Mathematical Sciences HE Curriculum Innovation Project

History of Mathematics in the Higher Education Curriculum

Edited by Mark McCartney

The Higher Education Academy Maths, Stats & OR Network

National HE STEM Programme

bshm British Society for the History of Mathematics

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A report by the working group on History of Mathematics in the Higher Education Curriculum, May 2012.

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Introduction

Mathematics is usually, and of course correctly, presented 'ready-made' to students, with techniques and applications presented systematically and in logical order. However, like any other academic subject, mathematics has a history which is rich in astonishing breakthroughs, false starts, misattributions, confusions and dead-ends. This history gives a narrative and human context which adds colour and context to the discipline. Indeed, it has been shown by Hagerty, Smith and Goodwin [1] that "the inclusion of historical modules caused positive changes in mathematical communication, student achievement and attitudes". This echoes the views of many other academics and educators (see, for example, [2]).

Setting historical context can motivate and enthuse learning, but it also enriches the curriculum, shows connections between different branches of the subject, and helps to produce students with a greater sense of the breadth and, what might be termed, the creative life of mathematics as a discipline. This report seeks to give examples of how history has been integrated into undergraduate mathematics teaching in higher education through case studies, in dedicated modules, parts of modules, to develop mathematical topics, in an education setting and through distance learning. A final section gives annotated suggestions of useful teaching resources in paper, DVD and web-based formats.

A tale of two approaches: using the history of mathematics in the curriculum

At the risk of oversimplification, it can be stated that there are two broad ways to integrate the history of mathematics into the undergraduate curriculum. The first might be described as the bite-sized approach. Here history is introduced by the lecturer as asides and anecdotes, either verbally, or perhaps even integrated into lecture notes as short sections. It is important to note that the term anecdote is not in any way meant pejoratively here: a brief story about Newton's time at the Mint, or the many famous 'Dirac Stories' can not only give a short break from the more serious work of the lecture in hand, but also give student windows on the men and women behind the mathematics which can quicken an interest in wider learning and reading. It is of course not possible to quantify how widely such informal inclusions occur, but it is almost certainly not uncommon, as many academics become inevitably become interested in the history of their discipline, and then naturally include the fruits of their own reading in their teaching.

The belief that many other academics would like to include some history in their teaching, but perhaps lacked the time to produce materials led Bradshaw, Mann and McCartney [3] to produce freely available resources [4] in the form of PDF and mp3 files giving short introductions to a range of topics in the history of mathematics. The use of mp3 files in this project is indicative of the change in learning styles of the 21st century student, many of whom use mobile devices to listen to podcasts and other resources while travelling to university and at other times. Podcasts such as *Math/Maths* [5] and *Travels in a Mathematical World* [6] have gathered a following amongst undergraduate mathematicians and frequently contain historical content. Students can also be directed to the range of excellent archive of podcasts on the history of mathematics and science resulting from BBC Radio 4's *In Our Time* series [7] or the Gresham College lecture archive [8] [9].

If bite-sized is the first way to use history in the curriculum, the second is to construct a full module in the history of mathematics and place it within a mathematics degree programme. This, though much more ambitious, is not uncommon within UK universities. Table 1 lists institutions within the UK which, in 2010, had such a module. The vast majority of the modules were taught in the third year of the programme. In the modules at Greenwich and London Met. the history of mathematics forms only a part of the module. NUI Galway, NUI Maynooth, University College Cork, and St. Patrick's College of Education, Drumcondra in the Republic of Ireland also have modules in the history of mathematics. Though this list is, at face value, encouraging, there is anecdotal evidence that as staff responsible for teaching history of mathematics modules retire some modules have disappeared from other universities.

University of Dundee
University of East Anglia
University of Exeter
University of Greenwich
University of Leeds
University of Leicester
University of Liverpool
King's College London
London Metropolitan University
University College London
University of Manchester
University of Oxford
University of Reading
University of Sheffield
Sheffield Hallam University
University of St Andrews
Swansea University
University of Warwick

Table 1: The 18 UK universities which, as of 2010, were known to have a course in the history of mathematics as part of their undergraduate mathematics degree programme. (Results obtained by web trawl. Other universities may offer modules without their being visible on an external-facing website.)

As noted in the introduction a broad benefit of including history is the enrichment and depth it brings to the curriculum. Full subject modules allow the student to engage seriously with the history of their subject, and provide a course of study which can be refreshingly orthogonal, if the phrase may be forgiven, to other modules. Not only is the content going to be different, but also the mode of learning and assessment. Thus instead of using modern techniques, they may learn original methods, or instead of a weekly problem sheet they may have a weekly reading list, and finally the students will have to do, what some of them may not have done before in their undergraduate maths degree; consider how to write an essay. In these regards a module in the history of mathematics can be seen as an important technique for embedding key skills in writing and communication in the curriculum.

Case studies

The division of approaches to teaching the history of mathematics into the two categories above is of course a simplification. However, rather than attempt to further classify and divide, it is perhaps more beneficial to give examples via case studies, showing how the history of mathematics functions at the module level in different university courses across the country. The selection indicates both the variety of approaches and the creative pedagogy used and is testimony to the vibrant role which history can and does play within the undergraduate curriculum.

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Teaching the history of mathematics at the University of St Andrews

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The School of Mathematics and Statistics at St Andrews runs one 15-credit undergraduate module on the history of mathematics. This runs every other year, and is taken by students in either their third or fourth year (of a typically four year degree). We also run a related master's level module, see the end of this study for separate comments.

The module is one of our most popular: for example it was taken by 75 students the last time it was run, out of around 140 who could have taken it (many of whom are joint honours students who only spend 50% of their time on mathematics). Anecdotally, students take the module because it will provide an interesting contrast with their other courses, and because it is viewed as good preparation for the main honours dissertation. Student feedback is usually excellent, and many students go on to do a historical topic for their main dissertation.

The module is delivered via 24 lectures, normally broken up into 4 blocks of 6 lectures each, delivered by three or four lecturers. In addition, there are around 10 tutorials. Around 6 to 8 of these are similar to standard mathematics tutorials, with the lecturer working through solutions to a problem sheet with the class. The remainder are devoted to the project, see below. There is no set reading.

Recent topics have included Archimedes, Euler, Poincare, Newton, the development of algebra from antiquity to Galois (this required 2 blocks!), a similar "development of algebra", but stopping at the solution of the quartic (1 block), the development of the modern number system, and many others (I have only lectured it twice so am limited in recall). Some topics include some general historical background, whilst others focus more closely on the mathematics. An effort is made to cover a broad historical and geographical spread each time the course is delivered, although the choice of topics is ultimately up to that year's lecturers to determine between them.

The course is assessed in a very different manner from other Honours modules. 50% of the final mark derives from an individual project, and 50% is made up from two 'class tests'. There is no final exam.

The class tests each take place during the lecture hour, and each test is on two of the blocks of lectures. The style of the test questions varies a little from year to year, but roughly speaking each question takes around 25 minutes to answer, and could be a short essay question, a request to carry out a calculation in a certain historical style, or a mixture of the two. Some recent questions are given at the end of this Case Study.

The project is usually around 10 sides of A4. Students are expected to think of their own topic, as well as finding appropriate sources and deciding how to write it up. There is a tutorial at the beginning of the course which includes some ideas on how to go about choosing a project, and also directs students towards various starting points for their research (e.g. the History of Mathematics website at St Andrews), whilst emphasising that these sources are only a beginning, and that students should follow up lists of references, and references within those references, etc. Students are encouraged to use journal articles, books in the library, and various online sources, however there are no specific resources that students are required to use, and no recommended reading other than the relevant topics from the History of Mathematics website. During the course there are typically two more tutorials on the project, where initially students ask questions that they think will be relevant to the whole class, and then lecturers deal with individual questions.

The module is in something of a state of flux at present, with all of the key staff involved in the History of Mathematics having retired over the past few years. The School is keen to retain it, but there is some discussion as to its format. There is a belief amongst some staff (myself included) that not only is the course popular, it also teaches students transferrable skills which they do not encounter during a standard mathematics module, and that we ought to develop the module more in this direction; for example by incorporating verbal presentations, close reading of historical texts, and possibly group work. Other members of staff would prefer the module to be more similar to other modules, with a standard written exam and more focus on both learning facts and timed essay writing. It is likely that these issues will be left up to the discretion of the lecturers actually teaching the course in any given session.

More information is available at www.mcs.st-and.ac.uk/ug/hon4/MT4501.shtml.

We also run a Level 5 History of Maths module which is only taken by one year MSc students (*not* by our MMath cohort, who take the BSc module). This is delivered via the same lectures and tutorials as the BSc module, and students take the same class tests, but the project is substantially longer, and is worth 66% of the final mark.

Some recent test questions

1. (a) Who, in your opinion, is most deserving of being called the father of Algebra? Justify your assertion.

   (b) Solve the cubic equation $x^3 - 18x = 35$ using essentially the method of Cardan in Ars Magna.

2. (a) "The French revolution of 1789 changed the course of mathematics". Discuss.

   (b) Copy the approach of Gauss' first proof of the Fundamental Theorem of Algebra to show that the equation $x^2 - 3ix + (-2 + 2i) = 0$ has two roots.

3. Describe how Archimedes used his "Mechanical method" to calculate areas and volumes.

Some recent project topics

- Fermat's last theorem.
- Mathematical games and recreations.
- Ian Stewart's popularization of mathematics.
- History of weather forecasting.
- Alan Turing.
- The development of the modern number system.
- Mathematics in 16th century Spain.
- Japanese Mathematics.
- The mathematics of the solar system.

History in the undergraduate mathematics curriculum – a case study from Greenwich

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Including material on the history of mathematics in teaching mathematics undergraduates has proved to be a useful way to motivate students and to provide additional interest for those whose mathematics background has already covered topics in the first year curriculum [1]. This case study discusses the use of history in a new first year mathematics module introduced by the authors in 2008/2009 [2].

Mathematical Technology and Thinking (often abbreviated to "MaTT", which students seem to like) is a core first-year module for all mathematics programmes at Greenwich. It contains several strands – a group modelling project, PDP material encouraging reflection on learning and planning, and computer skills including spreadsheet and Matlab programming – but the bulk of the course aims to provide brief overviews of topics across the whole range of mathematics, to give students awareness of the extent of the subject and some understanding of the different areas in which mathematicians work. Topics range from relativity and quantum theory through topology, algebra and number theory to mathematical logic, Gödel's Theorems and the philosophy of mathematics, each topic being the focus of a single workshop. The resources produced for the Mathematical Motivators project [3] are made available to students as additional material to support the workshops. These are a range of "bite-sized" audio and text files on topics in the history of mathematics, designed to support and complement the mathematics undergraduate curriculum.

Clearly in the time available the workshops have to provide an overview of the topic rather than explaining the mathematics in any level of detail. For many of these workshops therefore, it has been natural to present the subject from a historical perspective. For example, the class on relativity begins by presenting Maxwell's equations, discusses nineteenth-century worries about the existence of the ether, presents Einstein's special relativity and moves on to his general relativity, showing how the 1919 evidence subsequently brought the theory into public limelight. Students learn about Einstein's prominence as a public figure and reflect on the public image of mathematicians.

While some students might like MaTT to focus exclusively on technical mathematics – for example one student commented this year "*we learn about history and stupid subjects, i paid money to come to this uni to do MATH*" – comments such as "*I want to do maths because I am inspired by knowing about the people behind it"* are more representative[1]. This feedback shows that many students relate well to mathematics presented as a human subject, still developing, to which they themselves might contribute, and feel they have a better understanding through seeing mathematics in its context.

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[1]Both comments from anonymous student feedback.

Teaching History of Mathematics at King's College London

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Introduction

For the past twenty years (roughly) there has been a module available to third-year students at King's College London with the title of 'History and Development of Mathematics'. It has been an option for anyone with a maths component in their degree, and has counted for one final year credit (out of a normal eight). Initially, I think, I taught the course in a rather unstructured way. After some years it was taken over by David Robinson who gave it the format it has today; and about eight years ago I took it back, but have continued teaching to the same formula.

Structure

The students are taught by two hours of lectures per week in the second semester. I give them weekly handouts to supplement the lectures, and more recently Power Point presentations. (Since 2006 I have also been able to recommend my own book; it is expensive, but it's compact, corresponds well to the course, and quite a few of the students buy it.) The idea is that they should be given an idea of the mathematics of the past as it was done in different periods; and particularly, that they should be shown what in other cultures might be called 'mathematicians' were often working in different areas and solving different problems – even with a different idea of what constitutes a 'solution'. So you could say that the idea is to combine teaching 'history' in the basic what happened version with 'critical history' of a sort of historicist or culturalist kind. Or at least to raise the question of what a 'history' should be doing.

This is rather a lot to ask, perhaps; and moreover it's combined with the requirement that the students should write an essay of 2000-2500 words on a subject 'within history of mathematics', to be handed in by the end of the ten-week term. The subject can be their own choice (there are some suggestions). They are required to get the subject approved by me, but a fair number don't in practice, and it doesn't honestly result in essays that are substantially worse. They are given suggestions for reading, both books and the increasingly available websites.

This essay counts for 25% of the module mark, the remaining 75% being by a written exam in May. Again, the structure of the written exam is invariable; I think that rather than describe it I'll simply attach a past paper which speaks for itself. (Regrettably, the College's policy is against allowing the students to receive their essays with marks and comments – as happens in most other universities that I know.)

Evaluation

The number of students opting for the course has increased steadily, from about 20 ten years ago to more like fifty now; and this is without a loss of quality. The students don't necessarily absorb all the ideas I hope they will, and I find the weaker ones repeating my own statements uncritically. Hardly any of them fail, and the mean mark tends to be above 50 – I have to explain to them, since these days there seems to be a constant grade point anxiety, that they aren't likely to get more than 80. However, there are always a number who surprise me – choosing an unusual essay subject, researching well, even performing well under exam conditions. In a London university like Kings, the large Asian and Chinese intake are always attracted by the opportunity I give them (particularly in the essay) to explore their own traditions, even if I try to encourage a critical approach to some of the wilder dating in ancient India.

I have to say that the course has always been a less rigorous training for historians than the one David Fowler designed decades ago in Warwick (in which *explication de texte* played a central role). I was to some extent inspired by his ideas, but I felt it would be too difficult to teach on his lines. And now I'm too old to change. The college is happy with it, I think; it provides the usual opportunity for students who are mathematically underperforming to get marks in an essay-based subject. I would like to think that some of them emerge with an idea of how our mathematics changes across cultures – this, I suppose, was the point of the word 'Development' in the course's title- But I don't have any evidence for this.

Examples of Recent Examination Questions

1. Does the historical study of mathematics cast a helpful light on the modern subject, or is it simply to be considered interesting for its own sake?
2. What, if any, of the ideas and methods of Greek mathematics are still in use today?
3. With regard to any period you have studied, discuss the problem of accessing and interpreting source-material.
4. What practical problems contributed to the development of mathematics between 1500 and 1700, and in what way?
5. What was 'non-rigorous' about the calculus in the eighteenth century, and how far was it seen as a problem?
6. In what sense did the work of Lobachevsky and Bolyai lead to a 'revolution' in mathematics?
7. How would you explain the relatively small number of women who feature in the history of mathematics?
8. Do you consider the current teaching of history of mathematics shows `bias' in undervaluing the contributions of a particular culture or group? Justify your answer, with examples.

History for learning Analysis

Bob Burn, University Fellow, Exeter University.

Calculus in English sixth forms is based on an intuitive view of limits, tangents, areas and volumes, abetted by the binomial theorem. These predominantly geometrical concepts are enough to generate a form of the fundamental theorem of the calculus and so empower students to find areas and volumes and their centres of gravity, and to study linear and circular motion. Although this sixth form view of the calculus is enhanced by Euler's use of exponentials and logarithms, it stands on a 17th century understanding of the subject.

Today's undergraduate analysis, by contrast, is an arithmetised subject structured in the sequence: Definition, Theorem, Proof. During the 20th century it was not uncommon for diagrams to be treated as suspect. The combination of formal rigour with a lack of acknowledgement of the geometrical origins of the subject, has made conscientious students focus on the logical steps, and ignore the big picture. The proverb "take care of the pennies and the pounds will take care of themselves" can be applied by lecturers (and students) to themselves when their overriding consideration is the rigour of presentation. Yet the overview of the subject matter is what gives point to the enterprise, and ought to be at the heart of all motivation. A more detailed account of student difficulties on encountering a first course in analysis is given in the preface to my *Numbers and Functions* [1].

One might have expected studies in Mathematics Education to propose an improved pedagogy, but most of the published research in this area either addresses particular concepts all of which form a necessary component of any treatment, or else describes styles of student interaction, which again may be invoked however the subject is structured. Another possibility has been proposed by Otto Toeplitz (1926), and quoted in the preface to his *The Calculus – a genetic approach* [2].

> *Regarding all these basic topics in infinitesimal calculus which we teach today as canonical requisites, e.g. mean-value theorem, Taylor series, the concept of convergence, the definite integral, and the differential quotient itself, the question is never raised 'Why so?' or 'How does one arrive at them?' Yet all these matters must at one time have been goals of an urgent quest, answers to burning questions, at the time, namely, when they were created. If we were to go back to the origins of these ideas, they would lose that dead appearance of cut-and-dried facts and instead take on vibrant life again.*

Toeplitz went on to say

> *Nothing, indeed, is further from me than to give a course on the history of infinitesimal calculus. I myself, as a student, made my escape from a course of that kind. It is not history for its own sake in which I am interested, but the genesis, at its cardinal points, of problems, facts and proofs.*

That history can point to an alternative genesis for the subject distinct from a logical development from axioms is at least admissible as a possibility. The preface to *Analysis by its history*, by Hairer and Wanner [3], illustrates how a modern conventional account of the subject reverses the historical development. But just how historical awareness may shape a learning sequence today is not obvious and the difficulty emerges from particular cases:

The earliest logarithms, those of Napier and Bürgi, do not admit the equation $\log a + \log b = \log ab$. This requires $\log 1 = 0$, which is false in both these systems.

The origins of both Cauchy's and Riemann's view of the integral lie in circumscribed and inscribed rectangles as used by Fermat, who in turn recognised their potency by translating

Archimedes' quadrature of the spiral into algebra. Archimedes' spiral does not figure in analysis courses.

Barrow is cited as the first to publish the fundamental theorem of the calculus, but it is much harder to recognise the theorem in Barrow's diagram than from the sketches in the notebooks of Newton whose ideas are the basis of modern proofs.

Rolle's theorem first appeared as an intermediate value theorem for derivatives of polynomials. This is hardly an effective motivator.

There are moments identifiable in the literature, which can be motivators, for example:

- Briggs' meeting with the elderly Napier ([4], p. 297);
- Cauchy's declared avoidance of arguments from 'the generality of algebra';
- Dedekind's motivation for the construction of his 'cuts' ([5], p. 1);
- And there are of course a train of special cases which must form part of any course of analysis: The irrationality of $\sqrt{2}$; $\sum 1/n$ divergent (Oresme, c.1350); $\sum +1-1$ (Grandi, 1703); $(1 + 1/n)^n$ (Euler 1748); sin $1/x$ and exp $(-1/x^2)$ (Cauchy, 1821, 1823) [6]; Dirichlet's function (1829); x sin $1/x$ (Weierstrass 1874).

Having been a student who endured rather than understood undergraduate analysis, it was quite a challenge to shape an undergraduate course on the subject. I was aware that any modern course must embrace a Weierstrassian perspective, and searched in early Weierstrassian textbooks hoping that my student struggles might be reflected in the earliest textbooks with this perspective. Dini's was the first such (1878) [7] and Harnack's which soon followed (1881) [8] was translated into English in 1891. I looked again at the second edition of Jordan (1893) which had so illuminated G.H.Hardy and at other texts recommended by Ivor Grattan-Guinness. I then concentrated on the earliest publications in the UK and the USA which transmitted a Weierstrassian perspective. A significant number of young American mathematicians had studied in Germany in the later part of the 19th century, and the first decade of the 20th century saw a proliferation of their publications. One, in particular, transformed my awareness of the subject. W.F.Osgood had studied with Felix Klein 1887-1890, and was then appointed to Harvard. In 1907 he published *Lehrbuch der Funktionentheorie* in German [9]. In this text he developed the standard theorems on continuity on an intuitive basis and then declared that we could (at this point) prove none of them! So, how could they go wrong? I was electrified. The Intermediate value theorem might fail, since the function $\mathbb{Q} \to \mathbb{Q}$ defined by $x \mapsto x^2 - 2$ does not reach 0 on the interval [1, 2]. The function $\mathbb{Q} \to \mathbb{Q}$ defined by $x \mapsto 1/(x^2 - 2)$ is continuous on [1, 2] but is not bounded. And so on. The theorems fail without completeness. The weakness of one's geometric intuition is exposed. The monsters of Lakatos' *Proofs and Refutations* [10] had come alive!

This immediately raised two questions, a historical one and a pedagogical one. The historical question was how had earlier mathematicians, from Euclid to Euler (at least) managed without completeness. The pedagogical one was how to introduce completeness to beginners. I hoped that an answer to the historical question might illuminate the pedagogical question. How, without completeness could the theorems on areas and volumes in Euclid Book XII be established? And what about the far more extensive results of Archimedes on areas, volumes and centres of gravity? Archimedes repeatedly claimed his order axiom (given two positive quantities, some multiple of the smaller will exceed the greater) and Euclid used his Definition IV, Book V, in a similar way to establish Euclid X.1 (given two quantities, a larger and a smaller one, repeatedly halving the larger will result in a quantity less than the smaller one). From a modern perspective, the axiom of Archimedean Order holds in every subfield of the real numbers and was appealed to for some of the integrations in the 17th century; not of course for those arguments which used indivisibles. It was startling to realise just how much calculus *could* be done using Archimedean Order, but not completeness. The modern definitions of limit, of continuity and of derivative are coherent, without completeness, and with Archimedean Order, generate a host of useful theorems.

It is quite common for courses in Analysis to start with a least upper bound axiom from which Archimedean Order is deduced. This hides the historical genesis, and deprives students of the experience of working without completeness, which experience, in turn, heightens the significance and power of completeness when it is invoked.

For the record, W.F.Osgood used an idea of du Bois Reymond (1882) [11] to introduce completeness, namely that an infinite decimal is convergent, and deduced from that assumption that monotonic bounded sequences converge. The same argument was used by O.Toeplitz in the first chapter of The calculus – a genetic approach [2], and I decided to use it too. A whole chapter exploring the various equivalent forms of completeness followed, giving, time and again, the existence of real numbers for which no direct means of computation was available.

The first five chapters of this course were used as a one term introduction to analysis at Warwick University in the late 1990s. It is quite impossible to say what role the historical remarks played, since the whole of Numbers and Functions was structured as a sequence of just under 800 questions, to be pursued by student problem-solving rather than by lectures. The questions were about analysis, not about history, including the theorems as questions. Compared with a control group which followed a conventional lecture course the exam results were significant: the achievement of 40 problem class students was compared with that of 40 lectured students and the problem class scored about 10% more on average. The book included intermittent summaries of the central theorems. Original authors were cited where possible, and a chronological epilogue rounded off each chapter.

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History of Mathematics in a College of Education Context

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Since 2008, I have taught a module in the History of Mathematics (HoM) to students taking mathematics (and one other subject) in a three-year B.A. at St Patrick's College Drumcondra, a college of Dublin City University. Because the cohorts are relatively small, the module (of 7.5 ECTS credits) has been offered every other year to second and third year students combined. At the time of writing (May 2012), the module has just been completed for the third time. The module involves about 45 hours of contact with the students, delivered over a period of 14 weeks (excluding a two-week vacation period), and is assessed by a combination of continuous assessment (CA) and a terminal examination, accounting for 30% and 70% of credit, respectively. It is one of four modules, of equal size, taken by each cohort of students. In the current model, second year students take modules in Analysis, Linear Algebra and Number Theory, while third year students take Calculus of Several Variables, Abstract Algebra and Statistics. In alternate years, students take a module in Geometry.

The stated aim of the module was to provide students with a framework for appreciating the historical development of mathematics. To this end, the following seven learning outcomes were given:

1. Outline in very general terms the timeline of the development of mathematics;
2. Describe significant historical periods when key changes in mathematical thought occurred and new areas emerged;
3. Summarise some important contributions of prominent mathematicians;
4. Explain how topics arising in school mathematics developed historically;
5. Discuss important examples of cultural factors influencing the development of mathematics;
6. Discuss the technical details of specific mathematical problems pertinent to 2-5, above;
7. Situate points 2-5, above, in a broader historical context.

The anchor text for the module was *Unknown Quantity* by John Derbyshire (2006) [1]. This book was chosen because it is accessible (and affordable!); moreover it covers a broad canvass of the history of algebra as well as touching on aspects of geometry. Thus the emphasis in the module was on the history of algebra with several cross-references to geometry. In addition, over a three-week period, the history of the calculus was considered, drawing on *The Calculus Gallery* by William Dunham (2008) [2]. To complement these texts, extensive use was made of O'Connor and Robertson's *MacTutor History of Mathematics* [3]; moreover, standard HoM texts (such as Victor Katz') were recommended and available in the College Library. Some primary sources were used such as Rosen's translation of al-Khwarizmi's *al-jabr wa'l-muqabala*, Decartes' *Geometry*, and letters by W.R. Hamilton.

Resources were made available via the virtual learning environment, Moodle. These resources included web links, explanatory notes on particular topics, appropriate files in GeoGebra and Maple. Some elements of the module are outlined at: http://staff.spd.dcu.ie/oreillym/math_history.htm

Apart from the terminal 3-hour examination, CA combined three elements: extracts from a learning journal, setting an examination question (on the solution of cubic equations) and an essay on a topic taken from a list provided.

Students were asked to keep a learning journal (LJ) and were directed that this should indicate their engagement with the course, include reflection on this engagement and be expected to be a useful resource when preparing for the exam. Students were advised to "avoid merely repeating material (either mathematical or historical) covered in the lectures or in handouts. However, it does make sense to use the LJ to make such material 'your own', to include extra insights you have gleaned from elsewhere (e.g. the web, the library), to articulate questions that arise in your own study as you engage in the course, to pull together aspects of the course so as to enhance your own overview of it." Rather than assessing the entire LJ, students were asked to draw from it to address six specific questions throughout the semester. For example, for the final direction asked students to "submit an entry extracted from your LJ on reviewing *Unknown Quantity* focusing on its strengths and weaknesses as the anchor text for this module. Include at least one strength and at least one weakness".

Students (18 this year) were asked to set an examination question on the solution of the cubic equation combining both mathematical and historical aspects. They did this work in pairs, and their work was assessed on the basis of the appropriateness of their question for the final examination. The nine questions, modified as 'reasonable exam questions', were made available to all students and they were advised that one of them would appear on final exam.

Finally students were provided with a list of essay topics to write about 2200 words. Topics chosen included: Calculus in India; Viète and Harriot: 16th century maths in France and England; Omar Khayyam: poet and mathematician; Sophie Bryant and Alicia Boole Stott: two women mathematicians born in Ireland.

In feedback from 11 out of 18 students, 9 found the essay helpful for their learning, 8 found the text (by J Derbyshire) helpful, 7 found the resources on Moodle helpful, 6 found the learning journal helpful, 5 found the exam exercise helpful; 9 found the material difficult while only 5 found it interesting. Here are examples of what students liked: "learning about the origins and beginnings of mathematics", "learning about the history of the people", "learning about the history of maths", and that the course "dealt with both history and maths". They disliked the "overload of information in the lectures", having "to learn how to solve problems from 500 years ago when we have much more efficient and much easier methods today", and that "a lot of the material in the course was quite difficult".

As the lecturer delivering this module, I see this work as evolutionary. Every cohort of students differs; moreover, the constraints under which a module is delivered differ from one implementation to the next. I welcome informal contact from colleagues, both more and less experienced than I am, to discuss how to engage with teaching HoM more effectively.

References

1. Derbyshire, J. (2006). *Unknown Quantity: A Real And Imaginary History of Algebra*. Washington, D.C., U.S.A.: National Academies Press.
2. Dunham, W. (2005). *The Calculus Gallery: Masterpieces From Newton To Lebesgue*. Princeton University Press.
3. *MacTutor History of Mathematics*. Available via: www-history.mcs.st-and.ac.uk [last accessed May 2012].

Teaching the history of mathematics at the Open University

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Introduction

From 1987 to 2007 the Open University ran a nine-month 30-point course MA290, *Topics in the History of Mathematics*, covering the development of mathematics from earliest times to around 1900. Replacing an earlier course that had run for about twelve years, it was studied by about 300 students per year. Since 2008 a shorter course at a lower level, TM190, *The Story of Maths*, has been presented, attracting about 250 students per year.

Structure

Like other OU courses, MA290 was designed as a multi-media course for adult students studying at home. It comprised 17 correspondence texts, eight BBC television programmes and several cassette tapes (later sent out as DVDs), and other supplementary material. The correspondence texts, each about 36 pages long, were associated with the set book *The History of Mathematics: A Reader*, compiled and edited by two of the course authors, John Fauvel and Jeremy Gray [1].

The course consisted of four blocks, each with four or five associated texts:

1. Mathematics in the Ancient World [Early mathematics; Mathematics in the Greek world; The Greek concept of proof; The Greek study of curves]
2. From the Middle Ages to the Seventeenth Century [From the Greeks to the renaissance; The Renaissance of mathematical sciences in Britain; European mathematics in the early seventeenth century; Descartes: algebra and geometry]
3. The Seventeenth and Eighteenth Centuries [The route to the calculus; The development of the calculus; Mathematical physics and the system of the world; Style and formalism in the eighteenth century]
4. Topics in Nineteenth-Century Mathematics [Non-Euclidean geometry; Algebra and the profession of mathematics; Projective geometry and the axiomatisation of mathematics; Fundamentals; Topics in the history of computing]

The 25-minute television programmes, mainly filmed on location, covered a wide range of topics, from The emergence of Greek mathematics, via The vernacular tradition and The founding of the Royal Society, to Paris and the new mathematics and The liberation of algebra.

Assessment was via four tutor-marked assignments, in which all of the questions were essays, ranging from short ones of 500 words to a final one of up to 2000 words. A favourite type of question consisted in asking the student to comment on the context, content and significance of a specific mathematical text. The course concluded with a three-hour examination.

Since the course finished, a group of course-team members has been converting the correspondence texts into book form, to be published by the Mathematical Association of America.

The Story of Maths

In 2008, a 10-point OU course was introduced to accompany the BBC–Open University television series *The Story of Maths*, filmed around the world and presented by Prof. Marcus du Sautoy; the theme of the course is to ‘learn the mathematics behind the programmes’. The course lasts ten weeks and is presented twice per year. The students receive copies of the TV programmes and some especially written course notes. The assessment consists of a substantial number of computer-marked questions, testing the student’s grasp of the mathematical content of the course, and two short essays testing their overall understanding of the material. The retention rate for the course is high and most students who complete the course pass it. TM190 is expected to run for a further three years.

References

1. Fauvel, J. and Gray, J. (1987). *The History of Mathematics: A Reader*. Open University.

Suggested Resources

There are a wide range of resources available for teaching the history of mathematics. Below we give a selection of these. These lists are not intended to be exhaustive, nor are they authoritative, rather they are a selection which members of the working group have found useful and/or have used in their own teaching and scholarship.

Books

The range of books on the history of mathematics is (happily) vast, and we have divided books into the broad categories of *introductory* (useful to the novice or general reader), *intermediate* (useful in undergraduate teaching, as either background reading, or as a set text) and *advanced* (more detailed study of topics which may be studied at undergraduate or postgraduate research project level). The boundaries between these categories are soft and, just as there was some disagreement amongst the authors as to where particular books may fit best, we will not discontent if some readers choose to disagree with us on exact categorisation.

Introductory

Raymond Flood and Robin Wilson, *The Great Mathematicians* (Arcturus, 2011). A well-illustrated A4 format book giving readable stand-alone 2 page biographies of famous mathematicians from ancient times to the 20th century.

Clifford Pickover, *The Math Book* (Sterling, 2009). A highly illustrated tour through 250 milestones in the history of mathematics. A book which can be opened at random with profit.

Anne Rooney, *The Story of Mathematics* (Arcturus, 2009). A well-illustrated general history of mathematics.

Ian Stewart, *Taming the Infinite: The Story of Mathematics* (Quercus, 2009). Another well illustrated history. A nice feature of the book is its inclusion sections on ‘what particular topics did for us’ – giving contemporary application of mathematical discoveries.

Intermediate

John Fauvel and Jeremy Gray (eds.), *The History of Mathematics: A Reader*, (Macmillan, 1987). A useful source book, which is not only enjoyable to dip into, but also gives a wide range of mathematical texts from four millennia.

Luke Hodgkin, *A History of Mathematics*, (OUP, 2005). A relatively short course textbook as used by one of the case study authors (see page 15), which covers a broad range of topics.

Victor Katz, *A History of Mathematics* (3/e) (Addison-Wesley, 2009). A ‘standard’ text for history of mathematics courses.

Ian Stewart, *Why Beauty is Truth: The History of Symmetry* (Basic Books, 2008). An account of the development of the pure mathematics of symmetry and its influence on modern physics.

Dirk J Struik, *A Concise History of Mathematics* (4/e), (Dover, 1987). The chief advantage of this book is that it is both, as its name suggests, short, and relatively cheap making it a helpful introductory text, though admittedly not one which interacts with the most up-to-date scholarship.

Robin Wilson, *Four Colours Suffice* (Allen-Lane: Penguin, 2002). A readable account of the famous problem.

James Gleick, *Chaos: Making a New Science* (Vintage, 1997). A classic semi-popular introduction to chaos provided through its historical development.

Marcus du Sautoy, *Finding Moonshine* (Harper, 2009). A book about how contemporary mathematicians work.

Jacqueline Stedall, *The history of mathematics: a very short introduction* (OUP, 2012). A short, stimulating account of how historians approach the history of mathematics.

Jacqueline Stedall, *Mathematics Emerging: A Sourcebook* (1540-1900) (OUP, 2008). A useful source of primary material, showing mathematical breakthroughs in context.

Benjamin Wardhaugh, *How to read historical mathematics* (Princeton, 2010) A user friendly guide for students on how to approach the reading of mathematical texts in their original context.

Advanced

Raymond Flood, Adrian Rice and Robin Wilson (ed.), *Mathematics in Victorian Britain* (OUP, 2011). A readable and wide ranging survey of mathematics and mathematicians in Britain and beyond in the 19th century.

Loren Graham and Jean-Michel Kantor, *Naming infinity: A True Story of Religious Mysticism and Mathematical Creativity* (Belknap/Harvard, 2009). An example of modern scholarship which shows the human side of recent abstract mathematics.

Mary Jo Nye (ed.), *The Cambridge History of Science vol. 5: The Modern Physical and Mathematical Sciences* (CUP, 2003). Contains a wide range of 'stand-alone' articles on matters both physical and mathematical, showing both links between the disciplines, and breadth of modern historical scholarship.

Eleanor Robson and Jacqueline Stedall, *The Oxford Handbook of the History of Mathematics* (OUP, 2008). A collection of essays on many different aspects of the history of mathematics. A surprising and stimulating collection which shows the diversity of the topic.

Andy Warwick, *Masters of Theory: Cambridge and the Rise of Mathematical Physics* (Chicago, 2003). A heavyweight piece of scholarship showing the key role of Victorian Cambridge and the famous Tripos in the history of applied mathematics and theoretical physics.

DVD

Edward Burger, *Zero to Infinity: A History of Numbers* (The Teaching Company, 2007). A set of twenty four 30 minute lectures on a wide range of topics in the history of mathematics.

Marcus du Sautoy, *The Story of Mathematics* (OU Worldwide/BBC, 2008). This four part series gives an excellent short introduction to the history of mathematics.

Web Resources

The University of St Andrews MacTutor site: www-history.mcs.st-andrews.ac.uk [last accessed May 2012]. A large and ever increasing repository of entries on a whole range of people and topics within the history of mathematics. An excellent and, in terms of search engine ranking, almost inevitable first port of call.

The British Society for the History of Mathematics site: www.bshm.ac.uk [last accessed May 2012]. A site containing information on what is happening in the scholarly world of the history of mathematics. In particular the '*Resources*' section of the site contains an extensive list of other web resources.

Oral History Sites. Oral history provides a rich source of information on aspects of the recent history of mathematics. Coverage of the history of mathematics currently appears to be somewhat patchy. A number of repositories are available on the web including those at Princeton, Caltech, and the American Institute of Physics:
www.princeton.edu/~mudd/finding_aids/mathoral/pm02.htm [last accessed May 2012];
oralhistories.library.caltech.edu/view/subjects/sub.html [last accessed May 2012];
www.aip.org/history/oral_history/transcripts.html [last accessed May 2012].

Mathematics is usually, and of course correctly, presented 'ready-made' to students, with techniques and applications presented systematically and in logical order. However, like any other academic subject, mathematics has a history which is rich in astonishing breakthroughs, false starts, misattributions, confusions and dead-ends. This history gives a narrative and human context which adds colour and context to the discipline.

Setting historical context can motivate and enthuse learning, but it also enriches the curriculum, shows connections between different branches of the subject, and helps to produce students with a greater sense of the breadth and, what might be termed, the creative life of mathematics as a discipline. This report seeks to give examples of how history has been integrated into undergraduate mathematics teaching in higher education through case studies.

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