A^{-1}$, and*
3. $\displaystyle\sum_{A\in\Omega_R(p,m)} A = J_{p^m}$.

Remark 4.17. Let $H = GH(q,q^{d-1})$ be a generalized Hadamard matrix over $\Omega(p,m)$. Let H_{ij} be the (i,j)-entry of the matrix H. Based on Definition 1.11, for all distinct $i,j \in \{1,2,\ldots,q^d\}$ the multiset $\{H_{ik}^{-1}H_{jk} : 1 \le k \le q^d\}$ contains exactly q^{d-1} copies of each element of $\Omega(p,m)$. In $GH(q,q^{d-1})$, if we replace each $\omega \in \Omega(p,m)$ with $\omega R_p \in \Omega_R(p,m)$, the multiset $\{H_{ik}^{-1}H_{jk} : 1 \leq k \leq q^d\}$ will still contain exactly q^{d-1} copies of each element of $\Omega(p,m)$. This is true because for any $\omega_i, \omega_j \in \Omega(p,m)$, Lemma 4.14 shows

$$\begin{aligned}
\omega_i R_q (\omega_j R_q)^{-1} &= (\omega_i R_q)(\omega_j R_q) \\
&= \omega_i R_q R_q \omega_j^T \\
&= \omega_i \omega_j^T \\
&= \omega_i \omega_j^{-1}.
\end{aligned}$$

While using ωR_q instead of ω, $r_\alpha^* r_\alpha$ is equal to $r_\alpha^T r_\alpha$ because $(\omega R_q)^{-1} = \omega R_q$. By these changes, matrices generated in Definition 4.6 are converted to symmetric matrices.

Definition 4.18. Let $q = p^m$ be prime power, d a positive integer, $\varepsilon \in \{0,\ldots,q-1\}$, and $GH(q,q^\theta)$ a generalized Hadamard matrix over the group $\Omega(p,m) = \{\omega_0,\ldots,\omega_{p^m-1}\}$, where $0 \leq \theta \leq d-1$. We use the notation $\Delta'_\varepsilon(GH(q,q^\theta),d)$ to show the set

$$\{A(I_{q^{\theta+1}} \otimes R_q \otimes I_{q^{d-\theta-1}}) | A \in \Delta_\varepsilon(GH(q,q^\theta),d)\}$$

of $(0,1)$-matrices of size q^{d+1}. If $d = 1$, we may use the notation $\Delta'_\varepsilon(GH(q,q^\theta))$ instead of $\Delta'_\varepsilon(GH(q,q^\theta),1)$.

Lemma 4.9 implies the following lemma in which we prove $A_R(\varepsilon)$ and $A_R(\vartheta)$ are disjoint whenever $\varepsilon \neq \vartheta$.

Lemma 4.19. *Let $q = p^m$ be a prime power, d a positive integer, $\varepsilon \in \{0,\ldots,q-1\}$, R_q the back diagonal identity matrix of size q, and $GH(q,q^\theta)$ a generalized Hadamard matrix over the group $\Omega(p,m)$, where $0 \leq \theta \leq d-1$. For $\varepsilon \neq \vartheta$, let $A_R(\varepsilon) \in \Delta'_\varepsilon(GH(q,q^\theta),d)$ and $A_R(\vartheta) \in \Delta'_\vartheta(GH(q,q^\theta),d)$. Then, $A_R(\varepsilon)$ and $A_R(\vartheta)$ are disjoint.*

From Lemma 4.14, we have the following.

Lemma 4.20. *Let $q = p^m$ be a prime power, d a positive integer, $\varepsilon \in \{0,\dots,q-1\}$, and $GH(q,q^\theta)$ a generalized Hadamard matrix over the group $\Omega(p,m)$, where $0 \le \theta \le d-1$. All matrices in the set $\Delta'_\varepsilon(GH(q,q^\theta),d)$ are symmetric.*

Definition 4.21. Let $q = p^m$ be a prime power, d a positive integer, and $\varepsilon \in \{0,\dots,q-1\}$. Suppose $GH(q,1)$, ..., $GH(q,q^{d-1})$ are generalized Hadamard matrices over $\Omega(p,m)$. We use the notation $\Xi_R(q,d,\varepsilon)$ to show the set

$$\Delta'_\varepsilon(GH(q,1),d) \cup \ldots \cup \Delta'_\varepsilon(GH(q,q^{d-1}),d) \cup \{O\} \cup \{\omega_\varepsilon R_q \otimes J_{q^d}\}.$$

From Lemma 4.20, we know that all matrices in the set $\Xi_R(q,d,\varepsilon)$ are symmetric.

Lemma 4.22. *Let $q = p^m$ be a prime power, d a positive integer, $\varepsilon \in \{0,\dots,q-1\}$, and $GH(q,q^\theta)$ a generalized Hadamard matrix over the group $\Omega(p,m) = \{\omega_0,\omega_1,\dots,\omega_{p^m-1}\}$, where $0 \le \theta \le d-1$. The elements of the set $\Xi_R(q,d,\varepsilon)$ satisfy the following conditions:*

1. *For any two distinct elements $A \neq O$ and $B \neq O$ of the set $\Xi_R(q,d,\varepsilon)$, we have $AB^T = q^{d-1}J_{q^{d+1}}$;*
2. *For any nonzero element $A_R(\varepsilon) \in \Xi_R(q,d,\varepsilon)$, we have $A_R(\varepsilon)A_R(\varepsilon)^T = q^dA(0)$;*
3. $\displaystyle\sum_{A_R(\varepsilon)\in\Delta'_\varepsilon(GH(q,q^\theta),d)} A_R(\varepsilon) = q^{\theta+1}(I_{q^{\theta+1}} \otimes (\omega_\varepsilon R_q) \otimes J_{q^{d-\theta-1}}) + q^\theta((J_{q^{\theta+1}} - I_{q^{\theta+1}}) \otimes J_{q^{d-\theta}})$.

We have now a main theorem.

Theorem 4.23. *Let $q = p^m$ be prime power, $\varepsilon \in \{0,\dots,q-1\}$, d a positive integer, $r = (q^{d+1}-1)/(q-1)$, and L a symmetric Latin square over $\{1,\dots,r+1\}$. Insert the elements of $\Xi_R(q,d,\varepsilon)$ in L to make a $(0,1)$-matrix $N_R(\varepsilon)$ of size $(r+1)q^{d+1}$. Then, $N_R(\varepsilon)$ is symmetric and is the incidence matrix of a symmetric $((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$-design.*

Proof. By the proof of Theorem 4.11, $N_R(\varepsilon)$ is the incidence matrix of a symmetric $((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$-design. Since the Latin square L and all matrices in the sets $\Xi_R(q,d,\varepsilon)$ are symmetric, $N_R(\varepsilon)$ is also symmetric. □

In the following theorem, we prove $N_R(\varepsilon)$ and $N_R(\vartheta)$ have disjoint incidence matrices whenever $\varepsilon \neq \vartheta$.

Theorem 4.24. *Let $q = p^m$ be prime power, d a positive integer, $\varepsilon, \vartheta \in \{0, \ldots, q-1\}$, $\varepsilon \neq \vartheta$, $r = (q^{d+1}-1)/(q-1)$, and L a Latin square over $\{1, \ldots, r+1\}$. Insert the elements of $\Xi_R(q,d,\varepsilon)$ and $\Xi_R(q,d,\vartheta)$ in the Latin square L to construct two $(0,1)$-matrices $N_R(\varepsilon)$ and $N_R(\vartheta)$, respectively. Then, the matrices $N(\varepsilon)$ and $N(\vartheta)$ are disjoint.*

Proof. Since we use the same Latin square for $N(\varepsilon)$ and $N(\vartheta)$, Lemma 4.19 implies the result. □

We summarize the main conclusions of this section in the following two theorems.

Theorem 4.25. *Let $q = p^m$ be a prime power, d a positive integer, and $r = \dfrac{q^{d+1}-1}{q-1}$. There is a decomposition of the complete design with $(r+1)q^{d+1}$ points into q symmetric $((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$-designs, a symmetric $((r+1)q^{d+1}, 1, 0)$-design, and a $DDG((r+1)q^{d+1}, q^{d+1}-1, q^{d+1}-2, 0, r+1, q^{d+1})$.*

Proof. For each $\varepsilon \in \{0, \ldots, q-1\}$, Theorem 4.11 (or Theorem 4.23) generates the incidence matrix N_ε of a symmetric $((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$-design. Therefore, we have a set $\{N_0, \ldots, N_{q-1}\}$ of incidence matrices of symmetric $((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$-designs. Suppose L is a constant diagonal Latin square. If we use the same Latin square L for all elements of the set $\{N_0, \ldots, N_{q-1}\}$, Theorem 4.12 (or Theorem 4.24) proves $\{N_0, \ldots, N_{q-1}\}$ is a set of disjoint matrices.

Since diagonal blocks of all elements in $\{N_0, \ldots, N_{q-1}\}$ are the zero block, then so are the diagonal blocks of $\sum_{i=0}^{q-1} N_i$. Since each matrix N_i has rq^d ones in each row and since $\{N_0, \ldots, N_{q-1}\}$ is a set of disjoint matrices, all off diagonal blocks of $\sum_{i=0}^{q-1} N_i$ are equal to $J_{q^{d+1}}$. Therefore, $\sum_{i=0}^{q-1} N_i = (J_{r+1} - I_{r+1}) \otimes J_{q^{d+1}}$.

Let $N_q = I_{(r+1)q^{d+1}}$ and $N_{q+1} = I_{r+1} \otimes (J_{q^{d+1}} - I_{q^{d+1}})$. Then, we have $\sum_{i=0}^{q+1} N_i = J_{(r+1)q^{d+1}}$. Moreover, the matrix N_q is a symmetric $((r+1)q^{d+1}, 1, 0)$-design and the matrix N_{q+1}

is a $DDG((r+1)q^{d+1}, q^{d+1}-1, q^{d+1}-2, 0, r+1, q^{d+1})$. Hence, $\{N_0, \ldots, N_{q+1}\}$ decomposes the complete design with $(r+1)q^{d+1}$ points into $q+1$ symmetric designs and a $DDG((r+1)q^{d+1}, q^{d+1}-1, q^{d+1}-2, 0, r+1, q^{d+1})$. □

Theorem 4.26. *Let $q = p^m$ be a prime power, d a positive integer, and $r = \dfrac{q^{d+1}-1}{q-1}$. If $r+1$ is an even number, there is a decomposition of the complete graph on $(r+1)q^{d+1}$ vertices into q strongly regular graphs $SRG((r+1)q^{d+1}, rq^d, (r-1)q^{d-1}, (r-1)q^{d-1})$ and a $DDG((r+1)q^{d+1}, q^{d+1}-1, q^{d+1}-2, 0, r+1, q^{d+1})$.*

Proof. Since $r+1$ is even, Lemma 3.8 confirms there is a symmetric constant diagonal Latin square L of size $r+1$. For each $\varepsilon \in \{0, \ldots, q-1\}$, Theorem 4.23 generates the incidence matrix N_ε of a symmetric $((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$-design. The matrix generated in Theorem 4.23 is symmetric with zero diagonal. Hence, it is the adjacency matrix of a $SRG((r+1)q^{d+1}, rq^d, (r-1)q^{d-1}, (r-1)q^{d-1})$. Thus, the set $\{N_0, \ldots, N_{q-1}\}$ is a set of adjacency matrices of $SRG((r+1)q^{d+1}, rq^d, (r-1)q^{d-1}, (r-1)q^{d-1})$.

Theorem 4.24 proves $\{N_0, \ldots, N_{q-1}\}$ is a set of disjoint matrices. Similar to the proof of Theorem 4.25, we have $\sum_{i=0}^{q-1} N_i = (J_{r+1} - I_{r+1}) \otimes J_{q^{d+1}}$. Let $N_q = I_{r+1} \otimes (J_{q^{d+1}} - I_{q^{d+1}})$. Then, the matrix N_q is a $DDG((r+1)q^{d+1}, q^{d+1}-1, q^{d+1}-2, 0, r+1, q^{d+1})$. Moreover, we have $\sum_{i=0}^{q} N_i = J_{(r+1)q^{d+1}} - I_{(r+1)q^{d+1}}$. Hence, $\{N_0, \ldots, N_q\}$ decomposes the complete graph on $(r+1)q^{d+1}$ vertices into q strongly regular graphs $SRG((r+1)q^{d+1}, rq^d, (r-1)q^{d-1}, (r-1)q^{d-1})$ and a $DDG((r+1)q^{d+1}, q^{d+1}-1, q^{d+1}-2, 0, r+1, q^{d+1})$. □

4.3 Constructing Symmetric Designs Sharing Complete Tripartite Graphs

In this section, we apply a similar process of Section 4.2 to generate three symmetric $(3^{d+1}(3^{d+1}-1)/2, 3^d(3^{d+1}+1)/2, 3^d(3^d+1)/2)$-designs whose incidence matrices share complete tripartite graphs. We start by modifying Definition 3.27.

Definition 4.27. Let d be a positive integer and $\varepsilon \in \{0, 1, 2\}$.

Suppose $GH(3,1)$, $GH(3,3)$, $\ldots$, $GH(3, 3^{d-1})$ are generalized Hadamard matrices over

$\Omega(3,1)$. We define

$$\Xi'(3,d,\varepsilon) = \Delta_\varepsilon(GH(3,1),d) \cup \ldots \cup \Delta_\varepsilon(GH(3,3^{d-1}),d) \cup \{J_{3^{d+1}} - (I_3 \otimes J_{3^d})\}.$$

The number of elements are the same in the two sets $\Xi'(3,d,\varepsilon)$ and $\{1,\ldots,r\}$. We define a bijective function $M'_\varepsilon : \{1,\ldots,r\} \to \Xi'(3,d,\varepsilon)$ such that $M'_\varepsilon(r) = J_{3^{d+1}} - (I_3 \otimes J_{3^d})$. Now we let $M'_\varepsilon(1)$ to be an element in $\Xi'(3,d,\varepsilon) \setminus \{M'_\varepsilon(r)\}$. Then inductively we assign $M'_\varepsilon(i+1)$ to an element in $\Xi'(3,d,\varepsilon) \setminus \{M'_\varepsilon(r), M'_\varepsilon(k) | k = 1,\ldots,i\}$.

For each $\varepsilon \in \{0,1,2\}$, a symmetric $(3^{d+1}(3^{d+1}-1)/2, 3^d(3^{d+1}+1)/2, 3^d(3^d+1)/2)$-design is constructed in the following theorem.

Theorem 4.28. *Let d be a positive integer, $r = (3^{d+1}-1)/2$, and L a constant diagonal Latin square over $\{1,\ldots,r\}$ with r on the diagonal. For each $\varepsilon \in \{0,1,2\}$, substitute i with $M'_\varepsilon(i)$ in the Latin square L to make a $(0,1)$-matrix $N(\varepsilon)$ of size $r(3^{d+1})$. Then, $N(\varepsilon)$ is the incidence matrix of a symmetric*

$$(3^{d+1}(3^{d+1}-1)/2, 3^d(3^{d+1}+1)/2, 3^d(3^d+1)/2)\text{-design.}$$

Then, all matrices in $\{N(0),N(1),N(2)\}$ share r disjoint complete tripartite graphs $K^3_{3^d}$.

Proof. Since we use the same Latin square for all designs, Lemma 4.9 proves that all matrices in $\{N(0),N(1),N(2)\}$ share the adjacency matrix $I_r \otimes (J_{3^{d+1}} - (I_3 \otimes J_{3^d}))$, which consists of r disjoint complete tripartite graphs $K^3_{3^d}$. □

Toward generating symmetric designs with symmetric zero diagonal incidence matrices, we make the following definition.

Definition 4.29. Let d be a positive integer and $\varepsilon \in \{0,1,2\}$. Suppose $GH(3,1)$, $GH(3,3)$, $\ldots$, $GH(3,3^{d-1})$ are generalized Hadamard matrices over $\Omega(3,1)$. We define

$$\Xi'_R(3,d,\varepsilon) = \Delta'_\varepsilon(GH(3,1),d) \cup \ldots \cup \Delta'_\varepsilon(GH(3,3^{d-1}),d) \cup \{J_{3^{d+1}} - (I_3 \otimes J_{3^d})\}.$$

Let $r=(3^{d+1}-1)/2$ be even and L be a symmetric constant diagonal Latin square of size r. By using L and the sets $\Xi'_R(3,d,0), \Xi'_R(3,d,1)$, and $\Xi'_R(3,d,2)$ in Theorem 4.28, we construct three $SRG(3^{d+1}(3^{d+1}-1)/2, 3^d(3^{d+1}+1)/2, 3^d(3^d+1)/2, 3^d(3^d+1)/2)$ whose adjacency matrices share r disjoint complete tripartite graphs.

Theorem 4.30. *Let d be a positive integer, and $r=(3^{d+1}-1)/2$ be an even number. There are three $SRG(3^{d+1}(3^{d+1}-1)/2, 3^d(3^{d+1}+1)/2, 3^d(3^d+1)/2, 3^d(3^d+1)/2)$ whose adjacency matrices share r disjoint complete tripartite graphs $K^3_{3^d}$.*

Example 4.31. In this example, we induce three $SRG(36,15,6,6)$ whose adjacency matrices share four complete tripartite graphs K^3_3. For this purpose, we set $q=3$ and $d=1$. Since $d=1$, we only need the generalized Hadamard matrix $GH(3,1)=\begin{bmatrix}\omega_0&\omega_0&\omega_0\\ \omega_0&\omega_1&\omega_2\\ \omega_0&\omega_2&\omega_1\end{bmatrix}$. We also note that $R_3=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$. Using Definition 4.29, we first define the sets $\Xi'_R(3,1,0)$, $\Xi'_R(3,1,1)$, and $\Xi'_R(3,1,2)$. We know $\Omega_R(3,1)=\{\omega_0R_3=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}, \omega_1R_3=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}, \omega_2R_3=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\}$. Three 9×9 matrices of the set $\Delta'_0(GH(3,1))$ are generated below:

$$r_1^Tr_1=\begin{bmatrix}\omega_0\omega_0\omega_0\\ \omega_0\omega_0\omega_0\\ \omega_0\omega_0\omega_0\end{bmatrix}=\begin{bmatrix}001001001\\010010010\\100100100\\001001001\\010010010\\100100100\\001001001\\010010010\\100100100\end{bmatrix}, r_2^Tr_2=\begin{bmatrix}\omega_0\omega_1\omega_2\\ \omega_1\omega_2\omega_0\\ \omega_2\omega_0\omega_1\end{bmatrix}=\begin{bmatrix}001010100\\010100001\\100001010\\010100001\\100001010\\001010100\\100001010\\001010100\\010100001\end{bmatrix}, r_3^Tr_3=\begin{bmatrix}\omega_0\omega_2\omega_1\\ \omega_2\omega_1\omega_0\\ \omega_1\omega_0\omega_2\end{bmatrix}=\begin{bmatrix}001100010\\010001100\\100010001\\100010001\\001100010\\010001100\\010001100\\100010001\\001100010\end{bmatrix}.$$

Therefore, we have

$$\Xi'_R(3,1,0)=\left\{\begin{bmatrix}001001001\\010010010\\100100100\\001001001\\010010010\\100100100\\001001001\\010010010\\100100100\end{bmatrix},\begin{bmatrix}001010100\\010100001\\100001010\\010100001\\100001010\\001010100\\100001010\\001010100\\010100001\end{bmatrix},\begin{bmatrix}001100010\\010001100\\100010001\\100010001\\001100010\\010001100\\010001100\\100010001\\001100010\end{bmatrix},\begin{bmatrix}000111111\\000111111\\000111111\\111000111\\111000111\\111000111\\111111000\\111111000\\111111000\end{bmatrix}\right\}.$$

Similarly, we have

$$\Xi'_R(3,1,1)=\left\{\begin{bmatrix}010010010\\100100100\\001001001\\010010010\\100100100\\001001001\\010010010\\100100100\\001001001\end{bmatrix},\begin{bmatrix}010100001\\100001010\\001010100\\100001010\\001010100\\010100001\\001010100\\010100001\\100001010\end{bmatrix},\begin{bmatrix}010001100\\100010001\\001100010\\001100010\\010001100\\100010001\\100010001\\001100010\\010001100\end{bmatrix},\begin{bmatrix}000111111\\000111111\\000111111\\111000111\\111000111\\111000111\\111111000\\111111000\\111111000\end{bmatrix}\right\}$$

and

$$\Xi'_R(3,1,2) = \left\{ \begin{bmatrix} 100100100 \\ 001001001 \\ 010010010 \\ 100100100 \\ 001001001 \\ 010010010 \\ 100100100 \\ 001001001 \\ 010010010 \end{bmatrix}, \begin{bmatrix} 100001010 \\ 001010100 \\ 010100001 \\ 001010100 \\ 010100001 \\ 100001010 \\ 010100001 \\ 100001010 \\ 001010100 \end{bmatrix}, \begin{bmatrix} 100010001 \\ 001100010 \\ 010001100 \\ 010001100 \\ 100010001 \\ 001100010 \\ 001100010 \\ 010001100 \\ 100010001 \end{bmatrix}, \begin{bmatrix} 000111111 \\ 000111111 \\ 000111111 \\ 111000111 \\ 111000111 \\ 111000111 \\ 111111000 \\ 111111000 \\ 111111000 \end{bmatrix} \right\}.$$

For each $\varepsilon = 0,1,2$, substitute i with $M'_\varepsilon(i)$ in $L = \begin{bmatrix} 4 & 1 & 2 & 3 \\ 1 & 4 & 3 & 2 \\ 2 & 3 & 4 & 1 \\ 3 & 2 & 1 & 4 \end{bmatrix}$ to make a $(0,1)$-matrix $N_i(\varepsilon)$ of size 36 which is the adjacency matrix of a $SRG(36,15,6,6)$. All matrices N_0, N_1, and N_2 share four disjoint complete tripartite graphs K_3^3. These three matrices are presented in Appendix 5.

Chapter 5

A New Series of Complete Design Decompositions

In [10], Ionin and Kharaghani used balanced generalized weighing matrices and symmetric designs to generate new families of strongly regular graphs. In this Chapter, we use their method to generate an infinite class of decompositions.

5.1 McFarland Designs and a Generalization

Let $\mathcal{M}$ be a non-empty set of matrices and let G be a cyclic group of bijections $\mathcal{M} \to \mathcal{M}$. Suppose $W = [w_{ij}]$ is a balanced generalized weighing matrix over G, then, for $D \in \mathcal{M}$, $W \otimes D$ represents the block matrix $[w_{ij}D]$, where, for $w_{ij} \in G$, $w_{ij}D$ is the image of D under the bijection w_{ij}, [10]. By the abuse of language we call the operation mentioned above the Kronecker product of W and D.

Theorem 5.1 ([10, Theorem 3.1, p. 312])**.** *Suppose that a nonempty set $\mathcal{M}$ of incidence matrices of symmetric (ν, k, λ)-designs and a finite group S of bijections $\mathcal{M} \to \mathcal{M}$ satisfy conditions:*

1. $(\sigma M)(\sigma N)^T = MN^T$ *for all* $M, N \in \mathcal{M}$ *and all* $\sigma \in S$.

2. *For each* $M \in \mathcal{M}$*, the matrix* $\sum_{\sigma \in S} \sigma M$ *is a multiple of* J_ν.

If W is a $BGW(w, l, \mu)$ over S with $k^2\mu = \nu\lambda l$, then, for $N \in \mathcal{M}$, $W \otimes N$, as defined above, is an incidence matrix of a symmetric $(\nu w, kl, \lambda l)$-design.

Proof. Let $W = [\alpha_{ij}]$. The (i, j)-block of size ν of $W \otimes N$ is $\alpha_{ij}N$. We need to prove

$$(W \otimes N)(W \otimes N)^T = (kl - \lambda l)I_{\nu w} + (\lambda l)J_{\nu w}. \tag{5.1}$$

We first calculate the matrix $(W \otimes N)(W \otimes N)^T = [M_{ij}]$, where M_{ij} is the (i, j)-block of size ν of the matrix $(W \otimes N)(W \otimes N)^T$. For each $i \in \{1, 2, \dots, w\}$, there exist $\sigma_1, \sigma_2, \dots, \sigma_l \in S$ such that

$$\begin{aligned} M_{ii} &= \sum_{j=1}^{w} (\alpha_{ij}N)(\alpha_{ij}N)^T = \sum_{j=1}^{l} (\sigma_j N)(\sigma_j N)^T = \sum_{j=1}^{l} NN^T = l(NN^T) \\ &= l((k - \lambda)I_\nu + (\lambda)J_\nu) = (kl - \lambda l)I_\nu + \lambda l J_\nu. \end{aligned}$$

Therefore, diagonal blocks of $(W \otimes N)(W \otimes N)^T$ satisfy (5.1). Since each row of any element in $\mathcal{M}$ has k ones, and since the matrix $\sum_{\sigma \in S} \sigma M$ is a multiple of J, we conclude that all entries of the matrix $\sum_{\sigma \in S} \sigma M$ are equal to $(k|S|)/\nu$. For $i, j \in \{1, 2, \dots, w\}$ with $i \neq j$, we have

$$\begin{aligned} M_{ij} &= \sum_{k=1}^{w} (\alpha_{ik}N)(\alpha_{jk}N)^T = \sum_{k=1}^{w} (\alpha_{ik}\alpha_{jk}^{-1}N)N^T \\ &= \frac{\mu}{|S|}\left(\sum_{\sigma \in S} \sigma N\right)N^T = \frac{\mu}{|S|}\left(\frac{k|S|}{\nu}J_\nu\right)M^T \\ &= \frac{k^2\mu}{\nu}J_\nu \\ &= \frac{\nu\lambda l}{\nu}J_\nu \\ &= (\lambda l)J_\nu. \end{aligned}$$

Hence, we proved $(W \otimes N)(W \otimes N)^T = (kl - \lambda l)I_{\nu w} + (\lambda l)J_{\nu w}$ and so $W \otimes N$ is the incidence matrix of a symmetric $(\nu w, kl, \lambda l)$-design. $\square$

Lemma 5.2. *Let $q = p^m$ be a prime power, d a positive integer, and $r = (q^{d+1} - 1)/(q - 1)$. Then,* $\gcd(r, q) = 1$. *Moreover,* $\gcd(r + 1, q) = 1$ *when q is odd.*

Proof. We have

$$r = \frac{q^{d+1}-1}{q-1} = q^d + \cdots + q + 1.$$

Hence, we conclude $\gcd(r,q) = 1$. Moreover, $\gcd(r+1,q) = 1$ when q is odd. $\square$

Let $q = p^m$ be a prime power, d a positive integer, $r = \dfrac{q^{d+1}-1}{q-1}$, $C_{r+1} = circ(0,1,0,\ldots,0)$ a circulant matrix of size $r+1$, and L a Latin square on $\{0,1,\ldots,r\}$. For each ε, let $\Xi(q,d,\varepsilon) = \{A^{(0)}(\varepsilon),\ldots,A^{(r)}(\varepsilon)\}$.

For $\varepsilon \neq \vartheta$, we correspond an element $A^{(i)}(\varepsilon)$ of $\Xi(q,d,\varepsilon)$ to an element $A^{(i)}(\vartheta)$ of $\Xi(q,d,\vartheta)$ on Page 47 of this thesis. (For $\varepsilon \neq \vartheta$, there is an isomorphism ϕ between $\Delta_\varepsilon(GH(q,q^\theta),d)$ and $\Delta_\vartheta(GH(q,q^\theta),d)$ which corresponds each element of $\Delta_\varepsilon(GH(q,q^\theta),d)$ to an element of $\Delta_\vartheta(GH(q,q^\theta),d)$. Let $A(\varepsilon) = [(I_{q^{\theta+1}} \otimes \omega_\varepsilon)(r^*_\alpha r_\alpha)] \otimes J_{q^{d-\theta-1}}$ be an arbitrary element of the set $\Delta_\varepsilon(GH(q,q^\theta),d)$, where $\alpha \in \{1,\ldots,q^{\theta+1}\}$. Then the image of $A(\varepsilon)$ under the isomorphism is equal to $A(\vartheta) = [(I_{q^{\theta+1}} \otimes \omega_\vartheta)(r^*_\alpha r_\alpha)] \otimes J_{q^{d-\theta-1}}$. The same row of $GH(q,q^\theta)$ is used in both $A(\varepsilon)$ and $A(\vartheta)$.)

The set $\{A^{(i)}(0),\ldots,A^{(i)}(q-1)\}$ is a set of disjoint $(0,1)$-matrices by Lemma 4.9. Similarly, let $\Xi_R(q,d,\varepsilon) = \{A_R^{(0)}(\varepsilon),\ldots,A_R^{(r)}(\varepsilon)\}$, then, the set $\{A_R^{(i)}(0),\ldots,A_R^{(i)}(q-1)\}$ is a set of disjoint $(0,1)$-matrices by Lemma 4.19.

Now, we use the notation $\Upsilon(\vartheta,l,L)$ to show the symmetric design made by replacing i with $A^{(i)}(\vartheta)$ in the Latin square $C_{r+1}^l L$. Moreover, let $\Upsilon_R(\vartheta,l,L)$ be the symmetric design made by replacing i with $A_R^{(i)}(\vartheta)$ in $C_{r+1}^l L$. Then, an action g on $\Upsilon(\vartheta,l,L)$ is defined by

$$g(\Upsilon(\vartheta,l,L)) = \Upsilon(\vartheta+1 \pmod q, l+1 \pmod{r+1}, L).$$

We use the notation $G(q,d)$ to denote the cyclic group generated by g. We note that

$$g^2(\Upsilon(\vartheta,l,L)) = \Upsilon(\vartheta+2 \pmod q, l+2 \pmod{r+1}, L),$$

$$g^3(\Upsilon(\vartheta,l,L)) = \Upsilon(\vartheta+3 \pmod q), l+3 \pmod{r+1}, L),$$

$$g^4(\Upsilon(\vartheta,l,L)) = \Upsilon(\vartheta+4 \pmod q), l+4 \pmod{r+1}, L),$$

$$\vdots$$

$$g^t(\Upsilon(\vartheta,l,L)) = \Upsilon(\vartheta+t \pmod q), l+t \pmod{r+1}, L).$$

One may note that

$$g^{q(r+1)}(\Upsilon(\vartheta,l,L)) = \Upsilon(\vartheta+q(r+1) \pmod q), l+q(r+1) \pmod{r+1}, L) = \Upsilon(\vartheta,l,L).$$

Since $gcd(q,r+1)=1$, we conclude that $|G(q,d)| = q(r+1)$ and

$$G(q,d) =< g >= \{g^t | 1 \leq t \leq q(r+1)\}.$$

Similarly, we define

$$g(\Upsilon_R(\vartheta,l,L)) = \Upsilon_R(\vartheta+1 \pmod q), l+1 \pmod{r+1}, L).$$

Example 5.3. From Example 4.13, we have

$$\Xi(3,1,0) =$$

$$\left\{ \begin{bmatrix} 000000000 \\ 000000000 \\ 000000000 \\ 000000000 \\ 000000000 \\ 000000000 \\ 000000000 \\ 000000000 \\ 000000000 \end{bmatrix}, \begin{bmatrix} 111000000 \\ 111000000 \\ 111000000 \\ 000111000 \\ 000111000 \\ 000111000 \\ 000000111 \\ 000000111 \\ 000000111 \end{bmatrix}, \begin{bmatrix} 100100100 \\ 010010010 \\ 001001001 \\ 100100100 \\ 010010010 \\ 001001001 \\ 100100100 \\ 010010010 \\ 001001001 \end{bmatrix}, \begin{bmatrix} 100001010 \\ 010100001 \\ 001010100 \\ 010100001 \\ 001010100 \\ 100001010 \\ 001010100 \\ 100001010 \\ 010100001 \end{bmatrix}, \begin{bmatrix} 100010001 \\ 010001100 \\ 001100010 \\ 001100010 \\ 100010001 \\ 010001100 \\ 010001100 \\ 001100010 \\ 100010001 \end{bmatrix} \right\}$$

and

$$\Xi(3,1,1) =$$

$$\left\{ \begin{bmatrix} 000000000\\ 000000000\\ 000000000\\ 000000000\\ 000000000\\ 000000000\\ 000000000\\ 000000000\\ 000000000 \end{bmatrix}, \begin{bmatrix} 000111000\\ 000111000\\ 000111000\\ 000000111\\ 000000111\\ 000000111\\ 111000000\\ 111000000\\ 111000000 \end{bmatrix}, \begin{bmatrix} 010010010\\ 001001001\\ 100100100\\ 010010010\\ 001001001\\ 100100100\\ 010010010\\ 001001001\\ 100100100 \end{bmatrix}, \begin{bmatrix} 010100001\\ 001010100\\ 100001010\\ 001010100\\ 100001010\\ 010100001\\ 100001010\\ 010100001\\ 001010100 \end{bmatrix}, \begin{bmatrix} 010001100\\ 001100010\\ 100010001\\ 100010001\\ 010001100\\ 001100010\\ 001100010\\ 100010001\\ 010001100 \end{bmatrix} \right\}.$$

For elements of $\Xi(3,1,0)$, we have

$$A^{(0)}(0) = \begin{bmatrix} 000000000\\ 000000000\\ 000000000\\ 000000000\\ 000000000\\ 000000000\\ 000000000\\ 000000000\\ 000000000 \end{bmatrix},\ A^{(1)}(0) = \begin{bmatrix} 111000000\\ 111000000\\ 111000000\\ 000111000\\ 000111000\\ 000111000\\ 000000111\\ 000000111\\ 000000111 \end{bmatrix},\ A^{(2)}(0) = \begin{bmatrix} 100100100\\ 010010010\\ 001001001\\ 100100100\\ 010010010\\ 001001001\\ 100100100\\ 010010010\\ 001001001 \end{bmatrix},$$

$$A^{(3)}(0) = \begin{bmatrix} 100001010\\ 010100001\\ 001010100\\ 010100001\\ 001010100\\ 100001010\\ 001010100\\ 100001010\\ 010100001 \end{bmatrix},\ A^{(4)}(0) = \begin{bmatrix} 100010001\\ 010001100\\ 001100010\\ 001100010\\ 100010001\\ 010001100\\ 010001100\\ 001100010\\ 100010001 \end{bmatrix}.$$

For elements of $\Xi(3,1,1)$, we have

$$A^{(0)}(1) = \begin{bmatrix} 000000000\\ 000000000\\ 000000000\\ 000000000\\ 000000000\\ 000000000\\ 000000000\\ 000000000\\ 000000000 \end{bmatrix},\ A^{(1)}(1) = \begin{bmatrix} 000111000\\ 000111000\\ 000111000\\ 000000111\\ 000000111\\ 000000111\\ 111000000\\ 111000000\\ 111000000 \end{bmatrix},\ A^{(2)}(1) = \begin{bmatrix} 010010010\\ 001001001\\ 100100100\\ 010010010\\ 001001001\\ 100100100\\ 010010010\\ 001001001\\ 100100100 \end{bmatrix},$$

$$A^{(3)}(1) = \begin{bmatrix} 010100001\\ 001010100\\ 100001010\\ 001010100\\ 100001010\\ 010100001\\ 100001010\\ 010100001\\ 001010100 \end{bmatrix},\ A^{(4)}(1) = \begin{bmatrix} 010001100\\ 001100010\\ 100010001\\ 100010001\\ 010001100\\ 001100010\\ 001100010\\ 100010001\\ 010001100 \end{bmatrix}.$$

Let $C_5 = circ(0,1,0,0,0)$. If $L = \begin{bmatrix} 0&1&2&3&4\\ 4&0&1&2&3\\ 3&4&0&1&2\\ 2&3&4&0&1\\ 1&2&3&4&0 \end{bmatrix}$, we have $C_5 L = \begin{bmatrix} 4&0&1&2&3\\ 3&4&0&1&2\\ 2&3&4&0&1\\ 1&2&3&4&0\\ 0&1&2&3&4 \end{bmatrix}$. Then

$$g(\Upsilon(0,0,L)) = \Upsilon(1,1,L),$$

where

$$\Upsilon(0,0,L) = \begin{bmatrix} A^{(0)}(0) & A^{(1)}(0) & A^{(2)}(0) & A^{(3)}(0) & A^{(4)}(0) \\ A^{(4)}(0) & A^{(0)}(0) & A^{(1)}(0) & A^{(2)}(0) & A^{(3)}(0) \\ A^{(3)}(0) & A^{(4)}(0) & A^{(0)}(0) & A^{(1)}(0) & A^{(2)}(0) \\ A^{(2)}(0) & A^{(3)}(0) & A^{(4)}(0) & A^{(0)}(0) & A^{(1)}(0) \\ A^{(1)}(0) & A^{(2)}(0) & A^{(3)}(0) & A^{(4)}(0) & A^{(0)}(0) \end{bmatrix}$$

and

$$\Upsilon(1,1,L) = \begin{bmatrix} A^{(4)}(1) & A^{(0)}(1) & A^{(1)}(1) & A^{(2)}(1) & A^{(3)}(1) \\ A^{(3)}(1) & A^{(4)}(1) & A^{(0)}(1) & A^{(1)}(1) & A^{(2)}(1) \\ A^{(2)}(1) & A^{(3)}(1) & A^{(4)}(1) & A^{(0)}(1) & A^{(1)}(1) \\ A^{(1)}(1) & A^{(2)}(1) & A^{(3)}(1) & A^{(4)}(1) & A^{(0)}(1) \\ A^{(0)}(1) & A^{(1)}(1) & A^{(2)}(1) & A^{(3)}(1) & A^{(4)}(1) \end{bmatrix}.$$

Definition 5.4. Let $q = p^m$ be a prime power, d a positive integer, $r = \dfrac{q^{d+1}-1}{q-1}$, L a Latin square over $\{0,\ldots,r\}$, and $C_{r+1} = circ(0,1,0,\ldots,0)$ a circulant matrix of size $r+1$. Then, we use the notation $\mathcal{M}(q,d,L)$ to denote the set

$$\{\Upsilon(\vartheta,t,L) | \vartheta \in \{0,\ldots,q-1\}, 0 \le t \le r\}.$$

Also we use the notation $\mathcal{M}_R(q,d,L)$ to denote the set

$$\{\Upsilon_R(\vartheta,t,L) | \vartheta \in \{0,\ldots,q-1\}, 0 \le t \le r\}.$$

Example 5.5. In this Example, we generate $\mathcal{M}(3,1,L)$, where $L = \begin{bmatrix} 0&1&2&3&4 \\ 4&0&1&2&3 \\ 3&4&0&1&2 \\ 2&3&4&0&1 \\ 1&2&3&4&0 \end{bmatrix}$. We have $C_5^0 L = L$, $C_5 L = \begin{bmatrix} 4&0&1&2&3 \\ 3&4&0&1&2 \\ 2&3&4&0&1 \\ 1&2&3&4&0 \\ 0&1&2&3&4 \end{bmatrix}$, $C_5^2 L = \begin{bmatrix} 3&4&0&1&2 \\ 2&3&4&0&1 \\ 1&2&3&4&0 \\ 0&1&2&3&4 \\ 4&0&1&2&3 \end{bmatrix}$, $C_5^3 L = \begin{bmatrix} 2&3&4&0&1 \\ 1&2&3&4&0 \\ 0&1&2&3&4 \\ 4&0&1&2&3 \\ 3&4&0&1&2 \end{bmatrix}$, and $C_5^4 L = \begin{bmatrix} 1&2&3&4&0 \\ 0&1&2&3&4 \\ 4&0&1&2&3 \\ 3&4&0&1&2 \\ 2&3&4&0&1 \end{bmatrix}$, where $C_5 = circ(0,1,0,0,0)$. We know $\Upsilon_R(\vartheta,t,L)$ is the symmetric $(45,12,3)$-design made by replacing i with $A^{(i)}(\vartheta)$ in the Latin square $C^t L$, and we have

$$\mathcal{M}(3,1,L) = \{\Upsilon_R(\vartheta,t,L) | \vartheta \in \{0,1,2\}, 0 \le t \le 4\}.$$

The set $\mathcal{M}(3,1,L)$ consists of fifteen symmetric $(45,12,3)$-designs whose incidence matrices appear in Appendix 6.

The following theorem shows how $G(q,d)$ acts on two sets $\mathcal{M}(q,d,L)$ and $\mathcal{M}_R(q,d,L)$.

Theorem 5.6. *Let $q = p^m$ be a prime power, d a positive integer, $r = \dfrac{q^{d+1}-1}{q-1}$, and L a circulant Latin square over $\{0,\dots,r\}$. Then, the group $G(q,d)$ (Defined on Page 63) is a cyclic group of bijections on $\mathcal{M}(q,d,L)$, and also is a cyclic group of bijections on $\mathcal{M}_R(q,d,L)$.*

Proof. Let $\sigma = g^t$ be an arbitrary element in $G(q,d)$, for some $0 \le t \le r$. Then, the image of $\Upsilon(\vartheta,l,L)$ under σ is $\Upsilon(\vartheta + t \pmod q, l + t \pmod{r+1}, L)$. It follows that σ is a cyclic bijection from $\mathcal{M}(q,d,L)$ to $\mathcal{M}(q,d,L)$. Hence, the group $G(q,d)$ is a cyclic group of bijections from $\mathcal{M}(q,d,L)$ to $\mathcal{M}(q,d,L)$. Similarly, it is possible to prove that the group $G(q,d)$ is also a cyclic group of bijections from $\mathcal{M}_R(q,d,L)$ to $\mathcal{M}_R(q,d,L)$. □

We have the following lemma.

Lemma 5.7. *Let $q = p^m$ be a prime power, d a positive integer, $\varepsilon,\vartheta \in \{0,\dots,q-1\}$, $\vartheta \neq \varepsilon$, and $GH(q,q^\theta)$ a generalized Hadamard matrix over $\Omega(p,m) = \{\omega_0,\omega_1,\dots,\omega_{p^m-1}\}$, where $0 \le \theta \le d-1$. The matrices $A \in \Xi(q,d,\varepsilon)$ and $B \in \Xi(q,d,\vartheta)$ satisfy the following conditions:*

1. *If $A(\vartheta) = B$, we have $AB^T = q^d A(\varepsilon - \vartheta)$;*

2. *If $A(\vartheta) \neq B$, we have $AB^T = q^{d-1} J_{q^{d+1}}$.*

Similarly, for the matrices $A_R \in \Xi_R(q,d,\varepsilon)$ and $B_R \in \Xi_R(q,d,\vartheta)$ we have:

(i). If $A_R(\vartheta) = B_R$, we have $A_R B_R^T = q^d A(\varepsilon - \vartheta)$;

(ii). If $A_R(\vartheta) \neq B_R$, we have $A_R B_R^T = q^{d-1} J_{q^{d+1}}$.

The following theorem shows $G(q,d)$ and $\mathcal{M}(q,d,L)$ satisfy the 1st and 2nd conditions of Theorem 5.1 if L is circulant.

Theorem 5.8. *Let $q = p^m$ be a prime power, d a positive integer, $r = \dfrac{q^{d+1}-1}{q-1}$, and L a circulant Latin square over $\{0,\dots,r\}$. Then, we have*

1. $(\sigma M)(\sigma N)^T = MN^T$ *for all* $M,N \in \mathcal{M}(q,d,L)$ *and all* $\sigma \in G(q,d)$.

2. *For each* $M \in \mathcal{M}(q,d,L)$, *the matrix* $\sum_{\sigma\in G(q,d)} \sigma M$ *is a multiple of* $J_{(r+1)q^{d+1}}$.

Proof. 1. For arbitrary elements $M,N \in \mathcal{M}(q,d,L)$ and $\sigma \in G(q,d)$, let $M = \Upsilon(\varepsilon,s,L)$, $N = \Upsilon(\vartheta,t,L)$, and $\sigma = g^u$, for some $\vartheta,\varepsilon \in \{0,\ldots,q-1\}$, $s,t \in \{0,\ldots,r\}$, and $u \in \{0,\ldots,q(r+1)-1\}$. Note that

$$\Upsilon(\varepsilon,s,L) = (C^s \otimes I_{q^{d+1}})\Upsilon(\varepsilon,0,L)$$

and

$$\Upsilon(\varepsilon,s,L) = \Upsilon(\varepsilon,0,L)(C^s \otimes I_{q^{d+1}}).$$

We first calculate $\Upsilon(\varepsilon,0,L)\Upsilon(\vartheta,0,L)^T = [M_{ij}]$, where M_{ij} is (i,j)-block of size q^{d+1} of the matrix $\Upsilon(\varepsilon,0,L)\Upsilon(\vartheta,0,L)^T$. By Lemma 5.7, we have

$$\begin{aligned} M_{ii} = &\sum_{\substack{A(\varepsilon)\in\Xi(q,d,\varepsilon)\\ A(\vartheta)\in\Xi(q,d,\vartheta)}} A(\varepsilon)A(\vartheta) = q^d(\omega_{\varepsilon-\vartheta}\otimes J_{q^d})\\ &+q^d \sum_{A(\varepsilon-\vartheta)\in\Xi(q,d,\varepsilon-\vartheta)} A(\varepsilon-\vartheta)\\ &= q^d(\omega_{\varepsilon-\vartheta}\otimes J_{q^d})\\ &+q^d\sum_{\theta=0}^{d-1}\left[q^{\theta+1}(I_{q^{\theta+1}}\otimes\omega_{\varepsilon-\vartheta}\otimes J_{q^{d-\theta-1}})+q^{\theta}((J_{q^{\theta+1}}-I_{q^{\theta+1}})\otimes J_{q^{d-\theta}})\right]. \end{aligned}$$

For $i \neq j$, Lemma 5.7 shows that $M_{ij} = (r-1)q^{d-1}J_{q^{d+1}}$. From this, we conclude

$$\left[\Upsilon(\varepsilon_1,0,L)\Upsilon(\vartheta_1,0,L)^T = \Upsilon(\varepsilon_2,0,L)\Upsilon(\vartheta_2,0,L)^T\right] \Leftrightarrow \left[\varepsilon_1-\vartheta_1 = \varepsilon_2-\vartheta_2\right].$$

Now, we have

$$(\sigma M)(\sigma N)^T = \sigma(\Upsilon(\varepsilon,s,L))(\sigma(\Upsilon(\vartheta,t,L)))^T = \Upsilon(\varepsilon+u,s+u,L)\Upsilon(\vartheta+u,t+u,L)^T$$

$$
\begin{aligned}
&= \big((C^{s+u}\otimes I_{q^{d+1}})\Upsilon(\varepsilon+u,0,L)\big)\big((C^{t+u}\otimes I_{q^{d+1}})\Upsilon(\vartheta+u,0,L)\big)^T \\
&= ((C^{s+u}\otimes I_{q^{d+1}})\Upsilon(\varepsilon+u,0,L))\Upsilon(\vartheta+u,0,L)^T(C^{-t-u}\otimes I_{q^{d+1}}) \\
&= (C^{s+u}\otimes I_{q^{d+1}})(C^{-t-u}\otimes I_{q^{d+1}}))\Upsilon(\varepsilon+u,0,L)\Upsilon(\vartheta+u,0,L)^T \\
&= (C^{s-t}\otimes I_{q^{d+1}})\Upsilon(\varepsilon+u,0,L)\Upsilon(\vartheta+u,0,L)^T \\
&= (C^{s-t}\otimes I_{q^{d+1}})\Upsilon(\varepsilon,0,L)\Upsilon(\vartheta,0,L)^T \\
&= (C^{s}\otimes I_{q^{d+1}})\Upsilon(\varepsilon,0,L)((C^{t}\otimes I_{q^{d+1}})\Upsilon(\vartheta,0,L))^T \\
&= \Upsilon(\varepsilon,s,L)\Upsilon(\vartheta,t,L)^T \\
&= MN^T.
\end{aligned}
$$

2. Let $M=\Upsilon(\varepsilon,s,L)$, for some $\varepsilon\in\{0,\ldots,q-1\}$ and $0\le s\le r$. We have

$$
\begin{aligned}
\sum_{\sigma\in G(q,d)}\sigma M &= \sum_{\sigma\in G(q,d)}\sigma\Upsilon(\varepsilon,s,L) \\
&= \sum_{\rho\in\{0,\ldots,q-1\}}\sum_{u=0}^{r}\Upsilon(\varepsilon+\rho,s+u,L) \\
&= \sum_{N\in\mathcal{M}(q,d,L)} N.
\end{aligned}
$$

Let M_{ij} be the (i,j)-block of size q^{d+1} of the matrix $\sum_{N\in\mathcal{M}(q,d,L)}N$. By Definition 5.4 and Lemma 4.10, we have

$$
\begin{aligned}
M_{ij} &= \sum_{\varepsilon\in\{0,\ldots,q-1\}}\sum_{A\in\Xi(q,d,\varepsilon)} A \\
&= \sum_{\varepsilon\in\{0,\ldots,q-1\}}\left(\omega_\varepsilon\otimes J_{q^d}+\sum_{\theta=0}^{d-1}\left[q^{\theta+1}(I_{q^{\theta+1}}\otimes\omega_\varepsilon\otimes J_{q^{d-\theta-1}})+q^{\theta}((J_{q^{\theta+1}}-I_{q^{\theta+1}})\otimes J_{q^{d-\theta}})\right]\right) \\
&= (\sum_{\varepsilon\in\{0,\ldots,q-1\}}\omega_\varepsilon)\otimes J_{q^d}+\sum_{\theta=0}^{d-1}\left[q^{\theta+1}(I_{q^{\theta+1}}\otimes(\sum_{\varepsilon\in\{0,\ldots,q-1\}}\omega_\varepsilon)\otimes J_{q^{d-\theta-1}})\right. \\
&\quad \left. +q^{\theta+1}((J_{q^{\theta+1}}-I_{q^{\theta+1}})\otimes J_{q^{d-\theta}})\right] \\
&= J_q\otimes J_{q^d}+\sum_{\theta=0}^{d-1}\left[q^{\theta+1}(I_{q^{\theta+1}}\otimes J_q\otimes J_{q^{d-\theta-1}})+q^{\theta+1}((J_{q^{\theta+1}}-I_{q^{\theta+1}})\otimes J_{q^{d-\theta}})\right]
\end{aligned}
$$

$$= J_{q^{d+1}} + (\sum_{\theta=0}^{d-1} q^{\theta+1})J_{q^{d+1}}$$

$$= (\sum_{\theta=0}^{d} q^{\theta})J_{q^{d+1}}$$

$$= rJ_{q^{d+1}}.$$

□

The following theorem shows $G(q,d)$ and $\mathcal{M}_R(q,d,L)$ satisfy the 1st and 2nd conditions of Theorem 5.1 if L is back circulant. The proof of the following theorem is similar to the proof of Theorem 5.8.

Theorem 5.9. *Let $q = p^m$ be a prime power, d a positive integer, $r = \frac{q^{d+1}-1}{q-1}$, and L be a back circulant Latin square over $\{0,\dots,r\}$. Then, we have*

1. $(\sigma M)(\sigma N)^T = MN^T$ *for all $M,N \in \mathcal{M}_R(q,d,L)$ and all $\sigma \in G(q,d)$.*

2. *For each $M \in \mathcal{M}_R(q,d,L)$, the matrix $\sum_{\sigma\in G(q,d)} \sigma M$ is a multiple of $J_{(r+1)q^{d+1}}$.*

To be able to apply Theorem 5.1 for the set $\mathcal{M}(q,d,L)$, we review balanced generalized weighing matrices. As it is mentioned in [10], most of the known balanced generalized weighing matrices belong to the family $BGW(\frac{q^{d+1}-1}{q-1}, q^d, q^d - q^{d-1})$ over G, where q is a prime power, d is positive, and G is a cyclic group whose order divides $q-1$. In the following three theorems, we study balanced generalized weighing matrices.

Theorem 5.10 ([11, Proposition 10.1.17, p. 326])**.** *Let q be a prime power. Then, there is a zero diagonal $BGW(q+1,q,q-1)$ over a cyclic group G whose order $|G|$ divides $q-1$.*

Theorem 5.11 ([14, Lemma 1, p. 141])**.** *Let q be a prime power. There exists a symmetric zero diagonal $BGW(q+1,q,q-1)$ over a cyclic group G if $\frac{q-1}{|G|}$ is an even integer.*

Theorem 5.12 ([10, Theorem 2.3, p. 211])**.** *Let q be a prime power, n a divisor of $q-1$, and G a cyclic group of order n. If $\frac{q-1}{n}$ is even, then, for any positive integer d, there* *exists a symmetric balanced generalized weighing matrix*

$$BGW\left(\frac{q^{2d}-1}{q-1}, q^{2d-1}, q^{2d-1}-q^{2d-2}\right)$$

over G with all diagonal entries equal to 0.

In the following theorem, we show how any balanced generalized weighing matrices and the set $\mathcal{M}(q,d,L)$ form a symmetric design. Although the following theorem is not mentioned in [10], it is directly follows from [10, Theorem 5.1, p. 216].

Theorem 5.13. *Let q be an odd prime power and d a positive integer. If $r = \dfrac{q^{d+1}-1}{q-1}$ is a prime power, then, for any positive integer m, there exists a symmetric design with parameters*

$$\left(\frac{q^{d+1}(r^{4m}-1)}{r-1}, q^d r^{4m-1}, q^{d-1} r^{4m-2}(r-1)\right).$$

Proof. Let L be a circulant Latin square over $\{0,\dots,r\}$. Consider the set $\mathcal{M}(q,d,L)$ consisting of $q(r+1)$ symmetric $((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$-designs. Theorem 5.6 shows that the group $G(q,d)$ is a cyclic group of bijections $\mathcal{M}(q,d,L) \to \mathcal{M}(q,d,L)$. By Theorem 5.8, the group $G(q,d)$ and the set $\mathcal{M}(q,d,L)$ satisfy the first and second conditions of Theorem 5.1. For any positive integer m, Theorem 5.12 guarantees the existence of a symmetric zero diagonal BGW over the group $G(q,d)$ with parameters

$$BGW\left(\frac{r^{4m}-1}{r^2-1}, r^{4m-2}, r^{4m-2}-r^{4m-4}\right).$$

Let $(\nu,k,\lambda) = ((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$ and $(w,l,\mu) = (\dfrac{r^{4m}-1}{r^2-1}, r^{4m-2}, r^{4m-2}-r^{4m-4})$. To be able to use Theorem 5.1, we need to prove $k^2\mu = \nu\lambda l$:

$$\begin{aligned}
k^2\mu &= r^2 q^{2d}(r^{4m-2}-r^{4m-4}) \\
&= r^{4m-2} q^{2d}(r^2-1) \\
&= (r+1)q^{d+1}((r-1)q^{d-1})r^{4m-2}
\end{aligned}$$

$$= \nu\lambda l.$$

From Theorem 5.1, we conclude that there is a symmetric $(\nu w, kl, \lambda l)$-design or a symmetric $\left(\frac{q^{d+1}(r^{4m}-1)}{r-1}, q^d r^{4m-1}, q^{d-1} r^{4m-2}(r-1)\right)$-design. $\square$

Utilizing a similar proof to the proof of Theorem 5.13, we can prove the following theorem.

Theorem 5.14 ([10, Theorem 5.1, p. 216])**.** *Let q be an odd prime power and d a positive integer. If $r = \frac{q^{d+1}-1}{q-1}$ is a prime power, then, for any positive integer m, there exists a strongly regular graph with parameters*

$$\left(\frac{q^{d+1}(r^{4m}-1)}{r-1}, q^d r^{4m-1}, q^{d-1} r^{4m-2}(r-1), q^{d-1} r^{4m-2}(r-1)\right).$$

5.2 Spence Designs and a Generalization

Let d be a positive integer, $r = \frac{3^{d+1}-1}{2}$, and $\Omega(3,1) = \{\omega_0, \omega_1, \omega_2\}$. In Section 5.1, we use McFarland Designs and balanced generalized weighing matrices to make more symmetric designs. In this section, we apply a similar technique on the Spence designs made in Theorem 4.28. By Definition 4.27, the matrix $J_{3^{d+1}} - (I_3 \otimes J_{3^d})$ is common in all sets $\Xi'(3,d,0)$, $\Xi'(3,d,1)$, and $\Xi'(3,d,2)$. Since we also need the matrices $J_{3^{d+1}} - (\omega_1 \otimes J_{3^d})$ and $J_{3^{d+1}} - (\omega_2 \otimes J_{3^d})$, we define three sets as follow:

$$\Lambda(d,\varepsilon) = \Delta_\varepsilon(GH(3,1),d) \cup \ldots \cup \Delta_\varepsilon(GH(3,3^{d-1}),d) \cup \{J_{3^{d+1}} - (\omega_\varepsilon \otimes J_{3^d})\},$$

for each $\varepsilon \in \{0,1,2\}$. One may note that $\Lambda(d,0) = \Xi'(3,d,0)$, and for $\varepsilon \in \{0,1,2\} \setminus \{0\}$, we have

$$\Lambda(d,\varepsilon) = (\Xi'(3,d,0) \setminus \{J_{3^{d+1}} - (I_3 \otimes J_{3^d})\}) \cup \{J_{3^{d+1}} - (\omega_\varepsilon \otimes J_{3^d})\}.$$

Similarly, we have:

$$\Lambda_R(d,\varepsilon) = \Delta'_\varepsilon(GH(3,1),d) \cup \ldots \cup \Delta'_\varepsilon(GH(3,3^{d-1}),d) \cup \{J_{3^{d+1}} - (\omega_\varepsilon R_3 \otimes J_{3^d})\},$$

for each $\varepsilon \in \{0,1,2\}$.

Example 5.15.

$$\Lambda(1,0) = \left\{ \begin{bmatrix} 000111111 \\ 000111111 \\ 000111111 \\ 111000111 \\ 111000111 \\ 111000111 \\ 111111000 \\ 111111000 \\ 111111000 \end{bmatrix}, \begin{bmatrix} 100100100 \\ 010010010 \\ 001001001 \\ 100100100 \\ 010010010 \\ 001001001 \\ 100100100 \\ 010010010 \\ 001001001 \end{bmatrix}, \begin{bmatrix} 100001010 \\ 010100001 \\ 001010100 \\ 010100001 \\ 001010100 \\ 100001010 \\ 001010100 \\ 100001010 \\ 010100001 \end{bmatrix}, \begin{bmatrix} 100010001 \\ 010001100 \\ 001100010 \\ 001100010 \\ 100010001 \\ 010001100 \\ 010001100 \\ 001100010 \\ 100010001 \end{bmatrix} \right\},$$

$$\Lambda(1,1) = \left\{ \begin{bmatrix} 111000111 \\ 111000111 \\ 111000111 \\ 111111000 \\ 111111000 \\ 111111000 \\ 000111111 \\ 000111111 \\ 000111111 \end{bmatrix}, \begin{bmatrix} 010010010 \\ 001001001 \\ 100100100 \\ 010010010 \\ 001001001 \\ 100100100 \\ 010010010 \\ 001001001 \\ 100100100 \end{bmatrix}, \begin{bmatrix} 010100001 \\ 001010100 \\ 100001010 \\ 001010100 \\ 100001010 \\ 010100001 \\ 100001010 \\ 010100001 \\ 001010100 \end{bmatrix}, \begin{bmatrix} 010001100 \\ 001100010 \\ 100010001 \\ 100010001 \\ 010001100 \\ 001100010 \\ 001100010 \\ 100010001 \\ 010001100 \end{bmatrix} \right\},$$

$$\Lambda(1,2) = \left\{ \begin{bmatrix} 111111000 \\ 111111000 \\ 111111000 \\ 000111111 \\ 000111111 \\ 000111111 \\ 111000111 \\ 111000111 \\ 111000111 \end{bmatrix}, \begin{bmatrix} 001001001 \\ 100100100 \\ 010010010 \\ 001001001 \\ 100100100 \\ 010010010 \\ 001001001 \\ 100100100 \\ 010010010 \end{bmatrix}, \begin{bmatrix} 001010100 \\ 100001010 \\ 010100001 \\ 100001010 \\ 010100001 \\ 001010100 \\ 010100001 \\ 001010100 \\ 100001010 \end{bmatrix}, \begin{bmatrix} 001100010 \\ 100010001 \\ 010001100 \\ 010001100 \\ 001100010 \\ 100010001 \\ 100010001 \\ 010001100 \\ 001100010 \end{bmatrix} \right\}.$$

Let d be a positive integer, $r = \dfrac{3^{d+1}-1}{2}$, $C_r = circ(0,1,0,\ldots,0)$ a circulant matrix of size r, and L a Latin square on $\{0,\ldots,r-1\}$. For each ε, let

$$\Lambda(d,\varepsilon) = \{A^{(0)}(\varepsilon),\ldots,A^{(r-1)}(\varepsilon)\},$$

and

$$\Lambda_R(d,\varepsilon) = \{A_R^{(0)}(\varepsilon),\ldots,A_R^{(r-1)}(\varepsilon)\}.$$

Now, we use the notation $\Upsilon'(\varepsilon,l,L)$ to show the symmetric design made by replacing i with $A^{(i)}(\varepsilon)$ in the Latin square $C_r^l L$. Moreover, let $\Upsilon_R'(\varepsilon,l,L)$ be the symmetric design made by replacing i with $A_R^{(i)}(\varepsilon)$ in $C_r^l L$. Then, an action g on $\Upsilon'(\varepsilon,l,L)$ is defined by

$$g(\Upsilon'(\varepsilon,l,L)) = \Upsilon'(\varepsilon+1 \pmod 3, l+1 \pmod r, L).$$

We use the notation $G'(3,d)$ to denote the cyclic group generated by g. We note that

$$g^2(\Upsilon'(\varepsilon,l,L)) = \Upsilon'(\varepsilon+2 \pmod 3, l+2 \pmod r, L),$$

$$g^3(\Upsilon'(\varepsilon,l,L)) = \Upsilon'(\varepsilon+3 \pmod 3, l+3 \pmod r, L),$$

$$\vdots$$

$$g^t(\Upsilon'(\varepsilon,l,L)) = \Upsilon'(\varepsilon+t \pmod 3, l+t \pmod r, L).$$

It can be seen

$$g^{3r}(\Upsilon'(\varepsilon,l,L)) = \Upsilon'(\varepsilon+3r \pmod 3, l+3r \pmod r, L) = \Upsilon'(\varepsilon,l,L).$$

Since $gcd(3,r) = 1$, we conclude that $|G'(3,d)| = 3r$ and

$$G'(3,d) =< g >= \{g^t | 1 \le t \le 3r\}.$$

Similarly, we define

$$g(\Upsilon_R'(\varepsilon,l,L)) = \Upsilon_R'(\varepsilon+1 \pmod 3, l+1 \pmod r, L).$$

Definition 5.16. Let d be a positive integer, $r = \dfrac{3^{d+1}-1}{2}$, L a circulant Latin square over $\{0,\ldots,r-1\}$, and $C_r = circ(0,1,0,\ldots,0)$ a circulant matrix of size r. Then, we use the notation $\mathcal{M}'(3,d,L)$ to denote the set $\{\Upsilon'(\vartheta,t,L)|\vartheta \in \{0,1,2\}, 0 \le t \le r-1\}$. We also define $\mathcal{M}'_R(3,d,L) = \{\Upsilon'_R(\vartheta,t,L)|\vartheta \in \{0,1,2\}, 0 \le t \le r-1\}$.

Example 5.17. In this Example, we generate $\mathcal{M}'(3,1,L)$, where $L = \begin{bmatrix} 0&1&2&3\\ 3&0&1&2\\ 2&3&0&1\\ 1&2&3&0 \end{bmatrix}$. We have $C_4^0 L = L$, $C_4 L = \begin{bmatrix} 3&0&1&2\\ 2&3&0&1\\ 1&2&3&0\\ 0&1&2&3 \end{bmatrix}$, $C_4^2 L = \begin{bmatrix} 2&3&0&1\\ 1&2&3&0\\ 0&1&2&3\\ 3&0&1&2 \end{bmatrix}$, and $C_4^3 L = \begin{bmatrix} 1&2&3&0\\ 0&1&2&3\\ 3&0&1&2\\ 2&3&0&1 \end{bmatrix}$. We demonstrate the set $\mathcal{M}'(3,1,L) = \{\Upsilon'(\vartheta,t,L)|\vartheta \in \{0,1,2\}, 0 \le t \le 3\}$. The set $\mathcal{M}'(3,1,L)$ consists of twelve symmetric $(36,15,6)$-designs whose incidence matrices appear in Appendix 7.

A similar proof to the proof of Theorem 5.6 confirms the following theorem.

Theorem 5.18. *Let d be a positive integer, $r = \dfrac{3^{d+1}-1}{2}$, and L a circulant Latin square over $\{0,\ldots,r-1\}$. Then, the group $G'(3,d)$ (Defined on Page 74) is a cyclic group of bijections on $\mathcal{M}'(3,d,L)$, and also is a cyclic group of bijections on $\mathcal{M}'_R(3,d,L)$.*

The proof of Theorem 5.8 implies the following two theorems.

Theorem 5.19. *Let d be a positive integer, $r = \dfrac{3^{d+1}-1}{2}$, and L a circulant Latin square over $\{0,\ldots,r-1\}$. Then, we have*

1. $(\sigma M)(\sigma N)^T = MN^T$ *for all* $M,N \in \mathcal{M}'(3,d,L)$ *and all* $\sigma \in G'(3,d)$.

2. *For each* $M \in \mathcal{M}'(3,d,L)$*, the matrix* $\sum_{\sigma \in G'(3,d)} \sigma M$ *is a multiple of* $J_{r3^{d+1}}$.

Theorem 5.20. *Let d be a positive integer, $r = \dfrac{3^{d+1}-1}{2}$, and L a back circulant Latin square over $\{0,\ldots,r-1\}$. Then, we have*

1. $(\sigma M)(\sigma N)^T = MN^T$ *for all* $M,N \in \mathcal{M}'_R(3,d,L)$ *and all* $\sigma \in G'(3,d)$.

2. *For each* $M \in \mathcal{M}'_R(3,d,L)$*, the matrix* $\sum_{\sigma \in G'(3,d)} \sigma M$ *is a multiple of* $J_{r3^{d+1}}$.

In the following theorem, we show how a balanced generalized weighing matrix and the set $\mathcal{M}'(3,d,L)$ form a symmetric design. Although the following theorem is not mentioned in [10], it is directly follows from [10, Theorem 5.3, p. 216].

Theorem 5.21. *Let d be a positive integer. If $q = \frac{3^{d+1}+1}{2}$ is a prime power, then, for any positive integer m, there exists a symmetric design with parameters*

$$\left(\frac{3^{d+1}(q^{4m}-1)}{q+1}, 3^d q^{4m-1}, \frac{3^d(3^d+1)q^{4m-2}}{2}\right).$$

Proof. Let L be a circulant Latin square over $\{0,\ldots,r-1\}$, where $r = \frac{3^{d+1}-1}{2}$. One may note that $q-1 = r$. Consider the set $\mathcal{M}'(3,d,L)$ consisting of $3r$ symmetric

$$(3^{d+1}(3^{d+1}-1)/2, 3^d(3^{d+1}+1)/2, 3^d(3^d+1)/2)\text{-designs.}$$

By Theorem 5.19, the group $G'(3,d)$ and the set $\mathcal{M}'(3,d,L)$ satisfy the 1st and 2nd conditions of Theorem 5.1. For any positive integer m, Theorem 5.12 guarantees the existence of a symmetric zero diagonal BGW over the group $G'(3,d)$ with parameters

$$BGW(w,l,\mu) = BGW\left(\frac{q^{4m}-1}{q^2-1}, q^{4m-2}, q^{4m-2}-q^{4m-4}\right).$$

Let $(\nu,k,\lambda) = \left(\frac{3^{d+1}(3^{d+1}-1)}{2}, \frac{3^d(3^{d+1}+1)}{2}, \frac{3^d(3^d+1)}{2}\right)$. To be able to use Theorem 5.1, we need to prove $k^2\mu = \nu\lambda l$:

$$\begin{aligned}
k^2\mu &= 3^{2d}q^2(q^{4m-2}-q^{4m-4})\\
&= 3^{2d}q^{4m-2}(q^2-1)\\
&= 3^{d+1}(q-1)(3^{d-1}(q+1))q^{4m-2}\\
&= 3^{d+1}(q-1)\left(\frac{3^d(3^d+1)}{2}\right)q^{4m-2}\\
&= \nu\lambda l.
\end{aligned}$$

From Theorem 5.1, we conclude that there is a symmetric $(\nu w, kl, \lambda l)$-design, i.e. a symmetric $\left(\frac{3^{d+1}(q^{4m}-1)}{q+1}, 3^d q^{4m-1}, \frac{3^d(3^d+1)q^{4m-2}}{2}\right)$-design. $\square$

In the following theorem, a symmetric zero diagonal balanced generalized weighing matrix and the set $\mathcal{M}_R'(q,d,L)$ form a strongly regular graph. The proof of the following theorem is very similar to the proof of Theorem 5.21.

Theorem 5.22 ([10, Theorem 5.3, p. 216])**.** *Let d be a positive integer. If $q = \dfrac{3^{d+1}+1}{2}$ is a prime power, then, for any positive integer m, there exists a strongly regular graph with parameters*

$$\left(\frac{3^{d+1}(q^{4m}-1)}{q+1}, 3^d q^{4m-1}, \frac{3^d(3^d+1)q^{4m-2}}{2}, \frac{3^d(3^d+1)q^{4m-2}}{2}\right).$$

5.3 An Application of Balanced Generalized Weighing Matrices

In this section, we use the technique of the last section to decompose complete designs and complete graphs.

Theorem 5.23. *Let $q = p^m$ be an odd prime power, d a positive integer, $r = \dfrac{q^{d+1}-1}{q-1}$ a prime power, and W a symmetric zero diagonal $BGW\left(\dfrac{r^{4m}-1}{r^2-1}, r^{4m-2}, r^{4m-2}-r^{4m-4}\right)$ over the group $G(q,d)$. Then, for any positive integer m, there exists a decomposition of the complete design on $\dfrac{q^{d+1}(r^{4m}-1)}{r-1}$ points into*

- *q symmetric $\left(\dfrac{q^{d+1}(r^{4m}-1)}{r-1}, q^d r^{4m-1}, q^{d-1}r^{4m-2}(r-1)\right)$-designs,*
- *a $DDG\left(\dfrac{q^{d+1}(r^{4m}-1)}{r-1}, (q^{d+1}(r+1))-1, (q^{d+1}(r+1))-2, 0, \dfrac{r^{4m}-1}{r^2-1}, q^{d+1}(r+1)\right)$,*
- *$I_{\frac{q^{d+1}(r^{4m}-1)}{r-1}}$, and*
- *$W \otimes D$, where D is a $DDG((r+1)q^{d+1}, q^{d+1}, q^{d+1}, 0, (r+1), q^{d+1})$.*

Proof. Let L be a circulant Latin square over $\{0,\ldots,r\}$. The set

$$\{\Upsilon(\varepsilon, 0, L) | \varepsilon \in \{0,\ldots,q-1\}\}$$

is a set of q disjoint symmetric $((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$-designs whose diagonal blocks of size q^{d+1} are equal to the zero block. . Then, the set

$$\{W \otimes \Upsilon(\varepsilon, 0, L) | \varepsilon \in \{0, \ldots, q-1\}\}$$

is a set of q disjoint symmetric

$$\left(\frac{q^{d+1}(r^{4m}-1)}{r-1}, q^d r^{4m-1}, q^{d-1} r^{4m-2}(r-1)\right)\text{-designs.}$$

Let $D = J_{(r+1)q^{d+1}} - \sum_{\varepsilon=0}^{q-1} \Upsilon(\varepsilon, 0, L) = (J_{r+1} - I_{r+1}) \otimes J_{q^{d+1}}$ which is a

$$DDG((r+1)q^{d+1}, q^{d+1}, q^{d+1}, 0, (r+1), q^{d+1}).$$

We have

$$I_{\frac{q^{d+1}(r^{4m}-1)}{r-1}} + I_{\frac{r^{4m}-1}{r^2-1}} \otimes (J_{(r+1)q^{d+1}} - I_{(r+1)q^{d+1}}) + W \otimes D + \sum_{\varepsilon=0}^{q-1} (W \otimes \Upsilon(\varepsilon, t, L)) = J_{\frac{q^{d+1}(r^{4m}-1)}{r-1}}.$$

The matrix $I_{\frac{r^{4m}-1}{r^2-1}} \otimes (J_{(r+1)q^{d+1}} - I_{(r+1)q^{d+1}})$ is the adjacency matrix of a

$$DDG\left(\frac{q^{d+1}(r^{4m}-1)}{r-1}, (q^{d+1}(r+1)) - 1, (q^{d+1}(r+1)) - 2, 0, \frac{r^{4m}-1}{r^2-1}, q^{d+1}(r+1)\right).$$

Hence, we come to a decomposition of the complete design on $\frac{q^{d+1}(r^{4m}-1)}{r-1}$ points into q symmetric designs with parameters

$$\left(\frac{q^{d+1}(r^{4m}-1)}{r-1}, q^d r^{4m-1}, q^{d-1} r^{4m-2}(r-1)\right),$$

a $DDG\left(\frac{q^{d+1}(r^{4m}-1)}{r-1}, (q^{d+1}(r+1)) - 1, (q^{d+1}(r+1)) - 2, 0, \frac{r^{4m}-1}{r^2-1}, q^{d+1}(r+1)\right)$, $I_{\frac{q^{d+1}(r^{4m}-1)}{r-1}}$, and $W \otimes D$, where D is a $DDG((r+1)q^{d+1}, q^{d+1}, q^{d+1}, 0, (r+1), q^{d+1})$. □

Applying back circulant Latin square L and the set $\mathcal{M}_R(q,d,L)$ in Theorem 5.23, we conclude the following theorem:

Theorem 5.24. *Let q be an odd prime power, d a positive integer, $r = \dfrac{q^{d+1}-1}{q-1}$ a prime power, and W a symmetric zero diagonal $BGW\left(\dfrac{r^{4m}-1}{r^2-1}, r^{4m-2}, r^{4m-2}-r^{4m-4}\right)$ over the group $G(q,d)$. Then, for any positive integer m, there exists a decomposition of the complete graph on $\dfrac{q^{d+1}(r^{4m}-1)}{r-1}$ vertices into*

- *q SRG$\left(\dfrac{q^{d+1}(r^{4m}-1)}{r-1}, q^d r^{4m-1}, q^{d-1} r^{4m-2}(r-1), q^{d-1} r^{4m-2}(r-1)\right)$,*
- *a DDG$\left(\dfrac{q^{d+1}(r^{4m}-1)}{r-1}, (q^{d+1}(r+1))-1, (q^{d+1}(r+1))-2, 0, \dfrac{r^{4m}-1}{r^2-1}, q^{d+1}(r+1)\right)$, and*
- *$W \otimes D$, where D is a DDG$((r+1)q^{d+1}, q^{d+1}, q^{d+1}, 0, (r+1), q^{d+1})$.*

We now focus on Spence designs. In the following theorem, three symmetric designs with parameters

$$\left(\frac{3^{d+1}(q^{4m}-1)}{q+1}, 3^d q^{4m-1}, \frac{3^d(3^d+1)q^{4m-2}}{2}\right)$$

are made. These symmetric designs are not disjoint. In Theorem 5.26 below, three non-disjoint strongly regular graphs with parameters

$$\left(\frac{3^{d+1}(q^{4m}-1)}{q+1}, 3^d q^{4m-1}, \frac{3^d(3^d+1)q^{4m-2}}{2}, \frac{3^d(3^d+1)q^{4m-2}}{2}\right)$$

are constructed. Since they are not disjoint, they can not be elements of a decomposition of a complete graph. However, their existence is important because we use them to form a non-commutative association scheme in Chapter 6.

Theorem 5.25. *Let d be a positive integer, $q = \dfrac{3^{d+1}+1}{2}$ a prime power, and W a symmetric zero diagonal $BGW\left(\dfrac{q^{4m}-1}{q^2-1}, q^{4m-2}, q^{4m-2}-q^{4m-4}\right)$ over $G'(3,d)$. Then, for any positive* *integer m, there exists a set consisting of three symmetric designs with parameters*

$$\Big(\frac{3^{d+1}(q^{4m}-1)}{q+1}, 3^d q^{4m-1}, \frac{3^d(3^d+1)q^{4m-2}}{2}\Big).$$

Proof. Let L be a circulant Latin square over $\{0,\ldots,q-2\}$. Let $r=\frac{3^{d+1}-1}{2}$. One may note that $q-1=r$. Let the diagonal blocks of size 3^{d+1} of each matrix in $\{\Upsilon'(\varepsilon,0,L)|\varepsilon\in\{0,1,2\}\}$ be equal to $(J_3-I_3)\otimes J_{3^d}$. The set $\{\Upsilon'(\varepsilon,0,L)|\varepsilon\in\{0,1,2\}\}$ is a set of three symmetric

$$\Big(\frac{3^{d+1}(3^{d+1}-1)}{2}, \frac{3^d(3^{d+1}+1)}{2}, \frac{3^d(3^d+1)}{2}\Big)\text{-designs}$$

sharing complete tripartite graphs $K^3_{3^d}$. Then, the set $\{W\otimes\Upsilon'(\varepsilon,t,L)|\varepsilon\in\{0,1,2\}\}$ is a set of three symmetric designs with parameters

$$\Big(\frac{3^{d+1}(q^{4m}-1)}{q+1}, 3^d q^{4m-1}, \frac{3^d(3^d+1)q^{4m-2}}{2}\Big).$$

The elements of the set $\{W\otimes\Upsilon'(\varepsilon,t,L)|\varepsilon\in\{0,1,2\}\}$ are not disjoint because the elements of the set $\{\Upsilon'(\varepsilon,0,L)|\varepsilon\in\{0,1,2\}\}$ are not disjoint. $\square$

Applying a back circulant Latin square L and the set $\mathcal{M}'_R(3,d,L)$ in Theorem 5.25, we conclude the following theorem. The resulted matrix is a constant diagonal matrix because we use a zero diagonal balanced generalized weighing matrix.

Theorem 5.26. *Let d be a positive integer, $q=\frac{3^{d+1}+1}{2}$ a prime power, and W a symmetric zero diagonal BGW$\big(\frac{q^{4m}-1}{q^2-1}, q^{4m-2}, q^{4m-2}-q^{4m-4}\big)$ over $G'(3,d)$. Then, for any positive integer m, there are three*

$$SRG\Big(\frac{3^{d+1}(q^{4m}-1)}{q+1}, 3^d q^{4m-1}, \frac{3^d(3^d+1)q^{4m-2}}{2}, \frac{3^d(3^d+1)q^{4m-2}}{2}\Big)$$

sharing the matrix $W\otimes I_4\otimes(J_3-I_3)\otimes J_3$.

Chapter 6

Decomposing the Complete Designs into Association Schemes

One natural application of decompositions is in creating association schemes. In this Chapter, we introduce an infinite class of non-commutative association schemes. Furthermore, by the use of McFarland designs, we will present complete design decompositions, complete graph decompositions, and association schemes. The outcome is an infinite class of symmetric association schemes.

6.1 A New Series of Association Schemes

We first review the definition of an association scheme.

Definition 6.1 ([2, p. 52])**.** The pair $(X, \mathbf{R})$ is called a *d-class association scheme* if X is a set of cardinality n, and $\mathbf{R} = \{R_0, \ldots, R_d\}$ is a partition of $X \times X$ such that

1. $R_0 = \{(x,x) | x \in X\}$;
2. $R_i^t = R_{i'}$, for some $i' = 1, \ldots, d$, where $R_i^t = \{(x,y) | (y,x) \in R_i\}$;
3. For i, j, k in $\{0, 1, \ldots, d\}$ and for all $(x,y) \in R_k$, the number of $z \in X$ such that $(x,z) \in R_i$ and $(z,y) \in R_j$ is a constant number denoted by $p_{ij}^{(k)}$.

If $R_i^t = R_i$ for all $i = 1, \ldots, d$, the association scheme is called *symmetric*. The parameters $p_{ij}^{(k)}$, $0 \le i, j, k \le d$, are called the *intersection numbers* of the association scheme.

In Definition 6.1, if we add the following condition

4. $p_{ij}^k = p_{ji}^k$ for all i, j, and k,

the association scheme is called *commutative*.

Definition 6.2 ([2, p. 57])**.** Suppose $(X, \mathbf{R})$ is a d-class associate scheme with intersection numbers $p_{ij}^{(k)}$, $0 \leq i, j, k \leq d$. The *intersection matrices* $B_0, \ldots, B_d$ of $(X, \mathbf{R})$ are defined by $(B_i)_{j,k} = p_{ij}^{(k)}$.

Let $(X, \mathbf{R})$ be an association scheme with $\mathbf{R} = \{R_0, \ldots, R_d\}$. For each $k = \{0, 1, \ldots, d\}$, we define the matrix $A_k = [a_{ij}^{(k)}]$ as follows:

$$a_{ij}^{(k)} = \begin{cases} 1 & \text{if } (i, j) \in R_k, \\ 0 & \text{otherwise.} \end{cases}$$

The set $\{A_0, A_1, \ldots, A_d\}$ is called the *set of incidence matrices*. The following equivalent definition of association schemes relates to the incidence matrices.

Definition 6.3 ([2, p. 53])**.** A set of non-zero $(0,1)$-matrices $\{A_0, A_1, \ldots, A_d\}$ in $\mathbb{R}^{n \times n}$ is an *association scheme* with d classes on a finite set X of cardinality n if

1. $A_0 = I_n$;

2. $A_i^T \in \{A_0, A_1, \ldots, A_d\}$ for each $i = 1, \ldots, d$;

3. $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$, for all $i, j \in \{0, \ldots, d\}$;

4. $\sum_{i=0}^{d} A_i = J_n$.

If all adjacency matrices of an association scheme are symmetric, the association scheme is symmetric. An association scheme is commutative if we add the following property to Definition 6.3:

4. $A_i A_j = A_j A_i$ for all $A_i, A_j \in \{A_0, A_1, \ldots, A_d\}$.

For any two matrices A_i and A_j in a symmetric association scheme, we have

$$(A_i A_j)^T = A_j^T A_i^T = A_j A_i$$

and

$$\begin{aligned}(A_iA_j)^T &= (p_{ij}^0 I + p_{ij}^1 A_1 + \ldots + p_{ij}^d A_d)^T \\ &= p_{ij}^0 I^T + p_{ij}^1 A_1^T + \ldots + p_{ij}^d A_d^T \\ &= p_{ij}^0 I + p_{ij}^1 A_1 + \ldots + p_{ij}^d A_d \\ &= A_iA_j.\end{aligned}$$

Therefore, $A_iA_j = A_jA_i$. Thus, any symmetric association scheme is commutative . Suppose $\{A_0, A_1, \ldots, A_d\}$ is a commutative association scheme. The real vector space spanned by $\{A_0, A_1, \ldots, A_d\}$ is called the *Bose-Mesner algebra* of the association scheme.

Example 6.4. The following matrices are the adjacency matrices of a 3-class association scheme on 8 elements.

$$A_0 = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}, A_1 = \begin{bmatrix} 0&1&1&1&0&0&0&0\\ 1&0&1&1&0&0&0&0\\ 1&1&0&1&0&0&0&0\\ 1&1&1&0&0&0&0&0\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&1&0&1&1\\ 0&0&0&0&1&1&0&1\\ 0&0&0&0&1&1&1&0 \end{bmatrix}, A_2 = \begin{bmatrix} 0&0&0&0&0&1&1&1\\ 0&0&0&0&1&0&1&1\\ 0&0&0&0&1&1&0&1\\ 0&0&0&0&1&1&1&0\\ 0&1&1&1&0&0&0&0\\ 1&0&1&1&0&0&0&0\\ 1&1&0&1&0&0&0&0\\ 1&1&1&0&0&0&0&0 \end{bmatrix}, A_3 = \begin{bmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0 \end{bmatrix}.$$

We note that $A_0 = I_8$ and all adjacency matrices A_0 , …, A_3 are symmetric. Moreover, we have $\sum_{i=0}^{3} A_i = J_8$ and we also have

$$\begin{aligned}&A_0A_0 = A_0, \quad A_0A_1 = A_1, \quad A_0A_2 = A_2, \quad A_0A_3 = A_3,\\ &A_1A_0 = A_1, \quad A_1A_1 = 3A_0 + 2A_1, \quad A_1A_2 = 2A_2 + 3A_3, \quad A_1A_3 = A_2,\\ &A_2A_0 = A_2, \quad A_2A_1 = 2A_2 + 3A_3, \quad A_2A_2 = 3A_0 + 2A_1, \quad A_2A_3 = A_1,\\ &A_3A_0 = A_3, \quad A_3A_1 = A_2, \quad A_3A_2 = A_1, \quad A_3A_3 = A_0.\end{aligned}$$

The trivial association schemes is defined by $\{A_0 = I, A_1 = J - I\}$. Symmetric association schemes with two classes are equivalent to strongly regular graphs. The classes of any symmetric association scheme with two classes are a strongly regular graph and its complement. One of the important classes of association schemes is called *amorphous* association schemes.

Definition 6.5 ([17, p. 112])**.** Let $(X,\{R_i\}_{i=0}^{d})$ be an association scheme. Let $\{\P_0,\ldots,\P_n\}$ be a partition of $\{0,1,\ldots,d\}$ with $\P_0=\{0\}$. If $(X,\{\cup_{l\in\P_{l'}}R_l\}_{l'=0}^{n})$ is an association scheme, then it is called a *fusion scheme* of the association scheme $(X,\{R_i\}_{i=0}^{d})$.

Definition 6.6 ([17, p. 31])**.** An association scheme $(X,\{R_i\}_{i=0}^{d})$ is called *amorphous* if $(X,\{\cup_{l\in\P_{l'}}R_l\}_{l'=0}^{n})$ is a fusion scheme of $(X,\{R_i\}_{i=0}^{d})$ for any partition $\{\P_0,\ldots,\P_n\}$ with $\P_0=\{0\}$.

Now, we focus on strongly regular graphs made in Theorem 5.26 for $d=1$. We use them to produce an infinite class of non-commutative association schemes with 9 classes and an infinite class of non-commutative association schemes with 10 classes. Let

$$\Delta'_0(GH(3,1))\cup\{(J_3-R_3)\otimes J_3\}=\{A^{^{(0)}}(0),A^{^{(1)}}(0),A^{^{(2)}}(0),A^{^{(3)}}(0)\},$$

$$\Delta'_1(GH(3,1))\cup\{(J_3-R_3)\otimes J_3\}=\{A^{^{(0)}}(1),A^{^{(1)}}(1),A^{^{(2)}}(1),A^{^{(3)}}(1)\},$$

and

$$\Delta'_2(GH(3,1))\cup\{(J_3-R_3)\otimes J_3\}=\{A^{^{(0)}}(2),A^{^{(1)}}(2),A^{^{(2)}}(2),A^{^{(3)}}(2)\}$$

such that $A^{^{(3)}}(0)=A^{^{(3)}}(1)=A^{^{(3)}}(2)=(J_3-R_3)\otimes J_3$. For each ε, let N_ε be the symmetric $(36,15,6)$-design made by replaceing i with $A^{^{(i)}}(\varepsilon)$ in the Latin square $L=\begin{bmatrix}0&1&2&3\\1&2&3&0\\2&3&0&1\\3&0&1&2\end{bmatrix}$.

Theorem 6.7. *For $m\geq 1$, let W be a symmetric zero diagonal balanced generalized weighing matrix $BGW\left(\dfrac{25^{2m}-1}{24},25^{2m-1},25^{2m-1}-25^{2m-2}\right)$ over the group $G'(3,1)$, $L=\begin{bmatrix}0&1&2&3\\1&2&3&0\\2&3&0&1\\3&0&1&2\end{bmatrix}$, and N_0,N_1, and N_2 the symmetric $(36,15,6)$-designs made above. Then, $\{A_0,A_1,\ldots,A_{10}\}$ forms a non-commutative 10-class association scheme, where*

$$A_0 = I_{\frac{3(25^{2m}-1)}{2}},$$

$$A_1 = I_{\frac{(25^{2m}-1)}{2}} \otimes \omega_1,$$

$$A_2 = I_{\frac{(25^{2m}-1)}{2}} \otimes \omega_2,$$

$$A_3 = I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes (J_3 - I_3) \otimes J_3,$$

$$A_4 = I_{\frac{(25^{2m}-1)}{24}} \otimes (J_4 - I_4) \otimes J_9,$$

$$A_5 = W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3],$$

$$A_6 = W \otimes [R_{12} \otimes J_3],$$

$$A_7 = \Big[W \otimes N_0\Big] - \Big[W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3]\Big],$$

$$A_8 = \Big[W \otimes N_1\Big] - \Big[W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3]\Big],$$

$$A_9 = \Big[W \otimes N_2\Big] - \Big[W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3]\Big], \textit{ and}$$

$$A_{10} = J_{\frac{3(25^{2m}-1)}{2}} - W \otimes J_{36} - I_{\frac{(25^{2m}-1)}{24}} \otimes J_{36}.$$

Proof. We need to prove that the set $\mathcal{A} = \{A_0, A_1, \ldots, A_{10}\}$ satisfies all four axioms mentioned in Definition 6.3. We have $A_0 = I_{\frac{3(25^{2m}-1)}{2}}$, $A_1^T = A_2$, and A_i is symmetric for all $i \in \{3, \ldots, 10\}$. So $\mathcal{A}$ satisfies the first and second axioms. Now, we note

$$\begin{aligned} A_0 + A_1 + A_2 &= I_{\frac{(25^{2m}-1)}{2}} \otimes \omega_0 + I_{\frac{(25^{2m}-1)}{2}} \otimes \omega_1 + I_{\frac{(25^{2m}-1)}{2}} \otimes \omega_2 \\ &= I_{\frac{(25^{2m}-1)}{2}} \otimes (\omega_0 + \omega_1 + \omega_2) \\ &= I_{\frac{(25^{2m}-1)}{2}} \otimes J_3. \end{aligned}$$

This proves A_0, A_1, and A_2 are disjoint and all of their off diagonal blocks of size 3 are zero.

Moreover, we have

$$\begin{aligned}
\sum_{i=0}^{4} A_i &= I_{\frac{(25^{2m}-1)}{2}} \otimes J_3 + \left[I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes (J_3 - I_3) \otimes J_3 \right] + \\
&\left[I_{\frac{(25^{2m}-1)}{24}} \otimes (J_4 - I_4) \otimes J_9 \right] \\
&= \left[I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes I_3 \otimes J_3 \right] + \left[I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes (J_3 - I_3) \otimes J_3 \right] + \\
&\left[I_{\frac{(25^{2m}-1)}{24}} \otimes (J_4 - I_4) \otimes J_9 \right] \\
&= \left[I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes J_9 \right] + \left[I_{\frac{(25^{2m}-1)}{24}} \otimes (J_4 - I_4) \otimes J_9 \right] \\
&= I_{\frac{(25^{2m}-1)}{24}} \otimes J_{36}.
\end{aligned}$$

For $A, B \in \mathcal{M}'(3,1,L)$ and $\sigma \in G'(3,1)$, we have $\sigma(A+B) = \sigma A + \sigma B$. From this, we conclude $W \otimes (A+B) = W \otimes A + W \otimes B$. Since the matrices $W \otimes N_0$, $W \otimes N_1$, and $W \otimes N_2$ share the matrix $W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3]$, we come to the conclusion that five matrices A_5, A_6, A_7, A_8, and A_9 are disjoint, and we have

$$\begin{aligned}
\sum_{i=5}^{9} A_i &= \left(\sum_{i=0}^{2} \left[W \otimes N_i \right] - \left[W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3] \right] \right) + A_5 + A_6 \\
&= W \otimes \left[\sum_{i=0}^{2} N_i \right] - 3 \left[W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3] \right] + A_5 + A_6 \\
&= W \otimes \left[3R_4 \otimes (J_3 - R_3) \otimes J_3 + (J_4 - R_4) \otimes J_9 \right] - 3 \left[W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3] \right] + \\
&A_5 + A_6 \\
&= 3W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3] + W \otimes [(J_4 - R_4) \otimes J_9] - 3W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3] + \\
&A_5 + A_6 \\
&= W \otimes [(J_4 - R_4) \otimes J_9] + A_5 + A_6 \\
&= W \otimes [(J_4 - R_4) \otimes J_9] + W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3] + W \otimes [R_{12} \otimes J_3] \\
&= W \otimes [(J_4 - R_4) \otimes J_9] + W \otimes [R_4 \otimes J_9] = W \otimes J_{36}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}\sum_{i=0}^{10} A_i &= I_{\frac{(25^{2m}-1)}{24}} \otimes J_{36} + W \otimes J_{36} + J_{\frac{3(25^{2m}-1)}{2}} - W \otimes J_{36} - I_{\frac{(25^{2m}-1)}{24}} \otimes J_{36} \\ &= J_{\frac{3(25^{2m}-1)}{2}}.\end{aligned}$$

By this, we are left with the third axiom of Definition 6.3. We need to check if the multiplication of any two matrices in $\mathcal{A}$ is a linear combination of the elements of $\mathcal{A}$. We have $A_0A_i = A_iA_0 = A_i$ for all i. We also note that

$$\begin{aligned}A_1A_1 = (I_{\frac{(25^{2m}-1)}{2}} \otimes \omega_1)(I_{\frac{(25^{2m}-1)}{2}} \otimes \omega_1) &= I_{\frac{(25^{2m}-1)}{2}} \otimes (\omega_1\omega_1) \\ &= I_{\frac{(25^{2m}-1)}{2}} \otimes \omega_2 \\ &= A_2,\end{aligned}$$

$$\begin{aligned}A_2A_2 = (I_{\frac{(25^{2m}-1)}{2}} \otimes \omega_2)(I_{\frac{(25^{2m}-1)}{2}} \otimes \omega_2) &= I_{\frac{(25^{2m}-1)}{2}} \otimes (\omega_2\omega_2) \\ &= I_{\frac{(25^{2m}-1)}{2}} \otimes \omega_1 \\ &= A_1, \quad \text{and}\end{aligned}$$

$$\begin{aligned}A_1A_2 = A_2A_1 = (I_{\frac{(25^{2m}-1)}{2}} \otimes \omega_1)(I_{\frac{(25^{2m}-1)}{2}} \otimes \omega_2) &= I_{\frac{(25^{2m}-1)}{2}} \otimes (\omega_1\omega_2) \\ &= I_{\frac{(25^{2m}-1)}{2}} \otimes \omega_0 \\ &= A_0.\end{aligned}$$

Since $\omega_1 J_3 = \omega_2 J_3 = J_3$, we have

$$\begin{aligned}A_1A_3 = A_3A_1 = A_2A_3 = A_3A_2 &= (I_{\frac{(25^{2m}-1)}{2}} \otimes \omega_i)(I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes (J_3 - I_3) \otimes J_3) \\ &= I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes (J_3 - I_3) \otimes J_3\end{aligned}$$

$$= A_3,$$

$$\begin{aligned} A_1A_4 = A_4A_1 = A_2A_4 = A_4A_2 &= (I_{\frac{(25^{2m}-1)}{2}} \otimes \omega_i)(I_{\frac{(25^{2m}-1)}{24}} \otimes (J_4 - I_4) \otimes J_9) \\ &= I_{\frac{(25^{2m}-1)}{24}} \otimes (J_4 - I_4) \otimes J_9 \\ &= A_4, \end{aligned}$$

$$\begin{aligned} A_1A_5 = A_5A_1 = A_2A_5 = A_5A_2 &= (I_{\frac{(25^{2m}-1)}{2}} \otimes \omega_1)(W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3]) \\ &= W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3] \\ &= A_5, \end{aligned}$$

$$\begin{aligned} A_1A_6 = A_6A_1 = A_2A_6 = A_6A_2 &= (I_{\frac{(25^{2m}-1)}{2}} \otimes \omega_1)(W \otimes [R_{12} \otimes J_3]) \\ &= W \otimes [R_{12} \otimes J_3] \\ &= A_6, \quad \text{and} \end{aligned}$$

$$\begin{aligned} A_1A_{10} = A_{10}A_1 = A_2A_{10} = A_{10}A_2 &= (I_{\frac{(25^{2m}-1)}{2}} \otimes \omega_i)(J_{\frac{3(25^{2m}-1)}{2}} - W \otimes J_{36} - I_{\frac{(25^{2m}-1)}{24}} \otimes J_{36}) \\ &= J_{\frac{3(25^{2m}-1)}{2}} - W \otimes J_{36} - I_{\frac{(25^{2m}-1)}{24}} \otimes J_{36} \\ &= A_{10} \end{aligned}$$

Rules of the Kronecker product imply

$$\begin{aligned} A_3A_4 = A_4A_3 &= (I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes (J_3 - I_3) \otimes J_3)(I_{\frac{(25^{2m}-1)}{24}} \otimes (J_4 - I_4) \otimes J_9) \\ &= I_{\frac{(25^{2m}-1)}{24}} \otimes (J_4 - I_4) \otimes [((J_3 - I_3) \otimes J_3)J_9] \\ &= 6I_{\frac{(25^{2m}-1)}{24}} \otimes (J_4 - I_4) \otimes J_9, \end{aligned}$$

$$= 6A_4,$$

$$\begin{aligned}
A_3A_3 &= (I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes (J_3 - I_3) \otimes J_3)(I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes (J_3 - I_3) \otimes J_3) \\
&= 3I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes (J_3 + I_3) \otimes J_3 \\
&= 3I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes J_9 + 3I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes I_3 \otimes J_3 \\
&= 6A_0 + 6A_1 + 6A_2 + 3A_3,
\end{aligned}$$

$$\begin{aligned}
A_4A_4 &= (I_{\frac{(25^{2m}-1)}{24}} \otimes (J_4 - I_4) \otimes J_9)(I_{\frac{(25^{2m}-1)}{24}} \otimes (J_4 - I_4) \otimes J_9) \\
&= 9I_{\frac{(25^{2m}-1)}{24}} \otimes (2J_4 + I_4) \otimes J_9 \\
&= 18I_{\frac{(25^{2m}-1)}{24}} \otimes J_{36} + 9I_{\frac{(25^{2m}-1)}{6}} \otimes J_9 \\
&= 27A_0 + 27A_1 + 27A_2 + 27A_3 + 18A_4,
\end{aligned}$$

$$\begin{aligned}
A_3A_{10} = A_{10}A_3 &= (I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes (J_3 - I_3) \otimes J_3)(J_{\frac{3(25^{2m}-1)}{2}} - W \otimes J_{36} - I_{\frac{(25^{2m}-1)}{24}} \otimes J_{36}) \\
&= 6J_{\frac{3(25^{2m}-1)}{2}} - 6W \otimes J_{36} - 6I_{\frac{(25^{2m}-1)}{24}} \otimes J_{36} \\
&= 6A_{10}, \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
A_4A_{10} = A_{10}A_4 &= (I_{\frac{(25^{2m}-1)}{24}} \otimes (J_4 - I_4) \otimes J_9)(J_{\frac{3(25^{2m}-1)}{2}} - W \otimes J_{36} - I_{\frac{(25^{2m}-1)}{24}} \otimes J_{36}) \\
&= 27J_{\frac{3(25^{2m}-1)}{2}} - 27W \otimes J_{36} - 27I_{\frac{(25^{2m}-1)}{24}} \otimes J_{36} \\
&= 27A_{10}.
\end{aligned}$$

For any $t \in \{0, \ldots, r-1\}$ and any $i \in \{0, 1, 2\}$, we have

$$(I_4 \otimes (J_3 - I_3) \otimes J_3)(C^t R_4 \otimes (J_3 - \omega_i R_3) \otimes J_3) = 3(C^t R_4 \otimes (J_3 + \omega_i R_3) \otimes J_3).$$

So

$$\begin{aligned}
A_3A_5 = A_5A_3 &= (I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes (J_3 - I_3) \otimes J_3)(W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3]) \\
&= 3W \otimes [R_4 \otimes (J_3 + R_3) \otimes J_3] \\
&= 3W \otimes [R_4 \otimes J_9] + 3W \otimes [R_{12} \otimes J_3] \\
&= 3A_5 + 6A_6.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
A_3A_6 = A_6A_3 &= (I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes (J_3 - I_3) \otimes J_3)(W \otimes [R_{12} \otimes J_3]) \\
&= 3W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3] \\
&= 3W \otimes [R_4 \otimes J_9] - 3W \otimes [R_{12} \otimes J_3] \\
&= 3A_5 + 3A_6 - 3A_6 = 3A_5,
\end{aligned}$$

$$\begin{aligned}
A_4A_5 = A_5A_4 &= (I_{\frac{(25^{2m}-1)}{24}} \otimes (J_4 - I_4) \otimes J_9)(W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3]) \\
&= 6W \otimes [(J_4 - R_4) \otimes J_9] \\
&= 6W \otimes [J_{36}] - 6W \otimes [R_4 \otimes J_9] \\
&= 6\sum_{i=5}^{9} A_i - 6(A_5 + A_6) \\
&= 6\sum_{i=7}^{9} A_i,
\end{aligned}$$

and

$$\begin{aligned}
A_4A_6 = A_6A_4 &= (I_{\frac{(25^{2m}-1)}{24}} \otimes (J_4 - I_4) \otimes J_9)(W \otimes [R_{12} \otimes J_3]) \\
&= 3W \otimes [(J_4 - R_4) \otimes J_9] \\
&= 3W \otimes [J_{36}] - 3W \otimes [R_4 \otimes J_9]
\end{aligned}$$

$$
\begin{aligned}
&= 3\sum_{i=5}^{9} A_i - 3(A_5 + A_6) \\
&= 3\sum_{i=7}^{9} A_i.
\end{aligned}
$$

We first note that $(R_4 \otimes (J_3 - R_3) \otimes J_3)(R_4 \otimes (J_3 - R_3) \otimes J_3) = 3(I_4 \otimes (J_3 + I_3) \otimes J_3)$. The diagonal blocks of the matrix $(W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3])(W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3])$ all are equal to $3(25)^{2m-1}(I_4 \otimes (J_3 + I_3) \otimes J_3)$ and its off diagonal blocks all are equal to $3\left(\dfrac{25^{2m-1} - 25^{2m-2}}{12}\right)$ times of the sum

$$
\sum_{t=0}^{3}\sum_{j=0}^{2} C^t \otimes (J_3 + \omega_j) \otimes J_3 = 4J_{36}.
$$

So the off diagonal blocks all are equal to $(25^{2m-1} - 25^{2m-2})J_{36}$. Hence, we have

$$
\begin{aligned}
A_5A_5 &= (W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3])(W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3]) \\
&= 3(25)^{2m-1}\big(I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes (J_3 + I_3) \otimes J_3\big) \\
&+ (25^{2m-1} - 25^{2m-2})(J_{\frac{(25^{2m}-1)}{24}} - I_{\frac{(25^{2m}-1)}{24}}) \otimes J_{36} \\
&= 6(25)^{2m-1}\sum_{i=0}^{2} A_i + 3(25)^{2m-1}A_3 + (25^{2m-1} - 25^{2m-2})\sum_{i=5}^{10} A_i.
\end{aligned}
$$

We have

$$
\begin{aligned}
A_5A_6 = A_6A_5 &= (W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3])(W \otimes [R_{12} \otimes J_3]) \\
&= 3(25)^{2m-1}\big(I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes (J_3 - I_3) \otimes J_3\big) \\
&+ \left(\frac{25^{2m-1} - 25^{2m-2}}{2}\right)(J_{\frac{(25^{2m}-1)}{24}} - I_{\frac{(25^{2m}-1)}{24}}) \otimes J_{36} \\
&= 3(25)^{2m-1}\sum_{i=0}^{3} A_i - 3(25)^{2m-1}\sum_{i=0}^{2} A_i + \left(\frac{25^{2m-1} - 25^{2m-2}}{2}\right)\sum_{i=5}^{10} A_i \\
&= 3(25)^{2m-1}A_3 + \left(\frac{25^{2m-1} - 25^{2m-2}}{2}\right)\sum_{i=5}^{10} A_i,
\end{aligned}
$$

$$\begin{aligned}
A_6A_6 &= (W\otimes[R_{12}\otimes J_3])(W\otimes[R_{12}\otimes J_3]) \\
&= 3(25)^{2m-1}\big(I_{\frac{(25^{2m}-1)}{24}}\otimes I_{12}\otimes J_3\big) \\
&+\left(\frac{25^{2m-1}-25^{2m-2}}{4}\right)(J_{\frac{(25^{2m}-1)}{24}}-I_{\frac{(25^{2m}-1)}{24}})\otimes J_{36} \\
&= 3(25)^{2m-1}\sum_{i=0}^{2}A_i+\left(\frac{25^{2m-1}-25^{2m-2}}{4}\right)\sum_{i=5}^{10}A_i,
\end{aligned}$$

$$\begin{aligned}
A_5A_{10}=A_{10}A_5 &= (W\otimes[R_4\otimes(J_3-R_3)\otimes J_3])(J_{\frac{3(25^{2m}-1)}{2}}-W\otimes J_{36}-I_{\frac{(25^{2m}-1)}{24}}\otimes J_{36}) \\
&= 6(25)^{2m-1}J_{\frac{3(25^{2m}-1)}{2}}-(W\otimes[R_4\otimes J_9])(W\otimes J_{36})-6W\otimes J_{36} \\
&= 6(25)^{2m-1}J_{\frac{3(25^{2m}-1)}{2}}-6(25)^{2m-1}(I_{\frac{(25^{2m}-1)}{24}}\otimes J_{36}) \\
&-6(25^{2m-1}-25^{2m-2})((J_{\frac{(25^{2m}-1)}{24}}-I_{\frac{(25^{2m}-1)}{24}})\otimes J_{36})-6W\otimes J_{36} \\
&= 6(25)^{2m-1}((J_{\frac{(25^{2m}-1)}{24}}-I_{\frac{(25^{2m}-1)}{24}})\otimes J_{36}) \\
&-6(25^{2m-1}-25^{2m-2})((J_{\frac{(25^{2m}-1)}{24}}-I_{\frac{(25^{2m}-1)}{24}})\otimes J_{36})-6W\otimes J_{36} \\
&= 6(25)^{2m-2}((J_{\frac{(25^{2m}-1)}{24}}-I_{\frac{(25^{2m}-1)}{24}})\otimes J_{36})-6W\otimes J_{36} \\
&= 6(25)^{2m-2}\sum_{i=5}^{10}A_i-6\sum_{i=5}^{9}A_i,
\end{aligned}$$

and

$$A_6A_{10}=A_{10}A_6=3(25)^{2m-2}\sum_{i=5}^{10}A_i-3\sum_{i=5}^{9}A_i.$$

Let $\varepsilon\in\{0,1,2\}$ and $A_R(\varepsilon)$ be in $\Delta'_\varepsilon(GH(3,1))$. We may note $\omega_i\omega_jR_q=\omega_{i+j}R_q$ and $\omega_jR_q\omega_i=\omega_j\omega_i^TR_q=\omega_j\omega_i^{-1}R_q=\omega_{j-i}R_q$. This results in

$$(I_3\otimes\omega_1)A_R(\varepsilon)=A(\varepsilon+1),\qquad A_R(\varepsilon)(I_3\otimes\omega_1)=A(\varepsilon-1),$$

$$(I_3\otimes\omega_2)A_R(\varepsilon)=A(\varepsilon+2),\quad\text{and}\quad A_R(\varepsilon)(I_3\otimes\omega_2)=A(\varepsilon-2).$$

Hence

$$(I_{12}\otimes\omega_1)(N_0-[R_4\otimes(J_3-R_3)\otimes J_3])=(N_1-[R_4\otimes(J_3-R_3)\otimes J_3]),$$

which implies

$$\begin{aligned}
A_1A_7 &= (I_{\frac{(25^{2m}-1)}{2}}\otimes\omega_1)\Big[W\otimes N_0-W\otimes[R_4\otimes(J_3-R_3)\otimes J_3]\Big]\\
&= (I_{\frac{(25^{2m}-1)}{2}}\otimes\omega_1)\Big[W\otimes\big[N_0-[R_4\otimes(J_3-R_3)\otimes J_3]\big]\Big]\\
&= \Big[W\otimes\big[N_1-[R_4\otimes(J_3-R_3)\otimes J_3]\big]\Big]\\
&= \Big[W\otimes N_1\Big]-\Big[W\otimes[R_4\otimes(J_3-R_3)\otimes J_3]\Big]\\
&= A_8,\quad \text{and}
\end{aligned}$$

$$\begin{aligned}
A_7A_1 &= \Big[W\otimes N_0-W\otimes[R_4\otimes(J_3-R_3)\otimes J_3]\Big](I_{\frac{(25^{2m}-1)}{2}}\otimes\omega_1)\\
&= \Big[W\otimes\big[N_0-[R_4\otimes(J_3-R_3)\otimes J_3]\big]\Big](I_{\frac{(25^{2m}-1)}{2}}\otimes\omega_1)\\
&= \Big[W\otimes\big[N_2-[R_4\otimes(J_3-R_3)\otimes J_3]\big]\Big]\\
&= A_9.
\end{aligned}$$

Similarly, we have

$$A_1A_8=A_9,\quad A_8A_1=A_7,$$

$$A_1A_9=A_7,\quad A_9A_1=A_8,$$

$$A_2A_7=A_9,\quad A_7A_2=A_8,$$

$$A_2A_8=A_7,\quad A_8A_2=A_9,$$

$$A_2A_9=A_8,\quad A_9A_2=A_7.$$

For an arbitrary element $\sigma = g^t$ of $G'(3,d)$, suppose $t \equiv s \pmod 3$ and $t \equiv u \pmod r$. We have

$$\begin{aligned}
&(I_4 \otimes (J_3 - I_3) \otimes J_3)\sigma(N_0 - [R_4 \otimes (J_3 - R_3) \otimes J_3]) \\
&= (I_4 \otimes (J_3 - I_3) \otimes J_3)(\sigma N_0 - [C^u R_4 \otimes (J_3 - \omega_s R_3) \otimes J_3]) \\
&= 2(J_4 - C^u R_4) \otimes J_9 \\
&= 2\sigma\Big[(I_4 \otimes (J_3 - I_3) \otimes J_3)(N_0 - [R_4 \otimes (J_3 - R_3) \otimes J_3])\Big].
\end{aligned}$$

So

$$\begin{aligned}
&(I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes (J_3 - I_3) \otimes J_3)\Big[W \otimes (N_0 - [R_4 \otimes (J_3 - R_3) \otimes J_3])\Big] \\
&\quad = W \otimes \Big[(I_4 \otimes (J_3 - I_3) \otimes J_3)(N_0 - [R_4 \otimes (J_3 - R_3) \otimes J_3])\Big].
\end{aligned}$$

Thus

$$\begin{aligned}
A_3A_7 = A_7A_3 &= (I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes (J_3 - I_3) \otimes J_3)\Big[W \otimes N_0 - W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3]\Big] \\
&= (I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes (J_3 - I_3) \otimes J_3)\Big[W \otimes \big[N_0 - [R_4 \otimes (J_3 - R_3) \otimes J_3]\big]\Big] \\
&= 2\Big[W \otimes \big[(J_4 - R_4) \otimes J_9\big]\Big] \\
&= 2W \otimes J_{36} - 2W \otimes \big[R_4 \otimes J_9\big] \\
&= 2A_5 + 2A_6 + 2A_7 + 2A_8 + 2A_9 - 2(A_5 + A_6) \\
&= 2A_7 + 2A_8 + 2A_9.
\end{aligned}$$

In a similar way,

$$A_3A_8 = A_8A_3 = A_3A_9 = A_9A_3 = 2A_7 + 2A_8 + 2A_9.$$

We find that

$$A_4A_7 = A_7A_4 = A_4A_8 = A_8A_4 = A_4A_9 = A_9A_4 = 9A_5 + 9A_6 + 6A_7 + 6A_8 + 6A_9,$$

$$\begin{aligned}&A_5A_7 = A_7A_5 = A_5A_8 = A_8A_5 = A_5A_9 = A_9A_5 = \\ &2(25)^{2m-1}A_4 + 6\left(\frac{25^{2m-1} - 25^{2m-2}}{4}\right)\sum_{i=5}^{10} A_{10},\end{aligned}$$

and

$$\begin{aligned}&A_6A_7 = A_7A_6 = A_6A_8 = A_8A_6 = A_6A_9 = A_9A_6 = \\ &25^{2m-1}A_4 + 3\left(\frac{25^{2m-1} - 25^{2m-2}}{4}\right)\sum_{i=5}^{10} A_{10}.\end{aligned}$$

Now, we want to calculate the matrix

$$(N_\varepsilon - [R_4 \otimes (J_3 - R_3) \otimes J_3])(N_\vartheta - [R_4 \otimes (J_3 - R_3) \otimes J_3]) = [M_{ij}],$$

where M_{ij} is the block of size 9. For $i = j$, we have

$$\begin{aligned}M_{ii} &= \sum_{\substack{A_R(\varepsilon)\in\Delta'_\varepsilon(GH(3,1)) \\ A_R(\vartheta)\in\Delta'_\vartheta(GH(3,1))}} A_R(\varepsilon)A_R(\vartheta)^T \\ &= 3\sum_{\{A(\varepsilon-\vartheta)\in\Delta_{\varepsilon-\vartheta}(GH(3,1))\}} A(\varepsilon-\vartheta) \\ &= 9(I_3 \otimes (\omega_{\varepsilon-\vartheta})) + 3((J_3 - I_3) \otimes J_3).\end{aligned}$$

For $i \neq j$, we have $M_{ij} = 2J_9$. So

$$\begin{aligned}&(N_\varepsilon - [R_4 \otimes (J_3 - R_3) \otimes J_3])(N_\vartheta - [R_4 \otimes (J_3 - R_3) \otimes J_3]) \\ &= I_4 \otimes [9(I_3 \otimes (\omega_{\varepsilon-\vartheta})) + 3((J_3 - I_3) \otimes J_3)] + 2(J_4 - I_4) \otimes J_9,\end{aligned}$$

from which we evaluate $A_7A_8 = [M_{ij}]$, where M_{ij} is the block of size 36. For $i = j$, we have

$$\begin{aligned} M_{ii} = & \sum_{i=0}^{25^{2m-1}} \left[I_4 \otimes [9(I_3 \otimes \omega_2) + 3((J_3 - I_3) \otimes J_3)] + 2(J_4 - I_4) \otimes J_9\right] \\ &= 25^{2m-1}\left[I_4 \otimes [9(I_3 \otimes \omega_2) + 3((J_3 - I_3) \otimes J_3)] + 2(J_4 - I_4) \otimes J_9\right]. \end{aligned}$$

For two arbitrary elements $\sigma_1 = g^t$ and $\sigma_2 = g^w$, suppose $t \equiv \varepsilon \pmod 3$, $t \equiv u \pmod r$, $w \equiv \vartheta \pmod 3$, and $w \equiv k \pmod r$. We have

$$\begin{aligned} &\sigma_1(N_0 - [R_4 \otimes (J_3 - R_3) \otimes J_3])\sigma_2(N_1 - [R_4 \otimes (J_3 - R_3) \otimes J_3]) \\ &= C^{k-u} \otimes [9(I_3 \otimes (\omega_{2+\varepsilon-\vartheta})) + 3((J_3 - I_3) \otimes J_3)] + 2(J_4 - C^{k-u}) \otimes J_9. \end{aligned}$$

Suppose $a = \left(\frac{25^{2m-1} - 25^{2m-2}}{12}\right)$. For $i \neq j$, the last equality contributes to

$$\begin{aligned} M_{ij} &= a \sum_{\substack{i \in \{0,1,2,3\} \\ \varepsilon \in \{0,1,2\}}} \left[C^i \otimes [9(I_3 \otimes (\omega_{2+\varepsilon})) + 3((J_3 - I_3) \otimes J_3)] + 2(J_4 - C^i) \otimes J_9\right] \\ &= a \sum_{i \in \{0,1,2,3\}} \left[C^i \otimes [9(I_3 \otimes J_3) + 9((J_3 - I_3) \otimes J_3)] + 6(J_4 - C^i) \otimes J_9\right] \\ &= a \sum_{i \in \{0,1,2,3\}} \left[9C^i \otimes J_9 + 6(J_4 - C^i) \otimes J_9\right] \\ &= a\left[9J_4 \otimes J_9 + 6(4J_4 - J_4) \otimes J_9\right] \\ &= a\left[9J_4 \otimes J_9 + 18J_4 \otimes J_9\right] \\ &= a\left[27J_4 \otimes J_9\right] \\ &= 27\left(\frac{25^{2m-1} - 25^{2m-2}}{12}\right)J_{36}. \end{aligned}$$

Hence, we have

$$A_7A_8 = \left(\left[W \otimes N_0\right] - \left[W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3]\right]\right)$$

$$
\begin{aligned}
&\left(\left[W \otimes N_1\right] - \left[W \otimes [R_4 \otimes (J_3 - R_3) \otimes J_3]\right]\right) \\
&= 25^{2m-1} I_{\frac{(25^{2m}-1)}{24}} \otimes \left(I_4 \otimes [9(I_3 \otimes \omega_2) + 3((J_3 - I_3) \otimes J_3)] + 2(J_4 - I_4) \otimes J_9\right) \\
&+ 27\left(\frac{25^{2m-1} - 25^{2m-2}}{12}\right)(J_{\frac{(25^{2m}-1)}{24}} - I_{\frac{(25^{2m}-1)}{24}}) \otimes J_{36} \\
&= 9(25)^{2m-1}A_2 + 3(25)^{2m-1}A_3 + 2(25)^{2m-1}A_4 + 27\left(\frac{25^{2m-1} - 25^{2m-2}}{12}\right)\sum_{i=5}^{10} A_i.
\end{aligned}
$$

In this way, we have

$$
\begin{aligned}
&A_7A_7 = A_8A_8 = A_9A_9 = \\
&9(25)^{2m-1}A_0 + 3(25)^{2m-1}A_3 + 2(25)^{2m-1}A_4 + 27\left(\frac{25^{2m-1} - 25^{2m-2}}{12}\right)\sum_{i=5}^{10} A_i,
\end{aligned}
$$

$$
\begin{aligned}
&A_7A_9 = A_8A_7 = A_9A_8 = \\
&9(25)^{2m-1}A_1 + 3(25)^{2m-1}A_3 + 2(25)^{2m-1}A_4 + 27\left(\frac{25^{2m-1} - 25^{2m-2}}{12}\right)\sum_{i=5}^{10} A_i,
\end{aligned}
$$

and

$$
\begin{aligned}
&A_8A_9 = A_9A_7 = \\
&9(25)^{2m-1}A_2 + 3(25)^{2m-1}A_3 + 2(25)^{2m-1}A_4 + 27\left(\frac{25^{2m-1} - 25^{2m-2}}{12}\right)\sum_{i=5}^{10} A_i.
\end{aligned}
$$

Since $(N_0 - [R_4 \otimes (J_3 - R_3) \otimes J_3])J_{36} = 9J_{36}$, we conclude that the diagonal blocks of the matrix $(W \otimes \left[N_0 - [R_4 \otimes (J_3 - R_3) \otimes J_3]\right])(W \otimes J_{36})$ all are equal to $9(25)^{2m-1}J_{36}$, and off-diagonal blocks of this matrix all are equal to the matrix $9(25^{2m-1} - 25^{2m-2})J_{36}$. Hence, we have

$$
\begin{aligned}
&(W \otimes [N_0 - [R_4 \otimes (J_3 - R_3) \otimes J_3]])(W \otimes J_{36}) \\
&= 9(25)^{2m-1}(I_{\frac{(25^{2m}-1)}{24}} \otimes J_{36}) + 9(25^{2m-1} - 25^{2m-2})((J_{\frac{(25^{2m}-1)}{24}} - I_{\frac{(25^{2m}-1)}{24}}) \otimes J_{36}),
\end{aligned}
$$

which results in

$$\begin{aligned}
&A_7A_{10} = A_{10}A_7 = (W \otimes \left[N_0 - [R_4 \otimes (J_3 - R_3) \otimes J_3]\right]) \\
&(J_{\frac{3(25^{2m}-1)}{2}} - W \otimes J_{36} - I_{\frac{(25^{2m}-1)}{24}} \otimes J_{36}) \\
&= 9(25)^{2m-1} J_{\frac{3(25^{2m}-1)}{2}} - (W \otimes [N_0 - [R_4 \otimes (J_3 - R_3) \otimes J_3]])(W \otimes J_{36}) - 9W \otimes J_{36} \\
&= 9(25)^{2m-1} J_{\frac{3(25^{2m}-1)}{2}} - 9(25)^{2m-1}(I_{\frac{(25^{2m}-1)}{24}} \otimes J_{36}) \\
&- 9(25^{2m-1} - 25^{2m-2})((J_{\frac{(25^{2m}-1)}{24}} - I_{\frac{(25^{2m}-1)}{24}}) \otimes J_{36}) - 9W \otimes J_{36} \\
&= 9(25)^{2m-1}((J_{\frac{(25^{2m}-1)}{24}} - I_{\frac{(25^{2m}-1)}{24}}) \otimes J_{36}) \\
&- 9(25^{2m-1} - 25^{2m-2})((J_{\frac{(25^{2m}-1)}{24}} - I_{\frac{(25^{2m}-1)}{24}}) \otimes J_{36}) - 9W \otimes J_{36} \\
&= 9(25)^{2m-2}((J_{\frac{(25^{2m}-1)}{24}} - I_{\frac{(25^{2m}-1)}{24}}) \otimes J_{36}) - 9W \otimes J_{36} \\
&= 9(25)^{2m-2} \sum_{i=5}^{10} A_i - 9 \sum_{i=5}^{9} A_i.
\end{aligned}$$

Likewise, we have

$$A_8A_{10} = A_{10}A_8 = A_9A_{10} = A_{10}A_9 = 9(25)^{2m-2} \sum_{i=5}^{10} A_i - 9 \sum_{i=5}^{9} A_i.$$

Suppose (i, j)-entry $\overline{W}_{ij}$ of $\overline{W}$ is defined by (i, j)-entry W_{ij} of the matrix W as follows

$$\overline{W}_{ij} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } W_{ij} = 0 \text{ and } i \neq j \\ 0 & \text{if } W_{ij} \neq 0. \end{cases}$$

By this definition, we have $A_{10} = \overline{W} \otimes J_{36}$. The property of balanced generalized Hadamard matrices contribute

$$(\overline{W} + I)(\overline{W} + I)$$

$$
\begin{aligned}
&= \left(\frac{25^{2m}-1}{24} - 25^{2m-1}\right) I_{\frac{25^{2m}-1}{24}} \\
&+ \left(\frac{25^{2m}-1}{24} - 2(25)^{2m-1} + 25^{2m-1} - 25^{2m-2}\right)(J_{\frac{25^{2m}-1}{24}} - I_{\frac{25^{2m}-1}{24}}) \\
&= \left(\frac{25^{2m-1}-1}{24}\right) I_{\frac{25^{2m}-1}{24}} + \left(\frac{25^{2m-2}-1}{24}\right)(J_{\frac{25^{2m}-1}{24}} - I_{\frac{25^{2m}-1}{24}}), \quad \text{so}
\end{aligned}
$$

$$
\begin{aligned}
&(\overline{W})(\overline{W}) = (\overline{W}+I)(\overline{W}+I) - 2\overline{W} - I \\
&= \left(\frac{25^{2m-1}-25}{24}\right) I_{\frac{25^{2m}-1}{24}} + \left(\frac{25^{2m-2}-1}{24}\right)(J_{\frac{25^{2m}-1}{24}} - I_{\frac{25^{2m}-1}{24}}) - 2\overline{W}.
\end{aligned}
$$

All in all, we achieve

$$
\begin{aligned}
&A_{10}A_{10} = (\overline{W} \otimes J_{36})(\overline{W} \otimes J_{36}) \\
&= ((\overline{W})(\overline{W})) \otimes (J_{36}J_{36}) \\
&= 36\left[\left(\frac{25^{2m-1}-25}{24}\right) I_{\frac{25^{2m}-1}{24}} + \left(\frac{25^{2m-2}-1}{24}\right)(J_{\frac{25^{2m}-1}{24}} - I_{\frac{25^{2m}-1}{24}}) - 2\overline{W}\right] \otimes J_{36} \\
&= 36\left[\left(\frac{25^{2m-1}-25}{24}\right) I_{\frac{25^{2m}-1}{24}}\right] \otimes J_{36} + 36\left[\left(\frac{25^{2m-2}-1}{24}\right)(J_{\frac{25^{2m}-25}{24}} - I_{\frac{25^{2m}-1}{24}})\right] \otimes J_{36} \\
&- 72\overline{W} \otimes J_{36} \\
&= 36\left(\frac{25^{2m-1}-25}{24}\right)\sum_{i=0}^{4} A_i + 36\left(\frac{25^{2m-2}-1}{24}\right)\sum_{i=5}^{10} A_i - 72A_{10}.
\end{aligned}
$$

Hence, it is shown that the product A_iA_j, $0 \leq i, j \leq 10$, is a linear combination of the elements of the set $\{A_0, A_1, \ldots, A_{10}\}$. As a summary, the intersection matrices $B_0, \ldots, B_{10}$ are presented in Appendix 8. Since A_1 and A_2 are not symmetric, this association scheme is not symmetric. Moreover, this association scheme is not commutative because $A_1A_7 \neq A_7A_1$. We note that only when $m = 1$, the matrix A_{10} is equal to zero. In this case, we have a 9-class association scheme. □

In the following theorem, we present one of the fusion scheme (Definition 6.5) of the association scheme mentioned in Theorem 6.7.

Theorem 6.8. *Let the set $\{B_0, B_1, \ldots, B_{10}\}$ be the association scheme generated in Theorem 6.7 with $B_0 = I_{\frac{3(25^{2m}-1)}{2}}$. Then, the set $\{A_0, A_1, \ldots, A_8, A_9\}$ forms a non-commutative 9-class association scheme, where $A_i = B_i$, for $i = 0, \ldots, 4$, $A_5 = B_5 + B_6$, and $A_i =_{i+1}$, for $i = 6, \ldots, 9$.*

Proof. We need to prove that the set $\mathcal{A} = \{A_0, A_1, \ldots, A_9\}$ satisfies all four axioms in Definition 6.3. The proof of Theorem 6.7 shows that the set $\mathcal{A}$ satisfies the first, second, and fourth axioms. By this, we are left with only the third axiom of Definition 6.3.

Since the classes in $\mathcal{A} = \{A_0, A_1, \ldots, A_9\}$ are defined based on the classes in $\mathcal{A}' = \{B_0, B_1, \ldots, B_{10}\}$, it is really helpful to look at the intersection numbers $p_{ij}^{(k)}$ in the proof of the Theorem 6.7. In the association scheme $\mathcal{A}'$, we always have $p_{ij}^{(5)} = p_{ij}^{(6)}$ when $i, j \notin \{5, 6\}$. In other words, the coefficients of B_5 and B_6 in the sum $B_iB_j = \sum_{k=0}^{10} B_k$ are always equal as long as $i, j \notin \{5, 6\}$. This demonstrates that A_iA_j is a linear combination of the elements of the set $\mathcal{A}$ whenever $i, j \neq 5$. Therefore, to show that $\mathcal{A}$ is an association scheme, we only need to prove A_5A_i and A_iA_5 are linear combinations of the elements of the set $\mathcal{A}$, for $i = 0, 1, \ldots, 9$. Since $\omega_1 J_3 = \omega_2 J_3 = J_3$, we have

$$\begin{aligned}
A_1A_5 = A_5A_1 = A_2A_5 = A_5A_2 &= (I_{\frac{(25^{2m}-1)}{2}} \otimes \omega_1)(W \otimes [R_4 \otimes J_9]) \\
&= (W \otimes [R_4 \otimes J_9]) \\
&= A_5.
\end{aligned}$$

Since $(I_4 \otimes (J_3 - I_3) \otimes J_3)(C^s R_4 \otimes J_9) = 6C^s R_4 \otimes J_9$, we have

$$\begin{aligned}
A_3A_5 = A_5A_3 &= (I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes (J_3 - I_3) \otimes J_3)(W \otimes [R_4 \otimes J_9]) \\
&= 6W \otimes [R_4 \otimes J_9] \\
&= 6A_5.
\end{aligned}$$

The equality $((J_4 - I_4)\otimes J_9)(C^s R_4 \otimes J_9) = 9(J_4 - C^s R_4)\otimes J_9$ implies

$$\begin{aligned}
A_4A_5 = A_5A_4 &= (I_{\frac{(25^{2m}-1)}{24}} \otimes (J_4 - I_4)\otimes J_9)(W \otimes [R_4 \otimes J_9]) \\
&= 9W \otimes [(J_4 - R_4)\otimes J_9] \\
&= 9W \otimes [J_{36}] - 9W \otimes [R_4 \otimes J_9] \\
&= 9\sum_{i=5}^{8} A_i - 9A_5 \\
&= 9\sum_{i=6}^{8} A_i.
\end{aligned}$$

Since $(C^u R_4 \otimes J_9)(C^s R_4 \otimes J_9) = 9C^{u-s}\otimes J_9$, the diagonal blocks of the matrix

$$(W \otimes [R_4 \otimes J_9])(W \otimes [R_4 \otimes J_9])$$

all are equal to $9(25)^{2m-1}(I_4 \otimes J_9)$, and the off diagonal blocks of this matrix all are equal to $9\left(\frac{25^{2m-1}-25^{2m-2}}{4}\right)J_{36}$. Hence, we have

$$\begin{aligned}
A_5A_5 &= (W \otimes [R_4 \otimes J_9])(W \otimes [R_4 \otimes J_9]) \\
&= 9(25)^{2m-1} I_{\frac{(25^{2m}-1)}{24}} \otimes I_4 \otimes J_9 + 9\left(\frac{25^{2m-1}-25^{2m-2}}{4}\right)(J_{\frac{(25^{2m}-1)}{24}} - I_{\frac{(25^{2m}-1)}{24}}) \otimes J_{36} \\
&= 9(25)^{2m-1}\sum_{i=0}^{3} A_i + 9\left(\frac{25^{2m-1}-25^{2m-2}}{4}\right)\sum_{i=5}^{9} A_i.
\end{aligned}$$

Since we know $(R_4 \otimes J_9)J_{36} = 9J_{36}$, we conclude that the diagonal blocks of the matrix $(W \otimes [R_4 \otimes J_9])(W \otimes J_{36})$ all are equal to $9(25)^{2m-1}J_{36}$, and off-diagonal blocks of this matrix all are equal to the matrix $9(25^{2m-1} - 25^{2m-2})J_{36}$. Hence, we have

$$\begin{aligned}
&(W \otimes [R_4 \otimes J_9])(W \otimes J_{36}) \\
&= 9(25)^{2m-1}(I_{\frac{(25^{2m}-1)}{24}} \otimes J_{36}) + 9(25^{2m-1} - 25^{2m-2})((J_{\frac{(25^{2m}-1)}{24}} - I_{\frac{(25^{2m}-1)}{24}}) \otimes J_{36}),
\end{aligned}$$

which results in

$$\begin{aligned}
A_5A_9 = A_5A_9 &= (W \otimes [R_4 \otimes J_9])(J_{\frac{3(25^{2m}-1)}{2}} - W \otimes J_{36} - I_{\frac{(25^{2m}-1)}{24}} \otimes J_{36}) \\
&= 9(25)^{2m-1} J_{\frac{3(25^{2m}-1)}{2}} - (W \otimes [R_4 \otimes J_9])(W \otimes J_{36}) - 9W \otimes J_{36} \\
&= 9(25)^{2m-1} J_{\frac{3(25^{2m}-1)}{2}} - 9(25)^{2m-1}(I_{\frac{(25^{2m}-1)}{24}} \otimes J_{36}) \\
&- 9(25^{2m-1} - 25^{2m-2})((J_{\frac{(25^{2m}-1)}{24}} - I_{\frac{(25^{2m}-1)}{24}}) \otimes J_{36}) - 9W \otimes J_{36} \\
&= 9(25)^{2m-1}((J_{\frac{(25^{2m}-1)}{24}} - I_{\frac{(25^{2m}-1)}{24}}) \otimes J_{36}) \\
&- 9(25^{2m-1} - 25^{2m-2})((J_{\frac{(25^{2m}-1)}{24}} - I_{\frac{(25^{2m}-1)}{24}}) \otimes J_{36}) - 9W \otimes J_{36} \\
&= 9(25)^{2m-2}((J_{\frac{(25^{2m}-1)}{24}} - I_{\frac{(25^{2m}-1)}{24}}) \otimes J_{36}) - 9W \otimes J_{36} \\
&= 9(25)^{2m-2} \sum_{i=5}^{9} A_i - 9 \sum_{i=5}^{8} A_i.
\end{aligned}$$

For an arbitrary element $\sigma = g^t$ of $G'(3,d)$, suppose $t \equiv s \pmod 3$ and $t \equiv u \pmod r$. We have

$$\begin{aligned}
(C^k R_4 \otimes J_9)\sigma(N_0 - [R_4 \otimes (J_3 - I_3) \otimes J_3]) &= (C^k R_4 \otimes J_9)(\sigma N_0 - [C^u R_4 \otimes (J_3 - \omega_s) \otimes J_3]) \\
&= 3(J_4 - C^{k-u}) \otimes J_9.
\end{aligned}$$

By this, the diagonal blocks of the matrix

$$(W \otimes [R_4 \otimes J_9]) \Big[W \otimes N_0 - W \otimes [R_4 \otimes (J_3 - I_3) \otimes J_3] \Big]$$

all are equal to $3(25)^{2m-1}((J_4 - I_4) \otimes J_9)$, and all off diagonal blocks are equal to

$$\begin{aligned}
&3\left(\frac{25^{2m-1} - 25^{2m-2}}{4}\right) \sum_{i=0}^{3} (J_4 - C^i R_4) \otimes J_9 \\
&= 3\left(\frac{25^{2m-1} - 25^{2m-2}}{4}\right) (4J_4 - \sum_{i=0}^{3} C^i R_4) \otimes J_9
\end{aligned}$$

$$
\begin{aligned}
&= 3\left(\frac{25^{2m-1}-25^{2m-2}}{4}\right)(4J_4 - J_4)\otimes J_9\\
&= 9\left(\frac{25^{2m-1}-25^{2m-2}}{4}\right)J_4\otimes J_9\\
&= 9\left(\frac{25^{2m-1}-25^{2m-2}}{4}\right)J_{36}.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&A_5A_6 = A_6A_5 = (W\otimes[R_4\otimes J_9])\Big[W\otimes N_0 - W\otimes[R_4\otimes(J_3-I_3)\otimes J_3]\Big]\\
&= 3(25)^{2m-1}I_{\frac{(25^{2m}-1)}{24}}\otimes(J_4-I_4)\otimes J_9 + 9\left(\frac{25^{2m-1}-25^{2m-2}}{4}\right)(J_{\frac{(25^{2m}-1)}{24}} - I_{\frac{(25^{2m}-1)}{24}})\otimes J_9\\
&= 3(25)^{2m-1}A_4 + 9\left(\frac{25^{2m-1}-25^{2m-2}}{4}\right)\sum_{i=5}^{9}A_i.
\end{aligned}
$$

Likewise, we conclude

$$
A_5A_7 = A_7A_5 = A_5A_8 = A_8A_5 = 3(25)^{2m-1}A_4 + 9\left(\frac{25^{2m-1}-25^{2m-2}}{4}\right)\sum_{i=5}^{9}A_i.
$$

Since A_1 and A_2 are not symmetric, this association scheme is not symmetric. Furthermore, the inequality $A_1A_6 \neq A_6A_1$ demonstrates that this association is not commutative. Only when $m=1$, the matrix A_9 is equal to zero. In this case, we have an 8-class non-commutative association scheme. □

6.2 Decomposition of the Complete Designs with $2^{2m}(2^m+2)$ Points

The results of this section and the next section are presented in [16] where the authors present a decomposition of the complete design on $2^{2m}(2^m+2)$ points. Then, they show how to use this decomposition to induce an infinite class of symmetric association schemes. In the construction introduced in Section 4.2, set $p=2$ and $d=1$. By Definition 3.10, we have $\Omega(2,m) = \{A_1\otimes\ldots\otimes A_m | A_1,\ldots,A_m \in G_2\}$, where $G_2 = \{\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]\}$. In Example

3.13, we observe that $\Omega(2,1)=G_2$, and

$$\Omega(2,2)=\left\{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix},\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix},\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}\right\}.$$

For an odd prime power q and $\varepsilon\in\{1,\ldots,q-1\}$, Definition 4.6 defines the set $\Delta_\varepsilon(GH(q,1))$ consisting of q non-symmetric $(0,1)$-matrices of size q^2. However, the matrices of the set $\Delta_\varepsilon(GH(q,1))$ are all symmetric when $q=2^m$:

Lemma 6.9. *For $m\geq 1$ and $\varepsilon\in\{0,1,\ldots,2^m-1\}$, let $GH(2^m,1)$ be a generalized Hadamard matrix over the group $\Omega(2,m)=\{\omega_0,\ldots,\omega_{2^m-1}\}$. Then, A is symmetric, for each A in*

$$\Delta_\varepsilon(GH(2^m,1))=\{(I_{2^m}\otimes\omega_\varepsilon)r_\alpha^*r_\alpha|1\leq\alpha\leq 2^m,\ r_\alpha \textit{ is the } \alpha\textit{-th row of } GH(2^m,1)\}.$$

Proof. Let $A_\alpha=(I_q\otimes\omega_\varepsilon)r_\alpha^*r_\alpha$, for some $1\leq\alpha\leq 2^m$. The element of $\Omega(2,m)$ is generated by Kronecker product of two symmetric matrices $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ and $\begin{bmatrix}0&1\\1&0\end{bmatrix}$. Thus, the elements of $\Omega(2,m)$ are symmetric. Since the inverse of each element of the set $\Omega(2,m)$ is itself, the matrix $r_\alpha^*r_\alpha$ is symmetric. Hence, the matrix $A_\alpha=(I_q\otimes\omega_\varepsilon)r_\alpha^*r_\alpha$ is symmetric. $\square$

Theorem 6.10 ([16, Theorem 3.3, p. 361])**.** *Let $m\geq 1$. There is a decomposition of the complete graph on $2^{2m}(2^m+2)$ vertices into 2^m disjoint $SRG(2^{2m}(2^m+2),2^m(2^m+1),2^m,2^m)$ and one divisible design graph $DDG(2^{2m}(2^m+2),2^{2m}-1,2^{2m}-2,0,2^m+2,2^{2m})$.*

Proof. Since $r=\dfrac{2^{2m}-1}{2^m-1}=2^m+1$, $r+1$ is an even number. Therefore, by Lemma 3.8 there is a symmetric constant diagonal Latin square L over $\{0,1,\ldots,r\}$. For each $\varepsilon\in\{0,\ldots,2^m-1\}$, let $\Xi(2^m,1,\varepsilon)=\{A^{(0)}(\varepsilon),\ldots,A^{(r)}(\varepsilon)\}$. Let N_ε be the symmetric design made by replacing i with $A^{(i)}(\varepsilon)$ in L such that the zero block is on the diagonal.

For each $\varepsilon\in\{0,\ldots,2^m-1\}$, the matrix N_ε is symmetric with zero diagonal. Hence, N_ε is the adjacency matrix of a $SRG(2^{2m}(2^m+2),2^m(2^m+1),2^m,2^m)$.

Based on Theorem 4.12, two matrices N_ε and N_ϑ are disjoint when $\varepsilon\neq\vartheta$, which implies that the sum $\sum_{i=0}^{2^m-1}N_i$ is a $(0,1)$-matrix whose diagonal blocks of size 2^{2m} are equal to zero.

Since the matrix $\sum_{i=0}^{2^m-1} N_i$ contain $2^{2m}(2^m+1)$ one in each row, we conclude that the off-diagonal blocks of $\sum_{i=0}^{2^m-1} N_i$ all are equal to $J_{2^{2m}}$. Thus, we have

$$\sum_{i=0}^{2^m-1} N_i = (J_{2^m+2} - I_{2^m+2}) \otimes J_{2^{2m}}.$$

Therefore, the sum $I_{2^m+2} \otimes (J_{2^{2m}} - I_{2^{2m}}) + \sum_{i=0}^{2^m-1} N_i$ is equal to $J_{2^{2m}(2^m+2)} - I_{2^{2m}(2^m+2)}$ which is the adjacency matrix of the complete graph on $2^{2m}(2^m+2)$ vertices. One may note that the matrix $I_{2^m+2} \otimes (J_{2^{2m}} - I_{2^{2m}})$ is the adjacency matrix of a divisible design graph $DDG(2^{2m}(2^m+2), 2^{2m}-1, 2^{2m}-2, 0, 2^m+2, 2^{2m})$. Hence, we come to the conclusion that the set $\{N_0, \ldots, N_{2^m-1}\} \cup \{I_{2^m+2} \otimes (J_{2^{2m}} - I_{2^{2m}})\}$ forms a decomposition of the complete graph on $2^{2m}(2^m+2)$ vertices into 2^m disjoint $SRG(2^{2m}(2^m+2), 2^m(2^m+1), 2^m, 2^m)$ and a $DDG(2^{2m}(2^m+2), 2^{2m}-1, 2^{2m}-2, 0, 2^m+2, 2^{2m})$. $\square$

In [16, p. 363], one may find an example in which the complete graph on 16 vertices is decomposed into two $SRG(16,6,2,2)$ and a $DDG(16,3,2,0,4,4)$. Here, we move to the next case and decompose a complete graph on 96 vertices.

Example 6.11. In this example, we decompose the complete graph on 96 vertices into four $SRG(96,20,4,4)$ and a $DDG(96,15,14,0,6,16)$. Let $m=2$ and $q=2^m=4$. Following the method mentioned in the proof of Theorem 6.10, we first define four sets

$$\Xi(4,1,0), \ldots, \Xi(4,1,3).$$

For each $i = 0,1,2,3$, we know

$$\Xi(4,1,i) = \Delta_i(GH(4,1)) \cup \{0\} \cup \{\omega_i \otimes J_4\},$$

where $GH(4,1) = \begin{bmatrix} \omega_0 & \omega_0 & \omega_0 & \omega_0 \\ \omega_0 & \omega_1 & \omega_2 & \omega_3 \\ \omega_0 & \omega_2 & \omega_3 & \omega_1 \\ \omega_0 & \omega_3 & \omega_1 & \omega_2 \end{bmatrix}$ is a generalized Hadamard matrix over the group

$$\Omega(2,2) = \left\{ \omega_0 = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix},\ \omega_1 = \begin{bmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{bmatrix},\ \omega_2 = \begin{bmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{bmatrix},\ \omega_3 = \begin{bmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{bmatrix} \right\}.$$

Following Definition 4.6, we have

$\Xi(4,1,0) = \{0\} \cup$

$$\left\{ \begin{bmatrix} 1111000000000000 \\ 1111000000000000 \\ 1111000000000000 \\ 1111000000000000 \\ 0000111100000000 \\ 0000111100000000 \\ 0000111100000000 \\ 0000111100000000 \\ 0000000011110000 \\ 0000000011110000 \\ 0000000011110000 \\ 0000000011110000 \\ 0000000000001111 \\ 0000000000001111 \\ 0000000000001111 \\ 0000000000001111 \end{bmatrix}, \begin{bmatrix} 1000100010001000 \\ 0100010001000100 \\ 0010001000100010 \\ 0001000100010001 \\ 1000100010001000 \\ 0100010001000100 \\ 0010001000100010 \\ 0001000100010001 \\ 1000100010001000 \\ 0100010001000100 \\ 0010001000100010 \\ 0001000100010001 \\ 1000100010001000 \\ 0100010001000100 \\ 0010001000100010 \\ 0001000100010001 \end{bmatrix}, \begin{bmatrix} 1000010000100001 \\ 0100100000010010 \\ 0010000110000100 \\ 0001001001001000 \\ 0100100000010010 \\ 1000010000100001 \\ 0001001001001000 \\ 0010000110000100 \\ 0010000110000100 \\ 0001001001001000 \\ 1000010000100001 \\ 0100100000010010 \\ 0001001001001000 \\ 0010000110000100 \\ 0100100000010010 \\ 1000010000100001 \end{bmatrix}, \begin{bmatrix} 1000001000010100 \\ 0100000100101000 \\ 0010100001000001 \\ 0001010010000010 \\ 0010100001000001 \\ 0001010010000010 \\ 1000001000010100 \\ 0100000100101000 \\ 0001010010000010 \\ 0010100001000001 \\ 0100000100101000 \\ 1000001000010100 \\ 0100000100101000 \\ 1000001000010100 \\ 0001010010000010 \\ 0010100001000001 \end{bmatrix}, \begin{bmatrix} 1000000101000010 \\ 0100001010000001 \\ 0010010000011000 \\ 0001100000100100 \\ 0001100000100100 \\ 0010010000011000 \\ 0100001010000001 \\ 1000000101000010 \\ 0100001010000001 \\ 1000000101000010 \\ 0001100000100100 \\ 0010010000011000 \\ 0010010000011000 \\ 0001100000100100 \\ 1000000101000010 \\ 0100001010000001 \end{bmatrix} \right\},$$

$\Xi(4,1,1) = \{0\} \cup$

$$\left\{ \begin{bmatrix} 0000111100000000 \\ 0000111100000000 \\ 0000111100000000 \\ 0000111100000000 \\ 1111000000000000 \\ 1111000000000000 \\ 1111000000000000 \\ 1111000000000000 \\ 0000000000001111 \\ 0000000000001111 \\ 0000000000001111 \\ 0000000000001111 \\ 0000000011110000 \\ 0000000011110000 \\ 0000000011110000 \\ 0000000011110000 \end{bmatrix}, \begin{bmatrix} 0100010001000100 \\ 1000100010001000 \\ 0001000100010001 \\ 0010001000100010 \\ 0100010001000100 \\ 1000100010001000 \\ 0001000100010001 \\ 0010001000100010 \\ 0100010001000100 \\ 1000100010001000 \\ 0001000100010001 \\ 0010001000100010 \\ 0100010001000100 \\ 1000100010001000 \\ 0001000100010001 \\ 0010001000100010 \end{bmatrix}, \begin{bmatrix} 0100100000010010 \\ 1000010000100001 \\ 0001001001001000 \\ 0010000110000100 \\ 1000010000100001 \\ 0100100000010010 \\ 0010000110000100 \\ 0001001001001000 \\ 0001001001001000 \\ 0010000110000100 \\ 0100100000010010 \\ 1000010000100001 \\ 0010000110000100 \\ 0001001001001000 \\ 1000010000100001 \\ 0100100000010010 \end{bmatrix}, \begin{bmatrix} 0100000100101000 \\ 1000001000010100 \\ 0001010010000010 \\ 0010100001000001 \\ 0001010010000010 \\ 0010100001000001 \\ 0100000100101000 \\ 1000001000010100 \\ 0010100001000001 \\ 0001010010000010 \\ 1000001000010100 \\ 0100000100101000 \\ 1000001000010100 \\ 0100000100101000 \\ 0010100001000001 \\ 0001010010000010 \end{bmatrix}, \begin{bmatrix} 0100001010000001 \\ 1000000101000010 \\ 0001100000100100 \\ 0010010000011000 \\ 0010010000011000 \\ 0001100000100100 \\ 1000000101000010 \\ 0100001010000001 \\ 1000000101000010 \\ 0100001010000001 \\ 0010010000011000 \\ 0001100000100100 \\ 0001100000100100 \\ 0010010000011000 \\ 0100001010000001 \\ 1000000101000010 \end{bmatrix} \right\},$$

$\Xi(4,1,2) = \{0\} \cup$

$$\left\{ \begin{bmatrix} 0000000011110000 \\ 0000000011110000 \\ 0000000011110000 \\ 0000000011110000 \\ 0000000000001111 \\ 0000000000001111 \\ 0000000000001111 \\ 0000000000001111 \\ 1111000000000000 \\ 1111000000000000 \\ 1111000000000000 \\ 1111000000000000 \\ 0000111100000000 \\ 0000111100000000 \\ 0000111100000000 \\ 0000111100000000 \end{bmatrix}, \begin{bmatrix} 0010001000100010 \\ 0001000100010001 \\ 1000100010001000 \\ 0100010001000100 \\ 0010001000100010 \\ 0001000100010001 \\ 1000100010001000 \\ 0100010001000100 \\ 0010001000100010 \\ 0001000100010001 \\ 1000100010001000 \\ 0100010001000100 \\ 0010001000100010 \\ 0001000100010001 \\ 1000100010001000 \\ 0100010001000100 \end{bmatrix}, \begin{bmatrix} 0010000110000100 \\ 0001001001001000 \\ 1000010000100001 \\ 0100100000010010 \\ 0001001001001000 \\ 0010000110000100 \\ 0100100000010010 \\ 1000010000100001 \\ 1000010000100001 \\ 0100100000010010 \\ 0010000110000100 \\ 0001001001001000 \\ 0100100000010010 \\ 1000010000100001 \\ 0001001001001000 \\ 0010000110000100 \end{bmatrix}, \begin{bmatrix} 0010100001000001 \\ 0001010010000010 \\ 1000001000010100 \\ 0100000100101000 \\ 1000001000010100 \\ 0100000100101000 \\ 0010100001000001 \\ 0001010010000010 \\ 0100000100101000 \\ 1000001000010100 \\ 0001010010000010 \\ 0010100001000001 \\ 0001010010000010 \\ 0010100001000001 \\ 0100000100101000 \\ 1000001000010100 \end{bmatrix}, \begin{bmatrix} 0010010000011000 \\ 0001100000100100 \\ 1000000101000010 \\ 0100001010000001 \\ 0100001010000001 \\ 1000000101000010 \\ 0001100000100100 \\ 0010010000011000 \\ 0001100000100100 \\ 0010010000011000 \\ 0100001010000001 \\ 1000000101000010 \\ 1000000101000010 \\ 0100001010000001 \\ 0010010000011000 \\ 0001100000100100 \end{bmatrix} \right\},$$

$\Xi(4,1,3) = \{0\} \cup$

$$\left\{ \begin{bmatrix} 0000000000001111 \\ 0000000000001111 \\ 0000000000001111 \\ 0000000000001111 \\ 0000000011110000 \\ 0000000011110000 \\ 0000000011110000 \\ 0000000011110000 \\ 0000111100000000 \\ 0000111100000000 \\ 0000111100000000 \\ 0000111100000000 \\ 1111000000000000 \\ 1111000000000000 \\ 1111000000000000 \\ 1111000000000000 \end{bmatrix}, \begin{bmatrix} 0001000100010001 \\ 0010001000100010 \\ 0100010001000100 \\ 1000100010001000 \\ 0001000100010001 \\ 0010001000100010 \\ 0100010001000100 \\ 1000100010001000 \\ 0001000100010001 \\ 0010001000100010 \\ 0100010001000100 \\ 1000100010001000 \\ 0001000100010001 \\ 0010001000100010 \\ 0100010001000100 \\ 1000100010001000 \end{bmatrix}, \begin{bmatrix} 0001001001001000 \\ 0010000110000100 \\ 0100100000010010 \\ 1000010000100001 \\ 0010000110000100 \\ 0001001001001000 \\ 1000010000100001 \\ 0100100000010010 \\ 0100100000010010 \\ 1000010000100001 \\ 0001001001001000 \\ 0010000110000100 \\ 1000010000100001 \\ 0100100000010010 \\ 0010000110000100 \\ 0001001001001000 \end{bmatrix}, \begin{bmatrix} 0001010010000010 \\ 0010100001000001 \\ 0100000100101000 \\ 1000001000010100 \\ 0100000100101000 \\ 1000001000010100 \\ 0001010010000010 \\ 0010100001000001 \\ 1000001000010100 \\ 0100000100101000 \\ 0010100001000001 \\ 0001010010000010 \\ 0010100001000001 \\ 0001010010000010 \\ 1000001000010100 \\ 0100000100101000 \end{bmatrix}, \begin{bmatrix} 0001100000100100 \\ 0010010000011000 \\ 0100001010000001 \\ 1000000101000010 \\ 1000000101000010 \\ 0100001010000001 \\ 0010010000011000 \\ 0001100000100100 \\ 0010010000011000 \\ 0001100000100100 \\ 1000000101000010 \\ 0100001010000001 \\ 0100001010000001 \\ 1000000101000010 \\ 0001100000100100 \\ 0010010000011000 \end{bmatrix} \right\}.$$

For each $\varepsilon \in \{0,1,2,3\}$ to make N_ε, insert the elements of $\Xi(4,1,\varepsilon)$ in the Latin square $L = \begin{bmatrix} 5&1&2&3&4&0 \\ 1&5&3&4&0&2 \\ 2&3&5&0&1&4 \\ 3&4&0&5&2&1 \\ 4&0&1&2&5&3 \\ 0&2&4&1&3&5 \end{bmatrix}$ such that the zero block is on the diagonal. The matrices $N_1,\dots,N_3$ are the adjacency matrices of $SRG(96,20,4,4)$. Theorem 6.10 confirms that the set

$$\{N_0, N_1, N_2, N_3, N_4 = I_6 \otimes (J_{16} - I_{16})\}$$

decomposes the complete graph on 96 vertices into four $SRG(96,20,4,4)$ and a

$$DDG(96,15,14,0,6,16).$$

One may find all five matrices $N_0,\dots,N_4$ in Appendix 9.

6.3 Some Symmetric Association Schemes

Utilizing the decomposition studied in Theorem 6.10, the following theorem demonstrates an infinite class of symmetric association schemes. The following theorem uses Latin squares which are symmetric, constant diagonal with constant back diagonal. Since the size of these Latin squares are even, the existence of such objects follows from Lemma 3.8.

Theorem 6.12 ([16, Theorem 3.4, p. 362])**.** *Let $q = 2^m$, where $m \geq 1$. Let L be a symmetric constant diagonal Latin square having constant back diagonal over $\{0,2,\dots,q+1\}$. For each $\varepsilon \in \{0,\dots,2^m-1\}$, let $\Xi(2^m,1,\varepsilon) = \{A^{(0)}(\varepsilon),\dots,A^{(r)}(\varepsilon)\}$. Let N_ε be the symmetric* *design made by replacing i with* $A^{(i)}(\varepsilon)$ *in L such that the zero block is on the diagonal and the block* $\omega_\varepsilon \otimes J_q$ *is on the back diagonal. Then, the set*

$$\mathcal{A} = \{A_0, \ldots, A_{q-1}, B_1, \ldots, B_{q-1}, C_0, \ldots, C_{q-1}, D_0, \ldots, D_{q-1}\}$$

forms a symmetric $4q-2$*-class association scheme, where*

$A_i = I_{q(q+2)} \otimes \omega_i$, *for* $i = 0, \ldots, q-1$*;*

$B_i = I_{q+2} \otimes \omega_i \otimes J_q$, *for* $i = 1, \ldots, q-1$*;*

$C_i = R_{q+2} \otimes \omega_i \otimes J_q$, *for* $i = 0, \ldots, q-1$*; and*

$D_i = N_i - C_i$, *for* $i = 0, \ldots, q-1$.

Proof. We need to prove that the set $\mathcal{A}$ satisfies all four axioms mentioned in Definition 6.3. We first note that $A_0 = I_{q(q+2)} \otimes \omega_0 = I_{q(q+2)} \otimes I_q = I_{q^2(q+2)}$. By the proof of Theorem 6.10, N_i is a symmetric matrix, for each $i = 0, \ldots, q-1$. Since a Kronecker product of symmetric matrices is a symmetric matrix, C_i is symmetric. So D_i is symmetric, for each $0 \leq i \leq q-1$. Since the elements of the group $\Omega(2,m)$ are symmetric, we conclude A_i and B_j are symmetric, for all $i = 0, \ldots, q-1$ and $j = 1, \ldots, q-1$. So $\mathcal{A}$ satisfies the first and second axioms. To check the forth axiom, we note that

$$\begin{aligned}
\sum_{i=0}^{q-1} A_i + \sum_{i=1}^{q-1} B_i + \sum_{i=0}^{q-1} C_i + \sum_{i=0}^{q-1} D_i &= \sum_{i=0}^{q-1} I_{q(q+2)} \otimes \omega_i + \sum_{i=1}^{q-1} I_{q+2} \otimes \omega_i \otimes J_q + \sum_{i=0}^{q-1} N_i \\
&= I_{q(q+2)} \otimes J_q + I_{q+2} \otimes (J_q - I_q) \otimes J_q + \sum_{i=0}^{q-1} N_i \\
&= I_{q+2} \otimes J_{q^2} + \sum_{i=0}^{q-1} N_i \\
&= I_{q+2} \otimes J_{q^2} + (J_{q+2} - I_{q+2}) \otimes J_{q^2} \\
&= J_{q^2(q+2)}.
\end{aligned}$$

Hence, only the third axiom of Definition 6.3 is left to be proved. We need to show that the multiplication of any two matrices in $\mathcal{A}$ is a linear combination of the elements of $\mathcal{A}$. For $i, j \in \{0, \ldots, q-1\}$, we have

$$\begin{aligned} A_i A_j &= (I_{q(q+2)} \otimes \omega_i)(I_{q(q+2)} \otimes \omega_j) \\ &= I_{q(q+2)} \otimes (\omega_i)(\omega_j) \\ &= I_{q(q+2)} \otimes \omega_{i+j} \\ &= A_{i+j}. \end{aligned}$$

For $i, j \in \{1, \ldots, q-1\}$, if $i \neq j$ then we have

$$\begin{aligned} B_i B_j &= (I_{q+2} \otimes \omega_i \otimes J_q)(I_{q+2} \otimes \omega_j \otimes J_q) \\ &= q(I_{q+2} \otimes (\omega_i \omega_j) \otimes J_q) \\ &= q(I_{q+2} \otimes \omega_{i+j} \otimes J_q) \\ &= qB_{i+j}, \end{aligned}$$

while for $i = j$, we find

$$\begin{aligned} B_i B_i &= (I_{q+2} \otimes \omega_i \otimes J_q)(I_{q+2} \otimes \omega_i \otimes J_q) \\ &= q(I_{q+2} \otimes (\omega_i \omega_i) \otimes J_q) \\ &= q(I_{q+2} \otimes I_q \otimes J_q) \\ &= q \sum_{l=0}^{q-1} A_l. \end{aligned}$$

Moreover, for $i \neq j \in \{0, \ldots, q-1\}$, we have

$$\begin{aligned} C_i C_j &= (R_{q+2} \otimes \omega_i \otimes J_q)(R_{q+2} \otimes \omega_j \otimes J_q) \\ &= q(I_{q+2} \otimes (\omega_i \omega_j) \otimes J_q) \end{aligned}$$

$$= q(I_{q+2} \otimes \omega_{i+j} \otimes J_q)$$
$$= qB_{i+j},$$

while for $i = j$, we have

$$\begin{aligned} C_iC_i &= (R_{q+2} \otimes \omega_i \otimes J_q)(R_{q+2} \otimes \omega_i \otimes J_q) \\ &= q(I_{q+2} \otimes (\omega_i\omega_i) \otimes J_q) \\ &= q(I_{q+2} \otimes I_q \otimes J_q) \\ &= q\sum_{l=0}^{q-1} A_l. \end{aligned}$$

Furthermore, for $i, j \in \{0, \ldots, q-1\}$ and $1 \leq l \leq q-1$, we note

$$\begin{aligned} A_iB_l = B_lA_i &= (I_{q(q+2)} \otimes \omega_i)(I_{q+2} \otimes \omega_l \otimes J_q) \\ &= (I_{q+2} \otimes I_q \otimes \omega_i)(I_{q+2} \otimes \omega_l \otimes J_q) \\ &= I_{q+2} \otimes \omega_l \otimes J_q \\ &= B_l, \end{aligned}$$

$$\begin{aligned} A_iC_j = C_jA_i &= (I_{q(q+2)} \otimes \omega_i)(R_{q+2} \otimes \omega_j \otimes J_q) \\ &= (I_{q+2} \otimes I_q \otimes \omega_i)(R_{q+2} \otimes \omega_j \otimes J_q) \\ &= R_{q+2} \otimes \omega_j \otimes J_q \\ &= C_j, \quad \text{and} \end{aligned}$$

$$\begin{aligned} B_lC_j = C_jB_l &= (I_{q+2} \otimes \omega_l \otimes J_q)(R_{q+2} \otimes \omega_j \otimes J_q) \\ &= q(R_{q+2} \otimes (\omega_l\omega_j) \otimes J_q) \\ &= q(R_{q+2} \otimes \omega_{l+j} \otimes J_q) \end{aligned}$$

$$= qC_{l+j}.$$

Let $i, j \in \{0, \ldots, q-1\}$ and $A(i)$ be in $\Delta_i(GH(q,1))$. Since the elements of the group $\Omega(2,m)$ are symmetric, we have $\omega_i\omega_j = \omega_j\omega_i = \omega_{i+j}$. This results in

$$(I_{q(q+2)} \otimes \omega_j)A(i) = A(i)(I_{q(q+2)} \otimes \omega_j) = A(i+j).$$

All nonzero blocks of size q of the matrix D_i are elements of $\Delta_i(GH(q,1))$. Hence, we conclude

$$\begin{aligned} A_iD_j = D_jA_i &= (I_{q(q+2)} \otimes \omega_j)D_j \\ &= D_{i+j}. \end{aligned}$$

Now, let $0 \leq i \leq q-1$ and $1 \leq j \leq q-1$. All nonzero blocks of the size q of the matrix D_i are elements of $\Omega(2,m)$ and $J_q\omega_\varepsilon = J_q$, for all $\omega_\varepsilon \in \Omega(2,m)$. Therefore, we conclude

$$\begin{aligned} B_iD_j = D_jB_i &= (I_{q+2} \otimes \omega_i \otimes J_q)D_j \\ &= (J_{q+2} - I_{q+2} - R_{q+2}) \otimes J_{q^2} \\ &= \sum_{l=0}^{q-1} D_l, \quad \text{and} \\ C_iD_j = D_jC_i &= (R_{q+2} \otimes \omega_i \otimes J_q)D_j \\ &= (J_{q+2} - I_{q+2} - R_{q+2}) \otimes J_{q^2} \\ &= \sum_{l=0}^{q-1} D_l. \end{aligned}$$

For $i, j \in \{0, \ldots, q-1\}$, we calculate the matrix $D_iD_j = D_jD_i = [M_{kl}]$, where M_{kl} is the block of size q^2. For $k = l$, we have

$$\begin{aligned}
M_{kk} &= \sum_{\substack{A(i)\in\Delta_i(GH(q,1))\\ A(j)\in\Delta_j(GH(q,1))}} A(i)A(j)\\
&= q \sum_{\{A(i+j)\in\Delta_{i+j}(GH(q,1))\}} A(i+j)\\
&= q^2(I_q \otimes \omega_{i+j}) + q((J_q - I_q) \otimes J_q).
\end{aligned}$$

Hence, we find

$$\begin{aligned}
D_iD_j = D_jD_i &= I_{q+2} \otimes [q^2(I_q \otimes (\omega_{i+j})) + q((J_q - I_q) \otimes J_q)]\\
&\quad + \left[(q-2)(J_{q+2} - I_{q+2}) + 2R_{q+2}\right] \otimes J_{q^2}\\
&= q^2A_{i+j} + q\sum_{l=1}^{q-1} B_l + (q-2)\sum_{l=0}^{q-1}[D_l + C_l] + 2\sum_{l=0}^{q-1} C_l\\
&= q^2A_{i+j} + q\sum_{l=1}^{q-1} B_l + q\sum_{l=0}^{q-1} C_l + (q-2)\sum_{l=0}^{q-1} D_l.
\end{aligned}$$

□

Considering the decomposition studied in Example 6.11, we now generate a symmetric 14-class association scheme in the next example.

Example 6.13. In Example 6.11, the set $\{N_0, N_1, N_2, N_3, N_4 = I_6 \otimes (J_{16} - I_{16})\}$ decomposes the complete graph on 96 vertices into four $SRG(96, 20, 4, 4)$ and a $DDG(96, 15, 14, 0, 6, 16)$. Now, we define an association scheme $\{A_0, \ldots, A_3, B_1, B_2, B_3, C_0, \ldots, C_3, D_0, \ldots, D_3\}$ such that $N_4 = \sum_{i=1}^{3}[A_i + B_i]$ and $N_l = C_l + D_l$, for each $l = 0, 1, 2, 3$. We have

$A_i = I_{24} \otimes \omega_i$, for $i = 0, 1, 2, 3$;

$B_i = I_6 \otimes \omega_i \otimes J_4$, for $i = 1, 2, 3$;

$C_i = R_6 \otimes \omega_i \otimes J_4$, for $i = 0, 1, 2, 3$;

$D_i = N_i - C_i$, for $i = 0, 1, 2, 3$,

where $\omega_i \in \Omega(2,2)$. All elements of the set $\{A_0, \ldots, A_3, B_1, B_2, B_3, C_0, \ldots, C_3, D_0, \ldots, D_3\}$ appear in Appendix 10.

In the construction of designs and graphs in this thesis, we used Latin squares. One of the open questions is whether different Latin squares generate equivalent designs and graphs or not. Studying various properties of these designs and graphs may lead us to new combinatorial objects.

In section 6.1, we combined Spence designs and balanced generalized weighing matrices to generate infinite classes of non-commutative association schemes. As a part of the future research plan, one can use the same process for McFarland Designs.

There are some examples in which McFarland and Spence designs generate amorphous association scheme which can be generalized to an infinite class of amorphous association schemes.

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Appendix 1

The Symmetric $(45, 12, 3)$-Design constructed in Example 3.6

$$
N = \begin{bmatrix}
000000000 & 111000000 & 100100100 & 001010100 & 001100010 \\
000000000 & 111000000 & 001001001 & 010100001 & 010001100 \\
000000000 & 111000000 & 010010010 & 100001010 & 100010001 \\
000000000 & 000000111 & 100100100 & 010100001 & 100010001 \\
000000000 & 000000111 & 001001001 & 100001010 & 001100010 \\
000000000 & 000000111 & 010010010 & 001010100 & 010001100 \\
000000000 & 000111000 & 100100100 & 100001010 & 010001100 \\
000000000 & 000111000 & 001001001 & 001010100 & 100010001 \\
000000000 & 000111000 & 010010010 & 010100001 & 001100010 \\
111000000 & 100100100 & 001010100 & 001100010 & 000000000 \\
111000000 & 001001001 & 010100001 & 010001100 & 000000000 \\
111000000 & 010010010 & 100001010 & 100010001 & 000000000 \\
000000111 & 100100100 & 010100001 & 100010001 & 000000000 \\
000000111 & 001001001 & 100001010 & 001100010 & 000000000 \\
000000111 & 010010010 & 001010100 & 010001100 & 000000000 \\
000111000 & 100100100 & 100001010 & 010001100 & 000000000 \\
000111000 & 001001001 & 001010100 & 100010001 & 000000000 \\
000111000 & 010010010 & 010100001 & 001100010 & 000000000 \\
100100100 & 001010100 & 001100010 & 000000000 & 111000000 \\
001001001 & 010100001 & 010001100 & 000000000 & 111000000 \\
010010010 & 100001010 & 100010001 & 000000000 & 111000000 \\
100100100 & 010100001 & 100010001 & 000000000 & 000000111 \\
001001001 & 100001010 & 001100010 & 000000000 & 000000111 \\
010010010 & 001010100 & 010001100 & 000000000 & 000000111 \\
100100100 & 100001010 & 010001100 & 000000000 & 000111000 \\
001001001 & 001010100 & 100010001 & 000000000 & 000111000 \\
010010010 & 010100001 & 001100010 & 000000000 & 000111000 \\
001010100 & 001100010 & 000000000 & 111000000 & 100100100 \\
010100001 & 010001100 & 000000000 & 111000000 & 001001001 \\
100001010 & 100010001 & 000000000 & 111000000 & 010010010 \\
010100001 & 100010001 & 000000000 & 000000111 & 100100100 \\
100001010 & 001100010 & 000000000 & 000000111 & 001001001 \\
001010100 & 010001100 & 000000000 & 000000111 & 010010010 \\
100001010 & 010001100 & 000000000 & 000111000 & 100100100 \\
001010100 & 100010001 & 000000000 & 000111000 & 001001001 \\
010100001 & 001100010 & 000000000 & 000111000 & 010010010 \\
001100010 & 000000000 & 111000000 & 100100100 & 001010100 \\
010001100 & 000000000 & 111000000 & 001001001 & 010100001 \\
100010001 & 000000000 & 111000000 & 010010010 & 100001010 \\
100010001 & 000000000 & 000000111 & 100100100 & 010100001 \\
001100010 & 000000000 & 000000111 & 001001001 & 100001010 \\
010001100 & 000000000 & 000000111 & 010010010 & 001010100 \\
010001100 & 000000000 & 000111000 & 100100100 & 100001010 \\
100010001 & 000000000 & 000111000 & 001001001 & 001010100 \\
001100010 & 000000000 & 000111000 & 010010010 & 010100001
\end{bmatrix}
$$

Appendix 2

The Symmetric $(45, 12, 3)$-Design constructed in Example 3.26

$$N = \begin{bmatrix}
000000000 & 111000000 & 100100100 & 100010001 & 100001010 \\
000000000 & 111000000 & 010010010 & 010001100 & 010100001 \\
000000000 & 111000000 & 001001001 & 001100010 & 001010100 \\
000000000 & 000111000 & 100100100 & 010001100 & 001010100 \\
000000000 & 000111000 & 010010010 & 001100010 & 100001010 \\
000000000 & 000111000 & 001001001 & 100010001 & 010100001 \\
000000000 & 000000111 & 100100100 & 001100010 & 010100001 \\
000000000 & 000000111 & 010010010 & 100010001 & 001010100 \\
000000000 & 000000111 & 001001001 & 010001100 & 100001010 \\
100001010 & 000000000 & 111000000 & 100100100 & 100010001 \\
010100001 & 000000000 & 111000000 & 010010010 & 010001100 \\
001010100 & 000000000 & 111000000 & 001001001 & 001100010 \\
001010100 & 000000000 & 000111000 & 100100100 & 010001100 \\
100001010 & 000000000 & 000111000 & 010010010 & 001100010 \\
010100001 & 000000000 & 000111000 & 001001001 & 100010001 \\
010100001 & 000000000 & 000000111 & 100100100 & 001100010 \\
001010100 & 000000000 & 000000111 & 010010010 & 100010001 \\
100001010 & 000000000 & 000000111 & 001001001 & 010001100 \\
100010001 & 100001010 & 000000000 & 111000000 & 100100100 \\
010001100 & 010100001 & 000000000 & 111000000 & 010010010 \\
001100010 & 001010100 & 000000000 & 111000000 & 001001001 \\
010001100 & 001010100 & 000000000 & 000111000 & 100100100 \\
001100010 & 100001010 & 000000000 & 000111000 & 010010010 \\
100010001 & 010100001 & 000000000 & 000111000 & 001001001 \\
001100010 & 010100001 & 000000000 & 000000111 & 100100100 \\
100010001 & 001010100 & 000000000 & 000000111 & 010010010 \\
010001100 & 100001010 & 000000000 & 000000111 & 001001001 \\
100100100 & 100010001 & 100001010 & 000000000 & 111000000 \\
010010010 & 010001100 & 010100001 & 000000000 & 111000000 \\
001001001 & 001100010 & 001010100 & 000000000 & 111000000 \\
100100100 & 010001100 & 001010100 & 000000000 & 000111000 \\
010010010 & 001100010 & 100001010 & 000000000 & 000111000 \\
001001001 & 100010001 & 010100001 & 000000000 & 000111000 \\
100100100 & 001100010 & 010100001 & 000000000 & 000000111 \\
010010010 & 100010001 & 001010100 & 000000000 & 000000111 \\
001001001 & 010001100 & 100001010 & 000000000 & 000000111 \\
111000000 & 100100100 & 100010001 & 100001010 & 000000000 \\
111000000 & 010010010 & 010001100 & 010100001 & 000000000 \\
111000000 & 001001001 & 001100010 & 001010100 & 000000000 \\
000111000 & 100100100 & 010001100 & 001010100 & 000000000 \\
000111000 & 010010010 & 001100010 & 100001010 & 000000000 \\
000111000 & 001001001 & 100010001 & 010100001 & 000000000 \\
000000111 & 100100100 & 001100010 & 010100001 & 000000000 \\
000000111 & 010010010 & 100010001 & 001010100 & 000000000 \\
000000111 & 001001001 & 010001100 & 100001010 & 000000000
\end{bmatrix}$$

Appendix 3

The Symmetric $(36, 15, 6)$-Design constructed in Example 3.30

$$
N = \begin{bmatrix}
000111111 & 001001001 & 001010100 & 001100010 \\
000111111 & 010010010 & 010100001 & 010001100 \\
000111111 & 100100100 & 100001010 & 100010001 \\
111000111 & 001001001 & 010100001 & 100010001 \\
111000111 & 010010010 & 100001010 & 001100010 \\
111000111 & 100100100 & 001010100 & 010001100 \\
111111000 & 001001001 & 100001010 & 010001100 \\
111111000 & 010010010 & 001010100 & 100010001 \\
111111000 & 100100100 & 010100001 & 001100010 \\[4pt]
001001001 & 000111111 & 001100010 & 001010100 \\
010010010 & 000111111 & 010001100 & 010100001 \\
100100100 & 000111111 & 100010001 & 100001010 \\
001001001 & 111000111 & 100010001 & 010100001 \\
010010010 & 111000111 & 001100010 & 100001010 \\
100100100 & 111000111 & 010001100 & 001010100 \\
001001001 & 111111000 & 010001100 & 100001010 \\
010010010 & 111111000 & 100010001 & 001010100 \\
100100100 & 111111000 & 001100010 & 010100001 \\[4pt]
001010100 & 001100010 & 000111111 & 001001001 \\
010100001 & 010001100 & 000111111 & 010010010 \\
100001010 & 100010001 & 000111111 & 100100100 \\
010100001 & 100010001 & 111000111 & 001001001 \\
100001010 & 001100010 & 111000111 & 010010010 \\
001010100 & 010001100 & 111000111 & 100100100 \\
100001010 & 010001100 & 111111000 & 001001001 \\
001010100 & 100010001 & 111111000 & 010010010 \\
010100001 & 001100010 & 111111000 & 100100100 \\[4pt]
001100010 & 001010100 & 001001001 & 000111111 \\
010001100 & 010100001 & 010010010 & 000111111 \\
100010001 & 100001010 & 100100100 & 000111111 \\
100010001 & 010100001 & 001001001 & 111000111 \\
001100010 & 100001010 & 010010010 & 111000111 \\
010001100 & 001010100 & 100100100 & 111000111 \\
010001100 & 100001010 & 001001001 & 111111000 \\
100010001 & 001010100 & 010010010 & 111111000 \\
001100010 & 010100001 & 100100100 & 111111000
\end{bmatrix}
$$

Appendix 4

Three Symmetric $(45, 12, 3)$-Designs constructed in Example 4.13

$$
N_0 = \begin{pmatrix}
000000000 & 111000000 & 100100100 & 100010001 & 100001010 \\
000000000 & 111000000 & 010010010 & 010001100 & 010100001 \\
000000000 & 111000000 & 001001001 & 001100010 & 001010100 \\
000000000 & 000111000 & 100100100 & 001100010 & 010100001 \\
000000000 & 000111000 & 010010010 & 100010001 & 001010100 \\
000000000 & 000111000 & 001001001 & 010001100 & 100001010 \\
000000000 & 000000111 & 100100100 & 010001100 & 001010100 \\
000000000 & 000000111 & 010010010 & 001100010 & 100001010 \\
000000000 & 000000111 & 001001001 & 100010001 & 010100001 \\
100001010 & 000000000 & 111000000 & 100100100 & 100010001 \\
010100001 & 000000000 & 111000000 & 010010010 & 010001100 \\
001010100 & 000000000 & 111000000 & 001001001 & 001100010 \\
010100001 & 000000000 & 000111000 & 100100100 & 001100010 \\
001010100 & 000000000 & 000111000 & 010010010 & 100010001 \\
100001010 & 000000000 & 000111000 & 001001001 & 010001100 \\
001010100 & 000000000 & 000000111 & 100100100 & 010001100 \\
100001010 & 000000000 & 000000111 & 010010010 & 001100010 \\
010100001 & 000000000 & 000000111 & 001001001 & 100010001 \\
100010001 & 100001010 & 000000000 & 111000000 & 100100100 \\
010001100 & 010100001 & 000000000 & 111000000 & 010010010 \\
001100010 & 001010100 & 000000000 & 111000000 & 001001001 \\
001100010 & 010100001 & 000000000 & 000111000 & 100100100 \\
100010001 & 001010100 & 000000000 & 000111000 & 010010010 \\
010001100 & 100001010 & 000000000 & 000111000 & 001001001 \\
010001100 & 001010100 & 000000000 & 000000111 & 100100100 \\
001100010 & 100001010 & 000000000 & 000000111 & 010010010 \\
100010001 & 010100001 & 000000000 & 000000111 & 001001001 \\
100100100 & 100010001 & 100001010 & 000000000 & 111000000 \\
010010010 & 010001100 & 010100001 & 000000000 & 111000000 \\
001001001 & 001100010 & 001010100 & 000000000 & 111000000 \\
100100100 & 001100010 & 010100001 & 000000000 & 000111000 \\
010010010 & 100010001 & 001010100 & 000000000 & 000111000 \\
001001001 & 010001100 & 100001010 & 000000000 & 000111000 \\
100100100 & 010001100 & 001010100 & 000000000 & 000000111 \\
010010010 & 001100010 & 100001010 & 000000000 & 000000111 \\
001001001 & 100010001 & 010100001 & 000000000 & 000000111 \\
111000000 & 100100100 & 100010001 & 100001010 & 000000000 \\
111000000 & 010010010 & 010001100 & 010100001 & 000000000 \\
111000000 & 001001001 & 001100010 & 001010100 & 000000000 \\
000111000 & 100100100 & 001100010 & 010100001 & 000000000 \\
000111000 & 010010010 & 100010001 & 001010100 & 000000000 \\
000111000 & 001001001 & 010001100 & 100001010 & 000000000 \\
000000111 & 100100100 & 010001100 & 001010100 & 000000000 \\
000000111 & 010010010 & 001100010 & 100001010 & 000000000 \\
000000111 & 001001001 & 100010001 & 010100001 & 000000000
\end{pmatrix}
$$

$$N_1 = \begin{pmatrix}
000000000 & 000111000 & 010010010 & 010001100 & 010100001 \\
000000000 & 000111000 & 001001001 & 001100010 & 001010100 \\
000000000 & 000111000 & 100100100 & 100010001 & 100001010 \\
000000000 & 000000111 & 010010010 & 100010001 & 001010100 \\
000000000 & 000000111 & 001001001 & 010001100 & 100001010 \\
000000000 & 000000111 & 100100100 & 001100010 & 010100001 \\
000000000 & 111000000 & 010010010 & 001100010 & 100001010 \\
000000000 & 111000000 & 001001001 & 100010001 & 010100001 \\
000000000 & 111000000 & 100100100 & 010001100 & 001010100 \\
010100001 & 000000000 & 000111000 & 010010010 & 010001100 \\
001010100 & 000000000 & 000111000 & 001001001 & 001100010 \\
100001010 & 000000000 & 000111000 & 100100100 & 100010001 \\
001010100 & 000000000 & 000000111 & 010010010 & 100010001 \\
100001010 & 000000000 & 000000111 & 001001001 & 010001100 \\
010100001 & 000000000 & 000000111 & 100100100 & 001100010 \\
100001010 & 000000000 & 111000000 & 010010010 & 001100010 \\
010100001 & 000000000 & 111000000 & 001001001 & 100010001 \\
001010100 & 000000000 & 111000000 & 100100100 & 010001100 \\
010001100 & 010100001 & 000000000 & 000111000 & 010010010 \\
001100010 & 001010100 & 000000000 & 000111000 & 001001001 \\
100010001 & 100001010 & 000000000 & 000111000 & 100100100 \\
100010001 & 001010100 & 000000000 & 000000111 & 010010010 \\
010001100 & 100001010 & 000000000 & 000000111 & 001001001 \\
001100010 & 010100001 & 000000000 & 000000111 & 100100100 \\
001100010 & 100001010 & 000000000 & 111000000 & 010010010 \\
100010001 & 010100001 & 000000000 & 111000000 & 001001001 \\
010001100 & 001010100 & 000000000 & 111000000 & 100100100 \\
010010010 & 010001100 & 010100001 & 000000000 & 000111000 \\
001001001 & 001100010 & 001010100 & 000000000 & 000111000 \\
100100100 & 100010001 & 100001010 & 000000000 & 000111000 \\
010010010 & 100010001 & 001010100 & 000000000 & 000000111 \\
001001001 & 010001100 & 100001010 & 000000000 & 000000111 \\
100100100 & 001100010 & 010100001 & 000000000 & 000000111 \\
010010010 & 001100010 & 100001010 & 000000000 & 111000000 \\
001001001 & 100010001 & 010100001 & 000000000 & 111000000 \\
100100100 & 010001100 & 001010100 & 000000000 & 111000000 \\
000111000 & 010010010 & 010001100 & 010100001 & 000000000 \\
000111000 & 001001001 & 001100010 & 001010100 & 000000000 \\
000111000 & 100100100 & 100010001 & 100001010 & 000000000 \\
000000111 & 010010010 & 100010001 & 001010100 & 000000000 \\
000000111 & 001001001 & 010001100 & 100001010 & 000000000 \\
000000111 & 100100100 & 001100010 & 010100001 & 000000000 \\
111000000 & 010010010 & 001100010 & 100001010 & 000000000 \\
111000000 & 001001001 & 100010001 & 010100001 & 000000000 \\
111000000 & 100100100 & 010001100 & 001010100 & 000000000
\end{pmatrix}$$

$$N_2 = \begin{pmatrix}
000000000 & 000000111 & 001001001 & 001100010 & 001010100 \\
000000000 & 000000111 & 100100100 & 100010001 & 100001010 \\
000000000 & 000000111 & 010010010 & 010001100 & 010100001 \\
000000000 & 111000000 & 001001001 & 010001100 & 100001010 \\
000000000 & 111000000 & 100100100 & 001100010 & 010100001 \\
000000000 & 111000000 & 010010010 & 100010001 & 001010100 \\
000000000 & 000111000 & 001001001 & 100010001 & 010100001 \\
000000000 & 000111000 & 100100100 & 010001100 & 001010100 \\
000000000 & 000111000 & 010010010 & 001100010 & 100001010 \\
001010100 & 000000000 & 000000111 & 001001001 & 001100010 \\
100001010 & 000000000 & 000000111 & 100100100 & 100010001 \\
010100001 & 000000000 & 000000111 & 010010010 & 010001100 \\
100001010 & 000000000 & 111000000 & 001001001 & 010001100 \\
010100001 & 000000000 & 111000000 & 100100100 & 001100010 \\
001010100 & 000000000 & 111000000 & 010010010 & 100010001 \\
010100001 & 000000000 & 000111000 & 001001001 & 100010001 \\
001010100 & 000000000 & 000111000 & 100100100 & 010001100 \\
100001010 & 000000000 & 000111000 & 010010010 & 001100010 \\
001100010 & 001010100 & 000000000 & 000000111 & 001001001 \\
100010001 & 100001010 & 000000000 & 000000111 & 100100100 \\
010001100 & 010100001 & 000000000 & 000000111 & 010010010 \\
010001100 & 100001010 & 000000000 & 111000000 & 001001001 \\
001100010 & 010100001 & 000000000 & 111000000 & 100100100 \\
100010001 & 001010100 & 000000000 & 111000000 & 010010010 \\
100010001 & 010100001 & 000000000 & 000111000 & 001001001 \\
010001100 & 001010100 & 000000000 & 000111000 & 100100100 \\
001100010 & 100001010 & 000000000 & 000111000 & 010010010 \\
001001001 & 001100010 & 001010100 & 000000000 & 000000111 \\
100100100 & 100010001 & 100001010 & 000000000 & 000000111 \\
010010010 & 010001100 & 010100001 & 000000000 & 000000111 \\
001001001 & 010001100 & 100001010 & 000000000 & 111000000 \\
100100100 & 001100010 & 010100001 & 000000000 & 111000000 \\
010010010 & 100010001 & 001010100 & 000000000 & 111000000 \\
001001001 & 100010001 & 010100001 & 000000000 & 000111000 \\
100100100 & 010001100 & 001010100 & 000000000 & 000111000 \\
010010010 & 001100010 & 100001010 & 000000000 & 000111000 \\
000000111 & 001001001 & 001100010 & 001010100 & 000000000 \\
000000111 & 100100100 & 100010001 & 100001010 & 000000000 \\
000000111 & 010010010 & 010001100 & 010100001 & 000000000 \\
111000000 & 001001001 & 010001100 & 100001010 & 000000000 \\
111000000 & 100100100 & 001100010 & 010100001 & 000000000 \\
111000000 & 010010010 & 100010001 & 001010100 & 000000000 \\
000111000 & 001001001 & 100010001 & 010100001 & 000000000 \\
000111000 & 100100100 & 010001100 & 001010100 & 000000000 \\
000111000 & 010010010 & 001100010 & 100001010 & 000000000
\end{pmatrix}$$

Appendix 5

Three SRG$(36, 15, 6, 6)$ constructed in Example 4.31

$$N_0 = \begin{bmatrix}
000111111 & 001001001 & 001010100 & 001100010 \\
000111111 & 010010010 & 010100001 & 010001100 \\
000111111 & 100100100 & 100001010 & 100010001 \\
111000111 & 001001001 & 010100001 & 100010001 \\
111000111 & 010010010 & 100001010 & 001100010 \\
111000111 & 100100100 & 001010100 & 010001100 \\
111111000 & 001001001 & 100001010 & 010001100 \\
111111000 & 010010010 & 001010100 & 100010001 \\
111111000 & 100100100 & 010100001 & 001100010 \\
001001001 & 000111111 & 001100010 & 001010100 \\
010010010 & 000111111 & 010001100 & 010100001 \\
100100100 & 000111111 & 100010001 & 100001010 \\
001001001 & 111000111 & 100010001 & 010100001 \\
010010010 & 111000111 & 001100010 & 100001010 \\
100100100 & 111000111 & 010001100 & 001010100 \\
001001001 & 111111000 & 010001100 & 100001010 \\
010010010 & 111111000 & 100010001 & 001010100 \\
100100100 & 111111000 & 001100010 & 010100001 \\
001010100 & 001100010 & 000111111 & 001001001 \\
010100001 & 010001100 & 000111111 & 010010010 \\
100001010 & 100010001 & 000111111 & 100100100 \\
010100001 & 100010001 & 111000111 & 001001001 \\
100001010 & 001100010 & 111000111 & 010010010 \\
001010100 & 010001100 & 111000111 & 100100100 \\
100001010 & 010001100 & 111111000 & 001001001 \\
001010100 & 100010001 & 111111000 & 010010010 \\
010100001 & 001100010 & 111111000 & 100100100 \\
001100010 & 001010100 & 001001001 & 000111111 \\
010001100 & 010100001 & 010010010 & 000111111 \\
100010001 & 100001010 & 100100100 & 000111111 \\
100010001 & 010100001 & 001001001 & 111000111 \\
001100010 & 100001010 & 010010010 & 111000111 \\
010001100 & 001010100 & 100100100 & 111000111 \\
010001100 & 100001010 & 001001001 & 111111000 \\
100010001 & 001010100 & 010010010 & 111111000 \\
001100010 & 010100001 & 100100100 & 111111000
\end{bmatrix}$$

$$N_1 = \begin{bmatrix}
000111111 & 010010010 & 010100001 & 010001100 \\
000111111 & 100100100 & 100001010 & 100010001 \\
000111111 & 001001001 & 001010100 & 001100010 \\
111000111 & 010010010 & 100001010 & 001100010 \\
111000111 & 100100100 & 001010100 & 010001100 \\
111000111 & 001001001 & 010100001 & 100010001 \\
111111000 & 010010010 & 001010100 & 100010001 \\
111111000 & 100100100 & 010100001 & 001100010 \\
111111000 & 001001001 & 100001010 & 010001100 \\
010010010 & 000111111 & 010001100 & 010100001 \\
100100100 & 000111111 & 100010001 & 100001010 \\
001001001 & 000111111 & 001100010 & 001010100 \\
010010010 & 111000111 & 001100010 & 100001010 \\
100100100 & 111000111 & 010001100 & 001010100 \\
001001001 & 111000111 & 100010001 & 010100001 \\
010010010 & 111111000 & 100010001 & 001010100 \\
100100100 & 111111000 & 001100010 & 010100001 \\
001001001 & 111111000 & 010001100 & 100001010 \\
010100001 & 010001100 & 000111111 & 010010010 \\
100001010 & 100010001 & 000111111 & 100100100 \\
001010100 & 001100010 & 000111111 & 001001001 \\
100001010 & 001100010 & 111000111 & 010010010 \\
001010100 & 010001100 & 111000111 & 100100100 \\
010100001 & 100010001 & 111000111 & 001001001 \\
001010100 & 100010001 & 111111000 & 010010010 \\
010100001 & 001100010 & 111111000 & 100100100 \\
100001010 & 010001100 & 111111000 & 001001001 \\
010001100 & 010100001 & 010010010 & 000111111 \\
100010001 & 100001010 & 100100100 & 000111111 \\
001100010 & 001010100 & 001001001 & 000111111 \\
001100010 & 100001010 & 010010010 & 111000111 \\
010001100 & 001010100 & 100100100 & 111000111 \\
100010001 & 010100001 & 001001001 & 111000111 \\
100010001 & 001010100 & 010010010 & 111111000 \\
001100010 & 010100001 & 100100100 & 111111000 \\
010001100 & 100001010 & 001001001 & 111111000
\end{bmatrix}$$

$$
N_2 = \begin{bmatrix}
000111111 & 100100100 & 100001010 & 100010001 \\
000111111 & 001001001 & 001010100 & 001100010 \\
000111111 & 010010010 & 010100001 & 010001100 \\
111000111 & 100100100 & 001010100 & 010001100 \\
111000111 & 001001001 & 010100001 & 100010001 \\
111000111 & 010010010 & 100001010 & 001100010 \\
111111000 & 100100100 & 010100001 & 001100010 \\
111111000 & 001001001 & 100001010 & 010001100 \\
111111000 & 010010010 & 001010100 & 100010001 \\
100100100 & 000111111 & 100010001 & 100001010 \\
001001001 & 000111111 & 001100010 & 001010100 \\
010010010 & 000111111 & 010001100 & 010100001 \\
100100100 & 111000111 & 010001100 & 001010100 \\
001001001 & 111000111 & 100010001 & 010100001 \\
010010010 & 111000111 & 001100010 & 100001010 \\
100100100 & 111111000 & 001100010 & 010100001 \\
001001001 & 111111000 & 010001100 & 100001010 \\
010010010 & 111111000 & 100010001 & 001010100 \\
100001010 & 100010001 & 000111111 & 100100100 \\
001010100 & 001100010 & 000111111 & 001001001 \\
010100001 & 010001100 & 000111111 & 010010010 \\
001010100 & 010001100 & 111000111 & 100100100 \\
010100001 & 100010001 & 111000111 & 001001001 \\
100001010 & 001100010 & 111000111 & 010010010 \\
010100001 & 001100010 & 111111000 & 100100100 \\
100001010 & 010001100 & 111111000 & 001001001 \\
001010100 & 100010001 & 111111000 & 010010010 \\
100010001 & 100001010 & 100100100 & 000111111 \\
001100010 & 001010100 & 001001001 & 000111111 \\
010001100 & 010100001 & 010010010 & 000111111 \\
010001100 & 001010100 & 100100100 & 111000111 \\
100010001 & 010100001 & 001001001 & 111000111 \\
001100010 & 100001010 & 010010010 & 111000111 \\
001100010 & 010100001 & 100100100 & 111111000 \\
010001100 & 100001010 & 001001001 & 111111000 \\
100010001 & 001010100 & 010010010 & 111111000
\end{bmatrix}
$$

Appendix 6

Fifteen Symmetric $(45,12,3)$-Designs constructed in Example 5.5

$$\Upsilon(0,0,L) = \begin{bmatrix}
000000000 & 111000000 & 100100100 & 100001010 & 100010001 \\
000000000 & 111000000 & 010010010 & 010100001 & 010001100 \\
000000000 & 111000000 & 001001001 & 001010100 & 001100010 \\
000000000 & 000111000 & 100100100 & 010100001 & 001100010 \\
000000000 & 000111000 & 010010010 & 001010100 & 100010001 \\
000000000 & 000111000 & 001001001 & 100001010 & 010001100 \\
000000000 & 000000111 & 100100100 & 001010100 & 010001100 \\
000000000 & 000000111 & 010010010 & 100001010 & 001100010 \\
000000000 & 000000111 & 001001001 & 010100001 & 100010001 \\
100010001 & 000000000 & 111000000 & 100100100 & 100001010 \\
010001100 & 000000000 & 111000000 & 010010010 & 010100001 \\
001100010 & 000000000 & 111000000 & 001001001 & 001010100 \\
001100010 & 000000000 & 000111000 & 100100100 & 010100001 \\
100010001 & 000000000 & 000111000 & 010010010 & 001010100 \\
010001100 & 000000000 & 000111000 & 001001001 & 100001010 \\
010001100 & 000000000 & 000000111 & 100100100 & 001010100 \\
001100010 & 000000000 & 000000111 & 010010010 & 100001010 \\
100010001 & 000000000 & 000000111 & 001001001 & 010100001 \\
100001010 & 100010001 & 000000000 & 111000000 & 100100100 \\
010100001 & 010001100 & 000000000 & 111000000 & 010010010 \\
001010100 & 001100010 & 000000000 & 111000000 & 001001001 \\
010100001 & 001100010 & 000000000 & 000111000 & 100100100 \\
001010100 & 100010001 & 000000000 & 000111000 & 010010010 \\
100001010 & 010001100 & 000000000 & 000111000 & 001001001 \\
001010100 & 010001100 & 000000000 & 000000111 & 100100100 \\
100001010 & 001100010 & 000000000 & 000000111 & 010010010 \\
010100001 & 100010001 & 000000000 & 000000111 & 001001001 \\
100100100 & 100001010 & 100010001 & 000000000 & 111000000 \\
010010010 & 010100001 & 010001100 & 000000000 & 111000000 \\
001001001 & 001010100 & 001100010 & 000000000 & 111000000 \\
100100100 & 010100001 & 001100010 & 000000000 & 000111000 \\
010010010 & 001010100 & 100010001 & 000000000 & 000111000 \\
001001001 & 100001010 & 010001100 & 000000000 & 000111000 \\
100100100 & 001010100 & 010001100 & 000000000 & 000000111 \\
010010010 & 100001010 & 001100010 & 000000000 & 000000111 \\
001001001 & 010100001 & 100010001 & 000000000 & 000000111 \\
111000000 & 100100100 & 100001010 & 100010001 & 000000000 \\
111000000 & 010010010 & 010100001 & 010001100 & 000000000 \\
111000000 & 001001001 & 001010100 & 001100010 & 000000000 \\
000111000 & 100100100 & 010100001 & 001100010 & 000000000 \\
000111000 & 010010010 & 001010100 & 100010001 & 000000000 \\
000111000 & 001001001 & 100001010 & 010001100 & 000000000 \\
000000111 & 100100100 & 001010100 & 010001100 & 000000000 \\
000000111 & 010010010 & 100001010 & 001100010 & 000000000 \\
000000111 & 001001001 & 010100001 & 100010001 & 000000000
\end{bmatrix}$$

$$\Upsilon(1,1,L)=\begin{bmatrix}
010001100 & 000000000 & 000111000 & 010010010 & 010100001\\
001100010 & 000000000 & 000111000 & 001001001 & 001010100\\
100010001 & 000000000 & 000111000 & 100100100 & 100001010\\
100010001 & 000000000 & 000000111 & 010010010 & 001010100\\
010001100 & 000000000 & 000000111 & 001001001 & 100001010\\
001100010 & 000000000 & 000000111 & 100100100 & 010100001\\
001100010 & 000000000 & 111000000 & 010010010 & 100001010\\
100010001 & 000000000 & 111000000 & 001001001 & 010100001\\
010001100 & 000000000 & 111000000 & 100100100 & 001010100\\
010100001 & 010001100 & 000000000 & 000111000 & 010010010\\
001010100 & 001100010 & 000000000 & 000111000 & 001001001\\
100001010 & 100010001 & 000000000 & 000111000 & 100100100\\
001010100 & 100010001 & 000000000 & 000000111 & 010010010\\
100001010 & 010001100 & 000000000 & 000000111 & 001001001\\
010100001 & 001100010 & 000000000 & 000000111 & 100100100\\
100001010 & 001100010 & 000000000 & 111000000 & 010010010\\
010100001 & 100010001 & 000000000 & 111000000 & 001001001\\
001010100 & 010001100 & 000000000 & 111000000 & 100100100\\
010010010 & 010100001 & 010001100 & 000000000 & 000111000\\
001001001 & 001010100 & 001100010 & 000000000 & 000111000\\
100100100 & 100001010 & 100010001 & 000000000 & 000111000\\
010010010 & 001010100 & 100010001 & 000000000 & 000000111\\
001001001 & 100001010 & 010001100 & 000000000 & 000000111\\
100100100 & 010100001 & 001100010 & 000000000 & 000000111\\
010010010 & 100001010 & 001100010 & 000000000 & 111000000\\
001001001 & 010100001 & 100010001 & 000000000 & 111000000\\
100100100 & 001010100 & 010001100 & 000000000 & 111000000\\
000111000 & 010010010 & 010100001 & 010001100 & 000000000\\
000111000 & 001001001 & 001010100 & 001100010 & 000000000\\
000111000 & 100100100 & 100001010 & 100010001 & 000000000\\
000000111 & 010010010 & 001010100 & 100010001 & 000000000\\
000000111 & 001001001 & 100001010 & 010001100 & 000000000\\
000000111 & 100100100 & 010100001 & 001100010 & 000000000\\
111000000 & 010010010 & 100001010 & 001100010 & 000000000\\
111000000 & 001001001 & 010100001 & 100010001 & 000000000\\
111000000 & 100100100 & 001010100 & 010001100 & 000000000\\
000000000 & 000111000 & 010010010 & 010100001 & 010001100\\
000000000 & 000111000 & 001001001 & 001010100 & 001100010\\
000000000 & 000111000 & 100100100 & 100001010 & 100010001\\
000000000 & 000000111 & 010010010 & 001010100 & 100010001\\
000000000 & 000000111 & 001001001 & 100001010 & 010001100\\
000000000 & 000000111 & 100100100 & 010100001 & 001100010\\
000000000 & 111000000 & 010010010 & 100001010 & 001100010\\
000000000 & 111000000 & 001001001 & 010100001 & 100010001\\
000000000 & 111000000 & 100100100 & 001010100 & 010001100
\end{bmatrix}$$

$$\Upsilon(2,2,L) = \begin{bmatrix}
001010100 & 001100010 & 000000000 & 000000111 & 001001001 \\
100001010 & 100010001 & 000000000 & 000000111 & 100100100 \\
010100001 & 010001100 & 000000000 & 000000111 & 010010010 \\
100001010 & 010001100 & 000000000 & 111000000 & 001001001 \\
010100001 & 001100010 & 000000000 & 111000000 & 100100100 \\
001010100 & 100010001 & 000000000 & 111000000 & 010010010 \\
010100001 & 100010001 & 000000000 & 000111000 & 001001001 \\
001010100 & 010001100 & 000000000 & 000111000 & 100100100 \\
100001010 & 001100010 & 000000000 & 000111000 & 010010010 \\
001001001 & 001010100 & 001100010 & 000000000 & 000000111 \\
100100100 & 100001010 & 100010001 & 000000000 & 000000111 \\
010010010 & 010100001 & 010001100 & 000000000 & 000000111 \\
001001001 & 100001010 & 010001100 & 000000000 & 111000000 \\
100100100 & 010100001 & 001100010 & 000000000 & 111000000 \\
010010010 & 001010100 & 100010001 & 000000000 & 111000000 \\
001001001 & 010100001 & 100010001 & 000000000 & 000111000 \\
100100100 & 001010100 & 010001100 & 000000000 & 000111000 \\
010010010 & 100001010 & 001100010 & 000000000 & 000111000 \\
000000111 & 001001001 & 001010100 & 001100010 & 000000000 \\
000000111 & 100100100 & 100001010 & 100010001 & 000000000 \\
000000111 & 010010010 & 010100001 & 010001100 & 000000000 \\
111000000 & 001001001 & 100001010 & 010001100 & 000000000 \\
111000000 & 100100100 & 010100001 & 001100010 & 000000000 \\
111000000 & 010010010 & 001010100 & 100010001 & 000000000 \\
000111000 & 001001001 & 010100001 & 100010001 & 000000000 \\
000111000 & 100100100 & 001010100 & 010001100 & 000000000 \\
000111000 & 010010010 & 100001010 & 001100010 & 000000000 \\
000000000 & 000000111 & 001001001 & 001010100 & 001100010 \\
000000000 & 000000111 & 100100100 & 100001010 & 100010001 \\
000000000 & 000000111 & 010010010 & 010100001 & 010001100 \\
000000000 & 111000000 & 001001001 & 100001010 & 010001100 \\
000000000 & 111000000 & 100100100 & 010100001 & 001100010 \\
000000000 & 111000000 & 010010010 & 001010100 & 100010001 \\
000000000 & 000111000 & 001001001 & 010100001 & 100010001 \\
000000000 & 000111000 & 100100100 & 001010100 & 010001100 \\
000000000 & 000111000 & 010010010 & 100001010 & 001100010 \\
001100010 & 000000000 & 000000111 & 001001001 & 001010100 \\
100010001 & 000000000 & 000000111 & 100100100 & 100001010 \\
010001100 & 000000000 & 000000111 & 010010010 & 010100001 \\
010001100 & 000000000 & 111000000 & 001001001 & 100001010 \\
001100010 & 000000000 & 111000000 & 100100100 & 010100001 \\
100010001 & 000000000 & 111000000 & 010010010 & 001010100 \\
100010001 & 000000000 & 000111000 & 001001001 & 010100001 \\
010001100 & 000000000 & 000111000 & 100100100 & 001010100 \\
001100010 & 000000000 & 000111000 & 010010010 & 100001010
\end{bmatrix}$$

$$\Upsilon(0,3,L)=\begin{bmatrix}
100100100 & 100001010 & 100010001 & 000000000 & 111000000\\
010010010 & 010100001 & 010001100 & 000000000 & 111000000\\
001001001 & 001010100 & 001100010 & 000000000 & 111000000\\
100100100 & 010100001 & 001100010 & 000000000 & 000111000\\
010010010 & 001010100 & 100010001 & 000000000 & 000111000\\
001001001 & 100001010 & 010001100 & 000000000 & 000111000\\
100100100 & 001010100 & 010001100 & 000000000 & 000000111\\
010010010 & 100001010 & 001100010 & 000000000 & 000000111\\
001001001 & 010100001 & 100010001 & 000000000 & 000000111\\
111000000 & 100100100 & 100001010 & 100010001 & 000000000\\
111000000 & 010010010 & 010100001 & 010001100 & 000000000\\
111000000 & 001001001 & 001010100 & 001100010 & 000000000\\
000111000 & 100100100 & 010100001 & 001100010 & 000000000\\
000111000 & 010010010 & 001010100 & 100010001 & 000000000\\
000111000 & 001001001 & 100001010 & 010001100 & 000000000\\
000000111 & 100100100 & 001010100 & 010001100 & 000000000\\
000000111 & 010010010 & 100001010 & 001100010 & 000000000\\
000000111 & 001001001 & 010100001 & 100010001 & 000000000\\
000000000 & 111000000 & 100100100 & 100001010 & 100010001\\
000000000 & 111000000 & 010010010 & 010100001 & 010001100\\
000000000 & 111000000 & 001001001 & 001010100 & 001100010\\
000000000 & 000111000 & 100100100 & 010100001 & 001100010\\
000000000 & 000111000 & 010010010 & 001010100 & 100010001\\
000000000 & 000111000 & 001001001 & 100001010 & 010001100\\
000000000 & 000000111 & 100100100 & 001010100 & 010001100\\
000000000 & 000000111 & 010010010 & 100001010 & 001100010\\
000000000 & 000000111 & 001001001 & 010100001 & 100010001\\
100010001 & 000000000 & 111000000 & 100100100 & 100001010\\
010001100 & 000000000 & 111000000 & 010010010 & 010100001\\
001100010 & 000000000 & 111000000 & 001001001 & 001010100\\
001100010 & 000000000 & 000111000 & 100100100 & 010100001\\
100010001 & 000000000 & 000111000 & 010010010 & 001010100\\
010001100 & 000000000 & 000111000 & 001001001 & 100001010\\
010001100 & 000000000 & 000000111 & 100100100 & 001010100\\
001100010 & 000000000 & 000000111 & 010010010 & 100001010\\
100010001 & 000000000 & 000000111 & 001001001 & 010100001\\
100001010 & 100010001 & 000000000 & 111000000 & 100100100\\
010100001 & 010001100 & 000000000 & 111000000 & 010010010\\
001010100 & 001100010 & 000000000 & 111000000 & 001001001\\
010100001 & 001100010 & 000000000 & 000111000 & 100100100\\
001010100 & 100010001 & 000000000 & 000111000 & 010010010\\
100001010 & 010001100 & 000000000 & 000111000 & 001001001\\
001010100 & 010001100 & 000000000 & 000000111 & 100100100\\
100001010 & 001100010 & 000000000 & 000000111 & 010010010\\
010100001 & 100010001 & 000000000 & 000000111 & 001001001
\end{bmatrix}$$

$$
\Upsilon(1,4,L)=\begin{bmatrix}
000111000 & 010010010 & 010100001 & 010001100 & 000000000\\
000111000 & 001001001 & 001010100 & 001100010 & 000000000\\
000111000 & 100100100 & 100001010 & 100010001 & 000000000\\
000000111 & 010010010 & 001010100 & 100010001 & 000000000\\
000000111 & 001001001 & 100001010 & 010001100 & 000000000\\
000000111 & 100100100 & 010100001 & 001100010 & 000000000\\
111000000 & 010010010 & 100001010 & 001100010 & 000000000\\
111000000 & 001001001 & 010100001 & 100010001 & 000000000\\
111000000 & 100100100 & 001010100 & 010001100 & 000000000\\
000000000 & 000111000 & 010010010 & 010100001 & 010001100\\
000000000 & 000111000 & 001001001 & 001010100 & 001100010\\
000000000 & 000111000 & 100100100 & 100001010 & 100010001\\
000000000 & 000000111 & 010010010 & 001010100 & 100010001\\
000000000 & 000000111 & 001001001 & 100001010 & 010001100\\
000000000 & 000000111 & 100100100 & 010100001 & 001100010\\
000000000 & 111000000 & 010010010 & 100001010 & 001100010\\
000000000 & 111000000 & 001001001 & 010100001 & 100010001\\
000000000 & 111000000 & 100100100 & 001010100 & 010001100\\
010001100 & 000000000 & 000111000 & 010010010 & 010100001\\
001100010 & 000000000 & 000111000 & 001001001 & 001010100\\
100010001 & 000000000 & 000111000 & 100100100 & 100001010\\
100010001 & 000000000 & 000000111 & 010010010 & 001010100\\
010001100 & 000000000 & 000000111 & 001001001 & 100001010\\
001100010 & 000000000 & 000000111 & 100100100 & 010100001\\
001100010 & 000000000 & 111000000 & 010010010 & 100001010\\
100010001 & 000000000 & 111000000 & 001001001 & 010100001\\
010001100 & 000000000 & 111000000 & 100100100 & 001010100\\
010100001 & 010001100 & 000000000 & 000111000 & 010010010\\
001010100 & 001100010 & 000000000 & 000111000 & 001001001\\
100001010 & 100010001 & 000000000 & 000111000 & 100100100\\
001010100 & 100010001 & 000000000 & 000000111 & 010010010\\
100001010 & 010001100 & 000000000 & 000000111 & 001001001\\
010100001 & 001100010 & 000000000 & 000000111 & 100100100\\
100001010 & 001100010 & 000000000 & 111000000 & 010010010\\
010100001 & 100010001 & 000000000 & 111000000 & 001001001\\
001010100 & 010001100 & 000000000 & 111000000 & 100100100\\
010010010 & 010100001 & 010001100 & 000000000 & 000111000\\
001001001 & 001010100 & 001100010 & 000000000 & 000111000\\
100100100 & 100001010 & 100010001 & 000000000 & 000111000\\
010010010 & 001010100 & 100010001 & 000000000 & 000000111\\
001001001 & 100001010 & 010001100 & 000000000 & 000000111\\
100100100 & 010100001 & 001100010 & 000000000 & 000000111\\
010010010 & 100001010 & 001100010 & 000000000 & 111000000\\
001001001 & 010100001 & 100010001 & 000000000 & 111000000\\
100100100 & 001010100 & 010001100 & 000000000 & 111000000
\end{bmatrix}
$$

$$\Upsilon(2,0,L) = \begin{bmatrix}
000000000 & 000000111 & 001001001 & 001010100 & 001100010 \\
000000000 & 000000111 & 100100100 & 100001010 & 100010001 \\
000000000 & 000000111 & 010010010 & 010100001 & 010001100 \\
000000000 & 111000000 & 001001001 & 100001010 & 010001100 \\
000000000 & 111000000 & 100100100 & 010100001 & 001100010 \\
000000000 & 111000000 & 010010010 & 001010100 & 100010001 \\
000000000 & 000111000 & 001001001 & 010100001 & 100010001 \\
000000000 & 000111000 & 100100100 & 001010100 & 010001100 \\
000000000 & 000111000 & 010010010 & 100001010 & 001100010 \\
001100010 & 000000000 & 000000111 & 001001001 & 001010100 \\
100010001 & 000000000 & 000000111 & 100100100 & 100001010 \\
010001100 & 000000000 & 000000111 & 010010010 & 010100001 \\
010001100 & 000000000 & 111000000 & 001001001 & 100001010 \\
001100010 & 000000000 & 111000000 & 100100100 & 010100001 \\
100010001 & 000000000 & 111000000 & 010010010 & 001010100 \\
100010001 & 000000000 & 000111000 & 001001001 & 010100001 \\
010001100 & 000000000 & 000111000 & 100100100 & 001010100 \\
001100010 & 000000000 & 000111000 & 010010010 & 100001010 \\
001010100 & 001100010 & 000000000 & 000000111 & 001001001 \\
100001010 & 100010001 & 000000000 & 000000111 & 100100100 \\
010100001 & 010001100 & 000000000 & 000000111 & 010010010 \\
100001010 & 010001100 & 000000000 & 111000000 & 001001001 \\
010100001 & 001100010 & 000000000 & 111000000 & 100100100 \\
001010100 & 100010001 & 000000000 & 111000000 & 010010010 \\
010100001 & 100010001 & 000000000 & 000111000 & 001001001 \\
001010100 & 010001100 & 000000000 & 000111000 & 100100100 \\
100001010 & 001100010 & 000000000 & 000111000 & 010010010 \\
001001001 & 001010100 & 001100010 & 000000000 & 000000111 \\
100100100 & 100001010 & 100010001 & 000000000 & 000000111 \\
010010010 & 010100001 & 010001100 & 000000000 & 000000111 \\
001001001 & 100001010 & 010001100 & 000000000 & 111000000 \\
100100100 & 010100001 & 001100010 & 000000000 & 111000000 \\
010010010 & 001010100 & 100010001 & 000000000 & 111000000 \\
001001001 & 010100001 & 100010001 & 000000000 & 000111000 \\
100100100 & 001010100 & 010001100 & 000000000 & 000111000 \\
010010010 & 100001010 & 001100010 & 000000000 & 000111000 \\
000000111 & 001001001 & 001010100 & 001100010 & 000000000 \\
000000111 & 100100100 & 100001010 & 100010001 & 000000000 \\
000000111 & 010010010 & 010100001 & 010001100 & 000000000 \\
111000000 & 001001001 & 100001010 & 010001100 & 000000000 \\
111000000 & 100100100 & 010100001 & 001100010 & 000000000 \\
111000000 & 010010010 & 001010100 & 100010001 & 000000000 \\
000111000 & 001001001 & 010100001 & 100010001 & 000000000 \\
000111000 & 100100100 & 001010100 & 010001100 & 000000000 \\
000111000 & 010010010 & 100001010 & 001100010 & 000000000
\end{bmatrix}$$

$$
\Upsilon(0,1,L) = \begin{bmatrix}
100010001 & 000000000 & 111000000 & 100100100 & 100001010 \\
010001100 & 000000000 & 111000000 & 010010010 & 010100001 \\
001100010 & 000000000 & 111000000 & 001001001 & 001010100 \\
001100010 & 000000000 & 000111000 & 100100100 & 010100001 \\
100010001 & 000000000 & 000111000 & 010010010 & 001010100 \\
010001100 & 000000000 & 000111000 & 001001001 & 100001010 \\
010001100 & 000000000 & 000000111 & 100100100 & 001010100 \\
001100010 & 000000000 & 000000111 & 010010010 & 100001010 \\
100010001 & 000000000 & 000000111 & 001001001 & 010100001 \\
100001010 & 100010001 & 000000000 & 111000000 & 100100100 \\
010100001 & 010001100 & 000000000 & 111000000 & 010010010 \\
001010100 & 001100010 & 000000000 & 111000000 & 001001001 \\
010100001 & 001100010 & 000000000 & 000111000 & 100100100 \\
001010100 & 100010001 & 000000000 & 000111000 & 010010010 \\
100001010 & 010001100 & 000000000 & 000111000 & 001001001 \\
001010100 & 010001100 & 000000000 & 000000111 & 100100100 \\
100001010 & 001100010 & 000000000 & 000000111 & 010010010 \\
010100001 & 100010001 & 000000000 & 000000111 & 001001001 \\
100100100 & 100001010 & 100010001 & 000000000 & 111000000 \\
010010010 & 010100001 & 010001100 & 000000000 & 111000000 \\
001001001 & 001010100 & 001100010 & 000000000 & 111000000 \\
100100100 & 010100001 & 001100010 & 000000000 & 000111000 \\
010010010 & 001010100 & 100010001 & 000000000 & 000111000 \\
001001001 & 100001010 & 010001100 & 000000000 & 000111000 \\
100100100 & 001010100 & 010001100 & 000000000 & 000000111 \\
010010010 & 100001010 & 001100010 & 000000000 & 000000111 \\
001001001 & 010100001 & 100010001 & 000000000 & 000000111 \\
111000000 & 100100100 & 100001010 & 100010001 & 000000000 \\
111000000 & 010010010 & 010100001 & 010001100 & 000000000 \\
111000000 & 001001001 & 001010100 & 001100010 & 000000000 \\
000111000 & 100100100 & 010100001 & 001100010 & 000000000 \\
000111000 & 010010010 & 001010100 & 100010001 & 000000000 \\
000111000 & 001001001 & 100001010 & 010001100 & 000000000 \\
000000111 & 100100100 & 001010100 & 010001100 & 000000000 \\
000000111 & 010010010 & 100001010 & 001100010 & 000000000 \\
000000111 & 001001001 & 010100001 & 100010001 & 000000000 \\
000000000 & 111000000 & 100100100 & 100001010 & 100010001 \\
000000000 & 111000000 & 010010010 & 010100001 & 010001100 \\
000000000 & 111000000 & 001001001 & 001010100 & 001100010 \\
000000000 & 000111000 & 100100100 & 010100001 & 001100010 \\
000000000 & 000111000 & 010010010 & 001010100 & 100010001 \\
000000000 & 000111000 & 001001001 & 100001010 & 010001100 \\
000000000 & 000000111 & 100100100 & 001010100 & 010001100 \\
000000000 & 000000111 & 010010010 & 100001010 & 001100010 \\
000000000 & 000000111 & 001001001 & 010100001 & 100010001
\end{bmatrix}
$$

$$\Upsilon(1,2,L) = \begin{bmatrix}
010100001 & 010001100 & 000000000 & 000111000 & 010010010 \\
001010100 & 001100010 & 000000000 & 000111000 & 001001001 \\
100001010 & 100010001 & 000000000 & 000111000 & 100100100 \\
001010100 & 100010001 & 000000000 & 000000111 & 010010010 \\
100001010 & 010001100 & 000000000 & 000000111 & 001001001 \\
010100001 & 001100010 & 000000000 & 000000111 & 100100100 \\
100001010 & 001100010 & 000000000 & 111000000 & 010010010 \\
010100001 & 100010001 & 000000000 & 111000000 & 001001001 \\
001010100 & 010001100 & 000000000 & 111000000 & 100100100 \\
010010010 & 010100001 & 010001100 & 000000000 & 000111000 \\
001001001 & 001010100 & 001100010 & 000000000 & 000111000 \\
100100100 & 100001010 & 100010001 & 000000000 & 000111000 \\
010010010 & 001010100 & 100010001 & 000000000 & 000000111 \\
001001001 & 100001010 & 010001100 & 000000000 & 000000111 \\
100100100 & 010100001 & 001100010 & 000000000 & 000000111 \\
010010010 & 100001010 & 001100010 & 000000000 & 111000000 \\
001001001 & 010100001 & 100010001 & 000000000 & 111000000 \\
100100100 & 001010100 & 010001100 & 000000000 & 111000000 \\
000111000 & 010010010 & 010100001 & 010001100 & 000000000 \\
000111000 & 001001001 & 001010100 & 001100010 & 000000000 \\
000111000 & 100100100 & 100001010 & 100010001 & 000000000 \\
000000111 & 010010010 & 001010100 & 100010001 & 000000000 \\
000000111 & 001001001 & 100001010 & 010001100 & 000000000 \\
000000111 & 100100100 & 010100001 & 001100010 & 000000000 \\
111000000 & 010010010 & 100001010 & 001100010 & 000000000 \\
111000000 & 001001001 & 010100001 & 100010001 & 000000000 \\
111000000 & 100100100 & 001010100 & 010001100 & 000000000 \\
000000000 & 000111000 & 010010010 & 010100001 & 010001100 \\
000000000 & 000111000 & 001001001 & 001010100 & 001100010 \\
000000000 & 000111000 & 100100100 & 100001010 & 100010001 \\
000000000 & 000000111 & 010010010 & 001010100 & 100010001 \\
000000000 & 000000111 & 001001001 & 100001010 & 010001100 \\
000000000 & 000000111 & 100100100 & 010100001 & 001100010 \\
000000000 & 111000000 & 010010010 & 100001010 & 001100010 \\
000000000 & 111000000 & 001001001 & 010100001 & 100010001 \\
000000000 & 111000000 & 100100100 & 001010100 & 010001100 \\
010001100 & 000000000 & 000111000 & 010010010 & 010100001 \\
001100010 & 000000000 & 000111000 & 001001001 & 001010100 \\
100010001 & 000000000 & 000111000 & 100100100 & 100001010 \\
100010001 & 000000000 & 000000111 & 010010010 & 001010100 \\
010001100 & 000000000 & 000000111 & 001001001 & 100001010 \\
001100010 & 000000000 & 000000111 & 100100100 & 010100001 \\
001100010 & 000000000 & 111000000 & 010010010 & 100001010 \\
100010001 & 000000000 & 111000000 & 001001001 & 010100001 \\
010001100 & 000000000 & 111000000 & 100100100 & 001010100
\end{bmatrix}$$

$$
\Upsilon(2,3,L)=\begin{bmatrix}
001001001 & 001010100 & 001100010 & 000000000 & 000000111\\
100100100 & 100001010 & 100010001 & 000000000 & 000000111\\
010010010 & 010100001 & 010001100 & 000000000 & 000000111\\
001001001 & 100001010 & 010001100 & 000000000 & 111000000\\
100100100 & 010100001 & 001100010 & 000000000 & 111000000\\
010010010 & 001010100 & 100010001 & 000000000 & 111000000\\
001001001 & 010100001 & 100010001 & 000000000 & 000111000\\
100100100 & 001010100 & 010001100 & 000000000 & 000111000\\
010010010 & 100001010 & 001100010 & 000000000 & 000111000\\
000000111 & 001001001 & 001010100 & 001100010 & 000000000\\
000000111 & 100100100 & 100001010 & 100010001 & 000000000\\
000000111 & 010010010 & 010100001 & 010001100 & 000000000\\
111000000 & 001001001 & 100001010 & 010001100 & 000000000\\
111000000 & 100100100 & 010100001 & 001100010 & 000000000\\
111000000 & 010010010 & 001010100 & 100010001 & 000000000\\
000111000 & 001001001 & 010100001 & 100010001 & 000000000\\
000111000 & 100100100 & 001010100 & 010001100 & 000000000\\
000111000 & 010010010 & 100001010 & 001100010 & 000000000\\
000000000 & 000000111 & 001001001 & 001010100 & 001100010\\
000000000 & 000000111 & 100100100 & 100001010 & 100010001\\
000000000 & 000000111 & 010010010 & 010100001 & 010001100\\
000000000 & 111000000 & 001001001 & 100001010 & 010001100\\
000000000 & 111000000 & 100100100 & 010100001 & 001100010\\
000000000 & 111000000 & 010010010 & 001010100 & 100010001\\
000000000 & 000111000 & 001001001 & 010100001 & 100010001\\
000000000 & 000111000 & 100100100 & 001010100 & 010001100\\
000000000 & 000111000 & 010010010 & 100001010 & 001100010\\
001100010 & 000000000 & 000000111 & 001001001 & 001010100\\
100010001 & 000000000 & 000000111 & 100100100 & 100001010\\
010001100 & 000000000 & 000000111 & 010010010 & 010100001\\
010001100 & 000000000 & 111000000 & 001001001 & 100001010\\
001100010 & 000000000 & 111000000 & 100100100 & 010100001\\
100010001 & 000000000 & 111000000 & 010010010 & 001010100\\
100010001 & 000000000 & 000111000 & 001001001 & 010100001\\
010001100 & 000000000 & 000111000 & 100100100 & 001010100\\
001100010 & 000000000 & 000111000 & 010010010 & 100001010\\
001010100 & 001100010 & 000000000 & 000000111 & 001001001\\
100001010 & 100010001 & 000000000 & 000000111 & 100100100\\
010100001 & 010001100 & 000000000 & 000000111 & 010010010\\
100001010 & 010001100 & 000000000 & 111000000 & 001001001\\
010100001 & 001100010 & 000000000 & 111000000 & 100100100\\
001010100 & 100010001 & 000000000 & 111000000 & 010010010\\
010100001 & 100010001 & 000000000 & 000111000 & 001001001\\
001010100 & 010001100 & 000000000 & 000111000 & 100100100\\
100001010 & 001100010 & 000000000 & 000111000 & 010010010
\end{bmatrix}
$$

$$\Upsilon(0,4,L)=\begin{bmatrix}
111000000 & 100100100 & 100001010 & 100010001 & 000000000\\
111000000 & 010010010 & 010100001 & 010001100 & 000000000\\
111000000 & 001001001 & 001010100 & 001100010 & 000000000\\
000111000 & 100100100 & 010100001 & 001100010 & 000000000\\
000111000 & 010010010 & 001010100 & 100010001 & 000000000\\
000111000 & 001001001 & 100001010 & 010001100 & 000000000\\
000000111 & 100100100 & 001010100 & 010001100 & 000000000\\
000000111 & 010010010 & 100001010 & 001100010 & 000000000\\
000000111 & 001001001 & 010100001 & 100010001 & 000000000\\
000000000 & 111000000 & 100100100 & 100001010 & 100010001\\
000000000 & 111000000 & 010010010 & 010100001 & 010001100\\
000000000 & 111000000 & 001001001 & 001010100 & 001100010\\
000000000 & 000111000 & 100100100 & 010100001 & 001100010\\
000000000 & 000111000 & 010010010 & 001010100 & 100010001\\
000000000 & 000111000 & 001001001 & 100001010 & 010001100\\
000000000 & 000000111 & 100100100 & 001010100 & 010001100\\
000000000 & 000000111 & 010010010 & 100001010 & 001100010\\
000000000 & 000000111 & 001001001 & 010100001 & 100010001\\
100010001 & 000000000 & 111000000 & 100100100 & 100001010\\
010001100 & 000000000 & 111000000 & 010010010 & 010100001\\
001100010 & 000000000 & 111000000 & 001001001 & 001010100\\
001100010 & 000000000 & 000111000 & 100100100 & 010100001\\
100010001 & 000000000 & 000111000 & 010010010 & 001010100\\
010001100 & 000000000 & 000111000 & 001001001 & 100001010\\
010001100 & 000000000 & 000000111 & 100100100 & 001010100\\
001100010 & 000000000 & 000000111 & 010010010 & 100001010\\
100010001 & 000000000 & 000000111 & 001001001 & 010100001\\
100001010 & 100010001 & 000000000 & 111000000 & 100100100\\
010100001 & 010001100 & 000000000 & 111000000 & 010010010\\
001010100 & 001100010 & 000000000 & 111000000 & 001001001\\
010100001 & 001100010 & 000000000 & 000111000 & 100100100\\
001010100 & 100010001 & 000000000 & 000111000 & 010010010\\
100001010 & 010001100 & 000000000 & 000111000 & 001001001\\
001010100 & 010001100 & 000000000 & 000000111 & 100100100\\
100001010 & 001100010 & 000000000 & 000000111 & 010010010\\
010100001 & 100010001 & 000000000 & 000000111 & 001001001\\
100100100 & 100001010 & 100010001 & 000000000 & 111000000\\
010010010 & 010100001 & 010001100 & 000000000 & 111000000\\
001001001 & 001010100 & 001100010 & 000000000 & 111000000\\
100100100 & 010100001 & 001100010 & 000000000 & 000111000\\
010010010 & 001010100 & 100010001 & 000000000 & 000111000\\
001001001 & 100001010 & 010001100 & 000000000 & 000111000\\
100100100 & 001010100 & 010001100 & 000000000 & 000000111\\
010010010 & 100001010 & 001100010 & 000000000 & 000000111\\
001001001 & 010100001 & 100010001 & 000000000 & 000000111
\end{bmatrix}$$

$$\Upsilon(1,0,L)=\begin{bmatrix}
000000000 & 000111000 & 010010010 & 010100001 & 010001100\\
000000000 & 000111000 & 001001001 & 001010100 & 001100010\\
000000000 & 000111000 & 100100100 & 100001010 & 100010001\\
000000000 & 000000111 & 010010010 & 001010100 & 100010001\\
000000000 & 000000111 & 001001001 & 100001010 & 010001100\\
000000000 & 000000111 & 100100100 & 010100001 & 001100010\\
000000000 & 111000000 & 010010010 & 100001010 & 001100010\\
000000000 & 111000000 & 001001001 & 010100001 & 100010001\\
000000000 & 111000000 & 100100100 & 001010100 & 010001100\\
010001100 & 000000000 & 000111000 & 010010010 & 010100001\\
001100010 & 000000000 & 000111000 & 001001001 & 001010100\\
100010001 & 000000000 & 000111000 & 100100100 & 100001010\\
100010001 & 000000000 & 000000111 & 010010010 & 001010100\\
010001100 & 000000000 & 000000111 & 001001001 & 100001010\\
001100010 & 000000000 & 000000111 & 100100100 & 010100001\\
001100010 & 000000000 & 111000000 & 010010010 & 100001010\\
100010001 & 000000000 & 111000000 & 001001001 & 010100001\\
010001100 & 000000000 & 111000000 & 100100100 & 001010100\\
010100001 & 010001100 & 000000000 & 000111000 & 010010010\\
001010100 & 001100010 & 000000000 & 000111000 & 001001001\\
100001010 & 100010001 & 000000000 & 000111000 & 100100100\\
001010100 & 100010001 & 000000000 & 000000111 & 010010010\\
100001010 & 010001100 & 000000000 & 000000111 & 001001001\\
010100001 & 001100010 & 000000000 & 000000111 & 100100100\\
100001010 & 001100010 & 000000000 & 111000000 & 010010010\\
010100001 & 100010001 & 000000000 & 111000000 & 001001001\\
001010100 & 010001100 & 000000000 & 111000000 & 100100100\\
010010010 & 010100001 & 010001100 & 000000000 & 000111000\\
001001001 & 001010100 & 001100010 & 000000000 & 000111000\\
100100100 & 100001010 & 100010001 & 000000000 & 000111000\\
010010010 & 001010100 & 100010001 & 000000000 & 000000111\\
001001001 & 100001010 & 010001100 & 000000000 & 000000111\\
100100100 & 010100001 & 001100010 & 000000000 & 000000111\\
010010010 & 100001010 & 001100010 & 000000000 & 111000000\\
001001001 & 010100001 & 100010001 & 000000000 & 111000000\\
100100100 & 001010100 & 010001100 & 000000000 & 111000000\\
000111000 & 010010010 & 010100001 & 010001100 & 000000000\\
000111000 & 001001001 & 001010100 & 001100010 & 000000000\\
000111000 & 100100100 & 100001010 & 100010001 & 000000000\\
000000111 & 010010010 & 001010100 & 100010001 & 000000000\\
000000111 & 001001001 & 100001010 & 010001100 & 000000000\\
000000111 & 100100100 & 010100001 & 001100010 & 000000000\\
111000000 & 010010010 & 100001010 & 001100010 & 000000000\\
111000000 & 001001001 & 010100001 & 100010001 & 000000000\\
111000000 & 100100100 & 001010100 & 010001100 & 000000000
\end{bmatrix}$$

$$\Upsilon(2,1,L) = \begin{bmatrix}
001100010 & 000000000 & 000000111 & 001001001 & 001010100 \\
100010001 & 000000000 & 000000111 & 100100100 & 100001010 \\
010001100 & 000000000 & 000000111 & 010010010 & 010100001 \\
010001100 & 000000000 & 111000000 & 001001001 & 100001010 \\
001100010 & 000000000 & 111000000 & 100100100 & 010100001 \\
100010001 & 000000000 & 111000000 & 010010010 & 001010100 \\
100010001 & 000000000 & 000111000 & 001001001 & 010100001 \\
010001100 & 000000000 & 000111000 & 100100100 & 001010100 \\
001100010 & 000000000 & 000111000 & 010010010 & 100001010 \\
001010100 & 001100010 & 000000000 & 000000111 & 001001001 \\
100001010 & 100010001 & 000000000 & 000000111 & 100100100 \\
010100001 & 010001100 & 000000000 & 000000111 & 010010010 \\
100001010 & 010001100 & 000000000 & 111000000 & 001001001 \\
010100001 & 001100010 & 000000000 & 111000000 & 100100100 \\
001010100 & 100010001 & 000000000 & 111000000 & 010010010 \\
010100001 & 100010001 & 000000000 & 000111000 & 001001001 \\
001010100 & 010001100 & 000000000 & 000111000 & 100100100 \\
100001010 & 001100010 & 000000000 & 000111000 & 010010010 \\
001001001 & 001010100 & 001100010 & 000000000 & 000000111 \\
100100100 & 100001010 & 100010001 & 000000000 & 000000111 \\
010010010 & 010100001 & 010001100 & 000000000 & 000000111 \\
001001001 & 100001010 & 010001100 & 000000000 & 111000000 \\
100100100 & 010100001 & 001100010 & 000000000 & 111000000 \\
010010010 & 001010100 & 100010001 & 000000000 & 111000000 \\
001001001 & 010100001 & 100010001 & 000000000 & 000111000 \\
100100100 & 001010100 & 010001100 & 000000000 & 000111000 \\
010010010 & 100001010 & 001100010 & 000000000 & 000111000 \\
000000111 & 001001001 & 001010100 & 001100010 & 000000000 \\
000000111 & 100100100 & 100001010 & 100010001 & 000000000 \\
000000111 & 010010010 & 010100001 & 010001100 & 000000000 \\
111000000 & 001001001 & 100001010 & 010001100 & 000000000 \\
111000000 & 100100100 & 010100001 & 001100010 & 000000000 \\
111000000 & 010010010 & 001010100 & 100010001 & 000000000 \\
000111000 & 001001001 & 010100001 & 100010001 & 000000000 \\
000111000 & 100100100 & 001010100 & 010001100 & 000000000 \\
000111000 & 010010010 & 100001010 & 001100010 & 000000000 \\
000000000 & 000000111 & 001001001 & 001010100 & 001100010 \\
000000000 & 000000111 & 100100100 & 100001010 & 100010001 \\
000000000 & 000000111 & 010010010 & 010100001 & 010001100 \\
000000000 & 111000000 & 001001001 & 100001010 & 010001100 \\
000000000 & 111000000 & 100100100 & 010100001 & 001100010 \\
000000000 & 111000000 & 010010010 & 001010100 & 100010001 \\
000000000 & 000111000 & 001001001 & 010100001 & 100010001 \\
000000000 & 000111000 & 100100100 & 001010100 & 010001100 \\
000000000 & 000111000 & 010010010 & 100001010 & 001100010
\end{bmatrix}$$

$$
\Upsilon(0,2,L) = \begin{bmatrix}
100001010 & 100010001 & 000000000 & 111000000 & 100100100 \\
010100001 & 010001100 & 000000000 & 111000000 & 010010010 \\
001010100 & 001100010 & 000000000 & 111000000 & 001001001 \\
010100001 & 001100010 & 000000000 & 000111000 & 100100100 \\
001010100 & 100010001 & 000000000 & 000111000 & 010010010 \\
100001010 & 010001100 & 000000000 & 000111000 & 001001001 \\
001010100 & 010001100 & 000000000 & 000000111 & 100100100 \\
100001010 & 001100010 & 000000000 & 000000111 & 010010010 \\
010100001 & 100010001 & 000000000 & 000000111 & 001001001 \\
100100100 & 100001010 & 100010001 & 000000000 & 111000000 \\
010010010 & 010100001 & 010001100 & 000000000 & 111000000 \\
001001001 & 001010100 & 001100010 & 000000000 & 111000000 \\
100100100 & 010100001 & 001100010 & 000000000 & 000111000 \\
010010010 & 001010100 & 100010001 & 000000000 & 000111000 \\
001001001 & 100001010 & 010001100 & 000000000 & 000111000 \\
100100100 & 001010100 & 010001100 & 000000000 & 000000111 \\
010010010 & 100001010 & 001100010 & 000000000 & 000000111 \\
001001001 & 010100001 & 100010001 & 000000000 & 000000111 \\
111000000 & 100100100 & 100001010 & 100010001 & 000000000 \\
111000000 & 010010010 & 010100001 & 010001100 & 000000000 \\
111000000 & 001001001 & 001010100 & 001100010 & 000000000 \\
000111000 & 100100100 & 010100001 & 001100010 & 000000000 \\
000111000 & 010010010 & 001010100 & 100010001 & 000000000 \\
000111000 & 001001001 & 100001010 & 010001100 & 000000000 \\
000000111 & 100100100 & 001010100 & 010001100 & 000000000 \\
000000111 & 010010010 & 100001010 & 001100010 & 000000000 \\
000000111 & 001001001 & 010100001 & 100010001 & 000000000 \\
000000000 & 111000000 & 100100100 & 100001010 & 100010001 \\
000000000 & 111000000 & 010010010 & 010100001 & 010001100 \\
000000000 & 111000000 & 001001001 & 001010100 & 001100010 \\
000000000 & 000111000 & 100100100 & 010100001 & 001100010 \\
000000000 & 000111000 & 010010010 & 001010100 & 100010001 \\
000000000 & 000111000 & 001001001 & 100001010 & 010001100 \\
000000000 & 000000111 & 100100100 & 001010100 & 010001100 \\
000000000 & 000000111 & 010010010 & 100001010 & 001100010 \\
000000000 & 000000111 & 001001001 & 010100001 & 100010001 \\
100010001 & 000000000 & 111000000 & 100100100 & 100001010 \\
010001100 & 000000000 & 111000000 & 010010010 & 010100001 \\
001100010 & 000000000 & 111000000 & 001001001 & 001010100 \\
001100010 & 000000000 & 000111000 & 100100100 & 010100001 \\
100010001 & 000000000 & 000111000 & 010010010 & 001010100 \\
010001100 & 000000000 & 000111000 & 001001001 & 100001010 \\
010001100 & 000000000 & 000000111 & 100100100 & 001010100 \\
001100010 & 000000000 & 000000111 & 010010010 & 100001010 \\
100010001 & 000000000 & 000000111 & 001001001 & 010100001
\end{bmatrix}
$$

$$
\Upsilon(1,3,L)=\begin{bmatrix}
010010010 & 010100001 & 010001100 & 000000000 & 000111000\\
001001001 & 001010100 & 001100010 & 000000000 & 000111000\\
100100100 & 100001010 & 100010001 & 000000000 & 000111000\\
010010010 & 001010100 & 100010001 & 000000000 & 000000111\\
001001001 & 100001010 & 010001100 & 000000000 & 000000111\\
100100100 & 010100001 & 001100010 & 000000000 & 000000111\\
010010010 & 100001010 & 001100010 & 000000000 & 111000000\\
001001001 & 010100001 & 100010001 & 000000000 & 111000000\\
100100100 & 001010100 & 010001100 & 000000000 & 111000000\\
000111000 & 010010010 & 010100001 & 010001100 & 000000000\\
000111000 & 001001001 & 001010100 & 001100010 & 000000000\\
000111000 & 100100100 & 100001010 & 100010001 & 000000000\\
000000111 & 010010010 & 001010100 & 100010001 & 000000000\\
000000111 & 001001001 & 100001010 & 010001100 & 000000000\\
000000111 & 100100100 & 010100001 & 001100010 & 000000000\\
111000000 & 010010010 & 100001010 & 001100010 & 000000000\\
111000000 & 001001001 & 010100001 & 100010001 & 000000000\\
111000000 & 100100100 & 001010100 & 010001100 & 000000000\\
000000000 & 000111000 & 010010010 & 010100001 & 010001100\\
000000000 & 000111000 & 001001001 & 001010100 & 001100010\\
000000000 & 000111000 & 100100100 & 100001010 & 100010001\\
000000000 & 000000111 & 010010010 & 001010100 & 100010001\\
000000000 & 000000111 & 001001001 & 100001010 & 010001100\\
000000000 & 000000111 & 100100100 & 010100001 & 001100010\\
000000000 & 111000000 & 010010010 & 100001010 & 001100010\\
000000000 & 111000000 & 001001001 & 010100001 & 100010001\\
000000000 & 111000000 & 100100100 & 001010100 & 010001100\\
010001100 & 000000000 & 000111000 & 010010010 & 010100001\\
001100010 & 000000000 & 000111000 & 001001001 & 001010100\\
100010001 & 000000000 & 000111000 & 100100100 & 100001010\\
100010001 & 000000000 & 000000111 & 010010010 & 001010100\\
010001100 & 000000000 & 000000111 & 001001001 & 100001010\\
001100010 & 000000000 & 000000111 & 100100100 & 010100001\\
001100010 & 000000000 & 111000000 & 010010010 & 100001010\\
100010001 & 000000000 & 111000000 & 001001001 & 010100001\\
010001100 & 000000000 & 111000000 & 100100100 & 001010100\\
010100001 & 010001100 & 000000000 & 000111000 & 010010010\\
001010100 & 001100010 & 000000000 & 000111000 & 001001001\\
100001010 & 100010001 & 000000000 & 000111000 & 100100100\\
001010100 & 100010001 & 000000000 & 000000111 & 010010010\\
100001010 & 010001100 & 000000000 & 000000111 & 001001001\\
010100001 & 001100010 & 000000000 & 000000111 & 100100100\\
100001010 & 001100010 & 000000000 & 111000000 & 010010010\\
010100001 & 100010001 & 000000000 & 111000000 & 001001001\\
001010100 & 010001100 & 000000000 & 111000000 & 100100100
\end{bmatrix}
$$

$$
\Upsilon(2,4,L) = \begin{bmatrix}
000000111 & 001001001 & 001010100 & 001100010 & 000000000 \\
000000111 & 100100100 & 100001010 & 100010001 & 000000000 \\
000000111 & 010010010 & 010100001 & 010001100 & 000000000 \\
111000000 & 001001001 & 100001010 & 010001100 & 000000000 \\
111000000 & 100100100 & 010100001 & 001100010 & 000000000 \\
111000000 & 010010010 & 001010100 & 100010001 & 000000000 \\
000111000 & 001001001 & 010100001 & 100010001 & 000000000 \\
000111000 & 100100100 & 001010100 & 010001100 & 000000000 \\
000111000 & 010010010 & 100001010 & 001100010 & 000000000 \\
000000000 & 000000111 & 001001001 & 001010100 & 001100010 \\
000000000 & 000000111 & 100100100 & 100001010 & 100010001 \\
000000000 & 000000111 & 010010010 & 010100001 & 010001100 \\
000000000 & 111000000 & 001001001 & 100001010 & 010001100 \\
000000000 & 111000000 & 100100100 & 010100001 & 001100010 \\
000000000 & 111000000 & 010010010 & 001010100 & 100010001 \\
000000000 & 000111000 & 001001001 & 010100001 & 100010001 \\
000000000 & 000111000 & 100100100 & 001010100 & 010001100 \\
000000000 & 000111000 & 010010010 & 100001010 & 001100010 \\
001100010 & 000000000 & 000000111 & 001001001 & 001010100 \\
100010001 & 000000000 & 000000111 & 100100100 & 100001010 \\
010001100 & 000000000 & 000000111 & 010010010 & 010100001 \\
010001100 & 000000000 & 111000000 & 001001001 & 100001010 \\
001100010 & 000000000 & 111000000 & 100100100 & 010100001 \\
100010001 & 000000000 & 111000000 & 010010010 & 001010100 \\
100010001 & 000000000 & 000111000 & 001001001 & 010100001 \\
010001100 & 000000000 & 000111000 & 100100100 & 001010100 \\
001100010 & 000000000 & 000111000 & 010010010 & 100001010 \\
001010100 & 001100010 & 000000000 & 000000111 & 001001001 \\
100001010 & 100010001 & 000000000 & 000000111 & 100100100 \\
010100001 & 010001100 & 000000000 & 000000111 & 010010010 \\
100001010 & 010001100 & 000000000 & 111000000 & 001001001 \\
010100001 & 001100010 & 000000000 & 111000000 & 100100100 \\
001010100 & 100010001 & 000000000 & 111000000 & 010010010 \\
010100001 & 100010001 & 000000000 & 000111000 & 001001001 \\
001010100 & 010001100 & 000000000 & 000111000 & 100100100 \\
100001010 & 001100010 & 000000000 & 000111000 & 010010010 \\
001001001 & 001010100 & 001100010 & 000000000 & 000000111 \\
100100100 & 100001010 & 100010001 & 000000000 & 000000111 \\
010010010 & 010100001 & 010001100 & 000000000 & 000000111 \\
001001001 & 100001010 & 010001100 & 000000000 & 111000000 \\
100100100 & 010100001 & 001100010 & 000000000 & 111000000 \\
010010010 & 001010100 & 100010001 & 000000000 & 111000000 \\
001001001 & 010100001 & 100010001 & 000000000 & 000111000 \\
100100100 & 001010100 & 010001100 & 000000000 & 000111000 \\
010010010 & 100001010 & 001100010 & 000000000 & 000111000
\end{bmatrix}
$$

Appendix 7

Twelve Symmetric $(36, 15, 6)$-Designs Made in Example 5.17

$$D_0 = \begin{bmatrix}
000111111 \; 100100100 \; 100001010 \; 100010001 \\
000111111 \; 010010010 \; 010100001 \; 010001100 \\
000111111 \; 001001001 \; 001010100 \; 001100010 \\
111000111 \; 100100100 \; 010100001 \; 001100010 \\
111000111 \; 010010010 \; 001010100 \; 100010001 \\
111000111 \; 001001001 \; 100001010 \; 010001100 \\
111111000 \; 100100100 \; 001010100 \; 010001100 \\
111111000 \; 010010010 \; 100001010 \; 001100010 \\
111111000 \; 001001001 \; 010100001 \; 100010001 \\
100010001 \; 000111111 \; 100100100 \; 100001010 \\
010001100 \; 000111111 \; 010010010 \; 010100001 \\
001100010 \; 000111111 \; 001001001 \; 001010100 \\
001100010 \; 111000111 \; 100100100 \; 010100001 \\
100010001 \; 111000111 \; 010010010 \; 001010100 \\
010001100 \; 111000111 \; 001001001 \; 100001010 \\
010001100 \; 111111000 \; 100100100 \; 001010100 \\
001100010 \; 111111000 \; 010010010 \; 100001010 \\
100010001 \; 111111000 \; 001001001 \; 010100001 \\
100001010 \; 100010001 \; 000111111 \; 100100100 \\
010100001 \; 010001100 \; 000111111 \; 010010010 \\
001010100 \; 001100010 \; 000111111 \; 001001001 \\
010100001 \; 001100010 \; 111000111 \; 100100100 \\
001010100 \; 100010001 \; 111000111 \; 010010010 \\
100001010 \; 010001100 \; 111000111 \; 001001001 \\
001010100 \; 010001100 \; 111111000 \; 100100100 \\
100001010 \; 001100010 \; 111111000 \; 010010010 \\
010100001 \; 100010001 \; 111111000 \; 001001001 \\
100100100 \; 100001010 \; 100010001 \; 000111111 \\
010010010 \; 010100001 \; 010001100 \; 000111111 \\
001001001 \; 001010100 \; 001100010 \; 000111111 \\
100100100 \; 010100001 \; 001100010 \; 111000111 \\
010010010 \; 001010100 \; 100010001 \; 111000111 \\
001001001 \; 100001010 \; 010001100 \; 111000111 \\
100100100 \; 001010100 \; 010001100 \; 111111000 \\
010010010 \; 100001010 \; 001100010 \; 111111000 \\
001001001 \; 010100001 \; 100010001 \; 111111000
\end{bmatrix}$$

$$
D_1 = \begin{bmatrix}
010001100 & 111000111 & 010010010 & 010100001 \\
001100010 & 111000111 & 001001001 & 001010100 \\
100010001 & 111000111 & 100100100 & 100001010 \\
100010001 & 111111000 & 010010010 & 001010100 \\
010001100 & 111111000 & 001001001 & 100001010 \\
001100010 & 111111000 & 100100100 & 010100001 \\
001100010 & 000111111 & 010010010 & 100001010 \\
100010001 & 000111111 & 001001001 & 010100001 \\
010001100 & 000111111 & 100100100 & 001010100 \\
010100001 & 010001100 & 111000111 & 010010010 \\
001010100 & 001100010 & 111000111 & 001001001 \\
100001010 & 100010001 & 111000111 & 100100100 \\
001010100 & 100010001 & 111111000 & 010010010 \\
100001010 & 010001100 & 111111000 & 001001001 \\
010100001 & 001100010 & 111111000 & 100100100 \\
100001010 & 001100010 & 000111111 & 010010010 \\
010100001 & 100010001 & 000111111 & 001001001 \\
001010100 & 010001100 & 000111111 & 100100100 \\
010010010 & 010100001 & 010001100 & 111000111 \\
001001001 & 001010100 & 001100010 & 111000111 \\
100100100 & 100001010 & 100010001 & 111000111 \\
010010010 & 001010100 & 100010001 & 111111000 \\
001001001 & 100001010 & 010001100 & 111111000 \\
100100100 & 010100001 & 001100010 & 111111000 \\
010010010 & 100001010 & 001100010 & 000111111 \\
001001001 & 010100001 & 100010001 & 000111111 \\
100100100 & 001010100 & 010001100 & 000111111 \\
111000111 & 010010010 & 010100001 & 010001100 \\
111000111 & 001001001 & 001010100 & 001100010 \\
111000111 & 100100100 & 100001010 & 100010001 \\
111111000 & 010010010 & 001010100 & 100010001 \\
111111000 & 001001001 & 100001010 & 010001100 \\
111111000 & 100100100 & 010100001 & 001100010 \\
000111111 & 010010010 & 100001010 & 001100010 \\
000111111 & 001001001 & 010100001 & 100010001 \\
000111111 & 100100100 & 001010100 & 010001100
\end{bmatrix}
$$

$$D_2 = \begin{bmatrix}
001010100 & 001100010 & 111111000 & 001001001 \\
100001010 & 100010001 & 111111000 & 100100100 \\
010100001 & 010001100 & 111111000 & 010010010 \\
100001010 & 010001100 & 000111111 & 001001001 \\
010100001 & 001100010 & 000111111 & 100100100 \\
001010100 & 100010001 & 000111111 & 010010010 \\
010100001 & 100010001 & 111000111 & 001001001 \\
001010100 & 010001100 & 111000111 & 100100100 \\
100001010 & 001100010 & 111000111 & 010010010 \\
001001001 & 001010100 & 001100010 & 111111000 \\
100100100 & 100001010 & 100010001 & 111111000 \\
010010010 & 010100001 & 010001100 & 111111000 \\
001001001 & 100001010 & 010001100 & 000111111 \\
100100100 & 010100001 & 001100010 & 000111111 \\
010010010 & 001010100 & 100010001 & 000111111 \\
001001001 & 010100001 & 100010001 & 111000111 \\
100100100 & 001010100 & 010001100 & 111000111 \\
010010010 & 100001010 & 001100010 & 111000111 \\
111111000 & 001001001 & 001010100 & 001100010 \\
111111000 & 100100100 & 100001010 & 100010001 \\
111111000 & 010010010 & 010100001 & 010001100 \\
000111111 & 001001001 & 100001010 & 010001100 \\
000111111 & 100100100 & 010100001 & 001100010 \\
000111111 & 010010010 & 001010100 & 100010001 \\
111000111 & 001001001 & 010100001 & 100010001 \\
111000111 & 100100100 & 001010100 & 010001100 \\
111000111 & 010010010 & 100001010 & 001100010 \\
001100010 & 111111000 & 001001001 & 001010100 \\
100010001 & 111111000 & 100100100 & 100001010 \\
010001100 & 111111000 & 010010010 & 010100001 \\
010001100 & 000111111 & 001001001 & 100001010 \\
001100010 & 000111111 & 100100100 & 010100001 \\
100010001 & 000111111 & 010010010 & 001010100 \\
100010001 & 111000111 & 001001001 & 010100001 \\
010001100 & 111000111 & 100100100 & 001010100 \\
001100010 & 111000111 & 010010010 & 100001010
\end{bmatrix}$$

$$D_3 = \begin{bmatrix}
100100100 & 100001010 & 100010001 & 000111111 \\
010010010 & 010100001 & 010001100 & 000111111 \\
001001001 & 001010100 & 001100010 & 000111111 \\
100100100 & 010100001 & 001100010 & 111000111 \\
010010010 & 001010100 & 100010001 & 111000111 \\
001001001 & 100001010 & 010001100 & 111000111 \\
100100100 & 001010100 & 010001100 & 111111000 \\
010010010 & 100001010 & 001100010 & 111111000 \\
001001001 & 010100001 & 100010001 & 111111000 \\
000111111 & 100100100 & 100001010 & 100010001 \\
000111111 & 010010010 & 010100001 & 010001100 \\
000111111 & 001001001 & 001010100 & 001100010 \\
111000111 & 100100100 & 010100001 & 001100010 \\
111000111 & 010010010 & 001010100 & 100010001 \\
111000111 & 001001001 & 100001010 & 010001100 \\
111111000 & 100100100 & 001010100 & 010001100 \\
111111000 & 010010010 & 100001010 & 001100010 \\
111111000 & 001001001 & 010100001 & 100010001 \\
100010001 & 000111111 & 100100100 & 100001010 \\
010001100 & 000111111 & 010010010 & 010100001 \\
001100010 & 000111111 & 001001001 & 001010100 \\
001100010 & 111000111 & 100100100 & 010100001 \\
100010001 & 111000111 & 010010010 & 001010100 \\
010001100 & 111000111 & 001001001 & 100001010 \\
010001100 & 111111000 & 100100100 & 001010100 \\
001100010 & 111111000 & 010010010 & 100001010 \\
100010001 & 111111000 & 001001001 & 010100001 \\
100001010 & 100010001 & 000111111 & 100100100 \\
010100001 & 010001100 & 000111111 & 010010010 \\
001010100 & 001100010 & 000111111 & 001001001 \\
010100001 & 001100010 & 111000111 & 100100100 \\
001010100 & 100010001 & 111000111 & 010010010 \\
100001010 & 010001100 & 111000111 & 001001001 \\
001010100 & 010001100 & 111111000 & 100100100 \\
100001010 & 001100010 & 111111000 & 010010010 \\
010100001 & 100010001 & 111111000 & 001001001
\end{bmatrix}$$

$$D_4 = \begin{bmatrix}
111000111 & 010010010 & 010100001 & 010001100 \\
111000111 & 001001001 & 001010100 & 001100010 \\
111000111 & 100100100 & 100001010 & 100010001 \\
111111000 & 010010010 & 001010100 & 100010001 \\
111111000 & 001001001 & 100001010 & 010001100 \\
111111000 & 100100100 & 010100001 & 001100010 \\
000111111 & 010010010 & 100001010 & 001100010 \\
000111111 & 001001001 & 010100001 & 100010001 \\
000111111 & 100100100 & 001010100 & 010001100 \\
010001100 & 111000111 & 010010010 & 010100001 \\
001100010 & 111000111 & 001001001 & 001010100 \\
100010001 & 111000111 & 100100100 & 100001010 \\
100010001 & 111111000 & 010010010 & 001010100 \\
010001100 & 111111000 & 001001001 & 100001010 \\
001100010 & 111111000 & 100100100 & 010100001 \\
001100010 & 000111111 & 010010010 & 100001010 \\
100010001 & 000111111 & 001001001 & 010100001 \\
010001100 & 000111111 & 100100100 & 001010100 \\
010100001 & 010001100 & 111000111 & 010010010 \\
001010100 & 001100010 & 111000111 & 001001001 \\
100001010 & 100010001 & 111000111 & 100100100 \\
001010100 & 100010001 & 111111000 & 010010010 \\
100001010 & 010001100 & 111111000 & 001001001 \\
010100001 & 001100010 & 111111000 & 100100100 \\
100001010 & 001100010 & 000111111 & 010010010 \\
010100001 & 100010001 & 000111111 & 001001001 \\
001010100 & 010001100 & 000111111 & 100100100 \\
010010010 & 010100001 & 010001100 & 111000111 \\
001001001 & 001010100 & 001100010 & 111000111 \\
100100100 & 100001010 & 100010001 & 111000111 \\
010010010 & 001010100 & 100010001 & 111111000 \\
001001001 & 100001010 & 010001100 & 111111000 \\
100100100 & 010100001 & 001100010 & 111111000 \\
010010010 & 100001010 & 001100010 & 000111111 \\
001001001 & 010100001 & 100010001 & 000111111 \\
100100100 & 001010100 & 010001100 & 000111111
\end{bmatrix}$$

$$D_5 = \begin{bmatrix}
001100010 & 111111000 & 001001001 & 001010100 \\
100010001 & 111111000 & 100100100 & 100001010 \\
010001100 & 111111000 & 010010010 & 010100001 \\
010001100 & 000111111 & 001001001 & 100001010 \\
001100010 & 000111111 & 100100100 & 010100001 \\
100010001 & 000111111 & 010010010 & 001010100 \\
100010001 & 111000111 & 001001001 & 010100001 \\
010001100 & 111000111 & 100100100 & 001010100 \\
001100010 & 111000111 & 010010010 & 100001010 \\
001010100 & 001100010 & 111111000 & 001001001 \\
100001010 & 100010001 & 111111000 & 100100100 \\
010100001 & 010001100 & 111111000 & 010010010 \\
100001010 & 010001100 & 000111111 & 001001001 \\
010100001 & 001100010 & 000111111 & 100100100 \\
001010100 & 100010001 & 000111111 & 010010010 \\
010100001 & 100010001 & 111000111 & 001001001 \\
001010100 & 010001100 & 111000111 & 100100100 \\
100001010 & 001100010 & 111000111 & 010010010 \\
001001001 & 001010100 & 001100010 & 111111000 \\
100100100 & 100001010 & 100010001 & 111111000 \\
010010010 & 010100001 & 010001100 & 111111000 \\
001001001 & 100001010 & 010001100 & 000111111 \\
100100100 & 010100001 & 001100010 & 000111111 \\
010010010 & 001010100 & 100010001 & 000111111 \\
001001001 & 010100001 & 100010001 & 111000111 \\
100100100 & 001010100 & 010001100 & 111000111 \\
010010010 & 100001010 & 001100010 & 111000111 \\
111111000 & 001001001 & 001010100 & 001100010 \\
111111000 & 100100100 & 100001010 & 100010001 \\
111111000 & 010010010 & 010100001 & 010001100 \\
000111111 & 001001001 & 100001010 & 010001100 \\
000111111 & 100100100 & 010100001 & 001100010 \\
000111111 & 010010010 & 001010100 & 100010001 \\
111000111 & 001001001 & 010100001 & 100010001 \\
111000111 & 100100100 & 001010100 & 010001100 \\
111000111 & 010010010 & 100001010 & 001100010
\end{bmatrix}$$

$$D_6 = \begin{bmatrix}
100001010 & 100010001 & 000111111 & 100100100 \\
010100001 & 010001100 & 000111111 & 010010010 \\
001010100 & 001100010 & 000111111 & 001001001 \\
010100001 & 001100010 & 111000111 & 100100100 \\
001010100 & 100010001 & 111000111 & 010010010 \\
100001010 & 010001100 & 111000111 & 001001001 \\
001010100 & 010001100 & 111111000 & 100100100 \\
100001010 & 001100010 & 111111000 & 010010010 \\
010100001 & 100010001 & 111111000 & 001001001 \\[4pt]
100100100 & 100001010 & 100010001 & 000111111 \\
010010010 & 010100001 & 010001100 & 000111111 \\
001001001 & 001010100 & 001100010 & 000111111 \\
100100100 & 010100001 & 001100010 & 111000111 \\
010010010 & 001010100 & 100010001 & 111000111 \\
001001001 & 100001010 & 010001100 & 111000111 \\
100100100 & 001010100 & 010001100 & 111111000 \\
010010010 & 100001010 & 001100010 & 111111000 \\
001001001 & 010100001 & 100010001 & 111111000 \\[4pt]
000111111 & 100100100 & 100001010 & 100010001 \\
000111111 & 010010010 & 010100001 & 010001100 \\
000111111 & 001001001 & 001010100 & 001100010 \\
111000111 & 100100100 & 010100001 & 001100010 \\
111000111 & 010010010 & 001010100 & 100010001 \\
111000111 & 001001001 & 100001010 & 010001100 \\
111111000 & 100100100 & 001010100 & 010001100 \\
111111000 & 010010010 & 100001010 & 001100010 \\
111111000 & 001001001 & 010100001 & 100010001 \\[4pt]
100010001 & 000111111 & 100100100 & 100001010 \\
010001100 & 000111111 & 010010010 & 010100001 \\
001100010 & 000111111 & 001001001 & 001010100 \\
001100010 & 111000111 & 100100100 & 010100001 \\
100010001 & 111000111 & 010010010 & 001010100 \\
010001100 & 111000111 & 001001001 & 100001010 \\
010001100 & 111111000 & 100100100 & 001010100 \\
001100010 & 111111000 & 010010010 & 100001010 \\
100010001 & 111111000 & 001001001 & 010100001
\end{bmatrix}$$

$$D_7 = \begin{bmatrix}
010010010 & 010100001 & 010001100 & 111000111 \\
001001001 & 001010100 & 001100010 & 111000111 \\
100100100 & 100001010 & 100010001 & 111000111 \\
010010010 & 001010100 & 100010001 & 111111000 \\
001001001 & 100001010 & 010001100 & 111111000 \\
100100100 & 010100001 & 001100010 & 111111000 \\
010010010 & 100001010 & 001100010 & 000111111 \\
001001001 & 010100001 & 100010001 & 000111111 \\
100100100 & 001010100 & 010001100 & 000111111 \\
111000111 & 010010010 & 010100001 & 010001100 \\
111000111 & 001001001 & 001010100 & 001100010 \\
111000111 & 100100100 & 100001010 & 100010001 \\
111111000 & 010010010 & 001010100 & 100010001 \\
111111000 & 001001001 & 100001010 & 010001100 \\
111111000 & 100100100 & 010100001 & 001100010 \\
000111111 & 010010010 & 100001010 & 001100010 \\
000111111 & 001001001 & 010100001 & 100010001 \\
000111111 & 100100100 & 001010100 & 010001100 \\
010001100 & 111000111 & 010010010 & 010100001 \\
001100010 & 111000111 & 001001001 & 001010100 \\
100010001 & 111000111 & 100100100 & 100001010 \\
100010001 & 111111000 & 010010010 & 001010100 \\
010001100 & 111111000 & 001001001 & 100001010 \\
001100010 & 111111000 & 100100100 & 010100001 \\
001100010 & 000111111 & 010010010 & 100001010 \\
100010001 & 000111111 & 001001001 & 010100001 \\
010001100 & 000111111 & 100100100 & 001010100 \\
010100001 & 010001100 & 111000111 & 010010010 \\
001010100 & 001100010 & 111000111 & 001001001 \\
100001010 & 100010001 & 111000111 & 100100100 \\
001010100 & 100010001 & 111111000 & 010010010 \\
100001010 & 010001100 & 111111000 & 001001001 \\
010100001 & 001100010 & 111111000 & 100100100 \\
100001010 & 001100010 & 000111111 & 010010010 \\
010100001 & 100010001 & 000111111 & 001001001 \\
001010100 & 010001100 & 000111111 & 100100100
\end{bmatrix}$$

$$D_8 = \begin{bmatrix}
111111000 & 001001001 & 001010100 & 001100010 \\
111111000 & 100100100 & 100001010 & 100010001 \\
111111000 & 010010010 & 010100001 & 010001100 \\
000111111 & 001001001 & 100001010 & 010001100 \\
000111111 & 100100100 & 010100001 & 001100010 \\
000111111 & 010010010 & 001010100 & 100010001 \\
111000111 & 001001001 & 010100001 & 100010001 \\
111000111 & 100100100 & 001010100 & 010001100 \\
111000111 & 010010010 & 100001010 & 001100010 \\
001100010 & 111111000 & 001001001 & 001010100 \\
100010001 & 111111000 & 100100100 & 100001010 \\
010001100 & 111111000 & 010010010 & 010100001 \\
010001100 & 000111111 & 001001001 & 100001010 \\
001100010 & 000111111 & 100100100 & 010100001 \\
100010001 & 000111111 & 010010010 & 001010100 \\
100010001 & 111000111 & 001001001 & 010100001 \\
010001100 & 111000111 & 100100100 & 001010100 \\
001100010 & 111000111 & 010010010 & 100001010 \\
001010100 & 001100010 & 111111000 & 001001001 \\
100001010 & 100010001 & 111111000 & 100100100 \\
010100001 & 010001100 & 111111000 & 010010010 \\
100001010 & 010001100 & 000111111 & 001001001 \\
010100001 & 001100010 & 000111111 & 100100100 \\
001010100 & 100010001 & 000111111 & 010010010 \\
010100001 & 100010001 & 111000111 & 001001001 \\
001010100 & 010001100 & 111000111 & 100100100 \\
100001010 & 001100010 & 111000111 & 010010010 \\
001001001 & 001010100 & 001100010 & 111111000 \\
100100100 & 100001010 & 100010001 & 111111000 \\
010010010 & 010100001 & 010001100 & 111111000 \\
001001001 & 100001010 & 010001100 & 000111111 \\
100100100 & 010100001 & 001100010 & 000111111 \\
010010010 & 001010100 & 100010001 & 000111111 \\
001001001 & 010100001 & 100010001 & 111000111 \\
100100100 & 001010100 & 010001100 & 111000111 \\
010010010 & 100001010 & 001100010 & 111000111
\end{bmatrix}$$

$$D_9 = \begin{bmatrix}
100010001 & 000111111 & 100100100 & 100001010 \\
010001100 & 000111111 & 010010010 & 010100001 \\
001100010 & 000111111 & 001001001 & 001010100 \\
001100010 & 111000111 & 100100100 & 010100001 \\
100010001 & 111000111 & 010010010 & 001010100 \\
010001100 & 111000111 & 001001001 & 100001010 \\
010001100 & 111111000 & 100100100 & 001010100 \\
001100010 & 111111000 & 010010010 & 100001010 \\
100010001 & 111111000 & 001001001 & 010100001 \\
100001010 & 100010001 & 000111111 & 100100100 \\
010100001 & 010001100 & 000111111 & 010010010 \\
001010100 & 001100010 & 000111111 & 001001001 \\
010100001 & 001100010 & 111000111 & 100100100 \\
001010100 & 100010001 & 111000111 & 010010010 \\
100001010 & 010001100 & 111000111 & 001001001 \\
001010100 & 010001100 & 111111000 & 100100100 \\
100001010 & 001100010 & 111111000 & 010010010 \\
010100001 & 100010001 & 111111000 & 001001001 \\
100100100 & 100001010 & 100010001 & 000111111 \\
010010010 & 010100001 & 010001100 & 000111111 \\
001001001 & 001010100 & 001100010 & 000111111 \\
100100100 & 010100001 & 001100010 & 111000111 \\
010010010 & 001010100 & 100010001 & 111000111 \\
001001001 & 100001010 & 010001100 & 111000111 \\
100100100 & 001010100 & 010001100 & 111111000 \\
010010010 & 100001010 & 001100010 & 111111000 \\
001001001 & 010100001 & 100010001 & 111111000 \\
000111111 & 100100100 & 100001010 & 100010001 \\
000111111 & 010010010 & 010100001 & 010001100 \\
000111111 & 001001001 & 001010100 & 001100010 \\
111000111 & 100100100 & 010100001 & 001100010 \\
111000111 & 010010010 & 001010100 & 100010001 \\
111000111 & 001001001 & 100001010 & 010001100 \\
111111000 & 100100100 & 001010100 & 010001100 \\
111111000 & 010010010 & 100001010 & 001100010 \\
111111000 & 001001001 & 010100001 & 100010001
\end{bmatrix}$$

$$D_{10} = \begin{bmatrix}
010100001 & 010001100 & 111000111 & 010010010 \\
001010100 & 001100010 & 111000111 & 001001001 \\
100001010 & 100010001 & 111000111 & 100100100 \\
001010100 & 100010001 & 111111000 & 010010010 \\
100001010 & 010001100 & 111111000 & 001001001 \\
010100001 & 001100010 & 111111000 & 100100100 \\
100001010 & 001100010 & 000111111 & 010010010 \\
010100001 & 100010001 & 000111111 & 001001001 \\
001010100 & 010001100 & 000111111 & 100100100 \\[4pt]
010010010 & 010100001 & 010001100 & 111000111 \\
001001001 & 001010100 & 001100010 & 111000111 \\
100100100 & 100001010 & 100010001 & 111000111 \\
010010010 & 001010100 & 100010001 & 111111000 \\
001001001 & 100001010 & 010001100 & 111111000 \\
100100100 & 010100001 & 001100010 & 111111000 \\
010010010 & 100001010 & 001100010 & 000111111 \\
001001001 & 010100001 & 100010001 & 000111111 \\
100100100 & 001010100 & 010001100 & 000111111 \\[4pt]
111000111 & 010010010 & 010100001 & 010001100 \\
111000111 & 001001001 & 001010100 & 001100010 \\
111000111 & 100100100 & 100001010 & 100010001 \\
111111000 & 010010010 & 001010100 & 100010001 \\
111111000 & 001001001 & 100001010 & 010001100 \\
111111000 & 100100100 & 010100001 & 001100010 \\
000111111 & 010010010 & 100001010 & 001100010 \\
000111111 & 001001001 & 010100001 & 100010001 \\
000111111 & 100100100 & 001010100 & 010001100 \\[4pt]
010001100 & 111000111 & 010010010 & 010100001 \\
001100010 & 111000111 & 001001001 & 001010100 \\
100010001 & 111000111 & 100100100 & 100001010 \\
100010001 & 111111000 & 010010010 & 001010100 \\
010001100 & 111111000 & 001001001 & 100001010 \\
001100010 & 111111000 & 100100100 & 010100001 \\
001100010 & 000111111 & 010010010 & 100001010 \\
100010001 & 000111111 & 001001001 & 010100001 \\
010001100 & 000111111 & 100100100 & 001010100
\end{bmatrix}$$

$$D_{11} = \begin{bmatrix}
001001001 & 001010100 & 001100010 & 111111000 \\
100100100 & 100001010 & 100010001 & 111111000 \\
010010010 & 010100001 & 010001100 & 111111000 \\
001001001 & 100001010 & 010001100 & 000111111 \\
100100100 & 010100001 & 001100010 & 000111111 \\
010010010 & 001010100 & 100010001 & 000111111 \\
001001001 & 010100001 & 100010001 & 111000111 \\
100100100 & 001010100 & 010001100 & 111000111 \\
010010010 & 100001010 & 001100010 & 111000111 \\
111111000 & 001001001 & 001010100 & 001100010 \\
111111000 & 100100100 & 100001010 & 100010001 \\
111111000 & 010010010 & 010100001 & 010001100 \\
000111111 & 001001001 & 100001010 & 010001100 \\
000111111 & 100100100 & 010100001 & 001100010 \\
000111111 & 010010010 & 001010100 & 100010001 \\
111000111 & 001001001 & 010100001 & 100010001 \\
111000111 & 100100100 & 001010100 & 010001100 \\
111000111 & 010010010 & 100001010 & 001100010 \\
001100010 & 111111000 & 001001001 & 001010100 \\
100010001 & 111111000 & 100100100 & 100001010 \\
010001100 & 111111000 & 010010010 & 010100001 \\
010001100 & 000111111 & 001001001 & 100001010 \\
001100010 & 000111111 & 100100100 & 010100001 \\
100010001 & 000111111 & 010010010 & 001010100 \\
100010001 & 111000111 & 001001001 & 010100001 \\
010001100 & 111000111 & 100100100 & 001010100 \\
001100010 & 111000111 & 010010010 & 100001010 \\
001010100 & 001100010 & 111111000 & 001001001 \\
100001010 & 100010001 & 111111000 & 100100100 \\
010100001 & 010001100 & 111111000 & 010010010 \\
100001010 & 010001100 & 000111111 & 001001001 \\
010100001 & 001100010 & 000111111 & 100100100 \\
001010100 & 100010001 & 000111111 & 010010010 \\
010100001 & 100010001 & 111000111 & 001001001 \\
001010100 & 010001100 & 111000111 & 100100100 \\
100001010 & 001100010 & 111000111 & 010010010
\end{bmatrix}$$

Appendix 8

Intersection Matrices of Theorem 6.7

$$B_0 = I_{11}, \quad B_1 = \begin{bmatrix} 0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0&0&1&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0 \\ 6&6&6&3&0&0&0&0&0&0&0 \\ 0&0&0&0&6&0&0&0&0&0&0 \\ 0&0&0&0&0&3&6&0&0&0&0 \\ 0&0&0&0&0&3&0&0&0&0&0 \\ 0&0&0&0&0&0&0&2&2&2&0 \\ 0&0&0&0&0&0&0&2&2&2&0 \\ 0&0&0&0&0&0&0&2&2&2&0 \\ 0&0&0&0&0&0&0&0&0&0&6 \end{bmatrix},$$

$$B_4 = \begin{bmatrix} 0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&6&0&0&0&0&0&0 \\ 27&27&27&27&18&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&6&6&6&0 \\ 0&0&0&0&0&0&0&3&3&3&0 \\ 0&0&0&0&0&9&9&6&6&6&0 \\ 0&0&0&0&0&9&9&6&6&6&0 \\ 0&0&0&0&0&9&9&6&6&6&0 \\ 0&0&0&0&0&0&0&0&0&0&27 \end{bmatrix}.$$

Let $\phi = 25^{2m-1}$, $\psi = 25^{2m-1} - 25^{2m-2}$, and $\tau = 25^{2m-2}$.

$$B_5 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 6 & 6 & 0 \\
6\phi & 6\phi & 6\phi & 3\phi & 0 & \psi & \psi & \psi & \psi & \psi & \psi \\
0 & 0 & 0 & 3\phi & 0 & \frac{\psi}{2} & \frac{\psi}{2} & \frac{\psi}{2} & \frac{\psi}{2} & \frac{\psi}{2} & \frac{\psi}{2} \\
0 & 0 & 0 & 0 & 2\phi & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} \\
0 & 0 & 0 & 0 & 2\phi & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} \\
0 & 0 & 0 & 0 & 2\phi & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} \\
0 & 0 & 0 & 0 & 0 & 6\tau-6 & 6\tau-6 & 6\tau-6 & 6\tau-6 & 6\tau-6 & 6\tau
\end{bmatrix},$$

$$B_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 0 \\ 0 & 0 & 0 & 3\phi & 0 & \frac{\psi}{2} & \frac{\psi}{2} & \frac{\psi}{2} & \frac{\psi}{2} & \frac{\psi}{2} & \frac{\psi}{2} \\ 3\phi & 3\phi & 3\phi & 0 & 0 & \frac{\psi}{4} & \frac{\psi}{4} & \frac{\psi}{4} & \frac{\psi}{4} & \frac{\psi}{4} & \frac{\psi}{4} \\ 0 & 0 & 0 & 0 & \phi & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} \\ 0 & 0 & 0 & 0 & \phi & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} \\ 0 & 0 & 0 & 0 & \phi & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} \\ 0 & 0 & 0 & 0 & 0 & 3\tau-3 & 3\tau-3 & 3\tau-3 & 3\tau-3 & 3\tau-3 & 3\tau \end{bmatrix},$$

$$B_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 & 9 & 6 & 6 & 6 & 0 \\ 0 & 0 & 0 & 0 & 2\phi & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} \\ 0 & 0 & 0 & 0 & \phi & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} \\ 9\phi & 0 & 0 & 3\phi & 2\phi & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} \\ 0 & 0 & 9\phi & 3\phi & 2\phi & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} \\ 0 & 9\phi & 0 & 3\phi & 2\phi & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} \\ 0 & 0 & 0 & 0 & 0 & 9\tau-9 & 9\tau-9 & 9\tau-9 & 9\tau-9 & 9\tau-9 & 9\tau \end{bmatrix},$$

$$B_8 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 & 9 & 6 & 6 & 6 & 0 \\ 0 & 0 & 0 & 0 & 2\phi & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} \\ 0 & 0 & 0 & 0 & \phi & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} \\ 0 & 9\phi & 0 & 3\phi & 2\phi & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} \\ 9\phi & 0 & 0 & 3\phi & 2\phi & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} \\ 0 & 0 & 9\phi & 3\phi & 2\phi & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} \\ 0 & 0 & 0 & 0 & 0 & 9\tau-9 & 9\tau-9 & 9\tau-9 & 9\tau-9 & 9\tau-9 & 9\tau \end{bmatrix},$$

$$B_9 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 & 9 & 6 & 6 & 6 & 0 \\ 0 & 0 & 0 & 0 & 2\phi & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} & \frac{3\psi}{2} \\ 0 & 0 & 0 & 0 & \phi & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} & \frac{3\psi}{4} \\ 0 & 0 & 9\phi & 3\phi & 2\phi & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} \\ 0 & 9\phi & 0 & 3\phi & 2\phi & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} \\ 9\phi & 0 & 0 & 3\phi & 2\phi & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} & \frac{9\psi}{4} \\ 0 & 0 & 0 & 0 & 0 & 9\tau-9 & 9\tau-9 & 9\tau-9 & 9\tau-9 & 9\tau-9 & 9\tau \end{bmatrix}.$$

Let $\varsigma_1 = 25^{2m-1} - 25$ and $\varsigma_2 = 25^{2m-2} - 1$.

$$B_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 27 \\ 0 & 0 & 0 & 0 & 0 & 6\tau-6 & 6\tau-6 & 6\tau-6 & 6\tau-6 & 6\tau-6 & 6\tau \\ 0 & 0 & 0 & 0 & 0 & 3\tau-3 & 3\tau-3 & 3\tau-3 & 3\tau-3 & 3\tau-3 & 3\tau \\ 0 & 0 & 0 & 0 & 0 & 9\tau-9 & 9\tau-9 & 9\tau-9 & 9\tau-9 & 9\tau-9 & 9\tau \\ 0 & 0 & 0 & 0 & 0 & 9\tau-9 & 9\tau-9 & 9\tau-9 & 9\tau-9 & 9\tau-9 & 9\tau \\ 0 & 0 & 0 & 0 & 0 & 9\tau-9 & 9\tau-9 & 9\tau-9 & 9\tau-9 & 9\tau-9 & 9\tau \\ \frac{3\varsigma_1}{2} & \frac{3\varsigma_1}{2} & \frac{3\varsigma_1}{2} & \frac{3\varsigma_1}{2} & \frac{3\varsigma_1}{2} & \frac{3\varsigma_2}{2} & \frac{3\varsigma_2}{2} & \frac{3\varsigma_2}{2} & \frac{3\varsigma_2}{2} & \frac{3\varsigma_2}{2} & \frac{3\varsigma_2}{2} - 72 \end{bmatrix}.$$

Appendix 9

Decomposition Made in Example 6.11

$$
N_0 = \begin{pmatrix}
0000000000000000 & 1000100010001000 & 1000010000100001 & 1000001000010100 & 1000000101000010 & 1111000000000000 \\
0000000000000000 & 0100010001000100 & 0100100000010010 & 0100000100101000 & 0100001010000001 & 1111000000000000 \\
0000000000000000 & 0010001000100010 & 0010000110000100 & 0010100001000001 & 0010010000011000 & 1111000000000000 \\
0000000000000000 & 0001000100010001 & 0001001001001000 & 0001010010000010 & 0001100000100100 & 1111000000000000 \\
0000000000000000 & 1000100010001000 & 0100100000010010 & 0010100001000001 & 0001100000100100 & 0000111100000000 \\
0000000000000000 & 0100010001000100 & 1000010000100001 & 0001010010000010 & 0010010000011000 & 0000111100000000 \\
0000000000000000 & 0010001000100010 & 0001001001001000 & 1000001000010100 & 0100001010000001 & 0000111100000000 \\
0000000000000000 & 0001000100010001 & 0010000110000100 & 0100000100101000 & 1000000101000010 & 0000111100000000 \\
0000000000000000 & 1000100010001000 & 0010000110000100 & 0001010010000010 & 0100001010000001 & 0000000011110000 \\
0000000000000000 & 0100010001000100 & 0001001001001000 & 0010100001000001 & 1000000101000010 & 0000000011110000 \\
0000000000000000 & 0010001000100010 & 1000010000100001 & 0100000100101000 & 0001100000100100 & 0000000011110000 \\
0000000000000000 & 0001000100010001 & 0100100000010010 & 1000001000010100 & 0010010000011000 & 0000000011110000 \\
0000000000000000 & 1000100010001000 & 0001001001001000 & 0100000100101000 & 0010010000011000 & 0000000000001111 \\
0000000000000000 & 0100010001000100 & 0010000110000100 & 1000001000010100 & 0001100000100100 & 0000000000001111 \\
0000000000000000 & 0010001000100010 & 0100100000010010 & 0001010010000010 & 1000000101000010 & 0000000000001111 \\
0000000000000000 & 0001000100010001 & 1000010000100001 & 0010100001000001 & 0100001010000001 & 0000000000001111 \\[6pt]
1000100010001000 & 0000000000000000 & 1000001000010100 & 1000000101000010 & 1111000000000000 & 1000010000100001 \\
0100010001000100 & 0000000000000000 & 0100000100101000 & 0100001010000001 & 1111000000000000 & 0100100000010010 \\
0010001000100010 & 0000000000000000 & 0010100001000001 & 0010010000011000 & 1111000000000000 & 0010000110000100 \\
0001000100010001 & 0000000000000000 & 0001010010000010 & 0001100000100100 & 1111000000000000 & 0001001001001000 \\
1000100010001000 & 0000000000000000 & 0010100001000001 & 0001100000100100 & 0000111100000000 & 0100100000010010 \\
0100010001000100 & 0000000000000000 & 0001010010000010 & 0010010000011000 & 0000111100000000 & 1000010000100001 \\
0010001000100010 & 0000000000000000 & 1000001000010100 & 0100001010000001 & 0000111100000000 & 0001001001001000 \\
0001000100010001 & 0000000000000000 & 0100000100101000 & 1000000101000010 & 0000111100000000 & 0010000110000100 \\
1000100010001000 & 0000000000000000 & 0001010010000010 & 0100001010000001 & 0000000011110000 & 0010000110000100 \\
0100010001000100 & 0000000000000000 & 0010100001000001 & 1000000101000010 & 0000000011110000 & 0001001001001000 \\
0010001000100010 & 0000000000000000 & 0100000100101000 & 0001100000100100 & 0000000011110000 & 1000010000100001 \\
0001000100010001 & 0000000000000000 & 1000001000010100 & 0010010000011000 & 0000000011110000 & 0100100000010010 \\
1000100010001000 & 0000000000000000 & 0100000100101000 & 0010010000011000 & 0000000000001111 & 0001001001001000 \\
0100010001000100 & 0000000000000000 & 1000001000010100 & 0001100000100100 & 0000000000001111 & 0010000110000100 \\
0010001000100010 & 0000000000000000 & 0001010010000010 & 1000000101000010 & 0000000000001111 & 0100100000010010 \\
0001000100010001 & 0000000000000000 & 0010100001000001 & 0100001010000001 & 0000000000001111 & 1000010000100001 \\[6pt]
1000010000100001 & 1000001000010100 & 0000000000000000 & 1111000000000000 & 1000100010001000 & 1000000101000010 \\
0100100000010010 & 0100000100101000 & 0000000000000000 & 1111000000000000 & 0100010001000100 & 0100001010000001 \\
0010000110000100 & 0010100001000001 & 0000000000000000 & 1111000000000000 & 0010001000100010 & 0010010000011000 \\
0001001001001000 & 0001010010000010 & 0000000000000000 & 1111000000000000 & 0001000100010001 & 0001100000100100 \\
0100100000010010 & 0010100001000001 & 0000000000000000 & 0000111100000000 & 1000100010001000 & 0001100000100100 \\
1000010000100001 & 0001010010000010 & 0000000000000000 & 0000111100000000 & 0100010001000100 & 0010010000011000 \\
0001001001001000 & 1000001000010100 & 0000000000000000 & 0000111100000000 & 0010001000100010 & 0100001010000001 \\
0010000110000100 & 0100000100101000 & 0000000000000000 & 0000111100000000 & 0001000100010001 & 1000000101000010 \\
0010000110000100 & 0001010010000010 & 0000000000000000 & 0000000011110000 & 1000100010001000 & 0100001010000001 \\
0001001001001000 & 0010100001000001 & 0000000000000000 & 0000000011110000 & 0100010001000100 & 1000000101000010 \\
1000010000100001 & 0100000100101000 & 0000000000000000 & 0000000011110000 & 0010001000100010 & 0001100000100100 \\
0100100000010010 & 1000001000010100 & 0000000000000000 & 0000000011110000 & 0001000100010001 & 0010010000011000 \\
0001001001001000 & 0100000100101000 & 0000000000000000 & 0000000000001111 & 1000100010001000 & 0010010000011000 \\
0010000110000100 & 1000001000010100 & 0000000000000000 & 0000000000001111 & 0100010001000100 & 0001100000100100 \\
0100100000010010 & 0001010010000010 & 0000000000000000 & 0000000000001111 & 0010001000100010 & 1000000101000010 \\
1000010000100001 & 0010100001000001 & 0000000000000000 & 0000000000001111 & 0001000100010001 & 0100001010000001 \\[6pt]
1000001000010100 & 1000000101000010 & 1111000000000000 & 0000000000000000 & 1000010000100001 & 1000100010001000 \\
0100000100101000 & 0100001010000001 & 1111000000000000 & 0000000000000000 & 0100100000010010 & 0100010001000100 \\
0010100001000001 & 0010010000011000 & 1111000000000000 & 0000000000000000 & 0010000110000100 & 0010001000100010 \\
0001010010000010 & 0001100000100100 & 1111000000000000 & 0000000000000000 & 0001001001001000 & 0001000100010001 \\
0010100001000001 & 0001100000100100 & 0000111100000000 & 0000000000000000 & 0100100000010010 & 1000100010001000 \\
0001010010000010 & 0010010000011000 & 0000111100000000 & 0000000000000000 & 1000010000100001 & 0100010001000100 \\
1000001000010100 & 0100001010000001 & 0000111100000000 & 0000000000000000 & 0001001001001000 & 0010001000100010 \\
0100000100101000 & 1000000101000010 & 0000111100000000 & 0000000000000000 & 0010000110000100 & 0001000100010001 \\
0001010010000010 & 0100001010000001 & 0000000011110000 & 0000000000000000 & 0010000110000100 & 1000100010001000 \\
0010100001000001 & 1000000101000010 & 0000000011110000 & 0000000000000000 & 0001001001001000 & 0100010001000100 \\
0100000100101000 & 0001100000100100 & 0000000011110000 & 0000000000000000 & 1000010000100001 & 0010001000100010 \\
1000001000010100 & 0010010000011000 & 0000000011110000 & 0000000000000000 & 0100100000010010 & 0001000100010001 \\
0100000100101000 & 0010010000011000 & 0000000000001111 & 0000000000000000 & 0001001001001000 & 1000100010001000 \\
1000001000010100 & 0001100000100100 & 0000000000001111 & 0000000000000000 & 0010000110000100 & 0100010001000100 \\
0001010010000010 & 1000000101000010 & 0000000000001111 & 0000000000000000 & 0100100000010010 & 0010001000100010 \\
0010100001000001 & 0100001010000001 & 0000000000001111 & 0000000000000000 & 1000010000100001 & 0001000100010001 \\[6pt]
1000000101000010 & 1111000000000000 & 1000100010001000 & 1000010000100001 & 0000000000000000 & 1000001000010100 \\
0100001010000001 & 1111000000000000 & 0100010001000100 & 0100100000010010 & 0000000000000000 & 0100000100101000 \\
0010010000011000 & 1111000000000000 & 0010001000100010 & 0010000110000100 & 0000000000000000 & 0010100001000001 \\
0001100000100100 & 1111000000000000 & 0001000100010001 & 0001001001001000 & 0000000000000000 & 0001010010000010 \\
0001100000100100 & 0000111100000000 & 1000100010001000 & 0100100000010010 & 0000000000000000 & 0010100001000001 \\
0010010000011000 & 0000111100000000 & 0100010001000100 & 1000010000100001 & 0000000000000000 & 0001010010000010 \\
0100001010000001 & 0000111100000000 & 0010001000100010 & 0001001001001000 & 0000000000000000 & 1000001000010100 \\
1000000101000010 & 0000111100000000 & 0001000100010001 & 0010000110000100 & 0000000000000000 & 0100000100101000 \\
0100001010000001 & 0000000011110000 & 1000100010001000 & 0010000110000100 & 0000000000000000 & 0001010010000010 \\
1000000101000010 & 0000000011110000 & 0100010001000100 & 0001001001001000 & 0000000000000000 & 0010100001000001 \\
0001100000100100 & 0000000011110000 & 0010001000100010 & 1000010000100001 & 0000000000000000 & 0100000100101000 \\
0010010000011000 & 0000000011110000 & 0001000100010001 & 0100100000010010 & 0000000000000000 & 1000001000010100 \\
0010010000011000 & 0000000000001111 & 1000100010001000 & 0001001001001000 & 0000000000000000 & 0100000100101000 \\
0001100000100100 & 0000000000001111 & 0100010001000100 & 0010000110000100 & 0000000000000000 & 1000001000010100 \\
1000000101000010 & 0000000000001111 & 0010001000100010 & 0100100000010010 & 0000000000000000 & 0001010010000010 \\
0100001010000001 & 0000000000001111 & 0001000100010001 & 1000010000100001 & 0000000000000000 & 0010100001000001 \\[6pt]
1111000000000000 & 1000010000100001 & 1000000101000010 & 1000100010001000 & 1000001000010100 & 0000000000000000 \\
1111000000000000 & 0100100000010010 & 0100001010000001 & 0100010001000100 & 0100000100101000 & 0000000000000000 \\
1111000000000000 & 0010000110000100 & 0010010000011000 & 0010001000100010 & 0010100001000001 & 0000000000000000 \\
1111000000000000 & 0001001001001000 & 0001100000100100 & 0001000100010001 & 0001010010000010 & 0000000000000000 \\
0000111100000000 & 0100100000010010 & 0001100000100100 & 1000100010001000 & 0010100001000001 & 0000000000000000 \\
0000111100000000 & 1000010000100001 & 0010010000011000 & 0100010001000100 & 0001010010000010 & 0000000000000000 \\
0000111100000000 & 0001001001001000 & 0100001010000001 & 0010001000100010 & 1000001000010100 & 0000000000000000 \\
0000111100000000 & 0010000110000100 & 1000000101000010 & 0001000100010001 & 0100000100101000 & 0000000000000000 \\
0000000011110000 & 0010000110000100 & 0100001010000001 & 1000100010001000 & 0001010010000010 & 0000000000000000 \\
0000000011110000 & 0001001001001000 & 1000000101000010 & 0100010001000100 & 0010100001000001 & 0000000000000000 \\
0000000011110000 & 1000010000100001 & 0001100000100100 & 0010001000100010 & 0100000100101000 & 0000000000000000 \\
0000000011110000 & 0100100000010010 & 0010010000011000 & 0001000100010001 & 1000001000010100 & 0000000000000000 \\
0000000000001111 & 0001001001001000 & 0010010000011000 & 1000100010001000 & 0100000100101000 & 0000000000000000 \\
0000000000001111 & 0010000110000100 & 0001100000100100 & 0100010001000100 & 1000001000010100 & 0000000000000000 \\
0000000000001111 & 0100100000010010 & 1000000101000010 & 0010001000100010 & 0001010010000010 & 0000000000000000 \\
0000000000001111 & 1000010000100001 & 0100001010000001 & 0001000100010001 & 0010100001000001 & 0000000000000000
\end{pmatrix}
$$

$$
N_1 = \begin{pmatrix}
0000000000000000 & 0100010001000100 & 0100100000010010 & 0100000100101000 & 0100001010000001 & 0000111100000000 \\
0000000000000000 & 1000100010001000 & 1000010000100001 & 1000001000010100 & 1000000101000010 & 0000111100000000 \\
0000000000000000 & 0001000100010001 & 0001001001001000 & 0001010010000010 & 0001100000100100 & 0000111100000000 \\
0000000000000000 & 0010001000100010 & 0010000110000100 & 0010100001000001 & 0010010000011000 & 0000111100000000 \\
0000000000000000 & 0100010001000100 & 1000010000100001 & 0001010010000010 & 0010010000011000 & 1111000000000000 \\
0000000000000000 & 1000100010001000 & 0100100000010010 & 0010100001000001 & 0001100000100100 & 1111000000000000 \\
0000000000000000 & 0001000100010001 & 0010000110000100 & 0100000100101000 & 1000000101000010 & 1111000000000000 \\
0000000000000000 & 0010001000100010 & 0001001001001000 & 1000001000010100 & 0100001010000001 & 1111000000000000 \\
0000000000000000 & 0100010001000100 & 0001001001001000 & 0010100001000001 & 1000000101000010 & 0000000000001111 \\
0000000000000000 & 1000100010001000 & 0010000110000100 & 0001010010000010 & 0100001010000001 & 0000000000001111 \\
0000000000000000 & 0001000100010001 & 0100100000010010 & 1000001000010100 & 0010010000011000 & 0000000000001111 \\
0000000000000000 & 0010001000100010 & 1000010000100001 & 0100000100101000 & 0001100000100100 & 0000000000001111 \\
0000000000000000 & 0100010001000100 & 0010000110000100 & 1000001000010100 & 0001100000100100 & 0000000011110000 \\
0000000000000000 & 1000100010001000 & 0001001001001000 & 0100000100101000 & 0010010000011000 & 0000000011110000 \\
0000000000000000 & 0001000100010001 & 1000010000100001 & 0010100001000001 & 0100001010000001 & 0000000011110000 \\
0000000000000000 & 0010001000100010 & 0100100000010010 & 0001010010000010 & 1000000101000010 & 0000000011110000 \\[1ex]
0100010001000100 & 0000000000000000 & 0100000100101000 & 0100001010000001 & 0000111100000000 & 0100100000010010 \\
1000100010001000 & 0000000000000000 & 1000001000010100 & 1000000101000010 & 0000111100000000 & 1000010000100001 \\
0001000100010001 & 0000000000000000 & 0001010010000010 & 0001100000100100 & 0000111100000000 & 0001001001001000 \\
0010001000100010 & 0000000000000000 & 0010100001000001 & 0010010000011000 & 0000111100000000 & 0010000110000100 \\
0100010001000100 & 0000000000000000 & 0001010010000010 & 0010010000011000 & 1111000000000000 & 1000010000100001 \\
1000100010001000 & 0000000000000000 & 0010100001000001 & 0001100000100100 & 1111000000000000 & 0100100000010010 \\
0001000100010001 & 0000000000000000 & 0100000100101000 & 1000000101000010 & 1111000000000000 & 0010000110000100 \\
0010001000100010 & 0000000000000000 & 1000001000010100 & 0100001010000001 & 1111000000000000 & 0001001001001000 \\
0100010001000100 & 0000000000000000 & 0010100001000001 & 1000000101000010 & 0000000000001111 & 0001001001001000 \\
1000100010001000 & 0000000000000000 & 0001010010000010 & 0100001010000001 & 0000000000001111 & 0010000110000100 \\
0001000100010001 & 0000000000000000 & 1000001000010100 & 0010010000011000 & 0000000000001111 & 0100100000010010 \\
0010001000100010 & 0000000000000000 & 0100000100101000 & 0001100000100100 & 0000000000001111 & 1000010000100001 \\
0100010001000100 & 0000000000000000 & 1000001000010100 & 0001100000100100 & 0000000011110000 & 0010000110000100 \\
1000100010001000 & 0000000000000000 & 0100000100101000 & 0010010000011000 & 0000000011110000 & 0001001001001000 \\
0001000100010001 & 0000000000000000 & 0010100001000001 & 0100001010000001 & 0000000011110000 & 1000010000100001 \\
0010001000100010 & 0000000000000000 & 0001010010000010 & 1000000101000010 & 0000000011110000 & 0100100000010010 \\[1ex]
0100100000010010 & 0100000100101000 & 0000000000000000 & 0000111100000000 & 0100010001000100 & 0100001010000001 \\
1000010000100001 & 1000001000010100 & 0000000000000000 & 0000111100000000 & 1000100010001000 & 1000000101000010 \\
0001001001001000 & 0001010010000010 & 0000000000000000 & 0000111100000000 & 0001000100010001 & 0001100000100100 \\
0010000110000100 & 0010100001000001 & 0000000000000000 & 0000111100000000 & 0010001000100010 & 0010010000011000 \\
1000010000100001 & 0001010010000010 & 0000000000000000 & 1111000000000000 & 0100010001000100 & 0010010000011000 \\
0100100000010010 & 0010100001000001 & 0000000000000000 & 1111000000000000 & 1000100010001000 & 0001100000100100 \\
0010000110000100 & 0100000100101000 & 0000000000000000 & 1111000000000000 & 0001000100010001 & 1000000101000010 \\
0001001001001000 & 1000001000010100 & 0000000000000000 & 1111000000000000 & 0010001000100010 & 0100001010000001 \\
0001001001001000 & 0010100001000001 & 0000000000000000 & 0000000000001111 & 0100010001000100 & 1000000101000010 \\
0010000110000100 & 0001010010000010 & 0000000000000000 & 0000000000001111 & 1000100010001000 & 0100001010000001 \\
0100100000010010 & 1000001000010100 & 0000000000000000 & 0000000000001111 & 0001000100010001 & 0010010000011000 \\
1000010000100001 & 0100000100101000 & 0000000000000000 & 0000000000001111 & 0010001000100010 & 0001100000100100 \\
0010000110000100 & 1000001000010100 & 0000000000000000 & 0000000011110000 & 0100010001000100 & 0001100000100100 \\
0001001001001000 & 0100000100101000 & 0000000000000000 & 0000000011110000 & 1000100010001000 & 0010010000011000 \\
1000010000100001 & 0010100001000001 & 0000000000000000 & 0000000011110000 & 0001000100010001 & 0100001010000001 \\
0100100000010010 & 0001010010000010 & 0000000000000000 & 0000000011110000 & 0010001000100010 & 1000000101000010 \\[1ex]
0100000100101000 & 0100001010000001 & 0000111100000000 & 0000000000000000 & 0100100000010010 & 0100010001000100 \\
1000001000010100 & 1000000101000010 & 0000111100000000 & 0000000000000000 & 1000010000100001 & 1000100010001000 \\
0001010010000010 & 0001100000100100 & 0000111100000000 & 0000000000000000 & 0001001001001000 & 0001000100010001 \\
0010100001000001 & 0010010000011000 & 0000111100000000 & 0000000000000000 & 0010000110000100 & 0010001000100010 \\
0001010010000010 & 0010010000011000 & 1111000000000000 & 0000000000000000 & 1000010000100001 & 0100010001000100 \\
0010100001000001 & 0001100000100100 & 1111000000000000 & 0000000000000000 & 0100100000010010 & 1000100010001000 \\
0100000100101000 & 1000000101000010 & 1111000000000000 & 0000000000000000 & 0010000110000100 & 0001000100010001 \\
1000001000010100 & 0100001010000001 & 1111000000000000 & 0000000000000000 & 0001001001001000 & 0010001000100010 \\
0010100001000001 & 1000000101000010 & 0000000000001111 & 0000000000000000 & 0001001001001000 & 0100010001000100 \\
0001010010000010 & 0100001010000001 & 0000000000001111 & 0000000000000000 & 0010000110000100 & 1000100010001000 \\
1000001000010100 & 0010010000011000 & 0000000000001111 & 0000000000000000 & 0100100000010010 & 0001000100010001 \\
0100000100101000 & 0001100000100100 & 0000000000001111 & 0000000000000000 & 1000010000100001 & 0010001000100010 \\
1000001000010100 & 0001100000100100 & 0000000011110000 & 0000000000000000 & 0010000110000100 & 0100010001000100 \\
0100000100101000 & 0010010000011000 & 0000000011110000 & 0000000000000000 & 0001001001001000 & 1000100010001000 \\
0010100001000001 & 0100001010000001 & 0000000011110000 & 0000000000000000 & 1000010000100001 & 0001000100010001 \\
0001010010000010 & 1000000101000010 & 0000000011110000 & 0000000000000000 & 0100100000010010 & 0010001000100010 \\[1ex]
0100001010000001 & 0000111100000000 & 0100010001000100 & 0100100000010010 & 0000000000000000 & 0100000100101000 \\
1000000101000010 & 0000111100000000 & 1000100010001000 & 1000010000100001 & 0000000000000000 & 1000001000010100 \\
0001100000100100 & 0000111100000000 & 0001000100010001 & 0001001001001000 & 0000000000000000 & 0001010010000010 \\
0010010000011000 & 0000111100000000 & 0010001000100010 & 0010000110000100 & 0000000000000000 & 0010100001000001 \\
0010010000011000 & 1111000000000000 & 0100010001000100 & 1000010000100001 & 0000000000000000 & 0001010010000010 \\
0001100000100100 & 1111000000000000 & 1000100010001000 & 0100100000010010 & 0000000000000000 & 0010100001000001 \\
1000000101000010 & 1111000000000000 & 0001000100010001 & 0010000110000100 & 0000000000000000 & 0100000100101000 \\
0100001010000001 & 1111000000000000 & 0010001000100010 & 0001001001001000 & 0000000000000000 & 1000001000010100 \\
1000000101000010 & 0000000000001111 & 0100010001000100 & 0001001001001000 & 0000000000000000 & 0010100001000001 \\
0100001010000001 & 0000000000001111 & 1000100010001000 & 0010000110000100 & 0000000000000000 & 0001010010000010 \\
0010010000011000 & 0000000000001111 & 0001000100010001 & 0100100000010010 & 0000000000000000 & 1000001000010100 \\
0001100000100100 & 0000000000001111 & 0010001000100010 & 1000010000100001 & 0000000000000000 & 0100000100101000 \\
0001100000100100 & 0000000011110000 & 0100010001000100 & 0010000110000100 & 0000000000000000 & 1000001000010100 \\
0010010000011000 & 0000000011110000 & 1000100010001000 & 0001001001001000 & 0000000000000000 & 0100000100101000 \\
0100001010000001 & 0000000011110000 & 0001000100010001 & 1000010000100001 & 0000000000000000 & 0010100001000001 \\
1000000101000010 & 0000000011110000 & 0010001000100010 & 0100100000010010 & 0000000000000000 & 0001010010000010 \\[1ex]
0000111100000000 & 0100100000010010 & 0100001010000001 & 0100010001000100 & 0100000100101000 & 0000000000000000 \\
0000111100000000 & 1000010000100001 & 1000000101000010 & 1000100010001000 & 1000001000010100 & 0000000000000000 \\
0000111100000000 & 0001001001001000 & 0001100000100100 & 0001000100010001 & 0001010010000010 & 0000000000000000 \\
0000111100000000 & 0010000110000100 & 0010010000011000 & 0010001000100010 & 0010100001000001 & 0000000000000000 \\
1111000000000000 & 1000010000100001 & 0010010000011000 & 0100010001000100 & 0001010010000010 & 0000000000000000 \\
1111000000000000 & 0100100000010010 & 0001100000100100 & 1000100010001000 & 0010100001000001 & 0000000000000000 \\
1111000000000000 & 0010000110000100 & 1000000101000010 & 0001000100010001 & 0100000100101000 & 0000000000000000 \\
1111000000000000 & 0001001001001000 & 0100001010000001 & 0010001000100010 & 1000001000010100 & 0000000000000000 \\
0000000000001111 & 0001001001001000 & 1000000101000010 & 0100010001000100 & 0010100001000001 & 0000000000000000 \\
0000000000001111 & 0010000110000100 & 0100001010000001 & 1000100010001000 & 0001010010000010 & 0000000000000000 \\
0000000000001111 & 0100100000010010 & 0010010000011000 & 0001000100010001 & 1000001000010100 & 0000000000000000 \\
0000000000001111 & 1000010000100001 & 0001100000100100 & 0010001000100010 & 0100000100101000 & 0000000000000000 \\
0000000011110000 & 0010000110000100 & 0001100000100100 & 0100010001000100 & 1000001000010100 & 0000000000000000 \\
0000000011110000 & 0001001001001000 & 0010010000011000 & 1000100010001000 & 0100000100101000 & 0000000000000000 \\
0000000011110000 & 1000010000100001 & 0100001010000001 & 0001000100010001 & 0010100001000001 & 0000000000000000 \\
0000000011110000 & 0100100000010010 & 1000000101000010 & 0010001000100010 & 0001010010000010 & 0000000000000000
\end{pmatrix}
$$

$$N_2 = \begin{pmatrix}
0000000000000000\;0010001000100010\;0010000110000100\;0010100001000001\;0010010000011000\;0000000011110000 \\
0000000000000000\;0001000100010001\;0001001001001000\;0001010010000010\;0001100000100100\;0000000011110000 \\
0000000000000000\;1000100010001000\;1000010000100001\;1000001000010100\;1000000101000010\;0000000011110000 \\
0000000000000000\;0100010001000100\;0100100000010010\;0100000100101000\;0100001010000001\;0000000011110000 \\
0000000000000000\;0010001000100010\;0001001001001000\;1000001000010100\;0100001010000001\;0000000000001111 \\
0000000000000000\;0001000100010001\;0010000110000100\;0100000100101000\;1000000101000010\;0000000000001111 \\
0000000000000000\;1000100010001000\;0100100000010010\;0010100001000001\;0001100000100100\;0000000000001111 \\
0000000000000000\;0100010001000100\;1000010000100001\;0001010010000010\;0010010000011000\;0000000000001111 \\
0000000000000000\;0010001000100010\;1000010000100001\;0100000100101000\;0001100000100100\;1111000000000000 \\
0000000000000000\;0001000100010001\;0100100000010010\;1000001000010100\;0010010000011000\;1111000000000000 \\
0000000000000000\;1000100010001000\;0010000110000100\;0001010010000010\;0100001010000001\;1111000000000000 \\
0000000000000000\;0100010001000100\;0001001001001000\;0010100001000001\;1000000101000010\;1111000000000000 \\
0000000000000000\;0010001000100010\;0100100000010010\;0001010010000010\;1000000101000010\;0000111100000000 \\
0000000000000000\;0001000100010001\;1000010000100001\;0010100001000001\;0100001010000001\;0000111100000000 \\
0000000000000000\;1000100010001000\;0001001001001000\;0100000100101000\;0010010000011000\;0000111100000000 \\
0000000000000000\;0100010001000100\;0010000110000100\;1000001000010100\;0001100000100100\;0000111100000000 \\[1ex]
0010001000100010\;0000000000000000\;0010100001000001\;0010010000011000\;0000000011110000\;0010000110000100 \\
0001000100010001\;0000000000000000\;0001010010000010\;0001100000100100\;0000000011110000\;0001001001001000 \\
1000100010001000\;0000000000000000\;1000001000010100\;1000000101000010\;0000000011110000\;1000010000100001 \\
0100010001000100\;0000000000000000\;0100000100101000\;0100001010000001\;0000000011110000\;0100100000010010 \\
0010001000100010\;0000000000000000\;1000001000010100\;0100001010000001\;0000000000001111\;0001001001001000 \\
0001000100010001\;0000000000000000\;0100000100101000\;1000000101000010\;0000000000001111\;0010000110000100 \\
1000100010001000\;0000000000000000\;0010100001000001\;0001100000100100\;0000000000001111\;0100100000010010 \\
0100010001000100\;0000000000000000\;0001010010000010\;0010010000011000\;0000000000001111\;1000010000100001 \\
0010001000100010\;0000000000000000\;0100000100101000\;0001100000100100\;1111000000000000\;1000010000100001 \\
0001000100010001\;0000000000000000\;1000001000010100\;0010010000011000\;1111000000000000\;0100100000010010 \\
1000100010001000\;0000000000000000\;0001010010000010\;0100001010000001\;1111000000000000\;0010000110000100 \\
0100010001000100\;0000000000000000\;0010100001000001\;1000000101000010\;1111000000000000\;0001001001001000 \\
0010001000100010\;0000000000000000\;0001010010000010\;1000000101000010\;0000111100000000\;0100100000010010 \\
0001000100010001\;0000000000000000\;0010100001000001\;0100001010000001\;0000111100000000\;1000010000100001 \\
1000100010001000\;0000000000000000\;0100000100101000\;0010010000011000\;0000111100000000\;0001001001001000 \\
0100010001000100\;0000000000000000\;1000001000010100\;0001100000100100\;0000111100000000\;0010000110000100 \\[1ex]
0010000110000100\;0010100001000001\;0000000000000000\;0000000011110000\;0010001000100010\;0010010000011000 \\
0001001001001000\;0001010010000010\;0000000000000000\;0000000011110000\;0001000100010001\;0001100000100100 \\
1000010000100001\;1000001000010100\;0000000000000000\;0000000011110000\;1000100010001000\;1000000101000010 \\
0100100000010010\;0100000100101000\;0000000000000000\;0000000011110000\;0100010001000100\;0100001010000001 \\
0001001001001000\;1000001000010100\;0000000000000000\;0000000000001111\;0010001000100010\;0100001010000001 \\
0010000110000100\;0100000100101000\;0000000000000000\;0000000000001111\;0001000100010001\;1000000101000010 \\
0100100000010010\;0010100001000001\;0000000000000000\;0000000000001111\;1000100010001000\;0001100000100100 \\
1000010000100001\;0001010010000010\;0000000000000000\;0000000000001111\;0100010001000100\;0010010000011000 \\
1000010000100001\;0100000100101000\;0000000000000000\;1111000000000000\;0010001000100010\;0001100000100100 \\
0100100000010010\;1000001000010100\;0000000000000000\;1111000000000000\;0001000100010001\;0010010000011000 \\
0010000110000100\;0001010010000010\;0000000000000000\;1111000000000000\;1000100010001000\;0100001010000001 \\
0001001001001000\;0010100001000001\;0000000000000000\;1111000000000000\;0100010001000100\;1000000101000010 \\
0100100000010010\;0001010010000010\;0000000000000000\;0000111100000000\;0010001000100010\;1000000101000010 \\
1000010000100001\;0010100001000001\;0000000000000000\;0000111100000000\;0001000100010001\;0100001010000001 \\
0001001001001000\;0100000100101000\;0000000000000000\;0000111100000000\;1000100010001000\;0010010000011000 \\
0010000110000100\;1000001000010100\;0000000000000000\;0000111100000000\;0100010001000100\;0001100000100100 \\[1ex]
0010100001000001\;0010010000011000\;0000000011110000\;0000000000000000\;0010000110000100\;0010001000100010 \\
0001010010000010\;0001100000100100\;0000000011110000\;0000000000000000\;0001001001001000\;0001000100010001 \\
1000001000010100\;1000000101000010\;0000000011110000\;0000000000000000\;1000010000100001\;1000100010001000 \\
0100000100101000\;0100001010000001\;0000000011110000\;0000000000000000\;0100100000010010\;0100010001000100 \\
1000001000010100\;0100001010000001\;0000000000001111\;0000000000000000\;0001001001001000\;0010001000100010 \\
0100000100101000\;1000000101000010\;0000000000001111\;0000000000000000\;0010000110000100\;0001000100010001 \\
0010100001000001\;0001100000100100\;0000000000001111\;0000000000000000\;0100100000010010\;1000100010001000 \\
0001010010000010\;0010010000011000\;0000000000001111\;0000000000000000\;1000010000100001\;0100010001000100 \\
0100000100101000\;0001100000100100\;1111000000000000\;0000000000000000\;1000010000100001\;0010001000100010 \\
1000001000010100\;0010010000011000\;1111000000000000\;0000000000000000\;0100100000010010\;0001000100010001 \\
0001010010000010\;0100001010000001\;1111000000000000\;0000000000000000\;0010000110000100\;1000100010001000 \\
0010100001000001\;1000000101000010\;1111000000000000\;0000000000000000\;0001001001001000\;0100010001000100 \\
0001010010000010\;1000000101000010\;0000111100000000\;0000000000000000\;0100100000010010\;0010001000100010 \\
0010100001000001\;0100001010000001\;0000111100000000\;0000000000000000\;1000010000100001\;0001000100010001 \\
0100000100101000\;0010010000011000\;0000111100000000\;0000000000000000\;0001001001001000\;1000100010001000 \\
1000001000010100\;0001100000100100\;0000111100000000\;0000000000000000\;0010000110000100\;0100010001000100 \\[1ex]
0010010000011000\;0000000011110000\;0010001000100010\;0010000110000100\;0000000000000000\;0010100001000001 \\
0001100000100100\;0000000011110000\;0001000100010001\;0001001001001000\;0000000000000000\;0001010010000010 \\
1000000101000010\;0000000011110000\;1000100010001000\;1000010000100001\;0000000000000000\;1000001000010100 \\
0100001010000001\;0000000011110000\;0100010001000100\;0100100000010010\;0000000000000000\;0100000100101000 \\
0100001010000001\;0000000000001111\;0010001000100010\;0001001001001000\;0000000000000000\;1000001000010100 \\
1000000101000010\;0000000000001111\;0001000100010001\;0010000110000100\;0000000000000000\;0100000100101000 \\
0001100000100100\;0000000000001111\;1000100010001000\;0100100000010010\;0000000000000000\;0010100001000001 \\
0010010000011000\;0000000000001111\;0100010001000100\;1000010000100001\;0000000000000000\;0001010010000010 \\
0001100000100100\;1111000000000000\;0010001000100010\;1000010000100001\;0000000000000000\;0100000100101000 \\
0010010000011000\;1111000000000000\;0001000100010001\;0100100000010010\;0000000000000000\;1000001000010100 \\
0100001010000001\;1111000000000000\;1000100010001000\;0010000110000100\;0000000000000000\;0001010010000010 \\
1000000101000010\;1111000000000000\;0100010001000100\;0001001001001000\;0000000000000000\;0010100001000001 \\
1000000101000010\;0000111100000000\;0010001000100010\;0100100000010010\;0000000000000000\;0001010010000010 \\
0100001010000001\;0000111100000000\;0001000100010001\;1000010000100001\;0000000000000000\;0010100001000001 \\
0010010000011000\;0000111100000000\;1000100010001000\;0001001001001000\;0000000000000000\;0100000100101000 \\
0001100000100100\;0000111100000000\;0100010001000100\;0010000110000100\;0000000000000000\;1000001000010100 \\[1ex]
0000000011110000\;0010000110000100\;0010010000011000\;0010001000100010\;0010100001000001\;0000000000000000 \\
0000000011110000\;0001001001001000\;0001100000100100\;0001000100010001\;0001010010000010\;0000000000000000 \\
0000000011110000\;1000010000100001\;1000000101000010\;1000100010001000\;1000001000010100\;0000000000000000 \\
0000000011110000\;0100100000010010\;0100001010000001\;0100010001000100\;0100000100101000\;0000000000000000 \\
0000000000001111\;0001001001001000\;0100001010000001\;0010001000100010\;1000001000010100\;0000000000000000 \\
0000000000001111\;0010000110000100\;1000000101000010\;0001000100010001\;0100000100101000\;0000000000000000 \\
0000000000001111\;0100100000010010\;0001100000100100\;1000100010001000\;0010100001000001\;0000000000000000 \\
0000000000001111\;1000010000100001\;0010010000011000\;0100010001000100\;0001010010000010\;0000000000000000 \\
1111000000000000\;1000010000100001\;0001100000100100\;0010001000100010\;0100000100101000\;0000000000000000 \\
1111000000000000\;0100100000010010\;0010010000011000\;0001000100010001\;1000001000010100\;0000000000000000 \\
1111000000000000\;0010000110000100\;0100001010000001\;1000100010001000\;0001010010000010\;0000000000000000 \\
1111000000000000\;0001001001001000\;1000000101000010\;0100010001000100\;0010100001000001\;0000000000000000 \\
0000111100000000\;0100100000010010\;1000000101000010\;0010001000100010\;0001010010000010\;0000000000000000 \\
0000111100000000\;1000010000100001\;0100001010000001\;0001000100010001\;0010100001000001\;0000000000000000 \\
0000111100000000\;0001001001001000\;0010010000011000\;1000100010001000\;0100000100101000\;0000000000000000 \\
0000111100000000\;0010000110000100\;0001100000100100\;0100010001000100\;1000001000010100\;0000000000000000
\end{pmatrix}$$

$$
N_3 = \begin{pmatrix}
0000000000000000 & 0001000100010001 & 0001001001001000 & 0001010010000010 & 0001100000100100 & 0000000000001111 \\
0000000000000000 & 0010001000100010 & 0010000110000100 & 0010100001000001 & 0010010000011000 & 0000000000001111 \\
0000000000000000 & 0100010001000100 & 0100100000010010 & 0100000100101000 & 0100001010000001 & 0000000000001111 \\
0000000000000000 & 1000100010001000 & 1000010000100001 & 1000001000010100 & 1000000101000010 & 0000000000001111 \\
0000000000000000 & 0001000100010001 & 0010000110000100 & 0100000100101000 & 1000000101000010 & 0000000011110000 \\
0000000000000000 & 0010001000100010 & 0001001001001000 & 1000001000010100 & 0100001010000001 & 0000000011110000 \\
0000000000000000 & 0100010001000100 & 1000010000100001 & 0001010010000010 & 0010010000011000 & 0000000011110000 \\
0000000000000000 & 1000100010001000 & 0100100000010010 & 0010100001000001 & 0001100000100100 & 0000000011110000 \\
0000000000000000 & 0001000100010001 & 0100100000010010 & 1000001000010100 & 0010010000011000 & 0000111100000000 \\
0000000000000000 & 0010001000100010 & 1000010000100001 & 0100000100101000 & 0001100000100100 & 0000111100000000 \\
0000000000000000 & 0100010001000100 & 0001001001001000 & 0010100001000001 & 1000000101000010 & 0000111100000000 \\
0000000000000000 & 1000100010001000 & 0010000110000100 & 0001010010000010 & 0100001010000001 & 0000111100000000 \\
0000000000000000 & 0001000100010001 & 1000010000100001 & 0010100001000001 & 0100001010000001 & 1111000000000000 \\
0000000000000000 & 0010001000100010 & 0100100000010010 & 0001010010000010 & 1000000101000010 & 1111000000000000 \\
0000000000000000 & 0100010001000100 & 0010000110000100 & 1000001000010100 & 0001100000100100 & 1111000000000000 \\
0000000000000000 & 1000100010001000 & 0001001001001000 & 0100000100101000 & 0010010000011000 & 1111000000000000 \\[6pt]
0001000100010001 & 0000000000000000 & 0001010010000010 & 0001100000100100 & 0000000000001111 & 0001001001001000 \\
0010001000100010 & 0000000000000000 & 0010100001000001 & 0010010000011000 & 0000000000001111 & 0010000110000100 \\
0100010001000100 & 0000000000000000 & 0100000100101000 & 0100001010000001 & 0000000000001111 & 0100100000010010 \\
1000100010001000 & 0000000000000000 & 1000001000010100 & 1000000101000010 & 0000000000001111 & 1000010000100001 \\
0001000100010001 & 0000000000000000 & 0100000100101000 & 1000000101000010 & 0000000011110000 & 0010000110000100 \\
0010001000100010 & 0000000000000000 & 1000001000010100 & 0100001010000001 & 0000000011110000 & 0001001001001000 \\
0100010001000100 & 0000000000000000 & 0001010010000010 & 0010010000011000 & 0000000011110000 & 1000010000100001 \\
1000100010001000 & 0000000000000000 & 0010100001000001 & 0001100000100100 & 0000000011110000 & 0100100000010010 \\
0001000100010001 & 0000000000000000 & 1000001000010100 & 0010010000011000 & 0000111100000000 & 0100100000010010 \\
0010001000100010 & 0000000000000000 & 0100000100101000 & 0001100000100100 & 0000111100000000 & 1000010000100001 \\
0100010001000100 & 0000000000000000 & 0010100001000001 & 1000000101000010 & 0000111100000000 & 0001001001001000 \\
1000100010001000 & 0000000000000000 & 0001010010000010 & 0100001010000001 & 0000111100000000 & 0010000110000100 \\
0001000100010001 & 0000000000000000 & 0010100001000001 & 0100001010000001 & 1111000000000000 & 1000010000100001 \\
0010001000100010 & 0000000000000000 & 0001010010000010 & 1000000101000010 & 1111000000000000 & 0100100000010010 \\
0100010001000100 & 0000000000000000 & 1000001000010100 & 0001100000100100 & 1111000000000000 & 0010000110000100 \\
1000100010001000 & 0000000000000000 & 0100000100101000 & 0010010000011000 & 1111000000000000 & 0001001001001000 \\[6pt]
0001001001001000 & 0001010010000010 & 0000000000000000 & 0000000000001111 & 0001000100010001 & 0001100000100100 \\
0010000110000100 & 0010100001000001 & 0000000000000000 & 0000000000001111 & 0010001000100010 & 0010010000011000 \\
0100100000010010 & 0100000100101000 & 0000000000000000 & 0000000000001111 & 0100010001000100 & 0100001010000001 \\
1000010000100001 & 1000001000010100 & 0000000000000000 & 0000000000001111 & 1000100010001000 & 1000000101000010 \\
0010000110000100 & 0100000100101000 & 0000000000000000 & 0000000011110000 & 0001000100010001 & 1000000101000010 \\
0001001001001000 & 1000001000010100 & 0000000000000000 & 0000000011110000 & 0010001000100010 & 0100001010000001 \\
1000010000100001 & 0001010010000010 & 0000000000000000 & 0000000011110000 & 0100010001000100 & 0010010000011000 \\
0100100000010010 & 0010100001000001 & 0000000000000000 & 0000000011110000 & 1000100010001000 & 0001100000100100 \\
0100100000010010 & 1000001000010100 & 0000000000000000 & 0000111100000000 & 0001000100010001 & 0010010000011000 \\
1000010000100001 & 0100000100101000 & 0000000000000000 & 0000111100000000 & 0010001000100010 & 0001100000100100 \\
0001001001001000 & 0010100001000001 & 0000000000000000 & 0000111100000000 & 0100010001000100 & 1000000101000010 \\
0010000110000100 & 0001010010000010 & 0000000000000000 & 0000111100000000 & 1000100010001000 & 0100001010000001 \\
1000010000100001 & 0010100001000001 & 0000000000000000 & 1111000000000000 & 0001000100010001 & 0100001010000001 \\
0100100000010010 & 0001010010000010 & 0000000000000000 & 1111000000000000 & 0010001000100010 & 1000000101000010 \\
0010000110000100 & 1000001000010100 & 0000000000000000 & 1111000000000000 & 0100010001000100 & 0001100000100100 \\
0001001001001000 & 0100000100101000 & 0000000000000000 & 1111000000000000 & 1000100010001000 & 0010010000011000 \\[6pt]
0001010010000010 & 0001100000100100 & 0000000000001111 & 0000000000000000 & 0001001001001000 & 0001000100010001 \\
0010100001000001 & 0010010000011000 & 0000000000001111 & 0000000000000000 & 0010000110000100 & 0010001000100010 \\
0100000100101000 & 0100001010000001 & 0000000000001111 & 0000000000000000 & 0100100000010010 & 0100010001000100 \\
1000001000010100 & 1000000101000010 & 0000000000001111 & 0000000000000000 & 1000010000100001 & 1000100010001000 \\
0100000100101000 & 1000000101000010 & 0000000011110000 & 0000000000000000 & 0010000110000100 & 0001000100010001 \\
1000001000010100 & 0100001010000001 & 0000000011110000 & 0000000000000000 & 0001001001001000 & 0010001000100010 \\
0001010010000010 & 0010010000011000 & 0000000011110000 & 0000000000000000 & 1000010000100001 & 0100010001000100 \\
0010100001000001 & 0001100000100100 & 0000000011110000 & 0000000000000000 & 0100100000010010 & 1000100010001000 \\
1000001000010100 & 0010010000011000 & 0000111100000000 & 0000000000000000 & 0100100000010010 & 0001000100010001 \\
0100000100101000 & 0001100000100100 & 0000111100000000 & 0000000000000000 & 1000010000100001 & 0010001000100010 \\
0010100001000001 & 1000000101000010 & 0000111100000000 & 0000000000000000 & 0001001001001000 & 0100010001000100 \\
0001010010000010 & 0100001010000001 & 0000111100000000 & 0000000000000000 & 0010000110000100 & 1000100010001000 \\
0010100001000001 & 0100001010000001 & 1111000000000000 & 0000000000000000 & 1000010000100001 & 0001000100010001 \\
0001010010000010 & 1000000101000010 & 1111000000000000 & 0000000000000000 & 0100100000010010 & 0010001000100010 \\
1000001000010100 & 0001100000100100 & 1111000000000000 & 0000000000000000 & 0010000110000100 & 0100010001000100 \\
0100000100101000 & 0010010000011000 & 1111000000000000 & 0000000000000000 & 0001001001001000 & 1000100010001000 \\[6pt]
0001100000100100 & 0000000000001111 & 0001000100010001 & 0001001001001000 & 0000000000000000 & 0001010010000010 \\
0010010000011000 & 0000000000001111 & 0010001000100010 & 0010000110000100 & 0000000000000000 & 0010100001000001 \\
0100001010000001 & 0000000000001111 & 0100010001000100 & 0100100000010010 & 0000000000000000 & 0100000100101000 \\
1000000101000010 & 0000000000001111 & 1000100010001000 & 1000010000100001 & 0000000000000000 & 1000001000010100 \\
1000000101000010 & 0000000011110000 & 0001000100010001 & 0010000110000100 & 0000000000000000 & 0100000100101000 \\
0100001010000001 & 0000000011110000 & 0010001000100010 & 0001001001001000 & 0000000000000000 & 1000001000010100 \\
0010010000011000 & 0000000011110000 & 0100010001000100 & 1000010000100001 & 0000000000000000 & 0001010010000010 \\
0001100000100100 & 0000000011110000 & 1000100010001000 & 0100100000010010 & 0000000000000000 & 0010100001000001 \\
0010010000011000 & 0000111100000000 & 0001000100010001 & 0100100000010010 & 0000000000000000 & 1000001000010100 \\
0001100000100100 & 0000111100000000 & 0010001000100010 & 1000010000100001 & 0000000000000000 & 0100000100101000 \\
1000000101000010 & 0000111100000000 & 0100010001000100 & 0001001001001000 & 0000000000000000 & 0010100001000001 \\
0100001010000001 & 0000111100000000 & 1000100010001000 & 0010000110000100 & 0000000000000000 & 0001010010000010 \\
0100001010000001 & 1111000000000000 & 0001000100010001 & 1000010000100001 & 0000000000000000 & 0010100001000001 \\
1000000101000010 & 1111000000000000 & 0010001000100010 & 0100100000010010 & 0000000000000000 & 0001010010000010 \\
0001100000100100 & 1111000000000000 & 0100010001000100 & 0010000110000100 & 0000000000000000 & 1000001000010100 \\
0010010000011000 & 1111000000000000 & 1000100010001000 & 0001001001001000 & 0000000000000000 & 0100000100101000 \\[6pt]
0000000000001111 & 0001001001001000 & 0001100000100100 & 0001000100010001 & 0001010010000010 & 0000000000000000 \\
0000000000001111 & 0010000110000100 & 0010010000011000 & 0010001000100010 & 0010100001000001 & 0000000000000000 \\
0000000000001111 & 0100100000010010 & 0100001010000001 & 0100010001000100 & 0100000100101000 & 0000000000000000 \\
0000000000001111 & 1000010000100001 & 1000000101000010 & 1000100010001000 & 1000001000010100 & 0000000000000000 \\
0000000011110000 & 0010000110000100 & 1000000101000010 & 0001000100010001 & 0100000100101000 & 0000000000000000 \\
0000000011110000 & 0001001001001000 & 0100001010000001 & 0010001000100010 & 1000001000010100 & 0000000000000000 \\
0000000011110000 & 1000010000100001 & 0010010000011000 & 0100010001000100 & 0001010010000010 & 0000000000000000 \\
0000000011110000 & 0100100000010010 & 0001100000100100 & 1000100010001000 & 0010100001000001 & 0000000000000000 \\
0000111100000000 & 0100100000010010 & 0010010000011000 & 0001000100010001 & 1000001000010100 & 0000000000000000 \\
0000111100000000 & 1000010000100001 & 0001100000100100 & 0010001000100010 & 0100000100101000 & 0000000000000000 \\
0000111100000000 & 0001001001001000 & 1000000101000010 & 0100010001000100 & 0010100001000001 & 0000000000000000 \\
0000111100000000 & 0010000110000100 & 0100001010000001 & 1000100010001000 & 0001010010000010 & 0000000000000000 \\
1111000000000000 & 1000010000100001 & 0100001010000001 & 0001000100010001 & 0010100001000001 & 0000000000000000 \\
1111000000000000 & 0100100000010010 & 1000000101000010 & 0010001000100010 & 0001010010000010 & 0000000000000000 \\
1111000000000000 & 0010000110000100 & 0001100000100100 & 0100010001000100 & 1000001000010100 & 0000000000000000 \\
1111000000000000 & 0001001001001000 & 0010010000011000 & 1000100010001000 & 0100000100101000 & 0000000000000000
\end{pmatrix}
$$

$$N_4 = \begin{pmatrix}
0111111111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
1011111111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
1101111111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
1110111111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
1111011111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
1111101111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
1111110111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
1111111011111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
1111111101111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
1111111110111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
1111111111011111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
1111111111101111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
1111111111110111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
1111111111111011 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
1111111111111101 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
1111111111111110 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\[1ex]
0000000000000000 & 0111111111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 1011111111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 1101111111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 1110111111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 1111011111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 1111101111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 1111110111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 1111111011111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 1111111101111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 1111111110111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 1111111111011111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 1111111111101111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 1111111111110111 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 1111111111111011 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 1111111111111101 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 1111111111111110 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\[1ex]
0000000000000000 & 0000000000000000 & 0111111111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 1011111111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 1101111111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 1110111111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 1111011111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 1111101111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 1111110111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 1111111011111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 1111111101111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 1111111110111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 1111111111011111 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 1111111111101111 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 1111111111110111 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 1111111111111011 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 1111111111111101 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 1111111111111110 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\[1ex]
0000000000000000 & 0000000000000000 & 0000000000000000 & 0111111111111111 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 1011111111111111 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 1101111111111111 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 1110111111111111 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 1111011111111111 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 1111101111111111 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 1111110111111111 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111011111111 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111101111111 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111110111111 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111111011111 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111111101111 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111111110111 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111111111011 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111111111101 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111111111110 & 0000000000000000 & 0000000000000000 \\[1ex]
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0111111111111111 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1011111111111111 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1101111111111111 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1110111111111111 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111011111111111 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111101111111111 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111110111111111 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111011111111 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111101111111 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111110111111 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111111011111 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111111101111 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111111110111 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111111111011 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111111111101 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111111111110 & 0000000000000000 \\[1ex]
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0111111111111111 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1011111111111111 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1101111111111111 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1110111111111111 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111011111111111 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111101111111111 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111110111111111 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111011111111 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111101111111 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111110111111 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111111011111 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111111101111 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111111110111 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111111111011 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111111111101 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111111111110
\end{pmatrix}$$

Appendix 10

Classes of Scheme of Example 6.13

$$A_0 = \left(\begin{array}{cccccc}
1000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0100000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0010000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0001000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000100000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000010000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000001000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000100000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000010000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000001000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000100000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000010000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000001000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000100 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000010 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000001 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\[1ex]
0000000000000000 & 1000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0100000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0010000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0001000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000100000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000010000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000001000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000100000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000010000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000001000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000100000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000010000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000001000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000100 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000010 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000001 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\[1ex]
0000000000000000 & 0000000000000000 & 1000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0100000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0010000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0001000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000100000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000010000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000001000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000100000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000010000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000001000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000100000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000010000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000001000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000100 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000010 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000001 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\[1ex]
0000000000000000 & 0000000000000000 & 0000000000000000 & 1000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0100000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0010000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0001000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000100000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000010000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000001000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000100000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000010000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000001000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000100000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000010000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000001000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000100 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000010 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000001 & 0000000000000000 & 0000000000000000 \\[1ex]
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0100000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0010000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0001000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000100000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000010000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000001000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000100000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000010000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000001000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000100000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000010000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000001000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000100 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000010 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000001 & 0000000000000000 \\[1ex]
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0100000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0010000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0001000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000100000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000010000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000001000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000100000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000010000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000001000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000100000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000010000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000001000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000100 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000010 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000001
\end{array}\right)$$

$$A_1 = \begin{pmatrix}
0100000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
1000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0001000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0010000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000010000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000100000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000100000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000001000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000001000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000010000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000010000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000100000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000100 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000001000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000001 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000010 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0100000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 1000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0001000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0010000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000010000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000100000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000100000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000001000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000001000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000010000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000010000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000100000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000100 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000001000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000001 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000010 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0100000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 1000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0001000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0010000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000010000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000100000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000100000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000001000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000001000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000010000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000010000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000100000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000100 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000001000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000001 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000010 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0100000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 1000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0001000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0010000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000010000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000100000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000100000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000001000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000001000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000010000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000010000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000100000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000100 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000001000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000001 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000010 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0100000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0001000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0010000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000010000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000100000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000100000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000001000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000001000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000010000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000010000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000100000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000100 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000001000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000001 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000010 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0100000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0001000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0010000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000010000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000100000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000100000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000001000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000001000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000010000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000010000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000100000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000100 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000001000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000001 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000010
\end{pmatrix}$$

$$A_2 = \begin{pmatrix}
0010000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0001000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
1000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0100000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000001000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000100000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000100000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000010000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000100000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000010000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000010000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000001000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000010 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000001 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000001000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000100 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0010000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0001000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 1000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0100000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000001000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000100000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000100000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000010000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000100000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000010000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000010000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000001000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000010 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000001 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000001000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000100 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0010000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0001000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 1000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0100000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000001000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000100000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000100000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000010000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000100000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000010000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000010000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000001000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000010 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000001 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000001000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000100 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0010000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0001000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 1000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0100000000000000 & 0000000000000000 & 0000000000000000 \\
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0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000100000000 & 0000000000000000 & 0000000000000000 \\
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0000000000000000 & 0000000000000000 & 0000000000000000 & 0000010000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000100000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000010000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000010000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000001000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000010 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000001 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000001000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000100 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0010000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0001000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0100000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000001000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000100000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000100000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000010000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000100000 & 0000000000000000 \\
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0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000010000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000001000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000010 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000001 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000001000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000100 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0010000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0001000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 1000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0100000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000001000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000100000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000100000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000010000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000100000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000010000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000010000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000001000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000010 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000001 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000001000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000100
\end{pmatrix}$$

$$A_3 = \left(\begin{array}{cccccc}
0001000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0010000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0100000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
1000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000100000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000001000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000010000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000100000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000010000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000100000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000001000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000010000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000001 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
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0000000000000100 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
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0001000100010001 & 0000000000000000 & 0001010010000010 & 0001100000100100 & 0000000000000000 & 0001001001001000 \\
1000100010001000 & 0000000000000000 & 0010100001000001 & 0001100000100100 & 0000000000000000 & 0100100000010010 \\
0100010001000100 & 0000000000000000 & 0001010010000010 & 0010010000011000 & 0000000000000000 & 1000010000100001 \\
0010001000100010 & 0000000000000000 & 1000001000010100 & 0100001010000001 & 0000000000000000 & 0001001001001000 \\
0001000100010001 & 0000000000000000 & 0100000100101000 & 1000000101000010 & 0000000000000000 & 0010000110000100 \\
1000100010001000 & 0000000000000000 & 0001010010000010 & 0100001010000001 & 0000000000000000 & 0010000110000100 \\
0100010001000100 & 0000000000000000 & 0010100001000001 & 1000000101000010 & 0000000000000000 & 0001001001001000 \\
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0001000100010001 & 0000000000000000 & 1000001000010100 & 0010010000011000 & 0000000000000000 & 0100100000010010 \\
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1000010000100001 & 1000001000010100 & 0000000000000000 & 0000000000000000 & 1000100010001000 & 1000000101000010 \\
0100100000010010 & 0100000100101000 & 0000000000000000 & 0000000000000000 & 0100010001000100 & 0100001010000001 \\
0010000110000100 & 0010100001000001 & 0000000000000000 & 0000000000000000 & 0010001000100010 & 0010010000011000 \\
0001001001001000 & 0001010010000010 & 0000000000000000 & 0000000000000000 & 0001000100010001 & 0001100000100100 \\
0100100000010010 & 0010100001000001 & 0000000000000000 & 0000000000000000 & 1000100010001000 & 0001100000100100 \\
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0001001001001000 & 1000001000010100 & 0000000000000000 & 0000000000000000 & 0010001000100010 & 0100001010000001 \\
0010000110000100 & 0100000100101000 & 0000000000000000 & 0000000000000000 & 0001000100010001 & 1000000101000010 \\
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0100100000010010 & 1000001000010100 & 0000000000000000 & 0000000000000000 & 0001000100010001 & 0010010000011000 \\
0001001001001000 & 0100000100101000 & 0000000000000000 & 0000000000000000 & 1000100010001000 & 0010010000011000 \\
0010000110000100 & 1000001000010100 & 0000000000000000 & 0000000000000000 & 0100010001000100 & 0001100000100100 \\
0100100000010010 & 0001010010000010 & 0000000000000000 & 0000000000000000 & 0010001000100010 & 1000000101000010 \\
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0100000100101000 & 0100001010000001 & 0000000000000000 & 0000000000000000 & 0100100000010010 & 0100010001000100 \\
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0001010010000010 & 0001100000100100 & 0000000000000000 & 0000000000000000 & 0001001001001000 & 0001000100010001 \\
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1000001000010100 & 0100001010000001 & 0000000000000000 & 0000000000000000 & 0001001001001000 & 0010001000100010 \\
0100000100101000 & 1000000101000010 & 0000000000000000 & 0000000000000000 & 0010000110000100 & 0001000100010001 \\
0001010010000010 & 0100001010000001 & 0000000000000000 & 0000000000000000 & 0010000110000100 & 1000100010001000 \\
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0100000100101000 & 0001100000100100 & 0000000000000000 & 0000000000000000 & 1000010000100001 & 0010001000100010 \\
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0001100000100100 & 0000000000000000 & 0001000100010001 & 0001001001001000 & 0000000000000000 & 0001010010000010 \\
0001100000100100 & 0000000000000000 & 1000100010001000 & 0100100000010010 & 0000000000000000 & 0010100001000001 \\
0010010000011000 & 0000000000000000 & 0100010001000100 & 1000010000100001 & 0000000000000000 & 0001010010000010 \\
0100001010000001 & 0000000000000000 & 0010001000100010 & 0001001001001000 & 0000000000000000 & 1000001000010100 \\
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0100001010000001 & 0000000000000000 & 1000100010001000 & 0010000110000100 & 0000000000000000 & 0001010010000010 \\
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0010010000011000 & 0000000000000000 & 1000100010001000 & 0001001001001000 & 0000000000000000 & 0100000100101000 \\
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0000000000000000 & 0010000110000100 & 0001100000100100 & 0100010001000100 & 1000001000010100 & 0000000000000000 \\
0000000000000000 & 0100100000010010 & 1000000101000010 & 0010001000100010 & 0001010010000010 & 0000000000000000 \\
0000000000000000 & 1000010000100001 & 0100001010000001 & 0001000100010001 & 0010100001000001 & 0000000000000000
\end{pmatrix}
$$

$$
D_1 = \begin{pmatrix}
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0000000000000000 & 0010001000100010 & 0010000110000100 & 0010100001000001 & 0010010000011000 & 0000000000000000 \\
0000000000000000 & 0100010001000100 & 1000010000100001 & 0001010010000010 & 0010010000011000 & 0000000000000000 \\
0000000000000000 & 1000100010001000 & 0100100000010010 & 0010100001000001 & 0001100000100100 & 0000000000000000 \\
0000000000000000 & 0001000100010001 & 0010000110000100 & 0100000100101000 & 1000000101000010 & 0000000000000000 \\
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0000000000000000 & 1000100010001000 & 0010000110000100 & 0001010010000010 & 0100001010000001 & 0000000000000000 \\
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0000000000000000 & 0010001000100010 & 1000010000100001 & 0100000100101000 & 0001100000100100 & 0000000000000000 \\
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0000000000000000 & 1000100010001000 & 0001001001001000 & 0100000100101000 & 0010010000011000 & 0000000000000000 \\
0000000000000000 & 0001000100010001 & 1000010000100001 & 0010100001000001 & 0100001010000001 & 0000000000000000 \\
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0001000100010001 & 0000000000000000 & 0001010010000010 & 0001100000100100 & 0000000000000000 & 0001001001001000 \\
0010001000100010 & 0000000000000000 & 0010100001000001 & 0010010000011000 & 0000000000000000 & 0010000110000100 \\
0100010001000100 & 0000000000000000 & 0001010010000010 & 0010010000011000 & 0000000000000000 & 1000010000100001 \\
1000100010001000 & 0000000000000000 & 0010100001000001 & 0001100000100100 & 0000000000000000 & 0100100000010010 \\
0001000100010001 & 0000000000000000 & 0100000100101000 & 1000000101000010 & 0000000000000000 & 0010000110000100 \\
0010001000100010 & 0000000000000000 & 1000001000010100 & 0100001010000001 & 0000000000000000 & 0001001001001000 \\
0100010001000100 & 0000000000000000 & 0010100001000001 & 1000000101000010 & 0000000000000000 & 0001001001001000 \\
1000100010001000 & 0000000000000000 & 0001010010000010 & 0100001010000001 & 0000000000000000 & 0010000110000100 \\
0001000100010001 & 0000000000000000 & 1000001000010100 & 0010010000011000 & 0000000000000000 & 0100100000010010 \\
0010001000100010 & 0000000000000000 & 0100000100101000 & 0001100000100100 & 0000000000000000 & 1000010000100001 \\
0100010001000100 & 0000000000000000 & 1000001000010100 & 0001100000100100 & 0000000000000000 & 0010000110000100 \\
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0001000100010001 & 0000000000000000 & 0010100001000001 & 0100001010000001 & 0000000000000000 & 1000010000100001 \\
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0001001001001000 & 1000001000010100 & 0000000000000000 & 0000000000000000 & 0010001000100010 & 0100001010000001 \\
0001001001001000 & 0010100001000001 & 0000000000000000 & 0000000000000000 & 0100010001000100 & 1000000101000010 \\
0010000110000100 & 0001010010000010 & 0000000000000000 & 0000000000000000 & 1000100010001000 & 0100001010000001 \\
0100100000010010 & 1000001000010100 & 0000000000000000 & 0000000000000000 & 0001000100010001 & 0010010000011000 \\
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0010000110000100 & 1000001000010100 & 0000000000000000 & 0000000000000000 & 0100010001000100 & 0001100000100100 \\
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1000010000100001 & 0010100001000001 & 0000000000000000 & 0000000000000000 & 0001000100010001 & 0100001010000001 \\
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0100000100101000 & 0100001010000001 & 0000000000000000 & 0000000000000000 & 0100100000010010 & 0100010001000100 \\
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0001010010000010 & 0010010000011000 & 0000000000000000 & 0000000000000000 & 1000010000100001 & 0100010001000100 \\
0010100001000001 & 0001100000100100 & 0000000000000000 & 0000000000000000 & 0100100000010010 & 1000100010001000 \\
0100000100101000 & 1000000101000010 & 0000000000000000 & 0000000000000000 & 0010000110000100 & 0001000100010001 \\
1000001000010100 & 0100001010000001 & 0000000000000000 & 0000000000000000 & 0001001001001000 & 0010001000100010 \\
0010100001000001 & 1000000101000010 & 0000000000000000 & 0000000000000000 & 0001001001001000 & 0100010001000100 \\
0001010010000010 & 0100001010000001 & 0000000000000000 & 0000000000000000 & 0010000110000100 & 1000100010001000 \\
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0100000100101000 & 0001100000100100 & 0000000000000000 & 0000000000000000 & 1000010000100001 & 0010001000100010 \\
1000001000010100 & 0001100000100100 & 0000000000000000 & 0000000000000000 & 0010000110000100 & 0100010001000100 \\
0100000100101000 & 0010010000011000 & 0000000000000000 & 0000000000000000 & 0001001001001000 & 1000100010001000 \\
0010100001000001 & 0100001010000001 & 0000000000000000 & 0000000000000000 & 1000010000100001 & 0001000100010001 \\
0001010010000010 & 1000000101000010 & 0000000000000000 & 0000000000000000 & 0100100000010010 & 0010001000100010 \\[6pt]
0100001010000001 & 0000000000000000 & 0100010001000100 & 0100100000010010 & 0000000000000000 & 0100000100101000 \\
1000000101000010 & 0000000000000000 & 1000100010001000 & 1000010000100001 & 0000000000000000 & 1000001000010100 \\
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0010010000011000 & 0000000000000000 & 0100010001000100 & 1000010000100001 & 0000000000000000 & 0001010010000010 \\
0001100000100100 & 0000000000000000 & 1000100010001000 & 0100100000010010 & 0000000000000000 & 0010100001000001 \\
1000000101000010 & 0000000000000000 & 0001000100010001 & 0010000110000100 & 0000000000000000 & 0100000100101000 \\
0100001010000001 & 0000000000000000 & 0010001000100010 & 0001001001001000 & 0000000000000000 & 1000001000010100 \\
1000000101000010 & 0000000000000000 & 0100010001000100 & 0001001001001000 & 0000000000000000 & 0010100001000001 \\
0100001010000001 & 0000000000000000 & 1000100010001000 & 0010000110000100 & 0000000000000000 & 0001010010000010 \\
0010010000011000 & 0000000000000000 & 0001000100010001 & 0100100000010010 & 0000000000000000 & 1000001000010100 \\
0001100000100100 & 0000000000000000 & 0010001000100010 & 1000010000100001 & 0000000000000000 & 0100000100101000 \\
0001100000100100 & 0000000000000000 & 0100010001000100 & 0010000110000100 & 0000000000000000 & 1000001000010100 \\
0010010000011000 & 0000000000000000 & 1000100010001000 & 0001001001001000 & 0000000000000000 & 0100000100101000 \\
0100001010000001 & 0000000000000000 & 0001000100010001 & 1000010000100001 & 0000000000000000 & 0010100001000001 \\
1000000101000010 & 0000000000000000 & 0010001000100010 & 0100100000010010 & 0000000000000000 & 0001010010000010 \\[6pt]
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0000000000000000 & 1000010000100001 & 1000000101000010 & 1000100010001000 & 1000001000010100 & 0000000000000000 \\
0000000000000000 & 0001001001001000 & 0001100000100100 & 0001000100010001 & 0001010010000010 & 0000000000000000 \\
0000000000000000 & 0010000110000100 & 0010010000011000 & 0010001000100010 & 0010100001000001 & 0000000000000000 \\
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0000000000000000 & 0001001001001000 & 0100001010000001 & 0010001000100010 & 1000001000010100 & 0000000000000000 \\
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0000000000000000 & 0100100000010010 & 0010010000011000 & 0001000100010001 & 1000001000010100 & 0000000000000000 \\
0000000000000000 & 1000010000100001 & 0001100000100100 & 0010001000100010 & 0100000100101000 & 0000000000000000 \\
0000000000000000 & 0010000110000100 & 0001100000100100 & 0100010001000100 & 1000001000010100 & 0000000000000000 \\
0000000000000000 & 0001001001001000 & 0010010000011000 & 1000100010001000 & 0100000100101000 & 0000000000000000 \\
0000000000000000 & 1000010000100001 & 0100001010000001 & 0001000100010001 & 0010100001000001 & 0000000000000000 \\
0000000000000000 & 0100100000010010 & 1000000101000010 & 0010001000100010 & 0001010010000010 & 0000000000000000
\end{pmatrix}
$$

$$D_2 = \begin{pmatrix}
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0000000000000000 & 0100010001000100 & 0100100000010010 & 0100000100101000 & 0100001010000001 & 0000000000000000 \\
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0000000000000000 & 0001000100010001 & 0010000110000100 & 0100000100101000 & 1000000101000010 & 0000000000000000 \\
0000000000000000 & 1000100010001000 & 0100100000010010 & 0010100001000001 & 0001100000100100 & 0000000000000000 \\
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0000000000000000 & 0010001000100010 & 1000010000100001 & 0100000100101000 & 0001100000100100 & 0000000000000000 \\
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0000000000000000 & 0001000100010001 & 1000010000100001 & 0010100001000001 & 0100001010000001 & 0000000000000000 \\
0000000000000000 & 1000100010001000 & 0001001001001000 & 0100000100101000 & 0010010000011000 & 0000000000000000 \\
0000000000000000 & 0100010001000100 & 0010000110000100 & 1000001000010100 & 0001100000100100 & 0000000000000000 \\[1ex]
0010001000100010 & 0000000000000000 & 0010100001000001 & 0010010000011000 & 0000000000000000 & 0010000110000100 \\
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0100010001000100 & 0000000000000000 & 0100000100101000 & 0100001010000001 & 0000000000000000 & 0100100000010010 \\
0010001000100010 & 0000000000000000 & 1000001000010100 & 0100001010000001 & 0000000000000000 & 0001001001001000 \\
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0010001000100010 & 0000000000000000 & 0001010010000010 & 1000000101000010 & 0000000000000000 & 0100100000010010 \\
0001000100010001 & 0000000000000000 & 0010100001000001 & 0100001010000001 & 0000000000000000 & 1000010000100001 \\
1000100010001000 & 0000000000000000 & 0100000100101000 & 0010010000011000 & 0000000000000000 & 0001001001001000 \\
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0010000110000100 & 0010100001000001 & 0000000000000000 & 0000000000000000 & 0010001000100010 & 0010010000011000 \\
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1000000101000010 & 0000000000000000 & 0100010001000100 & 0001001001001000 & 0000000000000000 & 0010100001000001 \\
1000000101000010 & 0000000000000000 & 0010001000100010 & 0100100000010010 & 0000000000000000 & 0001010010000010 \\
0100001010000001 & 0000000000000000 & 0001000100010001 & 1000010000100001 & 0000000000000000 & 0010100001000001 \\
0010010000011000 & 0000000000000000 & 1000100010001000 & 0001001001001000 & 0000000000000000 & 0100000100101000 \\
0001100000100100 & 0000000000000000 & 0100010001000100 & 0010000110000100 & 0000000000000000 & 1000001000010100 \\[1ex]
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0000000000000000 & 1000010000100001 & 1000000101000010 & 1000100010001000 & 1000001000010100 & 0000000000000000 \\
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0000000000000000 & 1000010000100001 & 0010010000011000 & 0100010001000100 & 0001010010000010 & 0000000000000000 \\
0000000000000000 & 1000010000100001 & 0001100000100100 & 0010001000100010 & 0100000100101000 & 0000000000000000 \\
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0000000000000000 & 0100100000010010 & 1000000101000010 & 0010001000100010 & 0001010010000010 & 0000000000000000 \\
0000000000000000 & 1000010000100001 & 0100001010000001 & 0001000100010001 & 0010100001000001 & 0000000000000000 \\
0000000000000000 & 0001001001001000 & 0010010000011000 & 1000100010001000 & 0100000100101000 & 0000000000000000 \\
0000000000000000 & 0010000110000100 & 0001100000100100 & 0100010001000100 & 1000001000010100 & 0000000000000000
\end{pmatrix}$$

$$D_3 = \left(\begin{array}{cccccc}
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0000000000000000 & 0100010001000100 & 1000010000100001 & 0001010010000010 & 0010010000011000 & 0000000000000000 \\
0000000000000000 & 1000100010001000 & 0100100000010010 & 0010100001000001 & 0001100000100100 & 0000000000000000 \\
0000000000000000 & 0001000100010001 & 0100100000010010 & 1000001000010100 & 0010010000011000 & 0000000000000000 \\
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0000000000000000 & 1000100010001000 & 0010000110000100 & 0001010010000010 & 0100001010000001 & 0000000000000000 \\
0000000000000000 & 0001000100010001 & 1000010000100001 & 0010100001000001 & 0100001010000001 & 0000000000000000 \\
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0000000000000000 & 0100010001000100 & 0010000110000100 & 1000001000010100 & 0001100000100100 & 0000000000000000 \\
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0010001000100010 & 0000000000000000 & 0100000100101000 & 0001100000100100 & 0000000000000000 & 1000010000100001 \\
0100010001000100 & 0000000000000000 & 0010100001000001 & 1000000101000010 & 0000000000000000 & 0001001001001000 \\
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0001000100010001 & 0000000000000000 & 0010100001000001 & 0100001010000001 & 0000000000000000 & 1000010000100001 \\
0010001000100010 & 0000000000000000 & 0001010010000010 & 1000000101000010 & 0000000000000000 & 0100100000010010 \\
0100010001000100 & 0000000000000000 & 1000001000010100 & 0001100000100100 & 0000000000000000 & 0010000110000100 \\
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0100000100101000 & 0010010000011000 & 0000000000000000 & 0000000000000000 & 0001001001001000 & 1000100010001000 \\[6pt]
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\end{array}\right)$$