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Asymptotic behavior of solutions of a Fisher equation with free boundaries and nonlocal term

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Abstract. We study the asymptotic behavior of solutions of a Fisher equation with free boundaries and the nonlocal term (an integral convolution in space). This problem can model the spreading of a biological or chemical species, where free boundaries represent the spreading fronts of the species. We give a dichotomy result, that is, the solution either converges to 1 locally uniformly in $\mathbb{R}$, or to 0 uniformly in the occupying domain. Moreover, we give the sharp threshold when the initial data $u_0 = \sigma\phi$, that is, there exists $\sigma^* > 0$ such that spreading happens when $\sigma > \sigma^*$, and vanishing happens when $\sigma \le \sigma^*$.

Keywords: asymptotic behavior of solutions, free boundary problem, Fisher equation, nonlocal.

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1 Introduction

Consider the following free boundary problem with nonlocal term

$$\begin{cases} u_t = u_{xx} + (1-u)\displaystyle\int_{\mathbb{R}} k(x-y)u(t,y)dy, & g(t) < x < h(t),\ t > 0, \\ u(t,x) = 0, & x \in \mathbb{R} \setminus (g(t), h(t)),\ t > 0, \\ g'(t) = -\mu u_x(t, g(t)), & t > 0, \\ h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ -g(0) = h(0) = h_0,\ \ u(0,x) = u_0(x), & -h_0 \le x \le h_0, \end{cases} \tag{1.1}$$

where $x = g(t)$ and $x = h(t)$ are moving boundaries to be determined together with $u(t,x)$, $\mu > 0$ is a constant, $h_0 > 0$. We use the standard hypotheses on kernel $k(\cdot)$ as follows:

$$k \in C^1(\mathbb{R}) \text{ is nonnegative, symmetric and } \int_{\mathbb{R}} k(x-y)dy = 1 \text{ for any } x \in \mathbb{R}. \tag{1.2}$$

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The initial function u_0 belongs to $\mathscr{X}(h_0)$ for some $h_0 > 0$, where

$$\mathscr{X}(h_0) := \Big\{ \phi \in C^2([-h_0, h_0]) : \phi(-h_0) = \phi(h_0) = 0,\ \phi(x) \geq (\not\equiv) 0 \text{ in } (-h_0, h_0) \Big\}. \tag{1.3}$$

Recently, Problem (1.1) with Fisher–KPP nonlinearity, i.e., $u_t = u_{xx} + u(1-u)$ was studied by [9, 10], etc. They used this model to describe the spreading of a new or invasion species, with the free boundaries $h(t)$ and $g(t)$ representing the expanding fronts of the species whose density is represented by $u(t,x)$. They obtained a spreading-vanishing dichotomy result. Moreover, [10] also studied the bistable nonlinearity and obtained a trichotomy result. Later, [2–4] also considered Fisher–KPP equation with other free boundary conditions. [11, 13, 14] also studied the corresponding problem of (1.1) with Fisher–KPP nonlinearity in high dimensional spaces (without nonlocal term).

It is known that some species are distributed in space randomly. They typically interact with physical environment and other individuals in their spatial neighborhood, so some species u at a point x and time t usually depend on u in the neighborhood of the point x. And they even depend on u in the whole region. Therefore, we added the nonlocal term into the equation, i.e., $\int_{\mathbb{R}} k(x-y)u(t,y)dy$, instead of u to model the growth of the species. The reason why this is a global term is that the population are moving, and then, the growth of the species is related to the population in a neighborhood of the original position. Hence, the growth rate of the species can be represented as a spatial weighted average. For the above reasons, we added the nonlocal term into the equation. Such nonlocal interactions are also used in epidemic reaction-diffusion models, such as [7] studied the following system

$$\begin{cases} u_t = d\Delta u - au + \int_{\Omega} k(x,y)v(t,y)dy, & t > 0,\ x \in \Omega, \\ v_t = -bv + G(u), & t > 0,\ x \in \Omega \end{cases} \tag{1.4}$$

with conditions $\beta(x)\frac{\partial u}{\partial n} + \alpha(x)u = 0$ and $\beta(x)\frac{\partial v}{\partial n} + \alpha(x)v = 0$ for $t > 0$ and $x \in \partial\Omega$. $k(x,y) > 0$ is symmetric and $\int_{\mathbb{R}} k(x,y)dy = 1$. They studied the globally asymptotic stability of the trivial solution and the existence of the nontrivial equilibrium which is globally asymptotically stable. There are many other papers (cf. [16, 18, 19] and so on) studied nonlocal problem of reaction-diffusion systems in bounded/unbounded domain. Recently, some authors introduce free boundaries to such nonlocal problems, [15] studied (1.4) (with $k(x,y)$ is replaced by $k(x-y)$) for $x \in [g(t), h(t)]$ with free boundaries $g(t)$ and $h(t)$ satisfying $g'(t) = -u_x(t, g(t))$, $h'(t) = -u_x(t, h(t))$, they obtained some sufficient conditions for spreading and vanishing, when spreading happens, they also give the estimates of the asymptotic spreading speed. [6] also considered the nonlocal SIS epidemic model with free boundaries and obtained some sufficient conditions for spreading ($\lim_{t\to\infty} \|u(t,\cdot)\|_{C([g(t),h(t)])} > 0$) and vanishing.

In this paper, we will introduce the nonlocal term to the free boundary problem of the Fisher equation, i.e., the problem (1.1). We see that this problem indicates that the whole class exhibit themselves on the whole region $\mathbb{R}$. The individuals occupy the initial region $[-h_0, h_0]$ and invade further into the new environment from two ends of the initial region. The spreading fronts spread at a speed that is proportional to the population gradient at the fronts, that is, they satisfy the Stefan conditions $h'(t) = -\mu u_x(t, h(t))$ and $g'(t) = -\mu u_x(t, g(t))$. Since the individuals are moving, the nonlocal term in (1.1) means that the individuals at location x can contact with susceptible individuals in the neighborhood of location x or the whole class, this gives rise to the nonlocal effect. We will give more explicit asymptotic behavior of solutions and consider the effect of the nonlocal term on the spreading of the solution. We first give some sufficient conditions for spreading ($\lim_{t\to\infty} g(t) = -\infty$, $\lim_{t\to\infty} h(t) = +\infty$ and the solution u converges to 1) and vanishing ($0 < \lim_{t\to\infty} h(t) - \lim_{t\to\infty} g(t) < 2\ell^*$ and the solution u converges to 0, where the definition of $\ell^* > 0$ is given in Section 2), then obtain a spreading-vanishing dichotomy. We finally give the sharp threshold result: when the initial data $u_0 = \sigma\phi$, there is critical value $\sigma^* > 0$, when $\sigma \leq \sigma^*$, vanishing happens; when $\sigma > \sigma^*$, spreading happens.

We first give the existence of the solution of the problem (1.1) and some basic properties of $g(t)$ and $h(t)$. It follows from the arguments in [9] (with obvious modifications) that the problem (1.1) has a unique solution

$$(u, g, h) \in C^{1+\frac{\gamma}{2}, 1+\gamma}(\overline{D}) \times C^{1+\gamma/2}([0, +\infty)) \times C^{1+\gamma/2}([0, +\infty))$$

for any $\gamma \in (0, 1)$, where $D := \{(t, x) \in \mathbb{R}^2 : \ x \in [g(t), h(t)], t \in (0, +\infty)\}$. By the strong comparison principle and the Hopf lemma, we also have $h'(t) > 0$ and $g'(t) < 0$ for all $t > 0$. Hence

$$h_\infty := \lim_{t\to\infty} h(t) \in (0, +\infty] \quad \text{and} \quad g_\infty := \lim_{t\to\infty} g(t) \in [-\infty, 0)$$

exist. Moreover, as in the proof of [10, Lemma 2.8] one can show that

$$-2h_0 < g(t) + h(t) < 2h_0 \quad \text{for all } t > 0. \tag{1.5}$$

We conclude from this inequality that $I_\infty := (g_\infty, h_\infty)$ is either $\mathbb{R}$ or a finite interval.

The main purpose of this paper is to study the asymptotic behavior of bounded solutions of (1.1).

Theorem 1.1. *Assume that (u, g, h) is a time-global solution of (1.1). Then either*

(i) *spreading: $(g_\infty, h_\infty) = \mathbb{R}$ and*

$$\lim_{t\to\infty} u(t, x) = 1 \text{ locally uniformly in } \mathbb{R},$$

or

(ii) *vanishing: (g_∞, h_∞) is a finite interval with length no bigger than $2\ell^*$, and*

$$\lim_{t\to\infty} \max_{g(t)\leq x\leq h(t)} u(t, x) = 0,$$

where the definition of ℓ^* is given in Proposition 2.1.

Moreover, if $u_0 = \sigma\phi$ with $\phi \in \mathscr{X}(h_0)$, then there exists $\sigma^* = \sigma^*(h_0, \phi) \geq 0$ such that vanishing happens when $\sigma \leq \sigma^*$, and spreading happens when $\sigma > \sigma^*$.

In Section 2, we show some preliminary results including the eigenvalue problem and comparison principle theorem. In Section 3, we first give sufficient conditions of spreading and vanishing, then prove the main theorem. In Section 4, we give the Appendix to prove incomplete or unproved statements of this paper.

2 Some preliminary results

In this section, we first consider the eigenvalue problem and discuss the properties of the principal eigenvalue, which plays an important role in studying spreading and vanishing. We then give the comparison principle theorems. Finally, we consider the convergence of the solutions of elliptic equations, which is used to prove spreading.

2.1 Eigenvalue problem

We consider the following eigenvalue problem

$$\begin{cases} \phi_{xx} + \int_{-\ell}^{\ell} k(x-y)\phi(y)dy + \lambda\phi = 0, \quad x \in (-\ell, \ell), \\ \phi(-\ell) = \phi(\ell) = 0, \end{cases} \tag{2.1}$$

for any $\ell > 0$. The principal eigenvalue of (2.1) is the smallest eigenvalue, which is the only eigenvalue admitting a positive eigenfunction except for $x = \pm\ell$. It is well-known that (2.1) has a unique principal eigenvalue (cf. Lemma A.2 in Appendix), denoted by $\lambda_1(\ell)$, and there is a positive eigenfunction ϕ_1 with $\|\phi_1\|_{L^2([-\ell,\ell])} = 1$ corresponding to $\lambda_1(\ell)$, and $\lambda_1(\ell)$ can be characterized by

$$\lambda_1(\ell) = \inf_{\phi \in C_0([-\ell,\ell]), \phi \not\equiv 0} \left(\frac{\int_{-\ell}^{\ell} (\phi'(x))^2 dx - \int_{-\ell}^{\ell}\int_{-\ell}^{\ell} k(x-y)\phi(x)\phi(y)dxdy}{\int_{-\ell}^{\ell} \phi^2(x)dx} \right), \tag{2.2}$$

where $\phi \in C_0([-\ell,\ell])$ means that $\phi \in C([-\ell,\ell])$ with $\phi(\pm\ell) = 0$. Moreover, for any $\ell_1, \ell_2 > 0$, Corollary 2.3 in [5] says that $\lambda_1(\ell_1) > \lambda_1(\ell_2)$ when $\ell_1 < \ell_2$. (The proof of this conclusion is given in Appendix).

Proposition 2.1. *There is a $\ell^* \geq \pi/2$ such that $\lambda_1(\ell) > 0$ when $\ell < \ell^*$, $\lambda_1(\ell) = 0$ when $\ell = \ell^*$ and $\lambda_1(\ell) < 0$ when $\ell > \ell^*$.*

Proof. Due to

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}} k(x-y)\phi(x)\phi(y)dxdy \\ &\quad\leq \int_{\mathbb{R}}\int_{\mathbb{R}} k(x-y)\frac{\phi^2(x)+\phi^2(y)}{2}dxdy \\ &\quad= \frac{1}{2}\int_{\mathbb{R}} \phi^2(y)dy \int_{\mathbb{R}} k(x-y)dx + \frac{1}{2}\int_{\mathbb{R}} \phi^2(x)dx \int_{\mathbb{R}} k(x-y)dy \\ &\quad= \int_{-\ell}^{\ell} \phi^2(x)dx \quad (\text{note that } \int_{\mathbb{R}} k(x-y)dy = 1), \end{aligned} \tag{2.3}$$

we have

$$\begin{aligned} \lambda_1(\ell) &\geq \inf_{\phi \in C_0, \phi \not\equiv 0} \frac{\int_{-\ell}^{\ell}(\phi'(x))^2 dx}{\int_{-\ell}^{\ell}\phi^2(x)dx} - \sup_{\phi \in C_0, \phi \not\equiv 0} \frac{\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} k(x-y)\phi(x)\phi(y)dxdy}{\int_{-\ell}^{\ell}\phi^2(x)dx} \\ &\geq \frac{\pi^2}{4\ell^2} - 1. \end{aligned} \tag{2.4}$$

Here we have used the well known result: $\frac{\int_{-\ell}^{\ell}(\phi'(x))^2dx}{\int_{-\ell}^{\ell}\phi^2(x)dx}$ attains its minimum at $\phi(x) = \cos(\frac{\pi x}{2\ell})$. Hence, it follows from (2.4) that

$$\lambda_1(\ell) > 0 \quad \text{when} \quad \ell < \frac{\pi}{2}.$$

On the other hand, taking $\phi(x) = \cos(\frac{\pi x}{2\ell})$ in (2.2), we have

$$\begin{aligned} \lambda_1(\ell) &\leq \frac{\int_{-\ell}^{\ell} \frac{\pi^2}{4\ell^2}\cos^2(\frac{\pi x}{2\ell})dx}{\int_{-\ell}^{\ell}\cos^2(\frac{\pi x}{2\ell})dx} - \frac{\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} k(x-y)\cos(\frac{\pi x}{2\ell})\cos(\frac{\pi y}{2\ell})dxdy}{\int_{-\ell}^{\ell}\cos^2(\frac{\pi x}{2\ell})dx} \\ &\leq \frac{\pi^2}{4\ell^2} - \frac{\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} k(x-y)\cos(\frac{\pi x}{2\ell})\cos(\frac{\pi y}{2\ell})dxdy}{\int_{-\ell}^{\ell}\cos^2(\frac{\pi x}{2\ell})dx}. \end{aligned}$$

Moreover, by (1.2), when ℓ is sufficiently large, there exists a subset $I \subset (-\ell,\ell)\times(-\ell,\ell)$ ($I \subset \mathbb{R}^2$) such that, when $(x,y) \in I$, the inequality $k(x-y) \geq \varepsilon_0$ holds for some small $\varepsilon_0 > 0$, and the area of I (denoted by $\sigma(I)$) is independent of ℓ when ℓ becomes large. Since $x = \pm\ell \notin I$, there exists some $\delta_0 > 0$ such that

$$\cos\left(\frac{\pi x}{2\ell}\right) \geq \delta_0,\ \cos\left(\frac{\pi y}{2\ell}\right) \geq \delta_0 \quad \text{for } (x,y) \in I.$$

Hence

$$\begin{aligned}&\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} k(x-y)\cos\left(\frac{\pi x}{2\ell}\right)\cos\left(\frac{\pi y}{2\ell}\right)dxdy\\ &\quad\geq \iint_I k(x-y)\cos\left(\frac{\pi x}{2\ell}\right)\cos\left(\frac{\pi y}{2\ell}\right)dxdy\\ &\quad\geq \sigma(I)\delta_0^2\varepsilon_0.\end{aligned}$$

Therefore, $\lambda_1(\ell) < 0$ for sufficiently large $\ell > 0$. Combining this and the monotonicity of $\lambda_1(\ell)$ (cf. Corollary A.3), the equation $\lambda_1(\ell) = 0$ has a unique root $\ell^* \geq \frac{\pi}{2}$. Furthermore, $\lambda_1(\ell) > 0$ when $0 < \ell < \ell^*$, and $\lambda_1(\ell) < 0$ when $\ell > \ell^*$. □

2.2 Comparison principle and some basic results

We mainly consider the asymptotic behavior of solutions of the problem (1.1) by constructing some suitable upper and lower solutions, so the comparison principle is essential here. Therefore, we give the following comparison theorems which can be proved similarly as in [9, Lemma 3.5], for the readers' convenience, we give the proofs in the Appendix.

Lemma 2.2. *Suppose that $T \in (0,\infty)$, $\overline{g},\overline{h} \in C^1([0,T])$, $\overline{u} \in C(\overline{D}_T)\cap C^{1,2}(D_T)$ with $D_T = \{(t,x)\in\mathbb{R}^2 : 0 < t \leq T, \overline{g}(t) < x < \overline{h}(t)\}$, and*

$$\begin{cases}\overline{u}_t \geq \overline{u}_{xx} + (1-\overline{u})\displaystyle\int_{\mathbb{R}} k(x-y)\overline{u}(t,y)dy, & 0 < t \leq T,\ \overline{g}(t) < x < \overline{h}(t),\\ \overline{u} = 0,\quad \overline{g}'(t) \leq -\mu\overline{u}_x(t,x), & 0 < t \leq T,\ x = \overline{g}(t),\\ \overline{u} = 0,\quad \overline{h}'(t) \geq -\mu\overline{u}_x(t,x), & 0 < t \leq T,\ x = \overline{h}(t).\end{cases}$$

If $[-h_0,h_0] \subseteq [\overline{g}(0),\overline{h}(0)]$, $u_0(x) \leq \overline{u}(0,x)$ in $[-h_0,h_0]$, and if (u,g,h) is a solution of (1.1), then

$$g(t) \geq \overline{g}(t),\ h(t) \leq \overline{h}(t),\ u(t,x) \leq \overline{u}(t,x) \quad \text{for } t \in (0,T] \text{ and } x \in (g(t),h(t)).$$

Lemma 2.3. *Suppose that $T \in (0,\infty)$, $\overline{g},\overline{h} \in C^1([0,T])$, $\overline{u} \in C(\overline{D}_T)\cap C^{1,2}(D_T)$ with $D_T = \{(t,x)\in\mathbb{R}^2 : 0 < t \leq T, \overline{g}(t) < x < \overline{h}(t)\}$, and*

$$\begin{cases}\overline{u}_t \geq \overline{u}_{xx} + (1-\overline{u})\displaystyle\int_{\mathbb{R}} k(x-y)\overline{u}(t,y)dy, & 0 < t \leq T,\ \overline{g}(t) < x < \overline{h}(t),\\ \overline{u} \geq u, & 0 < t \leq T,\ x = \overline{g}(t),\\ \overline{u} = 0,\quad \overline{h}'(t) \geq -\mu\overline{u}_x(t,x), & 0 < t \leq T,\ x = \overline{h}(t),\end{cases}$$

with $\overline{g}(t) \geq g(t)$ in $[0,T]$, $h_0 \leq \overline{h}(0)$, $u_0(x) \leq \overline{u}(0,x)$ in $[\overline{g}(0),h_0]$, where (u,g,h) is a solution of (1.1). Then

$$h(t) \leq \overline{h}(t) \quad \text{in } (0,T],\quad u(t,x) \leq \overline{u}(t,x) \quad \text{for } t \in (0,T] \text{ and } \overline{g}(t) < x < h(t).$$

Remark 2.4. The pair $(\overline{u}, \overline{g}, \overline{h})$ or $\overline{u}$ is usually called an upper solution of the problem (1.1) and one can define a lower solution by reverting all the inequalities.

In order to study the spreading of the solution, we need study the following problem, whose solution will be used as a lower solution of the problem (1.1). For any $\ell > 0$, consider

$$\begin{cases} v_{xx} + (1-v)\int_{\mathbb{R}} k(x-y)v(y)dy = 0, & x \in (-\ell, \ell), \\ v = 0, & x \in \mathbb{R} \setminus (-\ell, \ell), \\ v > 0, & x \in (-\ell, \ell). \end{cases} \tag{2.5}$$

Lemma 2.5. *Assume* (1.2)*, when* $\ell > \ell^*$ *the problem* (2.5) *has a unique solution* $V_\ell(x)$ *satisfying* $0 < V_\ell(x) < 1$ *for* $x \in (-\ell, \ell)$ *and* $V_\ell(-\ell) = V_\ell(\ell) = 0$*. Moreover,* $V_\ell(x)$ *is increasing in* ℓ *and* $V_\ell(x) \to 1$ *as* $\ell \to \infty$ *uniformly on any compact set of* $\mathbb{R}$*; when* $\ell \leq \ell^*$*, the problem* (2.5) *has only zero solution.*

Proof. By Lemma A.4 in the Appendix, the problem (2.5) has the comparison principle, that is, for positive v_1, v_2 in $C^2([-\ell, \ell])$ satisfying

$$v_{1xx} + (1-v_1)\int_{\mathbb{R}} k(x-y)v_1(y)dy \leq 0 \leq v_{2xx} + (1-v_2)\int_{\mathbb{R}} k(x-y)v_2(y)dy, \quad x \in (-\ell, \ell)$$

and $v_1(x) \geq v_2(x)$ for $x = \pm\ell$, then $v_2(x) \leq v_1(x)$ for $x \in [-\ell, \ell]$.

The existence of the solution of the problem (2.5) follows from the upper and lower solution argument. Clearly any constant greater than or equal to 1 is an upper solution. Let λ be the principle eigenvalue of (2.1) for any fixed $\ell > \ell^*$ and $\phi(x)$ be a positive eigenfunction corresponding to λ. Then for all small $\varepsilon > 0$, $\varepsilon\phi < 1$ is a lower solution. Thus the upper and lower solution argument shows that there is at least one positive solution.

If v_1 and v_2 are two positive solutions of (2.1), applying the above comparison principle, we have $v_1 \leq v_2$ and $v_2 \leq v_1$ on $[-\ell, \ell]$. Hence $v_1 = v_2$. This proves the uniqueness.

Moreover, by the above comparison principle, we can derive that V_ℓ is increasing in ℓ. Finally, we prove that $V_\ell \to 1$ as $\ell \to \infty$. Obviously, for any small $\varepsilon > 0$, $\overline{v} := 1 + \varepsilon$ is an upper solution. We now construct a lower solution. For any $\ell > \ell^*$, let λ_ℓ be the principal eigenvalue of (2.1) and ϕ_ℓ be the corresponding eigenfunction with $\|\phi_\ell\|_{L^2([-\ell,\ell])} = 1$. We choose some $\delta > 0$ small such that $\phi < 1 - \varepsilon$ is very small in $[-\ell, -\ell+\delta] \cup [\ell-\delta, \ell]$, and $[-\ell, -\ell+\delta] \cap [-\ell/2, \ell/2] = \emptyset$, $[\ell-\delta, \ell] \cap [-\ell/2, \ell/2] = \emptyset$. Now we can choose a smooth function $\underline{v}$ on $[-\ell, \ell]$, such that $\underline{v} = \phi_\ell$ on $[-\ell, -\ell+\delta] \cup [\ell-\delta, \ell]$, $\underline{v} = 1 - \varepsilon$ on $[-\ell/2, \ell/2]$ and $1 - \varepsilon/2 > \underline{v} > 0$ on the rest of $[-\ell, \ell]$. It is easily seen that such $\underline{v}$ is a lower solution of our problem. Since $\overline{v} < \underline{v}$, we deduce that $\overline{v} \leq v < \underline{v}$. In particular,

$$1 + \varepsilon \geq V_\ell(x) \geq 1 - \varepsilon \quad \text{for } x \in [-\ell/2, \ell/2].$$

Letting $\ell \to +\infty$, then $V_\ell \to 1$ as $\ell \to +\infty$ locally uniformly in $\mathbb{R}$.

When $\ell \leq \ell^*$, we construct an upper solution $\overline{u} := \varepsilon\phi$ in $(-\ell, \ell)$, where ϕ is the eigenfunction of (2.1), the eigenvalue $\lambda \geq 0$, and $\varepsilon > 0$ can be arbitrary small. Then letting $\varepsilon \to 0$, we have $0 \leq V_\ell \leq \overline{u} \to 0$ in $[-\ell, \ell]$. This implies $V_\ell \equiv 0$. □

3 Proof of the main theorem

In this section, we give the proof of Theorem 1.1. Our proof is divided into two parts. In part 1, we present some sufficient conditions for spreading and vanishing, and give a dichotomy result, namely, when $h_0 < \ell^*$ the solution of (1.1) is either vanishing or spreading. When $h_0 \geq \ell^*$, we prove that only spreading happens. In part 2, we consider the dependence of the asymptotic behavior of solutions on the initial value and give a sharp result.

3.1 Conditions for vanishing and spreading

Lemma 3.1. *If* $0 < -g_\infty, h_\infty < +\infty$*, then* $0 < h_\infty - g_\infty \leq 2\ell^*$*, and*

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{C([g(t),h(t)])} = 0. \tag{3.1}$$

Proof. We divide the proof into two steps.

Step 1. We prove that $0 < -g_\infty, h_\infty < +\infty$ implies (3.1). Suppose on the contrary that $\limsup_{t\to\infty} \|u(t,\cdot)\|_{C([g(t),h(t)])} = \varepsilon_0 > 0$, then there is a sequence $(x_n, t_n) \in (g(t), h(t)) \times (0, \infty)$ such that $u(t_n, x_n) \geq \varepsilon_0/2$ for all $n \in \mathbb{N}$ and $t_n \to +\infty$ as $n \to \infty$. By $-\infty < g_\infty < g(t) < x_n < h(t) < h_\infty < +\infty$, there is a subsequence of $\{x_n\}$ (denote it by $\{x_n\}$ again) converges to some $x_0 \in (g_\infty, h_\infty)$ as $n \to \infty$. Since $u(t,x) \in C^{1+\frac{\gamma}{2}, 2+\gamma}([1,+\infty) \times [g(t), h(t)])$ for any $\gamma \in (0,1)$, there is a subsequence $\{t_{n_j}\}_{j=1}^{\infty}$ such that

$$u(t + t_{n_j}, x) \to v_1(t,x) \quad \text{locally uniformly in } (t,x) \in \mathbb{R} \times (g_\infty, h_\infty)$$

and v_1 is a solution of

$$\begin{cases} v_t = v_{xx} + (1-v)\displaystyle\int_{\mathbb{R}} k(x-y)v(t,y)dy, & x \in (-g_\infty, h_\infty),\ t \in \mathbb{R}, \\ v(t, g_\infty) = v(t, h_\infty) = 0, & t \in \mathbb{R}, \end{cases} \tag{3.2}$$

and

$$h'(t + t_{n_j}) = -\mu u_x(t + t_{n_j}, h(t + t_{n_j})) \to -\mu v_{1x}(t, h_\infty) \quad \text{as } j \to \infty, \tag{3.3}$$

$$g'(t + t_{n_j}) = -\mu u_x(t + t_{n_j}, g(t + t_{n_j})) \to -\mu v_{1x}(t, g_\infty) \quad \text{as } j \to \infty. \tag{3.4}$$

In particular, (3.3) and (3.4) are also valid when $t = 0$. Note that $v_1(0, x_0) > 0$, so $v_1(t,x) > 0$ for $x \in (g_\infty, h_\infty)$ and $t \in \mathbb{R}$. Letting $M := \|v(t,\cdot)\|_{L^\infty([g_\infty, h_\infty])}$, then applying the Hopf lemma to the equation $v_t \geq v_{xx} - Mv$ for $g_\infty < x < h_\infty$, we have

$$v_x(0, h_\infty) < 0 \quad \text{and} \quad v_x(0, g_\infty) > 0.$$

On the other hand, since $h(t) \in C^{1+\frac{\gamma}{2}}([1,+\infty))$, $g(t) \in C^{1+\frac{\gamma}{2}}([1,+\infty))$, combining these with $0 < -g_\infty, h_\infty < +\infty$, we have $h'(t) \to 0$ and $g'(t) \to 0$ as $t \to \infty$. Therefore, letting $t = 0$ in (3.3) and (3.4), we have $v_{1x}(0, g_\infty) = 0$ and $v_{1x}(0, h_\infty) = 0$, these are contradictions. Therefore, (3.1) holds.

Step 2. Suppose on the contrary that $h_\infty - g_\infty > 2\ell^*$, then there is $T > 0$, such that $2\ell_1 := h(T) - g(T) > 2\ell^*$. By the monotonicity of $h(t)$ and $g(t)$, we have

$$h(t) - g(t) > 2\ell_1 > 2\ell^* \quad \text{for all } t > T. \tag{3.5}$$

We now consider the problem (2.1) with $\ell := \ell_1$, then the Proposition 2.1 shows that the principal eigenvalue $\lambda_1(\ell_1) < 0$, let $\phi_{1,\ell_1}(x)$ be the positive eigenfunction corresponding to

$\lambda_1(\ell_1)$. We prove that $\underline{u}(x) := \varepsilon\phi_{1,\ell_1}(x - \frac{h(T)+g(T)}{2})$ $(x \in [h(T), g(T)])$ is a lower solution of (1.1) when $\varepsilon > 0$ is sufficiently small. A direct calculation shows that

$$\begin{aligned}
&\underline{u}_t - \underline{u}_{xx} - (1-\underline{u})\int_{\mathbb{R}} k(x-y)\underline{u}(y)dy \\
&= \lambda_1(\ell_1)\varepsilon\phi_{1,\ell_1} + \varepsilon\int_{-\ell_1}^{\ell_1} k\left(x-y-\frac{h(T)+g(T)}{2}\right)\phi_{1,\ell_1}(y)dy \\
&\quad - (1-\varepsilon\phi_{1,\ell_1})\int_{g(T)}^{h(T)} k(x-y)\varepsilon\phi_{1,\ell_1}\left(y-\frac{h(T)+g(T)}{2}\right)dy \\
&= \lambda_1(\ell_1)\varepsilon\phi_{1,\ell_1} + \varepsilon\phi_{1,\ell_1}\int_{-\ell_1}^{\ell_1} k\left(x-y-\frac{h(T)+g(T)}{2}\right)\varepsilon\phi_{1,\ell_1}(y)dy \\
&= \varepsilon\phi_{1,\ell_1}\left(\lambda_1(\ell_1) + \varepsilon\int_{-\ell_1}^{\ell_1} k\left(x-y-\frac{h(T)+g(T)}{2}\right)\phi_{1,\ell_1}(y)dy\right) \\
&< 0
\end{aligned} \tag{3.6}$$

for sufficiently small $\varepsilon > 0$. We have used the fact that $\lambda_1(\ell_1) < 0$ in the last inequality. Choose $\varepsilon > 0$ small such that

$$u(T,x) \geq \varepsilon\phi_{1,\ell_1}\left(x - \frac{h(T)+g(T)}{2}\right) \quad \text{for } x \in [g(T), h(T)]. \tag{3.7}$$

Moreover, $u(t, h(T)) > 0 = \underline{u}(h(T))$ and $u(t, g(T)) > 0 = \underline{u}(g(T))$ for all $t > T$. Combining these with (3.6)–(3.7) and Remark 2.4, one can show that $(\underline{u}, -\ell_1, \ell_1)$ is a lower solution for $t > T$, hence

$$u(t,x) > \varepsilon\phi_{1,\ell_1}\left(x - \frac{h(T)+g(T)}{2}\right) \quad \text{for } x \in [g(T), h(T)] \subset [g(t), h(t)] \text{ and } t > T.$$

However, it follows from step 1 that this is impossible when $0 < -g_\infty, h_\infty < +\infty$. This contradiction implies $h_\infty - g_\infty < 2\ell^*$. □

Lemma 3.2. *If $-g_\infty = h_\infty = +\infty$, then*

$$\lim_{t\to\infty} u(t,\cdot) = 1 \quad \textit{locally uniformly in } \mathbb{R}. \tag{3.8}$$

Proof. It follows from $-g_\infty = h_\infty = +\infty$ that, for any $\ell > \ell^*$, there is a $T > 0$ such that $h(t) > \ell$ and $g(t) < -\ell$ when $t \geq T$. Now, define a function as follows

$$\underline{u}(t,x) := (1 - Me^{-\delta t})V_\ell(x), \quad x \in (-\ell, \ell),$$

where V_ℓ is the unique positive solution of (2.5). We choose $M > 0$ such that $u(T,x) > \underline{u}(0,x) = (1-M)V_\ell(x) > 0$ for $x \in (-\ell, \ell) \subset [g(T), h(T)]$. Moreover, one can derive that

$$\begin{aligned}
&\underline{u}_t - \underline{u}_{xx} - (1-\underline{u})\int_{\mathbb{R}} k(x-y)\underline{u}(t,y)dy \\
&= Me^{-\delta t}V_\ell\left(\delta - \int_{\mathbb{R}} k(x-y)(1-Me^{-\delta t})V_\ell(y)dy\right) \\
&< 0
\end{aligned} \tag{3.9}$$

provided that $\delta > 0$ is sufficiently small. Moreover, by $g(t) < -\ell$, $h(t) > \ell$ $(t > T)$ and the definition of V_ℓ, we have

$$u(t+T, -\ell) > 0 = \underline{u}(t, -\ell) \quad \text{and} \quad u(t+T, \ell) > 0 = \underline{u}(t, \ell) \quad \text{for all } t > 0. \tag{3.10}$$

Then it follows from the comparison principle that

$$u(t+T,x) > (1-Me^{-\delta t})V_\ell(x), \quad x \in [-\ell,\ell] \subset [g(t),h(t)],\ t>0.$$

Hence

$$\liminf_{t\to\infty} u(t+T,x) \geq V_\ell(x) \quad \text{for all } \ x \in [-\ell,\ell]. \tag{3.11}$$

By Lemma 2.5 and our assumption $-g_\infty = h_\infty = +\infty$, letting $\ell \to \infty$ in (3.11), we have

$$\liminf_{t\to\infty} u(t,x) \geq 1 \quad \text{uniformly in any compact subset of } \mathbb{R}. \tag{3.12}$$

On the other hand, we construct an upper solution to prove that $u(t,x) \leq 1$ for all $t>0$ and $x \in \mathbb{R}$. Define

$$\overline{u}(t) := 1 + Ne^{-\gamma t} \quad \text{for all } t>0.$$

We choose $N>0$ large such that $\overline{u}(0) = 1+N > u(0,x)$ for all $x \in [-h_0,h_0]$. Moreover, a direct calculation shows that

$$\overline{u}_t - \overline{u}_{xx} - (1-\overline{u})\int_{\mathbb{R}} k(x-y)\overline{u}(t)dy > 0$$

provided $0<\gamma<1$. Then by the comparison principle we can obtain $u(t,x) < 1$ for all $x \in [g(t),h(t)]$ and $t>0$. Therefore, we have

$$\limsup_{t\to\infty} u(t,x) \leq 1 \quad \text{uniformly for } x \in \mathbb{R}. \tag{3.13}$$

Hence, (3.12)–(3.13) completes the proof of the desired result. □

Combining Lemma 3.1 and Lemma 3.2, we immediately have the following dichotomy result.

Lemma 3.3. *Let (u,g,h) be the solution of the problem (1.1). Then the following alternative holds. Either*

(i) *spreading: $-g_\infty = h_\infty = +\infty$ and $\lim_{t\to\infty} u(t,x) = 1$ locally uniformly in $\mathbb{R}$;*

or

(ii) *vanishing: $0 < h_\infty - g_\infty \leq 2\ell^*$ and $\lim_{t\to\infty} \|u(t,x)\|_{C([g(t),h(t)])} = 0$.*

Due to $h'(t)>0$ and $g'(t)<0$ for $t>0$, we must have, when $h_0 \geq \ell^*$, $h_\infty - g_\infty > 2\ell^*$, then Lemma 3.1-3.2 implies the following result.

Lemma 3.4. *If $h_0 \geq \ell^*$, then $-g_\infty = h_\infty = +\infty$, and spreading happens.*

We next give sufficient conditions for vanishing and spreading when $h_0 < \ell^*$.

Lemma 3.5. *Let $h_0 < \ell^*$ and $u_0 \in \mathscr{X}(h_0)$, then u vanishes if $\|u_0\|_{L^\infty([-h_0,h_0])}$ is sufficiently small.*

Proof. We will construct three upper solutions $\overline{u}_1(t,x)$, $\overline{u}_2(t,x)$ and $\overline{u}_3(t,x)$ to prove vanishing. We use $\overline{u}_1(t,x)$ and $\overline{u}_2(t,x)$ to prevent the spreading of two free boundaries respectively, $\overline{u}_3(t,x)$ to control the growth of the solution (see the following Fig. 3.1). We first construct an upper solution to present the spreading of $h(t)$. Let $\lambda_1 > 0$ be the principal eigenvalue of the problem (2.1) when $\ell := \ell_1$, where $0<\ell_1<\ell^*$, ϕ_1 is the eigenfunction corresponding to λ_1.

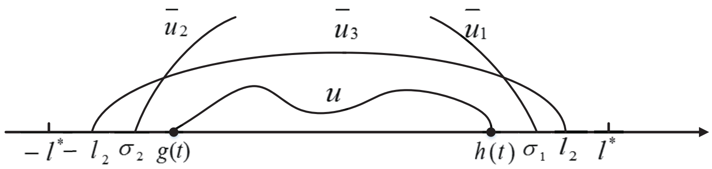

Figure 3.1: Three upper solutions $\overline{u}_1, \overline{u}_2$ and $\overline{u}_3$.

Set $\sigma_1(t) := h_0 + 2\gamma_1 - \gamma_1 e^{-\delta t}$, where $0 < \delta < \lambda_1$, $\gamma_1 > 0$ is small such that

$$h_0 - \gamma_1 > 0, \quad h_0 + 2\gamma_1 < \ell^*, \quad \phi_1'(x) < 0 \quad \text{for } x \in [\ell_1 - 2\gamma_1, \ell_1]. \tag{3.14}$$

Define

$$\overline{u}_1(t,x) := \varepsilon_1 e^{-\delta t}\phi_1(x - \sigma_1(t) + \ell_1) \quad \text{for } x \in [\sigma_1(t) - 2\gamma_1, \sigma_1(t)],$$

where $\varepsilon_1 > 0$ is sufficiently small such that

$$\gamma_1\delta \geq -\varepsilon_1\mu\phi_1'(\ell_1). \tag{3.15}$$

We now show that $(\overline{u}_1, \sigma_1(t) - 2\gamma_1, \sigma_1(t))$ is an upper solution of the problem (1.1). By (3.14), the definitions of $\overline{u}_1$ and ϕ_1 we have

$$\begin{aligned}
&\overline{u}_{1t} - \overline{u}_{1xx} - (1 - \overline{u}_1)\int_{\mathbb{R}} k(x-y)\overline{u}_1(t,y)dy \\
&= -\delta\varepsilon_1 e^{-\delta t}\phi_1 - \sigma_1'(t)\varepsilon_1 e^{-\delta t}\phi_1' - \varepsilon_1 e^{-\delta t}\phi_1'' \\
&\quad - (1 - \varepsilon_1 e^{-\delta t}\phi_1)\int_{\mathbb{R}} \varepsilon_1 e^{-\delta t} k(x-y)\phi_1(y - \sigma_1(t) + \ell_1)dy \\
&\geq \varepsilon_1 e^{-\delta t}(-\delta + \lambda_1)\phi_1 \\
&> 0.
\end{aligned} \tag{3.16}$$

Moreover,

$$\sigma_1(0) > h_0, \quad \sigma_1(t) - 2\gamma_1 \geq g(t) \quad \text{for all } \ t > 0, \tag{3.17}$$

and the condition (3.15) implies that

$$\sigma_1'(t) = \gamma_1\delta e^{-\delta t} \geq -\mu\overline{u}_{1x}(t, \sigma_1(t)) = -\varepsilon_1\mu e^{-\delta t}\phi_1'(\ell_1). \tag{3.18}$$

If

$$u(t, \sigma_1(t) - 2\gamma_1) < \overline{u}_1(t, \sigma_1(t) - 2\gamma_1) = \varepsilon_1 e^{-\delta t}\phi_1(\ell_1 - 2\gamma_1) \quad \text{for } t > 0, \tag{3.19}$$

then it follows from (3.16)–(3.19) and Remark 2.4 that $(\overline{u}_1, \sigma_1(t) - 2\gamma_1, \sigma_1(t))$ will be an upper solution of the problem (1.1) when the initial data satisfies

$$u_0(x) < \overline{u}_1(0,x) \quad \text{for } x \in (h_0 - \gamma_1, h_0). \tag{3.20}$$

Then by the comparison principle Lemma 2.3 we have

$$h(t) < \sigma_1(t) \quad \text{for all } t > 0. \tag{3.21}$$

Similarly, we define an upper solution $\overline{u}_2$ to prevent the spreading of $g(t)$ as follows:

$$\sigma_2(t) := -h_0 - 2\gamma_2 + \gamma_2 e^{-\delta t}, \quad \overline{u}_2(t,x) := \varepsilon_2 e^{-\delta t}\phi_1(x - \sigma_2(t) - \ell_1),$$

for $x \in [\sigma_2(t), \sigma_2(t)+2\gamma_2]$, where $\gamma_2 > 0$ is small such that

$$-h_0 + \gamma_2 < 0, \quad -h_0 - 2\gamma_2 > -\ell^*, \quad \phi_1'(x) > 0 \quad \text{for } x \in [-\ell_1, -\ell_1 + 2\gamma_2], \tag{3.22}$$

and $\varepsilon_2 > 0$ small satisfying $\delta\gamma_2 \geq \mu\varepsilon_2\phi_1'(-\ell_1)$. Then $(\overline{u}_2, \sigma_2(t), \sigma_2(t)+2\gamma_2)$ will be an upper solution when

$$u(t, \sigma_2(t)+2\gamma_2) < \overline{u}_2(t, \sigma_2(t)+2\gamma_2) = \varepsilon_2 e^{-\delta t}\phi_1(-\ell_1+2\gamma_2) \quad \text{for } t > 0 \tag{3.23}$$

and

$$u_0(x) < \overline{u}_2(0,x) \quad \text{for } x \in [-h_0, -h_0+\gamma_2]. \tag{3.24}$$

Therefore, Lemma 2.3 implies that

$$g(t) > \sigma_2(t) \quad \text{for all } t > 0. \tag{3.25}$$

We finally construct the third upper solution $\overline{u}_3$ to control the growth of the solution and ensure (3.19) and (3.23). Define

$$\overline{u}_3(t,x) := \varepsilon_3 e^{-\delta t}\phi_2(x), \; x \in [-\ell_2, \ell_2],$$

where $\varepsilon_3 > 0$ is very small, $\phi_2(x)$ is the eigenfunction of the problem (2.1) with $\ell := \ell_2$, and ℓ_2 satisfies

$$\max\{h_0 + 2\gamma_1, h_0 + 2\gamma_2\} < \ell_2 < \ell^*. \tag{3.26}$$

This is valid by (3.14) and (3.22). It follows from $\ell_2 < \ell^*$ that the principal eigenvalue $\lambda_2 > 0$. Moreover, (3.26), the definitions of σ_1 and σ_2 imply that

$$\sigma_1(t) < \ell_2 \quad \text{and} \quad \sigma_2(t) > -\ell_2 \quad \text{for all } t \geq 0. \tag{3.27}$$

A direct calculation shows that

$$\overline{u}_{3t} - \overline{u}_{3xx} - (1-\overline{u}_3)\int_{\mathbb{R}} k(x-y)\overline{u}_3(t,y)dy > \varepsilon_3 e^{-\delta t}(-\delta + \lambda_2)\phi_2 > 0 \tag{3.28}$$

provided that $\delta < \lambda_2$. In addition, when

$$g(t) > -\ell_2 \quad \text{and} \quad h(t) < \ell_2 \quad \text{for all } t > 0, \tag{3.29}$$

we have

$$\overline{u}_3(t, g(t)) > 0 = u(t, g(t)) \quad \text{and} \quad \overline{u}_3(t, h(t)) > 0 = u(t, h(t)). \tag{3.30}$$

Therefore, (3.28)–(3.30) imply that $\overline{u}_3$ is an upper solution when the initial data u_0 satisfies

$$u_0(x) < \overline{u}_3(0,x) = \varepsilon_3\phi_2(x), \quad x \in [-h_0, h_0] \subset [-\ell_2, \ell_2]. \tag{3.31}$$

Hence, it follows from the comparison principle that

$$u(t,x) < \overline{u}_3(t,x) \quad \text{for } x \in [g(t), h(t)], \; t > 0. \tag{3.32}$$

This can ensure (3.19) and (3.23) when $\varepsilon_3 > 0$ is sufficiently small. On the other hand, it follows from the definition of σ_1, σ_2 and the definition of ℓ_2 that (3.21) and (3.25) can ensure (3.29) (so (3.32) holds). To guarantee these conclusions, we now choose $u_0(x)$ sufficiently small such that (3.20), (3.24) and (3.31) hold, then $\overline{u}_1, \overline{u}_2$ and $\overline{u}_3$ are upper solutions at the same time for small $t > 0$, by the comparison principle and Lemma 2.3, we conclude that (3.19), (3.23) and (3.29) hold, $\overline{u}_1, \overline{u}_2$ and $\overline{u}_3$ are upper solutions for all $t > 0$. Therefore, by (3.21), (3.25) and (3.32), we obtain

$$h_\infty < \infty, \quad g_\infty > -\infty \quad \text{and} \quad \lim_{t\to\infty} \|u(t,\cdot)\|_{C([g(t),h(t)])} \to 0.$$

Then Lemma 3.1 implies that vanishing happens. □

Lemma 3.6. *Let $h_0 < \ell^*$ and $u_0 \in \mathscr{X}(h_0)$, then u spreads if $\mu > 0$ is sufficiently large.*

Proof. We first consider the case $\|u_0\|_{L^\infty([-h_0,h_0])} \le 1$, then we can derive from the comparison principle that $u(t,x) < 1$ for all $t > 0$ and $x \in [g(t), h(t)]$.

Direct calculation gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{g(t)}^{h(t)} u(t,x)dx &= \int_{g(t)}^{h(t)} u_t(t,x)dx + u(t,h(t))h'(t) - u(t,g(t))g'(t)\\ &= \int_{g(t)}^{h(t)} \left(u_{xx}(t,x) + (1-u)\int_{\mathbb{R}} k(x-y)u(t,y)dy\right)dx \\ &= \frac{g'(t)-h'(t)}{\mu} + \int_{g(t)}^{h(t)} (1-u)\int_{\mathbb{R}} k(x-y)u(t,y)dydx.\end{aligned} \tag{3.33}$$

Integrating from 0 to t yields

$$\begin{aligned}\int_{g(t)}^{h(t)} u(t,x)dx = &\int_{-h_0}^{h_0} u_0(x)dx + \frac{g(t)-h(t)+2h_0}{\mu}\\ &+ \int_0^t \Big[\int_{g(t)}^{h(t)} (1-u)\int_{\mathbb{R}} k(x-y)u(t,y)dydx\Big]dt, \quad t \ge 0.\end{aligned} \tag{3.34}$$

Since $0 < u(t,x) < 1$ for $t > 0$ and $x \in [g(t), h(t)]$, we have, for $t \ge 1$,

$$\int_0^t \Big[\int_{g(t)}^{h(t)} (1-u)\int_{\mathbb{R}} k(x-y)u(t,y)dydx\Big]dt > 0.$$

Assume $h_\infty \ne +\infty$ and $g_\infty \ne -\infty$, then Lemma 3.3 implies $h_\infty - g_\infty \le 2\ell^*$ and $\lim_{t\to\infty}\|u(t,\cdot)\|_{L^\infty([g(t),h(t)])} = 0$. Letting $t \to \infty$ in (3.34) gives

$$\int_{-h_0}^{h_0} u_0(x)dx \le \frac{2\ell^* - 2h_0}{\mu}, \tag{3.35}$$

this contradicts the assumption on μ.

For the case $\|u_0\|_{L^\infty([-h_0,h_0])} > 1$, we take $\underline{u}_0 = \frac{u_0(x)}{\|u_0\|_{L^\infty}}$. Let $(\underline{u}, \underline{g}, \underline{h})$ be the solution of the problem (1.1) with u_0 replaced by $\underline{u}_0$, then by Remark 2.4 we have that $(\underline{u}, \underline{g}, \underline{h})$ is a lower solution, so Lemma 2.2 implies that $g(t) < \underline{g}(t)$ and $h(t) > \underline{h}(t)$ for $t > 0$. On the other hand, by the first case, due to $\|\underline{u}_0\|_{L^\infty([-h_0,h_0])} = 1$ and our assumption on μ, we have $\lim_{t\to\infty} h(t) = +\infty$ and $\lim_{t\to\infty} g(t) = -\infty$. Then spreading happens. □

3.2 Sharp threshold

In this section, based on the previous results, we obtain sharp threshold behaviors between spreading and vanishing.

Proof of Theorem 1.1. According to Lemma 3.3, one can obtain spreading-vanishing dichotomy. In what follows, we will prove the sharp threshold behaviors.

When $h_0 < \ell^*$, Lemma 3.5 implies that in this case vanishing happens for all small $\sigma > 0$. Therefore denote

$$\sigma^* := \sigma^*(h_0, \phi) := \sup\{\sigma_0 : \text{ vanishing happens for } \sigma \in (0, \sigma_0]\} \in (0, +\infty].$$

If $\sigma^* = +\infty$, then there is nothing left to prove. So we now suppose $\sigma^* \in (0, +\infty)$. (i) We prove spreading when $\sigma > \sigma^*$. By the definition of σ^* and spreading-vanishing dichotomy, there is a sequence σ_n decreasing to σ^* such that spreading happens when $\sigma = \sigma_n$, $n = 1, 2, \dots$ For any $\sigma > \sigma^*$, there is $n_0 \geq 1$ such that $\sigma > \sigma_{n_0}$. Let $(u_{n_0}, g_{n_0}, h_{n_0})$ be the solution of the problem (1.1) with initial data $u_0 := \sigma_{n_0}\phi$, then by the comparison principle Lemma 2.2, we have $[g_{n_0}(t), h_{n_0}(t)] \subset [g(t), h(t)]$ and $u_{n_0}(t, x) \leq u(t, x)$. Hence spreading happens for such σ.

(ii) We show that vanishing happens when $\sigma \leq \sigma^*$. The definition of σ^* implies vanishing when $\sigma < \sigma^*$. We only need to prove vanishing when $\sigma = \sigma^*$. Otherwise spreading must happen when $\sigma = \sigma^*$, so we can find $t_0 > 0$ such that $h(t_0) - g(t_0) > 2\ell^*$. Due to the continuous dependence of the solution of the problem (1.1) on the initial values, we find that if $\varepsilon > 0$ is sufficiently small, then the solution of (1.1) with $u_0 = (\sigma - \varepsilon)\phi$, denote by $(u_\varepsilon, g_\varepsilon, h_\varepsilon)$, satisfies

$$h_\varepsilon(t_0) - g_\varepsilon(t_0) > 2\ell^*. \tag{3.36}$$

So $h_\infty - g_\infty > 2\ell^*$, by this and Lemma 3.3, we see that (3.36) implies spreading for $(u_\varepsilon, g_\varepsilon, h_\varepsilon)$, which contradicts the definition of σ^*.

When $h_0 \geq \ell^*$, it follows from Lemma 3.4 that spreading always happens for any solution of the problem (1.1), so $\sigma^*(h_0, \phi) = 0$ for any $\phi \in \mathscr{X}(h_0)$. □

A Appendix

Lemma A.1 (Krein–Rutman, [5, Theorem 2.12])**.** *Suppose that A is a compact linear operator on the ordered Banach space E with positive cone P. Suppose further that P has nonempty interior and that A is strongly positive. The eigenvalue problem $Au = \mu u$ admits a unique eigenvalue μ_1 which has a positive eigenvector u_1.*

We next prove the existence of the principal eigenvalue and the positive eigenfunction corresponding to it. One can use the similar method as in [8, Theorem 2.2] to prove the following lemma. Additionally, the existence of eigenvalue for (2.1) is equivalent to the existence of the eigenvalue for

$$\begin{cases} \phi_{xx} + \displaystyle\int_{-\ell}^{\ell} k(x-y)\phi(y)dy - a\phi + \lambda\phi = 0, \quad x \in (-\ell, \ell), \\ \phi(-\ell) = \phi(\ell) = 0, \end{cases} \tag{A.1}$$

where $a > 0$ is a constant. However, this problem is the same as the eigenvalue problem (13) in [6].

We now give the idea of the proof which is also similar with Example 1 on page 51 in [17].

Lemma A.2. *For any $\ell > 0$, (2.1) has a unique principal eigenvalue $\lambda_1(\ell)$ and a positive eigenfunction ϕ_1 corresponding to $\lambda_1(\ell)$.*

Proof. It is clear that (2.1) is equivalent to

$$\begin{cases} -\phi_{xx} - \displaystyle\int_{-\ell}^{\ell} k(x-y)\phi(y)dy + a\phi = (\lambda + a)\phi(x), \ x \in (-\ell, \ell), \\ \phi(-\ell) = \phi(\ell) = 0, \end{cases}$$

where the constant $a > 0$ is large and it will be chosen later. We first consider the eigenvalue problem

$$\begin{cases} -\phi_{xx} - \int_{-\ell}^{\ell} k(x-y)\phi(y)dy + a\phi = \mu\phi(x),\ x \in (-\ell,\ell), \\ \phi(-\ell) = \phi(\ell) = 0. \end{cases} \tag{A.2}$$

Then $\lambda = \mu - a$.

To prove the existence of eigenvalue, we define a linear operator A as follows

$$u = A\phi, \quad \phi \in C([-\ell,\ell]),$$

where u is the solution of linear problem

$$\begin{cases} -u_{xx} - \int_{-\ell}^{\ell} k(x-y)u(y)dy + au = \phi(x),\ x \in (-\ell,\ell), \\ u(-\ell) = u(\ell) = 0. \end{cases} \tag{A.3}$$

We now show that A is well defined. For any $\phi, u \in C([-\ell,\ell])$, by [8, Proposition 2.1], the problem

$$\begin{cases} -v_{xx} + av = \int_{-\ell}^{\ell} k(x-y)u(y)dy + \phi(x),\ x \in (-\ell,\ell), \\ v(-\ell) = v(\ell) = 0 \end{cases}$$

has a unique solution $v \in C^2([-\ell,\ell])$. Then define an operator $F(u) = v$, by Schauder fixed point theorem, the problem (A.3) has a solution $u \in C^2([-\ell,\ell])$. Moreover, by L^p estimates or the following method (the proof that A is strongly positive), one can show that the solution of (A.3) is unique. So A is well defined.

A maps the bounded set in $C([-\ell,\ell])$ onto bounded set in $C^2([-\ell,\ell])$, which is the relatively compact set belonging to $C([-\ell,\ell])$ (by the embedding theorem). Therefore, A is a compact operator over $C([-\ell,\ell])$.

Now, set $P = \text{closure}\{v : v \in C([-\ell,\ell]), v > 0 \text{ for } x \in (-\ell,\ell)\}$; $\text{int}\, P \neq \emptyset$. We next show that A is strongly positive, that is, when $\phi \in P\backslash\{\theta\}$, then $A\phi \in \text{int}\, P$. Assume on the contrary that u reaches a negative minimum at x_0, by the boundary condition, we have $x_0 \in (-\ell,\ell)$. Since $u(x_0) < 0$, we can take $a > 0$ large such that

$$-u_{xx}(x_0) - \int_{-\ell}^{\ell} k(x-y)u(y)dy + au(x_0) < 0,$$

this contradicts the choice of $\phi \in P\backslash\{\theta\}$. Hence $u(x) \geq 0$ for all $x \in (-\ell,\ell)$. Furthermore, we prove that $u(x) > 0$ for all $x \in (-\ell,\ell)$. Otherwise, by $u \in C^2$ and $u \geq 0$, there is some minimum point $x^* \in (-\ell,\ell)$ such that $u(x^*) = 0$ and $u_{xx}(x^*) > 0$, from these we have

$$-u_{xx}(x^*) - \int_{-\ell}^{\ell} k(x^*-y)u(y)dy + au(x^*) < 0,$$

this is also a contradiction. Therefore, $u \in \text{int}\, P$.

By Lemma A.1, $A\phi = \mu\phi$ admits a unique eigenvalue $\mu_1(\ell)$ which has a positive eigenfunction $\phi_1 \in P$ with $\|\phi_1\|_{L^2([-\ell,\ell])} = 1$, so the definition of A implies

$$-(\phi_1)_{xx} - \int_{-\ell}^{\ell} k(x-y)\phi_1(y)dy + a\phi_1 = \frac{1}{\mu_1(\ell)}\phi_1.$$

Obviously,

$$-(\phi_1)_{xx} - \int_{-\ell}^{\ell} k(x-y)\phi_1(y)dy = \lambda_1(\ell)\phi_1,$$

where $\lambda_1(\ell) = \frac{1}{\mu_1(\ell)} - a$. □

Corollary A.3. *For any $\ell > 0$, let $\lambda_1(\ell)$ be the principal eigenvalue of (2.1). Suppose that $\ell_2 > \ell_1 > 0$, then $\lambda_1(\ell_1) > \lambda_1(\ell_2)$.*

Proof. If $\phi \in C_0([-\ell_1, \ell_1])$, then ϕ can be extended to be zero on $[-\ell_2, \ell_2] \setminus (-\ell_1, \ell_1)$, and the resulting continuous function will belong to $C_0([-\ell_2, \ell_2])$. Take ϕ_1 to be the eigenfunction for $\lambda_1(\ell_1)$ and let $\widetilde{\phi}_1$ be the extension of ϕ_1 to $[-\ell_2, \ell_2]$ which is zero on $[-\ell_2, \ell_2] \setminus (-\ell_1, \ell_1)$. We have $\widetilde{\phi}_1 \in C_0([-\ell_2, \ell_2])$, and by (2.2),

$$\begin{aligned}
\lambda_1(\ell_2) &= \inf_{\phi \in C_0, \|\phi\|_{L^2}=1} \left(\int_{-\ell_2}^{\ell_2} (\phi'(x))^2 dx - \int_{-\ell_2}^{\ell_2} \int_{-\ell_2}^{\ell_2} k(x-y)\phi(x)\phi(y)dxdy \right) \\
&\leq \int_{-\ell_2}^{\ell_2} (\widetilde{\phi}_1'(x))^2 dx - \int_{-\ell_2}^{\ell_2} \int_{-\ell_2}^{\ell_2} k(x-y)\widetilde{\phi}_1(x)\widetilde{\phi}_1(y)dxdy \\
&= \int_{-\ell_1}^{\ell_1} (\phi_1'(x))^2 dx - \int_{-\ell_1}^{\ell_1} \int_{-\ell_1}^{\ell_1} k(x-y)\phi_1(x)\phi_1(y)dxdy \\
&= \lambda_1(\ell_1).
\end{aligned}$$

To obtain the strict inequality we note that the eigenfunction for $\lambda_1(\ell_2)$ is positive on $(-\ell_2, \ell_2)$, but $\widetilde{\phi}_1$ is not, so $\widetilde{\phi}_1$ cannot be the minimizer of the quotient for $\lambda_1(\ell_2)$. □

We only prove Lemma 2.2 since Lemma 2.3 can be proved similarly with some obvious modifications. However, the proof of Lemma 2.2 is very similar with Lemma 3.5 in [9]. For the readers' convenience, we list it here.

Proof of Lemma 2.2. For small $\varepsilon > 0$, let $(u_\varepsilon, g_\varepsilon, h_\varepsilon)$ denote the unique solution of (1.1) with h_0 replaced by $h_0^\varepsilon := h_0(1-\varepsilon)$, with μ replaced by $\mu_\varepsilon := \mu(1-\varepsilon)$, and with u_0 replaced by some $u_0^\varepsilon \in C^2([-h_0^\varepsilon, h_0^\varepsilon])$ satisfying

$$0 < u_0^\varepsilon(x) \leq u_0(x) \quad \text{in } [-h_0^\varepsilon, h_0^\varepsilon], \quad u_0^\varepsilon(-h_0^\varepsilon) = u_0^\varepsilon(h_0^\varepsilon) = 0,$$

and as $\varepsilon \to 0$,

$$u_0^\varepsilon \left(\frac{h_0}{h_0^\varepsilon} x \right) \to u_0(x)$$

in the $C^2([-h_0, h_0])$ norm.

We claim that $h_\varepsilon(t) < \overline{h}(t)$ and $g_\varepsilon(t) > \overline{g}(t)$ for all $t \in (0, T]$. Without loss of generality, we only prove $h_\varepsilon(t) < \overline{h}(t)$ for all $t \in (0, T]$. Clearly, this is true for small $t > 0$. If our claim is not true, then there exists a first $t^* < T$ such that $h_\varepsilon(t) < \overline{h}(t)$ for $t \in (0, t^*)$ and $h_\varepsilon(t^*) = \overline{h}(t^*)$,

$$h_\varepsilon'(t^*) \geq \overline{h}'(t^*). \tag{A.4}$$

We now compare u_ε and $\overline{u}$ over the region

$$\Omega := \{(t, x) \in \mathbb{R}^2 : 0 < t \leq t^*, 0 \leq x \leq h_\varepsilon(t)\}.$$

The strong maximum principle implies that $u_\varepsilon(t, x) < \overline{u}(t, x)$ in Ω. Hence $v(t, x) := \overline{u}(t, x) - u_\varepsilon(t, x) > 0$ in Ω with $v(t^*, h_\varepsilon(t^*)) = 0$, then $v_x(t^*, h_\varepsilon(t^*)) \leq 0$. Combining this,

$(u_\varepsilon)_x(t^*, h(t^*)) < 0$ and $\mu_\varepsilon < \mu$ we have $h'_\varepsilon(t^*) < \overline{h}'(t^*)$, which contradicts (A.4), so this proves our claim $h_\varepsilon(t) < \overline{h}(t)$. Similarly, we can prove $g_\varepsilon(t) > \overline{g}(t)$.

To complete the proof, we need to show that the unique solution of (1.1) depends continuously on the parameters in (1.1). Actually, since $(u_\varepsilon, g_\varepsilon, h_\varepsilon)$ is bounded in Hölder space $C^{1+\frac{\gamma}{2},2+\gamma}(\overline{D}) \times C^{1+\frac{\gamma}{2}}([0,T]) \times C^{1+\frac{\gamma}{2}}([0,T])$. For any sequence of $(u_\varepsilon, g_\varepsilon, h_\varepsilon)$, there is a subsequence converging to the solution (u, g, h) of the problem (1.1). By the uniqueness of the solution, we have $(u_\varepsilon, g_\varepsilon, h_\varepsilon) \to (u, g, h)$ as $\varepsilon \to 0$, the unique solution of (1.1). Hence the conclusion of Lemma 2.2 follows by letting $\varepsilon \to 0$ in the inequalities $u_\varepsilon < \overline{u}$, $h_\varepsilon < \overline{h}$ and $g_\varepsilon > \overline{g}$. □

Inspired by [12, Lemma 2.1], we now give the following comparison principle which used to prove the existence of the solution of the problem (2.5).

Lemma A.4. *Suppose $\ell > 0$. For positive u, v in $C^2([-\ell, \ell])$ satisfying*

$$u_{xx} + (1-u)\int_{\mathbb{R}} k(x-y)u(y)dy \le 0 \le v_{xx} + (1-v)\int_{\mathbb{R}} k(x-y)v(y)dy$$

for $x \in (-\ell, \ell)$, and $v(x) \le u(x)$ for $x = \pm\ell$, then $v(x) \le u(x)$ for $x \in [-\ell, \ell]$.

Proof. Suppose on the contrary that there are some intervals on which $v(x) > u(x)$. Denote

$$I_1 := \{x \in (-\ell, \ell) \mid v(x) > u(x)\}, \quad I_2 := \{y \in (-\ell, \ell) \mid v(y) < u(y)\},$$

then $u = v$ at the endpoints of I_1 and I_2. Suppose that $u > 0, v > 0$, then

$$uv_{xx} - vu_{xx} \ge v(1-u)\int_{\mathbb{R}} k(x-y)u(y)dy - u(1-v)\int_{\mathbb{R}} k(x-y)v(y)dy. \tag{A.5}$$

Integrate the above inequality on $D := I_1 \times [-\ell, \ell]$, integration by parts, we have

$$\int_{-\ell}^{\ell} dy \int_{I_1} [uv_{xx} - vu_{xx}]dx < 0. \tag{A.6}$$

Denote $D_1 := I_1 \times I_1$, $D_2 := I_1 \times I_2$. Integrate the right part of (A.5), we have

$$\begin{aligned} &\iint_D \left[v(1-u)\int_{\mathbb{R}} k(x-y)u(y)dy - u(1-v)\int_{\mathbb{R}} k(x-y)v(y)dy\right] dxdy \\ &= \iint_{D_1} \left[v(1-u)\int_{\mathbb{R}} k(x-y)u(y)dy - u(1-v)\int_{\mathbb{R}} k(x-y)v(y)dy\right] dxdy \\ &\quad + \iint_{D_2} \left[v(1-u)\int_{\mathbb{R}} k(x-y)u(y)dy - u(1-v)\int_{\mathbb{R}} k(x-y)v(y)dy\right] dxdy. \end{aligned} \tag{A.7}$$

Set $F(x,y) = v(x)u(y) - v(y)u(x)$, we have the property $F(x,y) = -F(y,x)$, combining this and the symmetric of k (see (1.2)), we have

$$\iint_{D_1} k(x-y)\left[v(x)u(y) - u(x)v(y)\right] dxdy = 0. \tag{A.8}$$

Therefore, by (A.8) and $v > u$ on I_1, we have

$$\begin{aligned} &\iint_{D_1} \left[v(1-u)\int_{\mathbb{R}} k(x-y)u(y)dy - u(1-v)\int_{\mathbb{R}} k(x-y)v(y)dy\right] dxdy \\ &= \iint_{D_1} k(x-y)\left[v(x)u(y) - u(x)v(y)\right] dxdy \\ &\quad + \iint_{D_1} k(x-y)\left[-v(x)u(x)u(y) + u(x)v(x)v(y)\right] dxdy \\ &> 0. \end{aligned} \tag{A.9}$$

Moreover,

$$\begin{aligned}&\iint_{D_2}\left[v(1-u)\int_{\mathbb{R}}k(x-y)u(y)dy-u(1-v)\int_{\mathbb{R}}k(x-y)v(y)dy\right]dxdy\\&\quad=\iint_{D_2}k(x-y)\left[v(x)u(y)(1-u(x))-u(x)v(y)(1-v(x))\right]dxdy\\&\quad>0,\end{aligned}\tag{A.10}$$

here, we have used the fact that $u(x) < v(x)$ and $u(y) > v(y)$ over D_2. By (A.7), (A.9) and (A.10) we can derive that the integration of the right part of (A.5) over D is positive, however, (A.6) implies that integration of the left part of (A.5) over D is negative, this contradiction proves $v(x) \leq u(x)$ for all $x \in [-\ell, \ell]$. □

Competing interests

The authors declare that they have no competing interests.

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