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Asymptotic properties of solutions to difference equations of Emden–Fowler type

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Abstract. We study the higher order difference equations of the following form

$$\Delta^m x_n = a_n f(x_{\sigma(n)}) + b_n.$$

We are interested in the asymptotic behavior of solutions x of the above equation. Assuming f is a power type function and $\Delta^m y_n = b_n$, we present sufficient conditions that guarantee the existence of a solution x such that

$$x_n = y_n + \mathrm{o}(n^s),$$

where $s \leq 0$ is fixed. We establish also conditions under which for a given solution x there exists a sequence y such that $\Delta^m y_n = b_n$ and x has the above asymptotic behavior.

Keywords: Emden–Fowler difference equation, asymptotic behavior.

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1 Introduction

We use the standard symbol $\mathbb{N}$ to denote the set of all positive integers. Analogously $\mathbb{R}$ denotes the set of all real numbers. In this paper we assume that

$$m \in \mathbb{N}, \qquad f : \mathbb{R} \to \mathbb{R}, \qquad \sigma : \mathbb{N} \to \mathbb{N}, \qquad \lim_{n\to\infty} \sigma(n) = \infty.$$

We consider difference equations of the form

$$\Delta^m x_n = a_n f(x_{\sigma(n)}) + b_n \tag{E}$$

where $a_n, b_n \in \mathbb{R}$. We say that a sequence $x : \mathbb{N} \to \mathbb{R}$ is a solution of (E) if there exists an index q such that (E) is satisfied for any $n \geq q$.

Asymptotic properties of solutions were investigated by many authors. Some classical results on asymptotic behavior of solutions of differential equations can be found in [8, 9, 12,

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14,22–25]. For the case of difference equations, see, e.g., [6,7,15,17,18,26–36]. With respect to dynamic equations on time-scales we refer the reader to [1–4].

The main purpose of this paper is to generalize the following theorem.

Theorem 1.1. *Assume $\sigma : \mathbb{N} \to \mathbb{N}$, $y : \mathbb{N} \to \mathbb{R}$, $\Delta^m y = b$, $s \in (-\infty, 0]$, $\lambda \in (0, \infty)$,*

$$\lim_{n\to\infty} \sigma(n) = \infty, \qquad \lim_{n\to\infty} y_n = \infty, \qquad \sum_{n=1}^{\infty} n^{m-1-s}|a_n| < \infty,$$

and f is continuous and bounded on $[\lambda, \infty)$. Then there exists a solution x of (E) such that $x_n = y_n + \mathrm{o}(n^s)$.

This theorem follows from [18, Theorem 4.7] or [20, Theorem 4.1]. Some specific cases have been proved in [15, Theorem 5.1], [16, Theorem 5.1], or [19, Theorem 4.1]. The generalization consists in replacing the condition "f is bounded on $[\lambda, \infty)$" by the condition "f is of power type on $[\lambda, \infty)$". The last condition means that there exists a constant $\mu \in [0, \infty)$ such that the function $t^{-\mu} f(t)$ is bounded on $[\lambda, \infty)$, i.e. $f(t) = \mathrm{O}(t^\mu)$. If $\sigma(n) = n$, $\mu \in [0, \infty)$, and $f(t) = t^\mu$ then equation (E) takes the form

$$\Delta^m x_n = a_n x_n^\mu + b_n$$

which is the "positive part" of discrete Emden–Fowler equation

$$\Delta^m x_n = a_n |x_n|^\mu \operatorname{sgn} x_n + b_n.$$

Asymptotic properties of solutions to discrete Emden–Fowler type equations were investigated, for example, in [5,10,11,13,27,35].

The paper is organized as follows. In Section 2, we introduce notation and terminology. In Section 3, in Theorem 3.1, we obtain our main result. In Theorem 3.1 the condition $\sum_{n=1}^{\infty} n^{m-1-s}|a_n| < \infty$ from Theorem 1.1 is replaced by a stronger condition

$$\sum_{n=1}^{\infty} n^{\alpha+m-1-s}|a_n| < \infty \tag{1.1}$$

where α is a non-negative constant dependent on the order of growth of f, the order of growth of y, and the order of growth of σ. In Section 4, in Theorem 4.2, we show that constant α is properly chosen, i.e. that condition (1.1) is not too strong. Section 5 is devoted to the problem of approximation of solutions. In Section 6 we give some remarks and additional results.

2 Preliminaries

We use the symbol $\mathbb{R}^{\mathbb{N}}$ to denote the space of all sequences $x : \mathbb{N} \to \mathbb{R}$. If $x \in \mathbb{R}^{\mathbb{N}}$, then $|x|$ denotes the sequence defined by $|x|(n) = |x_n|$. Moreover

$$\|x\| = \sup_{n\in\mathbb{N}} |x_n|, \qquad \mathrm{c}_0 = \{z : \mathbb{N} \to \mathbb{R} : \lim_{n\to\infty} z_n = 0\},$$

$$\mathrm{A}(m) := \left\{ x \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} n^{m-1}|x_n| < \infty \right\}, \qquad r^m : \mathrm{A}(m) \to \mathrm{c}_0,$$

$$r^m(x)(n) = \sum_{k=n}^{\infty} \binom{k-n+m-1}{m-1} x_k = \sum_{k=0}^{\infty} \binom{m+k-1}{m-1} x_{n+k}.$$

It is not difficult to see that $\mathrm{A}(m)$ is a linear subspace of c_0 and r^m is a linear operator. We will use the following two lemmas.

Lemma 2.1. *Assume* $s \in (-\infty, 0]$, $p \in \mathbb{N}$, *and* $\sum_{n=1}^{\infty} n^{m-1-s}|u_n| < \infty$. *Then*

$$r^m(|u|)(p) \le \sum_{k=p}^{\infty} k^{m-1}|u_k|, \quad r^m(u)(n) = \mathrm{o}(n^s), \quad \textit{and} \quad \Delta^m(r^m(u))(p) = (-1)^m u_p.$$

Proof. This lemma follows from [17, Lemma 4.1 and Lemma 4.2]. □

Let us recall that a topological space X has a fixed point property if every continuous map $U : X \to X$ has a fixed point.

Lemma 2.2 ([17, Lemma 4.7])**.** *Let y be an arbitrary real sequence and let γ be a sequence which is positive and convergent to zero. Then the space $\{u \in \mathbb{R}^{\mathbb{N}} : |u - y| \le \gamma\}$ with respect to the uniform convergence topology has a fixed point property.*

3 Solutions with prescribed asymptotic behavior

In this section, in Theorem 3.1, we present our main result. Next, in Corollary 3.5 we establish conditions under which there exist asymptotically polynomial solutions of equation (E).

Theorem 3.1. *Assume* $y : \mathbb{N} \to \mathbb{R}$, $\Delta^m y = b$, $\lim_{n\to\infty} y_n = \infty$, $s \in (-\infty, 0]$, $\lambda \in (0, \infty)$,

$$\mu \in [0, \infty), \quad \tau, \omega, \in (0, \infty), \quad f(t) = \mathrm{O}(t^\mu), \quad y_n = \mathrm{O}(n^\tau), \quad \sigma(n) = \mathrm{O}(n^\omega),$$

$$\sum_{n=1}^{\infty} n^{\mu\tau\omega+m-1-s}|a_n| < \infty, \tag{3.1}$$

and f is continuous on $[\lambda, \infty)$. Then there exists a solution x of (E) *such that*

$$x_n = y_n + \mathrm{o}(n^s).$$

Proof. There exist positive constants L, Q such that

$$|y_n| \le Qn^\tau, \qquad \sigma(n) \le Ln^\omega$$

for $n \in \mathbb{N}$. We may assume that there exists a positive constant K such that

$$|f(t)| \le Kt^\mu$$

for $t \in [\lambda, \infty)$. Define a constant M and sequences α, ρ by

$$M = (2QL^\tau)^\mu K, \quad \alpha_n = n^{\mu\tau\omega}|a_n|, \quad \rho = r^m(\alpha). \tag{3.2}$$

Then, by the conditions of the theorem and Lemma 2.1, we have $\alpha \in \mathrm{A}(m)$ and $\rho_n = \mathrm{o}(1)$. There exists an index $p_1 \in \mathbb{N}$ such that

$$M\rho_n \le 1, \quad \text{and} \quad y_n \ge \lambda + 1$$

for $n \ge p_1$. Let

$$S = \{x \in \mathbb{R}^{\mathbb{N}} : \ |x - y| \le M\rho \quad \text{and} \quad x_n = y_n \quad \text{for} \quad n < p_1\}.$$

There exists an index p_2 such that $p_2 \ge p_1$ and $\sigma(n) \ge p_1$ for $n \ge p_2$. If $x \in S$, $n \ge p_1$, then

$$|x_n - y_n| \le M\rho_n \le 1 \quad \text{and} \quad y_n \ge \lambda + 1.$$

Hence $x_n \geq y_n - 1 \geq \lambda + 1 - 1 = \lambda$. Therefore

$$x_{\sigma(n)} \geq \lambda \tag{3.3}$$

for any $x \in S$ and any $n \geq p_2$. Let $x \in S$. If $n \geq p_2$, then

$$\begin{aligned}|f(x_{\sigma(n)})| \leq Kx^{\mu}_{\sigma(n)} &= K(x_{\sigma(n)} - y_{\sigma(n)} + y_{\sigma(n)})^{\mu} \leq K(|x_{\sigma(n)} - y_{\sigma(n)}| + |y_{\sigma(n)}|)^{\mu} \\ &\leq K(M\rho_{\sigma(n)} + |y_{\sigma(n)}|)^{\mu} \leq K(1 + |y_{\sigma(n)}|)^{\mu} \leq K(2|y_{\sigma(n)}|)^{\mu} \\ &\leq K2^{\mu}(Q\sigma(n)^{\tau})^{\mu} \leq K2^{\mu}Q^{\mu}(Ln^{\omega})^{\tau\mu} = K2^{\mu}Q^{\mu}L^{\tau\mu}n^{\mu\tau\omega}.\end{aligned}$$

Hence

$$|f(x_{\sigma(n)})| \leq Mn^{\mu\tau\omega} \tag{3.4}$$

for any $x \in S$ and any $n \geq p_2$. Let $x \in \mathbb{R}^{\mathbb{N}}$, for $n \in \mathbb{N}$ let

$$x^*_n = a_n f(x_{\sigma(n)}). \tag{3.5}$$

Let $x \in S$. By (3.4), we have

$$|x^*_n| \leq |a_n||f(x_{\sigma(n)})| \leq Mn^{\mu\tau\omega}|a_n| = M\alpha_n$$

for $n \geq p_2$. By (3.1) and (3.2), we get $x^* \in \mathrm{A}(m)$. Hence we may define a sequence $U(x)$ by

$$U(x)(n) = \begin{cases} y_n & \text{for } n < p_2 \\ y_n + (-1)^m r^m(x^*)(n) & \text{for } n \geq p_2. \end{cases} \tag{3.6}$$

If $n \geq p_2$, then

$$|U(x)(n) - y_n| = |r^m(x^*)(n)| \leq r^m(|x^*|)(n) \leq Mr^m(\alpha)(n) = M\rho_n.$$

Therefore $U(S) \subset S$. We will show that the map U is continuous. Assume ε is a positive real number. By (3.1),

$$\sum_{n=1}^{\infty} n^{m-1}n^{\mu\tau\omega}|a_n| < \infty.$$

Select an index $q \geq p_2$ and a positive constant γ such that

$$2M\sum_{n=q}^{\infty} n^{\mu\tau\omega+m-1}|a_n| < \varepsilon \quad \text{and} \quad \gamma\sum_{n=1}^{q} n^{m-1}|a_n| < \varepsilon. \tag{3.7}$$

Let

$$x \in S, \quad \eta = 1 + \max(x_{\sigma(1)}, \dots, x_{\sigma(q)}), \quad W = [\lambda, \eta].$$

Since the function f is uniformly continuous on W, there exists a number $\delta \in (0,1)$ such that

$$|f(r) - f(t)| \leq \gamma$$

for any $r, t \in W$ such that $|r - t| < \delta$. Select a sequence $z \in S$ such that $\|z - x\| < \delta$. Then

$$\begin{aligned}\|U(x) - U(z)\| &= \sup_{n\geq p_2} |r^m(x^*)(n) - r^m(z^*)(n)| = \sup_{n\geq p_2} |r^m(x^* - z^*)(n)| \\ &\leq \sup_{n\geq p_2} r^m(|x^* - z^*|)(n) = r^m(|x^* - z^*|)(p_2) \leq \sum_{n=p_2}^{\infty} n^{m-1}|x^*_n - z^*_n| \\ &\leq \sum_{n=p_2}^{q} n^{m-1}|a_n||f(x_{\sigma(n)}) - f(z_{\sigma(n)})| + \sum_{n=q}^{\infty} n^{m-1}|a_n||f(x_{\sigma(n)}) - f(z_{\sigma(n)})| \\ &\leq \sum_{n=1}^{q} n^{m-1}|a_n|\gamma + \sum_{n=q}^{\infty} n^{m-1}|a_n|2Mn^{\mu\tau\omega} < 2\varepsilon.\end{aligned}$$

Hence U is continuous and, by Lemma 2.2, there exists a sequence $x \in S$ such that $U(x) = x$. Then, by (3.6),

$$x_n = y_n + (-1)^m r^m(x^*)(n)$$

for $n \geq p_2$. Hence, using Lemma 2.1 and (3.5), for $n \geq p_2$, we obtain

$$\Delta^m x_n = \Delta^m y_n + (-1)^m \Delta^m (r^m(x^*))(n) = b_n + x_n^* = a_n f(x_{\sigma(n)}) + b_n.$$

Moreover, since $x \in S$, we get $|x_n - y_n| \leq M\rho_n$ eventually. By Lemma 2.1, $\rho_n = \mathrm{o}(n^s)$. Hence $x_n - y_n = \mathrm{o}(n^s)$ and we obtain

$$x_n = y_n + \mathrm{o}(n^s). \qquad \square$$

If the sequence b is "sufficiently small", then in Theorem 3.1, in place of a solution y of the equation $\Delta^m y_n = b_n$ we can take a polynomial sequence. More precisely, we have the following result.

Corollary 3.2. *Assume φ_n is a polynomial sequence, $\lim_{n\to\infty} \varphi_n = \infty$, $s \in (-\infty, 0]$,*

$$\mu \in [0,\infty), \quad \omega \in (0,\infty), \quad \tau \in (0,m), \quad f(t) = \mathrm{O}(t^\mu), \quad \varphi_n = \mathrm{O}(n^\tau), \quad \sigma(n) = \mathrm{O}(n^\omega),$$

$$\sum_{n=1}^{\infty} n^{\mu\tau\omega+m-1-s}|a_n| < \infty, \qquad \sum_{n=1}^{\infty} n^{m-1-s}|b_n| < \infty, \tag{3.8}$$

$\lambda \in (0,\infty)$, and f is continuous on $[\lambda,\infty)$. Then there exists a solution x of (E) such that

$$x_n = \varphi_n + \mathrm{o}(n^s).$$

Proof. By [15, Lemma 2.3], there exists a sequence w such that $w_n = \mathrm{o}(n^s)$ and $\Delta^m w = b$. Let $y = \varphi + w$. Then $\Delta^m y = \Delta^m \varphi + \Delta^m w = b$ and $\lim_{n\to\infty} y_n = \infty$. By Theorem 3.1 there exists a solution x of (E) such that $x_n = y_n + \mathrm{o}(n^s)$. Hence

$$x_n = \varphi_n + w_n + \mathrm{o}(n^s) = \varphi_n + \mathrm{o}(n^s). \qquad \square$$

In the case of the classical Emden–Fowler discrete equation we obtain the following corollary.

Corollary 3.3. *Assume $\mu \in [0,\infty)$, $s \in (-\infty, 0]$,*

$$\sum_{n=1}^{\infty} n^{(\mu+1)(m-1)-s}|a_n| < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} n^{m-1-s}|b_n| < \infty. \tag{3.9}$$

Then for any polynomial sequence φ_n such that $\deg\varphi < m$, and $\lim_{n\to\infty}\varphi_n = \infty$, there exists a solution x of the equation

$$\Delta^m x_n = a_n x_n^\mu + b_n \tag{3.10}$$

such that $x_n = \varphi_n + \mathrm{o}(n^s)$.

Proof. Let $\tau = m-1$ and $\omega = 1$. Applying Corollary 3.2 to equation (3.10) we get the result. $\square$

In particular, in the linear case, we obtain the following corollary.

Corollary 3.4. *Assume* $s \in (-\infty, 0]$,

$$\sum_{n=1}^{\infty} n^{2m-2-s}|a_n| < \infty, \quad \textit{and} \quad \sum_{n=1}^{\infty} n^{m-1-s}|b_n| < \infty.$$

Then for any polynomial sequence φ_n, *such that* $\deg \varphi < m$, *and* $\lim_{n\to\infty} \varphi_n = \infty$, *there exists a solution* x *of the equation* $\Delta^m x_n = a_n x_n + b_n$ *such that* $x_n = \varphi_n + \mathrm{o}(n^s)$.

The proof of the next theorem is similar to the proof of Theorem 3.1, therefore it will be omitted.

Theorem 3.5. *Assume* $y : \mathbb{N} \to \mathbb{R}$, $\Delta^m y = b$, $\lim_{n\to\infty} y_n = -\infty$, $s \in (-\infty, 0]$, $\lambda \in (-\infty, 0)$,

$$\mu \in [0, \infty), \quad \tau, \omega, \in (0, \infty), \quad y_n = \mathrm{O}(n^\tau), \quad \sigma(n) = \mathrm{O}(n^\omega), \quad \sum_{n=1}^{\infty} n^{\mu\tau\omega+m-1-s}|a_n| < \infty,$$

the function f *is continuous on* $(-\infty, \lambda]$, *and the function* $|t|^{-\mu} f(t)$ *is bounded on* $(-\infty, \lambda]$. *Then there exists a solution* x *of* (E) *such that* $x_n = y_n + \mathrm{o}(n^s)$.

Of course, Corollary 3.2, Corollary 3.3, and Corollary 3.4 can be reformulated and proven in a similar way. We leave it to the reader.

4 Necessary conditions

In this section we show that the condition of strong convergence of the sequence a_n in Theorem 3.1 is not too strong. More precisely, we present conditions under which the condition (3.8) in Corollary 3.2 is necessary for the existence of asymptotically polynomial solution of equation (E). We present the main results in Theorem 4.2 (in the case $s = 0$) and in Theorem 4.6 (in the case $s < 0$).

Lemma 4.1. *If* $z_n = \mathrm{o}(1)$ *and the sequence* $\Delta^m z_n$ *is nonoscillatory, then*

$$\sum_{n=1}^{\infty} n^{m-1}|\Delta^m z_n| < \infty.$$

Proof. This statement is a consequence of [17, Lemma 4.1 (f) and (b)]. □

Theorem 4.2. *Assume* $\omega, K, L \in (0, \infty)$, $\mu \in [0, \infty)$,

$$\sigma(n) \geq L n^\omega, \quad a_n \geq 0, \quad b_n \geq 0 \quad \textit{for large } n, \quad f(t) \geq K t^\mu \quad \textit{for large } t,$$

φ *is a polynomial sequence,* $\deg \varphi = \tau < m$, $\lim_{n\to\infty} \varphi_n = \infty$, *and a solution* x *of* (E) *exists such that* $x_n = \varphi_n + \mathrm{o}(1)$. *Then*

$$\sum_{n=1}^{\infty} n^{\mu\tau\omega+m-1}|a_n| < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} n^{m-1}|b_n| < \infty.$$

Proof. There exists a positive number ε such that $x_n \geq \varepsilon n^\tau$ eventually. Select $p_1 \in \mathbb{N}$ such that

$$a_n \geq 0, \quad b_n \geq 0, \quad \sigma(n) \geq L n^\omega, \quad x_n \geq \varepsilon n^\tau$$

for $n \geq p_1$. Select $p_2 \geq p_1$ such that $\sigma(n) \geq p_1$ for $n \geq p_2$. For $n \in \mathbb{N}$ let

$$u_n = a_n f(x_{\sigma(n)}) + b_n.$$

Then

$$u_n = \Delta^m x_n = \Delta^m(\varphi_n + \mathrm{o}(1)) = \Delta^m(\mathrm{o}(1))$$

and u_n is nonoscillatory. Hence by Lemma 4.1, we get

$$\sum_{n=1}^{\infty} n^{m-1}|u_n| < \infty. \tag{4.1}$$

Since $a_n f(x_{\sigma(n)}) \geq 0$ for large n, we have $b_n \leq u_n$ for large n. By (4.1) we obtain

$$\sum_{n=1}^{\infty} n^{m-1}|b_n| < \infty.$$

If $n \geq p_2$, then

$$f(x_{\sigma(n)}) \geq K x^{\mu}_{\sigma(n)} \geq K(\varepsilon\sigma(n)^{\tau})^{\mu} \geq K(\varepsilon(Ln^{\omega})^{\tau})^{\mu} = K\varepsilon^{\mu}L^{\tau\mu}n^{\omega\tau\mu}.$$

Hence there exists a positive constant δ such that

$$f(x_{\sigma(n)}) \geq \delta n^{\mu\tau\omega}$$

for $n \geq p_2$. If $n \geq p_2$, then we get

$$n^{\mu\tau\omega}|a_n| \leq \delta^{-1}|a_n| f(x_{\sigma(n)}) \leq \delta^{-1}|u_n|.$$

Now, by (4.1), we obtain

$$\sum_{n=1}^{\infty} n^{\mu\tau\omega+m-1}|a_n| < \infty. \qquad \square$$

Using Theorem 4.2 to the case of classical discrete Emden–Fowler equation we get:

Corollary 4.3. *Assume $\mu \in [0, \infty)$, $a_n \geq 0$, $b_n \geq 0$ eventually, φ is a polynomial sequence of degree $m-1$, $\lim_{n\to\infty} \varphi_n = \infty$, and there exists a solution x of the equation*

$$\Delta^m x_n = a_n x_n^{\mu} + b_n$$

such that $x_n = \varphi_n + \mathrm{o}(1)$. Then

$$\sum_{n=1}^{\infty} n^{(\mu+1)(m-1)}|a_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} n^{m-1}|b_n| < \infty.$$

The case $s < 0$ is, unexpectedly, more difficult. In this case we get a slightly weaker result. First, we will prove two lemmas.

Lemma 4.4. *Assume $s \in (-\infty, 0]$ and $z_n = \mathrm{o}(n^s)$, then for every $\varepsilon > 0$ the series*

$$\sum_{n=1}^{\infty} \frac{\Delta z_n}{n^{s+\varepsilon}}$$

is convergent.

Proof. Let ε be a positive real number and let $t = -s - \varepsilon$. By [15, Theorem 2.2], we have $\Delta n^t = \mathrm{O}(n^{t-1})$. Hence

$$\begin{aligned} n^t \Delta z_n &= n^t z_{n+1} - n^t z_n = n^t z_{n+1} - (n+1)^t z_{n+1} + (n+1)^t z_{n+1} - n^t z_n \\ &= -z_{n+1}\Delta n^t + \Delta(n^t z_n) = \Delta(n^t z_n) + z_{n+1}\mathrm{O}(n^{t-1}) = \Delta(n^t z_n) + z_{n+1}n^{t-1}\mathrm{O}(1). \end{aligned}$$

Moreover

$$z_{n+1}n^{t-1} = z_{n+1}n^{-s-\varepsilon}n^{-1} = \frac{z_{n+1}}{(n+1)^s}\left(\frac{n+1}{n}\right)^s \frac{1}{n^{1+\varepsilon}} = \mathrm{o}(1)\mathrm{O}(1)\frac{1}{n^{1+\varepsilon}} = \mathrm{O}\left(\frac{1}{n^{1+\varepsilon}}\right).$$

Therefore the series

$$\sum_{n=1}^{\infty} z_{n+1}n^{t-1}\mathrm{O}(1) = \sum_{n=1}^{\infty} \mathrm{O}\left(\frac{1}{n^{1+\varepsilon}}\right)$$

is convergent. Since $n^t z_n = \mathrm{o}(1)$, the series

$$\sum_{n=1}^{\infty} \Delta(n^t z_n)$$

is convergent, too. Thus we obtain the convergence of the series

$$\sum_{n=1}^{\infty} \frac{\Delta z_n}{n^{s+\varepsilon}} = \sum_{n=1}^{\infty} n^t \Delta z_n = \sum_{n=1}^{\infty} \Delta(n^t z_n) + \sum_{n=1}^{\infty} z_{n+1}n^{t-1}\mathrm{O}(1).$$ □

Lemma 4.5. *Assume $m \in \mathbb{N}$, $s \in (-\infty, 0)$, $z_n = \mathrm{o}(n^s)$, and the sequence $\Delta^m z_n$ is nonoscillatory for large n. Then for every $\varepsilon > 0$ the series*

$$\sum_{n=1}^{\infty} \frac{\Delta^m z_n}{n^{s+\varepsilon-m+1}}$$

is convergent.

Proof. Induction on m. By Lemma 4.4, the assertion is true for $m = 1$. Assume it is true for some $m \geq 1$ and the sequence $\Delta^{m+1} z_n$ is nonoscillatory for large n. Moreover, assume that $\Delta^{m+1} z_n \geq 0$ for large n. Let

$$y_n = \Delta^m z_n.$$

Then $y_n = \mathrm{o}(1)$ and $\Delta y_n \geq 0$ for large n. Hence $\Delta^m z_n = y_n \leq 0$ for large n. Obviously, we may assume, that

$$\Delta^m z_n = y_n < 0 \tag{4.2}$$

eventually. Select an $\varepsilon > 0$ and let

$$\lambda \in (0, \varepsilon) \cap (0, -s). \tag{4.3}$$

By inductive hypothesis the series

$$\sum_{n=1}^{\infty} \frac{y_n}{n^{s+\lambda-m+1}}$$

is convergent. Let

$$t = m - 1 - s - \lambda.$$

By (4.3), $t > 0$. Using de l'Hospital Theorem we obtain the following known limit

$$\lim_{n\to\infty}\frac{\Delta n^{t+1}}{n^t} = \lim_{n\to\infty}\frac{(n+1)^{t+1}-n^{t+1}}{n^{-1}n^{t+1}} = \lim_{n\to\infty}\frac{(1+n^{-1})^{t+1}-1}{n^{-1}}$$
$$= \lim_{n\to\infty}\frac{(t+1)(1+n^{-1})^t(-n^{-2})}{-n^{-2}} = t+1.$$

Hence, by the Cesàro–Stolz theorem, we obtain

$$\lim_{n\to\infty}\frac{1^t+2^t+\cdots+n^t}{n^{t+1}} = \lim_{n\to\infty}\frac{(n+1)^t}{\Delta n^{t+1}} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^t\frac{n^t}{\Delta n^{t+1}} = \frac{1}{t+1}. \tag{4.4}$$

Let $u_0 = 0$, $c_1 = 0$, $S = \sum_{k=1}^{\infty} k^t y_k$. For $n \ge 1$ let

$$u_n = \sum_{k=1}^{n} k^t y_k, \quad b_n = y_n^{-1}, \quad c_{n+1} = \sum_{k=1}^{n}(b_{k+1}-b_k)u_k.$$

Then

$$\lim_{n\to\infty}\frac{c_{n+1}-c_n}{b_{n+1}-b_n} = \lim_{n\to\infty}\frac{(b_{n+1}-b_n)u_n}{b_{n+1}-b_n} = \lim_{n\to\infty}u_n = S. \tag{4.5}$$

Note that $\lim_{n\to\infty} y_n = 0$. Moreover, by assumption $\Delta y_n \ge 0$ eventually. By (4.2), $y_n < 0$ eventually. Hence the sequence b_n is eventually monotonic and $\lim_{n\to\infty} b_n = -\infty$. Using (4.5) and Cesàro–Stolz theorem we get

$$\lim_{n\to\infty} y_n c_n = \lim_{n\to\infty}\frac{c_n}{b_n} = S.$$

Since $b_k(u_k - u_{k-1}) = k^t$, we have

$$\begin{aligned} y_n\sum_{k=1}^{n} k^t &= y_n\sum_{k=1}^{n} b_k(u_k-u_{k-1}) \\ &= y_n(b_1(u_1-u_0)+b_2(u_2-u_1)+\cdots+b_n(u_n-u_{n-1})) \\ &= y_n(-b_1u_0-(b_2-b_1)u_1-(b_3-b_1)u_2+\cdots-(b_n-b_{n-1})u_{n-1}+b_nu_n) \\ &= y_nb_nu_n - y_nc_n = u_n - y_nc_n. \end{aligned}$$

Hence

$$\lim_{n\to\infty} y_n\sum_{k=1}^{n} k^t = S - S = 0.$$

Therefore, by (4.4) we get

$$\lim_{n\to\infty} n^{t+1}y_n = \lim_{n\to\infty}\frac{n^{t+1}}{\sum_{k=1}^{n}k^t}\left(\sum_{k=1}^{n}k^t\right)y_n = (t+1)0 = 0.$$

Thus $y_n = \mathrm{o}(n^{-t-1}) = \mathrm{o}(n^{s+\lambda-m})$ and, by Lemma 4.4, the series

$$\sum_{n=1}^{\infty}\frac{\Delta^{m+1}z_n}{n^{s+\varepsilon-m}} = \sum_{n=1}^{\infty}\frac{\Delta y_n}{n^{s+\lambda-m+(\varepsilon-\lambda)}}$$

is convergent. □

Using Lemma 4.5 in place of Lemma 4.1 in the proof of Theorem 4.2 we obtain the following result.

Theorem 4.6. *Assume $\omega, K, L \in (0,\infty)$, $\mu \in [0,\infty)$, $s \in (-\infty,0]$*

$$\sigma(n) \geq Ln^{\omega}, \quad a_n \geq 0, \quad b_n \geq 0 \quad \text{for large } n, \quad f(t) \geq Kt^{\mu} \quad \text{for large } t,$$

φ is a polynomial sequence, $\deg \varphi = \tau < m$, $\lim_{n\to\infty} \varphi_n = \infty$, and there exists a solution x of (E) such that $x_n = \varphi_n + \mathrm{o}(n^s)$. Then for any $\varepsilon > 0$ we have

$$\sum_{n=1}^{\infty} n^{\mu\tau\omega+m-1-s-\varepsilon}|a_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} n^{m-1-s-\varepsilon}|b_n| < \infty. \tag{4.6}$$

Remark 4.7. The problem of whether condition (4.6) in Theorem 4.6 can be replaced by

$$\sum_{n=1}^{\infty} n^{\mu\tau\omega+m-1-s}|a_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} n^{m-1-s}|b_n| < \infty.$$

remains open.

5 Approximation of solutions

In this section we establish conditions under which a given solution x of (E) can be approximated by solutions of the equation $\Delta^m y_n = b_n$. More precisely, we present conditions under which for a given solution x of (E) and a given nonpositive real number s there exists a sequence y such that $\Delta^m y_n = b_n$ and $x_n = y_n + \mathrm{o}(n^s)$.

Lemma 5.1. *Assume $b, x, u : \mathbb{N} \to \mathbb{R}$, $s \in (-\infty,0]$,*

$$\Delta^m x_n = \mathrm{O}(u_n) + b_n, \quad \text{and} \quad \sum_{n=1}^{\infty} n^{m-1-s}|u_n| < \infty.$$

Then there exists a sequence y such that $\Delta^m y_n = b_n$ and $x_n = y_n + \mathrm{o}(n^s)$.

Proof. This statement follows from [17, Lemma 3.11 (a)]. □

Theorem 5.2. *Assume $s \in (-\infty,0]$, $\alpha \in [0,\infty)$, and*

$$\sum_{n=1}^{\infty} n^{\alpha+m-1-s}|a_n| < \infty.$$

Then for any solution x of (E) such that $f(x_{\sigma(n)}) = \mathrm{O}(n^{\alpha})$ there exists a sequence y such that $\Delta^m y_n = b_n$ for any n and $x_n = y_n + \mathrm{o}(n^s)$.

Proof. Assume x is a solution of (E) such that $f(x_{\sigma(n)}) = \mathrm{O}(n^{\alpha})$. There exists a positive constant M such that the condition $|f(x_{\sigma(n)})| \leq Mn^{\alpha}$ is satisfied for any n. Define a sequence u by the formula

$$u_n = a_n f(x_{\sigma(n)}).$$

Then

$$\sum_{n=1}^{\infty} n^{m-1-s}|u_n| \leq M \sum_{n=1}^{\infty} n^{\alpha+m-1-s}|a_n| < \infty.$$

Since x is a solution of (E), we have $\Delta^m x_n = u_n + b_n$ eventually. Hence, $\Delta^m x_n = \mathrm{O}(u_n) + b_n$ and, by Lemma 5.1, there exists a sequence y such that $\Delta^m y_n = b_n$ and $x_n = y_n + \mathrm{o}(n^s)$. □

Corollary 5.3. *Assume* $s \in (-\infty, 0]$, $\mu \in [0, \infty)$, $\tau, \omega, \in (0, \infty)$,

$$f(t) = \mathrm{O}(t^\mu), \quad \sigma(n) = \mathrm{O}(n^\omega), \quad \text{and} \quad \sum_{n=1}^{\infty} n^{\mu\tau\omega+m-1-s}|a_n| < \infty.$$

Then for any solution x *of* (E) *such that* $\lim_{n\to\infty} x_n = \infty$ *and* $x_n = \mathrm{O}(n^\tau)$ *there exists a solution* y *of the equation* $\Delta^m y_n = b_n$ *such that* $x_n = y_n + \mathrm{o}(n^s)$.

Proof. It is easy to see that $f(x_{\sigma(n)}) = \mathrm{O}(n^{\mu\tau\omega})$. Taking $\alpha = \mu\tau\omega$ in Theorem 5.2 we obtain the result. □

Theorem 5.4. *Assume* $s \in (-\infty, 0]$, $\alpha \in [0, \infty)$,

$$\sum_{n=1}^{\infty} n^{\alpha+m-1-s}|a_n| < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} n^{m-1-s}|b_n| < \infty.$$

Then for any solution x *of* (E) *such that* $f(x_{\sigma(n)}) = \mathrm{O}(n^\alpha)$ *there exists a polynomial sequence* φ *such that* $\deg \varphi < m$ *and* $x_n = \varphi_n + \mathrm{o}(n^s)$.

Proof. Assume x is a solution of (E) such that $f(x_{\sigma(n)}) = \mathrm{O}(n^\alpha)$. Define a sequence u by $u_n = a_n f(x_{\sigma(n)}) + b_n$. Then $\Delta^m x_n = \mathrm{O}(u_n)$ and

$$\sum_{n=1}^{\infty} n^{m-1-s}|u_n| < \infty.$$

By Lemma 5.1, there exists a solution φ of the equation $\Delta^m \varphi_n = 0$ such that

$$x_n = \varphi_n + \mathrm{o}(n^s).$$

□

6 Remarks and additional results

The convergence of a series can be difficult to verify. In the classical mathematical analysis, various criteria are known to check the convergence of a given series. Some of these criteria have been generalized in [18]. These generalized criteria can be used to check the convergence of series (3.1) or (3.8). In this way we can get a number of new results. Four of them are presented below.

Corollary 6.1. *Assume* $y : \mathbb{N} \to \mathbb{R}$, $\Delta^m y = b$, $\lim_{n\to\infty} y_n = \infty$, $s \in (-\infty, 0]$, $\lambda \in (0, \infty)$,

$$\mu \in [0, \infty), \quad \tau, \omega, \in (0, \infty), \quad f(t) = \mathrm{O}(t^\mu), \quad y_n = \mathrm{O}(n^\tau), \quad \sigma(n) = \mathrm{O}(n^\omega),$$

$$\liminf_{n\to\infty} n\left(\frac{|a_n|}{|a_{n+1}|} - 1\right) > \mu\tau\omega + m - s, \tag{6.1}$$

and f *is continuous on* $[\lambda, \infty)$. *Then there exists a solution* x *of* (E) *such that*

$$x_n = y_n + \mathrm{o}(n^s).$$

Proof. By [18, Lemma 6.3] we get

$$\sum_{n=1}^{\infty} n^{\mu\tau\omega+m-1-s}|a_n| < \infty.$$

Hence the result follows from Theorem 3.1. □

Using [18, Lemma 6.3] and Corollary 3.2 we obtain the following result.

Corollary 6.2. *Assume φ_n is a polynomial sequence, $\lim_{n\to\infty} \varphi_n = \infty$, $s \in (-\infty, 0]$,*

$$\mu \in [0,\infty), \quad \omega \in (0,\infty), \quad \tau \in (0,m), \quad f(t) = \mathrm{O}(t^\mu), \quad \varphi_n = \mathrm{O}(n^\tau), \quad \sigma(n) = \mathrm{O}(n^\omega),$$

$$\liminf_{n\to\infty} n\left(\frac{|a_n|}{|a_{n+1}|} - 1\right) > \mu\tau\omega + m - s, \quad \liminf_{n\to\infty} n\left(\frac{|b_n|}{|b_{n+1}|} - 1\right) > m - s,$$

$\lambda \in (0,\infty)$, and f is continuous on $[\lambda, \infty)$. Then there exists a solution x of (E) such that

$$x_n = \varphi_n + \mathrm{o}(n^s).$$

Corollary 6.3. *Assume $s \in (-\infty, 0]$, $\alpha \in [0,\infty)$, and*

$$\liminf_{n\to\infty} n\left(\frac{|a_n|}{|a_{n+1}|} - 1\right) > \alpha + m - s.$$

Then for any solution x of (E) such that $f(x_{\sigma(n)}) = \mathrm{O}(n^\alpha)$ there exists a solution y of the equation $\Delta^m y_n = b_n$ such that $x_n = y_n + \mathrm{o}(n^s)$.

Proof. This corollary follows from [18, Lemma 6.3] and Theorem 5.2. □

Using [18, Lemma 6.4] and Corollary 3.3 we obtain the following corollary.

Corollary 6.4. *Assume $\mu \in [0,\infty)$, $s \in (-\infty, 0]$,*

$$\liminf_{n\to\infty} n \ln \frac{|a_n|}{|a_{n+1}|} > (\mu+1)(m-1) + 1 - s,$$

and

$$\liminf_{n\to\infty} n \ln \frac{|b_n|}{|b_{n+1}|} > m - s.$$

Then for any polynomial sequence φ_n, such that $\deg \varphi < m$, and $\lim_{n\to\infty} \varphi_n = \infty$, there exists a solution x of the equation $\Delta^m x_n = a_n x_n^\mu + b_n$ such that $x_n = \varphi_n + \mathrm{o}(n^s)$.

We can also receive some new consequences of Theorem 5.2.

Corollary 6.5. *Assume $s \in (-\infty, 0]$, $\alpha \in [0,\infty)$, and*

$$\limsup_{n\to\infty} \frac{\ln |a_n|}{\ln n} < s - m - \alpha.$$

Then for any solution x of (E) such that $f(x_{\sigma(n)}) = \mathrm{O}(n^\alpha)$ there exists a sequence y such that $\Delta^m y_n = b_n$ for any n and $x_n = y_n + \mathrm{o}(n^s)$.

Proof. Using [18, Lemma 6.2] and Theorem 5.2 we obtain the result. □

Corollary 6.6. *Assume $s \in (-\infty, 0]$, $\alpha \in [0,\infty)$, $\mu \in (0,\infty)$, $\beta = \alpha/\mu$, and*

$$\limsup_{n\to\infty} \frac{\ln |a_n|}{\ln n} < s - m - \alpha.$$

Then for any positive solution x of the discrete Emden–Fowler equation

$$\Delta^m x_n = a_n x_n^\mu + b_n$$

such that $x_n = \mathrm{O}(n^\beta)$ there exists a sequence y such that $\Delta^m y_n = b_n$ for any n and $x_n = y_n + \mathrm{o}(n^s)$.

Proof. If $x_n = \mathrm{O}(n^\beta)$, then $x_n^\mu = \mathrm{O}(n^\alpha)$. Hence the result follows from Corollary 6.5. $\square$

Corollary 6.7. *Assume* $s \in (-\infty, 0]$, $\mu \in [0, \infty)$, $\tau, \omega, \in (0, \infty)$,

$$f(t) = \mathrm{O}(t^\mu), \quad \sigma(n) = \mathrm{O}(n^\omega), \quad \text{and} \quad \limsup_{n\to\infty} \frac{\ln |a_n|}{\ln n} < s - m - \mu\tau\omega.$$

Then for any solution x *of* (E) *such that* $\lim_{n\to\infty} x_n = \infty$ *and* $x_n = \mathrm{O}(n^\tau)$ *there exists a solution* y *of the equation* $\Delta^m y_n = b_n$ *such that* $x_n = y_n + \mathrm{o}(n^s)$.

Proof. This corollary follows from [18, Lemma 6.2] and Corollary 5.3. $\square$

Remark 6.8. In Corollary 3.2, due to assumptions $\tau \in (0, m)$, $\lim_{n\to\infty} \varphi_n = \infty$ and $\varphi_n = \mathrm{O}(n^\tau)$, the degree m of equation (E) fulfills the condition $m > 1$. On the other hand, in Theorem 3.1, the case of $m = 1$ is not excluded.

Example 6.9. Assume $m = 1$, $f(t) = t^2$, $y_n = n$, $b_n = 1$, $\sigma(n) = n$, $\mu = 2$, $\tau = 1$, $\omega = 1$, $s = 0$, and

$$a_n = \frac{n}{(n+1)(n^4 - 2n + 1)}.$$

Then $\Delta y_n = b_n$ and $\mu\tau\omega + m - 1 - s = 2$. Hence, by Theorem 3.1 there exists a solution x of the equation

$$\Delta x_n = \frac{n x_n^2}{(n+1)(n^4 - 2n + 1)} + 1$$

such that $x_n = n + \mathrm{o}(1)$. We leave the reader to check that the sequence $x_n = n + 1/n$ is such a solution.

Our results in Sections 3 and 4 relate to unbounded solutions. Below we present three facts about bounded solutions.

Theorem 6.10. *Assume* y *is a bounded solution of the equation* $\Delta^m y_n = b_n$, $s \in (-\infty, 0]$,

$$q \in \mathbb{N}, \quad \alpha \in (0, \infty), \quad U = \bigcup_{n=q}^{\infty} [y_n - \alpha, y_n + \alpha], \quad \sum_{n=1}^{\infty} n^{m-1-s} |a_n| < \infty,$$

and f *is continuous on* U. *Then there exists a solution* x *of* (E) *such that* $x_n = y_n + \mathrm{o}(n^s)$.

Proof. The assertion is a consequence of [18, Corollary 4.8]. $\square$

Corollary 6.11. *Assume* $c \in \mathbb{R}$, $s \in (-\infty, 0]$, U *is a neighborhood of* c,

$$\sum_{n=1}^{\infty} n^{m-1-s} (|a_n| + |b_n|) < \infty,$$

and f *is continuous on* U. *Then there exists a solution* x *of* (E) *such that* $x_n = c + \mathrm{o}(n^s)$.

Proof. We omit the proof which is analogous to the proof of Corollary 3.2. $\square$

The next example shows that in the above corollary the continuity of f on U can not be replaced by the continuity at the point c.

Example 6.12. Let $\mathbb{Q}$ be the set of all rational numbers. Assume

$$m = 1, \quad s = 0, \quad \sigma(n) = n, \quad b_n = 0, \quad \alpha \in (0,1), \quad a_n = \alpha^{2^n}, \quad c = 1,$$

and f is defined by

$$f(t) = \begin{cases} 1 & \text{for } t \in \mathbb{Q} \\ t & \text{for } t \notin \mathbb{Q}. \end{cases}$$

Then f is continuous at the point c,

$$\sum_{n=1}^{\infty} n^{m-1-s}(|a_n| + |b_n|) = \sum_{n=1}^{\infty} \alpha^{2^n} < \infty,$$

but, by [21, Example 1], there is no solution to equation (E) convergent to c.

Theorem 6.13. *Assume $a_n \geq 0$ and $b_n \geq 0$ eventually, $c \in \mathbb{R}$, U is a neighborhood of c, $\delta \in (0, \infty)$, $f(t) \geq \delta$ for any $t \in U$, and there exists a solution x of* (E) *such that $x_n = c + \mathrm{o}(1)$. Then*

$$\sum_{n=1}^{\infty} n^{m-1}(|a_n| + |b_n|) < \infty.$$

Proof. For $n \in \mathbb{N}$ let

$$u_n = a_n f(x_{\sigma(n)}) + b_n.$$

Then $u_n = \Delta^m x_n = \Delta^m(c + \mathrm{o}(1)) = \Delta^m(\mathrm{o}(1))$ and u_n is nonoscillatory. By Lemma 4.1

$$\sum_{n=1}^{\infty} n^{m-1}|u_n| < \infty.$$

Since $0 \leq b_n \leq u_n$ for large n, we get $\sum_{n=1}^{\infty} n^{m-1}|b_n| < \infty$. Since $f(x_{\sigma(n)}) > \delta$ eventually, we get $|a_n| \leq \delta^{-1}|a_n| f(x_{\sigma(n)}) \leq \delta^{-1}|u_n|$. Hence $\sum_{n=1}^{\infty} n^{m-1}|a_n| < \infty$. □

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