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THREE-DIMENSIONAL MORTAR CONTACT FORMULATION: AN EFFICIENT AND ACCURATE NUMERICAL IMPLEMENTATION

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ABSTRACT

The mortar contact formulation is a well-established technique to tie non-conforming finite element meshes in domain decomposition and is also the basis of many well-known contact algorithms. Mortar contact formulation allows for a variationally consistent treatment of contact conditions including mesh tying, non-penetration, frictionless and frictional sliding leading to satisfaction of contact patch test. Efficient, accurate and robust numerical implementation of the interface coupling terms associated with the mortar contact formulation remains challenging, especially in three-dimensional case. The computational contact algorithm presented in this paper is carefully designed for accuracy, efficiency and robustness and making use of the cutting-edge third-party computational tools including Mesh-Oriented datABase (MOAB), Portable, Extensible Toolkit for Scientific Computation (PETSc), Boost and clipper libraries. The computational framework is designed to take advantage of distributed memory high-performance computing and hierarchic basis functions. The numerical implementation is validated with two non-conforming mesh tying examples, which, on the one hand, remove some of the complexities associated with actual unilateral contact formulation but, on the other hand, clarify many of the conceptual and implementational aspects of the contact mechanics.

Key Words: *finite element analysis; mortar contact formulation; mesh tying; numerical integration; hierarchical basis functions*

1. Introduction

Mechanical interaction between different bodies often referred as "contact or impact", is of great importance in many engineering applications including prosthetics in biomedical engineering, pneumatic tires in automotive engineering and adhesion or slip between concrete and reinforcing steel in civil engineering. In mechanical engineering applications, contact can be found in gears, bearings, metal forming and car crash test [1, 2]. At the minimum, contact mechanics involves searching for the contact area between interacting bodies and subsequent prevention of inter-penetration. Relative movement or slip including both frictionless and frictional, intermittent interaction and wear also comes under the umbrella of contact mechanics. Due to the associated nonlinearities and complicated nature of problems involving contact, special assumptions were used in the past for their solution. Advances in Computational modelling allows solving these problems to be solved numerically with sufficient accuracy for engineering analysis/design. On the other hand, as compared to the current state of the art nonlinear finite element technology including finite deformation kinematics, inelastic material behavior and linear and nonlinear equations solvers, contact mechanics is relatively immature. Therefore, the design of efficient, accurate and robust contact algorithms is still a challenge for practical engineering problems.

In this paper, one of the most commonly used contact discretisation technique for the solution of contact problem, i.e. the mortar contact formulation is adopted, which is also referred to as a special type of segment-to-segment approach in the literature [3]. Alternative methods, including node-to-segment and enforcing contact condition at specific finite element node, cannot guarantee the satisfaction of contact patch test and suffers from non-physical oscillation in the contact forces. As compared to the strong or point-wise satisfaction of the interface continuity condition in the aforementioned contact mechanics approaches, a weak or integral form is used in the mortar contact formulation. The computational contact algorithm presented in this paper is carefully designed for accuracy, efficiency and robustness and makes use of the cutting-edge third party computational tools including Mesh-Oriented datABase (MOAB), Portable, Extensible Toolkit for Scientific Computation (PETSc), Boost and clipper [4] libraries. The developed algorithm is implemented within our group's finite element code, MOFEM [5]. Relatively simple, linear-elastic two body problems with non-conforming meshes are used to validate the numerical implementation. Tetrahedral elements are used to discretised the two contacting bodies leading to triangular elements on the common interfaces. The clipper library is used to create an intersecting polygon between every pair of triangles belongs to the two contacting surfaces. These polygons are then triangulated, with their integration points projected back on to the parent triangles, which are used subsequently for the numerical integration of the mortar contact integral. For the convenient and efficient handling of the mesh data associated with the contacting surfaces, prisms are inserted between every pair of contacting triangles and are stored in the multi-index containers. Furthermore, the computational framework is designed to take advantage of distributed memory high-performance computing and hierarchic basis functions [6].

2. Problem formulation

In this paper, we limit ourselves to the application of mortar contact formulation for tying non-conforming meshes in linear-elasticity by removing some of the complexities associated with unilateral contact mechanics. Nonetheless, this will clarify many of the conceptual and implementational aspects of the computational contact mechanics [7]. Consider two subdomains $\Omega^{(i)} \subset \mathbb{R}^3$, $i = 1,2$ bounded by boundaries $\partial\Omega^{(i)}$. The subdomain boundary $\partial\Omega^{(i)}$ is divided into disjoint sets of Dirichlet boundary $\Gamma_u^{(i)}$, Neumann boundary $\Gamma_\sigma^{(i)}$ and mesh tying interface Γ_c. The body force acting over the individual subdomain is $\mathbf{b}^{(i)}$. The strong form of governing equations in the case of linear-elasticity for each subdomain is written as:

$$\mathrm{Div}\boldsymbol{\sigma}^{(i)} + \mathbf{b}^{(i)} = \mathbf{0} \qquad \text{in } \Omega^{(i)}, \tag{1}$$

where $\boldsymbol{\sigma}^{(i)}$ is the Cauchy stress tensor. The associated boundary conditions are the Dirichlet, Neumann and mesh tying constraint and are written as:

$$\mathbf{u}^{(i)} = \bar{\mathbf{u}}^{(i)} \qquad \text{on } \Gamma_u^{(i)}, \tag{2a}$$

$$\boldsymbol{\sigma}^{(i)} \cdot \mathbf{n}^{(i)} = \bar{\mathbf{t}}^{(i)} \qquad \text{on } \Gamma_\sigma^{(i)}, \tag{2b}$$

$$\mathbf{u}^{(1)} = \mathbf{u}^{(2)} \qquad \text{on } \Gamma_c. \tag{2c}$$

The weak formulation is subsequently derived together with Lagrange multipliers $\boldsymbol{\lambda}$ to enforce the mesh tying constraint. In Figure 1(a), $\Omega^{(2)}$ is considered as the slave side and $\Omega^{(1)}$ is considered as the master side. The final discretised set of equations is written as:

$$\begin{bmatrix} \mathbf{K}_{\mathcal{NN}} & \mathbf{K}_{\mathcal{NM}} & \mathbf{K}_{\mathcal{NS}} & \mathbf{0} \\ \mathbf{K}_{\mathcal{MN}} & \mathbf{K}_{\mathcal{MM}} & \mathbf{0} & -\mathbf{M}^T \\ \mathbf{K}_{\mathcal{SN}} & \mathbf{0} & \mathbf{K}_{\mathcal{SS}} & \mathbf{D}^T \\ \mathbf{0} & -\mathbf{M} & \mathbf{D} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{d}_{\mathcal{N}} \\ \mathbf{d}_{\mathcal{M}} \\ \mathbf{d}_{\mathcal{S}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_{\mathcal{N}} \\ \mathbf{F}_{\mathcal{M}} \\ \mathbf{F}_{\mathcal{S}} \\ \mathbf{0} \end{Bmatrix}, \tag{3}$$

where $\mathbf{d}_{\mathcal{S}}$ and $\mathbf{d}_{\mathcal{M}}$ are the degrees of freedom associated with slave and master contact surfaces and $\mathbf{d}_{\mathcal{N}}$ consists of all the remaining degrees of freedom. $\mathbf{K}$ and $\mathbf{F}$ are the standard stiffness matrices and force vectors respectively. Both slave and master surfaces are not connected directly and are tied using the mortar contact formulation leading to blocks of zeros for both $\mathbf{K}_{\mathcal{MS}}$ and $\mathbf{K}_{\mathcal{SM}}$. Constraint matrices $\mathbf{D}$ and $\mathbf{M}$ are written as:

$$\mathbf{D} = \int_{\Gamma_c} \mathbf{N}_\lambda \mathbf{N}_{\mathcal{S}} d\Gamma_c = \sum_{pr=1}^{n_{pr}} \left(\sum_{g=1}^{n_{gp}} w_g^{\mathcal{S}} \mathbf{N}_\lambda \left(\xi^{\mathcal{S}}(\xi_g), \eta^{\mathcal{S}}(\eta_g) \right) \mathbf{N}_{\mathcal{S}} \left(\xi^{\mathcal{S}}(\xi_g), \eta^{\mathcal{S}}(\eta_g) \right) J^{\mathcal{S}} \right), \tag{4a}$$

$$\mathbf{M} = \int_{\Gamma_c} \mathbf{N}_\lambda \mathbf{N}_{\mathcal{M}} d\Gamma_c = \sum_{pr=1}^{n_{pr}} \left(\sum_{g=1}^{n_{gp}} w_g^{\mathcal{M}} \mathbf{N}_\lambda \left(\xi^{\mathcal{S}}(\xi_g), \eta^{\mathcal{S}}(\eta_g) \right) \mathbf{N}_{\mathcal{M}} \left(\xi^{\mathcal{M}}(\xi_g), \eta^{\mathcal{M}}(\eta_g) \right) J^{\mathcal{M}} \right), \tag{4b}$$

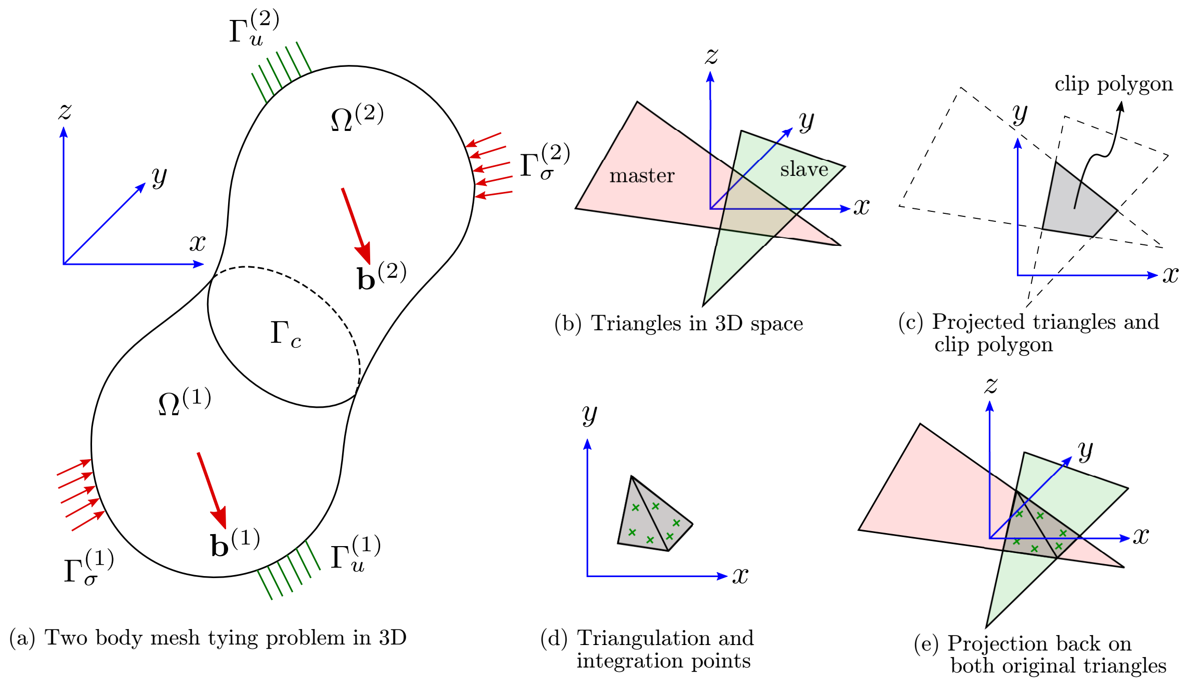

Figure 1: Mesh tying problem and steps involved in 3D mortar contact formulation for one pair of master and slave triangular elements

where $\mathbf{N}_{\mathcal{S}}$ and $\mathbf{N}_{\mathcal{M}}$ are matrices of shape functions for slave and master sides respectively. $\mathbf{N}_{\lambda}$ is a matrix of shape functions used for the discretisation of Lagrange multipliers, only exists on the slave side and is assumed the same as $\mathbf{N}_{\mathcal{S}}$. w and J are the weight and Jacobian associated with Gauss points. $\left(\xi^{\mathcal{S}}(\xi_g), \eta^{\mathcal{S}}(\eta_g)\right)$ and $\left(\xi^{\mathcal{M}}(\xi_g), \eta^{\mathcal{M}}(\eta_g)\right)$ represents the projection of Gauss points on the original slave and master triangles in local coordinates, explained below in detail.

A step-by-step procedure used for the 3D mortar contact formulation for one pair of slave and master triangles is shown in Figures 1(b-e). After projection of both triangles in xy-plane Figure (1(b)), the clipper library is used to determine the clip polygon (Figure 1(c)), which is subsequently triangulated (Figure 1(d)) with their integration points projected back on both the original master and slave triangles (Figure 1(e)). For the easy and efficient data handling, prisms are inserted between each intersecting slave and master triangles. In Equation (4), n_{pr} and n_{gp} are the number of prisms and Gauss points respectively.

3. Numerical Example

The two pairs of both non-conforming and conforming meshes, as shown in Figure 2, are used to validate the numerical implementation of the mortar contact formulation. The dimensions, loading, boundary conditions and coordinate system are also shown in Figure 2. Young's modulus and Poisson's ratio used in this case are $E = 10$ and $\nu = 0.5$ respectively. For both pairs, the conforming meshes, i.e. cases shown in Figures 2(b, d), are used for validation and is solved without mortar contact formulation. In Figure 2(a) and Figure 2(c) non-conforming meshes exists at their curved interfaces. The contours of z-component of displacement, i.e. u_z is also shown for all of the four cases. It can be seen that for each pair, the contours plots of u_z is exactly the same, demonstrating the correct implementation of the computational framework.

4. Conclusions

In this study, an efficient, accurate and robust numerical framework is presented for three-dimensional mortar contact formulation, which allows consistent treatment of the contact conditions. The computational framework is implemented in our group's finite element code, MOFEM, which uses state of

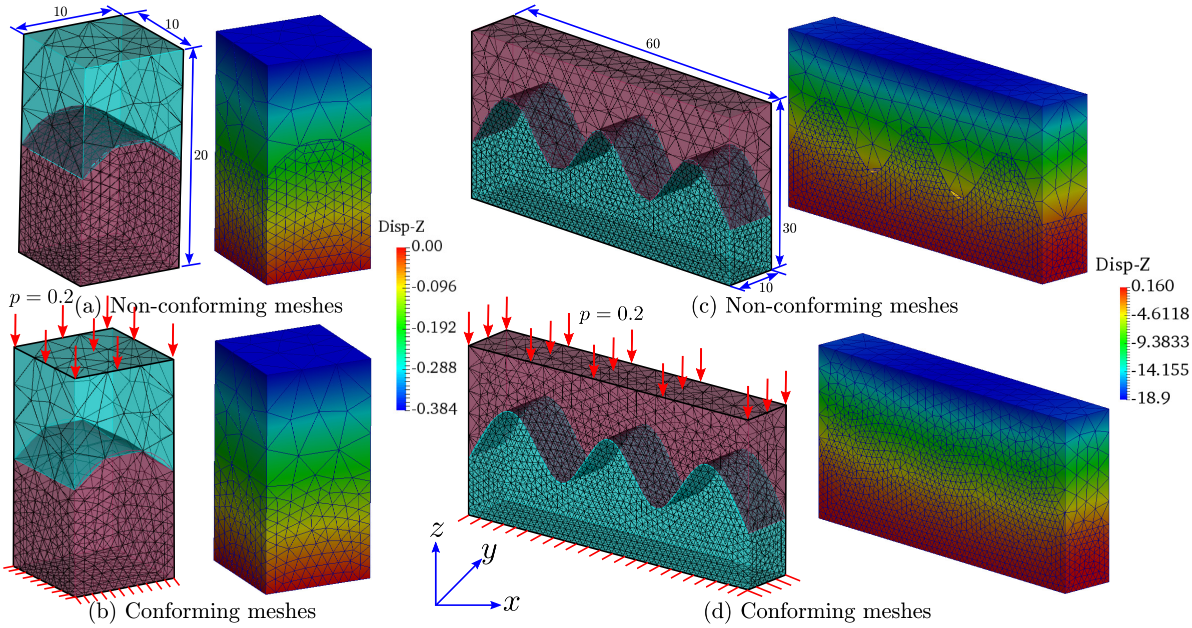

Figure 2: Two pairs of non-conforming and conforming meshes and corresponding contours of u_z

the art MOAB, PETSc and Boost libraries. The implementation is validated with two non-conforming mesh tying examples. Although, not demonstrated in this paper, the developed computational framework is designed to take advantages of the hierarchic basis functions and high-performance computing. This paper is restricted to linear-elastic problems but will be extended subsequently to include both material and geometric nonlinearities. The actual contact formulation including efficient contact search, non-penetration, frictionless and frictional sliding will be implemented next.

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