CONTROL AND INVERSE PROBLEMS FOR ONE DIMENSIONAL SYSTEMS

A

THESIS

Presented to the Faculty

of the University of Alaska Fairbanks

in Partial Fulfillment of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

By

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December 2009

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Abstract

The thesis is devoted to control and inverse problems (dynamical and spectral) for systems on graphs and on the half line.

In the first part we study the boundary control problems for the wave, heat, and Schrödinger equations on a finite graph. We suppose that the graph is a tree (i.e., it does not contain cycles), and on each edge an equation is defined. The control is acting through the Dirichlet condition applied to all or all but one boundary vertices. The exact controllability in L_2-classes of controls is proved and sharp estimates of the time of controllability are obtained for the wave equation. The null controllability for the heat equation and exact controllability for the Schrödinger equation in arbitrary time interval are obtained.

In the second part we consider the in-plane motion of elastic strings on a tree-like network, observed from the 'leaves.' We investigate the inverse problem of recovering not only the physical properties, i.e. the 'optical lengths' of each string, but also the topology of the tree which is represented by the edge degrees and the angles between branching edges. It is shown that under generic assumptions the inverse problem can be solved by applying measurements at all leaves, the root of the tree being fixed.

In the third part of the thesis we consider Inverse dynamical and spectral problems for the Schrödinger operator on the half line. Using the connection between dynamical (Boundary Control method) and spectral approaches (due to Krein, Gelfand-Levitan, Simon and Remling), we improved the result on the representation of so-called $A-$amplitude and derive the "local" version of the classical Gelfand–Levitan equations.

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Acknowledgements

This dissertation would not have been written without the support of many people. I would like to thank my advisor Prof. Sergei Avdonin for proposed research topics and permanent attention to my work. I am grateful to my graduate committee for their work and cooperation. My work would not have been possible without financial support from the Department of Mathematics and Statistics and the Graduate School. I thank all my fellow graduate students. I thank all my teachers who contributed to my growth. I thank my co-authors, Prof. Günter Leugering and Prof. Alexei Rybkin for their help.

General Introduction

0.1 Introduction.

This work is devoted to some control and inverse problems for dynamical systems on the half line and on finite trees. The work consists of three chapters. Each of them is self contained and preceded with its own introduction. The common subject of the problems under the consideration is that we are interested in the control and inverse problems for dynamical systems.

The control theory of distributed parameter systems is a broad area of mathematics and engineering (to mention just [70, 80, 8]). The general question is the following: having in hand some dynamical system, to determine if is it possible by applying some external force (control) to it to achieve a prescribed "state" of the system. Such control theory has a lot of applications in physics, engineering and mechanics. In the last fifty years several general approaches were developed, including the Hilbert Uniqueness method (see [70, 62]) and the approach via nonharmonic Fourier series (see [80, 81, 8]). The first one is based on the a priori estimates for the solutions of partial differential equations that describe the dynamical system. The second approach is based on the relation, given by the method of moments, between the control properties of the dynamical system and properties of corresponding exponential families.

Inverse problems of mathematical physics are problems of the following type: we have some physical process, usually described by partial differential equations over some bounded or unbounded domain or manifold. Assume that we can only measure some partial information (dynamical, spectral or scattering) related to this process. The question is whether it is possible to recover the whole physical process via the measurements. The problem is important in physics, engineering, geophysics and tomography. It has a lot of applications , including the inverse scattering method for KdV-equation and other nonlinear problems. The data for the inverse problems could be dynamical (e.g. dynamical Dirichlet-to-Neumann response operator), spectral (spectral data, or spectral function) or scattering (scattering matrix). The Inverse Problem theory has been developing from 1930s starting from works by Ambartsumyan, Borg and Marchenko on the inverse spectral problem for the Schrödinger operator on the interval. Since then several general approaches to inverse problems were developed (see [65, 26, 46, 54, 71]).

In the middle of '80s M. Belishev and his co-authors S. Avdonin, S. Ivanov, A. Kachalov, Y. Kurylev developed the general approach to multidimensional dynamical inverse problems, the boundary control method (BC method). The approach is based on the connection between the controllability and identification problems for partial differential equations. In many situations the BC method gives the affirmative answer on R. Kalman's hypothesis that the dynamical system is identificable if and only if it is controllable. The BC method has been successfully applied to wave [18, 19], Schrödinger [10, 11], Maxwell equations [27, 28], Lame-type system [24], problem of identification of manifold by dynamical or spectral data [20, 21], problems on graphs [9, 23, 35], etc. Surprisingly, the BC method is applicable even in problems where no boundary is present (see for example Chapter II of [43]). The approach is complex: it involves the control theory for corresponding dynamical systems and methods of the geometrical optics (propagation of singularities for wave equations). We remark that the approach is essentially dimension-independent, and, at least in one-dimensional situation, leads itself to straightforward algorithmic implementations and stable numerical schemes (see [6, 7, 34, 29, 33]). Another advantage of the BC method is its locality: the knowledge of the inverse data on some time interval allows one to recover the parameters of the dynamical system on a certain smaller interval, though to formulate the exact statement, we need to introduce so-called "optical distance" (see e.g. [21]).

Throughout the thesis the following idea on the connection of the control and inverse problems is emphasized: the controllability properties of the dynamical system are connected via the method of moments with the properties of the corresponding families of exponentials. Such an idea was used in Chapter I of [43] for the construction of sampling and interpolating sequences for multi-band signals. The Boundary control method is essentially based on the connections between controllability and identification problems for systems described by partial differential equations. So the progress in one of these three fields (control theory, inverse problems, nonharmonic

Fourier series) leads to the progress in two other fields.

The first part of the thesis deals with the problems on trees. The tree consists of edges connected at the vertices. Every edge is identified with an interval of the real line. The vertices can be considered as equivalence classes of the edge end points. The vertices with valency one are called boundary, all other are interior. We always assume that the tree is finite (we have only finite number of edges) and compact (no edges with infinite length present). On every edge of the tree a differential equation is given. Also the matching or coupling conditions are given at the interior vertices and some boundary conditions are given at the boundary of the tree.

Differential equations on graphs are used to describe many physical processes such as mechanical vibrations of multi-linked flexible structures (usually composed of flexible beams or strings), propagation of electro-magnetic waves in networks of optical fibers, heat flow in multi-link networks, and also electron flow in quantum mechanical circuits. Recently, quasi-one-dimensional structures (graphs), like quantum, atomic, and molecular wires, have become the subject of extensive experimental and theoretical studies. The simplest model is a wave equation on the planar graph. In the last few years dynamical control and inverse problems for the wave equation and spectral inverse problems for the Schrödinger equation on the planar graphs with no circles (trees) have received a lot of attention. The control problems on graphs (in different settings) were solved in [8, 22, 44, 61]. The dynamical and spectral inverse problems by BC method were considered in [9, 23, 35], different methods were used in [87, 40, 41].

In the first chapter we use results on the partial controllability of the wave equation on a tree [22], results on exponentials families [8], and on method of transmutation [73, 74] to show the exact controllability of the wave equation and null controllability of the parabolic and Schrödinger equations on the tree.

In the second chapter we consider the more complicated system on the tree: two-velocity dynamical system with constant densities. The difference with the wave equation on the tree is that on each edge the wave propagates in two channels and the coupling conditions at the interior vertices reflect the geometry of the graph (i.e. the angles between edges). We solve the inverse spectral problem for this system using the BC method.

The third part of the thesis is devoted to the inverse dynamical and spectral problems for the Schrödinger operator on the half-line. Using the ideas of boundary control approach we refine results due to F. Gesztesy and B. Simon [83, 49] and C. Remling [78] on the representation of so-called $A-$amplitude (response function in our terms). We also made a contribution to boundary control method: using dynamical approach and ideas of the BC method we derived the classical Gelfand-Levitan equations for the inverse problem.

0.2 Statement of contributions

In Chapter I we deal with the control problem for the wave equation on the finite tree. My advisor Prof. Avdonin stated the goal of proving the exact controllability result in the sharp time. He pointed out the way of proving Theorems 1 and 2, using ideas from [8] and the results on partial controllability from [22]. I have performed all work and proved also Theorem 3 on the exact controllability of the dynamical system governed by the Schrödinger equation.

In Chapter 2 we deal with the identification problem for the two-velocity system on the finite tree. The statement of the problem was formulated by Prof. S. Avdonin and Prof. G. Leugering. Also some of the results from paragraph 2.4 were obtained by Prof. S. Avdonin, who offered me opportunity to extend these results to the case of a star graph and arbitrary tree and to develop an algorithm of finding angles between edges. All results from paragraphs 2.5, 2,6 and the method of finding angles between edges in 2.4 were obtained by me.

In the third Chapter we study the Inverse dynamical and spectral problems for the Schrödinger operator on the half-line. Prof S. Avdonin formulated the problem to generalize the main results of the BC method to the case of L_1 potential and pointed out the connections of the BC method with other methods for inverse spectral and dynamical problems. I performed all work and obtained results from paragraphs 3.2.4 and 3.2.5.

So, the main results of the thesis obtained by me are the following:

1) Controllability of the Schrödinger equation on tree-like graphs (Chapter I).

2) Extension of the BC method to the two-velocity wave equation on trees, developing of the 'reduction' method (Chapter II).

3) Derivation of the local version of the Gelfand-Levitan equations using the BC approach (Chapter III).

4) New convergence result for the spectral representation of the response function and $A-$amplitude (Chapter III).

The main results of the thesis were presented at: Conference on Operator Theory, Analysis and Mathematical Physics (OTAMP), Lund, Sweden 2006; Petrovskii Conference (22-d meeting) Differential equations and related topics, Moscow State University, Moscow, Russia, 2007; ESF Mathematical Conference on Operator Theory, Analysis and Mathematical Physics, Poland, Bedlewo, 2008; Colloquiums Department of Mathematical Sciences, University of Alaska, Fairbanks, Spring 2006, Spring 2007, Spring 2008; Colloquium University of Tennessee, Knoxville, January 2009; V.I. Smirnov Seminar on Mathematical Physics at V.A. Steklov Mathematical Institute, St.-Petersburg, Russia, May 2009, and are published or submitted for publishing in [13, 14, 15].

Chapter 1

Controllability of partial differential equations on graphs

1.1 Introduction.

Controllability problems for multi-link flexible structures or, in other words, for the wave and beam equations on graphs were the subject of extensive investigations of many mathematicians (see, e.g. the review paper [2] and references therein). Lagnese, Leugering, and Schmidt in [61, 62] used the method of energy estimates together with the Hilbert Uniqueness Method to show that the exact controllability can be achieved in optimal time for tree-like graphs consisted of homogeneous strings, when all but one exterior nodes are controlled. Independently Avdonin and Ivanov [8, Ch. VII] applied the method of moments and the theory of vector-valued exponentials to study controllability problems on graphs for the wave equation. The authors have proved the exact controllability in the optimal time for the wave equation on the star-shaped graph of non-homogeneous strings. Belishev in [22, 23] using the propagation of singularities method obtained result on boundary controllability for a tree of non-homogeneous strings with respect to the first component (the shape) of the complete state.

The results on exact controllability fail as soon as cycles occur within the network, even if all nodes (including the interior nodes) are subjected to control. The reason for this effect is that eigenfunctions vanishing on certain edges can occur (see e.g. [8, Sec. VII.1]). However, the spectral controllability may be retained for many graphs with cycles (see [8, Ch. VII], [44, 62] for details). In [69] for the tree of homogeneous vibrating strings, the authors prove the exact controllability for some special class of initial/final data. Many interesting results on spectral controllability are obtained in [44].

In this chapter we prove the result on the exact controllability for the wave equation on a tree-like graph of non-homogeneous strings for controls acting through Dirichlet conditions applied to all or all but one boundary vertices. Our result generalizes the ones from [8] and [62]. Using the controllability of the wave equation and results from [8, 73, 74, 80], we prove also the null controllability of the parabolic and exact controllability of the Schrödinger equations on trees.

Controllability problems for partial differential equations on graphs have many important applications. They are also related to inverse problems on graphs [9, 22, 35] and to harmonic analysis [8, Ch. VII]. In this chapter we use some known and prove several new results describing connections between controllability of distributed parameter systems and properties of exponential families.

1.2 Statement of the problems and main results.

Let Ω be a finite connected compact graph without cycles (a tree). The graph consists of edges $E = \{e_1, \dots, e_N\}$ connected at the vertices $V = \{v_1 \dots, v_{N+1}\}$. Every edge $e_j \in E$ is identified with an interval (a_{2j-1}, a_{2j}) of the real line. The edges are connected at the vertices v_j which can be considered as equivalence classes of the edge end points $\{a_j\}$. The boundary $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ of Ω is a set of vertices having multiplicity one (the exterior nodes). We suppose that the graph is equipped with the density

$$\rho(x) \geqslant \text{const} > 0, \quad x \in \Omega \backslash V, \quad \rho \in C^1(\bar{e}_j),\ j = 1 \dots, N. \tag{1.2.1}$$

All the results of this paper are valid also for piecewise continuously differentiable functions ρ, because discontinuity of ρ or its derivative in the interior an edge is equivalent to the addition of an inner vertex of multiplicity two (see the compatibility conditions (1.2.3), (1.2.4) below).

Since the graph under consideration is a tree, for every $a, b \in \Omega$, $a \neq b$, there exist the unique path $\pi[a, b]$ connecting these points. The density determines the optical metric and the optical distance

$$d\sigma^2 := \rho(x)|dx|^2, \quad x \in \Omega \backslash V,$$
$$\sigma(a, b) = \int\limits_{\pi[a,b]} \sqrt{\rho(x)}|dx|, \quad a, b \in \Omega,$$

The optical diameter of the graph Ω is defined as

$$d(\Omega) = \max_{a,b \in \Gamma} \sigma(a, b).$$

The graph Ω and the optical metric determine the *metric graph* denoted by $\{\Omega, \rho\}$. For a rigorous definition of the metric graph see, e.g. [55, 58, 59, 72, 77]. The space of real valued functions on the graph, square integrable with the weight ρ is denoted by $L_{2,\rho}(\Omega)$.

1.2.1 Dirichlet spectral problem.

Let $\partial w(a_j)$ denotes the derivative of w at the vertex a_j taken along the corresponding edge in the direction toward the vertex. We associate the following spectral problem to the tree:

$$-\frac{1}{\rho}\frac{d^2 w}{dx^2} = \lambda w, \tag{1.2.2}$$

$$w \in C(\Omega), \tag{1.2.3}$$

$$\sum_{a_j \in v} \partial w(a_j) = 0 \quad \text{for } v \in V \backslash \Gamma, \tag{1.2.4}$$

$$w = 0 \quad \text{on } \Gamma. \tag{1.2.5}$$

The condition (1.2.4) is also known as the Kirchhoff rule, represent the conservation of flow trough the vertex. In different situations it could mean the conservation of charge, energy, etc.

It is well-known fact (for general compact graphs, see [37, 72, 84]) that the problem (1.2.2)–(1.2.5) has a discrete spectrum of eigenvalues $0 < \lambda_1 \leqslant \lambda_1 \leqslant \lambda_2 \ldots, \lambda_k \to +\infty$ and corresponding eigenfunctions $\phi_1, \phi_2, \ldots$ can be chosen so that $\{\phi_k\}_{k=1}^{\infty}$ forms an orthonormal basis in $\mathcal{H} := L_{2,\rho}(\Omega)$:

$$(\phi_i, \phi_j)_{\mathcal{H}} = \int_{\Omega} \phi_i(x)\phi_j(x)\rho(x)\,dx = \delta_{ij}\,.$$

Set $\varkappa_k(\gamma) = \partial\phi_k(\gamma)$, $\gamma \in \Gamma$. Let α_k be the m-dimensional column vector defined as $\alpha_k = \operatorname{col}\left(\frac{\varkappa_k(\gamma)}{\sqrt{\lambda_k}}\right)_{\gamma\in\Gamma}$.

Definition 1. *The set of pairs*

$$\{\lambda_k, \alpha_k\}_{k=1}^{\infty} \tag{1.2.6}$$

is called the Dirichlet spectral data of the tree $\{\Omega, \rho\}$.

1.2.2 Initial boundary value problems. Control from the whole boundary.

We associate three dynamical systems, described correspondingly by the wave, heat and Schrödinger equations, to the tree $\{\Omega, \rho\}$. The first one has the form:

$$\rho u_{tt} - u_{xx} = 0 \quad \text{in } \Omega \backslash V \times [0,T], \tag{1.2.7}$$

$$u|_{t=0} = u_t|_{t=0} = 0, \tag{1.2.8}$$

$$u(\cdot,t) \quad \text{satisfies (1.2.3) and (1.2.4) for all } t \in [0,T], \tag{1.2.9}$$

$$u = f \quad \text{on } \Gamma \times [0,T]. \tag{1.2.10}$$

Here $T > 0$, $f = f(\gamma, t)$, $\gamma \in \Gamma$, is the Dirichlet boundary control which belongs to $\mathcal{F}_\Gamma^T = L_2([0,T]; \mathbb{R}^m)$. The inner product in $\mathcal{F}_\Gamma^T$ is defined by

$$(f,g)_{\mathcal{F}_\Gamma^T} = \sum_{i=1}^m \int_0^T f(\gamma_i, t) g(\gamma_i, t)\, dt.$$

Let $D'(\Omega)$ be the set of distributions over the tree. We introduce the space

$$\mathcal{H}_{-1} = \left\{ g \in D'(\Omega) : \; g(x) = \sum_{k=1}^\infty g_k \phi_k(x), \; \left\{ \frac{g_k}{\sqrt{\lambda_k}} \right\}_{k=1}^\infty \in l_2 \right\}.$$

The initial boundary value problem (1.2.7)–(1.2.10) has a classical solution if $f \in C^2([0,T], \mathbb{R}^m)$. In our case when $f \in \mathcal{F}_\Gamma^T$, the solution to (1.2.7)–(1.2.10) is understood in weak (distributional sense). It can be proved (see [8, 35, 44, 62]) that the solution u^f satisfies the inclusion

$$u^f \in C([0,T]; \mathcal{H}) \cap C^1([0,T]; \mathcal{H}_{-1}).$$

This means that $u(\cdot,t) \in \mathcal{H}$, $u_t(\cdot,t) \in \mathcal{H}_{-1}$ for all $t \in [0,T]$, and both functions are continuous with respect to t in corresponding norms. In other words, the state of the dynamical system (1.2.7)–(1.2.10) $(u(\cdot,t), u_t(\cdot,t))$ is a point of $\mathcal{H} \times \mathcal{H}_{-1}$, and the trajectory of the system is a continuous curve in the state space $\mathcal{H} \times \mathcal{H}_{-1}$. This regularity result is sharp, see [8].

One of the main results of the present paper demonstrates the *exact controllability* of the system (1.2.7)–(1.2.10).

Theorem 1. *For arbitrary state $\{a,b\} \in \mathcal{H}\times\mathcal{H}_{-1}$, there exists such a control function $f(\gamma,t) \in \mathcal{F}_\Gamma^T$ with $T = d(\Omega)$, that solution of the initial boundary value problem (1.2.7)–(1.2.10) satisfies the equalities $u^f(\cdot,T) = a$, $u_t^f(\cdot,T) = b$.*

Another system we associate to the graph $\{\Omega,\rho\}$ is

$$\rho u_t - u_{xx} = 0 \quad \text{in } \Omega\backslash V \times [0,\tau], \tag{1.2.11}$$

$$u|_{t=0} = a, \tag{1.2.12}$$

$$u(\cdot,t) \quad \text{satisfies (1.2.3) and (1.2.4) for all } t \in [0,\tau], \tag{1.2.13}$$

$$u = f \quad \text{on } \Gamma \times [0,\tau], \tag{1.2.14}$$

where $\tau > 0$, $f \in \mathcal{F}_\Gamma^\tau$ and $a \in \mathcal{H}_{-1}$.

It is known (see e.g. [8, 38, 61]), that the solution u^f of the system (1.2.11)–(1.2.14) satisfies the inclusion

$$u^f \in C([0,\tau];\mathcal{H}_{-1}).$$

For the parabolic-type dynamical systems various types of controllability are considered in the literature (see [8, 62]). The following result demonstrates the *null controllability* of the system (1.2.11)–(1.2.14).

Theorem 2. *For arbitrary given state $a \in \mathcal{H}_{-1}$ and for arbitrary time interval $[0,\tau]$, $\tau > 0$, there exists a control $f \in \mathcal{F}_\Gamma^\tau$ such that the solution of the initial boundary value problem (1.2.11)–(1.2.14) satisfies the equality $u^f(\cdot,\tau) = 0$.*

The Schrödinger equation can also be associated to the graph $\{\Omega,\rho\}$:

$$\rho u_t + iu_{xx} = 0 \quad \text{in } \Omega\backslash V \times [0,\tau], \tag{1.2.15}$$

$$u|_{t=0} = a, \tag{1.2.16}$$

$$u(\cdot,t) \quad \text{satisfies (1.2.3) and (1.2.4) for all } t \in [0,\tau], \tag{1.2.17}$$

$$u = f \quad \text{on } \Gamma \times [0,\tau], \tag{1.2.18}$$

where $f = f(\gamma,t) \in \mathcal{F}_\Gamma^T$, $a \in \mathcal{H}_{-1}$. It is known (see, e.g. [10, 86]) that solution $u^f(x,t)$ of (1.2.15)–(1.2.18) satisfies the inclusion

$$u^f \in C([0,T];\mathcal{H}_{-1}).$$

For the dynamical system governed by the Schrödinger equation (1.2.15)–(1.2.18) the following result on the exact controllability holds. (Due to time reversibility, the exact and null controllability are equivalent for the Schrödinger equation.)

Theorem 3. *For arbitrary initial state $a \in \mathcal{H}_{-1}$ and for arbitrary time interval $[0, \tau]$, $\tau > 0$, there exists such a control $f \in \mathcal{F}_\Gamma^\tau$ that the solution to the initial boundary value problem (1.2.15)–(1.2.18) satisfies the equality $u^f(\cdot, \tau) = 0$.*

1.2.3 Initial boundary value problems. Control from a part of the boundary.

In the case when the graph is controlled from the whole boundary but contains cycles, the system (1.2.7)–(1.2.10) is not exactly controllable (see, e.g. [8, Sec. VII.1]). Similarly, if the graph is a tree, but the system is not controlled at two or more boundary points (the Dirichlet condition $u = 0$ is imposed there), the Theorem 1 fails; the corresponding example (in the case of homogeneous strings) is given in [44, Sec. 6.3] (see also [2]). Suppose that the graph is not controlled at one of the boundary points, say γ_1. Then one can introduce the length of the longest path from γ_1 to the rest of the boundary $\Gamma_1 = \Gamma \setminus \{\gamma_1\}$:

$$d_1(\gamma_1, \Omega) = \max_{\gamma \in \Gamma_1} \tau(\gamma_1, \gamma).$$

The boundary conditions for the system (1.2.7)–(1.2.9) have the form:

$$u(\gamma_1, t) = 0, \quad u(\gamma_i, t) = f(\gamma_i, t), \quad i = 2, \dots, N, \tag{1.2.19}$$

where $f \in \mathcal{F}_{\Gamma_1}^T = L_2([0, T]; \mathbb{R}^{m-1})$. In this situation the analog of Theorem 1 holds true:

Theorem 4. *For arbitrary state $\{a, b\} \in \mathcal{H} \times \mathcal{H}_{-1}$ there exists such a control function $f(\gamma, t) \in \mathcal{F}_{\Gamma_1}^T$ with $T = 2d_1(\gamma_1, \Omega)$ that the solution of the initial boundary value problem (1.2.7)–(1.2.9), (1.2.19) satisfies the equalities $u^f(\cdot, T) = a$, $u_t^f(\cdot, T) = b$.*

For the parabolic and Schrödinger type systems (1.2.11)–(1.2.13), (1.2.15)–(1.2.18), we can also consider the problem of the controllability from the part of the boundary, i.e., we add the boundary conditions (1.2.19) to the initial-value problem (1.2.11)–(1.2.13) and to the problem (1.2.15)–(1.2.17). In this case one can prove the analogs of Theorems 2 and 3:

Theorem 5. *For arbitrary given state $a \in \mathcal{H}_{-1}$ and for arbitrary time interval $[0, \tau]$, $\tau > 0$, there exists such a control $f \in \mathcal{F}^{\tau}_{\Gamma_1}$ that the solution of the initial boundary value problem (1.2.11)–(1.2.13), (1.2.19) satisfies the equality $u^f(\cdot, \tau) = 0$.*

Theorem 6. *For arbitrary initial state $a \in \mathcal{H}_{-1}$ and for arbitrary time interval $[0, \tau]$, $\tau > 0$, there exists such a control $f \in \mathcal{F}^{\tau}_{\Gamma_1}$ that the solution of the initial boundary value problem (1.2.15)–(1.2.17),* (1.2.19) *satisfies the equality $u^f(\cdot, \tau) = 0$.*

1.3 Auxiliary results.

In [22]–[35] the following result concerning the controllability with respect to the first component (the shape) of the complete state $\{u, u_t\}$ of the dynamical system (1.2.7)–(1.2.10) has been proved:

Theorem 7. *Let $T = d(\Omega)/2$, then for arbitrary $a \in \mathcal{H}$ there exists such a control $f(\gamma, t) \in \mathcal{F}^T_\Gamma$ that the solution of the initial boundary value problem (1.2.7)–(1.2.10) satisfies the equality $u^f(x, T) = a(x)$.*

In other words, the system (1.2.7)–(1.2.10) is controllable with respect to the shape for the time equal to the half optical diameter of the graph. Note that in general such a control is not unique.

To prove Theorem 7 the propagation of singularities method has been used and the controllability was reduced to solvability of the Volterra type equation. It was supposed in [22]–[35] that $\rho \in C^2$ on all edges, however, the method works for $\rho \in C^1$ as well. The same technique can be applied to obtain the result on the controllability of the system (1.2.7)–(1.2.10) with respect to the second component (the velocity) of the complete state:

Proposition 1. *If $T = d(\Omega)/2$ then for arbitrary $b \in \mathcal{H}_{-1}$, there exists such a control $f(\gamma, t) \in \mathcal{F}^T_\Gamma$, that the solution of the initial boundary value problem (1.2.7)–(1.2.10) satisfies the equality $u^f_t(x, T) = b(x)$.*

In the following two propositions we consider the case of boundary condition (1.2.19) for the system (1.2.7)–(1.2.9). The proof of the first proposition can be extracted from the proof of Theorem 7 [35, Sec. 2]. Let us introduce the 'optical center' of the graph Ω, i.e., such a point $\xi \in \Omega$, that $\max_{\gamma\in\Gamma} \tau(\xi,\gamma) = d(\Omega)/2 = T$. Since Ω is a tree, there can be only one optical center. Suppose that the final state $a(x)$ is supported in such a subtree $\Omega_1 \subset \Omega$ that $\xi \notin \Omega_1$. As it was shown in [22]–[35], to solve the control problem one need to use controls supported on the part of the boundary of the graph Ω which is the boundary of Ω_1. In other words, it is possible to construct such a control $f \in \mathcal{F}_\Gamma^T$ that $u^f(T,x) = a(x)$ and $f(\gamma,t) = 0$ for $\gamma \notin \Omega_1$. The authors offers the explicit procedure of the construction of such a control. If instead of the 'optical center' of the graph we take a boundary point γ_1 where the homogeneous Dirichlet condition $u(\gamma_1,t) = 0$ is imposed, we come to the following statements:

Proposition 2. *If $T = d_1(\gamma_1,\Omega)$, then for arbitrary $a \in \mathcal{H}$, there exists such a control $f \in \mathcal{F}_{\Gamma_1}^T$, that the solution of the boundary value problem (1.2.7)–(1.2.9), (1.2.19) satisfies the equality $u^f(x,T) = a(x)$.*

The same result holds true for the controllability with respect to the velocity:

Proposition 3. *If $T = d_1(\gamma_1,\Omega)$ then for arbitrary $b \in \mathcal{H}_{-1}$, there exists such a control $f \in \mathcal{F}_{\Gamma_1}^T$ that the solution of the boundary value problem (1.2.7)–(1.2.9), (1.2.19) satisfies the equality $u_t^f(x,T) = b(x)$.*

1.4 Proof of Theorem 1.

We begin with the reducing the problem of controllability of the dynamical system (1.2.7)–(1.2.10) to the moment problem in $\mathcal{F}_\Gamma^T$. Solving the initial boundary value problem (1.2.7)–(1.2.10) by the Fourier method and looking for the solution in the form

$$u^f(x,t) = \sum_{k=1}^{\infty} c_k^f(t)\phi_k(x), \tag{1.4.1}$$

we get the expression for the coefficients:

$$c_k^f(t) = \sum_{\gamma\in\Gamma} \frac{\varkappa_k(\gamma)}{\sqrt{\lambda_k}} \int_0^t \sin\sqrt{\lambda_k}(t-s) f(\gamma,s)\,ds.$$

Suppose that we are given the final state $\{a,b\} \in \mathcal{H} \times \mathcal{H}_{-1}$ at $t = T$, where the functions $a(x)$, $b(x)$ have the expansions

$$a(x) = \sum_{k=1}^{\infty} a_k \phi_k(x), \quad b(x) = \sum_{k=1}^{\infty} b_k \phi_k(x),$$

for some $\{a_k\}_{k=1}^{\infty} \in l_2$ and $\{\frac{b_k}{\sqrt{\lambda_k}}\}_{k=1}^{\infty} \in l_2$. Then for an unknown control $f \in \mathcal{F}_\Gamma^T$, the following moment equalities should hold at time $t = T$:

$$a_k = c_k^f(T) = \sum_{\gamma \in \Gamma} \frac{\varkappa_k(\gamma)}{\sqrt{\lambda_k}} \int_0^T \sin \sqrt{\lambda_k}(T-s) f(\gamma, s)\, ds, \quad k \in \mathbb{N}, \tag{1.4.2}$$

$$\frac{b_k}{\sqrt{\lambda_k}} = \frac{\dot{c}_k^f(T)}{\sqrt{\lambda_k}} = \sum_{\gamma \in \Gamma} \frac{\varkappa_k(\gamma)}{\sqrt{\lambda_k}} \int_0^T \cos \sqrt{\lambda_k}(T-s) f(\gamma, s)\, ds, \quad k \in \mathbb{N}. \tag{1.4.3}$$

Using Euler formulas for exponentials, we rewrite (1.4.2), (1.4.3) as

$$\frac{b_k}{\sqrt{\lambda_k}} \pm i a_k = \sum_{\gamma \in \Gamma} \frac{\varkappa_k(\gamma)}{\sqrt{\lambda_k}} \int_0^T e^{\pm i \sqrt{\lambda_k}(T-s)} f(\gamma, s)\, ds, \quad k \in \mathbb{N}. \tag{1.4.4}$$

Definition 2. *We call the moment problem* (1.4.4) *solvable (and $f(\gamma, t)$ a solution of the moment problem) in the time interval* $[0, T]$ *if, for arbitrary* $\{a_k\}_{k=1}^{\infty}$, $\{\frac{b_k}{\sqrt{\lambda_k}}\}_{k=1}^{\infty} \in l_2$, *there exist such a function $f \in \mathcal{F}_\Gamma^T$ that equalities* (1.4.4) *hold.*

We emphasize that the solvability of the moment problem (1.4.4) in the time interval $[0, T]$ for some $T > 0$ is equivalent to the controllability of the dynamical system (1.2.7)–(1.2.10) in the sense of Theorem 1 in the same time interval. This is a basic statement of the method of moments (see, e.g. [8, Ch. III], [80]).

We need a couple of definitions concerning vector families in arbitrary Hilbert space.

Definition 3. *The family $\{\xi_k\}_{k=1}^{\infty}$ in a Hilbert space H is called a Riesz basis, if it is an image of an orthonormal basis under the action of some linear isomorphism (bounded and boundedly invertible operator).*

Definition 4. *The family $\{\xi_k\}_{k=1}^{\infty}$ in a Hilbert space H is called an $\mathcal{L}$-basis, if it is a Riesz basis in the closure of the linear span of the family.*

The result on the controllability formulated in Theorem 7 implies the solvability of the moment problem (1.4.2) for $T = d(\Omega)/2$ for every $\{a_k\}_{k=1}^{\infty}$. The controllability result formulated in Proposition 1 implies the solvability of the moment problem (1.4.3) for $T = d(\Omega)/2$ for every $\{\frac{b_k}{\sqrt{\lambda_k}}\}_{k=1}^{\infty} \in l_2$. Our goal is to show that the solvability of the moment problems (1.4.2) and (1.4.3) for $T = d(\Omega)/2$ implies the solvability of the moment problem (1.4.4) for $T = d(\Omega)$. Let us put $T_* = d(\Omega)/2$ and introduce the families of vector valued functions

$$S_k(t) = \alpha_k \sin\sqrt{\lambda_k}t, \quad C_k(t) = \alpha_k \cos\sqrt{\lambda_k}t, \quad k \in \mathbb{N}.$$

According to Theorem III.3.3 of [8] the solvability of the moment problems (1.4.2) and (1.4.3) means that the families $\{S_k(t)\}_{k=1}^{\infty}$ and $\{C_k(t)\}_{k=1}^{\infty}$ form $\mathcal{L}$-bases in $L_2([0,T_*];\mathbb{R}^m)$.

Let us introduce subspaces of $L_2([0,T_*];\mathbb{R}^m)$:

$$\Xi_o = \bigvee\{S_k(t)\}_{k=1}^{\infty}, \quad \Xi_e = \bigvee\{C_k(t)\}_{k=1}^{\infty},$$

where $\bigvee$ denotes the closure of the linear span of a family. We extend the functions from Ξ_o to the interval $[-T_*, 0)$ in the odd way:

$$\widetilde{\varphi}(t) = \begin{cases} \varphi(t), & t \geqslant 0, \\ -\varphi(-t), & t < 0, \end{cases}, \quad -T_* \leqslant t \leqslant T_*, \quad \varphi \in \Xi_o,$$

and the functions from Ξ_e — in the even way:

$$\widetilde{\varphi}(t) = \begin{cases} \varphi(t), & t \geqslant 0, \\ \varphi(-t), & t < 0, \end{cases} \quad -T_* \leqslant t \leqslant T_*, \quad \varphi \in \Xi_e.$$

Let us denote the spaces of extended functions by $\widetilde{\Xi}_o$ and $\widetilde{\Xi}_e$ and notice that the extended families $\{\widetilde{S}_k(t)\}_{k=1}^{\infty}$ and $\{\widetilde{C}_k(t)\}_{k=1}^{\infty}$ are Riesz bases in $\widetilde{\Xi}_o$ and $\widetilde{\Xi}_e$ correspondingly. The orthogonality of the spaces $\widetilde{\Xi}_o$ and $\widetilde{\Xi}_e$ implies that the union

$$\left\{\widetilde{C}_k(t)\right\}_{k=1}^{\infty} \bigcup \left\{\widetilde{S}_k(t)\right\}_{k=1}^{\infty},$$

forms a Riesz basis in $\widetilde{\Xi}_o \bigoplus \widetilde{\Xi}_e \subset L_2([-T_*,T_*];\mathbb{R}^m)$. Introducing functions

$$E_{\pm k}(t) = C_k(t) \pm iS_k(t) = \alpha_k e^{\pm i\sqrt{\lambda_k}t}, \quad k \in \mathbb{N}, \tag{1.4.5}$$

we see that the set $\{E_{\pm k}\}_{k\in\mathbb{N}}$ forms an $\mathcal{L}$-basis in $L_2([-T_*, T_*]; \mathbb{C}^m)$. Shifting the argument, we come to the conclusion that the same family forms an $\mathcal{L}$-basis in $L_2([0, 2T_*]; \mathbb{C}^m)$. Then according to Theorem III.3.3 of [8], the moment problem (1.4.4) is solvable for time $T = 2T_* = d(\Omega)$. As we have already noticed, this implies the exact controllability of (1.2.7)–(1.2.10) in the time interval $[0, d(\Omega)]$. Theorem 1 is proved.

The proof of Theorem 4 is analogous to the previous one. We set α_k' to be the $(m-1)$-dimensional column vector defined as

$$\alpha_k' = \operatorname{col}\left(\frac{\varkappa(\gamma)}{\sqrt{\lambda_k}}\right)_{\gamma\in\Gamma_1}. \tag{1.4.6}$$

There naturally arise the families of vector functions in $L_2([0, T_1]; \mathbb{R}^{m-1})$ with $T_1 = d_1(\gamma_1, \Omega)$:

$$S_k^1(t) = \alpha_k' \sin\sqrt{\lambda_k}t, \quad C_k^1(t) = \alpha_k' \cos\sqrt{\lambda_k}t, \quad k\in\mathbb{N}.$$

One should perform the same procedure (using Propositions 2 and 3 instead of Theorem 7 and Proposition 1) as in the proof of Theorem 1, construct the family of vector exponentials

$$\left\{E_{\pm k}^1\right\}_{k\in N}, \quad E_{\pm k}^1(t) = \alpha_k' e^{\pm i\sqrt{\lambda_k}t}, \quad t\in(0, 2T_1), \quad k\in\mathbb{N}, \tag{1.4.7}$$

and use the connection between controllability and vector exponentials ([8], Theorem III.3.3).

In the proofs of Theorems 1 and 4 we have got important results which are of independent interest in Function Theory.

Proposition 4. *The family* $\{E_{\pm k}\}_{k\in\mathbb{N}}$ *(see* (1.4.5)*) constructed using the Dirichlet spectral data* (1.2.6) *is an* $\mathcal{L}$*-basis in* $L_2([0, d(\Omega)]; \mathbb{C}^m)$.

Suppose that we pick arbitrary boundary point of the graph (we keep the notation γ_1 for it), then we get

Proposition 5. *The family* $\left\{E_{\pm k}^1\right\}_{k\in\mathbb{N}}$ *(see* (1.4.7)*) constructed using the Dirichlet spectral data* (1.2.6), (1.4.6) *is an* $\mathcal{L}$*-basis in* $L_2([0, 2T_1]; \mathbb{C}^{m-1})$ *for* $T_1 = d_1(\gamma_1, \Omega)$.

It seems to be very difficult to obtain these results without using the control theoretic approach.

1.5 Proof of Theorem 2.

Looking for the solution of (1.2.11)–(1.2.14) in the form (1.4.1) for the fixed initial state $a \in \mathcal{H}_{-1}$ with the expansion

$$a(x) = \sum_{k=1}^{\infty} a_k \phi_k(x), \tag{1.5.1}$$

we come to the following formulas for the coefficients:

$$c_k^f(t) = a_k e^{-\lambda_k t} + \sum_{\gamma \in \Gamma} \varkappa_k(\gamma) \int_0^t e^{-\lambda_k (t-s)} f(\gamma, s)\, ds.$$

Solving the control problem associated with (1.2.11)–(1.2.14) in the time interval $[0, \tau]$, we need the equation $c_k^f(\tau) = 0,\ k \in \mathbb{N}$ to be satisfied. This leads to the following moment problem

$$0 = \frac{a_k}{\sqrt{\lambda_k}} e^{-\lambda_k \tau} + \sum_{\gamma \in \Gamma} \frac{\varkappa_k(\gamma)}{\sqrt{\lambda_k}} \int_0^{\tau} e^{-\lambda_k (\tau - s)} f(\gamma, s)\, ds, \quad k \in \mathbb{N}. \tag{1.5.2}$$

Definition 5. *The moment problem* (1.5.2) *is solvable in the time interval* $[0, \tau]$ *for some* $\tau > 0$ *if and only if, for arbitrary* $\left\{ \frac{a_k}{\sqrt{\lambda_k}} \right\}_{k=1}^{\infty} \in l_2$, *there exists* $f \in \mathcal{F}_\Gamma^\tau$ *such that equalities* (1.5.2) *hold.*

Note that solvability of the moment problem (1.5.2) is equivalent to the null controllability of the dynamical system (1.2.11)–(1.2.14).

Definition 6. *The family* $\{\xi_k\}_{k=1}^{\infty}$ *in a Hilbert space* H *is called minimal if, for every* $k \in \mathbb{N}$, *element* ξ_k *does not belong to the closure of the linear span of the remaining elements.*

Another equivalent characteristic of the minimal family $\{\xi_k\}_{k=1}^{\infty}$ in a Hilbert space H with the scalar product $< \cdot, \cdot >$ is the existence of the bi-orthogonal family $\{\xi_k'\}_{k=1}^{\infty} \subset H$ such that

$$< \xi_k, \xi_n' >= \delta_{k,n}, \quad k, n \in \mathbb{N}.$$

It is well known, that if a vector family is an $\mathcal{L}$-basis in H, it is minimal in H.

Proposition 4 states that the 'hyperbolic' family $\{E_{\pm k}\}_{k \in \mathbb{N}}$ defined by (1.4.5) forms an $\mathcal{L}$-basis in $L_2([0, d(\Omega)]; \mathbb{C}^m)$. Let us denote by $\{E_{\pm k}'\}_{k \in \mathbb{N}}$ the family bi-orthogonal to $\{E_{\pm k}\}_{k\in\mathbb{N}}$. There are connections between the 'hyperbolic' family (1.4.5) and the 'parabolic' one,

$$\{Q_k\}_{k=1}^{\infty}, \quad Q_k(t) = \alpha_k e^{-\lambda_k t}, \quad k \in \mathbb{N}, \tag{1.5.3}$$

first established by D.L. Russell [80]. We use his result in a slightly more general form, formulated in Theorem II.5.20 of [8], from which it follows that the 'parabolic' family $\{Q_k\}_{k=1}^{\infty}$ is minimal in $L_2([0,\tau],\mathbb{C}^m)$ for every $\tau > 0$ and for the members of the 'parabolic' bi-orthogonal family $\{Q'_k\}_{k=1}^{\infty}$ the following estimates hold:

$$\|Q'_k\|_{L_2([0,\tau],\mathbb{C}^m)} \leqslant C(\tau)\|E'_k\|_{L_2([0,d(\Omega)],\mathbb{C}^m)} e^{\beta\sqrt{|\lambda_n|}}, \quad k \in \mathbb{N}, \tag{1.5.4}$$

with positive constants $C(\tau)$ and β.

To prove Theorem 2, one needs to show the solvability of the moment problem (1.5.2) which can be rewritten as

$$-\frac{a_k}{\sqrt{\lambda_k}} e^{-\lambda_k \tau} = \sum_{\gamma\in\Gamma} \frac{\varkappa_k(\gamma)}{\sqrt{\lambda_k}} \int_0^{\tau} e^{-\lambda_k t} f(\gamma, \tau - t)\, dt, \quad k \in \mathbb{N},$$

or, shortly, as

$$-\frac{a_k}{\sqrt{\lambda_k}} e^{-\lambda_k \tau} = (Q_k, f^{\tau})_{\mathcal{F}_\Gamma^\tau}, \quad k \in \mathbb{N}, \tag{1.5.5}$$

where $f^{\tau}(\gamma,t) = f(\gamma,\tau-t)$. One can check that a formal solution of (1.5.5) has the form

$$f^{\tau}(\gamma,t) = -\sum_{k=1}^{\infty} a_k e^{-\lambda_k \tau} Q'_k(t). \tag{1.5.6}$$

Estimates (1.5.4) imply that $f^{\tau}(\gamma,t)$ defined by (1.5.6), belongs to $\mathcal{F}_\Gamma^\tau$ and therefore, the moment problem (1.5.2) is solvable. This completes the proof of Theorem 2.

The proof of Theorem 5 is similar. The corresponding family of exponentials that arise while reducing the control problem to the moment problem has the form:

$$\{Q_k^1\}_{k=1}^{\infty}, \quad Q_k^1(t) = \alpha'_k e^{-\lambda_k t} \quad k \in \mathbb{N}. \tag{1.5.7}$$

We conclude this section with results about families of vector exponentials that naturally appeared in the proofs.

Proposition 6. *The family $\{Q_k\}_{k=1}^{\infty}$ (see (1.5.3)) constructed using the Dirichlet spectral data (1.2.6) is minimal in $L_2([0,T];\mathbb{C}^m)$ for any $T > 0$.*

Suppose that we pick arbitrary boundary point of the graph (we keep the notation γ_1 for it), then the following statement is true.

Proposition 7. *The family* $\{Q_k^1\}_{k=1}^{\infty}$ *(see* (1.5.7)*) constructed using the Dirichlet spectral data* (1.2.6), (1.4.6) *is minimal in* $L_2([0,T];\mathbb{C}^{m-1})$ *for any* $T>0$.

We emphasize that an independent proof of Propositions 6, 7 without using the control theoretic approach would be very difficult.

1.6 Proof of Theorem 3.

To prove Theorem 3 we use the scheme proposed in [73]. We reformulate the initial boundary value problems (1.2.7)–(1.2.10) and (1.2.15)–(1.2.18) in the operator form. Results concerning the dependance of solutions to systems dual to (1.2.7)–(1.2.10), (1.2.15)–(1.2.18) on the initial data, allow us to use the Theorem 3.1 of [73] that derives the exact controllability of the first-order system (1.2.15)–(1.2.18) in the arbitrary time interval from the exact controllability of the second-order system (1.2.7)–(1.2.10) in some time interval.

Let us introduce the operator $A=-\frac{1}{\rho}\frac{d^2}{dx^2}$ in $H_0:=\mathcal{H}=L_{2,\,\rho}(\Omega)$. If the density ρ satisfies (1.2.1), the operator A is self-adjoint, positive definite and boundedly invertible with the domain

$$D(A)=\left\{a\in H_0,\, a|_{e_i}\in H^2(e_i),\, a \text{ satisfies (1.2.3), (1.2.4)}),\, a|_{\Gamma}=0\right\}.$$

This operator defines the scale H_p, $p\in\mathbb{Z}$, of Hilbert spaces. For $p>0$, integer, $H_p=D(A^{\frac{p}{2}})$ with the norm $\|x\|_p=\|A^{\frac{p}{2}}x\|$, H_{-p} is dual to H_p with respect to the scalar product in H_0. Another characterization of $H_{-p}(\Omega)$ is that it is the completion of H_0 with respect to the norm $\|x\|_{-p}=\|A^{-\frac{p}{2}}x\|$. By A' we denote the operator dual to A: it is the extension of A to H_{-2} with the domain H_0. Let $Y=\mathbb{R}^m$ and $C:H_2\mapsto Y$ be defined by:

$$Ca=\operatorname{col}\left(\partial a(\gamma)\right)_{\gamma\in\Gamma}.$$

Let the operator $B:Y\mapsto H_{-2}$ be dual to C. In this notations we can rewrite the dynamical system (1.2.15)–(1.2.18) as

$$u_t(t)-iA'u(t)=Bf(t),\quad u(0)=a\in H_0. \tag{1.6.1}$$

The dual observation system with output function y is defined by

$$u_t(t) - iAu(t) = 0, \quad u(0) = u_0 \in H_0, \quad y(t) = Cu(t). \tag{1.6.2}$$

The smoothness of the solution of (1.6.2) (see [10] for the case of one interval) guarantees that for the *observation* operator $\mathcal{C}_s : u_0 \mapsto y(t)$ the following estimate holds:

$$\|\mathcal{C}_s u_0\|_{\mathcal{F}^T} \leqslant K_T \|u_0\|_{H_0}, \quad u_0 \in H_2 \tag{1.6.3}$$

with $K_T > 0$.

System (1.2.7)–(1.2.10) can be rewritten as

$$u_{tt}(t) + A'u(t) = Bf(t), \quad u(0) = 0, \, u_t(0) = 0. \tag{1.6.4}$$

The dual observation system with the output function z has the form

$$u_{tt}(t) + Au(t) = 0, \quad u(0) = u_0 \in H_1, \, u_t(0) = u_1 \in H_0, \quad z(t) = Cu(t).$$

The *observation* operator $\mathcal{C}_w : \{u_0, u_1\} \mapsto z(t)$, satisfies the estimate:

$$\|\mathcal{C}_w\{u_0, u_1\}\|_{\mathcal{F}^T} \leqslant K_T^1(\|u_0\|_{H_1} + \|u_1\|_{H_0}) \tag{1.6.5}$$

with $K_T^1 > 0$ (see [64]). Now we can use Theorem 3.1 of [73], which says that if the dynamical system (1.6.4) is exactly controllable in some time interval (in our case it is controllable in the time interval $(0, d(\Omega))$, then the system (1.6.1) is exactly controllable in any time interval, provided observation operators satisfy inequalities (1.6.3), (1.6.5). This completes the proof of Theorem 3.

Remark 1. *The proof of Theorem 6 is similar, one should refer to Theorem 4 for the controllability of the corresponding second order dynamical system.*

Looking for the solution of (1.2.15)–(1.2.18) in the form (1.4.1) for the fixed initial state $a \in \mathcal{H}_{-1}$ with the expansion (1.5.1), we come to the following formulas for the coefficients:

$$c_k^f(t) = a_k e^{i\lambda_k t} + \sum_{\gamma \in \Gamma} \varkappa_k(\gamma) \int_0^t e^{i\lambda_k(t-s)} f(\gamma, s)\, ds.$$

Solving the control problem associated with (1.2.15)–(1.2.18) in the time interval $[0, \tau]$, we obtain the following moment problem

$$0 = \frac{a_k}{\sqrt{\lambda_k}} e^{i\lambda_k \tau} + \sum_{\gamma \in \Gamma} \frac{\varkappa_k(\gamma)}{\sqrt{\lambda_k}} \int_0^\tau e^{i\lambda_k(\tau - s)} f(\gamma, s)\, ds, \quad k \in \mathbb{N}. \tag{1.6.6}$$

Theorem 3 implies that the moment problem (1.6.6) is solvable for any $\tau > 0$. Using Theorem III.3.3 of [8] we deduce the result about family of vector valued exponentials that appeared in the moment problem (1.6.6).

Corollary 1. *The family*

$$\{D_k\}_{k=1}^{\infty}, \quad D_k(t) = \alpha_k e^{i\lambda_k t}, \quad k \in \mathbb{N},$$

constructed using the Dirichlet spectral data (1.2.6) is an $\mathcal{L}$-basis in $L_2([0,\tau];\mathbb{C}^m)$ for any $\tau > 0$.

Picking arbitrary boundary point of the graph (we keep the notation γ_1 for it) and using Theorem 6, we get

Corollary 2. *The family*

$$\left\{D_k^1\right\}_{k=1}^{\infty}, \quad D_k^1(t) = \alpha_k' e^{i\lambda_k t},$$

constructed using the Dirichlet spectral data (1.2.6), (1.4.6) is an $\mathcal{L}$-basis in $L_2([0,\tau];\mathbb{C}^{m-1})$ for any $\tau > 0$.

Chapter 2

On an inverse problem for tree-like networks of elastic strings

2.1 Introduction.

In many problems in science and engineering network-like structures play a fundamental role. The most classical area of applications consists of flexible structures made of strings, beams, cable and struts. Bridges, space-structures, antennas, transmission-line posts, steel-grid structures as reinforcements of buildings and other projects in civil engineering. See Lagnese, Leugering and Schmidt [62] for an account of multi-link structures. More recently applications also on a much smaller scale came into focus. In particular hierarchical materials like ceramic or metallic foams, percolation networks and even carbon nano-tubes have attracted much attention. In the latter context, the problem is understood as a quantum-tree-problem. See, e.g. Kuchment [58], Kostrykin and Schrader [55], Avdonin and Kurasov [9]. In all of these areas the topology of the underlying networks or graphs plays a dominant role. The understanding of the influence of the local topology and physical parameters, say at a given branching point, on the global mechanical or scattering properties is crucial in this area. Failure detection in mechanical multi-link structures by non-invasive methods as well as topological and material sensitivities with respect to an observer play an important role. Gaining this understanding is the central focus of this paper. Undoubtedly, the inverse problem for mechanical structural elements like membranes and plates has been discussed in the literature. The famous question by Kac [52]: "can one hear the shape of a drum" initiated major research in this direction. This question has been repeated in the literature regarding other structures, also for string-networks on a tree by Belishev and Vakulenko [23, 35], Brown and Weikard [40], and Avdonin and Kurasov [9]. However, in that work strings have been considered as deflecting out of the plane rather than in the plane. The important and in fact crucial difference is that such networks are insensitive for the geometry of the graph in the sense that the coupling conditions do not reflect the angles at which the strings are 'glued' together. Only in the case of in-plane motion are the coupling conditions dependent on the local geometry of the multiple joints. This observation is even more relevant for networks containing beams, a case that is subject to current investigation.

In a more abstract setting, where also electromagnetic or quantum effects are considered on graphs, one observes that such an in-plane modeling involves multi-channel and multi-velocity models for wave propagation in thin structures.

Tackling inverse problems involves the understanding Steklov-Poincaré operators just as in the case of domain decomposition. Such operators for problems on graphs have been investigated in Lagnese and Leugering [60]. Scattering matrices, indeed the Tichmarsh-Weyl function for in-plane-networks of strings, at that time called echo-analysis, have been investigated in Leugering [67]. In particular, the understanding of controllability properties of the underlying structures is crucial for a dynamic, and indeed real-time, detection of physical and geometrical properties. Again, exact controllability of networks of strings both in the out-of-the-plane and the more important in-plane mode has been investigated by Lagnese, Leugering and Schmidt [63, 61, 62], and by Avdonin and Ivanov [8], see also [2, 13]. There it has been shown that under generic assumptions, controllability of a rooted tree holds by controls at the leaves. Later Leugering and Zuazua [69] showed that under more refined assumptions on the nature of the out-of-the-plane string-tree exact controllability in refined spaces was even possible when the root was controlled only. This research has been extended considerably in Dager and Zuazua [44]. As it turned out in Belishev and Vakulenko [23, 35], Avdonin and Kurasov [9] exact controllability of both the state and the velocity appeared too demanding. Indeed, their 'boundary-control-approach' is based on controllability of the state only. The work on inverse problems by the way of the boundary-control-approach has by now become a major tool. The current paper is no exception in that direction. For inverse spectral problems on graphs see also the works of Freiling and Yurko [45, 87].

2.2 Forward dynamical and spectral problems for the two-velocity system on the tree.

Let Ω be a finite connected compact planar graph without cycles, i.e. a tree. The graph consists of edges $E = \{e_1, \dots, e_N\} = \{e_i | i \in \mathcal{I}\}$, where $\mathcal{I} := \{1, \dots, N\}$, connected at the vertices (nodes) $V = \{v_1 \dots, v_{N+1}\}$ (see Figure 2.1).

Every edge $e_i \in E$ is identified with an interval $[\beta_{2i-1}, \beta_{2i}]$ of the real line. The

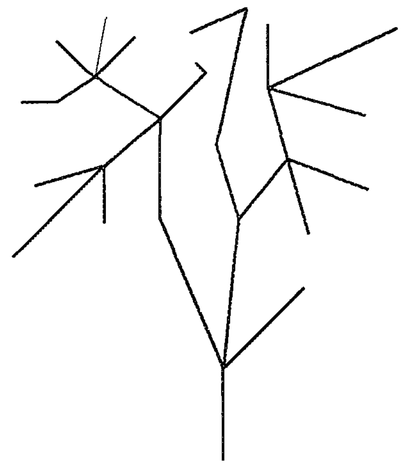

Figure 2.1: The tree

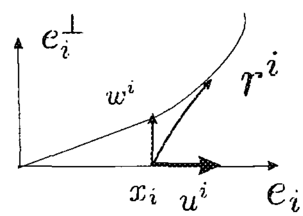

Figure 2.2: Representation of planar displacement

edges are connected at the vertices v_j which can be considered as equivalence classes of the edge end points $\{\beta_j\}$.

Once the geometry of the underlying graph is defined, one introduces displacements $r^i(x) \in \mathbf{R}^2$ at each point of the graph. As each edge carries along an individual coordinate system $\mathbf{e}_i$, $\mathbf{e}_i^\perp$ the displacement decomposes as follows:

$$r^i(x) := u^i(x)\mathbf{e}_i + w^i(x)\mathbf{e}_i^\perp$$

where $u^i(x)$ is the longitudinal (tangential) displacement and $w^i(x)$ the vertical (in the plane) or normal displacement $w^i(x)$ at the material point $x \in [\beta_{2i-1}, \beta_{2i}]$ (see Figure 2.2).

The corresponding strains decouple accordingly: $r^i_x(x) = u^i_x(x)\mathbf{e}_i + w^i_x(x)\mathbf{e}_i^\perp$, where the suffix x signifies a derivative with respect to the variable x. We assume linear Hookean material. The stiffness matrix can then be expressed as

$$K_i = k_{i1}^2\mathbf{e}_i\mathbf{e}_i^T + k_{i2}^2\mathbf{e}_i^\perp(\mathbf{e}_i^\perp)^T.$$

In this article we assume that the tangential and normal stiffness parameters are constant along each edge e_i. The local balance of forces along an individual edge then turns into the classical equation for the transient motion of a planar string:

$$r^i_{tt} - K_i r^i_{xx} = f_i,$$

where now the suffix t represents a time-derivative and f_i is an external force acting on edge e_i. In this notation the mass is incorporated into the stiffness parameter. Indeed, for $x \in e_i \subset \Omega \backslash V$ we may define the density $\rho(x) := \max\left\{\frac{1}{k_{i1}^2}, \frac{1}{k_{i2}^2}\right\}$.

Since the graph under consideration is a tree, for every $a, b \in \Omega$, $a \neq b$, there exist the unique path $\pi[a,b]$ connecting these points. The density ρ determines the optical metric and the optical distance

$$d\,\sigma := \sqrt{\rho(x)}\,|d\,x|, \quad x \in \Omega \backslash V,$$
$$\sigma(a,b) = \int\limits_{\pi[a,b]} \sqrt{\rho(x)}\,|d\,x|, \quad a, b \in \Omega,$$

The optical diameter of the graph Ω is defined as

$$d(\Omega) = \max_{a,\,b \in \Gamma} \sigma(a,b).$$

The graph Ω and the optical metric determine the *metric graph* denoted by $\{\Omega, \rho\}$. For a rigorous definition of the metric graph see, e.g. [55, 58, 59, 72, 77].

Once the equations of motions along an individual edge are given, we have to describe the coupling conditions across multiple joints. We are going to do that for general planar graphs rather than just for trees. To this end we introduce some additional notation. For the convenience of the reader we label vertices by upper case letters in order to distinguish clearly from edge labels which, in turn, are given by lower case letters. This convention is dropped in the analysis later because the tree structure implies a more direct labeling.

Given a node v_J we define $\mathcal{I}_J := \{i \in \mathcal{I} | e_i \text{ is incident at } v_J\}$ the incidence set, and $d_J = |\mathcal{I}_J|$ the edge degree of v_J. The set $\mathcal{J} = \{J | v_J \in V\}$ of node indices splits into $\mathcal{J}_S$ and $\mathcal{J}_M$ which correspond to simple and multiple nodes according to $d_J = 1$ and $d_J > 1$, respectively.

The set of simple (exterior) nodes is called the boundary of the graph Ω and is denoted by Γ. In this paper we suppose that external forces act only at the boundary of the graph through the non-homogeneous Dirichlet boundary conditions.

For $i \in \mathcal{I}_J$ we set

$$r^i(v_J) = \begin{cases} r^i(\beta_{2j-1}) & \text{if edge } i \text{ starts at } v_J, \\ r^i(\beta_{2j}) & \text{if edge } i \text{ ends at } v_J, \end{cases}$$

$$r^i_x(v_J) = \begin{cases} -r^i_x(\beta_{2j-1}) & \text{if edge } i \text{ starts at } v_J, \\ r^i_x(\beta_{2j}) & \text{if edge } i \text{ ends at } v_J. \end{cases}$$

This means that $r^i_x(v_J)$ is the derivative of r^i taken along the edge e_i at the endpoint corresponding to the vertex v_J in the direction towards the vertex.

We may then consider time dependent displacements $r^i(x,t)$, $t \in [0,T]$, (T is an arbitrary fixed positive number). The system of equations governing the full transient motion is given by

$$\begin{cases} r^i_{tt} - K_i r^i_{xx} = 0, \quad x \in (\beta_{2j-1}, \beta_{2j}),\ t \in (0,T),\ i \in \mathcal{I}, \\ r^i(v_J,t) = f_J(t), \quad i \in \mathcal{I}_J,\ J \in \mathcal{J}_S,\ t \in (0,T), \\ r^i(v_J,t) = r^j(v_J,t), \quad i,j \in \mathcal{I}_J,\ J \in \mathcal{J}_M,\ t \in (0,T), \\ \sum_{i \in \mathcal{I}_J} K_i r^i_x(v_J,t) = 0, \quad J \in \mathcal{J}_M,\ t \in (0,T), \\ r^i(x,0) = r^i_0,\ r^i_t(x,0) = r^i_1, \quad x \in (\beta_{2j-1}, \beta_{2j}),\ i \in \mathcal{I}. \end{cases} \tag{2.2.1}$$

It is important to understand the coupling conditions $(2.2.1)_{3,4}$. Indeed, the first of these conditions, namely $r^i(v_J,t) = r^j(v_J,t)$ for $i,j \in \mathcal{I}_J, J \in \mathcal{J}_M$, simply expresses the continuity of displacements across the vertex v_J. Without this condition the network falls apart. The second condition, namely $\sum_{i \in \mathcal{I}_J} K_i r^i_x(v_J,t) = 0$ for $J \in \mathcal{J}_M$, reflects the physical law that the forces at the vertex v_J, in the absence of additional external forces acting on v_J, should add up to zero. Notice that the coupling at multiple nodes v_J, those where $d_J > 1$, is a vectorial equation. This is in contrast to the out-of-the-plane model, where no such vectorial couplings occur which, in turn, makes the problem then independent of the particular geometry. In the case treated here the geometry, represented by the pairs $(\mathbf{e}_i, \mathbf{e}_i^{\perp})$ does play a crucial role.

In Leugering and Sokolowski [68] the static system, i.e. (2.2.1) without time-dependence, has been investigated with respect to topological sensitivities, such that the potential energy or other functionals are considered under variation of the edge-degree, nodal positions and edge-deletion. Like in this paper, the analysis of the Steklov-Poincaré operators plays a crucial role.

Another remark about model (2.2.1) is in order. If the system is static, the stiffness only active for longitudinal displacements, and if the state is edge-wise linear, then (2.2.1) comes down to a truss-model.

In the global Cartesian coordinate system, one can represent each edge by a rotation matrix

$$S_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

where $\mathbf{e} = (\cos\alpha, \sin\alpha)^T$. In fact, it is the global coordinate system that we will use throughout the paper, as we are going to identify the angles α_{ij} between two branching edges.

2.2.1 Spectral settings.

Now we introduce the spaces of real valued square integrable functions on the graph Ω:

$$L_2(\Omega) = \bigoplus_{i=1}^{N} L_2(e_i, \mathbb{R}^2). \tag{2.2.2}$$

For the element $U \in L_2(\Omega)$ we write

$$U = \{u, w\} = \left\{ \begin{pmatrix} u^i \\ w^i \end{pmatrix} \right\}_{i=1}^{N}, \quad u^i, w^i \in L_2(e_i). \tag{2.2.3}$$

We can reformulate the compatibility conditions in (2.2.1) at multiple nodes (vertices) v using global coordinates. For the sake of self-consistency in this framework we put this in the format of definitions. We denote by α_{ij} the angle between two edges e_i and e_j counting from e_i counterclockwise and introduce the matrices

$$D_i' = \begin{pmatrix} k_{i1}^2 & 0 \\ 0 & k_{i2}^2 \end{pmatrix}, \quad i \in \mathcal{I}. \tag{2.2.4}$$

We set $S_{ij} := S_{\alpha_{ij}}$.

Definition 1. *We say that the edge-wise continuous function U satisfy the first condition (continuity) at the (multiple node) internal vertex v_J if*

$$\begin{pmatrix} u^i(v_J) \\ w^i(v_J) \end{pmatrix} = S_{ij} \begin{pmatrix} u^j(v_J) \\ w^j(v_J) \end{pmatrix}, \quad i,j \in \mathcal{I}_J. \tag{2.2.5}$$

Let e_i be an edge incident at v_J.

Definition 2. *We say that the edge-wise continuously differentiable function U satisfies the second condition (force balance) at the internal vertex v_J if*

$$\sum_{j \in \mathcal{I}_J} S_{ij} D'_j \begin{pmatrix} u^j_x(v_J) \\ w^j_x(v_J) \end{pmatrix} = 0. \tag{2.2.6}$$

It is easy to check that if the condition (2.2.6) is satisfied for some $i \in \mathcal{I}_J$, then it is valid for any $i \in \mathcal{I}_J$.

We put $\{\phi, \psi\} = \left\{ \begin{pmatrix} \phi^i \\ \psi^i \end{pmatrix} \right\}_{i=1}^{N} \in L_2(\Omega)$ and associate the following spectral problem to the graph:

$$\left. \begin{aligned} -k_{i1}^2 \phi^i_{xx} &= \lambda \phi^i \\ -k_{i2}^2 \psi^i_{xx} &= \lambda \psi^i \end{aligned} \right\} \quad x \in e_i, \quad i \in \mathcal{I}, \tag{2.2.7}$$

$$\{\phi, \psi\} \text{ satisfies (2.2.5), (2.2.6) at all internal vertices } v_J,\ J \in \mathcal{J}_M, \tag{2.2.8}$$

$$\{\phi, \psi\} = 0 \quad \text{on the boundary } \Gamma \tag{2.2.9}$$

The last equality means that $\phi^i(v_J) = \psi^i(v_J) = 0$, for $i \in \mathcal{I}_J$, $J \in \mathcal{J}_S$.

Definition 3. *By* **S** *we denote the spectral problem described by (2.2.7)-(2.2.9).*

It is known that the problem **S** has a discrete spectrum of eigenvalues $0 < \lambda_1 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots, \lambda_k \to \infty$. Corresponding eigenfunctions $\{\phi, \psi\}$ can be chosen such that they form an orthonormal basis in $L_2(\Omega)$. Indeed, for scalar problems, i.e. out-of-the-plane displacements in the mechanical context or conductivity, the spectral behavior has been explored by von Below [37], Nicaise [76] and others. The in-plane case discussed here has been treated in Lagnese, Leugering and Schmidt [62].

2.2.2 Dynamical settings.

Along with the spectral, we consider the dynamical system, described by the two-velocity problem on the each edge of the graph:

$$\frac{1}{k_{i1}^2}u_{tt}^i - u_{xx}^i = 0, \quad t > 0, \ x \in e_i \tag{2.2.10}$$

$$\frac{1}{k_{i2}^2}w_{tt}^i - w_{xx}^i = 0, \quad t > 0, \ x \in e_i. \tag{2.2.11}$$

Here the coefficients k_{i1}, k_{i2} play the role of speeds of the wave propagation on the edge e_i, $i = 1, \ldots N$ in the first and second channels.

We assume that $|\mathcal{J}_S| = m$. By $\mathcal{F}_\Gamma^T = L_2([0,T], \mathbb{R}^{2m})$ we denote the space of controls acting on the boundary of the tree. For the element $F \in \mathcal{F}_\Gamma^T$ we write

$$F = \{f, g\} = \left\{ \begin{pmatrix} f^i \\ g^i \end{pmatrix} \right\}_{i=1}^m, \quad f^i, g^i \in L_2(0,T).$$

We will deal with the Dirichlet boundary conditions:

$$\{u, w\} = \{f, g\}, \quad \text{on } \Gamma \times [0,T]. \tag{2.2.12}$$

where $f, g \in \mathcal{F}_\Gamma^T$. The last equality means that $u^i(v_J) = f^i(v_J)$, $v^i(v_J) = g^i(v_J)$, for $i \in \mathcal{I}_J$, $J \in \mathcal{J}_S$.

Definition 4. *By* $\mathbf{D}$ *we denote the dynamical problem on the graph* Ω*, described by the equations on the edges (2.2.10), (2.2.11) which satisfies compatibility conditions (2.2.5), (2.2.6) at all internal vertexes for any* $t > 0$*, Dirichlet boundary condition (2.2.12) and zero initial conditions* $\{u(\cdot,0), w(\cdot,0)\} = \{0,0\}$*,* $\{u_t(\cdot,0), w_t(\cdot,0)\} = \{0,0\}$.

It is known that for any $T > 0$, $\{u, w\} \in C([0,T]; L^2(\Omega))$ if $F \in \mathcal{F}_\Gamma^T$ (see, e.g. [62, 60]).

2.3 Inverse dynamical and spectral problems. Connection of the inverse data.

We use the Titchmarsh-Weyl (TW) matrix as the data for the inverse spectral problem. For the spectral problem on an interval and on the half line the TW function is a classical object. For the inverse spectral problem on trees it was used in [9, 40, 87]. The general properties of the $M-$operator for self-adjoint operators are considered in [1, 42, 17].

Let us choose $\lambda \notin \mathbb{R}$. We define $\{\phi^1, \psi^1\}$, $\{\phi^2, \psi^2\}$ — two solutions of (2.2.7), (2.2.8) and the following boundary conditions:

$$\{\phi^1, \psi^1\} = \begin{pmatrix} (0,0) \\ \cdots\cdots \\ (1,0) \\ (0,0) \end{pmatrix}, \quad \{\phi^2, \psi^2\} = \begin{pmatrix} (0,0) \\ \cdots\cdots \\ (0,1) \\ (0,0) \end{pmatrix} \quad \text{on } \Gamma, \tag{2.3.1}$$

where nonzero elements are located at the $i-$th row. Then the TW matrix $\mathbf{M}(\lambda)$ is defined as $\mathbf{M}(\lambda) = \{M_{ij}(\lambda)\}_{i,j=1}^m$ where each $M_{ij}(\lambda)$ is a 2×2 matrix defined by

$$M_{ij}(\lambda) = \begin{pmatrix} \phi^1_x(v_j, \lambda) & \psi^1_x(v_j, \lambda) \\ \phi^2_x(v_j, \lambda) & \psi^2_x(v_j, \lambda) \end{pmatrix}, \quad 1 \leqslant i, j \leqslant m. \tag{2.3.2}$$

Let us consider the nonhomogeneous Dirichlet boundary condition

$$\{\phi, \psi\} = \{\zeta, \nu\} \text{ on } \Gamma, \tag{2.3.3}$$

where $\{\zeta, \nu\} \in \mathbb{R}^{2m}$, and let $\{\phi, \psi\}$ be the solution to (2.2.7), (2.2.8), (2.3.3). The Titchmarsh-Weyl matrix connects the values of $\{\phi, \psi\}$ on the boundary and the values of its derivative $\{\phi_x, \psi_x\}$ on the boundary:

$$\{\phi_x, \psi_x\} = \mathbf{M}(\lambda)\{\zeta, \nu\} \quad \text{on } \Gamma. \tag{2.3.4}$$

We set up the *inverse spectral problem* as follows: given the TW matrix $\mathbf{M}(\lambda)$, $\lambda \notin \mathbb{R}$, to recover the graph (lengths of edges, connectivity and angles between edges) and parameters of the system (2.2.7), i.e. the set of coefficients $\{k_{i1}, k_{i2}\}_{i=1}^N$.

Let $\{u, w\}$ be the solution to the problem $\mathbf{D}$ with the boundary control $\{f, g\} \in \mathcal{F}^T_\Gamma$. We introduce the *dynamical response operator* (the dynamical Dirichlet-to-Neumann map) to the problem $\mathbf{D}$ by the rule

$$R^T\{f, g\}(t) = \{u_x(\cdot, t), w_x(\cdot, t)\}\Big|_\Gamma, \quad t \in [0, T]. \tag{2.3.5}$$

The response operator has the form of a convolution:

$$\left(R^T\{f,g\}\right)(t) = \left(\mathbf{R} * \{f,g\}\right)(t)\,, t \in [0,T], \tag{2.3.6}$$

where $\mathbf{R}(t) = \{R_{ij}(t)\}_{i,j=1}^m$ and each $R_{ij}(t)$ is a 2×2 matrix. The entries $R_{ij}(t)$ are defined by the following procedure. We set up two dynamical problems defined by the equations (2.2.10), (2.2.11), (2.2.5), (2.2.6) and the boundary conditions given by

$$\{U^1(\cdot,t), W^1(\cdot,t)\} = \begin{pmatrix} (0,0) \\ \cdots\cdots \\ (\delta(t),0) \\ (0,0) \end{pmatrix}, \ \{U^2(\cdot,t), W^2(\cdot,t)\} = \begin{pmatrix} (0,0) \\ \cdots\cdots \\ (0,\delta(t)) \\ (0,0) \end{pmatrix} \text{ on } \Gamma. \tag{2.3.7}$$

In the above notations, the only nonzero rows is $i-$th. Then

$$R_{ij}(t) = \begin{pmatrix} U_x^1(v_j,t) & W_x^1(v_j,t) \\ U_x^2(v_j,t) & W_x^2(v_j,t) \end{pmatrix}. \tag{2.3.8}$$

So, to construct the entries of $\mathbf{R}$, we need to set up the boundary condition at $i-$th boundary point in the first and second channels, while having other boundary points fixed (impose homogeneous Dirichlet conditions there) and measure the response at $j-$th boundary point in the first and second channels.

We set up the *dynamical inverse problem* as follows: given the response operator R^T (2.3.5) (or what is equivalent, the matrix $\mathbf{R}(t)$, $t \in [0,T]$), for large enough T, to recover the graph (lengths of edges, connectivity and angles between edges) and parameters for the dynamical system (2.2.10), (2.2.11), i.e. speeds of the wave propagation on the edges.

The connection of the spectral and dynamical data is known and was used for studying the inverse spectral and dynamical problems, see for example [53, 9, 12]. Let $\{f,g\} \in \mathcal{F}_\Gamma^T \cap (C_0^\infty(0,+\infty))^{2m}$ and

$$\widehat{\{f,g\}}(k) := \int_0^\infty \{f(t),g(t)\}e^{ikt}\,dt$$

be its Fourier transform. The equations (2.2.10), (2.2.11) and (2.2.7) are connected by the Fourier transformation: going formally in (2.2.10), (2.2.11) over to the Fourier transform, we obtain (2.2.7) with $\lambda = k^2$. It is not difficult to check (see, e.g. [9, 12]) that the response matrix-function and Titchmarsh-Weyl matrix are connected by the same transform:

$$\mathbf{M}(k^2) = \int_0^\infty \mathbf{R}(t) e^{ikt}\, dt, \tag{2.3.9}$$

where this equality is understood in a weak sense.

2.4 Solution of the inverse problem. The case of two intervals.

We start with the inverse problem for two connected intervals. For the two-velocity system even this simple situation is nontrivial.

Suppose that a tree consists of two edges, e_1 and e_2 with the (unknown) lengths l_1 and l_2. The angle between edges we denote by $\alpha := \alpha_{12}$, the boundary Γ is $\{v_1, v_2\}$ and the only internal point is v_3. We suppose also that e_1 begins at v_1 and ends at v_3 and e_2 begins at v_3 and ends at v_2.

We consider the dynamical problem $\mathbf{D}$ on the tree and show that one needs to know the response operator or the TW function associated with one boundary vertex only to recover the graph.

Let us consider the initial boundary value problem (2.2.10), (2.2.11), (2.2.5), (2.2.6) with the boundary conditions given by

$$\{u, w\} = \begin{pmatrix} (\delta(t), 0) \\ (0, 0) \end{pmatrix}, \quad \text{on } \Gamma. \tag{2.4.1}$$

The solution of the above problem can be evaluated explicitly. For $0 \leqslant t \leqslant \frac{l_1}{k_{11}}$ it is given by

$$u^1(x,t) = \delta\left(t - \frac{x}{k_{11}}\right),$$
$$w^1(x,t) = 0.$$

On the time interval $\frac{l_1}{k_{11}} < t < \frac{2l_2}{\min\{k_{21}, k_{22}\}}$ on the first edge we have

$$u^1(x,t) = \delta\left(t - \frac{x}{k_{11}}\right) + a_1\delta\left(t + \frac{x}{k_{11}} - \frac{2l_1}{k_{11}}\right),$$
$$w^1(x,t) = b_1\delta\left(t + \frac{x}{k_{12}} - \gamma_{12}\right), \quad \gamma_{12} = \frac{l_1}{k_{11}} + \frac{l_1}{k_{12}},$$

and on the second edge

$$u^2(x,t) = a_2\delta\left(t - \frac{x}{k_{21}} - \gamma_{21}\right), \quad \gamma_{21} = \frac{l_1}{k_{11}} - \frac{l_1}{k_{21}},$$
$$w^2(x,t) = b_2\delta\left(t - \frac{x}{k_{22}} - \gamma_{22}\right), \quad \gamma_{22} = \frac{l_1}{k_{11}} - \frac{l_1}{k_{22}}.$$

In the formulas above, the coefficients a_1, a_2, b_1, b_2 are unknown.

From the condition (2.2.5) we obtain that

$$\begin{pmatrix} 1 + a_1 \\ b_1 \end{pmatrix} = S_\alpha \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}. \tag{2.4.2}$$

Condition (2.2.6) implies

$$D_1' \begin{pmatrix} -\frac{1}{k_{11}} + \frac{a_1}{k_{11}} \\ \frac{1}{k_{12}} \end{pmatrix} = S_\alpha D_2' \begin{pmatrix} -\frac{a_2}{k_{21}} \\ -\frac{b2}{k_{22}} \end{pmatrix}. \tag{2.4.3}$$

After introducing the notation

$$D_i = \begin{pmatrix} k_{i1} & 0 \\ 0 & k_{i2} \end{pmatrix}, \quad i = 1, \dots N,$$

one can rewrite (2.4.3) as

$$D_1 \begin{pmatrix} -1 + a_1 \\ b_1 \end{pmatrix} = -S_\alpha D_2 \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}. \tag{2.4.4}$$

Combining (2.4.2) and (2.4.4), we obtain

$$D_1 \begin{pmatrix} -1 + a_1 \\ b_1 \end{pmatrix} = -S_\alpha D_2 S_{-\alpha} \begin{pmatrix} 1 + a_1 \\ b_1 \end{pmatrix}. \tag{2.4.5}$$

Let us now consider the problem (2.2.10), (2.2.11), (2.2.5), (2.2.6) with the nonzero boundary condition at the second channel:

$$\{u, v\} = \begin{pmatrix} (0, \delta(t)) \\ (0, 0) \end{pmatrix}, \quad \text{on } \Gamma. \tag{2.4.6}$$

The computations similar to above show that the following condition must hold:

$$D_1 \begin{pmatrix} \widetilde{b}_1 \\ -1 + \widetilde{a}_1 \end{pmatrix} = -S_\alpha D_2 S_{-\alpha} \begin{pmatrix} \widetilde{b}_1 \\ 1 + \widetilde{a}_1 \end{pmatrix}, \tag{2.4.7}$$

where the coefficients $\widetilde{a}_1$, $\widetilde{a}_2$, $\widetilde{b}_1$, $\widetilde{b}_2$ are unknown.

The function $u^1(x,t)$, the first component of the solution to (2.2.10), (2.2.11), (2.2.5), (2.2.6), (2.4.1) has the following representation on the time interval $\frac{2l_1}{k_{11}} < t < \frac{2l_1}{k_{11}} + 2\min_{i,j}\{\frac{l_i}{k_{ij}}\}$

$$u^1(x,t) = \delta\left(t - \frac{x}{k_{11}}\right) + a_1\delta\left(t + \frac{x}{k_{11}} - \frac{2l_1}{k_{11}}\right) - a_1\delta\left(t - \frac{x}{k_{11}} - \frac{2l_1}{k_{11}}\right) \tag{2.4.8}$$

Thus (see the definition of the response operator (2.3.8)), for such t,

$$\{R_{11}\}_{11}(t) = u^1_x(0,t) = -\frac{1}{k_{11}}\delta'(t) + \frac{2a_1}{k_{11}}\delta'\left(t - \frac{2l_1}{k_{11}}\right). \tag{2.4.9}$$

A similar argument shows that for $\frac{l_1}{k_{11}} + \frac{l_1}{k_{12}} < t < \frac{l_1}{k_{11}} + \frac{l_1}{k_{12}} + 2\min_{i,j}\{\frac{l_i}{k_{ij}}\}$

$$\{R_{11}\}_{12}(t) = w^1_x(0,t) = 2\frac{b_1}{k_{12}}\delta'\left(t - \gamma_{12}\right). \tag{2.4.10}$$

Therefore, using the $\{R_{11}\}_{12}$, $\{R_{11}\}_{12}$ components of the response operator on the described time intervals, one can determine k_{11}, k_{12}, a_1, b_1, l_1. Applying the same argument to the problem (2.2.10), (2.2.11), (2.2.5), (2.2.6), (2.4.6), we conclude that $\{R_{11}\}_{21}$, $\{R_{11}\}_{22}$ components of the response function determine $\widetilde{a}_1$, $\widetilde{b}_1$.

Let us introduce the notations

$$\xi_1 = 1 + a_1, \quad \eta_1 = b_1,$$
$$\widetilde{\xi}_1 = \widetilde{b}_1, \quad \widetilde{\eta}_1 = 1 + \widetilde{a}_1,$$

and rewrite (2.4.5), (2.4.7) as

$$(S_\alpha D_2 S_{-\alpha} + D_1)\begin{pmatrix}\xi_1\\ \eta_1\end{pmatrix} = 2D_1\begin{pmatrix}1\\ 0\end{pmatrix}, \tag{2.4.11}$$

$$(S_\alpha D_2 S_{-\alpha} + D_1)\begin{pmatrix}\widetilde{\xi}_1\\ \widetilde{\eta}_1\end{pmatrix} = 2D_1\begin{pmatrix}0\\ 1\end{pmatrix}. \tag{2.4.12}$$

In the forward problem, the equations (2.4.11) and (2.4.12) can be used for the determination of the reflection and transmission coefficients a_1, b_1, $\widetilde{a}_1$, $\widetilde{b}_1$. Since the (given) matrixes $S_\alpha D_2 S_{-\alpha} + D_1$ and D_1 are positive definite (it is easy to check that

$S_\alpha D_j S_{-\alpha} \geqslant \min\{k_{j1}, k_{j2}\}I$), these coefficients are uniquely determined from (2.4.11), (2.4.12). Moreover, we necessarily have

$$\begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} \neq \begin{pmatrix} \widetilde{\xi}_1 \\ \widetilde{\eta}_1 \end{pmatrix}. \tag{2.4.13}$$

On the other hand, in the inverse problem we know D_1, ξ_1, η_1, $\widetilde{\xi}_1$, $\widetilde{\eta}_1$ (we can determine all these coefficients from R_{11} component of the response operator). Thus equations (2.4.11), (2.4.12) determine the matrix $A = S_\alpha D_2 S_{-\alpha}$. Using the invariants of matrix — the determinant and trace, we can find the matrix D_2 from the equations

$$k_{21} + k_{22} = \operatorname{tr} A, \quad k_{21}k_{22} = \det A. \tag{2.4.14}$$

The existence of angle α follows from the spectral theorem — S_α puts the operator A to a diagonal form.

Notice that if $D_2 = \gamma I$, there is no dependence on α in equations (2.4.11), (2.4.12).

We point out that we still need to determine the length of the second edge. For this aim we could analyze the representation of the solutions u^1, w^1 on a sufficiently large time interval. It would lead to an increasing number of terms in the representations (2.4.8), (2.4.9) and (2.4.10). Instead of that we will develop the method which works for general trees following the ideas of [9]. Let us consider the new tree, consisting of one edge: $\widetilde{\Omega} = e_2$. Idea of the method is to recalculate the TW matrix for the new tree using the TW matrix and response operator for the whole tree Ω and the data that we obtained on the first step, i.e. parameters of the first edge and the angle between edges.

Let $\{U, W\}$ be the solution of (2.2.7), (2.2.8) on Ω with boundary conditions

$$\{U, W\} = \{\zeta, \nu\}, \text{ at } v_1, \quad \{U, W\} = \{0, 0\}, \text{ at } v_2. \tag{2.4.15}$$

The compatibility conditions at the internal vertex v_3 are:

$$\begin{pmatrix} U^1(v_3, \lambda) \\ W^1(v_3, \lambda) \end{pmatrix} = S_\alpha \begin{pmatrix} U^2(v_3, \lambda) \\ W^2(v_3, \lambda) \end{pmatrix}, \tag{2.4.16}$$

$$D_1' \begin{pmatrix} U_x^1(v_3, \lambda) \\ W_x^1(v_3, \lambda) \end{pmatrix} = S_\alpha D_2' \begin{pmatrix} U_x^2(v_3, \lambda) \\ W_x^2(v_3, \lambda) \end{pmatrix}. \tag{2.4.17}$$

Let $\widetilde{M}(\lambda)$ be the TW matrix for the tree $\widetilde{\Omega}$. We see that $\widetilde{M}_{11}(\lambda)$ – component associated with the "new" boundary point v_3 satisfies the equation:

$$\begin{pmatrix} U^2_x(v_3,\lambda) \\ W^2_x(v_3,\lambda) \end{pmatrix} = \widetilde{M}_{11}(\lambda) \begin{pmatrix} U^2(v_3,\lambda) \\ W^2(v_3,\lambda) \end{pmatrix}. \tag{2.4.18}$$

From (2.4.16)–(2.4.18) it follows that

$$D_1' \begin{pmatrix} U^1_x(v_3,\lambda) \\ W^1_x(v_3,\lambda) \end{pmatrix} = S_\alpha D_2' \widetilde{M}_{11}(\lambda) S_{-\alpha} \begin{pmatrix} U^1(v_3,\lambda) \\ W^1(v_3,\lambda) \end{pmatrix}. \tag{2.4.19}$$

We emphasize that in (2.4.19) we know everything but the matrix $\widetilde{M}(\lambda)$. Choosing different boundary conditions for the problem in (2.4.15), we can get linear independent vectors $\begin{pmatrix} U^1(v_3,\lambda) \\ W^1(v_3,\lambda) \end{pmatrix}$ in (2.4.19). Thus (2.4.19) determines the matrix $\widetilde{M}_{11}(\lambda)$. The matrix $\widetilde{M}_{11}$ uniquely determines the corresponding component of the response operator $\widetilde{R}_{11}$ (see (2.3.9)). The latter operator in turn, allows us to find the parameters of the second edge, exactly as R_{11} determines the parameters of the first edge. (In our simple case of the two edge tree, the only parameter which we need to recover is the length of the second edge.)

We conclude the results of the present section in the following statement

Theorem 1. *Let Ω be the tree consisted of two edges. Then the tree and the parameters of the systems (2.2.10), (2.2.11) and (2.2.7) are determined by the 2×2 matrix $M_{11}(\lambda)$ — the diagonal element of the TW matrix, associated with the first boundary point.*

2.5 Solution of the inverse problem. The case of a star graph.

Suppose that the tree is a star graph with edges $e_1, \dots e_n$. We will show that to recover the graph and the parameters of the system (2.2.10), (2.2.11) it is sufficient to use the diagonal elements of the response operator (or diagonal elements of the Weyl matrix), associated to all but one boundary vertices.

Let us set up the initial-value problem (2.2.10), (2.2.11), (2.2.5), (2.2.6) with the boundary conditions given by the first equation in (2.3.7). We suppose that the only nonzero boundary conditions are given at the first channel of the $i-th$ boundary vertex, $i \neq n$. We suppose that e_i begins at v_i and ends at v_{n+1} and all other e_j, $j = 1, \ldots, n$, $j \neq i$ begin at v_{n+1} and end at v_j. Analyzing the solution $\{u, w\}$ of these problems in a way we did for the case of the graph of two edges, we obtain that on the time interval $\frac{l_i}{k_{i1}} < t < \frac{l_i}{k_{i1}} + 2\min_{i,j}\{\frac{l_i}{k_{ij}}\}$ on the i–th edge we have

$$u^i(x,t) = \delta\left(t - \frac{x}{k_{i1}}\right) + a_i^i\delta\left(t + \frac{x}{k_{i1}} - \frac{2l_i}{k_{i1}}\right),$$
$$w^i(x,t) = b_i^i\delta\left(t + \frac{x}{k_{i2}} - \gamma_{i2}\right), \quad \gamma_{i2} = \frac{l_i}{k_{i1}} + \frac{l_i}{k_{i2}}.$$

and on other edges ($j = 1 \ldots n$, $j \neq i$):

$$u^j(x,t) = a_j^i\delta\left(t - \frac{x}{k_{j1}} - \gamma_{j1}\right), \quad \gamma_{j1} = \frac{l_i}{k_{i1}} - \frac{l_i}{k_{j1}},$$
$$w^j(x,t) = b_j^i\delta\left(t - \frac{x}{k_{j2}} - \gamma_{j2}\right), \quad \gamma_{j2} = \frac{l_i}{k_{i1}} - \frac{l_i}{k_{j2}}.$$

Where a_j^i, b_j^i, $i = 1, \ldots, n-1$, $j = 1, \ldots, n$ are reflection and transmission coefficients associated with the $i-th$ vertex. Let us introduce new parameters

$$\xi_i = 1 + a_i^i, \quad \eta_i = b_i^i, \quad i = 1, \ldots, n-1. \tag{2.5.1}$$

The compatibility conditions (2.2.5), (2.2.6) at the internal vertex v_{n+1} (we need to rewrite them in a way we did for the case of two edges) lead to the following equalities (cf. (2.4.11)):

$$\left(\sum_{j=1,\, j\neq i}^{n} S_{ij}D_j(S_{ij})^{-1} + D_i\right)\begin{pmatrix}\xi_i\\ \eta_i\end{pmatrix} = 2D_i\begin{pmatrix}1\\0\end{pmatrix}, \quad i = 1, \ldots n-1. \tag{2.5.2}$$

Let us now set up the initial-value problem with the delta function in the second channel at i–th boundary point, $i \neq n$, which is given by (2.2.10), (2.2.11), (2.2.5), (2.2.6) and the boundary conditions given by the second equation in (2.3.7). Using the same orientation of edges as in the first case, we can obtain and analyze the representation for the solutions $\{\widetilde{u}, \widetilde{w}\}$ of these problems. Let $\widetilde{a}_i$, $\widetilde{b}_i$, $i = 1, \ldots, n-1$, $j = 1, \ldots, n$, be the reflection and transmission coefficients. Introducing new parameters

$$\widetilde{\xi}_i = \widetilde{b}_i^i, \quad \widetilde{\eta}_i = 1 + \widetilde{a}_i^i, \quad i = 1, \ldots n-1 \tag{2.5.3}$$

and making use the compatibility conditions (2.2.5), (2.2.6) at the internal vertex, we obtain the following equalities (cf. (2.4.12)):

$$\left(\sum_{j=1,\,j\neq i}^{n} S_{ij}D_j(S_{ij})^{-1}+D_i\right)\begin{pmatrix}\widetilde{\xi}_i\\ \widetilde{\eta}_i\end{pmatrix}=2D_i\begin{pmatrix}0\\1\end{pmatrix},\quad i=1,\dots n-1. \tag{2.5.4}$$

The matrices $\left(\sum_{j=1,\,j\neq i}^{n} S_{ij}D_j(S_{ij})^{-1}+D_i\right)$, $i=1,\dots,n-1$ are positive definite. If all angles between edges and all matrices D_j are known, the systems (2.5.2), (2.5.4) can be solved for ξ_i, η_i, $\widetilde{\xi}_i$, $\widetilde{\eta}_i$. Note that necessarily

$$\begin{pmatrix}\widetilde{\xi}_i\\ \widetilde{\eta}_i\end{pmatrix}\neq\begin{pmatrix}\xi_i\\ \eta_i\end{pmatrix},\quad i=1,\dots n-1.$$

In the situation of the inverse problem, using the diagonal elements $\{R_{ii}\}$, $i=1,\dots,n-1$, of the response operator, we can determine the reflection and transmission coefficients a^i_i, b^i_i, $\widetilde{a}^i_i$, $\widetilde{b}^i_i$, as well as l_i, D_i for $i=1,\dots,n-1$. Indeed, analyzing the solution to the dynamical system $\mathbf{D}$ with the boundary condition given by the delta function in the first channel at the $i-$th boundary vertex, it is easy to see (cf. (2.4.9), (2.4.10)) that:

$$\begin{aligned}\{R_{ii}\}_{11}(t)=u^i_x(0,t)&=-\frac{1}{k_{i1}}\delta'(t)+\frac{2a^i_i}{k_{i1}}\delta'\left(t-\frac{2l_i}{k_{i1}}\right),\\ &\qquad\frac{2l_i}{k_{i1}}<t<\frac{2l_i}{k_{i1}}+2\min_{i,j}\{\frac{l_i}{k_{ij}}\},\\ \{R_{ii}\}_{12}(t)=w^i_x(0,t)&=\frac{b^i_i}{k_{i2}}\delta'\left(t-\gamma_{i2}\right),\\ &\qquad\gamma_{i2}<t<\gamma_{i2}+2\min_{i,j}\{\frac{l_i}{k_{ij}}\}.\end{aligned}$$

The above representation allows one to determine a^i_i, b^i_i, l_i, k_{i1} for $i=1,\dots,n-1$. Analyzing the solutions to the dynamical system $\mathbf{D}$ with the boundary condition given by the delta function at the second channel of the $i-$th boundary vertex, we can determine $\widetilde{a}^i_i$, $\widetilde{b}^i_i$, k_{i2} for $i=1,\dots,n-1$.

Thus, since the vectors $\begin{pmatrix}\widetilde{\xi}_i\\ \widetilde{\eta}_i\end{pmatrix}$ and $\begin{pmatrix}\xi_i\\ \eta_i\end{pmatrix}$ in (2.5.2), (2.5.4) are known and necessarily different, the equations (2.5.2), (2.5.4) completely determine the matrixes

A_i,

$$A_i = \sum_{j=1,\, j\neq i}^{n} S_{ij} D_j (S_{ij})^{-1} + D_i, \quad i = 1 \dots n-1. \tag{2.5.5}$$

In (2.5.5) we do not know $n-1$ angles between edges and the matrix D_n. We use the system (2.5.5) to determine them in the same way we did it for the case of two intervals, but calculations are more involved.

Let us consider the condition (2.5.5) for $i = k$ and for $i = l$:

$$D_k + S_{kl} D_l (S_{kl})^{-1} + \sum_{j=1,\, j\neq k,\, l}^{n} S_{kj} D_j (S_{kj})^{-1} = A_k, \tag{2.5.6}$$

$$D_l + S_{lk} D_k (S_{lk})^{-1} + \sum_{j=1,\, j\neq k,\, l}^{n} S_{lj} D_j (S_{lj})^{-1} = A_l; \tag{2.5.7}$$

the matrices A_k and A_l here are known. Note that after the multiplication of (2.5.7) by S_{kl} from the left and by $(S_{kl})^{-1}$ from the right, and using that $S_{kl} = (S_{lk})^{-1}$, $S_{kl} S_{lj} = S_{kj}$, we obtain

$$A_k = S_{kl} A_l (S_{kl})^{-1}.$$

The angle α_{kl} can now be found using the spectral theorem. Repeating this procedure for various i, l we can determine all angles. After that we can use any of the conditions (2.5.5) to determine D_n. Indeed, taking $i = k$ we have

$$S_{kn} D_n (S_{kn})^{-1} = B_k \tag{2.5.8}$$

for some known matrix B_k. Then

$$k_{n1} + k_{n2} = \operatorname{tr} B_k, \quad k_{n1} k_{n2} = \det B_k. \tag{2.5.9}$$

The next step is crucial for solving the inverse problem: we have already recovered a part of the tree, and our next goal is to find the inverse data for the smaller "new" tree, using the initial inverse data and information that we obtained on the previous steps.

Let us consider the new tree, consisting of the one edge $\widetilde{\Omega} = e_n$. By $\{\Phi, \Psi\}$ we denote the solution to (2.2.7), (2.2.8) and the following boundary conditions

$$\{\Phi, \Psi\} = \{\zeta, \nu\}, \text{ at } v_1, \quad \{\Phi, \Psi\} = \{0, 0\}, \text{ at } v_i, \ 2 \leqslant i \leqslant n. \tag{2.5.10}$$

As in the case of a graph consisting of two intervals, our goal is to obtain the coefficient $\widetilde{M}_{11}$ of the TW-matrix for $\widetilde{\Omega}$, associated with the "new" boundary edge v_{n+1}. Note that we can assume that we have already recovered the information about all other edges and angles between them. So we have in hands the matrices D'_i, and α_{in} for $i = 1, \dots n$.

Note that solution to (2.2.7), (2.2.8), (2.5.10) on the edge e_1 solves the Cauchy problem

$$-k_{11}^2 \Phi^1_{xx} = \lambda \Phi^1, \quad -k_{12}^2 \Psi^1_{xx} = \lambda \Psi^1, \quad x \in e_1 \tag{2.5.11}$$

$$\{\Phi^1(v_1), \Psi^1(v_1)\} = \{\zeta, \nu\}, \tag{2.5.12}$$

$$\begin{pmatrix} \Phi^1_x(v_1) \\ \Psi^1_x(v_1) \end{pmatrix} = \{M_{11}(\lambda)\} \begin{pmatrix} \zeta \\ \nu \end{pmatrix}, \tag{2.5.13}$$

and on edges $e_2, \dots, e_{n-1}$ solves the Cauchy problems

$$-k_{i1}^2 \Phi^i_{xx} = \lambda \Phi^i, \quad -k_{i2}^2 \Psi^i_{xx} = \lambda \Psi^i, \quad x \in e_i \tag{2.5.14}$$

$$\{\Phi^1(v_i), \Psi^1(v_i)\} = \{0, 0\}, \tag{2.5.15}$$

$$\begin{pmatrix} \Phi^i_x(v_i) \\ \Psi^i_x(v_i) \end{pmatrix} = \{M_{1i}(\lambda)\} \begin{pmatrix} \zeta \\ \nu \end{pmatrix}. \tag{2.5.16}$$

Thus the function $\{\Phi, \Psi\}$ and its derivative is known on the edges $e_1, \dots, e_{n-1}$. At the internal vertex v_{n+1} the compatibility conditions hold:

$$\begin{pmatrix} \Phi^1(v_{n+1}, \lambda) \\ \Psi^1(v_{n+1}, \lambda) \end{pmatrix} = S_{in} \begin{pmatrix} \Phi^n(v_{n+1}, \lambda) \\ \Psi^n(v_{n+1}, \lambda) \end{pmatrix},$$

$$D'_1 \begin{pmatrix} \Phi^1_x(v_{n+1}, \lambda) \\ \Psi^1_x(v_{n+1}, \lambda) \end{pmatrix} = \sum_{j=2}^{n-1} S_{1j} D'_j \begin{pmatrix} \Phi^j_x(v_{n+1}, \lambda) \\ \Psi^j_x(v_{n+1}, \lambda) \end{pmatrix} + S_{1n} D'_n \begin{pmatrix} \Phi^n_x(v_{n+1}, \lambda) \\ \Psi^n_x(v_{n+1}, \lambda) \end{pmatrix}.$$

Using these conditions and the definition of the component of TW-matrix associated with the $n-$th edge:

$$\begin{pmatrix} \Phi^n_x(z_0, \lambda) \\ \Psi^n_x(z_0, \lambda) \end{pmatrix} = \widetilde{M}_{11}(\lambda) \begin{pmatrix} \Phi^n(v_{n+1}, \lambda) \\ \Psi^n(v_{n+1}, \lambda) \end{pmatrix},$$

we get the equations

$$D_i' \begin{pmatrix} \Phi_x^1(v_{n+1},\lambda) \\ \Psi_x^1(v_{n+1},\lambda) \end{pmatrix} = \sum_{j=2}^{n-1} S_{1j} D_j' \begin{pmatrix} \Phi_x^j(v_{n+1},\lambda) \\ \Psi_x^j(v_{n+1},\lambda) \end{pmatrix} + \\ S_{1n} D_n' \widetilde{M}_{11}(\lambda)(S_{1n})^{-1} \begin{pmatrix} \Phi^1(v_{n+1},\lambda) \\ \Psi^1(v_{n+1},\lambda) \end{pmatrix}. \tag{2.5.17}$$

Choosing the different boundary conditions at the $i-$th boundary point, we can get vectors $\begin{pmatrix} \Phi^i(z_0,\lambda) \\ \Psi^i(z_0,\lambda) \end{pmatrix}$ in (2.5.17) to be linearly independent. Since we know all other terms in (2.5.17), this equation determines $\widetilde{M}_{11}(\lambda)$. Using the connection of the dynamical and spectral data (2.3.9), we can recover the $\widetilde{R}_{11}$ component of the response function associated with $\widetilde{\Omega}$ and reduce our problem to the inverse problem for one edge. (Really we still need to recover only the length of the $n-$th edge.)

We combine all results of this section in

Theorem 2. *Let Ω be the a star graph consisted of n edges. Then the graph and the parameters of the systems (2.2.10), (2.2.11) and (2.2.7), are determined by the diagonal elements (2 × 2 matrices) $M_{ii}(\lambda)$ $1 \leqslant i \leqslant n-1$ of the TW matrix.*

2.6 Solution of the inverse problem. The case of an arbitrary tree.

Let Ω be a finite tree with m boundary points $\Gamma = \{v_1, \ldots, v_m\}$. Any boundary vertex of the tree can be taken as a root, so without loss of generality we can assume that the boundary vertex v_m is a root of the tree. We consider the dynamical problem **D** and the spectral problem **S** on Ω. Then the reduced response function $R(t) = \{R_{ij}(t)\}_{i,j=1}^{m-1}$ and the TW matrix $M(\lambda) = \{M_{ij}(\lambda)\}_{i,j=1}^{m-1}$ associated with all other boundary points are constructed in the same way as in the section 3.

Let us take two boundary edges, e_i with the length l_i and velocities in channels k_{i1}, k_{i2} and e_j with the length l_j and velocities in channels k_{j1}, k_{j2}. These two edges have one common point if and only if

$$\{R_{ij}\}_{11}(t) = \begin{cases} = 0, & \text{for } t < \frac{l_{i1}}{k_{i1}} + \frac{l_{j1}}{k_{j1}} \\ \neq 0, & \text{for } t > \frac{l_{i1}}{k_{i1}} + \frac{l_{j1}}{k_{j1}} \end{cases}, \quad 1 \leqslant i,j \leqslant m-1. \tag{2.6.1}$$

Note that one can use other components of R_{ij} to determine the connectivity of edges.

This method allows us to divide the boundary edges into groups, such that edges from one group have a common vertex. Let us take the first of such groups, say $e_1, \ldots, e_{m_0}$ with boundary vertices $v_1, \ldots, v_{m_0}$. These edges together with another edge e'_{m_0} form a star graph with the internal vertex $v_{m'_0}$, the subgraph of Ω. Note, that using the diagonal elements of the response operator $R_{ii}(t)$ or the diagonal elements of the TW-matrix $M_{ii}(\lambda)$, $i = 1, \ldots, m_0$ by the same method as in the case of star graph, we can determine angles and velocities for all edges $e_1, \ldots, e_{m_0}, e'_{m_0} = [v_{m'_0}, v_{m''_0}]$ and lengths all boundary edges $e_1, \ldots, e_{m_0}$.

We take the new tree $\widetilde{\Omega} = \Omega \setminus \bigcup_{i=1}^{m_0} e_i$. Our goal as in the previous cases is to calculate $\widetilde{M}(\lambda)$, the (reduced) TW-matrix associated with $\widetilde{\Omega}$.

Choose the orientation on the subgraph: the edge e_1 starts at v_1 and ends at $v_{m'_0}$, edges e_i start at $v_{m'_0}$ and end at v_i for $i = 2, \ldots, m_0$, e'_{m_0} starts at $v_{m'_0}$ and ends at $v_{m''_0}$. By $\{\Phi, \Psi\}$ we denote the solution to (2.2.7), (2.2.8) and the following boundary conditions

$$\{\Phi, \Psi\} = \{\zeta, \nu\}, \text{ at } v_1, \quad \{\Phi, \Psi\} = \{0, 0\}, \text{ at } v_i, \; 2 \leqslant i \leqslant m. \tag{2.6.2}$$

Note that the solution to (2.2.7), (2.2.8), (2.6.2) on the edge e_1 solves the Cauchy problem

$$-k_{11}^2 \Phi^1_{xx} = \lambda \Phi^1, \quad -k_{12}^2 \Psi^1_{xx} = \lambda \Psi^1, \quad x \in e_1 \tag{2.6.3}$$

$$\{\Phi^1(v_1), \Psi^1(v_1)\} = \{\zeta, \nu\}, \tag{2.6.4}$$

$$\begin{pmatrix} \Phi^1_x(v_1) \\ \Psi^1_x(v_1) \end{pmatrix} = \{M_{11}(\lambda)\} \begin{pmatrix} \zeta \\ \nu \end{pmatrix}, \tag{2.6.5}$$

and on the edges $e_2, \ldots, e_{m_0}$ solves

$$-k_{i1}^2 \Phi^i_{xx} = \lambda \Phi^i, \quad -k_{i2}^2 \Psi^i_{xx} = \lambda \Psi^i, \quad x \in e_i \tag{2.6.6}$$

$$\{\Phi^i(v_i), \Psi^i(v_i)\} = \{0, 0\}, \tag{2.6.7}$$

$$\begin{pmatrix} \Phi^i_x(v_i) \\ \Psi^i_x(v_i) \end{pmatrix} = \{M_{1i}(\lambda)\} \begin{pmatrix} \zeta \\ \nu \end{pmatrix}. \tag{2.6.8}$$

Thus the function $\{\Phi, \Psi\}$ and its derivative is known on edges $e_1, \ldots, e_{m_0}$.

At the internal vertex $v_{m_0'}$ compatibility conditions hold:

$$\begin{pmatrix} \Phi^1(v_{m_0'},\lambda) \\ \Psi^1(v_{m_0'},\lambda) \end{pmatrix} = S_{1m_0'} \begin{pmatrix} \Phi^{m_0'}(v_{m_0'},\lambda) \\ \Psi^{m_0'}(v_{m_0'},\lambda) \end{pmatrix}, \tag{2.6.9}$$

$$D_1' \begin{pmatrix} \Phi^1_x(v_{m_0}',\lambda) \\ \Psi^1_x(v_{m_0}',\lambda) \end{pmatrix} = \sum_{j=2}^{m_0} S_{1j} D_j' \begin{pmatrix} \Phi^j_x(v_{m_0'},\lambda) \\ \Psi^j_x(v_{m_0'},\lambda) \end{pmatrix} + \tag{2.6.10}$$

$$S_{1m_0'} D_{m_0'}' \begin{pmatrix} \Phi^{m_0'}_x(v_{m_0'},\lambda) \\ \Psi^{m_0'}_x(v_{m_0'},\lambda) \end{pmatrix}.$$

Using these conditions and the definition of the component of the (reduced) TW-matrix of the new graph $\widetilde{\Omega}$, associated with the edge e_{m_0}':

$$\begin{pmatrix} \Phi^{m_0'}_x(v_{m_0'},\lambda) \\ \Psi^{m_0'}_x(v_{m_0'},\lambda) \end{pmatrix} = \widetilde{M}_{m_0'm_0'}(\lambda) \begin{pmatrix} \Phi^{m_0'}(v_{m_0'},\lambda) \\ \Psi^{m_0'}(v_{m_0'},\lambda) \end{pmatrix}, \tag{2.6.11}$$

we obtain that

$$D_1' \begin{pmatrix} \Phi^1_x(v_{m_0'},\lambda) \\ \Psi^1_x(v_{m_0'},\lambda) \end{pmatrix} = \sum_{j=2}^{m_0'} S_{1j} D_j' \begin{pmatrix} \Phi^j_x(v_{m_0'},\lambda) \\ \Psi^j_x(v_{m_0'},\lambda) \end{pmatrix} + \tag{2.6.12}$$

$$S_{\alpha_{1m_0'}} D_{m_0'}' \widetilde{M}_{m_0'm_0'}(\lambda)(S_{1m_0'})^{-1} \begin{pmatrix} \Phi^1(v_{m_0'},\lambda) \\ \Psi^1(v_{m_0'},\lambda) \end{pmatrix}.$$

Equation (2.6.12) determines the matrix $\widetilde{M}_{m_0'm_0'}(\lambda)$. By the definition of the reduced TW-matrix we have

$$\{M_{1j}(\lambda)\} \begin{pmatrix} \zeta \\ \nu \end{pmatrix} = \begin{pmatrix} \Phi^j(v_j) \\ \Psi^j(v_j) \end{pmatrix}, \quad m_0 < j < m. \tag{2.6.13}$$

On the other hand, by the definition of the reduced TW-matrix for the new tree $\widetilde{\Omega}$,

$$\begin{pmatrix} \Phi^j(v_j) \\ \Psi^j(v_j) \end{pmatrix} = \{\widetilde{M}_{m_0'j}(\lambda)\} \begin{pmatrix} \Phi^{m_0'}(v_{m_0'}) \\ \Psi^{m_0'}(v_{m_0'}) \end{pmatrix}, \quad m_0 < j < m. \tag{2.6.14}$$

Thus $\{\widetilde{M}_{m_0'j}(\lambda)\}$ component of the TW matrix can be found from the equation

$$\{\widetilde{M}_{m_0'j}(\lambda)\} \begin{pmatrix} \Phi^{m_0'}(v_{m_0'}) \\ \Psi^{m_0'}(v_{m_0'}) \end{pmatrix} = \{M_{1j}(\lambda)\} \begin{pmatrix} \zeta \\ \nu \end{pmatrix}, \quad m_0 < j < m. \tag{2.6.15}$$

To find the components $\widetilde{M}_{im_0'}(\lambda)$, $m_0 < i < m$, we fix v_i and denote by $\{\Phi, \Psi\}$ the solution to (2.2.7), (2.2.8) with the boundary conditions

$$\{\Phi, \Psi\} = \{\zeta, \nu\}, \text{ at } v_i, \quad \{\Phi, \Psi\} = \{0, 0\}, \text{ at } v_j, \ j = 1, \dots m, j \neq i. \tag{2.6.16}$$

Note that on the edges $e_1, \dots, e_{m_0}$ $\{\Phi, \Psi\}$ satisfies the equations

$$-k_{j1}^2 \Phi_{xx}^j = \lambda \Phi^j, \quad -k_{j2}^2 \Psi_{xx}^j = \lambda \Psi^j, \quad x \in e_j \tag{2.6.17}$$

$$\{\Phi^j(v_j), \Psi^j(v_j)\} = \{0, 0\}, \tag{2.6.18}$$

$$\begin{pmatrix} \Phi_x^j(v_j) \\ \Psi_x^j(v_j) \end{pmatrix} = \{M_{ij}(\lambda)\} \begin{pmatrix} \zeta \\ \nu \end{pmatrix}. \tag{2.6.19}$$

Thus, the function $\{\Phi, \Psi\}$ and its derivative are known on the edges $e_1, \dots, e_{m_0}$. Using the compatibility conditions at the internal vertex $v_{m_0'}$, for every $\begin{pmatrix} \zeta \\ \nu \end{pmatrix}$ we can find the vectors $\begin{pmatrix} \Phi^{m_0'}(v_{m_0'}, \lambda) \\ \Psi^{m_0'}(v_{m_0'}, \lambda) \end{pmatrix}$, $\begin{pmatrix} \Phi_x^{m_0'}(v_{m_0'}, \lambda) \\ \Psi_x^{m_0'}(v_{m_0'}, \lambda) \end{pmatrix}$. We emphasize that the function $\{\Phi, \Psi\}$ does not satisfy zero Dirichlet conditions at $v_{m_0'}$. Components $\widetilde{M}_{im_0'}(\lambda)$, $i = m_0 + 1, \dots, m - 1$ can be obtained from the equations

$$\begin{pmatrix} \Phi_x^{m_0'}(v_{m_0'}, \lambda) \\ \Psi_x^{m_0'}(v_{m_0'}, \lambda) \end{pmatrix} - \widetilde{M}_{m_0'm_0'}(\lambda) \begin{pmatrix} \Phi^{m_0'}(v_{m_0'}, \lambda) \\ \Psi^{m_0'}(v_{m_0'}, \lambda) \end{pmatrix} = \widetilde{M}_{im_0'}(\lambda) \begin{pmatrix} \zeta \\ \nu \end{pmatrix}, \tag{2.6.20}$$

The procedure described reduces the initial problem to the inverse problem on the smaller subgraph. By repeating these steps a sufficiently many times we recover the whole graph and all parameters. We conclude this section with

Theorem 3. *Let Ω be an arbitrary tree. Then the tree and the parameters of the systems (2.2.10), (2.2.11) and (2.2.7), are determined by the elements (2×2 matrices) $M_{ij}(\lambda)$, $1 \leqslant i, j \leqslant m - 1$, of the TW matrix.*

Chapter 3

The boundary control approach to inverse spectral theory

3.1 Introduction.

In the Chapter I of this thesis we consider the Schrödinger operator

$$H = -\partial_x^2 + q(x) \tag{3.1.1}$$

on $L^2(\mathbb{R}_+)$, $\mathbb{R}_+ := [0, \infty)$, with a real-valued locally integrable potential q and Dirichlet boundary condition at $x = 0$. Let $d\rho(\lambda)$ be the *spectral measure* corresponding to H, and $m(z)$ be the (principal or Dirichlet) *Titchmarsh-Weyl m-function.*

In this section we give a brief review of five different approaches to inverse problems for the operator (3.1.1): the Gelfand–Levitan theory, the Krein method, the Simon theory, the Remling approach and the Boundary Control method. In the next section we describe the Boundary Control method in more detail and establish its connections with the other approaches (see also [82]).

3.1.1 Gelfand–Levitan theory.

Determining the potential q from the spectral measure is the main result of the seminal paper by Gelfand and Levitan [47]. To formulate the result let us define the following functions:

$$\sigma(\lambda) = \begin{cases} \rho(\lambda) - \frac{2}{3\pi}\lambda^{\frac{3}{2}}, & \lambda \geqslant 0, \\ \rho(\lambda), & \lambda < 0 \end{cases} \tag{3.1.2}$$

$$F(x,t) = \int_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}x \sin\sqrt{\lambda}t}{\lambda} d\sigma(\lambda). \tag{3.1.3}$$

Let $\varphi(x, \lambda)$ be a solution to the equation

$$-\varphi'' + q(x)\varphi = \lambda\varphi, \quad x > 0, \tag{3.1.4}$$

with the Cauchy data

$$\varphi(0, \lambda) = 0, \quad \varphi'(0, \lambda) = 1. \tag{3.1.5}$$

The so-called transformation operator transforms the solutions of (3.1.4), (3.1.5) with zero potential to the functions $\varphi(x, \lambda)$:

$$\varphi(x, \lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \int_0^x K(x,t)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}} dt. \tag{3.1.6}$$

The kernel $K(x,t)$ solves the Goursat problem

$$\begin{cases} K_{tt}(x,t) - K_{xx}(x,t) + q(x)K(x,t) = 0, \\ K(x,0) = 0, \quad \frac{d}{dx}K(x,x) = \frac{1}{2}q(x). \end{cases} \tag{3.1.7}$$

It was proved in [47] that $K(x,t)$ satisfies also the integral (Gelfand–Levitan) equation

$$F(x,t) + K(x,t) + \int_0^x K(x,s)\, F(s,t)\, ds = 0\,, \quad 0 \leqslant t < x. \tag{3.1.8}$$

The potential can be recovered from the solution of this equation by the rule

$$q(x) = 2\frac{d}{dx}K(x,x). \tag{3.1.9}$$

3.1.2 The Krein method.

In the beginning of fifties M. Krein developed an approach (see [56, 57]) to spectral inverse problems for the string equation which is different from the Gelfand-Levitan theory. Using the method of directing functionals developed by himself in the forties, Krein reduced the inverse problem to solving linear integral equations. Later this equation was derived by Blagoveschenskii [39] and independently by Gopinath and Sondhi [50, 51] using the dynamical approach.

3.1.3 Simon approach.

In [83] Barry Simon proposed a new approach to inverse spectral theory which has got a further development in the paper by Gesztesy and Simon [49] (see also an excellent survey paper [48]). As the data of inverse problem they used the Titchmarsh–Weyl m-function which is known to be in one-to-one correspondence with the spectral measure. It was shown in [83] that there exists a unique real valued function $A \in L^1_{loc}(\mathbb{R}_+)$ (the $A-$*amplitude*) such that

$$m(-k^2) = -k - \int_0^\infty A(t)e^{-2tk}\, dt\,. \tag{3.1.10}$$

The absolute convergence of the integral was proved for $q \in L^1(\mathbb{R}_+)$ and $q \in L^\infty(\mathbb{R}_+)$ in [49] for sufficiently large $\Re k$. In general situation one has an asymptotic equality

$$m(-k^2) = -k - \int_0^a A(t)e^{-2tk}\, dt + \mathcal{O}(e^{-2ak}) \tag{3.1.11}$$

(see [83, 49] for details).

Simon [83] put forward the local approach to solving the inverse problems (locality means that the $A-$amplitude on $[0, a]$ completely determines q on the same interval and vice versa). First, based on (3.1.11) Simon proved the local version of the Borg–Marchenko uniqueness theorem: $m_1(-k^2)-m_2(-k^2) = \mathcal{O}(e^{-2ak})$ if and only if $q_1(x) = q_2(x)$ for $x \in [0, a]$.

Second, he described how to recover the potential from the $A-$amplitude. If $A(\cdot, x)$ denotes the $A-$amplitude of the problem on $[x, \infty)$, then this family satisfies the nonlinear integro-differential equation

$$\frac{\partial A(t,x)}{\partial x} = \frac{\partial A(t,x)}{\partial t} + \int_0^t A(s,x)A(t-s,x)\,ds = 0\,. \tag{3.1.12}$$

If one solve this equation with the initial condition $A(t, 0) = A(t)$ in the domain $\{(x,t) : 0 \le x \le a,\ 0 \le t \le a - x\}$, then the potential on $[0, a]$ is determined by

$$\lim_{t \downarrow 0} A(t,x) = q(x)\,,\ 0 \le x \le a. \tag{3.1.13}$$

The $A-$amplitude has the explicit representation through the spectral measure by the formula derived in [49]:

$$A(t) = -2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}} e^{-\varepsilon\lambda} \frac{\sin(2t\sqrt{\lambda})}{\sqrt{\lambda}}\, d\rho(\lambda) \quad a.e. \tag{3.1.14}$$

Without the Abelian regularization the integral need not be convergent (even conditionally) [49].

3.1.4 Remling approach.

Motivated by Simon, Remling [78, 79] proposed another local approach to inverse spectral problems based on the theory of de Branges spaces. He introduced the integral operator $\mathcal{K}$ acting in the space $\mathcal{F}^T := L^2(0,T)$,

$$(\mathcal{K}f)(x) = \int_0^T k(x,t)\, f(t)\, dt\,, \tag{3.1.15}$$

where

$$k(x,t) = \frac{1}{2}[\phi(x-t) - \phi(x+t)]\,,\ \ \phi(x) = \int_0^{|x|/2} A(t)\, dt\,. \tag{3.1.16}$$

Remling proved that given a function $A \in L^1(0,T)$, there exists a unique $q \in L^1(0,T)$ such that A is the $A-$amplitude of this q if and only if the operator $I + \mathcal{K}$ is positive definite in $\mathcal{F}^T$. The same positivity condition was proved in [79] to be necessary and sufficient for solvability of the equation (3.1.12).

He proved the following representation of the $A-$amplitude through the regularized spectral measure $d\sigma$:

$$A(t) = -2\int_{\mathbb{R}} \frac{\sin(2t\sqrt{\lambda})}{\sqrt{\lambda}}\, d\sigma(\lambda) \tag{3.1.17}$$

with the convergence in the sense of distributions.

Remling derived two linear integral equations,

$$y(x,t) + \int_0^x k(t,s)y(x,s)\, ds = t\,, \tag{3.1.18}$$

$$z(x,t) + \int_0^x k(t,s)z(x,s)\, ds = \psi(t)\,, \tag{3.1.19}$$

where $0 \le t \le x \le T$ and $\psi(t) = -1 - \int_0^t \phi(s)\, ds$. The potential $q(x)$ on $[0,T]$ is uniquely determined by any of the functions y or z:

$$q(x) = \frac{\frac{d^2}{dx^2}y(x,x)}{y(x,x)}, \quad q(x) = \frac{\frac{d^2}{dx^2}z(x,x)}{z(x,x)}. \tag{3.1.20}$$

3.1.5 The Boundary Control method.

The Boundary Control (BC) method in inverse problems was developed about two decades ago by M. Belishev and his colleagues [18, 32, 31, 25, 5]. As well as methods of Simon and Remling, the BC method provides the local approach to inverse problems developing ideas of A. Blagoveshchenskii [39] who was a pioneer of the local approach to the 1d wave equation. It is worth to notice that the papers by Simon, Gesztesy and Remling (and also by Krein [56, 57]) are based on the spectral approach, and locality is proved there using sophisticated analytical tools. In the BC method, locality naturally follows from the finite speed of the wave propagation.

The main idea of the BC method is to study the dynamic Dirichlet-to-Neumann map $R : u(0,t) \mapsto u_x(0,t)$ for the wave equation associated with the operator (3.1.1):

$$u_{tt} - u_{xx} + q(x)u = 0, \quad x > 0,\ t > 0, \tag{3.1.21}$$

with zero initial conditions and boundary conditions $u(0,t) = f(t)$. Operator R has the form

$$(Rf)(t) = -f'(t) + \int_0^t r(s)f(t-s)\,ds, \tag{3.1.22}$$

and function $r(t)$ is considered as inverse data. Let us introduce the operator acting in $L_2(0,T)$:

$$(C^T f)(t) = f(t) + \int_0^T [p(2T-t-s) - p(t-s)]f(s)\,ds\,,\ 0 < t < T\,, \tag{3.1.23}$$

where

$$p(t) := \frac{1}{2}\int_0^{|t|} r\,(s)\ ds. \tag{3.1.24}$$

It is proved (see, e.g. [5]) that one can recover the potential using the unique solution to any of the equations

$$C^T f_0^T(t) = T - t\,,\ t \in [0,T], \tag{3.1.25}$$

$$(C^T f_1^T)(t) = -((R^T)^* \varkappa^T)(t)\,,\ t \in [0,T], \tag{3.1.26}$$

where the operator R^T in the second equation is determined by $r(t)$, $t \in [0,T]$ and $\varkappa(t) = T - t$. Then

$$q(T) = \frac{\frac{d^2}{dT^2} f_j^T(+0)}{f_j^T(+0)}, \quad j = 1,2. \tag{3.1.27}$$

It is important to note that the Krein equation, the Remling equation and the equation of the BC method can be reduced to each other by simple changes of variables. More exactly, Krein in [56, 57] considered the problem with Neumann boundary conditions at $x = 0$, and one of the equations derived in [78] can be reduced to the Krein equation. Equations (3.1.25), (3.1.26) and (3.1.18), (3.1.18) concerning Dirichlet conditions can be easily transformed to each other.

The main goal of this paper is to demonstrate the connections between all approaches mentioned above. We provide a new proof of the Gelfand–Levitan equations which demonstrates their local character. We describe in detail relations between dynamical and spectral approaches, in particular, we prove convergence a.e. in formula (3.1.17).

3.2 The Boundary Control approach.

The BC method uses the deep connection between inverse problems of mathematical physics, functional analysis and control theory for partial differential equations and offers an interesting and powerful alternative to previous identification techniques based on spectral or scattering methods. This approach has several advantages, namely: (i) it maintains linearity (does not introduce spurious nonlinearities); (ii) it is applicable to a wide range of linear point and/or distributed systems and reconstruction situations; (iii) it can identify coefficients occurring in highest order terms; (iv) it is, in principle, dimension-independent; and, finally, (v) it lends itself to straightforward algorithmic implementations. Being originally proposed for solving the boundary inverse problem for the multidimensional wave equation, the BC method has been successfully applied to all main types of linear equations of mathematical physics (see the review papers [20, 21] and references therein). In this paper we use this method in 1d situation applying it to inverse problems for the operator (3.1.1) and demonstrate its connections with the methods described above. We consider here Dirichlet boundary condition and note that our approach works also for other boundary conditions (see, e.g. [10] for Neumann condition and [16] for a non-self-adjoint condition).

3.2.1 The initial boundary value problem, Goursat problem.

Let us consider the initial boundary value problem for the 1d wave equation:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + q(x)u(x,t) = 0, \quad x > 0,\ t > 0, \\ u(x,0) = u_t(x,0) = 0,\ u(0,t) = f(t). \end{cases} \tag{3.2.1}$$

Here $q \in L^1_{loc}(\mathbb{R}_+)$ and f is an arbitrary $L^2_{loc}(\mathbb{R}_+)$ function referred to as a *boundary control.* The solution $u^f(x,t)$ of the problem (3.2.1) can be written in terms of the integral kernel $w(x,s)$ which is the unique solution to the Goursat problem:

$$\begin{cases} w_{tt}(x,t) - w_{xx}(x,t) + q(x)w(x,t) = 0, \quad 0 < x < t, \\ w(0,t) = 0,\ w(x,x) = -1/2 \int_0^x q(s)\, ds. \end{cases} \tag{3.2.2}$$

The properties of the solution to the Goursat problem are given in Appendix.

Let us consider now the dynamical system (3.2.1) on the time interval $[0,T]$ for some $T > 0$.

Proposition 1. *a) If $q \in C^1(\mathbb{R}_+)$, $f \in C^2(\mathbb{R}_+)$ and $f(0) = f'(0) = 0$, then*

$$u^f(x,t) = \begin{cases} f(t-x) + \int_x^t w(x,s) f(t-s)\, ds, & x \le t, \\ 0, & x > t. \end{cases} \tag{3.2.3}$$

is a classical solution to (3.2.1).

b) If $q \in L_{1,loc}(\mathbb{R}_+)$ and $f \in L_2(0,T)$, then formula (3.2.3) *represents a unique generalized solution to the initial-boundary value problem* (3.2.1) *$u^f \in C([0,T]; \mathcal{H}^T)$, where*

$$\mathcal{H} = L^2_{loc}(0,\infty) \quad \text{and} \quad \mathcal{H}^T := \{u \in \mathcal{H} : \ supp\ u \subset [0,T]\}.$$

First statement of the proposition can be checked by direct calculations. The proof of the second one follows from Propositions 5 and 6 (see Appendix).

3.2.2 The main operators of the BC method.

The **response operator** (the dynamical Dirichlet-to-Neumann map) R^T for the system (3.2.1) is defined in $\mathcal{F}^T := L_2(0,T)$ by

$$(R^T f)(t) = u_x^f(0,t), \ t \in (0,T), \tag{3.2.4}$$

with the domain $\{f \in C^2([0,T]) : \ f(0) = f'(0) = 0\}$. According to (3.2.3) it has a representation

$$(R^T f)(t) = -f'(t) + \int_0^t r(s) f(t-s)\, ds, \tag{3.2.5}$$

where $r(t) := w_x(0,t)$ is called the **response function.**

The response operator R^T is completely determined by the response function on the interval $[0,T]$, and the dynamical inverse problem can be formulated as follows. Given $r(t),\ t \in [0,2T]$, find $q(x),\ x \in [0,T]$.

Notice that from (3.2.2) one can derive the formula

$$r(t) = -\frac{1}{2} q\Big(\frac{t}{2}\Big) - \frac{1}{2} \int_0^t q\Big(\frac{t-\zeta}{2}\Big) v(\zeta, t)\, d\zeta., \tag{3.2.6}$$

where

$$v(\xi,\eta) = w\Big(\frac{\eta-\xi}{2}, \frac{\eta+\xi}{2}\Big).$$

To solve the dynamical inverse problem by the BC method let us introduce a couple more operators. Proposition 1 implies in particular that the **control operator** W^T,

$$W^T : \mathcal{F}^T \mapsto \mathcal{H}^T, \; W^T f = u^f(\cdot, T),$$

is bounded. From (3.2.3) it follows that

$$(W^T f)(x) = f(T - x) + \int_x^T w(x, \tau) f(T - \tau)\, d\tau \tag{3.2.7}$$

The next statement claims that the operator W^T is boundedly invertible.

Proposition 2. *Let $q \in L^1_{loc}(\mathbb{R}_+)$ and $T > 0$, then for any function $z \in \mathcal{H}^T$, there exists a unique control $f \in \mathcal{F}^T$ such that*

$$u^f(x, T) = z(x). \tag{3.2.8}$$

Proof. According to (3.2.7), condition (3.2.8) is equivalent to the following integral Volterra equation of the second kind

$$z(x) = f(T - x) + \int_x^T w(x, \tau) f(T - \tau)\, d\tau \quad x \in (0, T)\,. \tag{3.2.9}$$

The kernel $w(x, t)$ is continuous and therefore equation (3.2.9) is uniquely solvable, which proves the proposition. □

The **connecting operator** $C^T : \mathcal{F}^T \mapsto \mathcal{F}^T$, plays a central role in the BC method. It connects the outer space (the space of controls) of the dynamical system (3.2.1) with the inner space (the space of waves) being defined by its bilinear product:

$$\langle C^T f, g \rangle_{\mathcal{F}^T} = \langle u^f(\cdot, T), u^g(\cdot, T) \rangle_{\mathcal{H}^T} \tag{3.2.10}$$

In other words,

$$C^T = (W^T)^* W^T\,, \tag{3.2.11}$$

and Propositions 1, 2 imply that this operator is positive definite, bounded and boundedly invertible on $\mathcal{F}^T$.

Let $q_n \in C^\infty(\mathbb{R}_+)$, $n = 1, 2, \ldots$ and $q_n \to q$ in $L_{1, loc}(\mathbb{R}_+)$. We denote by the $r_n(t)$ the response function, corresponding to q_n. Formula (3.2.6) and Proposition 6 yields

$$r_n \xrightarrow{L^1_{loc}} r, \quad \text{as } n \to \infty. \tag{3.2.12}$$

Along with W^T and C^T we consider operators, W_n^T and C_n^T, corresponding smooth potentials q_n, $n = 1, 2, \dots$.

Lemma 1. *Let W_n^T, C_n^T be as described above, then:*

$$\|W_n^T - W^T\| \to 0, \quad \textit{as } n \to \infty, \tag{3.2.13}$$

$$\|C_n^T - C^T\| \to 0, \quad \textit{as } n \to \infty. \tag{3.2.14}$$

Proof. Let us take arbitrary $f \in \mathcal{F}^T$, then from (3.2.7) we see that

$$\|(W - W_n)(f)\|^2 \leqslant \sup_{0<x<s<T} |w(x,s) - w_n(x,s)| T^2 \|f\|^2,$$

Using (3.3.16) we obtain the first statement of the lemma. The second statement follows from the first one and the representation of C^T (3.2.11). □

The remarkable fact is that C^T can be explicitly expressed through R^{2T} (or through $r(t)$, $t \in [0, 2T]$).

Proposition 3. *For $q \in L^1_{loc}(0, \infty)$ and $T > 0$, operator C^T has the form*

$$(C^T f)(t) = f(t) + \int_0^T c^T(t,s) f(s)\, ds\,, \; 0 < t < T\,, \tag{3.2.15}$$

where

$$c^T(t,s) = [p(2T - t - s) - p(t - s)]. \tag{3.2.16}$$

and $p(t)$ is defined in (3.1.24).

Proof. For smooth potentials formula (3.1.23) is well known (see e.g [5]), therefore we give here only a sketch of the proof. One can easily check that for $q \in C^\infty(\mathbb{R}_+)$ and any $f, g \in C_0^\infty(0,T)$ the function $U(s,t) := \big(u^f(\cdot,s), u^g(\cdot,t)\big)_{\mathcal{H}}$ satisfies the equation

$$U_{tt} - U_{ss} = (R^T f(s) g(t) - f(s)(R^T g)(t)\,, \quad s, t > 0\,,$$

with the boundary and initial conditions

$$U(0,t) = 0\,, \;\; U(s,0) = U_t(s,0) = 0\,.$$

Using the D'Alambert formula gives representation (3.1.23). Making use of the results on the convergence of operators (3.2.14) and response functions (3.2.12), we can claim that representation 3.1.23 is valid also for $q \in L_{1,loc}(\mathbb{R}_+)$. □

3.2.3 The Krein type equations.

Let us suppose that $q \in C^\infty(\mathbb{R}_+)$ and consider the Cauchy problem:

$$-y'' + q(x)y = 0, \quad x > 0; \quad y(0) = \alpha\,,\ y'(0) = \beta\,. \tag{3.2.17}$$

Let f^T be a solution of the control problem

$$(W^T f^T)(x) = \begin{cases} y(x), \ 0 < x < T, \\ 0, \ x > T. \end{cases} \tag{3.2.18}$$

For any $g \in C_0^\infty(0,T)$ the identity

$$u^g(x,T) = \int_0^T \varkappa^T(t) u^g_{tt}(x,t)\,dt\,, \quad \varkappa^T(t) := T - t$$

is valid, and we have

$$\begin{aligned}(C^T f^T, g) = \int_0^T y(x) u^g(x,T)\,dx = \int_0^T y(x) \int_0^T \varkappa^T(t) u^g_{tt}(x,t)\,dt\,dx \\ = \int_0^T \varkappa^T(t)\left[y(x)u^g_x(x,T) - y_x(x)u^g(x,T)\right]_0^T)\,dt \\ = \int_0^T \beta\varkappa^T(t)g(t) - \alpha\varkappa^T(t)(R^T g)(t)\,dt = (\beta\varkappa^T - \alpha(R^T)^*\varkappa^T, g).\end{aligned}$$

Here $(R^T)^*$ is the operator adjoint to R^T in $\mathcal{F}^T$:

$$((R^T)^* f)(t) = f'(t) + \int_t^T r(s-t) f(s)\,ds. \tag{3.2.19}$$

We have used the fact that the solution $u^g(x,t)$ is classical and $u^g(T,T) = u^g_x(T,T) = 0$ (see (3.2.3)).

Let us denote by y_i, f_i^T, $i = 0, 1$, the functions corresponding the cases $\alpha = 0$, $\beta = 1$ and $\alpha = 1$, $\beta = 0$ respectively. Since g is an arbitrary smooth function, the functions f_0^T and f_1^T satisfy the equations

$$(C^T f_0^T)(t) = T - t\,, \quad (C^T f_1^T)(t) = -((R^T)^*\varkappa^T)(t)\,, \quad t \in [0,T]\,. \tag{3.2.20}$$

Using (3.1.23) these equations can be rewritten in more detail:

$$f_0^T(t) + \int_0^T c^T(t,s)\, f_0^T(s)\,ds \ = T - t\,,\ t \in [0,T]\,, \tag{3.2.21}$$

$$f_1^T(t) + \int_0^T c^T(t,s)\, f_1^T(s)\, ds \, = 1 - \int_t^T r(s-t)\,(T-s)\, ds\,, \ t \in [0,T]\,, \qquad (3.2.22)$$

Function c^T is defined in (3.2.16), and from (3.2.21), (3.2.22) it follows that functions f_j^T, $j = 0, 1$ possess additional regularity: $f_j^T \in H^1(0,T)$.

Remark 1. *Taking into account* (3.2.14) *and* (3.2.12) *we can claim that* (3.2.21), *(3.2.22) hold for $q \in L^1_{loc}(\mathbb{R}_+)$ as well.*

Using any of functions one can easily find the potential q in the following way. From equation (3.2.3) it follows that $u^f(t-0,t) = f(+0)$, and in particular, $y_i(T) = f_i^T(+0)$. Let us denote $f_i^T(+0)$ by $\mu_i(T)$. Then

$$q(T) = \frac{\mu_i''(T)}{\mu_i(T)}\,. \qquad (3.2.23)$$

Equations (3.2.20)–(3.2.23) were obtained for a matrix valued q of a class C^1 in [5].

In [12] we showed that the Titchmarsh–Weyl m-function (the spectral Dirichlet-to-Neumann map) and the response operator (the dynamical Dirichlet-to-Neumann map) are connected by the Laplace (or Fourier) transform and established the relation between the $A-$amplitude and the response function:

$$A(t) = -2r(2t)\,. \qquad (3.2.24)$$

Using this relation it is easy to check that the positivity condition of Remling's operator $I + \mathcal{K}$ is equivalent to the fact that the operator C^T is positive definite. Equations (3.2.21), (3.2.22) are reduced by simple changes of variables to equations (3.1.18), (3.1.19).

The fact that the positivity of C^T give the necessary and sufficient conditions of the solvability of the inverse problem was known in the BC community for a long time. A. Blagoveshchenskii [39] in 1971 obtained the necessary and sufficient conditions of the solvability of the inverse problem for the 1d wave equation (with smooth density) which are equivalent to the positivity of C^T. (Certainly these conditions were in other terms — the BC method and the operator C^T were proposed fifteen years later). Belishev and Ivanov [30] considered the two velocity system with smooth matrix-valued potential. In a particular case when two velocities are equal, their necessary and sufficient condition is the positivity of C^T. In [4] necessary and sufficient condition for solvability of a nonselfadjoint inverse problem with a matrix-valued potential in terms of C^T was formulated.

The equivalent necessary and sufficient conditions for the solvability of the inverse spectral problem for the string equation (in the form of positivity of certain integral operator) were obtained by Krein [56], [57].

The method proposed in [30] works also for non smooth potentials which leads to the following result. For given $r \in L^1(0, 2T)$, there exists a unique $q \in L^1(0, T)$ such that r is the response function corresponding to the problem (3.2.1) with this q if and only if the operator C^T constructed by this r according to (3.1.23) is positive definite. The fact that r and q belong to the same functional class is confirmed by formula (3.2.6).

3.2.4 Spectral representation of r and c^T.

The aim of the present section is to obtain the representation for the kernel of the integral part of the operator C^T, the function $c^T(t,s)$ and response function $r(t)$ in terms of the spectral measure of operator (3.1.1).

We consider the Schrödinger operator with a real valued potential $q \in L_{1,\ loc}(\mathbb{R}_+)$ and Dirichlet boundary condition:

$$-\phi_{xx} + q\phi = \lambda\phi, \quad x \in [0, +\infty), \tag{3.2.25}$$

$$\phi(0) = 0. \tag{3.2.26}$$

By $\varphi(x, \lambda)$ we denote the solution to (3.2.25) satisfying the initial conditions

$$\varphi(0, \lambda) = 0, \quad \varphi'(0, \lambda) = 1. \tag{3.2.27}$$

It is known that there exist a spectral measure $d\rho(\lambda)$, such that for all $f, g \in L_2(\mathbb{R}_+)$:

$$\int_0^\infty f(x)g(x)\,dx = \int_{-\infty}^\infty (Ff)(\lambda)(Fg)(\lambda)\,d\rho(\lambda), \tag{3.2.28}$$

$$(Ff)(\lambda) = \int_0^\infty f(x)\varphi(x, \lambda)\,dx. \tag{3.2.29}$$

The so-called inverse transformation operator transforms solutions of (3.2.25), (3.2.26), solutions of (3.2.25), (3.2.26) with $q \equiv 0$, (cf. (3.1.6)) :

$$\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} = \varphi(x,\lambda) + \int_0^x L(x,t)\varphi(t,\lambda)\,dt =: (I_x + L_x)\varphi, \tag{3.2.30}$$

where the kernel $L(x,t)$ satisfy the Goursat problem (see, e.g [65, 75]):

$$\begin{cases} L_{tt}(x,t) - L_{xx}(x,t) - q(t)L(x,t) = 0, \quad 0 < t < x, \\ L(x,0) = 0, \ \frac{d}{dx}L(x,x) = -\frac{1}{2}q(x). \end{cases} \tag{3.2.31}$$

Comparing (3.2.2) and (3.2.31) we conclude that $w(x,t) = L(t,x)$, and thus

$$\varphi(s,\lambda) + \int_0^s w(x,s)\varphi(x,\lambda)\,dx = \frac{\sin\sqrt{\lambda}s}{\sqrt{\lambda}}. \tag{3.2.32}$$

Let us introduce functions

$$\Phi_n(s,t) = \int_{-\infty}^n \frac{\sin\sqrt{\lambda}t\sin\sqrt{\lambda}s}{\lambda}\,d\sigma(\lambda), \tag{3.2.33}$$

where $\sigma(\lambda)$ is defined in (3.1.2). The fact they are well-defined follows from the proof of the lemma below. The following result seemed to be classical, although we have not been able to find it in the literature for the case of Dirichlet boundary condition. The case of Neumann boundary condition is considered in [65], [75] where the convergence of corresponding analogues of Φ_n was proven. We provide the proof here for the sake of completeness.

Lemma 2. *The sequence of functions $\Phi_n(s,t)$ converges to a continuous function $\Phi(s,t)$ differentiable outside the diagonal uniformly on every bounded set in $\mathbb{R}^2$ as $n \to \infty$.*

Proof. We follow the scheme proposed in [65], Lemma 2.2.2. In [66] it is shown that the sequence of functions

$$\Psi_n(t,s) = \int_{-\infty}^n \varphi(t,\lambda)\varphi(s,\lambda)\,d\rho(\lambda) - \int_0^n \frac{\sin\sqrt{\lambda}t\sin\sqrt{\lambda}s}{\lambda}\,d\left(\frac{2\pi}{3}\lambda^{\frac{3}{2}}\right), \tag{3.2.34}$$

converges uniformly on every bounded set to the differentiable outside the diagonal function as n tends to infinity. Applying operators $(I_s + L_s)(I_t + L_t)$ to (3.2.34) we have:

$$(I_s + L_s)(I_t + L_t)\Psi_n(t,s) = \Phi_n(s,t) - \tag{3.2.35}$$
$$\int_0^n \left(\int_0^t L(t,\tau) \frac{\sin\sqrt{\lambda}\tau}{\sqrt{\lambda}} \, d\tau \right) \frac{\sin\sqrt{\lambda}s}{\sqrt{\lambda}} \, d\left(\frac{2\pi}{3}\lambda^{\frac{3}{2}}\right) -$$
$$\int_0^n \left(\int_0^t L(s,\tau) \frac{\sin\sqrt{\lambda}\tau}{\sqrt{\lambda}} \, d\tau \right) \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}} \, d\left(\frac{2\pi}{3}\lambda^{\frac{3}{2}}\right) -$$
$$\int_0^n \left(\int_0^t L(t,\tau) \frac{\sin\sqrt{\lambda}\tau}{\sqrt{\lambda}} \, d\tau \right) \left(\int_0^t L(s,\tau) \frac{\sin\sqrt{\lambda}\tau}{\sqrt{\lambda}} \, d\tau \right) d\left(\frac{2\pi}{3}\lambda^{\frac{3}{2}}\right).$$

The sum of the last three terms in the right hand side of the above expression converges to $-L(s,t) - L(t,s) - \int_0^{\min\{s,t\}} L(s,\tau)L(t,\tau)\,d\tau$. This fact and the convergence of the left hand side of (3.2.35) imply the statement of the Lemma. □

The following theorem gives an expression for the integral part of the kernel of the operator C^T in terms of the spectral measure.

Theorem 1. *The kernel $c^T(s,t)$ admits the following representation:*

$$c^T(s,t) = \int_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}(T-t)\sin\sqrt{\lambda}(T-s)}{\lambda} \, d\sigma(\lambda), \quad s,t \in [0,T], \tag{3.2.36}$$

where the integral in the right-hand side of (3.2.36) converges uniformly on $[0,T] \times [0,T]$.

Proof. Let us take arbitrary $f, g \in \mathcal{F}^T$. Using (3.2.28), (3.2.29) we rewrite $(C^T f, g)_{\mathcal{F}^T}$ as

$$(C^T f, g)_{\mathcal{F}^T} = \int_0^T u^f(x,T)u^g(x,T)\,dx = \int_{-\infty}^{\infty} (Fu^f)(\lambda,T)(Fu^g)(\lambda,T)\,d\rho(\lambda). \tag{3.2.37}$$

Here (see also (3.2.3))

$$(Fu^f)(\lambda,T) = \int_0^T \varphi(x,\lambda)u^f(x,T)\,dx =$$
$$\int_0^T \varphi(x,\lambda) \left(f(T-x) + \int_0^T w(x,s)f(T-s)\,ds \right) dx.$$

Changing the order of integration and using the fact that $w(x,s)=0$ for $s<x$, we arrive at

$$(Fu^f)(\lambda,T)=\int_0^T f(T-s)\left(\varphi(s,\lambda)+\int_0^s w(x,s)\varphi(x,\lambda)\,dx\right)ds. \qquad (3.2.38)$$

Making use of (3.2.38), (3.2.32), we can rewrite (3.2.37) as

$$(C^T f,g)=\int_{-\infty}^{\infty}\int_0^T\int_0^T \frac{\sin\sqrt{\lambda}(T-t)\sin\sqrt{\lambda}(T-s)}{\lambda}f(t)g(s)\,dt\,ds\,d\rho(\lambda).$$

Comparing the last formula with (3.1.23), we see that

$$\int_{-\infty}^{\infty}\int_0^T\int_0^T \frac{\sin\sqrt{\lambda}(T-t)\sin\sqrt{\lambda}(T-s)}{\lambda}f(t)g(s)\,dt\,ds\,d\rho(\lambda)= \qquad (3.2.39)$$
$$\int_0^T f(s)g(s)\,ds+\int_0^T\int_0^T c^T(s,t)g(s)f(t)\,dt\,ds.$$

Now we make use of the sin transform: for all $h,j\in L_2(\mathbb{R}_+)$:

$$\widehat{h}(\lambda)=\int_0^\infty h(x)\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}\,dx,\quad h(x)=\int_0^\infty \widehat{h}(\lambda)\sin(\sqrt{\lambda}x)\,d\left(\frac{2}{3\pi}\lambda^{\frac{3}{2}}\right),$$
$$\int_0^\infty h(x)j(x)\,dx=\int_0^\infty \widehat{h}(\lambda)\widehat{j}(\lambda)\,d\left(\frac{2}{3\pi}\lambda^{\frac{3}{2}}\right).$$

Let us suppose that $f(t)=g(t)=0$ for $t>T$ and $t<0$ and use the notation $f_T(s)=f(T-s)$. Then we can rewrite the first term in the right hand side of (3.2.39) as

$$\int_0^T f(t)g(t)\,dt=\int_0^\infty f(T-s)g(T-s)\,ds= \qquad (3.2.40)$$
$$\int_0^\infty \widehat{f_T}(\lambda)\widehat{g_T}(\lambda)\,d\left(\frac{3}{2\pi}\lambda^{\frac{3}{2}}\right)=$$
$$\int_0^\infty\int_0^T\int_0^T \frac{\sin\sqrt{\lambda}(T-t)\sin\sqrt{\lambda}(T-s)}{\lambda}f(t)g(s)\,dt\,ds\,d\left(\frac{2}{3\pi}\lambda^{\frac{3}{2}}\right).$$

Plugging (3.2.40) in (3.2.39), we have that

$$\int_{-\infty}^{\infty}\int_0^T\int_0^T \frac{\sin\sqrt{\lambda}(T-t)\sin\sqrt{\lambda}(T-s)}{\lambda}f(t)g(s)\,dt\,ds\,d\sigma(\lambda)= \qquad (3.2.41)$$
$$\int_0^T\int_0^T c^T(s,t)f(t)g(s)\,dt\,ds.$$

In the last formula the function

$$C(s,t) := \int_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}(T-t)\sin\sqrt{\lambda}(T-s)}{\lambda}\, d\sigma(\lambda)$$

is a distributional kernel, whose action on functions f, g is defined by the left hand side of (3.2.41). On the other hand, comparing $C(s,t)$ with $\Phi(s,t)$, we see that $C(s,t) = \Phi(T-s, T-t)$ and according to Lemma 2, $C(s,t)$ is a continuous function on $[0,T] \times [0,T]$. Since (3.2.41) holds for arbitrary $f, g \in \mathcal{F}^T$, we deduce that

$$c^T(s,t) = C(s,t) = \int_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}(T-t)\sin\sqrt{\lambda}(T-s)}{\lambda}\, d\sigma(\lambda), \quad t,s \in [0,T]. \quad (3.2.42)$$

□

Using the representation for $c^T(t,s)$ obtained in Theorem 1, we can derive the formula for the response function:

Theorem 2. *The representation for the response function* r

$$r(t) = \int_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\, d\sigma(\lambda), \quad (3.2.43)$$

holds for almost all $t \in [0, +\infty)$.

Proof. Let us note that

$$\Phi(s,t) = \int_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}t\sin\sqrt{\lambda}s}{\lambda}\, d\sigma(\lambda) = c^T(T-t, T-s), \quad t,s \in [0,T]. \quad (3.2.44)$$

Using (3.1.23) we have

$$c^T(T-t, T-s) = \frac{1}{2}\int_{|t-s|}^{t+s} r(\tau)\, d\tau, \quad t,s \in [0,T]. \quad (3.2.45)$$

The integral in (3.2.44) can be rewritten as

$$\Phi(s,t) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{(\cos\sqrt{\lambda}(s+t) - 1 - (\cos\sqrt{\lambda}|s-t| - 1)}{\lambda}\, d\sigma(\lambda) = \quad (3.2.46)$$

$$\frac{1}{2}\int_{-\infty}^{\infty}\int_{|t-s|}^{t+s} \frac{\sin\sqrt{\lambda}\theta}{\sqrt{\lambda}}\, d\theta\, d\sigma(\lambda), \quad t,s \in [0,T].$$

Equating the expressions in Thus (3.2.45) and (3.2.46) for $t = s$ we get

$$2c^T(T-t,T-t) = \int_0^{2t} r(\tau)\,d\tau = \int_{-\infty}^{\infty}\int_0^{2t} \frac{\sin\sqrt{\lambda}\theta}{\sqrt{\lambda}}\,d\theta\,d\sigma(\lambda), \quad t \in [0,T] \tag{3.2.47}$$

According to (3.2.5) $r \in L_1(0,T)$, so we can use the Lebesgue theorem and differentiate the last equation. We obtain the following equality almost everywhere on $(0,2T)$

$$r(t) = \int_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\,d\sigma(\lambda). \tag{3.2.48}$$

Since the parameter T in consideration can be chosen arbitrary large, the last formula proves the statement of the proposition. $\square$

A direct consequence of this theorem is that integral in formula (3.1.17) converges for almost all $t \in [0,+\infty)$.

The finite speed of the wave propagation (equal to one) in the equation (3.2.1) implies the local nature of the response function $r(t)$: the values of $r(t)$, $t \in (0,2T)$ are determined by the potential $q(x)$, $x \in (0,T)$. This implies that if we are interested in the spectral representation of $c^T(s,t)$ for $s,t \in (0,T)$ and of $r(t)$ for $t \in (0,2T)$ in the formulas (3.2.36), (3.2.43) we can replace (for example) the regularized spectral function $\sigma(\lambda)$ by any of the following functions:

$$\sigma_{\mathrm{tr}}(\lambda) = \begin{cases} \rho_{\mathrm{tr}}(\lambda) - \frac{2}{3\pi}\lambda^{\frac{3}{2}}, & \lambda \geqslant 0, \\ \rho_{\mathrm{tr}}(\lambda), & \lambda < 0. \end{cases}, \quad \sigma_d(\lambda) = \begin{cases} \rho_d(\lambda) - \rho_0(\lambda), & \lambda \geqslant 0, \\ \rho_d(\lambda), & \lambda < 0. \end{cases} \tag{3.2.49}$$

Here ρ_{tr} is the spectral function corresponding to the truncated potential $q_T(x) = q(x)$, when $0 < x < T$ and $q_T(x) = \tilde{q}(x)$, when $x > T$, $\tilde{q} \in L_{1,\mathrm{loc}}(t,+\infty)$; $\rho_d(\lambda)$ is the spectral function associated to the discrete problem on the interval $(0,T)$ with the potential $q_d(x) = q(x)$, $x \in (0,T)$ and $\rho_0(\lambda)$ is the spectral function associated to the discrete problem on $(0,T)$ with zero potential (and with any self-adjoint boundary conditions at $x = T$.)

3.2.5 Gelfand-Levitan equations.

In this section, using the BC approach we derive the local version of the classical Gelfand-Levitan equations (3.1.8). The proof is based on the fact that the kernel K

of the transformation operator (3.1.6) satisfies a Goursat problem (3.1.7). We show that the kernel v of the operator $(W^T)^{-1}$ (which is inverse to the control operator W^T) satisfies a similar Goursat problem. We observe that the operator $(W^T)^{-1}$: $\mathcal{H}^T \mapsto \mathcal{F}^T$ can be constructed in the following way: we consider the initial-boundary value problem.

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + q(x)u(x,t) = 0, \quad 0 < x,\ 0 < t < T, \\ u(x,T) = a(x); \quad u(x,t) = 0,\ x > T, \end{cases} \tag{3.2.50}$$

and denote by $u^a(x,t)$ the solution of this problem. Basing on the uniqueness of the solutions to the initial boundary value problems (3.2.1) and (3.2.50) one can check that

$$\left(\left(W^T\right)^{-1} a\right)(t) = u^a(0,t), \quad 0 < t < T,$$

(see, e.g. [3] for more details). When $q \in C^1_{loc}(\mathbb{R}_+)$ and $a \in C^1(0,T)$, $a(0) = 0$, $u^a(x,t)$ is a classical solution and admits the representation

$$u^a(x,t) = \begin{cases} a(x-t+T) + \int_0^{t-x} v(T-x,s,T-t)a(T-s)\,ds, \quad x \leqslant t, \\ 0, \quad x > t. \end{cases} \tag{3.2.51}$$

in terms of the solution $v(x,s,t)$ to the following Goursat problem:

$$\begin{cases} v_{tt}(x,s,t) - v_{ss}(x,s,t) + q(T-s)v(x,s,t) = 0, \quad 0 < s < x-t, \\ v(x,s,0) = 0,\ \frac{d}{dt}v(x,t,x-t) = \frac{1}{2}q(T-x+t). \end{cases} \tag{3.2.52}$$

By the analogy with Propositions 5, 1 one can show that formula (3.2.51) gives a generalized solution for the case of non-smooth potential q and boundary condition a. From representation (3.2.51), the formula for $(W^T)^{-1}$ immediately follows:

$$\left(\left(W^T\right)^{-1} a\right)(t) = a(T-t) + \int_0^t V(y,t)a(T-y)\,dy, \tag{3.2.53}$$

Here the kernel $V(s,t)$ satisfies the Goursat problem

$$\begin{cases} V_{tt}(s,t) - V_{ss}(s,t) + q(T-s)V(s,t) = 0, \quad 0 < s < t, \\ V(s,T) = 0,\ \frac{d}{dt}V(t,t) = \frac{1}{2}q(t). \end{cases} \tag{3.2.54}$$

Let us introduce the following operators

$$J_T : L_2(0,T) \mapsto L_2(0,T), \quad (J_T a)(y) = a(T-y),$$
$$K : L_2(0,T) \mapsto L_2(0,T), \quad (Ka)(t) = \int_0^t V(y,t)a(y)\,dy, \quad t \in (0,T),$$
$$K^* : L_2(0,T) \mapsto L_2(0,T), \quad (K^*b)(t) = \int_t^T V(t,y)b(y)\,dy, \quad t \in (0,T).$$

Using these definitions, we can rewrite (3.2.53) as

$$\left(W^T\right)^{-1} a = (I+K)J_T a. \tag{3.2.55}$$

Proposition 2 and formula $(W^T)^* = J_T^*(I+K^*)$ yield

Proposition 4. *The operator $I + K^* : L_2(0,T) \mapsto L_2(0,T)$ is boundedly invertible.*

For arbitrary $f, g \in \mathcal{F}^T$, by the definition of C^T we have:

$$(C^T f, g)_{\mathcal{F}^T} = (W^T f, W^T g)_{\mathcal{H}^T}. \tag{3.2.56}$$

Let us put $f = (W^T)^{-1}a$, $g = (W^T)^{-1}b$, $a, b \in \mathcal{H}^T$ and rewrite (3.2.56) as

$$(C^T(I+K)J_T a, (I+K)J_T b)_{\mathcal{F}^T} = (a,b)_{\mathcal{H}^T} = (J_T a, J_T b)_{\mathcal{H}^T}, \tag{3.2.57}$$

Since (3.2.57) holds for all $a, b \in \mathcal{H}^T$, this leads to the following operator equation

$$(I+K)^* C^T (I+K) = I. \tag{3.2.58}$$

Introducing the operator

$$(C_T f)(t) = \int_0^T c^T(s,t) f(s)\,ds,$$

and using (3.2.15) we can rewrite (3.2.58) as

$$K^* + (I+K^*)(K + C_T + C_T K) = 0. \tag{3.2.59}$$

The function $V(y,t)$ was defined in (3.2.54) for $0 \leqslant y \leqslant t \leqslant T$, let us continue it by zero in the domain $t < y \leqslant T$ and introduce the function $\phi_y(t)$, $y, t \in [0,T]$ by the rule

$$\phi_y(t) = V(y,t) + c^T(y,t) + \int_0^T c^T(t,s)V(y,s)\,ds. \tag{3.2.60}$$

The equality (3.2.59) implies

$$V(t,y) + \phi_y(t) + \int_0^T V(t,z)\phi_y(z)\,dz = 0, \quad x, t \in (0,T). \tag{3.2.61}$$

Since $V(t,y) = 0$ for $0 < y < t < T$, we obtain that

$$\phi_y(t) + \int_t^T V(t,z)\phi_y(z)\,dz = 0, \quad 0 < y < t < T. \tag{3.2.62}$$

Rewriting this equation as

$$((I + K^*)\phi_y)(t) = 0, \quad 0 < y < t < T, \tag{3.2.63}$$

and taking into account the invertibility of $I + K^*$ (see Proposition 4), we get

$$\phi_y(t) = V(y,t) + c^T(y,t) + \int_y^T c^T(t,s)V(y,s)\,ds = 0, \quad 0 < y < t < T. \tag{3.2.64}$$

Let us formulate this result as

Theorem 3. *The kernels of operators C^T and K satisfy the following integral equation*

$$V(y,t) + c^T(y,t) + \int_y^T c^T(t,s)V(y,s)\,ds = 0, \quad 0 < y < t < T. \tag{3.2.65}$$

Solving the equation (3.2.65) for all $y \in (0,T)$ we can recover the potential using

$$q(y) = 2\frac{d}{dx}V(y,y).$$

It is easy to see that the kernel V is connected with the kernel of the transformation operator (3.1.6) by the rule $V(T-y, T-t) = K(y,s)$ and c^T is similarly related to F defined in (3.1.3): $c^T(T-x, T-t) = F(x,t)$. Therefore, equations (3.2.65) can be rewritten in a classical form (3.1.8). On the other hand, equations (3.2.65) have clearly a local character since $V(y,t)$ and $c^T(y,t)$ are completely determined by $q(y)$ on the interval $[0,T]$.

3.3 Appendix.

The Goursat problem was studied in [85, Sec. II.4] for smooth q, but the method works for $q \in L_1(0,a)$ as well (see [10, 11, 12, 9]).

Proposition 5. *a) If $q \in L^1_{loc}(\mathbb{R}_+)$, then the generalized solution $w(x,s)$ to the Goursat problem* (3.2.2) *is a continuous function and*

$$|w(x,s)| \leqslant \Big(\frac{1}{2}\int_0^{\frac{s+x}{2}} |q(\alpha)|\,d\alpha\Big)\exp\Big\{\frac{s-x}{4}\int_0^{\frac{s+x}{2}} |q(\alpha)|\,d\alpha\Big\}, \tag{3.3.1}$$

$$w_x(\cdot,s), w_s(\cdot,s), w_x(x,\cdot), w_s(x,\cdot) \in L_{1,loc}(\mathbb{R}_+). \tag{3.3.2}$$

Partial derivatives in (3.3.2) *continuously in $L^1_{loc}(\mathbb{R}_+)$ depend on parameters x, s. The equation in* (3.2.2) *holds almost everywhere and the boundary conditions are satisfied in the classical sense.*

b) If $q \in C_{loc}(\mathbb{R}_+)$, then the generalized solution to the Goursat problem (3.2.2) *is C^1-smooth, equation and boundary conditions are satisfied in the classical sense.*

c) If $q \in C^1_{loc}(\mathbb{R}_+)$, then the solution to the Goursat problem (3.2.2) *is classical, all its derivatives up to the second order are continuous.*

Proof. By setting $\xi = s - x$, $\eta = s + x$, and

$$v(\xi,\eta) = w\Big(\frac{\eta-\xi}{2}, \frac{\eta+\xi}{2}\Big) \tag{3.3.3}$$

equation (3.2.2) reduces to

$$\begin{cases} v_{\xi\eta} - \frac{1}{4}q(\frac{\eta-\xi}{2})v = 0, \quad 0<\xi<\eta, \\ v(\eta,\eta) = 0, \;\; v(0,\eta) = -\frac{1}{2}\int_0^{\eta/2} q(\alpha)\,d\alpha. \end{cases} \tag{3.3.4}$$

Boundary value problem (3.3.4) is equivalent to the integral equation

$$v(\xi,\eta) = -\frac{1}{2}\int_{\xi/2}^{\eta/2} q(\alpha)\,d\alpha - \frac{1}{4}\int_0^{\xi} d\xi_1 \int_{\xi}^{\eta} d\eta_1 q\Big(\frac{\eta_1-\xi_1}{2}\Big)v(\xi_1,\eta_1). \tag{3.3.5}$$

Introduce a new function

$$Q(\xi,\eta) = -\frac{1}{2}\int_{\xi/2}^{\eta/2} q(\alpha)\,d\alpha \tag{3.3.6}$$

and the operator $K : C(\mathbb{R}^2) \mapsto C(\mathbb{R}^2)$ by the rule

$$\Big(Kv\Big)(\xi,\eta) = \frac{1}{4}\int_0^{\xi} d\xi_1 \int_{\xi}^{\eta} d\eta_1 q\Big(\frac{\eta_1-\xi_1}{2}\Big)v(\xi_1,\eta_1). \tag{3.3.7}$$

Rewriting (3.3.5) as

$$v = Q - Kv \tag{3.3.8}$$

and formally solving it by iterations, we get

$$v(\xi,\eta) = Q(\xi,\eta) + \sum_{n=1}^{\infty}(-1)^n (K^n Q)(\xi,\eta). \tag{3.3.9}$$

To prove the convergence of (3.3.9) we need suitable estimates for $|K^n Q|(\xi,\eta)$. Observe that

$$|Q(\xi,\eta)| \leqslant \frac{1}{2}\int_0^{\eta/2} |q(\alpha)|\, d\alpha =: S(\eta). \tag{3.3.10}$$

For the first iteration we have

$$\begin{aligned} |(KQ)(\xi,\eta)| &\leqslant \frac{1}{4}\int_0^{\xi} d\xi_1 \int_{\xi}^{\eta} d\eta_1 \Big|q\Big(\frac{\eta_1-\xi_1}{2}\Big)\Big| S\Big(\eta_1\Big) \\ &\leqslant \frac{S(\eta)}{2}\int_0^{\xi} d\xi_1 \int_{\frac{\xi-\xi_1}{2}}^{\frac{\eta-\xi_1}{2}} |q(\tau)|\, d\tau \leqslant \frac{S^2(\eta)}{2}\xi. \end{aligned}$$

Easy induction arguments yield the following estimate

$$|(K^n Q)(\xi,\eta)| \leqslant \frac{S^{n+1}(\eta)}{2^n}\frac{\xi^n}{n!}, \quad n \in \mathbb{N}. \tag{3.3.11}$$

Combining (3.3.9) and (3.3.11) one has

$$|v(\xi,\eta)| \leqslant S(\eta)\exp\Big\{S(\eta)\frac{\xi}{2}\Big\}, \tag{3.3.12}$$

which due to (3.3.3) implies (3.3.1). Differentiating (3.3.5) we can obtain formulas for the derivatives of v:

$$v_\eta(\xi,\eta) = -\frac{1}{4}q\Big(\frac{\eta}{2}\Big) - \frac{1}{4}\int_0^{\xi} q\Big(\frac{\eta-\zeta}{2}\Big)v(\zeta,\eta)\, d\zeta, \tag{3.3.13}$$

$$\begin{aligned} v_\xi(\xi,\eta) = \frac{1}{4}q\Big(\frac{\xi}{2}\Big) - \frac{1}{4}\int_{\xi}^{\eta} q\Big(\frac{\zeta-\xi}{2}\Big)v(\xi,\zeta)\, d\zeta\, + \\ \frac{1}{4}\int_0^{\xi} q\Big(\frac{\xi-\zeta}{2}\Big)v(\zeta,\xi)\, d\zeta, \end{aligned} \tag{3.3.14}$$

which lead to (3.3.2).

When $q \in C_{loc}(\mathbb{R}_+)$, we can differentiate one more time in (3.3.13) with respect to ξ which proves that $v(\xi, \eta)$ is a classical solution of (3.3.4). For $w(x,t)$ this implies that $w_{tt} - w_{xx} \in C_{loc}(\mathbb{R}_+ \times \mathbb{R}_+)$ and equation in (3.2.2) holds in the classical sense.

When $q \in C^1_{loc}(\mathbb{R}_+)$, we can use (3.3.13), (3.3.14) to show that w has all continuous derivatives of the first and second order and thus is classical. □

Let us emphasize the following simple observation:

Remark 2. *Solution to the boundary value problem (3.2.2) is unique.*

Let $\{q_n\}_{n=1}^{\infty} \subset C^{\infty}(\mathbb{R}_+)$ be such that

$$q_n \xrightarrow{L^1_{loc}} q \quad \text{as } n \to \infty; \tag{3.3.15}$$

by $w_n(x,s)$ we denote the solution of (3.2.2) corresponding to the potential q_n.

Proposition 6. *For solutions w_n, w the following holds:*

$$w_n \xrightarrow{C_{loc}} w, \quad \text{as } n \to \infty, \tag{3.3.16}$$

$$\frac{\partial}{\partial t} w_n \xrightarrow{L^1_{loc}} \frac{\partial}{\partial t} w, \quad \text{as } n \to \infty, \tag{3.3.17}$$

$$\frac{\partial}{\partial x} w_n \xrightarrow{L^1_{loc}} \frac{\partial}{\partial x} w, \quad \text{as } n \to \infty. \tag{3.3.18}$$

Proof. It is sufficient to prove only (3.3.16), since (3.3.17) and (3.3.18) follows from this and formulas for derivatives (3.3.13), (3.3.14). We prove the convergence for the sequence $\{v_n\}_{n=1}^{\infty}$, that is obtained from $\{w_n\}_{n=1}^{\infty}$ by the change of variables (3.3.3).

Let us set $\Omega_N = [0,N] \times [0,N]$ and take arbitrary subsequence from $\{v_n\}_{n=1}^{\infty}$, we keep the same notations for it. It is straightforward to check that sequence $\{v_n\}_{n=1}^{\infty}$, being restricted to the compact Ω_N, satisfies the conditions of the Arzela-Ascoli theorem in $C(\Omega_N)$. Then there exist such a $\widetilde{v} \in C(\Omega_N)$ that for some subsequence

$$v_{n_k} \to \widetilde{v}, \quad \text{in } C(\Omega_N), \tag{3.3.19}$$

as $k \to \infty$. We rewrite (3.3.8) as

$$v = \mathbf{Q}q - \mathbf{K}(q)v, \tag{3.3.20}$$

where $\mathbf{Q} : L_1(0, 2N) \mapsto C(\Omega_N)$ is defined by (3.3.6) and $\mathbf{K} : L_1(0, N) \times C(\Omega_N) \mapsto C(\Omega_N)$ is defined by (3.3.7). We have

$$v - v_{n_k} = \mathbf{Q}(q - q_{n_k}) - \mathbf{K}(q - q_{n_k})(v + v_{n_k}) - \mathbf{K}(q_{n_k})v + \mathbf{K}(q)v_{n_k}. \qquad (3.3.21)$$

Going to the limit in (3.3.21) we get that

$$v - \widetilde{v} = -\mathbf{K}(q)v + \mathbf{K}(q)\widetilde{v}. \qquad (3.3.22)$$

Thus $\widetilde{v}$ satisfies the equation

$$\widetilde{v} = \mathbf{Q}q - \mathbf{K}(q)\widetilde{v}. \qquad (3.3.23)$$

Since the solution to (3.3.8) is unique (see remark 2), $\widetilde{v} \equiv v$. Thus, every subsequence of $\{v_n\}_{n=1}^{\infty}$ contains a subsequence, convergent to v in $C(\Omega_N)$. It implies that the very sequence converges to v. Since N is arbitrary, we arrive at (3.3.16). $\square$

Chapter 4
Conclusions

In this work we investigated some inverse and control problems for dynamical systems on the half line and on finite trees.

The general concept that we bore in mind was the following ideas on the connection of the inverse and control problems: the controllability properties of the dynamical system are connected (via the method of moments) with the properties of the corresponding families of exponentials. On the other hand, the Boundary Control method is based on the connections between controllability and identification problems for system described by partial differential equations. Thus, the progress in one of the fields (control, and inverse problems, and families of exponentials) could potentially lead to the progress in one or both of the other fields.

In the first Chapter the exact controllability for the wave equation on the finite tree has been proved. The control acts through the Dirichlet boundary data, where one of boundary vertices could be "clamped". Using the methods from [8] and [73, 74] the result on the null controllability for the parabolic and Schrödinger equations on the same tree was derived. The approach we used allowed us to obtain the new results on the families of exponentials associated with the wave, parabolic and Schrödinger equations on the tree (see also [8]). The partial (with respect to the shape) controllability result for the wave equation is interesting in itself. It is of great importance for dynamical inverse problems (see e.g. [22, 23, 35]) and could be considered as an intermediate step for solving the dynamical inverse problems. The extension of partial and exact controllability results on the new types of equations on trees, for example for the two-velocity dynamical system, Euler and Timoshenko beams and others could be of interest for studing the inverse problems (an application of BC method) and for the theory of functions (exponential families).

In the second Chapter we have investigated the inverse problem of recovering the material properties and the topology of a tree constituted by linear elastic two-velocity channels or in-plane-models of elastic strings. For a rooted tree this problem can be solved using measurements at all leaves (besides the root). The most remarkable novelty is the detection of the angles between two consecutive elements. Problems of the same type with variable coefficients (densities) is of great interest, as well as the problems involving frames of Euler-Bernoulli and Timoshenko beams. We believe that the method developed in the second chapter (see also [9, 35]) will help us to extend the local approach to inverse problems to other types of equations on graphs (trees).

In the third Chapter we have shown that the boundary control method (see [20, 21, 4, 5]) offers powerful tools for solving the dynamical and spectral inverse problems for the Schrödinger operator with the potential on the half line. Moreover, it is shown that the central objects of Gelfand-Levitan [47], Krein [56, 57], Simon [83, 49, 48] and Remling [78, 79] approaches to the spectral inverse problem naturally appear in the BC method. The BC method offers the elegant way to show the connection between the dynamical and spectral data (see also [53, 49]), this connection could be important for solving the inverse problems. By means of the boundary control method we derived the classical Gelfand-Levitan equations and improved the results of Simon and Remling on the convergence of $A-$amplitude, which is an important contribution to the BC method and spectral theory. The extension of our results to the new type of equations (first order hyperbolic systems, Hamilton system, etc.) and new type of problems (e.g. inverse scattering, see [36]) seems to be a promising direction in the development of the BC-method.

Everything that was said above allows us to conclude that the method of moments, families of exponentials and the Boundary Control method based on the the connection between controllability and identification properties of the dynamical system combined together provide the powerful tools for solving the control and inverse dynamical and spectral problems. Application of these methods to the new types of equations as well as to the new types of objects (for example, graphs with circles, infinite graphs, etc.) is a promising direction in the theory of Control and Inverse Problems and theory of vector valued exponential functions.

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