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DIFFUSION OF MAGNETOSPHERIC PROTONS
BY TURBULENT DRIFT WAVES.

University of Alaska, Ph.D., 1971
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DIFFUSION OF MAGNETOSPHERIC PROTONS BY TURBULENT DRIFT WAVES

A

DISSERTATION

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University of Alaska in partial fulfillment
of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

by

James E. Maggs, B.S., M.S.
College, Alaska

May 1971

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DIFFUSION OF MAGNETOSPHERIC PROTONS

BY TURBULENT DRIFT WAVES

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Abstract

The diffusion of charged particles by turbulent electrostatic oscillations in the magnetosphere is investigated as a possible mechanism for energizing and transporting ring current protons. A simple model of the magnetosphere is used to carry out the investigation. In the model the geomagnetic field is taken to be an azimuthally symmetric axial field, the ionosphere is taken to be perfectly conducting, and the ring current and plasmasphere plasmas are assumed to have Maxwellian distributions of velocities. The diffusion process is assumed to preserve the magnetic moment or first adiabatic invariant of the particles. The behavior of low frequency (<< ion gyrofrequency), long wave length (>> ion gyroradius) electrostatic oscillations is studied. The electrostatic oscillations are treated as natural modes of the magnetospheric cavity driven by fluctuations in the electric potential on the magnetopause, the magnetospheric boundary. It is concluded that diffusion of ring current protons by this process is not important in the magnetosphere.

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Diffusion of Magnetospheric

Protons by Turbulent Drift

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Chapter I

The purpose of this paper is to investigate the diffusion of protons in turbulent drift waves as a possible mechanism for energizing and transporting ring current particles. Since the ring current seems to play a basic role in the magnetospheric substorm as both a particle and energy reservoir, an understanding of the mechanism which supplies ring current particles and energy is basic to an understanding of the magnetospheric storm. A discussion of the magnetospheric storm and the concept of a substorm is given by S.-I. Akasofu (1968). Much of the following general material on magnetospheric storms is drawn from this reference.

The magnetospheric storm is apparently the product of the interaction of the solar wind with the magnetosphere. The magnetospheric storm can be divided into two parts, an initial phase and a number of relatively short magnetospheric substorms. The ring current plays a central role in the magnetospheric storm in that the phase or progress of a storm can be related to the behavior of the particles in the ring current. Originally, the ring current was concieved as a current system surrounding the earth outside the ionosphere which accounted for the main phase decrease of the magnetospheric storm. The ring current was considered a mathematically convenient representation of the main phase decrease mechanism and not necessarily a physical reality. However, a physical entity called the storm time radiation belt which behaves very much like the ring current has been detected by satellite measurements. The storm time radiation belt consists

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primarily of 1-50 Kev protons. The term "ring current" is now generally used interchangeably with the term "storm time radiation belt" to refer to this group of magnetospheric protons. The distribution and total intensity of ring current particles in the magnetosphere is greatly effected by the number and duration of previous magnetospheric substorms.

The concept of the magnetospheric substorm encompasses a number of apparently interrelated phenomena which occur sporadically and last for 1-3 hours. The auroral substorm, polar magnetic substorm, micropulsation substorm, ionospheric substorm, x-ray substorm, proton aurora substorm and the VLF emission substorm are all considered to be part of a larger whole, the magnetospheric substorm. During a magnetospheric substorm the total number of protons with energies between 1-50 Kev increases inside the trapping region. The increase in the total number of particles is not uniform in local time. Indeed, the effect of the magnetospheric substorm seems to be to increase both the asymmetry and the total number of ring current particles.

During the course of a magnetospheric storm, the total number of protons in the ring current builds up fairly rapidly via the mechanism of the magnetospheric substorm and then decays slowly back to pre-sub-storm levels. For a moderate storm the build up of ring current particles lasts about 10 hours. The decay takes about 5-15 days depending upon the size of the storm. The decay in the total number of ring current particles may be interrupted by the occurrence of additional substorms. The formation of the storm time radiation belt involves the transport and/or energization of large numbers of protons in the magnetosphere on the time scale of the magnetospheric substorm.

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Many transport and energization mechanisms that might be functioning in the magnetosphere have been investigated. Some of the mechanisms proposed and studied are: magnetohydrodynamic convection of particles due to a viscous interaction between the solar wind and the magnetosphere (Axford and Hines 1961); convection of particles due to a static electric field across the magnetosphere (Kellogg, 1959; Taylor and Hones, 1965); rotational and gradient drifts (Swift, 1971); magnetospheric tail instability (Axford 1967, Dungey 1968, Piddington 1968); diffusion of particles due to stochastic changes in the geomagnetic field (Parker 1960, Nakada and Mead 1965). None of the mechanisms proposed to date have been successful in explaining the formation of the storm time radiation belt. However, some of the mechanisms studied undoubtedly operate to some degree during a magnetospheric substorm. Most likely, the magnetospheric substorm invokes a complex combination of transport mechanisms.

The transport of particles in the magnetosphere due to diffusion in random electric and magnetic fields has been extensively investigated. Many investigators (Parker 1960, Davis and Chang 1962, Dungey 1965, Nakada and Mead 1965, Falthammar 1965, 1968; Conrath 1967, Schultz and Eviatar 1969) have studied diffusion driven by pressure changes in the solar wind which subsequently cause changes in the position of the magnetopause. Radial diffusion of particles occurs through violation of the third (flux) adiabatic invariant. The differences in the treatment of the problem are usually the method for obtaining the diffusion equation and the conditions imposed upon the magnetospheric plasma. A discussion of the work of Nakada and Mead (1965) and Schulz and Eviatar (1969) illustrates the different approaches to the problem.

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Nakada and Mead use the Mead (1964) model of the geomagnetic field. The effect on the main field of the position of the magnetopause is taken into account in this model of the field. The diffusion mechanism is assumed to work in the following manner. A compression of the magnetosphere (i.e. an inward motion of the magnetopause) is assumed to take place on a time scale short in comparison with the time it takes an equatorially trapped (J=0) particle to drift around the earth. Depending upon the particle's position in local time when the compression occurs, particles which were drifting on the same shell of constant B will follow the line of force to a different value of B and drift on different shells of constant B - thus violating the third adiabatic invariant. If the decay of the field to its original configuration is slow in comparison with the time it takes a particle to drift around the earth, the third invariant will be conserved and the particle will remain on its new shell of constant B. The total effect is to spread out particles on neighboring shells of constant B. That is, the particles undergo radial diffusion.

Such sudden compressions in the field with slow recovery do indeed occur. They are called sudden impulses or sudden commencements. Nakada and Mead calculated a dispersion coefficient by using an ensemble of these impulses compiled from actual observations.

Nakada and Mead use a Fokker-Plank equation to describe the evolution of the distribution of particles trapped in the magnetosphere. Particle energy losses due to Coulomb scattering and charge exchange collisions with neutrals are taken into account in the solution of the equation. The coefficient of dynamical friction is assumed to be proportional to the dispersion coefficient. The value used

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for the dispersion coefficient was .031 (L^{10}/L_o^8) (earth radii)^2/day. L_o is the distance from the center of the earth to the magnetopause along the earth sun line. The equation was solved using measured values of particle fluxes > 100 Kev for an initial distribution. Although some qualitative agreement with measured equilibrium particle fluxes was obtained for high energy particles (>50 Kev), a larger dispersion coefficient than the one calculated is needed to obtain quantitative agreement between the theory and observation.

It was implicit in the discussion of the diffusion mechanism above that the compression of the magnetopause was so sudden that all particles on a field line moved with it, regardless of their μ value (μ is the magnetic moment). This, of course, is not strictly true and a diffusion equation and diffusion coefficient for which this assumption is not made has been derived by Schulz and Eviatar (1969). They do, however, assume J=0 and $\vec{E}\cdot\vec{B}=0$. The evolution of the distribution function is given by a diffusion equation rather than the Fokker-Plank equation. The diffusion equation is derived using the Mead (1964) model of the field, but the unperturbed drift orbits of the particle around the earth are assumed to be non-circular. The diffusion coefficient depends upon the power spectrum of the fluctuations in the stand-off distance of the magnetopause (L_o). If an inverse power law is assumed for the power spectrum the expression for the diffusion coefficient is similar to Nakada and Mead's for $L \ll L_o$.

A different mechanism for diffusion of particles in the magnetosphere has been investigated by Birmingham (1969). He suggests that diffusion is not driven by the induced electric field caused by fluctuations in the geomagnetic field but by variations in the

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electric potential fields ($\vec{\nabla} \times \vec{E} = 0$) governed by the dynamics of low energy (<8Kev) trapped particles. The potential has wave lengths on the order of the dimensions of the magnetosphere. That is, Birmingham assumes the low energy component of trapped particle fluxes produces an electric potential V such that $\vec{E} = -\vec{\nabla} V = -\frac{\vec{v}}{c} \times \vec{B}$ where $\vec{v}$ is the hydromagnetic flow velocity of the low energy plasma. Birmingham uses the cut-off energy of 8 Kev because particles below this energy are dominated by electric field drifts of "typical" fields found in the magnetosphere while particles above 8 Kev are dominated by gradient and curvature drifts.

Birmingham uses a diffusion equation developed by Birmingham, Northrop and Fälthammar (1967). The equation is valid for particles with arbitrary values of μ and J. Again diffusion occurs as a result of violation of the flux invariant. A simple model is used to derive an explicit expression for the diffusion coefficients. A dipole magnetic field is assumed and a potential of the form $V = A(t)\, r \sin\phi / \sin^2\theta$ is used to describe the electric potential in spherical coordinates. The flow pattern of the low energy component of the plasma is similar to the convective flow patterns envisaged by Levy et al. (1964) and, to some extent, other convective flow models (e.g. Axford and Hines 1961). It is shown that the diffusion coefficient is proportional to L^6 and the value of the power spectrum of the time dependent coefficient A(t) evaluated at the azimuthal drift frequency. If the autocorrelation function of A(t) (i.e. the inverse Fourier transform of the power spectrum) is assumed to have the form $C \exp(-t^2/\tau_c^2)$ where τ_c is the correlation time, then the diffusion coefficient is proportional

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to $L^6\tau_c \exp(-\omega_D^2\tau_c^2/4)$. For particles with azimuthal drift frequencies ω_D much less than $\frac{1}{\tau_c}$ it is proportional to $L^6\tau_c$ and thus independant of μ and J.

Birmingham solves a one dimensional diffusion equation with no loss terms using the approximation to the diffusion coefficient with $\omega_D^2\tau_c^2 << 1$. Thus the results are applicable to particles with drift periods $\frac{2\pi}{\omega_D}$ ($\simeq 4.4\times10^4$ (L x [energy in Kev])$^{-1}$ minutes) much larger than the correlation time. Birmingham uses a value of one hour for the correlation time. Choosing a value of $(2\times10^{-4}$ v/m$)^2$ for the coefficient C in the correlation function, Birmingham shows that an initial delta function distribution in number density at 8 R_e (earth radii) will yield a significant number of particles at 5 R_e in about 18 hours (see Birmingham 1969, figure 1). A similar analysis using the diffusion coefficient given by Nakada and Mead takes over 100 days to produce the same results. Thus a substantial increase in the diffusion rates can be obtained by using the variations in the convective electric field associated with the flow of the low energy component of the magnetospheric plasma. The diffusion rates appear, however, to be still too slow to transport particles several earth radii on the time scale of the magnetospheric substorm. If diffusion processes are to be important during a substorm an increase in the diffusion rate is necessary.

Most of the studies of diffusion processes in the magnetosphere have concentrated on very low frequency variations in the magnetic field. The frequencies are low enough that the second adiabatic invariant of the particles (J) can be assumed to be conserved. This study will concentrate on electrostatic oscillations whose

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frequencies are near the bounce frequency of a typical ring current proton but much less than its gyrofrequency. The interaction of a proton with electrostatic oscillations in this frequency range will violate the second (longitudinal) and third adiabatic invariants of the particle. The first invariant, μ, is assumed to be conserved.

Such electrostatic oscillations which can exist in plasmas with number and/or temperature gradients, such as the magnetospheric plasma, are generally called drift waves. Drift waves characteristically have wave lengths parallel to the magnetic field, $\vec{B}$, much larger than their wave lengths perpendicular to $\vec{B}$. This is an ideal property in the respect that the requirement of conserving the first adiabatic invariant demands that the parallel electric field be small but for strong diffusion the electric field perpendicular to the magnetic field must be large so that the $\vec{E}\times\vec{B}$ drift of the particles is large. The equation for the electric potential associated with the drift waves will be developed under the assumption that the wave length perpendicular to $\vec{B}$ is larger than an ion gyroradius and the wave frequency is less than the ion gyrofrequency but greater than $\ell\omega_D$ (ℓ is the azimuthal wave number in that $\frac{2\pi r}{\ell}$ is the wave length of the oscillation in the azimuthal direction and ω_D is the azimuthal ion drift frequency since $\frac{2\pi}{\omega_D}$ is the time it takes an ion to drift around the earth.) The energy source for the diffusion is assumed to be near steady (in time) fluctuations in the electric potential at the magnetospheric boundary, the magnetopause. Thus the solutions of primary interest will be undamped or only slightly damped oscillations which have significant electric fields throughout a large part of the magnetosphere.

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Chapter 2

A Model Magnetosphere

The interaction of the solar wind with the earth's magnetic field creates a magnetic cavity. The confinement of the earth's field by the solar plasma was investigated theoretically as early as 1931 by Chapman and Ferraro. With the advent of the space age and satellite borne magnetometers the confinement of the earth's field in space was demonstrated.

Intensive investigation of the particles and fields surrounding the earth has revealed most of the salient features of the magnetospheric cavity. However, there is still disagreement over whether the cavity is open (geomagnetic field lines connected to interplanetary field lines) or closed (no connection). Further, some regions of the magnetosphere have yet to be investigated. Several general features of the magnetosphere have been firmly established, however.

The magnetosphere is generally divided up into various regions such as the magnetotail, the trapping region, the plasmasphere, etc. Regions in the magnetosphere are identified by changes in the character of the magnetic field or by various characteristics of the local particle population or energy spectrum. Surrounding the magnetosphere is a region called the magnetosheath. The magnetosheath is essentially a transition between the region dominated by the geomagnetic field and the ordered flow of the solar wind. The

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magnetosheath is characterized by the turbulent nature of the magnetic field whose fluctuations can be as large as 100% of the average field (Smith and Davis, 1970). The bow shock marks the outer boundary of the magnetosheath on the sunward side of the magnetosphere. The bow shock is an essentially permanent, collisionless, standing shock wave whose thinness (much less than an ion cyclotron radius (Heppner et al., 1967)) suggests that it is an electrostatically generated phenomenon. The inner boundary of the magnetosheath, the magnetopause, separates the disordered field of the magnetosheath from the ordered field of the earth. On the day side of the magnetosphere the magnetopause is easily identified by a change in field magnitude and direction over a distance of a few hundred kilometers. The magnetopause is more difficult to identify in the tail region of the magnetosphere.

The geomagnetic field close to the earth is almost dipolar but deviates considerably from a dipole field near the magnetopause and in the magnetotail. On the day side the field is compressed and lines of constant B in the equatorial plane are displaced outward (away from the earth) from those of a dipole field. In the antisolar direction the field is inflated and lines of constant B are displaced inward from those of a dipole field. Near the magnetospheric equatorial plane the field changes direction abruptly forming a region called the neutral sheet. The neutral sheet extends across most of the magnetospheric tail in the magnetospheric equatorial plane. The inner edge of the neutral sheet begins at about 10-15 R_e

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in the antisolar direction. The high latitude field lines all extend in the antisolar direction and constitute most of the magnetic field in the tail.

Particle populations in and near the magnetosphere are identified by the shape of their energy spectra and their number densities. Usually, the internal energy of a plasma can be used to define a temperature ($\frac{3kT}{2}$ = internal energy). The actual particle distribution function can then be approximated by a Maxwellian distribution of velocities corresponding to this temperature. The actual measured plasma velocity distribution can then be compared to the associated Maxwellian distribution. For example, the magnetosheath electron velocity distribution is broader and has fewer low energy particles than its associated Maxwellian distribution. The magnetosheath proton distribution similarly lacks low energy particles and also has a small peak in the number of particles located at about 2-3 times the average particle energy. The average proton energy is a few hundred eV and the average electron energy 1-2 hundred eV (Montgomery et al., 1970). The transition from the solar wind plasma to the magnetosheath plasma which occurs at or near the bow shock is also discussed by Montgomery et al.

Frank (1970) has identified two characteristic proton energy spectra in the magnetotail. One spectrum is quite similar to that of magnetosheath protons which suggests that magnetosheath particles have access to the tail. The other group of particles identified by Frank has a much broader characteristic spectrum with an average

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energy near 5 Kev. Both of these groups of ions are generally taken to be members of the plasma sheet. The plasma sheet is a particle population centered about the neutral sheet. In the tail region it is about 4-6 R_e thick, measured perpendicular to the neutral sheet. It extends near the earth into the dawn and dusk sectors where it is 6-12 R_e thick. The inner edge of the plasma sheet is usually identified by the behavior of the electrons. The electron velocity distribution is roughly Maxwellian but has a high energy tail. The average electron energy is near 1 Kev but can range up or down by an order of magnitude (Vette, 1970). The best indicator of the inner edge seems to be a decrease in the total energy density or a decrease in the number density of all particles with energies greater than 700 eV. The determination of the inner edge depends upon the energy of the electrons observed (Schield and Frank, 1970). The inner edge of the plasma sheet has not been observed to penetrate the plasmasphere even during disturbed times. However, the character of the electron energy density and the position of the inner edge is apparently quite different during quiet and disturbed times. Vasylinnas (1968) puts the inner edge of the plasma sheet during quiet times at 11 ± 1 R_e. Schield and Frank place the inner edge 1-5 R_e from the plasmapause. The difference in the two results may be due to a latitudinal dependence (Schield and Frank, 1970).

The plasmasphere is a cold dense plasma (compared to the ring current) that corotates with the earth. The outer boundary of the plasmasphere, the plasmapause, is most easily identified by an abrupt change in the

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number density of plasmasphere ions. The change in the number density is typically over an order of magnitude. Several examples of plasmapause crossings and representative number density profiles are given by Taylor et al. (1968) and Chappell et al. (1970). The density of cold ions (10^3-10^4°K) just outside the plasmapause is typically 1-10/cm^3 and inside the density rises abruptly to 100-1000/cm^3 and continues to increase at a slower rate with decreasing L value. The studies by Taylor and Chappell both indicated that the plasmapause tended to move inwards with increasing magnetic activity. Recently, Russell and Thorne (1970) have reported that the plasmapause actually coincides very closely with the location of the maximum density of the storm time radiation belt.

The development of sensitive differential energy analysers by L.A. Frank has enabled him to study very low energy protons in the energy range 200 eV to 50 Kev (Frank 1967, 1970). Protons in this energy range were subsequently found to be an important particle population in the magnetosphere. The storm time radiation belt, primarily comprised of particles in the 1-50 Kev range, is largely responsible for the main phase decrease during magnetic storms (Frank, 1967) and thus can be closely identified with the ring current. The ring current protons in the 1-50 Kev range are recognized by their characteristic energy spectrum and radial number density profile. The differential energy spectrum hardens slightly as L decreases. The spectrum at 6 or 7 R_e is almost flat from 3 to 30 Kev. The average energy is typically between 5-20 Kev. The number density of these low energy protons

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depends upon the magnetic activity and the local time. Satellite and ground based magnetometer measurements (Cummings et al. 1968) indicate that during a magnetospheric substorm there may be a decrease of ring current particles in the midnight to dawn sector (2400-0600 local time) while in the early evening to midnight sector (1600-2400 local time) an increase of ring current particles has been directly measured by satellite (Frank and Owens, 1970). Furthermore, Frank (1970) has measured the number density at different local times during various phases of a magnetic storm and found there exists an asymmetry in the number density during all phases of a storm. The asymmetry appears to be largest during the early development of the main phase. Frank and Owens (1970) have investigated the proton distribution in the midnight sector and found the persistent presence of a "quiet time ring current" with peak proton energy densities near L=6.5. Several typical flux profiles i.e. the L dependence of the flux are given by Frank (1967b) for various energy ranges. These profiles all have the same qualitative behavior as functions of L. Since the shape of the differential energy spectrum changes only slightly with L and is nearly flat, the number density as a function of L will be proportional to the flux profile for energy ranges near the average energy. Thus the flux profile for the energy range 16-25 Kev should have the same L dependence as the number density. Peak number densities in the storm time belt vary with the magnetic activity. Frank (1967) reports a measurement of the peak number density of 8 $\pm$ 2 protons/cm^3 at magnetic latitude 27° for the main phase of a moderate storm. A

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slightly larger value is expected for the equatorial number density. The electron energy density is only about 25% of the proton energy density in the storm time radiation belt.

The upper regions of the earth's atmosphere are partially ionized by solar radiation and energetic particle bombardment forming a conducting layer known as the ionosphere. The conductivity of the ionosphere must be expressed as a tensor. The components of the conductivity tensor are functions of electron density, collision frequency, gyrofrequency, latitude and local time. Rather than use a complex model of the ionosphere, it will be represented in the model magnetosphere as a perfect conductor.

Ideally a model of the magnetosphere should reflect all of the significant features of the actual magnetosphere but still be simple enough to work with. Models of the geomagnetic field in popular use are formed by a dipole, image dipole and tail current and account for the gross features of the magnetospheric field, the magnetopause and the magnetotail. However, such effects as field inflation due to particle populations are not taken into account.

To facilitate the study of diffusion in the magnetosphere it is best to chose the simplest model possible which adequately describes the physical situation. Thus Nakada and Mead as well as Schulz and Eviatar chose a model which accounts for the magnetopause since they are considering diffusion driven by induced electric fields produced by changes in the magnetopause position. Birmingham, however, chooses a dipole field model since he is considering diffusion driven by an internally generated electric potential. The

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determining factor in choosing a model suitable for treating cavity modes is the ease with which boundary conditions can be specified. This factor eliminates the consideration of numerical models and discourages the use of complex geometries.

The model that will be used for calculating the functional dependence of the cavity modes is roughly equivalent to a tin can filled with a lossy dielectric. The model is best described in terms of the usual cylindrical coordinates (r, θ, z). An azimuthally symmetric axial magnetic field is assumed. The field is taken to be in the z direction and thus its magnitude is a function of r only. The magnetopause is represented by the cylindrical surface, $r = 10\ R_e$ (earth radii). The ionosphere is represented by perfectly conducting endplates at $z = \pm \Lambda\ R_e$. The plasmasphere is an inner core of cold, Maxwellian, electron-proton plasma with T_i (the ion temperature) = T_e (the electron temperature) = 10^4 °K. The storm time belt is represented by a hot, Maxwellian, electron-proton plasma with $T_i = 10^8$ °K.

The plasmapause is assumed to be located at $r = 5.9\ R_e$ and is represented by a sharp decrease in the number density of the cold plasma. The hot ring current plasma number density has a maximum value at 6 R_e. The behavior of the ring current number density on r between the location of the peak number density (6 R_e) and the magnetopause (10 R_e) is given by either an exponential model ($k_1 \exp -(r-6R_e)^2/a^2$) or a power law model ($k_1/(r^2)^p$). The use of these models is discussed in Chapter 5. A schematic representation of the behavior of the ring current and plasmasphere number densities is given in Figure 2.1.

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Figure 2.1 A diagram of the model used to represent the magnetosphere. The model is azimuthally symmetric. The field lines are straight and directed perpendicular to the ionospheres. The ionospheres are represented by perfectly conducting disks at either end of the cylinder. The outer cylindrical surface represents the magnetopause and the inner cylindrical surfaces indicate the location of the plasmapause. The cavity is 2Λ long and 20 R_e in diameter. The schematic dependence of the number densities of the ring current and plasmasphere plasmas on radial distance are given in semi-log plots.

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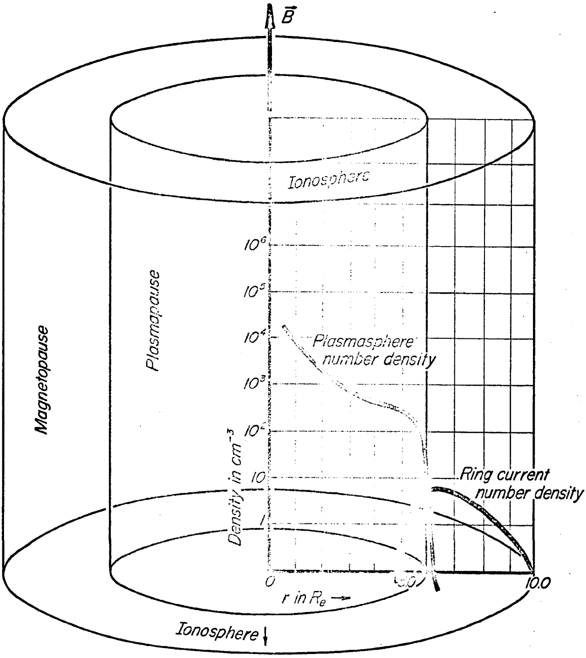

Figure 2.1

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The cylindrical geometry of the model makes it very easy to apply boundary conditions. Since the magnitude of the magnetic field depends upon r only, the gradient drift is in the azimuthal direction only. Since the field lines are straight, there is no curvature drift nor mirror force on the particles. The mirroring motion of particles in the magnetosphere is simulated by considering the ionosphere as perfectly elastic. That is, particles are reflected at $z = \pm \Lambda$ with no loss of energy. The question of whether or not this model adequately represents the essential features of any diffusion processes that might be driven by drift modes in the magnetosphere is best taken up after this simple model has been investigated.

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Chapter 3

The Diffusion Equation

Diffusion in the magnetosphere is assumed to result from the plasma's interaction with wave turbulence present in the plasma. The turbulence in the plasma is assumed to be weak enough that it can be treated as a perturbation effect on the particle's behavior in the absence of turbulence. Changes in the phase space density of particles due to perturbations in the particle's orbits can be expressed by means of a diffusion equation in a manner similar to that used by Birmingham, Northrop and Fälthammar (1967). The actual wave turbulence in the magnetosphere is represented by a statistical model in which the phases of the various Fourier components of the perturbation wave fields are assumed to vary randomly with time. The phase space orbit of a test particle is expressed in terms of the amplitudes and phases of the perturbing wave fields and the diffusion coefficients used in the diffusion equation can then be expressed in terms of the random phase variables. A final expression for the diffusion coefficients is obtained by ensemble averaging.

The diffusion equation will be developed restricting the discussion to the magnetic field of the model magnetosphere and electrostatic oscillations. The development of the equation which is based on Birmingham, Northrop and Fälthammar (1967) is outlined here and the details are given in Appendix A-II. First the magnetic field, $B(r)\,\vec{1}_z$, is expressed in the usual Euler coordinates α and β: $\vec{B}(r) = \vec{\nabla}\alpha \times \vec{\nabla}\beta$. The restriction that the magnetic moment of the particle, μ, be conserved is used to obtain the Hamiltonian (see appendix A-I)

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$$H = p_s^2/2m + \mu B + e\Phi + \frac{e}{c}\alpha\frac{\partial\beta}{\partial t} + p_s\left(\frac{\partial s}{\partial t} + e\,\frac{\vec{E}\times\vec{B}}{B^2}\cdot\vec{\nabla}s\right) + \frac{mc^2E_\perp^2}{2B^2}$$

s is the measure of distance along a field line from a fixed reference plane. The field line is given by the intersection of surfaces of constant α and β. p_s is the momentum conjugate to s. The continuity equation in the phase space (α,β,s,p_s,μ) for the density of guiding centers Q

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial\alpha}(\dot{\alpha}Q) + \frac{\partial}{\partial\beta}(\dot{\beta}Q) + \frac{\partial}{\partial s}(\dot{s}Q) + \frac{\partial}{\partial p_s}(\dot{p}_sQ) = 0$$

can be written as a Liouville equation

$$\frac{\partial Q}{\partial t} + \dot{\alpha}\frac{\partial Q}{\partial\alpha} + \dot{\beta}\frac{\partial Q}{\partial\beta} + \dot{s}\frac{\partial Q}{\partial s} + \dot{p}_s\frac{\partial Q}{\partial p_s} = 0$$

since α, β and s, p_s are canonical variables (see appendix A-I).

If Q, $\frac{\partial Q}{\partial\alpha}$, $\dot{\alpha}$, etc. are considered as random variables, they can be conveniently written in the form $x = \langle x\rangle + \delta x$. The brackets $\langle\ \rangle$ indicate the ensemble average and δ the fluctuating part of the variables. If the perturbations in the particle's phase space orbits during a time interval which is less than or equal to the correlation time of the random process are assumed small compared to the scale size of the average phase space density $\langle Q\rangle$, it can be shown (see appendix A-II) that $\langle Q\rangle$ satisfies a diffusion equation:

$$\frac{\partial}{\partial t}\langle Q\rangle + \sum_{i=1}^{4}\langle\dot{x}_i\rangle\frac{\partial}{\partial x_i}\langle Q\rangle = \sum_{i=1}^{4}\sum_{j=1}^{4}\frac{\partial}{\partial x_i}\left(D_{x_ix_j}\frac{\partial}{\partial x_j}\langle Q\rangle\right)$$

where the notation $x_1=\alpha$; $x_2=\beta$; $x_3=s$; $x_4=p_s$ is used to denote the phase space coordinates. The diffusion coefficients $D_{x_ix_j}$ are given by the expression

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$$D_{x_i x_j} = \left\langle \int_0^t dt' \; \delta\dot{x}_i(\alpha,\beta,s,p_s,t) \; \delta\dot{x}_j(w_1,w_2,w_3,w_4,t') \right\rangle$$

The unperturbed or zero order orbits of the particle in phase space as a function of the parameter $t'-t = \tau$ which measures time "backwards" along the orbit are denoted by w_i. The w_i must satisfy certain conditions:

at $t' = t$ $\quad w_1 = \alpha;\ w_2 = \beta;\ w_3 = s;\ w_4 = p_s$

and $\dot{w}_1 = \langle\dot{\alpha}\rangle;\ \dot{w}_2 = \langle\dot{\beta}\rangle;\ \dot{w}_3 = \langle\dot{s}\rangle;\ \dot{w}_4 = \langle\dot{p}_s\rangle$

The notation $\dot{w}$ indicates differentiation with respect to the time argument. Since the fluctuations $\delta\dot{x}$ can be expressed in terms of the Hamiltonian (for example, $\delta\dot{\alpha} = -\frac{c}{e}\left[\frac{\partial H}{\partial\beta} - \frac{\partial}{\partial\beta}\langle H\rangle\right]$), the diffusion coefficients can be expressed in terms of the Hamiltonian. Before proceeding with the calculation of the diffusion coefficients, a statistical model of the turbulence must be developed.

Suppose we are considering the interaction of a group of particles with a coherent electrostatic plane wave of wave length $\lambda = 2\pi/k$ and frequency ω. Suppose also that the particles have a velocity distribution $g(v)$ and $g(\omega/k) \neq 0$. Then some of the particles will undergo substantial $\vec{E}\times\vec{B}$ drifts since they see the wave as an almost constant electric field. After a time T, the number of particles that undergo a substantial drift is roughly $g(\omega/k)\ \lambda/T$. So the number of particles that see the wave as constant becomes arbitrarily small after a sufficiently long time. As a result essentially no particles, on the average, are transported by the wave. Turbulence effectively puts an upper limit T_c on T, so that even for long times

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a group of roughly $g(\frac{\omega}{k})\,\frac{\lambda}{T_c}$ particles interacts with the wave and is transported by it. This effect can be reproduced by using a statistical model to represent the turbulence. A common practice is to assume that the phase of the wave is changing randomly. An example of a representation of turbulence with a random wave phase statistical model is given in a discussion of plasma heating by stochastic electric fields by Puri (1966). Puri assumes that the wave phase can change by an arbitrary amount at intervals of time which are randomly distributed. The statistical model employed here will be similar in that the wave phase will be considered a random variable but the phase will be assumed to change only a small amount in a small time.

It will be shown later that the electric potential associated with the electrostatic fluctuations in the model magnetosphere can be written as

$$\phi(r,\theta,z,t) = \sum_{\ell,m,n} A(\ell,m,n,r)\exp i(\omega t+\ell\theta-kz+\psi(\ell,m,n)) \qquad 3\text{-}1$$

$k = \frac{\pi m}{2\Lambda}$ is the parallel wave number, $\omega = \omega(\ell,m,n)$ is the wave frequency and $\psi(\ell,m,n)$ is the phase of the wave. It is convenient to introduce the shorthand notation

$$\sum_L f(L) \text{ and } g(L) \quad \text{for} \quad \sum_{\ell,m,n} f(\ell,m,n) \text{ and } g(\ell,m,n)$$

The phase $\psi(L)$ is considered a random variable. The random process can be completely described by giving explicit expressions for the hierarchy of probability distributions

$$W_n(\psi_1(1),\psi_1(2),\psi_1(3)\cdots t_1;\ \psi_2(1),\psi_2(2),\psi_2(3)\cdots t_2;$$
$$\psi_n(1),\psi_n(2),\psi_n(3)\cdots t_n)$$

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(see Wang and Uhlenbeck, 1945 as a general reference). Again it is convenient to introduce a short hand notation and $R\psi_m$ will be used to denote

$$\psi_m(1), \psi_m(2); \psi_m(3) \cdots \psi_m(\infty)$$

It will be assumed, as is usual, that the entire process is stationary and Markoff. So the heirarchy of probability densities can be expressed in terms of the second order probability density $W_2(R\psi_1, t_1;\ R\psi_2, t_2)$. Also, it is assumed that the second order probability density for the whole process can be written as a product of the second order probability densities for individual wave components:

$$W_2(R\psi_1, t_1;\ R\psi_2, t_2) = \prod_i W_2(\psi_1(i),\ t_1; \psi_2(i), t_2) \qquad 3\text{-}2$$

$W_2(\psi_1(L), t_1; \psi_2(L), t_2) d\psi_1(L)\ d\psi_2(L)$ is the probability that the phase of the L^{th} component has a value between $\psi_1(L)$ and $\psi_1(L)+d\psi_1(L)$ at the time t_1 and a value between $\psi_2(L)$ and $\psi_2(L)+d\psi_2(L)$ at time t_2. Physically the assumption 3-2 means the particles see the phase of each component independently of any other. Let $\psi_1, \psi_2, \psi_3, \cdots$ denote the phase angles of any particular component at $t_1, t_2, t_3 \cdots$ respectively. Then it follows from assumption 3-2 and the assumption that the entire process is Markoff that the process for an individual wave component is also Markoff, since

$$W_3(\psi_1, t_1;\ \psi_2, t_2;\ \psi_3, t_3) = \underline{\int} W_3(R\psi_1, t_1; R\psi_2, t_2; R\psi_3, t_3)$$

$$= \underline{\int} W_2(R\psi_1, t_1; R\psi_2, t_2) P_2(R\psi_2, t_2 \mid R\psi_3, t_3)$$

$$= W_2(\psi_1, t_1; \psi_2, t_2) P_2(\psi_2, t_2 \mid \psi_3, t_3)$$

where $\underline{\int}$ denotes integration over the phase variables of all components except ψ_1, ψ_2, ψ_3 and $P_2(|)$ is the second order conditional

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probability density. $P(\psi_1,t_1|\psi_2,t_2)d\psi_2$ is the probability that the phase has a value between ψ_2 and $\psi_2+d\psi_2$ at time t_2 given that the phase had the value ψ_1 at time t_1. A similar relation holds for all $W_n(\psi_1,t_1 \cdots \psi_n,t_n)$ and thus the entire heirarchy of probability densities for an individual wave component can be expressed in terms of $W_2(\psi_1,t_1;\psi_2,t_2)$ and so the process is Markoff. The entire process can be described once the second order conditional probability density $P_2(\psi_1,t_1;\psi_2,t_2)$ is given. Since $W_2(\psi_1,t_1;\psi_2,t_2)$ is Markoff, $P_2(\psi_1,t_1;\psi_2,t_2)$ must satisfy the Smoluchowski equation as well as the condition

$$P_2(\psi_1,t_1;\psi_2,t_2) \rightarrow \delta(\psi_2-\psi_1) \text{ as } t_2 \rightarrow t_1 \qquad 3\text{-}3$$

If it is assumed that in a time interval of length δt the phase can change by only 0 or $\pm\,\Delta$ then the Smoluchowski equation becomes a familiar partial differential equation (Wang and Uhlenbeck, 1945)

$$\frac{\partial}{\partial T} P_2(\psi_1,t_1 \mid \psi_2,t_2) = D \frac{d^2}{d\Psi^2} P_2(\psi_1,t_1 \mid \psi_2,t_2)$$

where $t = t_2-t_1$ and $\Psi = \psi_2-\psi_1$ and D is given by $\lim\limits_{\substack{\Delta\to 0\\ \delta t\to 0}} \frac{\Delta^2}{2\delta t}$

The solution to the above equation which satisfies condition 3-3 is

$$P_2(\psi_1,t_1 \mid \psi_2,t_2) = \frac{1}{(4\pi Dt)^{1/2}} \exp - \frac{\psi^2}{4Dt} \qquad 3\text{-}4$$

Equation 3-4 is sometimes called the random walk distribution because the solution to the random walk problem in which a step of fixed size is taken at fixed intervals of time goes over into 3-4 in the continuous case. Using the random walk as an analogy, the phase is considered to change by $\Delta\psi$ every step and each step occurs every $1/\kappa$ seconds and $D = \frac{(\Delta\psi)^2\kappa}{2}$. So κ will be called the dephasing frequency and is an indicator of the strength of the turbulence.

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$W_1(\psi_1,t_1)d\psi_1$ is the probability that the phase ψ has a value between ψ_1 and $\psi_1+d\psi_1$ at time t_1. If the phase is evenly distributed between 0 and 2π

$$W_1(\psi_1,t_1) = 1/2\pi$$

and
$$W_2(\psi_1,t_1;\psi_2,t_2) = W_1(\psi_1,t_1)P_2(\psi_1,t_1 \mid \psi_2,t_2)$$
$$= \frac{1}{(2\pi)^{3/2}\Delta\psi\sqrt{\kappa t}} \exp \frac{-\psi^2}{2(\Delta\psi)^2\kappa t} \qquad 3\text{-}5$$

In the expression 3-5 the phase at t_2 can have any value between $-\infty$ and ∞. Usually the phase angle of a wave is restricted between 0 and 2π. An expression for W_2 which is appropriate for this restriction is

$$W_2(\psi_1,t_1;\psi_2,t_2) = \frac{1}{(2\pi)^{3/2}\Delta\psi\sqrt{\kappa t}} \sum_{n=-\infty}^{\infty} \exp \frac{-(\psi+2n\pi)^2}{2\kappa t(\Delta\psi)^2} \qquad 3\text{-}6$$

When ensemble averaging the expression 3-5 is more convenient to use and gives the same results as 3-6 as long as the integration over ψ_2 is carried out before the integration over ψ_1. Using 3-5 and 3-2 the second order probability density for the entire process can be written

$$W_2(R\psi_1,t_1;R\psi_2,t_2) = \frac{1}{(2\pi)^{3/2}} \prod_L \frac{\exp(-(\psi_2(L)-\psi_1(L))^2/2\Delta\psi^2(L)\kappa(L)t)}{\Delta\psi(L)\sqrt{\kappa(L)t}} \qquad 3\text{-}7$$

The diffusion coefficient $D_{\alpha\alpha}$ for the model magnetosphere can now be calculated using 3-7 as the probability density for the ensemble. The details of the calculation are given in appendix A-III. For particles whose bounce periods are somewhat larger than the inverse of the dephasing frequency the diffusion coefficient $D_{\alpha\alpha}$ can be written for a steady state solution as (see equation A-III-18)

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$$D_{\alpha\alpha} = \frac{c^2}{2} \sum_{\ell,m,n} \ell^2 |C(\ell,m,n,\alpha)|^2 \, G(\ell,m,n,v_{\parallel},s) \qquad 3\text{-}8$$

The coefficients $C(\ell,m,n,\alpha)$ are the Fourier coefficients of the electric potential defined by

$$\Phi(\alpha,\beta,s,t) = \sum_{\ell,m,n} C(\ell,m,n,\alpha) \exp i(\omega t+\ell\beta-ks+\psi)$$

The $G(\ell,m,n,v_{\parallel},s)$'s are complicated functions of the initial conditions of the particle. The position of a particle on a field line is an important factor in the diffusion coefficient because the particle will see an abrupt change in the phase of the wave when it reflects from the end plates in the model. Also, since the dephasing frequency is assumed to be higher than the bounce frequency, the particle will have "seen" the wave at all phase angles in a few bounce periods. Thus the first one or two bounces of the particle will make the largest contribution to the diffusion coefficient. $G(\ell,m,n,v_{\parallel},s)$ is given by

$$G(\ell,m,n,v_{\parallel},s) = \frac{\kappa'(\ell,m,n)}{\kappa'^2(\ell,m,n)+\omega_-^2(\ell,m,n)} \left\{ 1 - \frac{2kv_{\parallel}e^{-\kappa' t_o}}{\kappa'^2+\omega_+^2} \left[2\omega(\cos(\omega_- t_o) - e^{-\kappa' T}\cos(\omega_+ T+\omega_- t_o) + \left(\frac{\omega_+\omega_- - \kappa'^2}{\kappa'}\right) [\sin(\omega_- t_o) - e^{-\kappa T}\sin(\omega_+ T+\omega_- t_o)] \right] \right\}$$

where $T = 2\Lambda/|v_{\parallel}|$ is the particle's bounce period, $t_o = \Lambda/|v_{\parallel}|-s/v_{\parallel}$ is the time to the first reflection of the particle from the ionosphere, and

$$\kappa'(\ell,m,n) = \kappa(\ell,m,n)\Delta\psi^2(\ell,m,n)/2$$

$$\omega_+(\ell,m,n) = \omega(\ell,m,n) + \ell\langle\omega_D(\alpha)\rangle + kv_{\parallel}$$

$$\omega_-(\ell,m,n) = \omega(\ell,m,n) + \ell\langle\omega_D(\alpha)\rangle - kv_{\parallel}$$

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The dephasing frequency κ and the phase change $\Delta\psi$ have been written so that they can be different for each wave component. Equation 3-8 is given under the assumption that $\kappa'(\ell,m,n) > T^{-1}$ for all ℓ,m,n.

The diffusion coefficients can be expressed in terms of the amplitude of the fluctuations of the potential on the magnetopause once the electric potential in the model magnetosphere is calculated. The calculation of the electric potential will be the subject of the next few chapters.

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CHAPTER 4

DRIFT MODES IN THE MODEL MAGNETOSPHERE

The theoretical investigation of electrostatic oscillations in plasmas with number and temperature gradients has been developed to the point where general magnetic configurations can be treated (Krall and Rosenbluth, 1965; Rutherford and Frieman, 1968). The simple magnetic field configuration used in the model magnetosphere allows a differential equation for the electric potential to be developed without resorting to the procedures used in the consideration of general field configurations. A differential equation for the potential associated with electrostatic oscillations in a cylindrical plasma column with an axial, azimuthally symmetric constant magnetic field has been developed by Swift (1967). The method used by Swift to develop the equation can be easily modified under certain restrictions to allow for the magnitude of the field to be a function of radius. The development of the differential equation for the electric potential and the assumptions under which it is derived are given in Appendix B-1.

A differential equation for the electric potential, Φ, is derived by solving the collisionless Vlasov equation and Poisson's equation. The term containing the electric field in the Vlasov equation is treated as a perturbation. The perturbation electric field is assumed to be electrostatic in nature and thus arises from a potential determined by Poisson's equation. The electrostatic acceleration $\frac{e}{m}|\vec{\nabla}\phi|$ is considered to be small compared to the Lorentz acceleration $\frac{e}{m}|\vec{v}\times\vec{B}|$.

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The electric potential is assumed to vary on a scale size larger than the ion gyroradius so that the expansion of equation B-1-6 is valid. The inflation of the magnetic field caused by the diamagnetic current arising from the gradients in the plasma is ignored. The wave frequency is assumed to be between the ion gyrofrequency and $\ell\omega_D$ (ℓ is the azimuthal wave number and ω_D is the azimuthal drift frequency of the ions). The electric potential is expressed as

$$\Phi(r,\theta,z,t) = \sum_{\ell,m} \int d\omega\, A(\ell,m,r,\omega)\, \exp i(\omega t+\ell\theta-kz) \qquad 4\text{-}1$$

Since the ionosphere of the model magnetosphere is represented by a perfectly conducting disk at $z = \pm\Lambda$, k can be written as $\frac{\pi m}{2\Lambda}$ (where m is an integer).

Before giving the differential equation which determines Φ, a discussion of the notation used is helpful in understanding the equation. The subscript i will be used to denote the particle species (i=1 for ions and i=2 for electrons). The subscript j will be used to denote whether the variable refers to the ring current plasma or the plasmasphere plasma (j=1 denotes the plasmasphere and j=2 denotes the ring current). The zero order or unperturbed number density will be denoted by n_{oj} and its derivative with respect to r^2 is denoted by n'_{oj}. A graph of the unperturbed number densities for the plasmasphere and the ring current as a function of radius is given in figure 2.1. The velocity space distribution of both the plasmasphere and ring current plasmas is assumed to be Maxwellian. The plasmas are allowed to have different temperatures parallel and perpendicular to the magnetic field. The parallel temperature of

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the plasma is denoted by $T_{\parallel ij}$ and the perpendicular temperature is given by $T_{\perp ij}$. The parallel thermal velocities of the particles will be given by V_{ij}. The charge, the sign of the charge, the mass and the gyrofrequency of the particles are given by e, ε_i, m_i, and Ω_i, respectively. $<\rho_j^2>$ denotes the ion gyroradius and $<\omega_D>_{ij}$ denotes the azimuthal drift frequency of the particles ($<\omega_D>_{ij} = \varepsilon_i \frac{m_i k_b T_{\perp ij}}{2rB^2} \frac{dB}{dr}$ where k_b denotes Boltzmann's constant). The ratio of the parallel phase velocity of the wave to the parallel thermal ion velocity is an important parameter and is denoted by zeta ($\zeta_{ij} = \frac{\omega}{kV_{ij}}$). $Z(\zeta_{ij})$ is the plasma dispersion function analytically continued from the lower half of the complex ω plane.

The differential equation for the electric potential with $T_{\parallel ij} = T_{\perp ij} = T_{ij}$ is given by equation 4-2. The equation appropriate for $T_{\parallel ij} \neq T_{\perp ij}$ is given by equation B-1-12.

$$\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} A(\ell,m,r,\omega) = \left[\frac{\ell^2}{r^2} + \frac{N(r)}{D(r)}\right] A(\ell,m,r,\omega) = U(r)A(\ell,m,r,\omega)$$

$$N(r) = k^2 + 4\pi e^2 \sum_i \sum_j \Big\{ \frac{2\ell\varepsilon_i n'_{oj}}{m_i \Omega_i k V_{ij}} \left(Z(\zeta_{ij}) + \frac{\ell<\omega_D>_{ij}}{kV_{ij}} Z'(\zeta_{ij})\right)$$

$$+ \frac{n_{oj}}{k_b T_{ij}} \left(1 + \zeta_{ij} Z(\zeta_{ij}) + \frac{\ell<\omega_D>_{ij}}{kV_{ij}} \zeta_{ij} Z'(\zeta_{ij})\right)\Big\} \qquad 4\text{-}2$$

$$D(r) = 1 - 4\pi e^2 \sum_j \frac{<\rho_j^2>}{4} \left[\frac{2\ell}{m_i \Omega_i k V_{ij}} \left(Z(\zeta_{ij}) + \frac{2\ell<\omega_D>_{ij}}{kV_{ij}} Z'(\zeta_{ij})\right)\right.$$

$$\left. + \frac{n_{oj}}{k_b T_{ij}} \left(1 + \zeta_{ij} Z(\zeta_{ij}) + \frac{2\ell<\omega_D>_{ij}}{kV_{ij}} \zeta_{ij} Z'(\zeta_{ij})\right)\right]$$

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Equation 4-2 can be put in the form of a familiar differential equation, the Schrodinger equation, by the transformation $r \rightarrow e^x$. The singularity in equation 4-2 at $r = 0$ is removed to $-\infty$ by the transformation and 4-2 becomes

$$\left\{\frac{d^2}{dx^2} - (\ell^2 + \bar{U}(x))\right\} \bar{A}(\ell,m,\omega,x) = 0 \qquad 4\text{-}3$$

where $\bar{U}(x) = \frac{N(e^x)}{D(e^x)} e^{2x}$. If $\frac{d}{dx} \ln \bar{U}(x)$ is small for all x, the W.K.B. or short wave length approximation may be used to solve 4-3. The solutions to equation 4-3 obtained by using the W.K.B. method and a brief discussion of their behavior when $[\ell^2 + \bar{U}(x)]$ changes phase or goes through zero is given in appendix B-II. The solutions to 4-2 obtained by using the short wave length approximation have the form

$$A(\ell,m,r,\omega) = A_o(\ell,m,\omega) \frac{1}{r\sqrt{q}} \exp \int q \, dr \qquad 4\text{-}4$$

where $q^2 = -U(r)$. Equation 4-4 actually gives two solutions to 4-2 since the phase of q has two values.

The behavior of the solutions to equation 4-2 is determined by the behavior of U(r). The variations in U(r) for a given ℓ and m are determined mainly by the changes in n_o and n_o'. Except for the inner edge of the ring current and at the plasmapause n_o and n_o' change significantly only over distances on the order of 1 R_e. In the derivation of equation 4-2 it was assumed that the wave length of the solution was several times larger than the ion gyroradius. In the magnetosphere near the magnetopause the ion gyroradius can be as large as .02 R_e. Since in the short wave length approximation Φ is assumed to vary more rapidly than U(r) the solutions to equation 4-2 can be expected to have wave lengths in the range of about .05 to .5 R_e.

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A study of the behavior of $U(r)$ is facilitated by writing $U(r) = A+iB$ where A and B are real functions of r. The function $q(r) = (-U(r))^{1/2}$ can be written as $q = C+iD$ (where C and D are also real functions of r) and

$$C = \frac{\pm 1}{\sqrt{2}} \left(\sqrt{A^2+B^2} - A\right)^{1/2} \; ; \quad D = \frac{\mp B}{\sqrt{2}\left(\sqrt{A^2+B^2} - A\right)^{1/2}} \qquad 4\text{-}5$$

Fractional powers of positive real quantities are taken to be positive. The sign notation means that if the plus sign is chosen for C, then the minus sign must be used for D and vice versa. From the form of the solutions given by equation 4-4, it is apparent that C determines the radial wave length of the solutions and D determines the "skin depth" or radial damping factor.

From equation 4-2, it can be seen that for small values of ℓ and for typical values of k (so that kV_{12} is on the order of $10^{-2} - 10^{-1}$) the magnitude of U is roughly

$$\frac{T_{\shortparallel 12}}{T_{\shortparallel 22}} \, \frac{1}{\langle\rho_2^2\rangle}$$

in the frequency range $0 < \omega < 10\ kV_{12}$. In order to satisfy the requirement that the scale size of Φ be larger than an ion gyroradius, $\langle\rho_i\rangle$, (so that equation 4-2 is a valid description of the behavior of the potential) ℓ, m and ω must be varied so that the magnitude of U is small enough that C and D, as given by equation 4-5, satisfy the requirements $C\langle\rho_i\rangle \le 1$ and $D\langle\rho_i\rangle \le 1$. This implies that an ℓ, m and ω must be chosen such that $U(r) = 0$ at, say r_o, so that hopefully $C\langle\rho_i\rangle$ and $D\langle\rho_i\rangle$ will be less than 1 for a wide range of r about r_o.

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Thus it is desirable to know which values of ℓ, m and ω give U a value of zero at a given r and how the magnitude of U changes as r is varied. Also from equation 4-2 it can be seen that, for k^2 small enough that it is negligible in the expression for N(r), the functions N(r) and D(r) have the same values at ℓ, m, ω and $n\ell$, nm, $n\omega$ (where n is some integer small enough that n^2k^2 is still negligible compared to the other terms of N(r)). So, if at some $r = r_o$, U has a zero for $\ell = \ell_o$, $m = m_o$, and $\omega = \omega_o$ then it also has a zero at $n\ell_o$, nm_o, $n\omega_o$ as long as $\frac{n^2\ell_o^2}{r_o^2}$ is negligible in comparison with $\frac{N(r_o)}{D(r_o)}$.

It is not always possible to chose an ℓ, m and ω such that for the $Im(\omega)=0$, $U(r) = 0$ at a given r since ℓ, m and ω are limited by the assumptions made in deriving equation 4-2. The limitations on ℓ arise from the requirement that the perpendicular wave length be larger than the ion gyroradius and $\ell\langle\omega_D\rangle < \omega$. The perpendicular wave number is $k_\perp = \frac{\ell}{r}$ so the restriction $k_\perp\langle\rho_i\rangle < 1$ puts an upper limit on the magnitude of ℓ. For small values of k (i.e. large values of Λ and small values of m) the requirement $\omega^2 \gg \ell^2\langle\omega_D\rangle^2$ can be more restrictive on ℓ than the requirement $\frac{\ell}{r}\langle\rho_i\rangle < 1$. For a given k and a fixed parallel ion temperature ζ is limited by the requirement that $\omega^2 \ll \Omega_i^2$ (this is a necessary restriction on the wave frequency if the magnetic moment is to be conserved). Since $\zeta = \frac{\omega}{kV}$ this requirement becomes $\zeta^2 \ll (\Omega_i kV)^2$. So the largest value of ζ for which U is investigated depends upon the magnitude of k. A lower limit is put on k by the value chosen for 2Λ which is the distance between the ionospheres in the model magnetosphere. 2Λ should be a characteristic

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field line length for the magnetosphere and thus is a difficult parameter to choose.

Solutions to equation 4-2 which satisfy the assumptions made in deriving it are sought by looking for zeros of U at fixed values of r. The restrictions on the parameters ℓ, m and ω limit the region of parameter space which must be investigated in attempting to find a zero of U. The methods used to find the zeros of U and the behavior of U as a function of r, ℓ, m, ω are discussed in the next chapter.

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CHAPTER 5

The Solutions

The diffusion of protons in the magnetosphere by electrostatic waves requires that the electric potential given by the solutions to equation 4-2 yield substantial electric fields throughout most of the ring current region. Using the W.K.B. solutions to equation 4-2 this requires that D (the imaginary part of q(r)) be small (on the order of 1/several earth radii). From equation 4-5 it can be seen that the requirement that D be small can be met by making both A, the Re(-U(r)), and B, the Im(-U(r)), small; or B small and A negative. The differential equation 4-2 is an adequate description of the electric potential only if the potential has perpendicular wave lengths, in both the azimuthal and radial directions, several times larger than the ion gyroradius. Thus both A and B must be small enough to satisfy this requirement. As discussed at the end of Chapter 4, the most likely regions of the complex frequency plane in which the potential is adequately described by equation 4-2 is near the zeros of U(r). For these reasons, it is necessary to find values of the parameters ω, $k = \frac{\pi m}{2\Lambda}$, and ℓ for which U(r) is zero.

Since U(r) is such a complicated function of r, ω, ℓ, and m (see equation 4-2) a computer program was used to find the zeros of U. A zero of U is sought in the complex zeta ($\zeta = \frac{\omega}{kV}$) plane at fixed values of r, ℓ, and m by varying the value of ω. The program is designed to test the values of U at points of a grid set up to cover a certain range of the complex zeta plane. Since the magnitude

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of U can vary considerably in the zeta plane, the method used to find a zero of U was to look for a simultaneous sign change in both the real and imaginary parts of U from one grid point to the next. When a simultaneous sign change is detected for some region of the zeta plane, the grid is scaled down to the size of an original grid square and the area of the zeta plane where the sign change was detected is searched using this smaller grid. The process can be repeated until the zero of U is located to within any desired accuracy. Once a zero of U is found for a particular value of the wave frequency the behavior of U as a function of r is obtained by using another program. The details of the computer programs, employing an exponential number density model for the ring current and the expression for U(r) given by equation B-I-12, are given in appendix C.

The solutions to equation 4-2 depend upon the real and imaginary parts of the wave frequency. The diffusion process in the magnetosphere is hypothesized to be a steady state (in time) process. The potential fluctuations at the magnetopause are assumed to be the energy source for the process and will be denoted by $\Phi_B(\theta,z,t)$. Thus the initial conditions of the problem will not be important and the solutions to 4-2 which are of interest will be those with purely real frequency. Equation 4-2 has two solutions which can be categorized as growing and decaying. The decaying solution has been chosen as the proper solution for two reasons. First, since the plasmasphere is a cold dense plasma it can be expected to act like a good conductor and the electric field should decay as the plasmapause is crossed. Second

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the decaying solution also yields a decrease in the wave energy flux as r decreases, which is consistent with the energy source for the process being at the magnetopause. The boundary condition imposed upon the solution is

$$\Phi(r=r_m,\ \theta,\ z,\ t) = \Phi_B(\theta,z,t)$$

where r_m denotes the position of the magnetopause. Writing

$$\Phi_B = \sum_{\ell',m'} \int d\omega\ B(\ell,m,\omega)\ \exp i(\omega t+\ell'\theta-m'\pi z/2\Lambda)$$

this requires, using the expression for $\Phi(r,\theta,z,t)$ from equation 4-1, that $A(r_m,\ell,m,\omega) = B(\ell,m,\omega)$. Using the W.K.B. solutions given by equation 4-4, this requirement becomes

$$A_o(\ell,m,\omega)\ \frac{1}{r_m\sqrt{q(r_m)}} = B(\ell,m,\omega)$$

where the integral in the solutions has been written

$$\int_r q(r)dr = \int_r^{r_m} q(r)dr$$

So

$$A(\ell,m,\omega,r) = B(\ell,m,\omega)\ \frac{r_m\sqrt{q(r_m)}}{r\sqrt{q(r)}}\ \exp i \int_r^{r_m} q(r)dr$$

where $B(\ell,m,\omega)$ is the Fourier coefficient of the boundary fluctuations.

The first attempt to find cavity modes that exist in the model magnetosphere was made using the exponential model of the ring current number density as shown in Figure 5.9 and $T_{\parallel 22} = T_{\perp 22} = .3\times10^{8}\,{}^{\circ}K$. The field line length (2Λ) and the behavior of the magnetic field magnitude as a function of r are discussed below. In deriving the equations that govern particle motion in the magnetosphere, the effect of the particle currents on the static permanent magnetic field were ignored. Actually, of course,

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the number gradients and temperature gradients in the plasma create diamagnetic currents which tend to inflate the geomagnetic field. The effect of field inflation and boundary currents on the geomagnetic field is to change the dependence of the magnetic field magnitude from $1/r^3$ for a dipole field to something more like $1/r^2$ in the ring current region of the magnetosphere (see figure 5.1). In the following calculations the magnetic field magnitude was chosen to vary as $1/r^2$. The field at the magnetopause was taken to be 100 gamma (10^{-3} gauss).

The field line length of a dipole field line is about 3.5 times the equatorial crossing distance. So at the location of the maximum number density of ring current particles (6 R_e) a dipole field line has a length of about 21 R_e and at the magnetopause (10 R_e) the field line has a length of about 35 R_e. Since the geomagnetic field deviates from a dipole field, the maximum field line length is probably somewhat larger than 35 R_e and will arbitrarily be chosen to be 50 R_e. Thus the minimum value of $k = \frac{\pi m}{2\Lambda}$ is 10^{-10} cm^{-1} for m=1.

For the magnetic field used in the calculations the largest ion gyroradius occurs at the magnetopause and is 1.34×10^7 cm. So, in order for the azimuthal wave length ($\frac{2\pi r}{\ell}$) to be sufficiently larger than the ion gyroradius at the magnetopause, ℓ must be less than 10^3. Since the perpendicular temperature of the plasma is taken to be independent of r, the azimuthal drift frequency of the particles ($\omega_D = \varepsilon \frac{c}{e} \mu \frac{1}{rB} \frac{dB}{dr}$) is independent of r for a magnetic field with a $1/r^2$ dependence. The azimuthal drift frequency for ring current protons with velocities equal to the thermal velocity ($\sqrt{\frac{2k_b T_\perp}{m}}$) is

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Figure 5.1 The magnitude of the magnetic field of the model magnetosphere is taken to vary as $1/r^2$ rather than $1/r^3$ (as for a dipole field magnitude). This figure illustrates that in the ring current region of the magnetosphere the magnetic field does behave more like $1/r^2$ than $1/r^3$. The measured field was obtained from a satellite pass almost along the earth-sun line (after Freeman, 1964).

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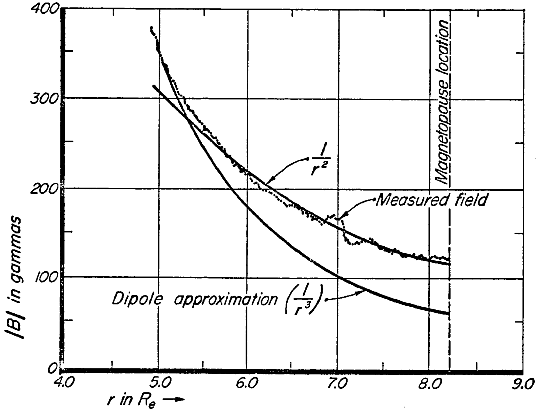

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4.28×10^{-5} Hz. The smallest ion gyrofrequency occurs at the magnetopause and is 9.6 Hz.

The results of the calculations for an exponential number density indicate that no solutions exist to equation 4-2 which have substantial electric fields throughout the ring current region of the magnetosphere. For example with $m = 5$ ($k = .5 \times 10^{-9}$) the function U(r) does not have any zeros for $6\ R_e < r < 10\ R_e$, $\ell < 10^3$ and ω real ($\ell\langle\omega_D\rangle < \omega < \Omega_i$). The values of U(r) and q(r) shown for $\ell = -1$ and $\zeta = 1.6$ in figure 5.2 are typical of the behavior of these functions in the ring current region of the magnetosphere. If it is assumed that equation 4-2 correctly describes the electric potential in the magnetosphere, then this means the ring current plasma acts as a very good conductor for waves of this frequency and wave length. However, the assumptions made in deriving the equation are not satisfied in this case. Thus it can only be concluded that no solutions with perpendicular wave lengths larger than the ion gyroradius exist for these parameter values.

A zero of U can be found for real values of the frequency with $m = 1$. The zeros of U occur near the magnetopause and an example is shown in figure 5.3. The differential equation 4-2 adequately describes the behavior of the potential (i.e. $C\langle\rho_i\rangle < 1$ and $D\langle\rho_i\rangle < 1$) only over a small range of r values (8.5 - 10 R_e). The solutions obtained, therefore, adequately describe the potential only over a local region. However, since the solutions (as shown in figure 5.6a) are confined to the region where equation 4-2 adequately describes the

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Figure 5.2 This figure illustrates the typical behavior of the functions U(r) and q(r) (defined in chapter 4) for the exponential number density model and values of m greater than 1. The values of the parameters used to obtain the functions shown were $\ell = -1$, $\zeta = \omega/kv_{\parallel} = 1.6$, and $m = 5$. The angular wave length is given by $2\pi r/\ell$ and m gives the number of antinodes of the electric potential along the field line. Note that the functions C and D are too large to satisfy the criteria given for the validity of equation 4.2 ($C\langle\rho_i\rangle < 1$ and $D\langle\rho_i\rangle < 1$). In the model magnetosphere the plasmapause is assumed to be at 5.9 R_e, the peak ring current density at 6 R_e and the magnetopause at 10 R_e. Thus this figure illustrates the behavior of the functions across the region of the magnetosphere dominated by the ring current plasma.

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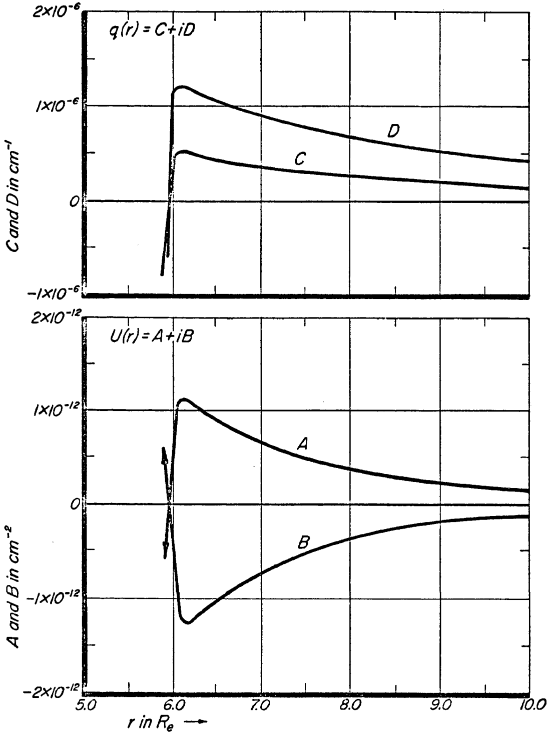

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Figure 5.3 This figure illustrates the behavior of the functions U(r) and q(r) for parameter values for which a turning point of U was found near the magnetopause. The exponential number density model of the ring current was used. The parameter values used were ℓ = -100, m = 1, and ζ = 1.52. The W.K.B. solution for this case is shown in figure 5.6a. The unit on the ordinate scale is changed to a larger value at 9 R_e so that the behavior of the function at smaller r values can be easily illustrated. In the plot of U it is changed from $.5 \times 10^{-14}$ to 1×10^{-12} and for the plot of q it is changed from 10^{-7} to 10^{-6}.

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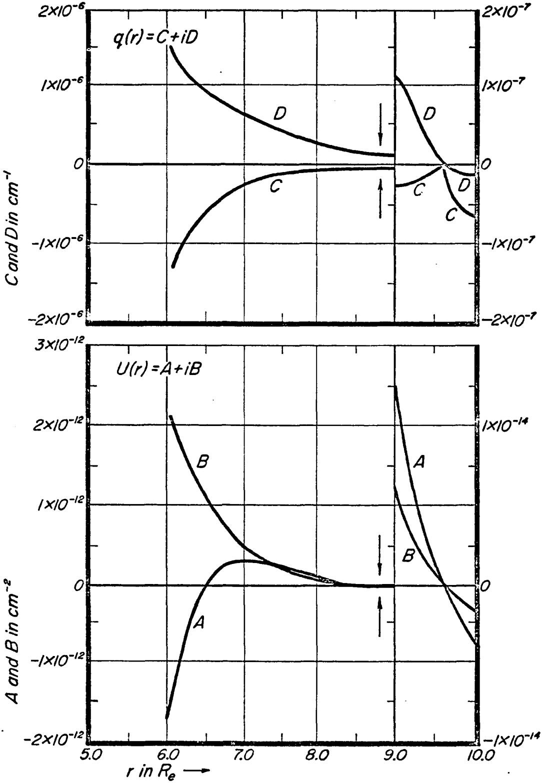

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potential they can be considered to be valid. The solutions of this type will not produce any significant diffusion of particles because these solutions produce substantial electric fields only over a localized area of the magnetosphere and are negligible throughout most of the ring current region.

The values of the parameters which result in a zero of U using the exponential number density model can be used to determine a number density model for which the solutions to 4-2 will produce diffusion throughout most of the ring current region. The zero of U(r) for the case shown in figure 5.3 occurs because the real part of N(r) due to the electrons (i.e. the $\frac{n_o}{k_b T_\parallel}$ $(1 + \zeta_e Z(\zeta_e))$ term) plus $\frac{\ell^2}{r^2}$ is balanced by the real part of N(r) due to the

$$\frac{2\ell n_o'}{m\Omega kV} \; Z(\zeta + \frac{\ell\langle\omega_D\rangle}{kV})$$

term of the ions. The imaginary part of N(r) due to the n_o and n_o' ion term and the n_o electron term is compensated for by the n_o' electron term. The solutions obtained using the exponential number density model are confined to a small region of the magnetosphere because the ratio between n_o and $\frac{n_o'}{\Omega}$ changes considerably as r changes. It is shown below that if n_o is chosen to be proportional to $\frac{n_o'}{\Omega}$ then $\frac{N(r)}{D(r)}$ is essentially independent of changes in n_o and the values of $\frac{N(r)}{D(r)}$ change by less than an order of magnitude across the ring current. The requirement that $n_o = \text{constant} \cdot \frac{n_o'}{\Omega}$ can be satisfied by a power law number density model ($n_o = k_1/(r^2)^p$). If $n_o(r)$ is chosen to be a power law and k^2 is assumed to be negligible in the expression for

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$N(r)$ and I is negligible in comparison with the other terms of $D(r)$ then it can be seen from equation 4-2 that

$$\frac{N(r)}{D(r)} \doteq \frac{1}{\langle \rho_i^2 \rangle} [C_1 + iC_2]$$

With the magnetic field varying as $1/r^2$, C_1 and C_2 are independent of r. Since $\frac{1}{\langle \rho_i^2 \rangle} \sim \frac{1}{r^4}$, $\frac{N(r)}{D(r)}$ varies as $1/r^4$. So A, the real part of U, has the form: $A \sim \frac{\ell^2}{r^2} + \frac{C_1'}{r^4}$ and B, the imaginary part of U, has the form: $B \sim \frac{C_2'}{r^4}$. In the ring current from 6 R_e to 10 R_e the real and imaginary parts of $\frac{N(r)}{D(r)}$ change by a factor of only $(10/6)^4 \approx 8$. The two constants in the power law expression for n_o (k_1 and p) are determined by specifying a value of n_o and n_o' at some $r = r_p$. Denoting the specified values of n_o and n_o' as $n_o(r_p)$ and $n_o'(r_p)$ respectively, the exponent p in the power law can be determined from $p = -(n_o'(r_p)/n_o(r_p))r_p^2$.

The solutions to equation 4-2 which will give a substantial potential across the ring current are those with small D ($q = C + iD$). Since $A \sim \frac{\ell^2}{r^2} + \frac{C_1'}{r^4}$ it can change considerably in the ring current region (when $\frac{\ell^2}{r^2} \approx \frac{C_1'}{r^4}$) whereas $B \sim \frac{C_2'}{r^4}$ changes by less than an order of magnitude. The solutions with small D throughout the ring current will be those with small B and somewhat larger negative A. For, in that case, C varies as $\sqrt{A}$ and D as $B/\sqrt{2A}$ (see equation 4-5) and so D is small throughout the ring current region. If, on the other hand, A is positive, C varies as $B/2\sqrt{A}$ and D as $\sqrt{2A}$. In this case D will be large a short distance from the magnetopause since $R_e\left(\frac{N(r_m)}{D(r_m)}\right)$ is on the order of $\frac{\ell^2}{r^2}$. For example, if the values of the parameters

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(ℓ, n_o, n_o', ω, m) near the zero of U shown in figure 5.3 are used along with a power law number density distribution, the function U(r) behaves as shown in figure 5.4 and leads to solutions that are substantial across the entire ring current. The values of the parameters n_o and n_o' near the turning point in the exponential number density model require that p equal 20 for the power law number density. This value of the exponent not only leads to extremely high number densities in the ring current but, if the ionosphere is not perfectly conducting, the power law number density distribution with such a large value of p would be unstable to the interchange (flute) instability (Chang et al., 1965; Swift, 1967). With the magnetic field chosen for the model ($B \sim \frac{1}{r^2}$) the maximum allowable value of p for stability in the magnetohydromagnetic limit with a zero conductivity ionosphere is 2.0 (i.e. $n_o \sim \frac{1}{r^4}$).

A smaller value of p can be used if $n_o'(r_p)$ is reduced or $n_o(r_p)$ is increased. From the behavior of $Z(\zeta)$ shown in figure 5.5 it can be seen that $Z(\zeta + \frac{\ell\langle\omega_D\rangle}{kV})$ will remain nearly constant if ζ is decreased as ℓ is increased. Then n_o' can be decreased and ℓ increased by the same amount and if ζ is decreased sufficiently the values of $R_e(\frac{N}{D})$ and $Im(\frac{N}{D})$ used in the example above will remain unchanged since $\frac{2\ell n_o'}{m\Omega kV}$ and the other terms in $\frac{N}{D}$ remain the same. It should be noted that if ℓ is increased the approximation used in deriving B-1-11 ($Z(\zeta + \frac{\ell\omega_D}{kV}) \simeq Z(\zeta) + \frac{\ell\omega_D}{kV} Z'(\zeta)$) is not adequate and the average over the perpendicular velocities indicated in equation B-1-9 must be performed numerically. It is easy to see from figure 5.5 that for

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Figure 5.4 This is a plot of U(r) and q(r) for a power law number density and parameter values such that B is small and A is small and negative. The units of the ordinate scale are different for A and B and for C and D so that both functions can be plotted together. The ordinate unit for B is 10^{-17} and for A it is 10^{-14}. The ordinate unit for C is 10^{-7} and for D it is 10^{-10}.

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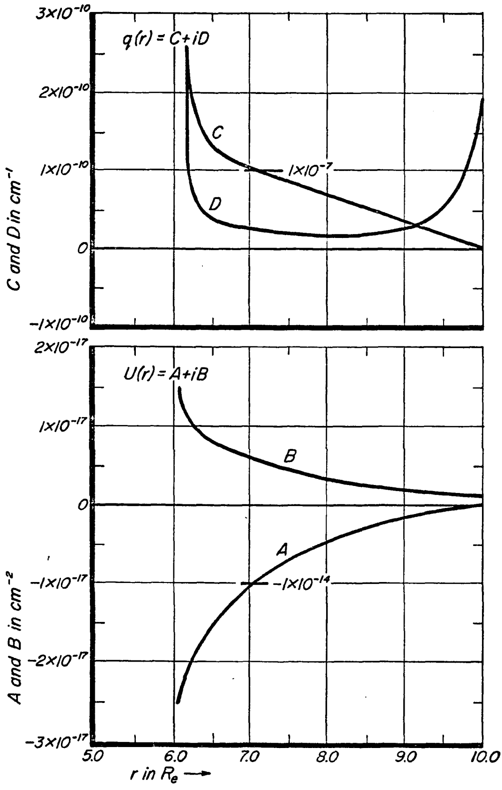

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values of ℓ large enough, ζ must be made so small to keep the imaginary part of $\langle Z(\zeta+\frac{\ell\omega_D}{kV})\rangle_{V_\perp}$ ($\langle\ \rangle_{V_\perp}$ is used to indicate the average over perpendicular velocities) constant that at some value of ℓ the real part of $\langle Z(\zeta+\frac{\ell\omega_D}{kV})\rangle_{V_\perp}$ decreases. Also $\frac{\ell^2}{r^2}$ increases as ℓ is increased. Thus, for some value of ℓ, A will become positive and solutions for larger values of ℓ decay substantially in a few tenths of an earth radius. So there is a maximum value of ℓ (and thus a minimum value of p) for which solutions of the type discussed can be obtained. The maximum value of ℓ which produces a function U(r) as shown in figure 5.4 is 280 (with $\zeta = 1.65$) and thus $p \simeq 7.0$. This value of p is still much too large for the ring current to be stable against the interchange instability.

Slowly decaying radial solutions $A(\ell,m,\omega,r)$ can not be found to equation 4.2 for a power law number density distribution with $p < 7.0$ because A is positive when the imaginary part of U is small and the imaginary part of U is large when A is negative. If a temperature anisotropy ($T_\parallel \neq T_\perp$) is assumed to exist in the ring current, another term is introduced into the expression for $\frac{N}{D}$ (see equation B-1-12). For large ℓ the ion temperature anisotropy term is comparable to the other terms in $\frac{N}{D}$ while the electron anisotropy term is negligible since it is much smaller. Introducing a temperature anisotropy in the ions with $T_{\parallel i} < T_{\perp i}$ will add a term with a negative real part to $\frac{N}{D}$ (assuming ℓ is negative) but it also introduces a fairly large negative contribution to the imaginary part of $\frac{N}{D}$. This would then give only rapidly decaying radial solutions unless ℓ were decreased

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Figure 5.5 This figure illustrates the behavior of the real and imaginary parts of the plasma dispersion function Z and its derivitive Z' as functions of a real argument. Zeta (ζ) is the ratio of the parallel phase velocity of the wave to the ion parallel velocity.

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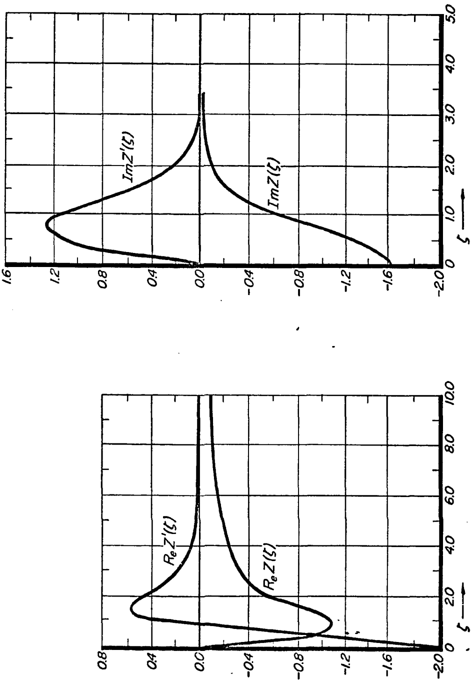

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and n_o' increased, which leads to larger rather than smaller values of p. If $T_{\parallel i} > T_{\perp i}$ the contribution from the ion anisotropy term makes A an even larger positive number. Increasing the parallel electron temperature, on the other hand, allows A to be negative when B is small if n_o' is decreased and n_o is increased (i.e. p is made smaller).

Using a distribution with p = 2 (so that it is just stable to the flute instability) and a number density of $.2/cm^3$ at the magnetopause, solutions of the form shown in figure 5.6b and c can be obtained when the parallel electron temperature is increased by a factor of 4. The dependence of the functions

$$\left| \frac{A(\ell,m,\omega,r)}{r_m \sqrt{q(r_m)}\, B(\ell,m,\omega)} \right|$$

on the azimuthal wave number and the wave frequency ω at r = 8 R_e is shown in figure 5.7. The behavior of the radial solutions shown in figure 5.7 indicates that energy can be transferred into 8 R_e over the frequency range $\sim .5 \times 10^{-2}$ Hz to $\sim 4.5 \times 10^{-2}$ Hz by various drift modes. However, the radial wave length of most of the higher ℓ modes is so short that $C\langle\rho_i\rangle > 1$ (although the wave length is still several gyroradii) and equation 4.2 is a questionable description of the potential in these cases. The behavior of the radial and azimuthal wave lengths of the various modes at 8 R_e is shown in figure 5.8. Solutions for smaller values of ℓ and thus longer azimuthal wave lengths can be obtained for the p = 2 distribution used above by increasing the parallel electron temperature even further.

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Figure 5.6 Radial wave functions for the electric potential:

a. A plot of the W.K.B. solution for the radial part of the electric potential for a case where U has a turning point near the magnetopause and the exponential number density model is used for the ring current. U(r) and q(r) for this case are shown in figure 5.3.

b. A plot of the absolute value of the radial wave function for various values of the parameter zeta ($\zeta = \omega/kv_{\parallel}$) and $\ell = -260$, m = 1 for the case in which the power law number density is used and the electron temperature is increased by a factor of four. The behavior of U(r) and q(r) for $\zeta = 1.875$ in this case and $\zeta = 2.0$ in the case illustrated below is very much like that shown in figure 5.4.

c. A plot of the radial wave function as in 5.6b but for $\ell = -250$.

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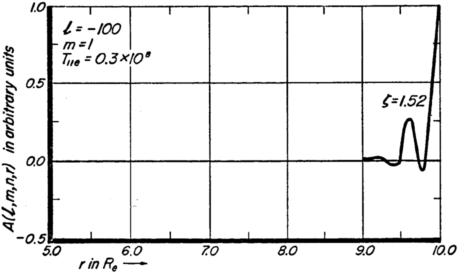

Figure 5.6a

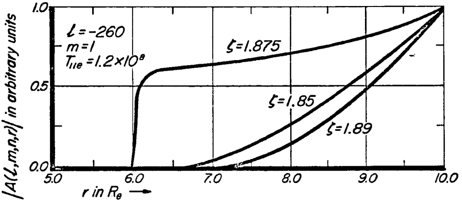

Figure 5.6b

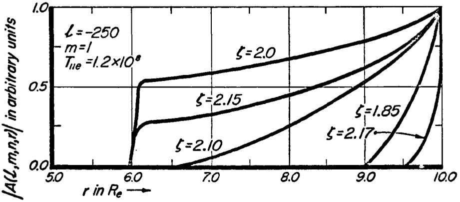

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Figure 5.7 The ratio of the absolute value of the radial wave function for the electric potential at 8 R_e to its value at the magnetopause (10 R_e) is plotted as a function of wave frequency for various values of ℓ and m. The radial wave functions plotted are for the case in which a power law number density was used for the ring current and the parallel electron temperature was increased by a factor of four over measured values. This case was specifically considered so that slowly decaying radial wave functions could be obtained. The potential at the magnetopause for a given ℓ and m is assumed to be constant for all values of ω. Thus the figure illustrates the response of the cavity to an impulse (in time) driving function. m is the number of antinodes of the potential along the field line and ℓ gives the value of the azimuthal or angular wave length $(2\pi r/\ell)$. The function plotted here is denoted by $M(8R_e,\ell,m,\omega)$ in chapter 6.

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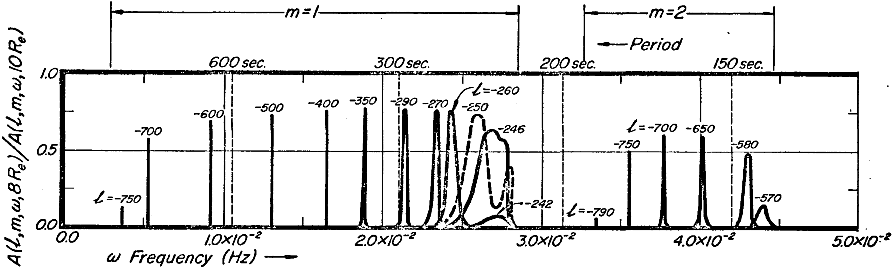

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Figure 5.8 The behavior of the radial and azimuthal wave lengths of the electric potential for the solutions shown in figure 5.7 as functions of ℓ and m for r = 8 R_e. Solutions with wave lengths below the line $C\langle\rho_i\rangle = 1$ fail to satisfy the criteria given in chapter 4 for the validity of the differential equation used to obtain the electric potential ($C\langle\rho_i\rangle < 1$). The relation of the wave lengths to the ion gyroradius at 8 R_e is indicated by the line $4\langle\rho_i\rangle$ which indicates the value of four times the ion gyroradius.

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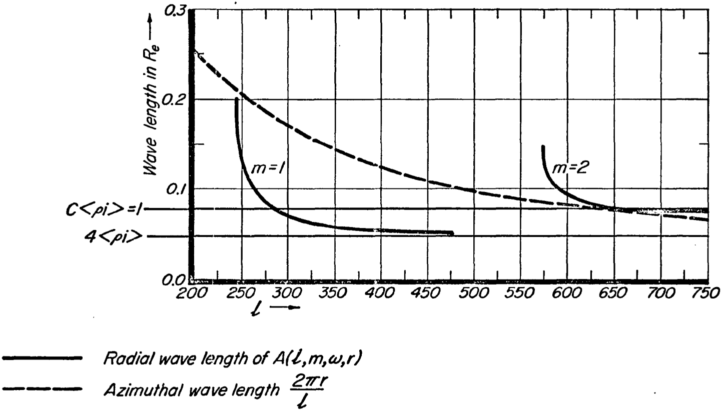

Figure 5.8

In summary, the exponential number density model does not have any solutions to equation 4-2 which will lead to diffusion of ring current protons. The introduction of a power law number density distribution for the ring current allows solutions to equation 4-2 which will diffuse ring current protons but the minimum value of the power law exponent is so large that the distribution would be unstable to the interchange instability. Slowly decaying radial solutions can be found for a stable power law distribution if it is assumed that the parallel electron temperature is much greater than has been measured. Also the power law distributions that are stable to the interchange instability have peak number densities somewhat lower than that assumed in Chapter 2. The exponential number density model and two power law distributions are compared in figure 5.9. The model used to obtain the solutions shown in figures 5.7 and 5.8 has a peak number density of $1.5/cm^3$.

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Figure 5.9 A comparison of the models used to represent the ring current number density. The exponential number density illustrated has a peak value of $7.5/cm^3$ at 6 R_e and a value of $0.1/cm^3$ at the magnetopause (10 R_e). The power law number density model used to obtain the solutions illustrated in figures 5.6b and c, 5.7, and 5.8 was chosen so that the ring current would be stable against the interchange instability and is given by the $1/r^4$ curve. It has a peak value of $1.5/cm^3$ at 6 R_e and a value of $0.2/cm^3$ at the magnetopause. As indicated, a power law of roughly $1/r^9$ must be used to obtain the same peak values and magnetopause values of the number density that were used in the exponential model.

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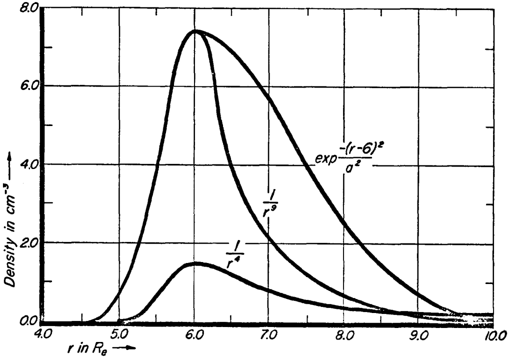

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CHAPTER 6

Results and Conclusions

The results of the calculations described in Chapter 5 indicate that electrostatic oscillations with wave lengths perpendicular to the magnetic field of several ion gyroradii or more and frequencies well below the ion gyrofrequency are not very effective in transporting energy into the magnetosphere. For the exponential number density model of the ring current, energy from the magnetopause could only be transported in a few tenths of an earth radius. To obtain radial wave functions that transport wave energy across most of the ring current region, it was necessary to carefully choose the functional behavior of the plasma number density. In the model used, for example, the use of a power law number density gives slowly decaying radial wave functions. The power law dependence does not arise from any of the physical properties inherent to the magnetosphere but works because of certain properties of the model. The assumptions that the temperature is constant across the magnetosphere, the magnetic field varies as $1/r^2$ and the field line length is constant are necessary in order that a power law number density yield solutions as shown in figure 5.7. These assumptions are not necessarily accurate representations of the magnetosphere. Furthermore, in order to obtain a number density distribution for the ring current which is stable to the interchange instability it was necessary to use a parallel electron temperature four times greater than has been directly measured by satellite. These considerations indicate the model of the magnetosphere used to obtain radial wave functions which yield substantial

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electric fields across most of the ring current is not an accurate representation of the magnetosphere.

For a diffusion mechanism to be important in the substorm process, it must have a diffusion coefficient about equal to the Bohm diffusion coefficient. Bohm diffusion is a diffusion process in which particles are moved on the average of one ion gyroradius in one gyroperiod. So the Bohm diffusion coefficient is approximately $(\frac{V_\perp}{\Omega})^2 \frac{\Omega}{2\pi}$ or $(5\ R_e)^2$/day for 5 Kev particles. It is interesting to note that even if the parameters of the model magnetosphere are specifically chosen to obtain slowly decaying radial wave functions, the diffusion driven by the propagating modes is very ineffective in transporting ring current protons. To illustrate this the value of the radial diffusion coefficient at 8 R_e will be estimated and the diffusion will be approximated by considering a one dimensional diffusion equation for the radial diffusion.

One procedure for obtaining a one dimensional diffusion equation from equation A-II-7 would be to average equation A-II-7 over $V_\perp$, $V_\parallel$, θ, z weighting the averages with the initial distribution function. Then it turns out that the averaged diffusion coefficient $D_{\alpha\alpha}$ is much larger than the other averaged diffusion coefficients. Since the average of $D_{\alpha\alpha}$ over the initial distribution function is not easy to do and only an approximation to the radial diffusion is being considered a rough estimate of the radial diffusion coefficient will suffice. From the basic definition of a diffusion coefficient and from equation A-III-1, $D_{\alpha\alpha}$ can be directly related to D_{rr}

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$$D_{\alpha\alpha} = \left\langle \frac{(\Delta\alpha)^2}{2\Delta t} \right\rangle = r^2B^2 \left\langle \frac{(\Delta r)^2}{2\Delta t} \right\rangle = r^2B^2D_{rr}$$

Since the radial wave functions of Chapter 5 are given as functions of r the diffusion coefficient D_{rr} will be calculated.

$$D_{rr} = \frac{c^2}{2} \frac{1}{r^2B^2} \sum_{\ell,m,n} |A(\ell,m,n,r)|^2 \ell^2 G(\ell,m,n,v_{\parallel},s) \qquad 6\text{-}1$$

Using the expression for $G(\ell,m,n,v_{\parallel},s)$ given in equation A-III-18 and assuming that $\kappa' \approx 10^{-2}$ so that $e^{-\kappa' T}$ is fairly small ($\sim$.05),

$$G(\ell,m,n,v_{\parallel},s) \approx \frac{\kappa'}{\kappa'^2 + \omega_-^2}$$

For the solutions shown in figure 5.7, ω is between $.5 \times 10^{-2}$ Hz and 4.5×10^{-2} Hz and so $\omega - kv_{\parallel}$ can range between 3×10^{-2} Hz and -1×10^{-2} Hz. Since $\ell\langle\omega_D\rangle$ is positive for protons and ranges between 1.5×10^{-2} Hz and 3.5×10^{-2} Hz, $\omega_- = \omega + \ell\langle\omega_D\rangle - kv_{\parallel}$ has a minimum value of about 2.5×10^{-2} Hz. So a representative value of G for most values of ℓ is about 15 Hz^{-1}.

Precise information on the behavior of the electric potential at the magnetopause in the frequency ranges shown in figure 5.7 ($.5 \times 10^{-2}$ Hz to 4.5×10^{-2} Hz) is not available from satellite measurements and so a value for the coefficients in equation 6-1 will have to be taken arbitrarily. It is assumed that the potential fluctuations on the magnetopause correspond to an infinite (in time) cisodal ($e^{i\omega_o t}$) wave train oscillating at some frequency ω_o in the range .5 to 4.5×10^{-2} Hz. The oscillation is assumed to be such that for each value of ℓ and m shown in figure 5.7 the amplitude of

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the electric field produced by the potential fluctuation is .1 mv/m ($\frac{10^{-8}}{3}$ statvolts/cm). This is a potential equivalent to about 15 kilovolts across the model magnetosphere and the electric field gives an $\vec{E}\times\vec{B}$ drift rate of 1 km/sec at the magnetopause. Thus it is assumed for each ℓ and m

$$\frac{\ell}{r} \int d\omega\ B(\ell,m,\omega)\ e^{i\omega t} = \frac{10^{-8}}{3}\ e^{i\omega_0 t} \quad \text{statvolts/cm}$$

so that

$$B(\ell,m,\omega) = \frac{r}{\ell}\ \frac{10^{-8}}{3}\ \delta(\omega - \omega_o) \quad \text{statvolts (cm-Hz)}^{-1}$$

Then the coefficients $A(\ell,m,n,r) = \int d\omega A(\ell,m,\omega,r)$ are

$$A(\ell,m,n,r) = \int d\omega\ B(\ell,m,\omega) \left[\frac{r_m\sqrt{q(r_m)}}{r\sqrt{q}}\ \exp\ i\int_r^{r_m} q\ dr\right] \equiv \qquad 6\text{-}2$$

$$\int d\omega\ B(\ell,m,\omega)\ M(r,\omega,\ell,m) = \frac{10^{-8}}{3}\ \frac{r}{\ell}\ M(r,\omega_o,\ell,m) \quad \text{statvolts/cm}$$

The behavior of the function $M(8R_e,\omega,\ell,m)$ is shown in figure 5.7. $M(8R_e,\omega,\ell,m)$ is a very sharply peaked function of ω for most ℓ so that if ω_o is in the range of .5 to 2×10^{-2} Hz the integral in equation 6-2 will be zero except for one value of ℓ and m. If ω_o is about 2.5×10^{-2} Hz the largest number of ℓ values will contribute to the diffusion coefficient since values of ℓ between -245 and -260 have non-zero integrals. So in the case where ω_o is between .5 and 2×10^{-2} Hz

$$D_{rr} = \frac{c^2}{2}\ \frac{15}{(1.5\times10^{-3})^2} \left(\frac{10^{-8}}{3}\right)^2 (.75)^2 = 1.5\times10^{10}\ \text{cm}^2/\text{sec} =$$

$$(.06\ R_e)^2\ /\ \text{day}$$

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For $\omega_o = 2.5 \times 10^{-2}$ Hz

$$D_{rr} = \frac{c^2}{2} \frac{15}{(1.5\times10^{-3})^2} \left(\frac{10^{-8}}{3}\right)^2 \sum_{\ell=-245}^{-260} |M(8,\omega_o,\ell,1)|^2 \approx$$

$$1.2 \times 10^{11} \text{ cm}^2/\text{sec} = (.2\, R_e)^2 \,/\, \text{day}$$

Both of these values of D_{rr} are well below the Bohm diffusion coefficient. If the function $M(8R_e,\omega,\ell,m)$ were not such a sharply peaked function of ω for most ℓ values, the diffusion coefficient could be larger.

The solution to the diffusion equation with a constant diffusion coefficient and a delta function initial distribution is

$$\frac{1}{2\sqrt{\pi D_{rr} t}} \exp - \frac{r^2}{4D_{rr} t}$$

So with the largest value of D_{rr} ($\sim(.2\, R_e)^2$/day) an initial "spike" of particles at 8 R_e in the magnetosphere would spread out in a Gaussian shaped distribution with a distance between the e-folding points of about .8 R_e in one day. That is, less than half of the particles initially at 8 R_e are moved inwards an average distance of about .4 R_e in a day. Diffusion of this sort is much too slow to be of any significance during the rapid build up of ring current particles during a substorm. Thus even in the unrealistic model used to obtain solutions which transport energy into the magnetosphere from the magnetopause the resulting diffusion process has a small diffusion coefficient.

Since long wave length electrostatic oscillations do not produce diffusion in the magnetosphere the question might arise as to whether the short wave length oscillations are important in producing diffusion.

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Although the diffusion coefficient given in equation 3-8 does not indicate it, the diffusion coefficient should be very small for most particles when the wave length of the oscillation is less than the ion gyroradius. The reason for this is that when the wave length of the electrostatic oscillation is on the order of the ion gyroradius or smaller, the great majority of the ions "see" the wave electric field as rapidly fluctuating simply because their gyration motion carries them over one wave length of the wave every gyroperiod. For the $\vec{E}\times\vec{B}$ drift to move a particle a large distance the particle must "see" the wave as nearly constant. Only a very small number of particles with a narrow range of phase velocities would see the short wave length oscillations as nearly constant and so most of the particles would not be diffused by the wave. Equation 3-8 for the diffusion coefficient $D_{\alpha\alpha}$ does not show this effect because it has been derived under the assumption that the wave length of the wave is much longer than the ion gyroradius. So the long wave length oscillations investigated are the ones which would produce significant diffusion for most of the particles in the plasma.

Although it seems reasonable to assume that long wave length, low frequency electrostatic oscillations would be very effective in transporting and energizing ring current protons it has been found that this is not the case because energy can not be transported into the hot ring current plasma by these waves. Even adjusting the parameters of the model so that energy at the magnetopause is

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transported into the magnetosphere it has been found that the waves transport very little of the energy available at the magnetopause into the magnetosphere and thus are not effective in rapidly diffusing particles. Any diffusion process of the nature proposed which operates in the magnetosphere should also operate in the simple model used to represent the magnetosphere. Since diffusion is not effective even in the simple model it is concluded from this study that diffusion of particles by turbulent electrostatic oscillations in the magnetospheric cavity driven by potential fluctuations on the magnetopause is not an energization and transportation mechanism for ring current particles.

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APPENDIX A-I

In this appendix the Hamiltonian appropriate to describe the motion of a charged particle in the electric and magnetic fields of the magnetosphere will be derived. The electric and magnetic fields are assumed to vary slowly enough that the first adiabatic invarient of the particle is conserved. The development of the Hamiltonian is patterned after the general approach of Gardner (1959). The Lagrangian in c.g.s. units for a particle of charge e and mass m in an electromagnetic field with electric potential $\tilde{\Phi}$ and vector potential $\vec{A}$ can be written in Cartesian coordinates (ρ_1, ρ_2, ρ_3) as

$$L = \frac{1}{2} m v^2 - e\tilde{\Phi} + \frac{e}{c} \vec{A} \cdot \vec{v}$$

$\vec{v} = \dot{\vec{\rho}}$ is the particle's velocity. Thus a component of the canonical momentum is written

$$\bar{p}_i \equiv \frac{\partial L}{\partial \dot{\rho}_i} = m \dot{\rho}_i + \frac{e}{c} A_i$$

or

$$v^2 = \dot{\vec{\rho}} \cdot \dot{\vec{\rho}} = \frac{1}{m} (\vec{\bar{p}} - \frac{e}{c} \vec{A})^2$$

The Hamiltonian for the particle is

$$K \equiv \sum_i \dot{\rho}_i \bar{p}_i - L = \frac{1}{2m} (\vec{\bar{p}} - \frac{e}{c} \vec{A})^2 + e\tilde{\Phi}$$

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It is assumed that the spatial variations in the electric and magnetic fields are small over distances on the order of the particle's gyroradius and the temporal variations in the fields are small during a gyroperiod. These requirements may be more explicitly written

$$\vec{A} = \frac{\vec{A}'}{\varepsilon}(\varepsilon\vec{\rho}, \varepsilon t); \quad \tilde{\Phi} = \frac{\tilde{\Phi}'}{\varepsilon}(\varepsilon\vec{\rho}, \varepsilon t)$$

ε is a dimensionless parameter which scales the variations in the fields. For example, ε can be taken to be the largest of the ratios between the gyroradius and the field scale size and the gyroperiod to the wave period.

The problem is easiest to handle in the usual magnetic coordinates (α, β, s). α, β determine the magnetic field and vector potential

$$\vec{\nabla}\alpha \times \vec{\nabla}\beta = \vec{B} \quad ; \quad \vec{A} = \alpha\vec{\nabla}\beta$$

and s is the measure of distance along a field line. In the simple magnetic field configuration used in the model magnetosphere any point in the space (ρ_1,ρ_2,ρ_3) is uniquely related at any time to a single point in the space (α,β,s) and vice versa. Thus α,β,s can be used as spatial coordinates and are written in terms of $\vec{\rho}$ and t as

$$\alpha = \frac{\alpha'}{\varepsilon}(\varepsilon\vec{\rho}, \varepsilon t) \; ; \quad \beta = \frac{\beta'}{\varepsilon}(\varepsilon\vec{\rho}, \varepsilon t) \; ; \quad s = \frac{s'}{\varepsilon}(\varepsilon\vec{\rho}, \varepsilon t)$$

The motion of the particle can be separated into a guiding center motion and a gyration by a judicious choice of a generating function for a canonical transformation from the phase space $(\vec{\rho}, \vec{\bar{p}})$ into a new phase space $(\vec{q}, \vec{p})$. The generating function to use is

$$F(\vec{\rho},\vec{p}) = \sqrt{\frac{e}{c}}\,\frac{\alpha'}{\varepsilon}\,p_3 + \sqrt{\frac{e}{c}}\,\frac{\beta'}{\varepsilon}\,p_1 + \frac{s'}{\varepsilon}\,p_2 - p_1 p_3$$

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The transformation equation $q_i = \frac{\partial F}{\partial p_i}$ gives

$$\sqrt{\frac{e}{c}}\,\alpha' = \epsilon p_1 + \epsilon q_3 \quad ; \quad \sqrt{\frac{e}{c}}\,\beta' = \epsilon q_1 + \epsilon p_3 \quad ; \quad s' = \epsilon q_2 \qquad \text{A-1-1}$$

The transformation equation $\bar{p}_i = \frac{\partial F}{\partial \rho_i}$ and the equations

$$\vec{A} = \alpha \vec{\nabla}\beta = \frac{1}{\epsilon^2}\alpha' \vec{\nabla}\beta' \quad ; \quad p_1 = \sqrt{\frac{e}{c}}\;\frac{\alpha'}{\epsilon} - q_3$$

give

$$\vec{v} = \vec{\bar{p}} - \frac{e}{c}\vec{A} = p_2 \frac{1}{\epsilon}\vec{\nabla}s' + p_3 \frac{1}{\epsilon'}\vec{\nabla}\alpha' + q_3 \frac{1}{\epsilon'}\vec{\nabla}\beta' \qquad \text{A-1-2}$$

where $\epsilon' = \sqrt{\frac{c}{e}}\,\epsilon$. Finally the new Hamiltonian, $\bar{H}$, is obtained from the last of the transformation equations $\bar{H} = K + \frac{\partial F}{\partial t}$ or

$$\bar{H} = e\,\Phi + \frac{1}{2m\epsilon}\left[p_2\vec{\nabla}s' + p_3\vec{\nabla}\alpha' - q_3\vec{\nabla}\beta'\right] + \frac{p_2}{\epsilon}\frac{\partial s}{\partial t} + \frac{\epsilon p_1}{\epsilon'}\frac{\partial \beta}{\partial t} + \frac{p_3}{\epsilon'}\frac{\partial \alpha'}{\partial t} \qquad \text{A-1-3}$$

where $\Phi\left(\frac{\alpha'}{\epsilon}, \frac{\beta'}{\epsilon}, \frac{s'}{\epsilon}, \epsilon t\right) = \tilde{\Phi}(\epsilon\vec{\rho}, \epsilon t)$ is the electric potential expressed in magnetic coordinates. α', β', s' are given in terms of $\vec{q}$ and $\vec{p}$ through the relationships given in equation A-1-1. It is apparent from the equations A-1-2 that p_2, p_3 and q_3 are essentially velocity components and thus can be considered to be of zero order in ϵ, or of order $1 = \epsilon^0$. While from A-1-1 it is apparent that ϵp_1, ϵq_1, ϵq_2 are essentially spatial coordinates with a small perturbation in the first two given by ϵq_3, ϵp_3. Thus A-1-3 can be expanded in powers of ϵ by using the approximations:

$$\sqrt{\frac{e}{c}}\,\alpha_0' = \epsilon p_1 \quad ; \quad \sqrt{\frac{e}{c}}\,\beta_0' = \epsilon q_1 ; \quad s_0' = \epsilon q_2$$

It will become apparent that α_0', β_0', s_0' are the guiding center

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coordinates of the particle while q_3, p_3 represent the gyration motion. With the definition

$$\vec{v}_{\perp} = \frac{p_3}{\varepsilon}\vec{\nabla}\alpha_o' - \frac{q_3}{\varepsilon}\vec{\nabla}\beta_o' + \frac{1}{m}\left[\vec{\nabla}\frac{s_o'}{\varepsilon} - \frac{\vec{B}}{B}\right]p_2 - c\,\frac{\vec{E}\times\vec{B}}{B^2} \qquad \text{A-1-4}$$

(where B is evaluated at the guiding center $\alpha_o', \beta_o', s_o'$) and using

$\vec{E} = -\vec{\nabla}\tilde{\Phi} - \frac{\partial}{\partial t}\alpha\vec{\nabla}\beta$ with the expansion $\tilde{\Phi}(\varepsilon\vec{\rho},\varepsilon t) = \Phi(\frac{\alpha_o'}{\varepsilon}, \frac{\beta_o'}{\varepsilon}, \frac{s_o'}{\varepsilon})$

$+ \varepsilon q_3 \frac{\partial\Phi}{\partial\alpha_o'} + \varepsilon p_3 \frac{\partial\Phi}{\partial\beta_o'}$, $\overline{H}$ can be expanded in the form

$\overline{H} = \frac{1}{\varepsilon}H_{-1} + H_o + \varepsilon H_1 + \ldots.$ When $\overline{H}$ is expanded in this form

$$H_{-1} = e\varepsilon\Phi + \frac{e}{c}\frac{\alpha_o'}{\varepsilon}\frac{\partial\beta_o'}{\partial t}$$

and

$$H_o = \frac{m}{2}v_{\perp}^2 - \frac{m}{2}\frac{c^2}{B^4}(\vec{E}\times\vec{B})^2 + \frac{1}{2m}\cdot$$

$$\left[p_2^{\,2} + \frac{2mp_2}{\varepsilon}\left(\frac{\partial s_o'}{\partial t} + \frac{c(\vec{E}\times\vec{B})}{B^2}.\vec{\nabla}s_o'\right)\right]$$

Writing $v_{\perp}$ in the form

$$\vec{v}_{\perp} = -\frac{\vec{\nabla}\beta_o'}{\varepsilon}\left[q_3 + \frac{1}{m\varepsilon}[p_2\frac{\vec{B}}{B} + \frac{c}{B^2}(\vec{E}\times\vec{B})]\cdot\frac{\vec{\nabla}\beta_o'}{B}\right] + \frac{\vec{\nabla}\alpha_o'}{\varepsilon}$$

$$\left[p_3 + \frac{1}{m\varepsilon}[p_2\frac{\vec{B}}{B} + \frac{c}{B^2}(\vec{E}\times\vec{B})]\cdot\frac{\vec{\nabla}\alpha_o'}{B}\right]$$

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or

$$\vec{v}_\perp = -\frac{\vec{\nabla}\beta'_o}{\varepsilon}[q_3+h(\varepsilon p_1,\varepsilon q_1,\varepsilon q_2,p_2,\varepsilon t)] + \frac{\vec{\nabla}\alpha'_o}{\varepsilon}[p_3+g(\varepsilon p_1,\varepsilon q_1,\varepsilon q_2,p_2,\varepsilon t)]$$

indicates that $\frac{1}{\varepsilon} H_{-1} + H_o$ is constant on an ellipse in the q_3, p_3 plane (i.e. when q_1, q_2, p_1, p_2 are held constant). The ellipses on which $\frac{1}{\varepsilon} H_{-1} + H_o$ is constant have the equations

$$(\vec{\nabla}\beta'_o)^2 (q_3+h)^2 - 2\, \vec{\nabla}\alpha'_o \cdot \vec{\nabla}\beta'_o (q_3+h)(p_3+g) + (\vec{\nabla}\alpha'_o)^2 (p_3+g)^2 = \text{constant}$$

Another canonical transformation from the phase space $(\vec{q}, \vec{p})$ to the phase space $(\vec{q}', \vec{p}')$ which is almost the identity transformation can be made so that the new Hamiltonian is constant on concentric circles in the q_3', p_3' plane. The new Hamiltonian, H, will depend only on the quantity $q_3'^2 + p_3'^2$ and the transformation from $(\vec{q}, \vec{p})$ to $(\vec{q}', \vec{p}')$ is equivalent to averaging over the angle ϕ in velocity space $(v_\perp, v_{||}, \phi)$. Under the transformation the new coordinates q_1', q_2', p_1', p_2' will be the same as the old coordinates q_1, q_2, p_1, p_2, to order ε. Also, the new Hamiltonian will have the same coefficients H_{-1}, H_o as the old Hamiltonian, to order ε. In constructing the generating function for the transformation it is assumed that $\vec{\nabla}\alpha'_o \cdot \vec{\nabla}\beta'_o = 0$. This is not a necessary assumption and is made simply because the calculations are less cumbersome. The generating function used is

$$F'(\vec{q}, \vec{p}') = q_1 p_1' + q_2 p_2' + \sqrt{\frac{|\vec{\nabla}\beta'_o|}{|\vec{\nabla}\alpha'_o|}}\,(q_3 + h)\, p_3' - g q_3$$

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The transformation equations

$$q_i' = \frac{\partial F'}{\partial p_i'} \quad ; \quad p_i = \frac{\partial F'}{\partial q_i} \quad ; \quad H = \overline{H} + \frac{\partial F'}{\partial t}$$

are used to obtain expressions of the new coordinates in terms of the old. For example

$$q_1' = q_1 - p_3' \frac{\partial h}{\partial p_1'} - q_3 \frac{\partial g}{\partial p_1'} = q_1 - \varepsilon p_3' \frac{\partial h}{\partial \varepsilon p_1'} - \varepsilon q_3 \frac{\partial g}{\partial \varepsilon p_1'}$$

However, since h is a function of $\varepsilon p_1'$, $\frac{\partial h}{\partial \varepsilon p_1'}$ is of order 1 as is $\frac{\partial g}{\partial \varepsilon p_1'}$. Thus $\varepsilon q_1' = \varepsilon q_1 + O(\varepsilon^2)$. From similar arguments

$$\varepsilon q_2' = \varepsilon q_2 + O(\varepsilon^2) \quad ; \quad \varepsilon p_1' = \varepsilon p_1 + O(\varepsilon^2) \quad ; \quad p_2' = p_2 + O(\varepsilon)$$

Finally q_3' and p_3' can be expressed in terms of q_3 and p_3 as

$$q_3' = \sqrt{\frac{|\vec{\nabla}\beta_o'|}{|\vec{\nabla}\alpha_o'|}} \, (q_3 + h) \quad ; \quad p_3' = \sqrt{\frac{|\vec{\nabla}\alpha_o'|}{|\vec{\nabla}\beta_o'|}} \, (p_3 + g)$$

so that

$$v_\perp^2 = |\vec{\nabla}\alpha_o'| \; |\vec{\nabla}\beta_o'| \; \frac{1}{\varepsilon^2} \; (p_3'^2 + q_3'^2) = B \, (q_3'^2 + p_3'^2) = \frac{2\mu B}{m}$$

Since h and g are functions of εt, $\frac{\partial F'}{\partial t} = \varepsilon \frac{\partial F}{\partial \varepsilon t}$ is of order ε and $H = \frac{1}{\varepsilon} H_{-1} + H_o$, to order ε. Thus, to order ε,

$$\frac{d}{dt} q_3' = \dot{q}_3' = \frac{\partial H}{\partial p_3'} = m p_3' \frac{\partial H}{\partial \mu} \quad ; \quad \dot{p}_3' = -\frac{\partial H}{\partial q_3'} = -m q_3' \frac{\partial H}{\partial \mu}$$

and

$$\frac{d}{dt} \left(\frac{v_\perp^2}{B}\right) = 2(q_3' \dot{q}_3' + p_3' \dot{p}_3') = 2\, m \left(q_3' p_3' \frac{\partial H}{\partial \mu} - p_3' q_3' \frac{\partial H}{\partial \mu}\right) = 0$$

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So $\mu = \frac{1}{2} m \frac{v_\perp^2}{B}$ is conserved to order ε (i.e. $\dot{\mu} = 0$, to order ε).

The expression for the perpendicular velocity is given in A-1-4 and is the usual expression for the particle's velocity of gyration.

Let $\alpha = \frac{\alpha_o'}{\varepsilon}$ and $\beta = \frac{\beta_o'}{\varepsilon}$ and $s = \frac{s_o'}{\varepsilon}$, then to order ε the Hamiltonian is (with $p_s \equiv p_2$)

$$H = e\Phi + \frac{e}{c} \alpha \frac{\partial \beta}{\partial t} + \mu B + \frac{p_s^2}{2m} + p_s \left(\frac{\partial s}{\partial t} + \frac{c(\vec{E} \times \vec{B})}{B^2} \cdot \vec{\nabla} s\right) + \frac{mc^2}{2} \frac{(\vec{E} \times \vec{B})^2}{B^4}$$

and since

$$\dot{p}_1 = \frac{-\partial H}{\partial q_1} \text{ or } \frac{d}{dt}\left(\sqrt{\frac{e}{c}} \frac{\alpha_o'}{\varepsilon}\right) = \frac{-\partial H}{\partial \sqrt{\frac{e}{c}} \frac{\beta_o'}{\varepsilon}}, \quad \dot{\alpha} = -\frac{c}{e} \frac{\partial H}{\partial \beta} . \qquad \text{A-1-5}$$

Similarly expressions for $\dot{\beta}$, $\dot{p}_s$, $\dot{s}$ can be obtained from Hamilton's equations, and

$$\dot{\alpha} = -\frac{c}{e} \frac{\partial H}{\partial \beta} \quad ; \quad \dot{\beta} = \frac{c}{e} \frac{\partial H}{\partial \alpha} \qquad \text{A-1-6}$$

$$\dot{p}_s = -\frac{\partial H}{\partial s} \quad ; \quad \dot{s} = \frac{\partial H}{\partial p_s}$$

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APPENDIX A-II

In this appendix a diffusion equation for the average guiding center density of particles in phase space will be developed under the assumption that the first adiabatic invarient of the particles is conserved. The method used is a slight generalization of the development of a diffusion equation governing the guiding center density by Birmingham, Northrop and Falthammar (1967) under the assumption that the first two adiabatic invarients of the particles are conserved. The starting point for the development is the continuity equation for the guiding center density Q $(\alpha,\beta,s,p_s,\mu,t)$ in the phase space given by α,β,s,p_s,μ

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial \alpha}(\dot{\alpha}Q) + \frac{\partial}{\partial \beta}(\dot{\beta}Q) + \frac{\partial}{\partial s}(\dot{s}Q) + \frac{\partial}{\partial p_s}(\dot{p}_s Q) + \frac{\partial}{\partial \mu}(\dot{\mu}Q) = 0$$

In Appendix A-I it is shown that to the order μ is conserved (i.e. $\dot{\mu}=0$) a Hamiltonian function H can be found such that:

$$\dot{\alpha} = \frac{-c}{e}\frac{\partial H}{\partial \beta} \; ; \; \dot{\beta} = \frac{c}{e}\frac{\partial H}{\partial \alpha} \; ; \; \dot{p}_s = \frac{-\partial H}{\partial s} \; ; \; \dot{s} = \frac{\partial H}{\partial p_s} \; .$$

The above equations imply

$$\frac{\partial}{\partial \alpha}(\dot{\alpha}) = \frac{-c}{e}\frac{\partial^2 H}{\partial\alpha\partial\beta} = -\frac{\partial}{\partial \beta}\left(\frac{c}{e}\frac{\partial H}{\partial \alpha}\right) = -\frac{\partial}{\partial \beta}(\dot{\beta})$$

and similarly

$$\frac{\partial}{\partial s}(\dot{s}) = \frac{-\partial}{\partial p_s}(\dot{p}_s)$$

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So the continuity equation becomes a Liouville equation

$$\frac{\partial Q}{\partial t} + \dot{\alpha}\,\frac{\partial Q}{\partial \alpha} + \dot{\beta}\,\frac{\partial Q}{\partial \beta} + \dot{s}\,\frac{\partial Q}{\partial s} + \dot{p}_s\,\frac{\partial Q}{\partial p_s} = 0$$

The phase space density and phase space velocities are treated as randomly fluctuating variables and it is convenient to write them as $x = \langle x \rangle + \delta x$. The symbol $\langle x \rangle$ denotes an ensemble average over the ensemble of fluctuations and δ denotes the variation of the variable from its average value.

With the notation:

$$x_1 = \alpha; \quad x_2 = \beta; \quad x_3 = s; \quad x_4 = p_s;$$

the Liouville equation can be written

$$\frac{\partial}{\partial t}\langle Q\rangle + \frac{\partial}{\partial t}\delta Q + \sum_{i=1}^{4}(\langle \dot{x}_i\rangle + \delta\dot{x}_i)\,\frac{\partial}{\partial x_i}\langle Q\rangle + \sum_{i=1}^{4}(\langle \dot{x}_i\rangle + \delta\dot{x}_i)\,\frac{\partial \delta Q}{\partial x_i} = 0$$

A-II-1

The ensemble average of this equation and the observation $\langle \delta\dot{x}\rangle = 0$ gives

$$\frac{\partial}{\partial t}\langle Q\rangle + \sum_{i=1}^{4}\langle \dot{x}_i\rangle\,\frac{\partial \langle Q\rangle}{\partial x_i} = -\sum_{i=1}^{4}\langle \delta\dot{x}_i\,\frac{\partial}{\partial x_i}\delta Q\rangle \qquad \text{A-II-2}$$

Subtracting equation II-2 from equation II-1 gives a differential equation for δQ, namely

$$\frac{\partial}{\partial t}\delta Q + \sum_{i=1}^{4}\langle \dot{x}_i\rangle\,\frac{\partial\,\delta Q}{\partial x_i} = \sum_{i=1}^{4}\left\{\langle \delta\dot{x}_i\frac{\partial}{\partial x_i}\delta Q\rangle - \delta\dot{x}_i\,\frac{\partial}{\partial x_i}\delta Q\right\} - \sum_{i=1}^{4}\delta\dot{x}_i\,\frac{\partial}{\partial x_i}\langle Q\rangle$$

A-II-3

If the fluctuating parts of the variables $\delta\dot{x}$ are considered small compared to $\langle \dot{x}\rangle$ so that the first term on the right hand side of

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equation II-3 is small in comparison to the second term, equation II-3 can be written

$$\frac{\bar{d}}{dt}\,\delta Q = -\sum_{i=1}^{4} \delta\dot{x}_i \frac{\partial}{\partial x_i} \langle Q\rangle \qquad \text{A-II-4}$$

The symbol $\frac{\bar{d}}{dt}$ on the left hand side denotes a convective derivitive taken along the phase space orbit with the velocity vector

$$\vec{u} = \sum_{i=1}^{4} \langle\dot{x}_i\rangle \; \vec{I}_{x_i}$$

So, assuming the fluctuations are turned on at t=0,

$$\delta Q = -\int_0^t dt'(\Sigma\, \delta\dot{x}_i(\vec{w},t') \frac{\partial}{\partial x_i} \langle Q(\vec{w},t')\rangle)$$

where the notation $(\vec{w},t')$ means that the quantity is evaluated at t' and along a specific phase space orbit where $\vec{w} = \vec{u}$. The components of the proper phase space orbit are solutions to the family of equations $\frac{dy_i}{dt} = \langle\dot{x}_i\rangle$ with the proper initial condition that at $t' = t$ the component of the orbit have the value $x_i(t)$. The components of the proper orbit are denoted by $w_i(t')$, where:

$$w_i(t') = x_i(t) + y_i(t') - y_i(t)$$

Using this solution for δQ in equation II-2 gives

$$\frac{\partial\langle Q\rangle}{\partial t} + \Sigma\langle\dot{x}_i\rangle \frac{\partial}{\partial x_i}\langle Q\rangle = \Sigma \left\langle \delta\dot{x}_i \frac{\partial}{\partial x_i} \int_0^t dt'\, \Sigma\delta\dot{x}_i(\vec{w},t') \frac{\partial\langle Q(\vec{w},t')\rangle}{\partial x_i} \right\rangle \qquad \text{A-II-5}$$

Since

$$\frac{\partial}{\partial\alpha}(\dot{\alpha}) = \frac{-\partial}{\partial\beta}(\dot{\beta}) \quad \text{and} \quad \dot{x}_i = \langle\dot{x}_i\rangle + \delta\dot{x}_i$$

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$$\frac{\partial}{\partial\alpha}\langle\dot{\alpha}\rangle + \frac{\partial}{\partial\alpha}\delta\dot{\alpha} = \frac{-\partial}{\partial\beta}\langle\dot{\beta}\rangle \frac{-\partial}{\partial\beta}\delta\dot{\beta}$$

Equating the fluctuating parts gives

$$\frac{\partial}{\partial\alpha}\delta\dot{\alpha} = \frac{-\partial}{\partial\beta}\delta\dot{\beta}$$

similarly

$$\frac{\partial}{\partial p_s}\delta\dot{p}_s = \frac{-\partial}{\partial s}\delta\dot{s} \quad \text{or} \quad \sum_{i=1}^{4}\frac{\partial}{\partial x_i}\delta\dot{x}_i = 0$$

Thus

$$\sum_i \frac{\partial}{\partial x_i}\left\langle\int_0^t \delta\dot{x}_i(t)\delta\dot{x}_j(t')dt'\right\rangle = \sum_i\left\langle\frac{\partial(\delta\dot{x}_i)}{\partial x_i}\int_0^t \delta\dot{x}_j(t')dt'\right\rangle$$

$$+ \sum_i\left\langle\delta\dot{x}_i \frac{\partial}{\partial x_i}\int_0^t \delta\dot{x}_j(t')dt'\right\rangle = \sum_i\left\langle\delta\dot{x}_i\frac{\partial}{\partial x_i}\int_0^t \delta\dot{x}_j(t')dt'\right\rangle$$

So equation II-5 becomes

$$\frac{\overline{d}}{dt}\langle Q\rangle = \sum_i \sum_j \frac{\partial}{\partial x_i}\left\langle\int_0^t \delta\dot{x}_i(t)\ \delta\dot{x}_j(\vec{w},t')\frac{\partial}{\partial x_j}\ \langle Q(\vec{w},t')\rangle\ dt'\right\rangle \qquad \text{A-II-6}$$

The quantity $x = \langle x\rangle + \delta x$ can be viewed as developing on two separate time scales. The ensemble average quantity $\langle x\rangle$ will change as the result of many small cumulative changes in δx. The quantity δx changes on the same time scale as the fluctuations. Thus $\langle x\rangle$ will change much more slowly than δx or the fluctuations. The time integral in A-II-6 is effectively limited to times on the order of the correlation time of the fluctuations, τ_c, because $\int_0^t dt'\delta x_i(t)\delta x_j(t')$ is effectively a cross correlation function and correlation functions are small for times larger than the correlation time. Since $\langle Q\rangle$ varies on a much larger scale, both temporally and spatially, than δQ or $\delta\dot{x}$

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the integral in A-II-6 will be dominated by the behavior of $\delta\dot{x}_i\ \delta\dot{x}_j$. So the quantity $\frac{\partial}{\partial x_j}$ $\langle Q(\vec{w},t)\rangle$ inside the integral sign in A-II-6 can be replaced by $\frac{\partial}{\partial x_j}\langle Q(\alpha,\beta,s,p_s,\mu,t)\rangle$ and equation A-II-6 can then be written

$$\frac{\partial\langle Q\rangle}{\partial t} + \Sigma\ \dot{x}_{oi}\ \frac{\partial\langle Q\rangle}{\partial x_i} = \sum_{i,j} \frac{\partial}{\partial x_i} D_{x_i x_j} \frac{\partial}{\partial x_j} \langle Q\rangle$$

where

$$D_{x_i x_j} = \left\langle \int_0^t \delta\dot{x}_i(\alpha,\beta,s,p_s,\mu,t)\ \delta\dot{x}_j(w_1,w_2,w_3,w_4,\mu,t')dt' \right\rangle \qquad \text{A-II-7}$$

and

$$w_i(t') = x_i(t) + (t'-t)\ \dot{x}_{oi} \qquad \text{A-II-8}$$

where $\dot{x}_{oi}$ denotes the value of $\dot{x}_i$ in the absence of fluctuations. The $\delta\dot{x}_i$ can be expressed in terms of the Hamiltonian given in Appendix A-I as:

$$\delta\dot{\alpha} = \dot{\alpha} - \langle\dot{\alpha}\rangle = \frac{-c}{e}\left(\frac{\partial H}{\partial\beta} - \frac{\partial\langle H\rangle}{\partial\beta}\right)\ ;\ \delta\dot{\beta} = \frac{c}{e}\ \frac{\partial}{\partial\alpha}(H-\langle H\rangle)$$

$$\delta\dot{p}_s = \frac{-\partial}{\partial s}(H-\langle H\rangle)\ ;\qquad \delta\dot{s} = \frac{\partial}{\partial p_s}(H-\langle H\rangle) \qquad \text{A-II-9}$$

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APPENDIX A-III

In this appendix the diffusion coefficient $D_{\alpha\alpha}$ will be calculated in detail to illustrate the technique used in calculating all of the coefficients. The expression given for the diffusion coefficients in A-II-7 requires expressions for the zero order orbit components w_i given in equation A-II-8 in terms of the magnetic coordinates α, β, s. The calculation of $D_{\alpha\alpha}$ will be restricted to the geometry of the model magnetosphere. In the model magnetosphere the magnetic field is taken to be static and so the magnetic coordinates α, β, s do not depend explicitly on time. Thus $\frac{\partial\alpha}{\partial t} = \frac{\partial\beta}{\partial t} = \frac{\partial s}{\partial t} = 0$. The Euler potentials α and β are such that $\vec{\nabla}\alpha \times \vec{\nabla}\beta = B(r)\vec{1}_z$. Since the magnetic field is only a function of r and points in the z direction a judicious choice of β is, $\beta \equiv \Theta$. Then α can be chosen as a function of r only and is obtained from the differential equation

$$\frac{d}{dr}\alpha = r\,B(r) \qquad \text{A-III-1}$$

The solution to A-III-1 can also be used to obtain r as a function of α. The other magnetic coordinate s is taken to be identically z ($s \equiv z$). Since B is a function of r only the gradient drift will have only an azimuthal component and

$$\dot{\alpha}_o = 0; \quad \dot{\beta}_o = \mu\frac{c}{e}\,\frac{1}{rB}\,\frac{dB}{dr} = \omega_D(\alpha)$$

Thus the first two components of the zero order orbit are

$$w_1(t') = \alpha; \quad w_2(t') = \beta + \omega_D(t'-t) \qquad \text{A-III-2}$$

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In the model magnetosphere the particles are assumed to be elastically reflected from the ionospheres at $s = \pm \Lambda$. The velocity of a particle will change abruptly from $v_{||}$ to $-v_{||}$ whenever it reaches the ionosphere. Since s is taken to be identically z, the momentum conjugate to s is given by

$$p_s = mv_{||} + m\frac{\partial s}{\partial t} + mc\,\frac{\vec{E}\times\vec{B}}{B^2}\cdot\vec{\nabla}s = mv_{||}$$

Thus w_4 which represents the zero order behavior of p_s is given by

$$w_4(t') = \begin{cases} mv_{||} & 0 < t' < t_0 \\ (-1)^n\, mv_{||} & (n-1)T + t_0 < t' < nT + t_0 \\ (-1)^N\, mv_{||} & (N-1)T + t_0 < t' < t \end{cases} \qquad \text{A-III-3}$$

where $v_{||}$ is the velocity of the particle at $t' = 0$. The time it takes a particle to reach the ionosphere from its position, s_o, at $t' = 0$ is given by $t_o = \frac{\Lambda}{|v_{||}|} - \frac{s_o}{v_{||}}$. At time t_o the particle reflects from the ionosphere and its parallel velocity changes sign. After a time T the particle again reflects from the ionosphere and the sign of its parallel velocity is again changed. The time T is called the bounce period and depends upon the parallel velocity of a particle and the distance between ionospheres: $T = \frac{2\Lambda}{v_{||}}$. The bouncing motion of the particle is represented by equation A-III-3 from $t' = 0$ up to $t' = t$. The integer N is the largest positive integer such that $t-(N-1)T-t_o>0$, and thus gives the correct sign for the parallel velocity at $t' = t$.

The zero order motion of the particle along the field can be

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found from A-III-3 and is given by

$$w_3(t') = \begin{cases} s_o + v_{||}t' & 0<t'<t_o \\ s_n + (-1)^n v_{||}t' & (n-1)T + t_o < t' < nT + t_o \\ s_n + (-1)^N v_{||}t' & (N-1)T + t_o < t' < t \end{cases} \qquad \text{A-III-4}$$

where s_o is the position of the particle along the field line at $t'=0$ and $s_n = -((n-1)T + t_o + \frac{\Lambda}{|v_{||}|})(-1)^n v_{||}$. w_3 always has a value between $+\Lambda$ and $-\Lambda$ since the particle is confined between the ionospheres.

The variations in the particle velocities, $\delta\dot{x}$, due to the fluctuations can be expressed in terms of the Hamiltonian by using equation A-II-9. To use the equations A-II-9 $\langle H \rangle$, and thus $\langle \Phi \rangle$, $\langle \mu B \rangle$ and $\langle E_\perp^2 \rangle$, are needed. In calculating the ensemble average quantities it is convenient to use the shorthand notation introduced in chapter 3. That is, the triple (ℓ, m, n) is replaced by L and $\sum_{\ell,m,n}$ is replaced by $\sum_L$. Furthermore the notation $\prod_L \int dx(L) f(L)$ will be used to denote $\int dx(1) \int dx(2) \int dx(3) \cdots \prod f(\ell, m, n)$. Also the statement $L' = L$ will mean $\ell' = \ell$ and $m' = m$ and $n' = n$.

From the definition of $W_2(R\psi_1, t_1; R\psi_2, t_2)$ given in equation 3-7 and from equation 3-1

$$\langle \Phi \rangle = \sum_L \prod_{L'} \left\{ \frac{1}{(2\pi)^{3/2}} \int_0^{2\pi} d\psi(L') \int_{-\infty}^{\infty} d\psi'(L') W_2(\psi(L'), t; \psi'(L'), t') \right. \\ \left. A(L,r) \exp i(\omega(L)\hat{t} + \ell\theta - kz + \psi'(L)) \right\} \qquad \text{A-III-5}$$

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$\psi(L)$ is the phase of the L component of the potential at time t, $\psi'(L)$ is the phase at t' and $\hat{t}$ = t'-t. Terms in the integrand of equation A-III-5 for which $L' \neq L$ are of the form

$$\frac{1}{(2\pi)^{3/2}\, a}\int_0^{2\pi} d\psi \int_{-\infty}^{\infty} d\psi' \exp - (\psi'-\psi)^2/_{2a^2} = 1.$$

The parameter a(L) is defined by $a^2(L) = \kappa(\ell,m,n)\Delta\psi^2(\ell,m,n)\hat{t}$. a(L) is allowed to depend upon L so that it can be taken to be different for each component of the wave. Equation A-III-5 can be written

$$\langle\Phi\rangle = \sum_L \frac{A\ (L,r)}{(2\pi)^{3/2}a(L)} \exp i\omega(L)\hat{t} + \ell\Theta - kz \int_0^{2\pi} d\psi(L) \int_{-\infty}^{\infty} d\psi'(L) e^{\frac{-(\psi'(L)-\psi(L))^2}{2a(L)^2}} e^{i\psi'(L)}$$

The integration over $\psi'(L)$ gives $e^{-a^2(L)/2} e^{i\psi(L)}$ and $\int_0^{2\pi} d\psi e^{i\psi} = 0$.

Thus the ensemble average over the potential is zero : $\langle\Phi\rangle = 0$. This is consistent with the assumption that there is no zero order electric field.

To calculate $\langle E_\perp^2\rangle$, first note that $E_\perp^2$ is of the form $Re(A_1 e^{i\phi_1}) \cdot Re(A_2 e^{i\phi_2})$. The notation "Re" means "the real part of"; "Im" will be used to mean "the imaginary part of"; and A* will denote the complex conjugate of A. From the definition of the complex conjugate it follows that

$$Re(A_1 e^{i\phi_1})Re(A_2 e^{i\phi_2}) = \tfrac{1}{2}(A_1 e^{i\phi_1} + A_1^* e^{-i\phi_1^*})\tfrac{1}{2}(A_2 e^{i\phi_2} + A_2^* e^{-i\phi_2^*})$$

$$= \frac{1}{4}(A_1A_2 e^{i(\phi_1+\phi_2)} + A_1^* A_2^* e^{-i(\phi_1+\phi_2)} + A_1^* A_2 e^{-i(\phi_1^*-\phi_2)} + A_1 A_2^* e^{i(\phi_1-\phi_2^*)}) \qquad \text{A-III-6}$$

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Since the electric field is given by $\vec{E} = -\vec{\nabla}\Phi$, $\vec{E}$ can be written in the form

$$\vec{E} = \sum_L (E_\alpha(L,\alpha)\vec{T}_\alpha + E_\beta(L,\alpha)\vec{T}_\beta \qquad \text{A-III-7}$$

$$+ E_s(L,\alpha)\vec{T}_s) \exp i(\omega(L)t + \ell\beta - ks + \psi(L))$$

and so

$$\langle E_\perp^2 \rangle = \prod_{L''} \int_0^{2\pi} d\psi(L'') \int_{-\infty}^{\infty} d\psi'(L'') \qquad \text{A-III-8}$$

$$\mathrm{Re}\left\{ \sum_L E_\alpha(L,\alpha) \exp i(\hat{\omega}(L)\hat{t} + \ell\beta - ks + \psi'(L)) \right\}$$

$$\mathrm{Re}\left\{ \sum_{L'} E_\alpha(L',\alpha) \exp i(\omega(L')\hat{t} + \ell'\beta - k's + \psi'(L')) \right\} W_2(\psi(L''),t;\psi'(L''),t')$$

+ a similar term for E_β.

According to the equation A-III-6, the integrand in A-III-7 must have terms of the form

$$\frac{1}{(2\pi)^{3/2}a} \int d\psi \int d\psi' \exp i(\psi'(L) \pm \psi'(L')) \exp -\frac{(\psi'-\psi)^2}{2a^2} = \begin{cases} 0 & \text{for + sign} \\ \delta_{LL'}, & \text{for - sign} \end{cases}$$

where $\delta_{LL'} = \delta_{\ell\ell'}\delta_{mm'}\delta_{nn'}$. Thus

$$\langle E_\perp^2 \rangle = \frac{1}{2} \sum_L \left\{ |E_\alpha(L,\alpha)|^2 + |E_\beta(L,\alpha)|^2 \right\} e^{-2\gamma(L)\hat{t}} \qquad \text{A-III-9}$$

where $\gamma(L) = \mathrm{Im}(\omega(L))$. The imaginary part of the wave frequency, $\gamma(L)$, gives the damping or growth rate of the L component of the wave. To calculate $D_{\alpha\alpha}$ only the quantity $\delta\dot{\alpha}$ is needed and $\delta\dot{\alpha} = -\frac{c}{e}\frac{\partial}{\partial\beta}(H - \langle H \rangle)$. Since B depends only on α and μ is an independant variable, $\frac{\partial}{\partial\beta}(\mu B) = 0$. From equation A-III-9 it can be seen that $\langle E_\perp^2 \rangle$ is a function of α only and thus $\frac{\partial}{\partial\beta}\langle E_\perp^2 \rangle = 0$. So

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$$\delta\dot{\alpha} = \frac{c}{e}\left[\frac{m}{2}\frac{c^2}{B^2}\frac{\partial}{\partial\beta}E_\perp^2 - e\frac{\partial}{\partial\beta}\Phi\right] \qquad \text{A-III-10}$$

Writing the expansion of the potential Φ in α,β,s space as

$\Phi = \sum_L C(L,\alpha)\exp i(\omega(L)t + \ell\beta - ks + \psi(L)) = \sum_L C(L,\alpha)e^{i\phi(L)}$ gives

$$\frac{\partial}{\partial\beta}\Phi = \Sigma i\ell\, C(L,\alpha)\exp i\phi(L) \qquad \text{A-III-11}$$

where $\phi(L) = \omega(L)t + \ell\beta - ks + \psi(L)$

Using equations A-III-6 and 7 to write $E_\perp^2$ gives

$$\frac{\partial E_\perp^2}{\partial\beta} = \frac{1}{4}\sum_L\sum_{L'}\Big\{ i(\ell+\ell')E_\alpha(L)E_\alpha(L')e^{i\phi_+(t)} - i(\ell+\ell')\cdot$$

$$E_\alpha^*(L)E_\alpha^*(L')e^{-i\phi_+^*(t)} - i(\ell-\ell')E_\alpha^*(L)E_\alpha(L')e^{-i\phi_-(t)} + i(\ell-\ell')E_\alpha(L)E_\alpha(L')e^{i\phi_-^*(t)}\Big\} \qquad \text{A-III-12}$$

+ similar terms for E_β

where $\phi_+(t) = (\omega(L) + \omega(L'))t + (\ell+\ell')\beta - (k+k')s + \psi(L) + \psi(L')$

$\phi_-(t) = (\omega^*(L) - \omega(L'))t + (\ell-\ell')\beta - (k-k')s + \psi(L) - \psi(L')$

Using the expansions for $w_i(t')$ from equations A-III-2,3 and 4 together with the observation that $w_1 = \alpha$ implies that any function of w_1 is simply a function of α, $\delta\dot{\alpha}(w_1,w_2,w_3,w_4,t') = \delta\dot{\alpha}(\vec{w},t')$ can be written from equations A-III-10, 11 and 12 as

$$\delta\dot{\alpha}(\vec{w},t') \equiv \delta\dot{\alpha}(t') = \frac{mc^3}{8eB^2}\Big[\sum_L\sum_{L'}\Big\{i(\ell+\ell')E_\alpha(L,\alpha)E_\alpha(L',\alpha)e^{i\phi_+(t')}$$

$$-i(\ell+\ell')E_\alpha^*(L,\alpha)E_\alpha^*(L',\alpha)e^{-i\phi_+^*(t')} - i(\ell-\ell')E_\alpha^*(L,\alpha)E_\alpha(L',\alpha)e^{-i\phi_-(t')}$$

$$+i(\ell-\ell')E_\alpha(L,\alpha)E_\alpha^*(L',\alpha)e^{i\phi_-^*(t')}\Big\} + \text{a similar term for } E_\beta\Big] \qquad \text{A-III-13}$$

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$$-c \sum_{L} i\ell C(L,\alpha)\exp\, i(\omega(L)t' + \ell\beta + \ell\omega_D\,(t'-t) - kw_3(t') + \psi'(L))$$

where

$$\phi_+(t') = (\omega(L) + \omega(L'))t' + (\ell+\ell')\left[\beta+\omega_D\,(t'-t)\right] - (k+k')w_3(t') + \psi'(L) + \psi'(L')$$

$$\phi_-(t') = (\omega^*(L) - \omega(L'))t' + (\ell-\ell')\left[\beta+\omega_D(t'-t)\right. - (k-k')w_3(t') + \psi'(L) - \psi'(L')$$

In calculating $D_{\alpha\alpha}$ it is convenient to use the following theorem (Komolgoroff) to interchange the ensemble average and time integration. Theorem: For a bounded ensemble Y(t) all of whose members, y(t), are Riemann integrable

$$\left\langle \int_0^t f\,[y(t')]\,dt' \right\rangle = \int_0^t \left\langle f\,[y(t')] \right\rangle dt'$$

provided that $\langle f[y(t')] \rangle$ is Riemann integrable.

Certainly the variations in the phase space velocities, $\delta\dot{x}$, are bounded and Riemann integrable and $D_{\alpha\alpha}$ is then given by

$$D_{\alpha\alpha} = \int_0^t \left\langle \delta\dot{\alpha}(\alpha,\beta,s,v_{||},\mu,t)\,\delta\dot{\alpha}(w_1,w_2,w_3,w_4,\mu,t') \right\rangle dt'$$

$$= \int_0^t \left\langle \delta\dot{\alpha}(t)\,\delta\dot{\alpha}(t') \right\rangle dt' \qquad \text{A-III-14}$$

The two terms needed for $\delta\dot{\alpha}(t)$ are given by equations A-III-11 and 12; $\delta\dot{\alpha}(t')$ is given by equation A-III-13. The expression $\langle \delta\dot{\alpha}(t)\delta\dot{\alpha}(t') \rangle$ has integrals of the form:

1) From multiplying the $\dfrac{\partial E_1^2}{\partial \beta}$ terms

$$\prod_{I} \int d\phi(I) \int d\psi'(I)\, \exp\left\{ i([\psi(L)+\psi(L')] \pm [\psi'(L'')+\psi(L''')]) \right\}$$

$$W_2(\psi(I),t;\psi'(I),t') = 0$$

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where $W_2(\psi(I),t;\ \psi'(I),t') = \dfrac{1}{(2\pi)^{3/2}\ a(I)} \exp - \dfrac{(\psi'(I)-\psi(I))^2}{2a^2(I)}$

and $a^2(I) = \kappa(i,j,k)\ (\Delta\psi(i,j,k))^2(t'-t)$; $\psi(I)=\psi(i,j,k)$

In the following integrals the index I will be left out for the sake of brevity but expressions such as $W_2(\psi,t;\psi',t')$ should be understood to stand for $W_2(\psi(I),t;\ \psi'(I),t')$. Other intergrals from the $\dfrac{\partial}{\partial\beta} E_\perp^2$ terms are

$$\Pi \int d\psi \int d\psi' \exp\left\{ i([\psi(L)-\psi(L')]\pm[\psi'(L'')+\psi'(L''')])\right\} W_2(\psi,t;\ \psi',t') = 0$$

$$\Pi \int d\psi \int d\psi' \exp\left\{ i([\psi(L)+\psi(L')] \pm [\psi'(L'')-\psi'(L''')])\right\} W_2(\psi,t;\psi',t') = 0$$

and

$$\Pi \int d\psi \int d\psi' \exp\left\{ i([\psi(L)- \psi(L')] \pm [\psi'(L'')-\psi'(L''')])\right\} W_2(\psi,t;\psi',t') = \delta_{LL'}\delta_{L'L''}\delta_{L''L'''}$$

2) From the cross terms between $\frac{\partial}{\partial\beta} E_\perp^2$ and $\frac{\partial}{\partial\beta}\Phi$

$$\Pi \int d\psi \int d\psi' \exp \left\{ i[\pm\psi(L) \pm (\psi'(L') \pm \psi'(L''))]\right\} W_2(\psi,t;\psi',t') = 0$$

for any of the eight possible combinations of signs, and

$$\Pi \int d\psi \int d\psi' \exp\left\{ i[\pm(\psi(L) \pm \psi(L')) \pm \psi'(L'')]\right\} W_2(\psi,t;\psi',t') = 0$$

3) From the term involving $\frac{\partial}{\partial\beta}\Phi$

$$\Pi \int d\psi \int d\psi' \exp \left\{ i[\psi'(L)\pm\psi'(L')]\right\} W_2(\psi,t;\psi',t')=e^{\frac{-a^2(L)}{2}} \begin{cases} 0 & \text{for + sign} \\ \delta_{LL'} & \text{for - sign} \end{cases}$$

Denoting the quantity $s(t)-w_3(t')$ by $s(\tau)$ with $\tau=t-t'$, the quantity $\langle\delta\dot{\alpha}(t)\ \delta\dot{\alpha}(t')\rangle$ can be written

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$$\left\langle \delta\dot{\alpha}(t)\delta\dot{\alpha}(t')\right\rangle = \frac{c^2}{4} \sum_L e^{\frac{-a^2(L)}{2}} \ell^2 |C(L,\alpha)|^2 \Big\{ \exp[-i(\omega(L)\tau+\ell\omega_D(\alpha)\tau-ks(\tau))]$$

$$+ \exp[i(\dot{\omega}(L)\tau + \ell\omega_D(\alpha)\tau - ks(\tau))] \Big\} e^{\gamma(L)\tau} e^{-2\gamma(L)t}$$

$$= \frac{c^2}{4} \sum_L e^{-2\gamma t} e^{\frac{-a^2}{2}} \ell^2 |C(L,\alpha)|^2 e^{\gamma\tau} \cos(\omega'(L)\tau - ks(\tau))$$

where $\omega'(L) = \omega(L) + \ell\omega_D(\alpha)$.

The integration over time in equation A-III-14 is best divided up into time intervals when the particle is between the ionospheres. Equation A-III-14 can then be written

$$D_{\alpha\alpha} = \frac{c^2}{2} \sum_L e^{-2\gamma(L)t} \ell^2 |C(L,\alpha|^2 \Big\{ \int_0^{t_o} d\tau\, e^{\frac{-a^2}{2}} e^{\gamma\tau} \cos[(\dot{\omega}'-kv_{||})\tau]$$

$$+ \sum_{n=1}^{N} \int_{(n-1)T+t_o}^{nT+t_o} d\tau e^{\frac{-a^2}{2}} e^{\gamma\tau} \cos[\omega'\tau - k(s_n + (-1)^n v_{||}\tau) + ks] + \int_{NT+t_0}^{t} d\tau\, e^{\frac{-a^2}{2}} e^{\gamma\tau}$$

$$\cos[\omega'(L)\tau - k(s_N + (-1)^N v_{||}\tau) + ks] \Big\} \qquad \text{A-III-15}$$

The integrals in equation A-III-15 are all of the form

$$\int_{t_1}^{t_2} e^{\tau d} \cos(b\tau + c) d\tau = \frac{1}{d^2+b^2} \Big\{ e^{t_2 d} [d\cos(bt_2+c) + b\sin(bt_2+c)]$$

$$- e^{\tau_1 d} [d\cos(bt_1+c) + b\sin(bt_1+c)] \Big\}$$

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In equation A-III-15

$$\tau d = -\frac{a^2}{2} + \gamma\tau = \frac{(-\kappa(L)\ \Delta\psi^2(L)}{2} + \gamma(L))\tau = -(\kappa'(L)-\gamma(L))\tau$$

$$b = \omega'(L) - (-1)^n kv_{||}$$

$$c = -ks_n - ks = k((n-1)T + t_o + \frac{\Lambda}{|v_{||}|})(-1)^n v_{||} - ks$$

The terms in A-III-15 can be combined after the time integration is performed since the summation contains terms like:

$$\frac{e^{-(\kappa'-\gamma)((n-1)T+t_o)}(\kappa'-\gamma)}{(\kappa'-\gamma)^2 + (\omega' - (-1)^n kv_{||})^2} \Big\{ \cos[(\omega' - (-1)^n kv_{||})((n-1)T+t_o)$$

$$+ (-1)^n kv_{||}((n-1)T + t_o + \frac{\Lambda}{|v_{||}|}) + ks] \qquad \text{A-III-16}$$

$$-\cos[(\omega'-(-1)^{n-1}kv_{||})((n-1)T+t_o)+(-1)^{n-1}kv_{||}((n-2)T+t_o+\frac{\Lambda}{|v_{||}|})+ks]\Big\}$$

where the first term is from the lower limit of the time integration of the n^{th} term of the summation and the second term is from the upper limit of the $(n-1)^{th}$ term. Equation A-III-16 can be written

$$4(-1)^n(\kappa'-\gamma)\omega kv_{||}\frac{e^{-(\kappa'-\gamma)((n-1)T+to)}\cos[\omega'((n-1)T+to)+(-1)^n\sigma k\Lambda+ks]}{[(\kappa'-\gamma)^2+(\omega'-kv_{||})^2][(\kappa'-\gamma)^2 + (\omega+kv_{||})^2]}$$

where $\sigma = \frac{v_{||}}{|v_{||}|}$ is the sign of the parallel velocity. Similarly the terms in the summation in equation A-III-15 which contain sine terms

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after the time integration is performed combine as:

$$\frac{(-1)^n kv_{||}(\omega^2-(kv_{||})^2-(\kappa'-\gamma)^2)e^{-(\kappa'-\gamma)((n-1)T+to)}}{[(\kappa'-\gamma)^2+(\omega'-kv_{||})^2][(\kappa'-\gamma)^2+(\omega'+kv_{||})^2]}\sin[\omega'(L)((n-1)T+to)$$

$$+(-1)^n \sigma k\Lambda + ks]$$

Using the notation

$$d' = \kappa'(L)-\gamma(L) \quad ; \quad k((-1)^n \sigma\Lambda+s) = kr_n \quad ; \quad nT+to = t_n$$

$$\omega_- = \omega'-kv_{||} \quad ; \quad \omega_+ = \omega' + kv_{||}$$

equation A-III-15 can be written

$$D_{\alpha\alpha} = \frac{c^2}{2} \sum_L \Bigg[e^{-2\gamma(L)t} \ell^2|C(L,\alpha)|^2 \Bigg\{ \frac{d'}{d'^2+\omega_-^2}$$

$$-\frac{2kv_{||}}{(d'^2+\omega_-^2)(d'^2+\omega_+^2)}\Big[2d'\omega \sum_{n=0}^{N-1} (-1)^n e^{-d't_n} \cos(\omega't_n-kr_n)+(\omega_+\omega_--d'^2)$$

$$\sum_{n=o}^{N-1} (-1)^n e^{-d't_n}\sin(\omega't_n-kr_n)\Big] - \frac{e^{-d't}(d'\cos(\omega't-kr_N)}{d'^2+(\omega'+(-1)^N kv_{||})^2}$$

$$+ (\omega'+(-1)^N kv_{||}) \sin(\omega't-kr_N))\Bigg\}\Bigg] \qquad \text{A-III-17}$$

Equation A-III-17 holds for any value of κ' and γ. A convenient approximation to A-III-17 can be made when $\gamma=0$ (steady state case) and $\exp{-\kappa' T} \ll 1$. These requirements on γ and κ' mean that the wave components are undamped and the wave turbulence is such that the

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particles see a phase change of approximately $\Delta\psi$ in a time somewhat less than the bounce period. Under these restrictions A-III-17 can be written

$$D_{\alpha\alpha} = \frac{c^2}{2} \sum_L \ell^2 |C(L,\alpha)|^2 \; G(L,v_{||},s)$$

where

$$G(L,v_{||},s) = \frac{\kappa'}{\kappa'^2+\omega_-^2} - \frac{2k\, v_{||} e^{-\kappa' t_o}}{[\kappa'^2+\omega_-^2][\kappa'^2+\omega_+^2]} \Big\{ 2\kappa'\omega[\cos(\omega_- t_o)$$

$$- e^{-\kappa' T} \cos(\omega_+ T + \omega_- t_o)] + (\omega_+\omega_- - \kappa'^2)[\sin(\omega_- t_o) - e^{-\kappa' T} \sin(\omega_+ T + \omega_- t_o)] \Big\}$$

Equation A-III-18 shows that, for t_o large (i.e. almost equal to a bounce period), $D_{\alpha\alpha}$ is largest for $\frac{\omega}{kv_{||}} = 1$. This means, of course, that particles with parallel velocities such that $\frac{\omega}{k} = v_{||}$ see the wave as almost constant for nearly a bounce period. If t_0 is small the particle is quickly reflected from the ionosphere and particles with parallel velocities such that $\frac{\omega}{kv_{||}} = -1$ see the wave most favorably. The initial conditions of the particle are very important in equation A-III-18 because of the value of κ'. κ' is taken to be so large in deriving A-III-18 that the particle motion is completely randomized in a very few bounce periods. If κ' is taken to be small so that $e^{-\kappa' T}$ is nearly unity then from equation A-III-17 it is clear that most terms in $D_{\alpha\alpha}$ will have the denominator $[\kappa'^2 + \omega_-^2]\cdot[\kappa'^2 + \omega_+^2]$. Thus the initial relationship of the particle to the wave is relatively unimportant and $D_{\alpha\alpha}$ doesn't peak sharply about any particular value of $\frac{\omega}{kv_{||}}$. Finally in the case in which γ is on the order of the bounce frequency of a particle ($2\pi/T$), $D_{\alpha\alpha}$ is quite small in a few bounce periods due to the factor $\exp -\gamma t$ which appears in every term of A-III-17.

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Appendix B-I

In this appendix electrostatic oscillations in a cylindrical geometry (r,θ,z) with a magnetic field given by $\vec{B} = B(r)\vec{1}_z$ will be investigated. The treatment will be restricted to the frequency range $\ell\langle\omega_D\rangle_I < \omega < \Omega_I$. The ion gyrofrequency is denoted by Ω_I, $\langle\omega_D\rangle_I$ is the azimuthal drift frequency of an ion with a velocity $= \sqrt{\frac{2k_b T_\perp}{m_i}}$ and ℓ is the azimuthal wave number. k_b is used to denote Boltzmann's constant. Both the ion gyroradius and the azimuthal drift frequency can be functions of r and the frequency restriction must be specified for a given range of r. The magnetic field is a function of r but is assumed to vary slowly over distances on the order of an ion gyroradius. The development of the differential equation for the electric potential will closely parallel the work of Swift (1967) who did similar calculations in a cylindrical geometry but with B = constant. The problem is solved by using a perturbation technique to find the single particle distribution function from the collisionless Vlasov equation with the electric potential being given by Poisson's equation. The electrostatic acceleration is assumed to be of higher order (i.e. smaller) than the Lorentz acceleration. The zero order or unperturbed distribution function can be chosen to be an arbitrary function of the three constants of the zero order motion v_z, $v_\perp^2 = v_r^2 + v_\theta^2$, and $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mrv_\theta + \frac{\varepsilon e}{c} r A(r)$. The momentum conjugate to θ, p_θ, is constant since the Lagrangian, L, of the particle does not depend upon θ (i.e. θ is cyclic). In the model magnetosphere a two component Maxwellian plasma with perpendicular and parallel temperatures given

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by $T_\perp$ and $T_\parallel$ is assumed so the zero order distribution function is written as

$$f_o = \left(\frac{m}{2\pi k_b T_\perp}\right)\left(\frac{m}{2\pi k_b T_\parallel}\right)^{1/2} g(rA(r) + \frac{rv_\theta}{\frac{\varepsilon e}{mc}}) \exp - (\frac{mv_\perp^2}{2k_b T_\perp} + \frac{mv_z^2}{2k_b T_\parallel}) \qquad \text{B-1-1}$$

Since the zero order number density of the plasma is obtained by integrating B-1-1 over velocity space g is related to n_o. This relationship will be specified later. The first order distribution function can be expressed as

$$f_1(t) = \frac{\varepsilon e}{m} \int_{t_o}^{t} \vec{\nabla}\phi \cdot \frac{\partial f_o}{\partial \vec{v}} \, dt' \qquad \text{B-1-2}$$

The integral in equation B-1-2 is evaluated along the zero order orbits of the particle in phase space. The lower limit of integration, t_o, is taken as $-\infty$ for growing oscillations ($\text{Im}\ \omega < 0$). The analytic continuation of $f_1(t)$ into the upper half of the complex ω plane gives the solutions for damped oscillations.

As discussed in appendix A-1 the motion of the particles to first order can be represented as a guiding center drift and a gyration motion about the guiding center. Since the magnitude of the magnetic field is a function of r only, the gradient drift will be in the azimuthal direction. So at time τ the particle is located at the cartesian coordinates

$$x(\tau) = \bar{r}\cos\bar{\theta} - \frac{\varepsilon v_\perp}{\Omega}\sin\tilde{\phi}$$

$$y(\tau) = -\bar{r}\sin\bar{\theta} + \frac{\varepsilon v_\perp}{\Omega}\cos\tilde{\phi} \qquad \text{B-1-3}$$

$$z(\tau) = -v_z\tau + z(0)$$

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where $\tau = t-t'$; $\bar{\theta} = -\omega_D\tau + \theta_o$; $\tilde{\phi} = \phi + \epsilon\Omega\tau$. The guiding center coordinates are given by $\bar{r}$, $\bar{\theta}$ and z. The gyroradius is defined by $\rho = \frac{v_\perp}{\Omega}$, θ_o and ϕ are initial phases and ϵ denotes the particle sign. In the following, unless specified otherwise, ω_D and Ω are assumed to be evaluated at the guiding center. The position of the particle at time t', r(t') is obtained from equations B-1-3

$$r^2(t') \equiv r'^2 = x^2(t') + y^2(t') = \bar{r}^2 + \epsilon \frac{v_\perp \bar{r}}{\Omega} \sin(\bar{\theta}-\tilde{\phi}) + \frac{v_\perp^2}{\Omega^2}$$

Considering $\frac{\rho}{r} << 1$ and using the expansion $(1+x)^2 \approx 1+2x$

$$r' = \bar{r}\left(1 + \epsilon \frac{\rho}{r} \sin(\bar{\theta}-\tilde{\phi}) + \frac{1}{2}\frac{\rho^2}{\bar{r}^2} \cos^2(\bar{\theta}-\tilde{\phi})\right) \qquad \text{B-1-4}$$

The velocity $v_r(t')$ is obtained from B-1-4 by differentiation

$$v_r' = v_r(t') = v_\perp \cos(\bar{\theta}-\tilde{\phi}) \left[\left(1 - \epsilon \frac{\omega_D}{\Omega} \left(1+\epsilon\frac{\rho}{\bar{r}} \sin(\bar{\theta}-\tilde{\phi}\right)\right)\right]$$

The azimuthal velocity, v_θ', can be obtained from $v_\theta' = -v_x'\sin\theta' + v_y'\cos\theta'$ and $\sin\theta' = \frac{y'}{r'}$; $\cos\theta' = \frac{x'}{r'}$ by struggling through some algebra

$$v_\theta' = \frac{\omega_D}{\bar{r}} - v_\perp \sin(\bar{\theta}-\tilde{\phi}) - \epsilon \frac{\rho}{\bar{r}} v_\perp \cos^2(\bar{\theta}-\tilde{\phi}) \qquad \text{B-1-5}$$

The electric potential will be written as

$$\Phi(r,\theta,z,t) = \int d\omega \sum_{\ell,m} A(\ell,m,r',\omega) \exp i(\omega t - kz' + \ell\theta')$$

and the individual components of the potential are expanded in terms of $\frac{\rho}{\bar{r}}$ for use in equation B-1-1.

$$A(\ell,m,r',\omega) = A(\ell,m,\bar{r},\omega) + [r'-\bar{r}] \frac{d}{d\bar{r}} A(\ell,m,\bar{r},\omega) + \frac{1}{2}[r'-\bar{r}]^2 \frac{d^2}{d\bar{r}^2} A(\ell,m,\bar{r},\omega) \qquad \text{B-1-6}$$

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Using equation B-1-4 and the notation $\delta = \frac{\varepsilon\rho}{\bar{r}}$ gives (retaining the δ^2 terms only) the following expression for the component $A(\ell,m,r',\omega)$

$$A(\ell,m,r',\omega) = A(\ell,m,\bar{r},\omega) + [\delta\sin(\bar{\theta}-\tilde{\phi}) + 1/2\ \delta^2 \cos^2(\bar{\theta}-\tilde{\phi})]\ \bar{r}\frac{d}{d\bar{r}}\ A(\ell,m,\bar{r},\omega) + 1/2\ \bar{r}^2\ \delta^2\ \sin^2(\bar{\theta}-\tilde{\phi})\ \frac{d^2}{d\bar{r}^2}\ A(\ell,m,\bar{r},\omega)$$

To expand $e^{i\ell\theta'}$ the relation

$$e^{i\theta'} = \frac{x'}{r'} + i\,\frac{y'}{r'} = e^{i\bar{\theta}}\,[1 - 1/2\ \delta^2 + i\delta e^{i(\tilde{\phi}-\bar{\theta})}\ (1-\delta\sin(\bar{\theta}-\tilde{\phi})) - \delta\sin(\bar{\theta}-\tilde{\phi})\ (1-3/2\ \delta^2)]$$

is used along with the approximation $(1-x)^n \approx 1 - nx + n(n-1)x^2 \quad [x^2 \ll 1]$ to obtain

$$e^{i\ell\theta'} = e^{i\ell\bar{\theta}}\ [1 + i\ell\delta\cos(\bar{\theta}-\tilde{\phi}) + \ell\delta^2\cos(\bar{\theta}-\tilde{\phi})[-i\sin(\bar{\theta}-\tilde{\phi}) - \frac{\ell}{2}\cos(\bar{\theta}-\tilde{\phi})]]$$

The integrand in equation B-1-2 can now be written as

$$\vec{\nabla}\phi\cdot\frac{\partial f_o}{\partial\vec{v}} = f_o e^{i(\omega t'-kz')}\left[\frac{ikv_z m}{k_b T_{\parallel}} - \frac{m}{k_b T_{\perp}}\frac{d}{dt'} + i\frac{\ell mc}{e}\,\frac{g'}{g}\right] A(\ell,m,r',\omega)e^{i\ell\theta'} \qquad \text{B-1-7}$$

where $\frac{d}{dt'}\ A(\ell,m,r',\omega)e^{i\ell\theta'} = [v'_r\ \frac{d}{dr'}\ A(\ell,m,r',\omega) + \frac{i\ell v'_\theta}{r'}\ A(\ell,m,r',\omega)]e^{i\ell\theta'}$ and g' is the derivitive of g with respect to its argument. The first order number density is obtained by integrating the first order distribution given by equation B-1-2 over velocity space. Since the average of $A(\ell,m,r',\omega)e^{i\ell\theta'}$ over the phase ϕ in velocity space gives

$$\int_0^{2\pi} A(\ell,m,r',\omega)e^{i\ell\theta'}\,d\phi = 2\pi e^{i\ell\bar{\theta}}(A(\ell,m,\bar{r},\omega) + \frac{1}{4}\rho^2\ [\frac{1}{\bar{r}}\frac{d}{d\bar{r}}\ \bar{r}\ \frac{d}{d\bar{r}} - \frac{\ell^2}{\bar{r}^2}\ ]\ A(\ell,m,\bar{r},\omega)\ )$$

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and

$$\frac{d}{dt'} \int_0^{2\pi} A(\ell,m,r',\omega)e^{i\ell\theta'} d\phi = 2\pi i\ell\omega_D \int_0^{2\pi} A(\ell,m,r',\omega)e^{i\ell\theta'} d\phi$$

the first order perturbation in the number density can be written

$$n_1 = 2i\frac{\varepsilon e}{m} \int_{-\infty}^{t} dt' \int_{-\infty}^{\infty} dv_z \int_0^{\infty} v_\perp dv_\perp \, e^{i(\omega t'-kz'-\ell\bar{\theta})}$$
$$\left\{\frac{kmv_z}{k_bT_\parallel} - \frac{m\ell\omega_D}{k_bT_\perp} + \frac{\ell me}{c}\frac{g'}{g}\right\} f_o \left[1+\frac{\rho^2}{4}\left(\frac{1}{\bar{r}}\frac{d}{d\bar{r}}\bar{r}\frac{d}{d\bar{r}}\right.\right.$$
$$\left.\left. - \frac{\ell^2}{\bar{r}^2}\right)\right] A(\ell,m,\bar{r},\omega) \qquad \text{B-1-8}$$

In evaluating the first order perturbation of the number density the time integration is done first, the integration over the parallel velocities second and the integration over the perpendicular velocities last. Using the expressions $z' = v_z(t'-t)+z$ and $\bar{\theta} = \omega_D(t'-t)+\theta$ the time integration yields terms of the form

$$\frac{-i \exp i(\omega t-kv_z+\ell\theta)}{(\omega+\ell\omega_D-kv_z)}$$

assuming that the $\mathrm{Im}\,\omega < 0$. The integration over the parallel velocities then gives

$$n_1 = \frac{\varepsilon e}{\pi k_bT_\parallel} \int_0^{\infty} v_\perp dv_\perp \, e^{-\frac{mv_\perp^2}{2k_bT_\perp}} \left\{\left[\frac{\ell\varepsilon}{kv}\frac{mc}{e} g' - g\frac{\ell\omega_D}{kv}\frac{m}{k_bT_\perp}\right] Z\left(\frac{\omega+\ell\omega_D}{kv}\right)\right. \qquad \text{B-1-9}$$
$$\left. + \frac{m}{k_bT_\parallel}\left(1+\left(\frac{\omega+\ell\omega_D}{kv}\right) Z\left(\frac{\omega+\ell\omega_D}{kv}\right)\right) g\right\} \left[1+\frac{\rho^2}{4} L_{\bar{r}}\right] A(\ell,m,\bar{r},\omega)$$

where the differential operator $L_{\bar{r}}$ is defined by $L_{\bar{r}} = \frac{1}{\bar{r}}\frac{d}{d\bar{r}}\bar{r}\frac{d}{d\bar{r}} - \frac{\ell^2}{\bar{r}^2}$; $v \equiv v_{\parallel\text{thermal}}$ and $Z(\zeta) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \frac{dx\, e^{-x^2}}{x-\zeta}$ is the plasma dispersion

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function defined for $\mathrm{Im}\,\omega < 0$.

Before carrying out the integration over the perpendicular velocities the dependence of g and Z on the perpendicular velocity is investigated. The vector potential in the argument of g satisfies the relation

$$\frac{A(r)}{r} + \frac{d}{dr}A(r) = \frac{1}{r}\frac{d}{dr}rA(r) = B(r) = 2\frac{d}{dr^2}(rA(r)) \qquad \text{B-1-10}$$

The assumptions that A(r) varies slowly over distances on the order of ρ and $\frac{dA}{dr} \neq 0$ give

$$A(r') = A(\bar{r}) + \varepsilon\rho\,\sin(\bar{\theta}-\tilde{\phi})\,\frac{d}{d\bar{r}}\,A(\bar{r})$$

and

$$r'A(r') = \bar{r}\,A(\bar{r}) + \bar{r}\,\frac{\varepsilon v_\perp}{\Omega}\sin(\bar{\theta}-\tilde{\phi})[A(\bar{r})+\bar{r}\,\frac{d}{d\bar{r}}\,A(\bar{r})]$$

or using equation B-1-10 $r'A(r') = \bar{r}A(\bar{r}) + \bar{r}\varepsilon v_\perp \frac{mc}{e}\sin(\bar{\theta}-\tilde{\phi})$. Using the expansion $r'v_\theta' = -v_\perp \sin(\bar{\theta}-\tilde{\phi}) - \frac{v_\perp^2}{\Omega\varepsilon}$ which is obtained by multiplying equation B-1-4 by B-1-5, the argument of g can be written as $r'A(r') + \frac{mc}{\varepsilon e}r'v_\theta' = \bar{r}A(\bar{r}) - \frac{v_\perp^2 mc}{\Omega e}$. So that

$$g(r'A(r')+\frac{mc}{\varepsilon e}r'v_\theta') = g(\bar{r}A(\bar{r})) - \frac{v_\perp^2 mc}{\Omega e}) \approx$$

$$g(\bar{r}A(\bar{r})) - \frac{2v_\perp^2}{\Omega^2}\frac{d^2}{d\bar{r}^2}g(\bar{r}A(\bar{r}) \approx g(\bar{r}A(\bar{r}))$$

is independent of the perpendicular velocities to the first order in ρ and $g(\bar{r}A(\bar{r})) = n_o(\bar{r})$. Also the derivitive of g with respect to its argument can be expressed as

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$$g' = \frac{d}{d\bar{r}A(\bar{r})} \quad g = \frac{d\bar{r}^2}{d\bar{r}A(\bar{r})} \frac{dg}{d\bar{r}^2} = \frac{2}{B(\bar{r})} \frac{d}{d\bar{r}^2} \quad g(\bar{r}A(\bar{r})) =$$

$$\frac{2}{B(\bar{r})} \frac{d}{d\bar{r}^2} \quad n_o(\bar{r})$$

The plasma dispersion function, Z, is expanded in powers of $\frac{\ell\omega_D}{kv}$ under the assumption that $(\ell\omega_D)^2 << \omega^2$

$$Z(\frac{\omega+\ell\omega_D}{kv}) \approx Z(\frac{\omega}{kv}) + \frac{\ell\omega_D}{kv} \quad Z'(\frac{\omega}{kv})$$

For the imaginary part of ω equal to zero both the real and imaginary part of Z' are about the same magnitude as the real and imaginary parts of Z. Also over much of the range of $\frac{\omega}{kv}$ the dispersion function does not vary rapidly. Thus the approximation made should be a good representation of $Z(\omega+\ell\omega_D)$ for $(\frac{\ell\omega_D}{\omega})^2 << 1$. For values of $\ell\omega_D$ near ω the approximation may still be fairly good if Z varies slowly in the range of $\frac{\omega}{kv}$ being investigated. The first order number density involves a term $\frac{\omega+\ell\omega_D}{kv} Z(\frac{\omega+\ell\omega_D}{kv})$ which is expanded as $\frac{\omega}{kv} Z(\frac{\omega}{kv})$ + $\frac{\omega}{kv} \frac{\ell\omega_D}{kv} Z'(\frac{\omega}{kv}) + \frac{\ell\omega_D}{kv} Z(\frac{\omega}{kv})$.

Using the above expansions for g and Z the integration over the perpendicular velocities can now be easily performed by noting that

$$\frac{m}{2\pi k_b T} \int_0^\infty v_\perp^{2p+1} \, e^{-mv_\perp^2/2k_bT} \, dv_\perp = \frac{p!}{2\pi} \left(\frac{2k_bT}{m} \right)^p$$

The first order number density can then be written

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$$n_1 = \frac{\varepsilon e}{m}\Bigg[\Bigg\{ \frac{2\ell\varepsilon}{k\Omega v}\left(Z(\tfrac{\omega}{kv}) + \frac{\ell<\omega_D>}{kv} Z'(\tfrac{\omega}{kv})\right) n_o' + \frac{n_o \ell<\omega_D>}{kv}\Bigg(Z(\tfrac{\omega}{kv})$$

$$+ \frac{\ell<\omega_D>}{kv} Z'(\tfrac{\omega}{kv})\Bigg) \frac{m}{k_b T_{\parallel}} [1 - \frac{T_{\parallel}}{T_{\perp}}] + \frac{m}{k_b T_{\parallel}} n_o\Bigg(1+\frac{\omega}{kv}(Z(\tfrac{\omega}{kv}) +$$

$$\frac{\ell<\omega_D>}{kv} Z'(\tfrac{\omega}{kv}))\Bigg)\Bigg\} A(\ell,m,\bar{r},\omega) + \frac{<\rho^2>}{4}\Bigg\{ \frac{2\ell\varepsilon n_o'}{k\Omega v} (Z(\tfrac{\omega}{kv})$$

$$+ \frac{2\ell<\omega_D>}{kv} Z'(\tfrac{\omega}{kv})) + \frac{2\ell<\omega_D>}{kv} n_o(Z(\tfrac{\omega}{kv}) + \frac{2\ell<\omega_D>}{kv} Z'(\tfrac{\omega}{kv}))$$

$$\cdot \frac{m}{k_b T_{\parallel}} [1 - \frac{T_{\parallel}}{T_{\perp}}] + \frac{m}{k_b T_{\parallel}} n_o\Bigg(1 + \frac{\omega}{kv} (Z(\tfrac{\omega}{kv}) + \frac{2\ell<\omega_D>}{kv}$$

$$\cdot Z'(\tfrac{\omega}{kv}))\Bigg) \Bigg\} L_{\bar{r}} \; A(\ell,m,\bar{r},\omega)\Bigg] \qquad \text{B-1-11}$$

where $n_o' = \frac{d}{d\bar{r}^2} n_o$.

Using equation B-1-11 and the fact that the electron gyroradius is much smaller than the ion gyroradius, Poisson's equation for a single component of the electric potential written in the form

$$(L_{\bar{r}} - k^2)\, A(\ell,m,\bar{r},\omega) = -4\pi e\, [n_1(\text{ions}) - n_1(\text{electrons})]$$

can be expressed as (replacing $\bar{r}$ by r)

$$\frac{1}{r}\frac{d}{dr}\, r\, \frac{d}{dr}\, A(\ell,m,r,\omega) = (\frac{\ell^2}{r^2} + \frac{N(r)}{D(r)})\, A(\ell,m,r,\omega)$$

Using the notation introduced in chapter 2 where the subscript i denotes the particle species and the subscript j refers to the plasma type, N(r) and D(r) are

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$$N(r) = k^2 + 4\pi e^2 \sum_i \sum_j \left\{ \frac{2\ell \varepsilon_i n_{oj}'}{m_i \Omega_i k v_{ij}} \left(Z(\zeta_{ij}) + \frac{\ell \langle\omega_D\rangle_{ij}}{k v_{ij}} Z'(\zeta_{ij}) \right) \right.$$

$$+ n_{oj} \frac{\ell\langle\omega_D\rangle_{ij}}{k v_{ij}} \left(Z(\zeta_{ij}) + \frac{\ell\langle\omega_D\rangle_{ij}}{k v_{ij}} Z'(\zeta_{ij}) \right) \frac{1}{k_b T_{\parallel ij}} [1 - \frac{T_{\parallel ij}}{T_{\perp ij}}]$$

$$\left. + \frac{n_{oj}}{k_b T_{\parallel ij}} \left(1 + \zeta_{ij} \left(Z(\zeta_{ij}) + \frac{\ell\langle\omega_D\rangle_{ij}}{k v_{ij}} Z'(\zeta_{ij}) \right) \right) \right\}$$

B-1-12

$$D(r) = 1 - 4\pi e^2 \sum_j \frac{\langle \rho_j^2 \rangle}{4} \left\{ \frac{2\ell n_{oj}'}{m_i \Omega_i k v_{ij}} \left(Z(\zeta_{ij}) + \frac{2\ell\langle\omega_D\rangle_{ij}}{k v_{ij}} Z'(\zeta_{ij}) \right) \right.$$

$$+ n_{oj} \frac{2\ell\langle\omega_D\rangle_{ij}}{k v_{ij}} \left(Z(\zeta_{ij}) + \frac{2\ell\langle\omega_D\rangle_{ij}}{k v_{ij}} Z'(\zeta_{ij}) \right) \frac{1}{k_b T_{\parallel ij}} [1 - \frac{T_{\parallel ij}}{T_{\perp ij}}]$$

$$\left. + \frac{n_{oj}}{k_b T_{\parallel ij}} \left(1 + \zeta_{ij} \left(Z(\zeta_{ij}) + \frac{2\ell\langle\omega_D\rangle_{ij}}{k v_{ij}} Z'(\zeta_{ij}) \right) \right) \right\}$$

The zero order distribution function has been written as

$$f_o = \sum_{i,j} n_{oj}(r) \left(\frac{m}{2\pi k_b T_{\parallel ij}} \right)^{1/2} \left(\frac{m}{2\pi k_b T_{\perp ij}} \right) e^{\frac{-m v_z^2}{2 k_b T_{\parallel ij}}} e^{\frac{-m v_\perp^2}{2 k_b T_{\perp ij}}}$$

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APPENDIX B-II

The object of this appendix is to review the method for obtaining solutions to the differential equation 4-3. The method employed is the familiar W.K.B. approximation but the procedure merits review because the function $q^2 = -(\ell^2 + \bar{U}(x))$ in equation 4-3 (which is usually a real function of x since it corresponds to the potential energy in the Schrodinger equation) is complex. Assuming the solutions to the equation have the form k exp i S(x) equation 4-3 can be regarded as an equation for S(x).

$$i\, S''(x) - [S'(x)]^2 + q^2(x) = 0 \qquad \text{B-II-1}$$

The assumption that S(x) is an almost linear function of x so that S"(x) is negligible gives a first approximation to S(x): $S_o(x) = \int q(x)dx$. The function $q(x) \equiv [-(\ell^2 + \bar{U}(x))]^{1/2}$ has two possible phases. In the following discussion a particular value of the phase of q is assumed to have been chosen and used consistently throughout. Using this first approximation to express S" in terms of q(x) gives another equation for S(x)

$$iq'(x) - [S'(x)]^2 = -q^2(x)$$

which has the solution $S_1(x) = \int \sqrt{q^2(x) + i\, q'(x)}\, dx$. The function $q^2(x)$ is assumed to be slowly varying so that $|q'(x)| << |q^2(x)|$ and the radical in the expression for S_1 can be expanded to give $S_1(x) = \int [q + \frac{i}{2}\frac{q'}{q}]\, dx = \int q dx + \frac{i}{2}\ \ln\, q$. Thus the solutions to equation 4-3 can be written as

$$\psi_{\pm} = \frac{1}{\sqrt{q}}\ \exp \pm i \int q(x)dx \qquad \text{B-II-2}$$

The plus and minus sign are used to denote the two independant solutions to equation 4-3 that arise from the two possible choices of the phase of q.

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These solutions are singular at the values of x for which $q^2(x) = 0$ (these values of x will be called the turning points of q^2 in analogy to the usual terminology). The singularities in the solutions are artifically created by the approximation used since the differential equation is regular at the values of x for which $q^2(x) = 0$. Thus the actual solutions to the equation near the turning points may be found by using a different approximation. Suppose that $q^2(x) = 0$ at $x = x_o$. Since $q^2(x)$ is assumed to be slowly varying it is approximately linear in the vicinity of a turning point and $q^2(x) = \eta(x-x_o)$. The slope of $q^2(x)$ evaluated at x_o is given by $\eta = \gamma + i\delta = \frac{d}{dx}[q^2(x)]|_{x=x_o}$. So near the turning points the solutions to equation 4-3 are governed by the equation $\frac{d^2}{dx^2}\psi + \eta(x-x_o)\psi = 0$ which has the solutions $\sqrt{x-x_o}\, J_{1/3}(\frac{2}{3}\sqrt{\eta}(x-x_o)^{3/2})$ and $\sqrt{x-x_o}\, J_{-1/3}(\frac{2}{3}\sqrt{\eta}(x-x_o)^{3/2})$. These solutions which are valid in the vicinity of the turning points can be related to those given by B-II-2 which are valid elsewhere. Noting that $\int_{x_o}^{x} q(x)dx$ reduces to $\frac{2}{3}\sqrt{\eta}\,(x-x_o)^{3/2}$ as $x \to x_o$ since $q^2 \to \eta(x-x_o)$,

the solutions valid near the turning points can be written in a conveniently expandable form:

$$\sqrt{f/q}\,(k_1 J_{1/3}(f) + k_2 J_{-1/3}(f)) \qquad \text{B-II-3}$$

where $f = \int_{x_o}^{x} q(x)dx$. The asymtotic expansions of the solutions given in B-II-3 depend upon the phase of f in the vicinity of the turning point (and thus on the phase of q(x)). The asymtotic expansion for the Bessel function $J_\nu(z)$ for various ranges of the phase ϕ_z, of z is

$$J_\nu(z) \to e^{in\pi(\nu+1/2)} \sqrt{\frac{2}{\pi z}} \cos\left[z - (-1)^n \frac{\pi}{2}(\nu+1/2)\right]$$

$$(n-1/2)\pi \leq \phi_z \leq (n+1/2)\pi \qquad \text{B-II-4}$$

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where n is an integer. The apparent difference in the phase term of the cosine in equation B-II-4 for the upper limit of the phase range for n=m and the lower limit of the phase range for n=m+1 is taken care of by the dominance of the term $\exp[(-1)^{n+1} i(z - (-1)^n \frac{\pi}{2}(\nu+1/2)]$ in the cosine at values of $\phi_z = (n+1/2)\pi$. The appropriate solutions in the regions $x >> x_o$ can be related to those in the region $x << x_o$ by using the asymtotic expansion of the approximate solution valid near the turning point x_o. Since the asymtotic expansions of the solutions depends upon the phase of q at the turning point the phase of q must be chosen and held fixed.

To illustrate the method used to connect the solutions through a turning point it is best to consider a specific example. For the example the phase of $\sqrt{\eta}$ (and thus q) for $x > x_o$ is chosen to lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Then, under the convention that the phase of a fractional power of a positive quantity is zero, the phase of f is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ for $x > x_o$ and

$$f = \int_{x_o}^{x} q(x)dx = \frac{2}{3}\sqrt{\eta}\,(x-x_o)^{3/2}$$

for $x < x_o$

$$f = \int_{x_o}^{x} \frac{2}{3}\sqrt{\eta}\,(x-x_o)^{1/2}\,dx = \frac{2}{3}\sqrt{\eta}\,(x-x_o)^{3/2} =$$

$$e^{\frac{\pm 3\pi}{2} i}\ \frac{2}{3}\sqrt{\eta}\,(x_o-x)^{3/2}$$

For the example the phase of f for $x < x_o$ is chosen to be between π and 2π. Since the slope of q^2 in the vicinity of the turning point

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has been written as $\eta = \gamma + i\,\delta$ (where γ and δ are real), the real and imaginary parts of $\sqrt{\eta}$ are given by

$$\mathrm{Re}(\sqrt{\eta}) = \pm \frac{1}{\sqrt{2}} \left(\gamma + \sqrt{\gamma^2+\delta^2}\right)^{1/2} \; ; \quad \mathrm{Im}(\sqrt{\eta}) = \frac{\pm\ \delta}{\sqrt{2}\left(\gamma + \sqrt{\gamma^2+\delta^2}\right)^{1/2}}$$

The sign convention is simply that the $\mathrm{Re}(\sqrt{\eta}) > 0$ means that the plus sign is used to find the $\mathrm{Im}(\sqrt{\eta})$ and $\mathrm{Re}(\sqrt{\eta}) < 0$ means that the minus sign is used to find $\mathrm{Im}(\sqrt{\eta})$. So choosing $\mathrm{Re}(\sqrt{\eta}) > 0$ is sufficient to guarantee the phase of $\sqrt{\eta}$ (and thus f) is between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$. The sign of δ, the imaginary part of η, further restricts the phase of $\sqrt{\eta}$. For $\delta > 0$ the phase of $\sqrt{\eta}$ is between 0 and $\pi/2$ and for $\delta < 0$ the phase of $\sqrt{\eta}$ between $-\pi/2$ and 0. The asymtotic solutions for $x > x_0$ are obtained from B-II-4 with $n = 0$ and $\nu = \pm \frac{1}{3}$

$$\psi = \sqrt{\frac{2}{\pi q}} \left[k_1 \cos(f - \tfrac{5\pi}{12}) + k_2 \cos(f - \tfrac{\pi}{12})\right] \qquad x >> x_o$$

since δ has been chosen less than zero the asymtotic expansion for $x < x_o$ is obtained from B-II-4 with $n=1$ and $\nu = \pm 1/3$

$$\psi = \sqrt{\frac{2}{\pi q}} \left[e^{i\frac{5\pi}{6}} k_1 \cos(f + \tfrac{5\pi}{12}) + e^{i\frac{\pi}{6}} k_2 \cos(f + \tfrac{\pi}{12})\right] \qquad x << x_o$$

The phase of f in the regions away from the turning points will depend upon the behavior of q^2 in these regions. The possible changes in the phase of q when the real or imaginary parts of q^2 change sign (but not simultaneously) is discussed later. The solution for $x << x_o$ can be written

$$\psi = \sqrt{\frac{2}{\pi q}} \left(e^{i\frac{\pi}{4}} (k_2 - k_1) e^{if} + (k_1 e^{i\frac{5\pi}{12}} + k_2 e^{i\frac{\pi}{12}})\, e^{-if} \right) \qquad \text{B-II-5}$$

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Appropriate choices of the constants k_1 and k_2 can be made to satisfy certain restrictions that may be placed on the solutions. For example if it is desired that no exponentially growing terms be included in the solution for the region $x < x_o$ then the e^{if} term in equation B-II-5 must be excluded. This can be accomplished simply by setting $k_1 = k_2$. Then the solution for $x > x_o$ which is related to this particular solution has the form

$$\psi = \sqrt{\frac{2}{\pi q}}\ k \cos \pi/6 \cos(f-\pi/4)$$

This procedure can be used to establish other connections through the turning points for particular requirements on the solutions. The techniques used in the example can be applied to any given situation to find the proper asymtotic expansions of the solution valid near the turning point.

It is also of interest to investigate the case where $Re(q^2)$ or $Im(q^2)$ equals zero but $|q^2|$ is not equal to zero. In this case the solutions obtained by the standard W.K.B. method are not singular and thus can be taken to be valid solutions to equation 4-3. The phase of q can change at these points and it is important to determine just how it changes. With the definition $q^2 = -(\ell^2 + \overline{U}(x)) = -(A' + iB')$ the real and imaginary parts of q for $A' \neq 0$ and $B' \neq 0$ can be written

$$Re(q) = \pm \frac{1}{\sqrt{2}} (\sqrt{A'^2+B'^2} - A')^{1/2};\quad Im(q) = \mp \frac{B'}{\sqrt{2}(\sqrt{A'^2+B'^2}-A')^{1/2}} \qquad \text{B-II-6}$$

If $B'=0$ the real and imaginary parts of q are

$$Re(q)=0;\ Im(q)=\pm\sqrt{A'}\quad \text{if } A'>0:\ Re(q)=\pm\sqrt{|A'|};\ Im(q)=0 \text{ if } A'<0 \qquad \text{B-II-7}$$

and if $A' = 0$ they are

$$Re(q) = \pm\sqrt{\left|\frac{B'}{2}\right|};\ Im(q) = \mp\sqrt{\left|\frac{B'}{2}\right|}\ \text{ if } B'>0;\ Re(q) = \pm\sqrt{\left|\frac{B'}{2}\right|};$$
$$Im(q) = \pm\sqrt{\left|\frac{B'}{2}\right|}\ \text{ if } B' < 0 \qquad \text{B-II-8}$$

So, once a particular pair of signs in B-II-6 are chosen and the phase of q thus specified, the sign of the imaginary part of q can change

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only if B' goes through zero and $A' < 0$. The real part of q^2 going through zero does not affect the signs of the parts of q nor does the imaginary part of q^2 going through zero if $A' > 0$. Thus the solutions to equation 4-3 change from a growing to a decaying solution, or vice versa, only if B' changes sign when $A' < 0$.

The solutions discussed in this appendix as functions of x are easily viewed as functions of r by using the relationship $x = \ln r$ and the definition $q^2(r) = -\left(\frac{\ell^2}{r^2} + \frac{N(r)}{D(r)}\right)$ to obtain

$$\int_{x_o}^{x} q(x)dx = \int_{x_o}^{x} (-\ell^2 - \bar{U}(x))^{1/2}dx = \int_{r_o}^{r} \left(-\frac{\ell^2}{r^2} - \frac{N(r)}{D(r)}\right)^{1/2}dr =$$

$$\int_{r_o}^{r} q(r)dr \quad \text{and} \quad \eta \equiv \left[\frac{d}{dx}\,(q^2(x)|_{x=x_o}\right] = \left[r\frac{d}{dr}\,(r^2q^2(r))|_{r=r_o}\right]$$

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APPENDIX C

In this appendix the computer programs that were used to investigate the behavior of the function $U(r,\ell,m,\omega)$ given in equation 4-2 are given in detail. The first program given below is used to calculate the values of U for various values of the dimensionless complex parameter zeta for fixed values of r,ℓ,m. As discussed in Chapter 4 the zeros of U are the points of interest. The zeros of U occur at the points where the numerator of U is zero. The program searches for zeros of U in the complex zeta plane. The grid has a starting point specified in the program by (ZEROWR, ZEROWI). The real and imaginary parts of zeta can be incremented from this point in any of the four possible directions by specifying the parameters QR and QI. If the real part of zeta is to be increased from the starting point value, QR is specified as +1. If it is to be decreased QR is given as a -1. QI controls the increment of the imaginary part in a similar fashion. The total number of grid "squares" or meshes is $IEND^2$. The size of a grid mesh is specified by the parameter STEP. The increment of the real part of zeta in one grid mesh is .1·(STEP) and the increment of the imaginary part is .1·(STEP)·(RATIO). In the program the grid is traversed by incrementing the real part of zeta until the end of the grid is reached and then increasing the imaginary part by one step and again running through the grid row by incrementing the real part of zeta, etc. At each grid point the sign of the imaginary part of the numerator of U is tested for a sign change. If a sign change is found the real part of the numerator of U is tested for a sign change between the two grid points at which the imaginary

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part of the numerator of U had a sign change. If a sign change in both the real and imaginary parts of the numerator of U is detected a smaller grid with as many squares as the original grid but the size of one grid square is set up in this vicinity of the complex zeta plane and U is tested for zeros in the same manner as before but using this smaller grid. This procedure is repeated (JUMP-1) times. The values of the numerator of U at each grid point are printed for reference.

The second program given below is used to calculate the behavior of U as a function of r. The values of ℓ,m, and ω are held fixed. The program also searches for turning points of U (i.e. where both the real and imaginary parts of U are equal to zero) and gives the location of any turning points that are found. The variable names used in both programs and the procedures used are explained by comment cards in the programs.

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```
C
C
C                                    MAINPROGRAM  1
C
C                                    **************
C
C   IN THIS PROGRAM THE NUMERATOR OF U IS TESTED FOR ZEROS BY LOOKING
C   FOR SIMULTANEOUS SIGN CHANGES IN THE REAL AND IMAGINARY PARTS OF
C   THE NUMERATOR OF U.
C   THE SIGN CHANGES IN THE PARTS OF U(ZETA) ARE ASSUMED TO OCCUR ON A SCALE
C  SIZE GREATER THAN  2*STEP  IN THE COMPLEX ZETA PLANE.  A GRID HAVING
C  A MESH SIZE DETERMINED BY STEP IS USED TO LOOK FOR SIGN CHANGES IN U.
C  IF A SUSPECTED ZERO OF U IS FOUND, A SMALLER MESH IS SET UP ABOUT THIS
C  POINT, AND SILMULTANEOUS SIGN CHANGES ARE AGAIN SOUGHT.  THE NUMBRE OF
C  TIMES THIS SMALLER MESH IS SET UP IS = (JUMP-1).  THERE IS ASSUMMED TO
C  BE,AT MOST, ONE ZERO OF U IN EACH GRID SQUARE.
C
C  THIS PROGRAM CHECKS THE IMAGINARY PART OF U FIRST FOR A SIGN CHANGE
C
C    THE MEANINGS OF THE VARIABLE NAMES ARE GIVEN WHEN THEY  ARE READ IN.
C  THE ORDER OF THE DATA CARDS IS
C  TEMP     190 FORMAT(4E10.3)
C   TEMP IS THE PARALLEL TEMPERATURE OF THE PLASMA
C  ISGN    191 FORMAT (2I3)
C   ISGN IS THE SIGN OF THE PARTICLE'S ELECTRONIC CHARGE
C  PMASS    192 FORMAT (2E10.4)
C   PMASS IS THE PARTICLE'S MASS
C  TPERP    12 FORMAT (4E10.3)
C   TPERP IS THE PERPENDICULAR TEMPERATURE OF THE PLASMA
C CHARG,POWER,BFLO,SPD     7 FORMAT (E10.3,F6.2,2E10.3)
C   CHARG IS THE ELECTRONIC CHAGE IN E.S.U.  BFLD IS THE VALUE OF THE
C   MAGNETIC FIELD AT THE MAGNETOPAUSE AND THE FIELD IS ASSUMED TO VARY
C   AS ( 1/R)**POWER.   SPD IS THE SPEED OF LIGHT IN C.G.S. UNITS
C  KAY,KAPPA     6 FORMAT (2E10.4)
C   KAY IS BOLTZMANN'S CONSTANT
C   KAPPA IS THE PARALLEL WAVE NUMBER
C  N1,N2,REP  30 FORMAT (2F6.1,F8.3)
C   N1 IS THE VALUE OF THE PLASMASPHERE NUMBER DENSITY AT THE MAGNETOPAUSE
C   N2 IS THE VALUE OF THE PEAK RING CURRENT NUMBER DENSITY.
C  IRAD1          8 FORMAT (5I3)
C   IRAD1 GIVES THE VALUE OF R AT WHICH U IS TO BE TESTED FOR ZEROS
C   THE FOLLOWING ARE INITIAL VALUES OF PARAMETERS USED TO CONTROL
C   THE PROGRAM AND THEIR FUNCTIONS ARE EXPLAINED WHEN THEY FIRST OCCUR
C   IN THE PROGRAM.
C  MONE1          8 FORMAT (5I3)
C  MTWO           8 FORMAT (5I3)
C  MTHREE         8 FORMAT (5I3)
C  MFOUR          8 FORMAT (5I3)
C  ZEROWR      18 FORMAT(5G10.4)
C  ZEROWI      18 FORMAT(5G10.4)
C  ICNT   8 FORMAT(5I3)
C  JCNT   8 FORMAT(5I3)
C  IJEND       60 FORMAT(I3)
C   IJEND GIVES THE NUMBER OF THE VALUES OF R AT WHICH U WILL BE INVESTIGATED
C  FOR EACH VALUE OF IJEND NEW VALUES OF THE FOLLOWING GROUP OF PARAMETERS
```

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```
C  ARE READ IN AND THUS THERE MUST BE IJEND GROUPS OF THESE CARDS.
C     STEP , JUMP  10 FORMAT(F5.1,I2)
C    IEND, IEND1    8 FORMAT (5I3)
C    QR,QI    45 FORMAT (2F3.0)
C       L        61 FORMAT (I5)
C    L IS THE AZIMUTHAL WAVE NUMBER.  2*PI*R/L IS THE AZIMUTHAL WAVE LENGTH
C     RATIO     6 FORMAT (2F10.4)
C    ARRAYS ARE GIVEN WITH THE RADIAL INDEX FIRST ( IF ONE IS NEEDED),THE
C    PARTICLE INDEX SECOND, AND THE PLASMA TYPE INDEX LAST.
      IMPLICIT COMPLEX (C,Z), REAL (K)
      EXTERNAL DISP,ZET
      REAL N1,N2
      REAL CHARG
      REAL ZEROWR(5), ZEROWI(5)
      COMPLEX YOU,DISP
      DIMENSION  ICNT(5), JCNT(5)
      DIMENSION G(100,2),GP(100,2),TEMP(2,2),ISGN(2),PMASS(2),RAD(200)
      DIMENSION  GYRO(100,2), GYRAD(100,2),COMEGA(100)
      DIMENSION MONE(5), MTWO(5)
      DIMENSION  MTHREE(5), MFOUR(5)
      DIMENSION IRAD1(5)
      DIMENSION VPERP(2,2)
      COMMON /COM1/VEL(2,2),ZETA(2,2)
      COMMON /COM2/ G,GP,PMASS, GYRO,GYRAD,TEMP,RAD,KAY,KAPPA,ISGN,L
      COMMON /COM3/ TOOBIG
      COMMON /COM5/ OMEGAD(100,2,2), TPERP(2,2)
      DATA COMEGA/100*(0.0,0.0)/
C    THE PARALLEL TEMPERATURE OF THE PLASMA IS GIVEN BY TEMP(J,I).
C    J=1 DENOTES IONS    J=2 , ELECTRONS
C    I=1 DENOTES PLASMASPHERE PARTICLES    I=2 , RING CURRENT PARTICLES
C    TEMP IS A 2 X 2 ARRAY IN FORMAT  4E10.3 CONTAINING THE PARALLEL TEMPERATURE
      READ(1,190) TEMP
  190 FORMAT(4E10.3)
C    ISGN IS AN ARRAY OF SIZE 2 IN FORMAT 2I3 CONTAINING THE PARTICLE SIGNS
      READ(1,191) ISGN
  191 FORMAT(2I3)
C    PMASS IS A SIZE 2 ARRAY IN FORMAT 2E10.4 CONTAINING THE PARTICLE MASSES
      READ(1,192) PMASS
  192 FORMAT(2E10.4)
C    THE PERPENDICULAR TEMPERATURE OF THE PLASMA IS GIVEN BY TPERP
C TPERP IS A 2X2 ARRAY IN FORMAT 4E10.3 CONTAINING THE PERPENDICULAR TEMPERATURE
      READ(1,12) TPERP
   12 FORMAT(4E10.3)
      WRITE(3,12) TPERP
    9 FORMAT (4E8.1,2I2,2E10.4/4E10.2)
C   CHARG AND SPD ARE THE ELECTRONIC CHARGE AND THE SPEED OF LIGHT IN CGS
C   UNITS.  BFLD IS THE VALUE OF THE MAGNETIC FIELD AT THE MAGNETOPAUSE IN GAUSS
C     B IS ASSUMED TO VARY AS (1/R)**POWER
      READ (1,7) CHARG,POWER,BFLD,SPD
    7 FORMAT (E10.3,F6.2,2E10.3)
C   TOOBIG IS USED TO MODULATE THE VALUE OF THE ARGUMENT OF THE SINE AND COSINE
C   IN THE SUBROUTINE DISP TO KEEP THEM FROM BECOMING TOO LARGE
      TOOBIG = .823549E+06
      KON = SPD/CHARG
C    KAY,KAPPA ARE THE PARALLEL WAVELENGTH AND BOLTZMANN'S CONSTANT IN FORMAT
```

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```
C   2E10.4
       READ (1,6) KAY,KAPPA
     6 FORMAT (2E10.4)
C   N1 IS THE PLASMASPHERE NUMBER DENSITY AT ITS OUTER BOUNDRY  N2 IS
C   THE PEAK RING CURRENT NUMBER DENSITY.  THE PEAK IS ASSUMED TO OCCUR
C    AT 6 RE IN THIS MODEL.  REP IS THE RADIAL DISTANCE TO THE BEGINNING
C   OF THE PLASMA PAUSE GIVEN IN RE.  IN THIS MODEL A VALUE OF 5.9 RE IS
C   USED SINCE RUSSEL AND THORNE HAVE FOUND THAT THE PLASMAPAUSE CLOSELY
C   COINCIDES WITH THE PEAK RING CURRENT NUMBER DENSITY
       READ(1,30) N1,N2,REP
    30 FORMAT (2F6.1,F8.2)
       WRITE (3,17)
    17 FORMAT ('1','   THE FOLLOWING IS A PRINT OUT OF DATA CARDS 5-7 AND
      1 1-3, THE ARRAY VEL (I,J), AND DATA CARDS 8-17, IN THAT ORDER'///)
       WRITE (3,7) CHARG,POWER,BFLD,SPD
       WRITE (3,6) KAY,KAPPA
       WRITE(3,30) N1,N2,REP
C   THE THERMAL VELOCITIES ARE CALCULATED
       DO 15 I=1,2
       DO 15 J = 1,2
C   VEL GIVES THE PARALLEL VELOCITIES AND VPERP THE PERPENDICULAR VELOCITIES
C   WITH THE FIRST INDEX GIVING THE PARTICLE SPIECE AND THE SECOND GIVING
C   THE PLASMA GENRE
       VEL (J,I) = SQRT ( 2*KAPPA*TEMP(J,I)/PMASS(J))
       VPERP(J,I)= SQRT(2*KAPPA*TPERP(J,I)/PMASS(J))
    15 CONTINUE
C   THE GYRORADIUS FOR THE IONS AND THE GYROFREQUENCIES ARE CALCULATED
C     B IS ASSUMED TO VARY AS (1/R)**POWER
       BO = BFLD*((6.35E9)**POWER)
       DO 5 I4 = 1,100
       R= (6.350E7)*I4
       RD= R**POWER
       BF = BO/RD
       DO 5 I5=1,2
       GYRO(I4,I5)= CHARG*BF/(PMASS(I5)*SPD)
       GYRAD(I4,I5)= VPERP(1,I5)/GYRO(I4,1)
       DO 5 I6=1,2
C   OMEGAD IS A 100 X 2 X 2  ARRAY WHICH GIVES THE AZIMUTHAL DRIFT FREQUECIES
C   OF THE PARTICLES
     5 OMEGAD(I4,I6,I5)=(-.5)*ISGN(I6)*(VPERP(I6,I5)**2)*PMASS(I6)*(1/BF)
      1*KON*(POWER/(R**2))
       WRITE(3,9) TEMP,ISGN,PMASS,VEL
C   FROM THIS POINT TO STATEMENT NUMBER 150
C    THE MODEL NUMBER DENSITY, G , AND ITS DERIVITIVE WITH RESPECT TO
C    R**2 , GP , ARE GENERATED
       PEAK=6*(6.350E8)
       RE2=(6.35E8)**2
       A2=(16./ALOG(10.*N2))*RE2
       A=SQRT(A2)
       B2=(1./ALOG(10.))*RE2
       B=SQRT(B2)
       REP = REP*(6.35E8)
       D=(.5/ALOG(10.*N1))*RE2
       REP2=REP*REP
       DO 150 I3=1,100
```

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```
      RAD(I3)=(6.350E7)*I3
      EX=-(RAD(I3)-PEAK)
      EX2=EX*EX
      IF (EX) 50,50,51
   50 G(I3,2)=N2*EXP(-EX2/A2)
      GP(I3,2)=2*EX*G(I3,2)/(A2*RAD(I3))
      GO TO 53
   51 G(I3,2)=N2*EXP(-EX2/B2)
      GP(I3,2)=2*EX*G(I3,2)/(B2*RAD(I3))
   53 IF(RAD(I3)-REP) 54,54,55
   54 G(I3,1)=N1*REP2/(RAD(I3)**2)
      GP(I3,1)= -1*G(I3,1)/(RAD(I3)**2)
      GO TO 150
   55 IF(RAD(I3)-PEAK) 58,58,59
   59  G(I3,1)=0.0
      GP(I3,1)= 0.0
      GO TO 150
   58 EXX=-(RAD(I3)-REP)
      EXX2=EXX*EXX
      G(I3,1)=N1*EXP(-EXX2/D)
      GP(I3,1) = 2*EXX*G(I3,1)/(D*RAD(I3))
  150 CONTINUE
C  ZEROS OF U ARE SOUGHT AT FIXED VALUES OF R.  IRAD1 IS AN ARRAY IN WHICH
C  THE VALUES OF R ARE GIVEN IN TENTHS OF EARTH RADII. E.G. 5.5 RE IS
C  GIVEN AS 55. IJEND MUST EQUAL THE DIMENSION OF IRAD1 (.LE. 5)
C  THIS ALLOWS U TO BE INVESTIGATED AT SEVERAL VALUES OF R.
      READ(1,8) IRAD1
    8 FORMAT(5I3)
C  M1 AND M2 DETERMINE AT WHICH POINT IN THE GRID OF MESH SIZE  STEP
C  THE SEARCH FOR ZEROS IS TO BEGIN.  M2*STEP*.1= REAL PART OF ZETA
C  M1*STEP*.1= IMAGINARY PART OF ZETA.  THE VALUES OF M1 AND M2 ARE STORED IN
C   THE ARRAYS MONE AND MTWO.
      READ (1,8) MONE
      READ (1,8) MTWO
      WRITE (3,8)  MONE
      WRITE (3,8) MTWO
C  MTHREE AND MFOUR ARE USED TO INSURE THAT ANY ZERO FOUND IN THE  FIRST GRID
C  ELEMENT WILL BE RECORDED.  THEY SHOULD INITIALLY HAVE THE VALUES OF
C  MONE AND MTWO.
      READ(1,8) MTHREE
      WRITE (3,8) MTHREE
      READ (1,8) MFOUR
      WRITE (3,8) MFOUR
C  ZEROWR AND ZEROWI ARE THE REAL PART AND IMAGINARY PART, RESPECTIVELY,
C  OF THE ORIGIN OR STARTING POINT OF THE GRID IN THE COMPLEX ZETA PLANE.
      READ (1,18) ZEROWR
      WRITE (3,18) ZEROWR
      READ(1,18) ZEROWI
      WRITE(3,18) ZEROWI
   18 FORMAT (5G10.4)
C  ICNT IS A SIZE 5 ARRAY WHICH CONTAINS THE INITIAL VALUES OF ICOUNT FOR
C  EACH RUN THAT IS TO BE MADE AT A DIFFERENT VALUE OF R.
C  ICOUNT COUNTS THE NUMBER OF TIMES THAT THE GRID HAS BEEN REDUCED IN SIZE
C  FOR A GIVEN TEST OF THE SIGNS OF U
C  IF ICOUNT=0 THEN ZEROW SHOULD EQUAL (ZEROWR,ZEROWI)
```

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```
      READ(1,8) ICNT
      WRITE (3,8) ICNT
C   JCNT IS THE INITIAL VALUE OF JCOUNT, THE COUNTER OF THE NUMBER OF
C   ZEROS OF U FOUND ON A GIVEN RUN
      READ (1,8) JCNT
      WRITE (3,8) JCNT
C  IJEND GIVES THE NUMBER OF DIFFERENT VALUES OF R FOR WHICH U IS TO
C  BE INVESTIGATED.  THE MAXIMUM VALUE THAT IT CAN HAVE IS 5.
      READ (1,60) IJEND
   60 FORMAT (I3)
      DO 210  IJ= 1,IJEND
      IRAD=IRAD1(IJ)
C  STEP IS THE SIZE OF THE REAL SIDE OF A GRID ELEMENT.  THAT IS, THE AMOUNT
C  THE REAL PART OF ZETA IS INCREASED IN EACH GRID ELEMENT.
C  STEP MUST BE EVENLY DIVISIBLE INTO IEND.
C  JUMP GIVES THE NUMBER OF TIMES THE GRID IS REDUCED AND THUS THE ACCURACY
C  TO WHICH A ZERO OF U IS FOUND CAN BE EXPRESSED IN TERMS OF JUMP.
C  A ZERO OF U LIES SOMEWHERE WITHIN AN AREA OF THE COMPLEX ZETA PLANE
C  OF STEP/(IEND**(JUMP-1)) BY STEP*RATIO/((IEND-1)**(JUMP-1))
      READ(1,10) STEP,JUMP
   10 FORMAT(F5.1,I2)
      READ (1,8) IEND,IEND1
C  QR GIVES THE  SIGN OF THE REAL PART OF ZETA, QI THE SIGN OF THE
C  IMAGINARY PART. E.G. QR=-1. MEANS THE REAL PART OF ZETA IS NEGATIVE
C  NOTE THAT A QR AND A QI IS READ FOR EACH VALUE OF R.
      READ (1,45) QR,QI
   45 FORMAT ( 2F3.0)
C   L IS THE AZIMUTHAL WAVE NUMBER
      READ(1,61) L
   61 FORMAT(I5)
      WRITE (3,14) L
   14 FORMAT('1','   THE VALUE OF L FOR THIS RUN IS ',I5///)
      WRITE(3,11)
   11 FORMAT('   THE DATA SET PERTINATE TO THIS RUN IS GIVEN IN THE ORDE
     1R   STEP,JUMP   IEND,IEND1   RATIO   QR,QI*RATIO. '///)
      WRITE(3,10) STEP,JUMP
      WRITE (3,8) IEND,IEND1
C    RATIO  IS THE RATIO OF THE IMAGINARY EXTENT OF A GRID ELEMENT TO THE
C    REAL EXTENT OF THE ELEMENT
C    IEND*STEP IS THE LARGEST REAL VALUE OF ZETA AND (IEND1-1)*STEP*RATIO
C    IS THE LARGEST IMAGINARY VALUE OF ZETA AT WHICH U IS TESTED FOR ZEROS
      READ(1,6) RATIO
      WRITE(3,6) RATIO
      QIR=QI
      QI= QI*RATIO
      WRITE (3,6) QR,QI
      EL=(L/RAD(IRAD))**2
      WRITE (3,16)
   16 FORMAT ('1')
C    THE PARAMETERS FOR A GIVEN VALUE OF R ARE INITIALIZED.
      ICOUNT=ICNT(IJ)
      JCOUNT =JCNT(IJ)
      ZEROW= CMPLX ( ZEROWR(IJ),ZEROWI(IJ))
      M1= MONE(IJ)
      M2= MTWO(IJ)
```

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```
      M3= MTHREE(IJ)
      M4= MFOUR(IJ)
C   SCALER AND SCALEI ARE SCALE FACTORS THAT REDUCE THE WHOLE GRID TO THE SIZE
C   OF A SINGLE GRID ELEMENT SO THAT U CAN BE TESTED FOR ZEROS ON A FINER
C   SCALE.  SCALER REDUCES THE REAL VALUES, SCALEI THE IMAGINARY VALUES
  102 SCALER= STEP/(IEND**ICOUNT)
      SCALEI = STEP/((IEND1-1)**ICOUNT)
C  A GRID IS SET UP IN THE COMPLEX ZETA PLANE.
  105 DO 202 II= M1,IEND1
      I=II-1
      ZAP = ZEROW+ I*(0.,.1)*QI*SCALEI + (M2-1)*QR*(.1,0.)*SCALER
      ZT=ZAP
C  THE VALUE OF THE NUMERATOR OF U AT GRID POINT (I,M2-1) IS OBTAINED
C  YOU IS A SUBROUTINE WHICH CALCULATES THE VALUES OF THE NUMERATOR OF U.
C  DISP IS A SUBROUTINE WHICH CALCULATES THE PLASMA DISPERSION FUNCTION.
C  ZET IS A SUBROUTINE WHICH CALCULATES THE VALUE OF ZETA FOR EACH PARTICLE
      ZTEST=YOU(ZET,DISP,ZT,IRAD)
      WRITE(3,400) ZTEST , ZT
  400 FORMAT('   THE NUMERATOR OF U HAS THE VALUE   ',G12.5,'   ',G12.5,
     1'   FOR ZETA=  ',2G10.4)
      TEST= AIMAG(ZTEST)
      IF (ICOUNT .EQ. 0) M3=II
C  THIS JUST INSURES THAT THE REAL PART OF ZETA IS NOT OUTSIDE THE RANGE OF THE
C GRID
      IF (M2-IEND .GT. 0) GO TO 204
      DO 200 J=M2,IEND
      ZETA2I= (.1,0.)*QR*J*SCALER + (0.,.1)*QI*I*SCALEI  + ZEROW
      ZT=ZETA2I
      ZTEST=YOU(ZET,DISP,ZT,IRAD)
C  THE VALUE OF THE NUMERATOR OF U AT EACH GRID POINT IS PRINTED OUT
C FOR REFFERENCE.
      WRITE(3,400) ZTEST , ZT
      BETA = AIMAG(ZTEST)
      IF (TEST .EQ. 0 ) GO TO 101
C  THE VALUE OF THE IMAGINARY PART OF THE NUMERATOR OF U AT GRID POINT (J,I)
C  IS COMPARED TO ITS VALUE AT THE GRID POINT (J+1,I)
      IF (BETA/TEST .GT. 0) GO TO 200
  101 IF ( ICOUNT .EQ.0 ) M4= J
C  IF THERE IS A SIGN CHANGE IN THE IMAGINARY PART OF THE NUMERATOR OF U
C  THE REAL PART OF THE NUMERATOR OF U IS TESTED FOR A SIGN CHANGE IN THE
C SAME GRID SQUARE AT  ELEVEN DIFFERENT POINTS ALONG THE GRID ELEMENT.
      DO 201 I2 = 1,11
      ZET1 = ZETA2I - QR*(I2-1)*(.01,0.)*SCALER - QI* (0.,.05)*SCALEI
      ZET2 = ZET1 + QI*(0.,.1)*SCALEI
      ALPH1=EL+REAL(YOU(ZET,DISP,ZET1,IRAD))
      ALPH2=EL+REAL(YOU(ZET,DISP,ZET2,IRAD))
       IF(ALPH1.EQ.0) GO TO 300
C  IF A SIGN CHANGE IS FOUND A FINER GRID IS SET UP IN THIS AREA OF THE COMPLEX
C  ZETA PLANE
      IF (ALPH2/ALPH1.LE.0) GO TO 300
  201 CONTINUE
C   THIS IS THE END OF THE DO LOOP WHICH INCREMENTS THE REAL PART OF ZETA
  200 TEST = BETA
      IF ( ICOUNT .NE.0) GO TO 202
  204 M2=1
```

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C   THIS IS THE END OF THE DO LOOP IN WHICH THE IMAGINARY PART OF
C   ZETA IS INCREMENTED.
  202 CONTINUE
C   THIS  TESTS WHETHER THE ENTIRE GRID HAS BEEN SEARCHED.
  211 IF (M3-IEND1) 212,213,213
  212 ICOUNT = 0
C   THE FOLLOWING STEPS ALLOW THE ORIGINAL GRID TO BE REENTERED AT THE
C   NEXT GRID POINT AFTER THE ONE AT WHICH IT WAS LEFT.
      M1=M3
      M2= M4+1
      ZEROW = CMPLX(ZEROWR(IJ),ZEROWI(IJ))
      SCALER = STEP
      SCALEI = STEP
      GO TO 105
  300 ICOUNT= ICOUNT+1
      M1=1
      M2=1
      ZEROW = ZETA2I- QI*(0.,.05) * SCALEI - QR*(.1,0.)*SCALER
      WRITE (3,301)  ALPH1,ALPH2,ZET1,ZET2
  301 FORMAT(//,'   THE REAL PARTS OF U DENOTING A SIGN CHANGE WERE  ',
     1G12.6,'  AND  ',G12.6,/,'   AT ZETA=  ',G10.4,'  ,  ',G10.4,'  AND
     2  ',G10.4,'  ,  ',G10.4//)
      IF(ICOUNT-JUMP) 102,230,230
  160 WRITE (3,1) QR,QIR
    1 FORMAT('    NO ZEROS FOR THE IMAGINARY PART OF U WERE FOUND FOR
     1QR=        AND QI=   ',T62,F4.1,T77,F4.1)
      GO TO 210
  230 JCOUNT=JCOUNT+1
C  THE PROGRAM IS TERMINATED IF MORE THAN 100 ZEROS OF U HAVE BEEN FOUND
      IF (JCOUNT.GT.100) GO TO 151
       ICOUNT=0
      M1=M3
      M2= M4+1
      ZEROW = CMPLX(ZEROWR(IJ),ZEROWI(IJ))
C  COMEGA SIMPLY STORES THE VALUES OF ZETA AT WHICH A ZERO OF U HAS BEEN FOUND
      COMEGA(JCOUNT) = ZET1 + QI* (0.,.05)* SCALEI
      WRITE (3,29) COMEGA(JCOUNT)
   29 FORMAT ('   A ROOT OF U IS AT OMEGA EQUAL TO  '/2G12.4)
      SCALER = STEP
      SCALEI = STEP
      GO TO 203
  203 IF(M1-IEND1) 105,105,210
  151 WRITE(3,33) QR,QIR,L,ZEROW
   33 FORMAT(1H1, ' THERE ARE MORE THAN 100 ROOTS OF U IN QUADRANT     ,
     1   FOR L=     . THE VALUE OF ZETA WAS ',T49,F3.0,T52,F3.0,T62,I3,
     2T91,2G12.6)
      GO TO 210
  213 IF (JCOUNT) 210,160,210
C   THIS IS THE END OF THE DO LOOP FOR A GIVEN VALUE OF R.  THAT IS
C   A NEW VALUE OF R IS READ IN
  210 CONTINUE
  152 STOP
      END
C
C                                  **************
```

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```
C
C
C                                   SUBPROGRAM    DISP
C
C   THIS SUBROUTINE IS A GEOPHYSICAL INSTITUTE LIBRARY SUBROUTINE
C  THIS SUBROUTINE GIVES THE PLASMA DISPERSION FUNCTION FOR VARIOUS VALUES
C  OF ZETA.  THE FUNCTION IS SIMILAR TO THE ONE TABULATED BY FRIED AND CONTE
C  BUT THE FUNCTION IS ANALYTICALLY CONTINUED FROM THE LOWER HALF OF THE
C ZETA PLANE.
C   A POWER SERIES EXPANSION IS USED TO CALCULATE THE DISPERSION FUNCTION
C   FOR VALUES OF ZETA LESS THAN OR EQUAL TO 3.  AN ASYMTOTIC EXPANSION IS USED
C  FOR VALUES OF ZETA LARGER THAN 3.  THE FUNCTION IS CALCULATED TO AN ACCURACY
C  OF .1 (
      COMPLEX FUNCTION DISP(ZO)
      COMMON/COM3/ TOOBIG
      COMPLEX ZO
        REAL X(2),A(2),B(2),C(2),D
      X1=REAL(ZO)
      X2= AIMAG(ZO)
      X(1)=X1
      X(2)=X2
      Y=X1*X1
      U=X2*X2
      YU=X1*X2
      A(1)=2.*(Y-U)
      A(2)=4.*YU
      RX=Y+U
      EX=-Y+U
      AR=2.*YU
      AR=AMOD(AR,TOOBIG)
    3 IF(EX-100.) 30,30,31
   31 PXP=1.E30
      GO TO 32
   30 PXP=1.772454*EXP(EX)
   32 Z1=-PXP*SIN(AR)
      Z2=-PXP*COS(AR)
      IF (RX-9.) 13,13,14
   14 IF(X(2)) 16,17,18
   16 SIGMA=0.
      GO TO 19
   17 SIGMA=1.
      GO TO 19
   18 SIGMA=2.
   19 C(1)=1.
      C(2)=0.
      Z1=Z1*SIGMA
      Z2=Z2*SIGMA
      CALL CDIV (C,A,B,KE)
      IF(KE-1)7,8,7
    8 WRITE(3,20)
      CALL EXIT
    7 C(1)=B(1)*5.+1.
      C(2)=B(2)*5.
      A(1)=(B(1)*C(1)-B(2)*C(2))*3.+1.
      A(2)=(B(2)*C(1)+C(2)*B(1))*3.
```

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```
      C(1)=B(1)*A(1)-B(2)*A(2)+1.
      C(2)=B(2)*A(1)+A(2)*B(1)
      CALL CDIV(C,X,A,KE)
      IF (KE-1)15,8,15
   15 Z1=-A(1)+Z1
      Z2=-A(2)+Z2
      GO TO 6
   13 B(1)=-2.*X(1)
      B(2)=-2.*X(2)
      D=3.
      Z1=Z1+B(1)
      Z2=Z2+B(2)
    4 C(1)=-(B(1)*A(1)-B(2)*A(2))/D
      C(2)=-(B(1)*A(2)+B(2)*A(1))/D
      Z1=Z1+C(1)
      Z2=Z2+C(2)
      IF ((C(1)*C(1)+C(2)*C(2))-1.E-6)6,6,5
    5 D=D+2.
      B(1)=C(1)
      B(2)=C(2)
      GO TO 4
   20 FORMAT(' DIVIDE BY ZERO RESULTING IN PROGRAM TERMINATION')
    6 DISP=CMPLX(Z1,Z2)
      RETURN
      END
C  THIS LITTLE SUBROUTINE IS USED TO DIVIDE TWO COMPLEX NUMBERS.  THIS
C   SUBPROGRAM WAS WRITTEN BEFORE THIS FEATURE WAS AVAILABLE  IN THE
C   FORTRAN COMPILER.
      SUBROUTINE CDIV(A,B,C,KE)
      REAL A,B,C,R
      DIMENSION A(2),B(2),C(2)
      R=B(1)**2+B(2)**2
      IF(R)1,1,2
    1 KE=1
      GO TO 3
    2 KE=2
      C(1)=(A(1)*B(1)+A(2)*B(2))/R
      C(2)=(A(2)*B(1)-A(1)*B(2))/R
    3 RETURN
      END
C
C                                      **************
C
C
C                                      SUBPROGRAM      ZET
C
C  THIS SUBROUTINE CALCULATES THE RATIO OF THE PHASE VELOCITY OF THE WAVE
C  PARALLEL TO THE MAGNETIC FIELD  TO THE  PARALLEL THERMAL VELOCITIES OF THE
C PARTICLES (  PARALLEL VELOCITIES= SQRT( 2*KAPPA* TEMP (I,J)/PMASS))
      SUBROUTINE ZET(X)
      COMMON /COM1/V(2,2),ARRAY(2,2)
      COMPLEX X,ARRAY
      DO 10 I=1,2
      DO 10 J=1,2
   10 ARRAY(I,J)= X*V(1,2)/V(I,J)
```

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```
      RETURN
      END
C
C                                  **************
C
C
C                                    SUBPROGRAM      YOU
C
C  THIS SUBROUTINE CALCULATES THE VALUES OF THE NUMERATOR OF THE FUNCTION
C  U FOR A FIXED VALUE OF R AND A VARIABLE VALUE OF ZETA.  THE FORMULA
C  USED IS GIVEN IN EQUATION B-I-12
      COMPLEX FUNCTION YOU(ZET,DISP,X,IRAD)
      IMPLICIT COMPLEX (C,Z), REAL (K)
      COMPLEX DISP,X
      COMMON /COM2/ G(100,2),GP(100,2)/COM1/VEL(2,2),ZETA(2,2)/COM2/
     1PMASS(2),GYRO(100,2),GYRAD(100,2),TEMP(2,2),RAD(200),KAY,KAPPA,
     2ISGN(2),L
      COMMON /COM5/ OMEGAD(100,2,2), TPERP(2,2)
      KONST=4.0*3.14159*(4.802E-10)**2
      CNUMU= KAY**2
      CALL ZET(X)
      DO 11 I=1,2
      DO 11 J=1,2
      ZO=ZETA(J,I)
      CN1= DISP(ZO)
      CN2=-2*(1+ZO*CN1)
      CN3= KAY*VEL(J,I)
      CN4= L*OMEGAD(IRAD,J,I)/CN3
      CNUM1= KONST* 2*L*ISGN(J)*GP(IRAD,I)*(CN1+CN4*CN2)/(PMASS(J)*GYRO(
     1IRAD,J)*CN3)
      CNUM2=KONST*G(IRAD,I)*(1+ZO*CN1+CN4*ZO*CN2)/(KAPPA*TEMP(J,I))
      CNUM3=KONST*G(IRAD,I)*CN4*(CN1+CN4*CN2)*(1-TEMP(J,I)/TPERP(J,I))/(
     1KAPPA*TEMP(J,I))
   11 CNUMU=CNUMU+CNUM1+CNUM2+CNUM3
      YOU = CNUMU
      RETURN
      END
```

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```
C
C
C                          MAINPROGRAM 2
C
C
C   THIS PROGRAM CALCULATES THE COMPLEX FUNCTION U AS A FUNCTION OF R FOR
C   A GIVEN VALUE OF ZETA ( THE RATIO OF THE WAVE PARALLEL PHASE VELOCITIY
C   TO THE PARALLEL ION VELOCITY).
C   THE  SUBPROGRAMS CALLED BY THIS MAINPROGRAM ARE DISP, CDIV, ZET AND YOU1
C   MOST OF THE VARIABLE NAMES USED IN THIS PROGRAM  ARE THE SAME AS IN
C   MAINPROGRAM 1
C   THE PROGRAM SEARCHES FOR TURNING POINTS OF U ( IE  WHEN BOTH THE REAL
C   PART AND THE IMAGINARY PART OF U ARE EQUAL TO ZERO) AND GIVES THE VALUES
C   OF R AT WHICH THEY OCCUR.
C   THE PROGRAM CALCULATES THE VALUES OF U(R) AT INTERVALS OF ( ONE EARTH
C   RADIUS)*SKALE .  THE PROGRAM TESTS FOR ZEROS OF U BY LOOKING FOR
C   SIMULTANEOUS SIGN CHANGES IN BOTH THE REAL AND IMAGINARY PART OF U
C   BETWEEN TWO ADJACENT VALUES OF R.   IF A SIMULTANEOUS SIGN CHANGE IS
C   FOUND U IS CALCULATED AT  NUMBRE  OF EVENLY SPACED VALUES OF R BETWEEN
C   THE TWO VALUES OF R FOR WHICH THE SIGN CHANGE WAS DETECTED.  IF
C   ANOTHER SIMULTANEOUS SIGN CHANGE IS DETECTED THE PROCESS IS
C   REPEATED   LIMIT  TIMES.
C   U(R) CAN BE CALCULATED FOR SEVERAL VALUES OF ZETA.  THE NUMBER OF VALUES OF
C   ZETA TO BE READ IN ARE SPECIFIED BY JEND.
C
C
      IMPLICIT COMPLEX(C,Z),REAL(K)
      COMPLEX YOU1,DISP
      REAL CHARG,N1,N2
      COMMON /COM1/ VEL(2,2),ZETA(2,2)
      DIMENSION GYRO(100,2),GYRAD(100,2)
      DIMENSION VPERP(2,2)
      COMMON /COM2/ G(100,2),GP(100,2),PMASS(2),GYRO,GYRAD,TEMP(2
     1,2),RAD(200),KAY,KAPPA,ISGN(2),L
      COMMON /COM3/ TOOBIG
      COMMON /COM4/ CFREQ(2,2), CDISP (2,2)
      COMMON /COM5/ OMEGAD(100,2,2), TPERP(2,2)
C   THE PARALLEL TEMPERATURE OF THE PLASMA IS GIVEN BY TEMP(J,I).
C   J=1 DENOTES IONS   J=2 , ELECTRONS
C   I=1 DENOTES PLASMASPHERE PARTICLES   I=2 , RING CURRENT PARTICLES
C   TEMP IS A 2 X 2 ARRAY IN FORMAT  4E10.3 CONTAINING THE PARALLEL TEMPERATURE
      READ(1,190) TEMP
  190 FORMAT(4E10.3)
C   ISGN IS AN ARRAY OF SIZE 2 IN FORMAT 2I3 CONTAINING THE PARTICLE SIGNS
      READ(1,191) ISGN
  191 FORMAT(2I3)
C   PMASS IS A SIZE 2 ARRAY IN FORMAT 2E10.4 CONTAINING THE PARTICLE MASSES
      READ(1,192) PMASS
  192 FORMAT(2E10.4)
C   THE PERPENDICULAR TEMPERATURE OF THE PLASMA IS GIVEN BY TPERP
C TPERP IS A 2X2 ARRAY IN FORMAT 4E10.3 CONTAINING THE PERPENDICULAR TEMPERATURE
      READ(1,12) TPERP
   12 FORMAT(4E10.3)
      WRITE(3,12) TPERP
    9 FORMAT (4E8.1,2I2,2E10.4/4E10.2)
```

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```
C  CHARG AND SPD ARE THE ELECTRONIC CHARGE AND THE SPEED OF LIGHT IN CGS
C   UNITS.  BFLD IS THE VALUE OF THE MAGNETIC FIELD AT THE MAGNETOPAUSE IN GAUSS
C    B IS ASSUMED TO VARY AS (1/R)**POWER
      READ (1,7) CHARG,POWER,BFLD,SPD
    7 FORMAT (E10.3,F6.2,2E10.3)
C  TOOBIG IS USED TO MODULATE THE VALUE OF THE ARGUMENT OF THE SINE AND COSINE
C  IN THE SUBROUTINE DISP TO KEEP THEM FROM BECOMING TOO LARGE
      TOOBIG = .823549E+06
      KON = SPD/CHARG
C   KAY IS BOLTZMANN'S CONSTANT
C   KAPPA IS THE PARALLEL WAVE NUMBER
C   L IS THE AZIMUTHAL WAVE NUMBER
      READ(1,6) KAY,KAPPA,L
    6 FORMAT (2E10.4/I5)
C  N1 IS THE PLASMASPHERE NUMBER DENSITY AT ITS OUTER BOUNDRY  N2 IS
C  THE PEAK RING CURRENT NUMBER DENSITY.  THE PEAK IS ASSUMED TO OCCUR
C   AT 6 RE IN THIS MODEL.  REP IS THE RADIAL DISTANCE TO THE BEGINNING
C  OF THE PLASMA PAUSE GIVEN IN RE.  IN THIS MODEL A VALUE OF 5.9 RE IS
C  USED SINCE RUSSEL AND THORNE HAVE FOUND THAT THE PLASMAPAUSE CLOSELY
C  COINCIDES WITH THE PEAK RING CURRENT NUMBER DENSITY
      READ(1,30) N1,N2,REP
   30 FORMAT (2F6.1,F8.2)
      WRITE(3,7) CHARG,POWER,BFLD,SPD
      WRITE(3,6) KAY,KAPPA,L
      WRITE(3,30) N1,N2,REP
C  THE THERMAL VELOCITIES ARE CALCULATED
      DO 15 I=1,2
      DO 15 J = 1,2
C  VEL GIVES THE PARALLEL VELOCITIES AND VPERP THE PERPENDICULAR VELOCITIES
C  WITH THE FIRST INDEX GIVING THE PARTICLE SPIECE AND THE SECOND GIVING
C  THE PLASMA GENRE
      VEL (J,I) = SQRT ( 2*KAPPA*TEMP(J,I)/PMASS(J))
      VPERP(J,I)= SQRT(2*KAPPA*TPERP(J,I)/PMASS(J))
   15 CONTINUE
C  THE GYRORADIUS FOR THE IONS AND THE GYROFREQUENCIES ARE CALCULATED
C    B IS ASSUMED TO VARY AS (1/R)**POWER
      BO = BFLD*((6.35E9)**POWER)
      DO 5 I4 = 1,100
      R= (6.350E7)*I4
      RD= R**POWER
      BF = BO/RD
      DO 5 I5=1,2
      GYRO(I4,I5)= CHARG*BF/(PMASS(I5)*SPD)
      GYRAD(I4,I5)= VPERP(1,I5)/GYRO(I4,1)
      DO 5 I6=1,2
C  OMEGAD IS A 100 X 2 X 2  ARRAY WHICH GIVES THE AZIMUTHAL DRIFT FREQUECIES
C  OF THE PARTICLES
    5 OMEGAD(I4,I6,I5)=(-.5)*ISGN(I6)*(VPERP(I6,I5)**2)*PMASS(I6)*(1/BF)
     1*KON*(POWER/(R**2))
      WRITE(3,9) TEMP,ISGN,PMASS,VEL
C  FROM THIS POINT TO STATEMENT NUMBER 150
C   THE MODEL NUMBER DENSITY, G , AND ITS DERIVITIVE WITH RESPECT TO
C   R**2 , GP , ARE GENERATED
C   AN EXPONENTIAL MODEL IS USED FOR THE NUMBER DENSITY
      PEAK=6*(6.350E8)
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```
      RE2=(6.35E8)**2
      A2=(16./ALOG(10.*N2))*RE2
      A=SQRT(A2)
      B2=(1./ALOG(10.))*RE2
      B=SQRT(B2)
      REP = REP*(6.35E8)
      D=(.5/ALOG(10.*N1))*RE2
      REP2=REP*REP
      DO 150 I3=1,100
      RAD(I3)=(6.350E7)*I3
      EX=-(RAD(I3)-PEAK)
      EX2=EX*EX
      IF (EX) 50,50,51
   50 G(I3,2)=N2*EXP(-EX2/A2)
      GP(I3,2)=2*EX*G(I3,2)/(A2*RAD(I3))
      GO TO 53
   51 G(I3,2)=N2*EXP(-EX2/B2)
      GP(I3,2)=2*EX*G(I3,2)/(B2*RAD(I3))
   53 IF(RAD(I3)-REP) 54,54,55
   54 G(I3,1)=N1*REP2/(RAD(I3)**2)
      GP(I3,1)= -1*G(I3,1)/(RAD(I3)**2)
      GO TO 150
   55 IF(RAD(I3)-PEAK) 58,58,59
   59  G(I3,1)=0.0
      GP(I3,1)= 0.0
      GO TO 150
   58 EXX=-(RAD(I3)-REP)
      EXX2=EXX*EXX
      G(I3,1)=N1*EXP(-EXX2/D)
      GP(I3,1) = 2*EXX*G(I3,1)/(D*RAD(I3))
  150 CONTINUE
      READ (1,1060) SKALE, NUMBRE,LIMIT,JEND
 1060 FORMAT (F6.2,I3,I2,I2)
      WRITE (3,1060) SKALE,NUMBRE,LIMIT, JEND
      DO 1051 NN=1,JEND
      READ (1,1061) COMEGA
      WRITE (3,1061 ) COMEGA
 1061 FORMAT ( 2G12.6)
C   IHOLD, ITAB AND NILCNT ARE SIMPLY COUNTERS USED TO KEEP TRACK
C   OF THE LOOP THAT SEARCHES FOR ZEROS OF U.
      IHOLD=1
      ITAB=0
      KNIL=0.
      NILCNT=0
      DO 1080 I=1,2
      DO 1080 J=1,2
      CALL ZET(COMEGA)
      CFREQ(J,I)=ZETA(J,I)
      CDISP(J,I)=DISP(CFREQ(J,I))
 1080 CONTINUE
 1001 CYOU1=YOU1(KNIL)
      STPSIZ=(6.35E8)*SKALE
      L1= IHOLD
      KNIL=0.
      MEND=IFIX(10/SKALE)
```

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```
      ITAB=0
      GO TO 1002
 1013 L1=1
 1002 DO 1021 N=L1,MEND
      ARE=KNIL +STPSIZ*N
      CYOU2=YOU1(ARE)
      ARE1 = ARE/6.35E8
      WRITE ( 3,20) CYOU2,ARE1
   20 FORMAT ('    U HAS THE VALUE                ,                     AT R= '
     1,T20,G12.6,T36,G12.6,T57,G10.5,T69,'R.E.')
      UI1= AIMAG(CYOU1)
      UI2=AIMAG(CYOU2)
      IF (UI2) 1003,1009,1003
 1009 IF(REAL(CYOU2)) 1005,1030,1005
 1030 IHOLD=IHOLD+1
      GO TO 1033
 1003 SIN1= UI1/UI2
C   THE IMAGINARY PART OF U IS TESTED FOR A SIGN CHANGE
      IF (SIN1) 1005,1010,1020
 1010 IF (REAL(CYOU1)) 1005,1031,1005
 1031 IHOLD=IHOLD+1
      GO TO 1032
 1005 UR1=REAL(CYOU1)
      UR2= REAL(CYOU2)
      IF (UR2) 1007,1020,1007
C   THE REAL PART OF U IS TESTED  FOR A SIGN CHANGE  IF ONE HAS BEEN FOUND
C  FOR THE IMAGINARY PART OF U.
 1007 SIN2=UR1/UR2
      IF (SIN2) 1011,1020,1020
C   THE NEXT FEW STEPS SET UP THE PROGRAM SO THAT U CAN BE TESTED
C   FOR A ZERO BETWEEN THE R VALUES WHERE A SIMULTANEOUS SIGN CHANGE HAS
C   BEEN FOUND
 1011 ITAB =ITAB+1
      KNIL = ARE-STPSIZ
      STPSIZ=STPSIZ/NUMBRE
      MEND = NUMBRE
      IF(ITAB -LIMIT ) 1013,1013,1030
 1020 IF (ITAB .GT. 0) GO TO 1022
      CYOU1=CYOU2
      IHOLD =N+1
      GO TO 1021
 1022 CYOU1=CYOU2
 1021 CONTINUE
      IF ( ITAB .GT. 0 )  IHOLD = IHOLD +1
      GO TO 1050
 1033 WRITE (3,1040) ARE
 1040 FORMAT ('     U HAS A TURNING POINT AT R=  ',E10.5)
      NILCNT=NILCNT+1
      GO TO 1050
 1032 WRITE (3,1040) KNIL
      NILCNT=NILCNT +1
 1050 IF ( IHOLD- 10/SKALE) 1052,1052,1051
 1052 KNIL = (IHOLD-1)*(6.35E8)* SKALE
      GO TO 1001
 1051 CONTINUE
```

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```
      STOP
      END
C
C                              **************
C
C
C
C                         SUBPROGRAM  YOU1
C
C
C   THE SUBROUTINE YOU1 CALCULATES THE VALUES OF THE FUNCTION U(R) FOR A
C   VARIABLE VALUE OF R AND A FIXED VALUE OF ZETA.
C
      COMPLEX FUNCTION YOU1(R)
      IMPLICIT COMPLEX (C,Z),REAL(K)
      COMMON /COM1/ VEL(2,2),ZETA(2,2)
      DIMENSION GYRO(100,2),GYRAD(100,2)
      COMMON /COM2/ G(100,2),GP(100,2),PMASS(2),GYRO,GYRAD,TEMP(2
     1,2),RAD(200),KAY,KAPPA,ISGN(2),L
      COMMON /COM4/ CFREQ(2,2), CDISP (2,2)
      COMMON /COM5/ OMEGAD(100,2,2), TPERP(2,2)
      DIMENSION X(2),Y(2)
      KONST = 4.0*3.14159*(4.802E-10)**2
      IF(R.LT. 6.35E7) GO TO 5
      CNUMU= (L/R)**2 + KAY**2
      CDENU=1.
      IRAD = IFIX(R/6.35E7)
      FRAC = AMOD (R,6.35E7)/6.35E7
      DO 10 I= 1,2
C   A LINEAR INTERPOLATION IS USED TO FIND THE VALUES OF G AND GP
C   WHEN R IS NOT AN INTEGRAL MULTIPLE OF ONE TENTH OF AN EARTH RADIUS.
C   THIS INTERPOLATION MUST BE USED SINCE G AND GP ARE ARRAYS IN WHICH THE
C   VALUES OF THE NUMBER DENSITY AND ITS DERIVITIVE ARE STORED  FOR  100
C   VALUES OF R EVENLY SPACED BETWEEN .1 AND 10 EARTH RADII  THAT IS,
C   EVERY TENTH OF AN EARTH RADIUS
      X(I)= G(IRAD,I)-(G(IRAD,I)-G(IRAD+1,I))*FRAC
      Y(I)= GP (IRAD,I)-(GP(IRAD,I)-GP(IRAD+1,I))*FRAC
      DO 11 J=1,2
      ZO=CFREQ(J,I)
      CN1=CDISP(J,I)
      CN2=-2*(1+ZO*CN1)
      CN3= KAY*VEL(J,I)
      CN4= L*OMEGAD(IRAD,J,I)/CN3
      CNUM1=KONST*2*L*ISGN(J)*Y(I)*(CN1+CN4*CN2)/(PMASS(J)*GYRO(IRAD,J)
     1*CN3)
      CNUM2=KONST*X(I)*(1+ZO*CN1+CN4*ZO*CN2)/(KAPPA*TEMP(J,I))
      CNUM3=KONST*X(I)*CN4*(CN1+CN4*CN2)*(1-TEMP(J,I)/TPERP(J,I))/(KAPPA
     1*TEMP(J,I))
   11 CNUMU=CNUMU+CNUM1+CNUM2+CNUM3
      ZO=CFREQ(1,I)
      CD1= CDISP(1,I)
      CD2= -2*(1+ZO*CD1)
      CD3=KAY*VEL(1,I)
      CD4= L*OMEGAD(IRAD,1,I)/CD3
      CDEN1=KONST*2*L*Y(I)*(CD1+2*CD4*CD2)/(PMASS(1)*GYRO(IRAD,1)*CD3)
```

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```
      CDEN2=KONST*X(I)*(1+Z0*(CD1+2*CD4*CD2))/(KAPPA*TEMP(1,I))
      CDEN3=KONST*2*X(I)*CD4*(CD1+2*CD4*CD2)*(1-TEMP(1,I)/TPERP(1,I))/
     1(KAPPA*TEMP(1,I))
   10 CDENU=CDENU-(.25)*(GYRAD(IRAD,I)**2)*(CDEN1+CDEN2+CDEN3)
      YOU1= CNUMU/CDENU
      GO TO 6
C     THIS VALUE IS ASSIGNED TO YOU  IF R IS LESS THAN ONE TENTH OF AN
C     EARTH RADIUS.  THIS IS  DONE SINCE THE VALUE OF  YOU AT ZERO IS INFINITY
C     AND THE FUNCTION CAN NOT BE INTERPOLATED IN THIS REGION.
    5 YOU1= (-1.,0.)
    6 RETURN
      END
```

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