Open mathematical problems which cannot be stated formally as they refer to intuitive meanings of mathematical formulae and the current mathematical knowledge

Apoloniusz Tyszka

Abstract

Let $\beta = ((24!)!)!$, and let $\mathcal{P}_{n^2+1}$ denote the set of all primes of the form n^2+1. Let $\mathcal{M}$ denote the set of all positive multiples of elements of the set $\mathcal{P}_{n^2+1} \cap (\beta, \infty)$. The set $\mathcal{X} = \{0, \ldots, \beta\} \cup \mathcal{M}$ satisfies the following conditions: (1) $\mathrm{card}(\mathcal{X})$ is greater than a huge positive integer and it is conjectured that $\mathcal{X}$ is infinite, (2) we do not know any algorithm deciding the finiteness of $\mathcal{X}$, (3) a known and short algorithm for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$, (4) a known and short algorithm returns an integer n such that $\mathcal{X}$ is infinite if and only if $\mathcal{X}$ contains an element greater than n. The following problem is open: *define a set $\mathcal{X} \subseteq \mathbb{N}$ such that $\mathcal{X}$ satisfies conditions* (1)-(4) *and a known and simple formula $\varphi(x)$ satisfies $\mathcal{X} = \{n \in \mathbb{N} : \varphi(n)\}$, where $\varphi(n)$ has the same intuitive meaning for every $n \in \mathbb{N}$* (5). The statements $\varphi(n)$ in condition (5) have always the same intuitive meaning, if the predicate $\varphi(x)$ expresses a *natural property*, the term propounded by the philosopher David Lewis (1941-2001). Let $f(3) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 3$. For an integer $n \geqslant 3$, let Ψ_n denote the following statement: *if a system of equations $\mathcal{S} \subseteq \left\{x_i! = x_{i+1} : 1 \leqslant i \leqslant n-1\right\} \cup \left\{x_i \cdot x_j = x_{j+1} : 1 \leqslant i \leqslant j \leqslant n-1\right\}$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant f(n)$.* We prove that for every statement Ψ_n the bound $f(n)$ cannot be decreased. The author's guess is that the statements $\Psi_3, \ldots, \Psi_9$ are true. We prove that the statement Ψ_9 implies that the set $\mathcal{X}$ of all non-negative integers n whose number of digits belongs to $\mathcal{P}_{n^2+1}$ satisfies conditions (1)-(5).

Key words and phrases: computable set $\mathcal{X} \subseteq \mathbb{N}$ whose finiteness remains conjectured, computable set $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness remains conjectured, David Lewis's notion of a natural property, huge integers for which arithmetical operations cannot be performed by any physical process, intuitive meaning of a mathematical formula, Zenkin's super-induction.

1 Introduction and basic definitions and lemmas

In this article, we discuss open problems on computable sets $\mathcal{X} = \{n \in \mathbb{N} : \varphi(n)\}$ which cannot be stated formally as they require that the finiteness (infiniteness) of $\mathcal{X}$ remains conjectured and $\varphi(n)$ has the same intuitive meaning for every $n \in \mathbb{N}$.

Definition 1. *Let $\beta = ((24!)!)!$.*

Lemma 1. $\beta \approx 10^{10^{10^{25.16114896940657}}}$.

Proof. We ask Wolfram Alpha at `http://wolframalpha.com`. □

Definition 2. *We say that an integer $m \geqslant -1$ is a threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if $\mathcal{X}$ is infinite if and only if $\mathcal{X}$ contains an element greater than m, cf. [11] and [12].*

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer $m \geqslant -1$ is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $\{\max(\mathcal{X}), \max(\mathcal{X})+1, \max(\mathcal{X})+2, \ldots\}$.

Definition 3. *We say that a non-negative integer m is a weak threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if $\mathcal{X}$ is infinite if and only if* $\mathrm{card}(\mathcal{X}) > m$.

Theorem 1. *For every $\mathcal{X} \subseteq \mathbb{N}$, if an integer $m \geqslant -1$ is a threshold number of $\mathcal{X}$, then $m+1$ is a weak threshold number of $\mathcal{X}$.*

Proof. For every $\mathcal{X} \subseteq \mathbb{N}$, if $m \in [-1, \infty) \cap \mathbb{Z}$ and $\mathrm{card}(\mathcal{X}) > m+1$, then $\mathcal{X} \cap [m+1, \infty) \neq \emptyset$. □

We do not know any weak threshold number of the set of all primes of the form n^2+1. The same is true for the sets

$$\left\{n \in \mathbb{N} : 2^{2^n}+1 \text{ is composite}\right\}$$

and

$$\{n \in \mathbb{N} : n!+1 \text{ is a square}\}$$

Lemma 2. *For every positive integers x and y, $x! \cdot y = y!$ if and only if*

$$(x+1=y) \vee (x=y=1)$$

Lemma 3. *(Wilson's theorem, [2, p. 89]). For every integer $x \geqslant 2$, x is prime if and only if x divides* $(x-1)!+1$.

2 Open Problems 1 and 2

The following two open problems cannot be stated formally as they refer to intuitive meanings of mathematical formulae and the current mathematical knowledge.

Open Problem 1. *Define a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies the following conditions:*
(1) $\mathrm{card}(\mathcal{X})$ *is greater than a huge positive integer and it is conjectured that $\mathcal{X}$ is infinite,*
(2) *we do not know any algorithm deciding the finiteness of $\mathcal{X}$,*
(3) *a known and short algorithm for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$,*
(4•) *a known and short algorithm returns an integer n such that $\mathcal{X}$ is infinite if and only if* $\mathrm{card}(\mathcal{X}) > n$,
(5) *a known and simple formula $\varphi(x)$ satisfies $\mathcal{X} = \{n \in \mathbb{N} : \varphi(n)\}$, where $\varphi(n)$ has the same intuitive meaning for every $n \in \mathbb{N}$.*

Open Problem 2. *Define a set $\mathcal{X} \subseteq \mathbb{N}$ such that $\mathcal{X}$ satisfies conditions* (1)-(3), (5), *and a known and short algorithm returns an integer n such that $\mathcal{X}$ is infinite if and only if $\mathcal{X}$ contains an element greater than n* (4).

The statements $\varphi(n)$ in condition (5) have always the same intuitive meaning, if the predicate $\varphi(x)$ expresses David Lewis's *natural property*. For the meaning of this term, the reader is referred to [1].

Theorem 2. *Open Problem 2 claims more than Open Problem 1.*

Proof. By Theorem 1, condition (4) implies condition (4•). □

3 Two partial solutions to Open Problem 2

Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of all primes of the form n^2+1 is infinite, see [5, pp. 37–38] and [8]. Let $\mathcal{M}$ denote the set of all positive multiples of elements of the set $\mathcal{P}_{n^2+1} \cap (\beta, \infty)$.

Theorem 3. *The set $\mathcal{X} = \{0, \ldots, \beta\} \cup \mathcal{M}$ satisfies conditions* (1)-(4).

Proof. Condition (1) holds as $\mathrm{card}(\mathcal{X}) > \beta$ and the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than β. Thus condition (2) holds. Condition (3) holds trivially. Since the set $\mathcal{M}$ is empty or infinite, the integer β is a threshold number of $\mathcal{X}$. Thus condition (4) holds. □

Let $[\cdot]$ denote the integer part function.

Lemma 4. *For every non-negative integer n, $\left[\dfrac{3n-3\beta+3}{3n-3\beta+2}\right]$ equals 0 or 1. The first case holds when $n \leqslant \beta - 1$. The second case holds when $n \geqslant \beta$.*

Lemma 5. *The function*

$$\mathbb{N} \cap [\beta, \infty) \ni n \xrightarrow{\theta} \beta + n - \left[\sqrt{n}\right]^2 \in \mathbb{N} \cap [\beta, \infty)$$

takes every integer value $k \geqslant \beta$ infinitely many times.

Proof. Let $t = k - \beta$. The equality $\theta(n) = k$ holds for every

$$n \in \left\{(t+0)^2 + t,\ (t+1)^2 + t,\ (t+2)^2 + t, \ldots)\right\} \cap [\beta, \infty)$$

□

Theorem 4. *The set* $\mathcal{X} = \left\{n \in \mathbb{N} : 2 + \left[\dfrac{3n-3\beta+3}{3n-3\beta+2}\right] \cdot \left(\left(\beta + n - \left[\sqrt{n}\right]^2\right)^2 - 1\right) \text{ is prime}\right\}$ *satisfies conditions* `(1)-(4)`.

Proof. Condition `(3)` holds trivially. By Lemma 4, $\mathcal{X} = \{0, \ldots, \beta - 1\} \cup \mathcal{H}$, where

$$\mathcal{H} = \left\{n \in \mathbb{N} \cap [\beta, \infty) : \left(\beta + n - \left[\sqrt{n}\right]^2\right)^2 + 1 \text{ is prime}\right\}$$

By Lemma 5, the set $\mathcal{H}$ is empty or infinite. The second case holds when

$$\exists\, k \in \mathbb{N} \cap [\beta, \infty) \;\; k^2 + 1 \text{ is prime} \tag{6}$$

The equality $\mathcal{X} = \{0, \ldots, \beta - 1\} \cup \mathcal{H}$ and the last two sentences imply that $\beta - 1$ is a threshold number of $\mathcal{X}$ and conditions `(1)` and `(4)` hold. Condition `(2)` holds as due to known physics we are not able to confirm statement (6) by a direct computation. □

4 The statements Ψ_n which seem to be true for every $n \in \{3, \ldots, 9\}$

Let $f(3) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 3$. For an integer $n \geqslant 3$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} \forall i \in \{1, \ldots, n-1\} \setminus \{2\}\ x_i! &= x_{i+1} \\ x_1 \cdot x_2 &= x_3 \\ x_2 \cdot x_2 &= x_3 \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system $\mathcal{U}_n$.

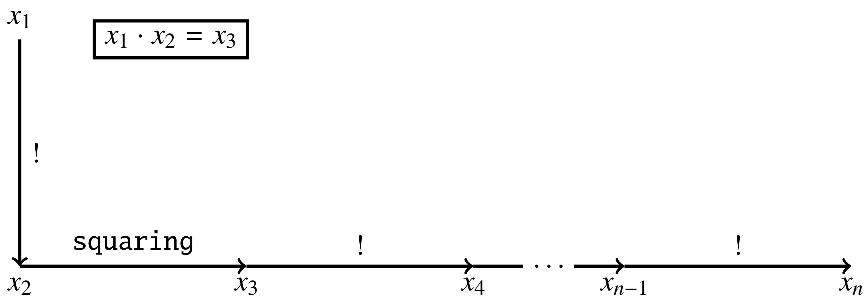

Fig. 1 Construction of the system $\mathcal{U}_n$

Lemma 6. *For every integer $n \geqslant 3$, the system $\mathcal{U}_n$ has exactly two solutions in positive integers, namely* $(1, \dots, 1)$ *and* $\big(2, 2, f(3), \dots, f(n)\big)$.

Let

$$B_n = \Big\{x_i! = x_{i+1} :\ 1 \leqslant i \leqslant n-1\Big\} \cup \Big\{x_i \cdot x_j = x_{j+1} :\ 1 \leqslant i \leqslant j \leqslant n-1\Big\}$$

For an integer $n \geqslant 3$, let Ψ_n denote the following statement: *if a system of equations $\mathcal{S} \subseteq B_n$ has only finitely many solutions in positive integers $x_1, \dots, x_n$, then each such solution $(x_1, \dots, x_n)$ satisfies $x_1, \dots, x_n \leqslant f(n)$.* The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible. The author's guess is that the statements $\Psi_3, \dots, \Psi_9$ are true.

Theorem 5. *Every statement Ψ_n is true with an unknown integer bound that depends on n.*

Proof. For every positive integer n, the system B_n has a finite number of subsystems. □

Theorem 6. *For every statement Ψ_n, the bound $f(n)$ cannot be decreased.*

Proof. It follows from Lemma 6 because $\mathcal{U}_n \subseteq B_n$. □

5 The statement Ψ_9 solves Open Problem 2

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{cases}$$

Lemma 2 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

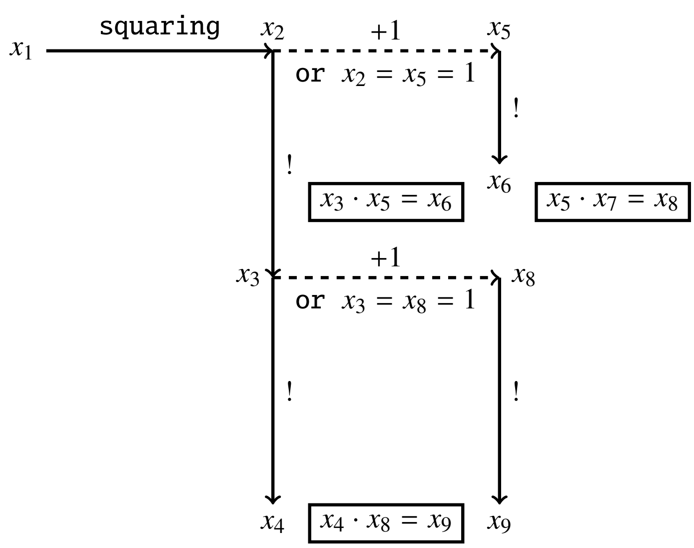

Fig. 2 Construction of the system $\mathcal{A}$

Lemma 7. *For every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$
\begin{array}{rcl}
x_2 & = & x_1^2 \\
x_3 & = & (x_1^2)! \\
x_4 & = & ((x_1^2)!)! \\
x_5 & = & x_1^2 + 1 \\
x_6 & = & (x_1^2 + 1)! \\
x_7 & = & \dfrac{(x_1^2)! + 1}{x_1^2 + 1} \\
x_8 & = & (x_1^2)! + 1 \\
x_9 & = & ((x_1^2)! + 1)!
\end{array}
$$

Proof. By Lemma 2, for every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 7 follows from Lemma 3. □

Lemma 8. *There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ which solve the system $\mathcal{A}$ and satisfy $x_1 = 1$.*

Proof. If a tuple $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ solves the system $\mathcal{A}$ and $x_1 = 1$, then $x_1, \ldots, x_9 \leqslant 2$. Indeed, $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leqslant 2$. □

Theorem 7. *The statement Ψ_9 proves the following implication: if there exists an integer $x_1 \geqslant 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set $\mathcal{P}_{n^2+1}$ is infinite.*

Proof. Suppose that the antecedent holds. By Lemma 7, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{A}$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geqslant f(7)$. Hence, $(x_1^2)! \geqslant f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geqslant (f(8) + 1)! > f(8)! = f(9)$$

Since $\mathcal{A} \subseteq B_9$, the statement Ψ_9 and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 7 and 8 the set $\mathcal{P}_{n^2+1}$ is infinite. □

Let $\mathcal{F}$ denote the set of all non-negative integers k whose number of digits belongs to $\mathcal{P}_{n^2+1}$.

Lemma 9. $\mathrm{card}(\mathcal{F}) \geqslant 9 \cdot 10^{9 \cdot 4^{747}}$.

Proof. The following `PARI/GP` ([7]) command

```
(12:26) gp > isprime(1+9*4^747,{flag=2})
%1 = 1
```

is shown together with its output. This command performs the `APRCL` primality test, the best deterministic primality test algorithm ([10, p. 226]). It rigorously shows that the number $\left(3 \cdot 2^{747}\right)^2 + 1$ is prime. Since $9 \cdot 10^{9 \cdot 4^{747}}$ non-negative integers have $1 + 9 \cdot 4^{747}$ digits, the desired inequality holds. □

Theorem 8. *The statement Ψ_9 implies that $\mathcal{X} = \mathcal{F}$ satisfies conditions* `(1)-(5)`.

Proof. Suppose that the antecedent holds. Since the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite, Lemma 9 implies condition `(1)`. Conditions `(3)` and `(5)` hold trivially. By Theorem 7, $\underbrace{9 \ldots 9}_{\beta \text{ digits}}$ is a threshold number of $\mathcal{X}$. Thus condition `(4)` holds. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $f(7) = ((24!)!)! = \beta$. Thus condition `(2)` holds. □

6 Open Problems 3 and 4

Definition 4. *Let* (1⋄) *denote the following condition:* card($\mathcal{X}$) *is greater than a huge positive integer and it is conjectured that* $\mathcal{X} = \mathbb{N}$.

Definition 5. *Let* (2⋄) *denote the following condition: we do not know any algorithm deciding the equality* $\mathcal{X} = \mathbb{N}$.

The following two open problems cannot be stated formally as they refer to intuitive meanings of mathematical formulae and the current mathematical knowledge.

Open Problem 3. *Define a set* $\mathcal{X} \subseteq \mathbb{N}$ *that satisfies conditions* (1⋄)-(2⋄), (2)-(3), (4•), *and* (5).

Open Problem 3 claims more than Open Problem 1 as condition (1⋄) implies condition (1).

Open Problem 4. *Define a set* $\mathcal{X} \subseteq \mathbb{N}$ *that satisfies conditions* (1⋄)-(2⋄) *and* (2)-(5).

Open Problem 4 claims more than Open Problem 2 as condition (1⋄) implies condition (1).

Theorem 9. *Open Problem 4 claims more than Open Problem 3.*

Proof. By Theorem 1, condition (4) implies condition (4•). □

7 A partial solution to Open Problem 4

Let $\mathcal{V}$ denote the set of all positive multiples of elements of the set

$$\{n \in \{\beta+1,\ \beta+2,\ \beta+3,\ \ldots\} : 2^{2^n}+1 \text{ is composite}\}$$

Theorem 10. *The set* $\mathcal{X} = \{0, \ldots, \beta\} \cup \mathcal{V}$ *satisfies conditions* (1⋄)-(2⋄) *and* (2)-(4).

Proof. The inequality card($\mathcal{X}$) $> \beta$ holds trivially. Most mathematicians believe that $2^{2^n}+1$ is composite for every integer $n \geqslant 5$, see [3, p. 23]. These two facts imply conditions (1⋄) and (2⋄). Condition (3) holds trivially. Since the set $\mathcal{V}$ is empty or infinite, the integer β is a threshold number of $\mathcal{X}$. Thus condition (4) holds. The question of finiteness of the set $\{n \in \mathbb{N} : 2^{2^n}+1 \text{ is composite}\}$ remains open, see [4, p. 159]. Hence, the question of emptiness of the set

$$\{n \in \{\beta+1,\ \beta+2,\ \beta+3,\ \ldots\} : 2^{2^n}+1 \text{ is composite}\}$$

remains open. Therefore, the question of finiteness of the set $\mathcal{V}$ remains open. Consequently, the question of finiteness of the set $\mathcal{X}$ remains open and condition (2) holds. □

8 Open Problems 5 and 6

Definition 6. *Let* (1*) *denote the following condition:* card($\mathcal{X}$) *is greater than a huge positive integer and it is conjectured that* $\mathcal{X}$ *is finite.*

The following two open problems cannot be stated formally as they refer to intuitive meanings of mathematical formulae and the current mathematical knowledge.

Open Problem 5. *Define a set* $\mathcal{X} \subseteq \mathbb{N}$ *that satisfies conditions* (1*), (2)-(3), (4•), *and* (5).

Open Problem 6. *Define a set* $\mathcal{X} \subseteq \mathbb{N}$ *that satisfies conditions* (1*) *and* (2)-(5).

Theorem 11. *Open Problem 6 claims more than Open Problem 5.*

Proof. By Theorem 1, condition (4) implies condition (4•). □

9 A partial solution to Open Problem 6

A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the equation $x! + 1 = y^2$, see [6].

Lemma 10. *([9, p. 297]). It is conjectured that* $x! + 1$ *is a square only for* $x \in \{4, 5, 7\}$.

Let $\mathcal{W}$ denote the set of all integers x greater than β such that $x! + 1$ is a square.

Theorem 12. *The set*

$$\mathcal{X} = \{0, \ldots, \beta\} \cup \{k \cdot x : (k \in \mathbb{N} \setminus \{0\}) \wedge (x \in \mathcal{W})\}$$

satisfies conditions `(1*)` *and* `(2)-(4)`.

Proof. Condition `(1*)` holds as $\mathrm{card}(\mathcal{X}) > \beta$ and the set $\mathcal{W}$ is conjecturally empty by Lemma 10. Condition `(3)` holds trivially. We do not know any algorithm that decides the emptiness of $\mathcal{W}$ and the set

$$\mathcal{Y} = \{k \cdot x : (k \in \mathbb{N} \setminus \{0\}) \wedge (x \in \mathcal{W})\}$$

is empty or infinite. Thus condition `(2)` holds. Since the set $\mathcal{Y}$ is empty or infinite, the integer β is a threshold number of $\mathcal{X}$. Thus condition `(4)` holds. □

10 The statement Ψ_6 solves Open Problem 6

Let C denote the following system of equations:

$$\begin{cases} x_1! &= x_2 \\ x_2! &= x_3 \\ x_5! &= x_6 \\ x_4 \cdot x_4 &= x_5 \\ x_3 \cdot x_5 &= x_6 \end{cases}$$

Lemma 2 and the diagram in Figure 3 explain the construction of the system C.

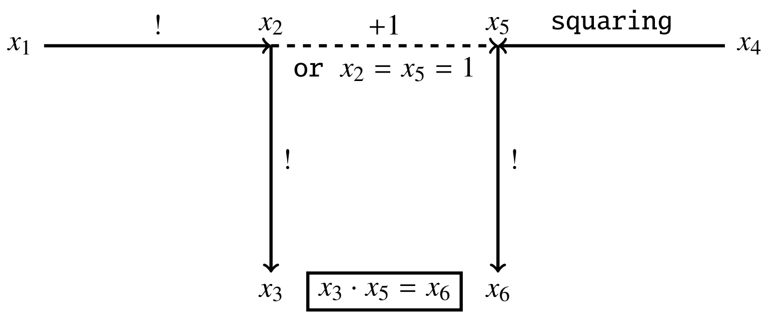

Fig. 3 Construction of the system C

Lemma 11. *For every* $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$*, the system* C *is solvable in positive integers* x_2, x_3, x_5, x_6 *if and only if* $x_1! + 1 = x_4^2$*. In this case, the integers* x_2, x_3, x_5, x_6 *are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \end{aligned}$$

Proof. It follows from Lemma 2. □

Theorem 13. *If the equation $x_1! + 1 = x_4^2$ has only finitely many solutions in positive integers, then the statement Ψ_6 guarantees that each such solution (x_1, x_4) belongs to the set* $\{(4,5),(5,11),(7,71)\}$.

Proof. Suppose that the antecedent holds. Let positive integers x_1 and x_4 satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0,1\}$. By Lemma 11, the system C is solvable in positive integers x_2, x_3, x_5, x_6. Since $C \subseteq B_6$, the statement Ψ_6 implies that $x_6 = (x_1! + 1)! \leqslant f(6) = f(5)!$. Hence, $x_1! + 1 \leqslant f(5) = f(4)!$. Consequently, $x_1 < f(4) = 24$. If $x_1 \in \{1, \ldots, 23\}$, then $x_1! + 1$ is a square only for $x_1 \in \{4,5,7\}$. □

Theorem 14. *Let $\mathcal{X}$ denote the set of all non-negative integers n which have* $(((k!)!)!)!$ *digits for some* $k \in \{m \in \mathbb{N} : m! + 1$ is a square$\}$. *We claim that the statement Ψ_6 implies that $\mathcal{X}$ satisfies conditions* `(1*)` *and* `(2)-(5)`.

Proof. Let $d = (((7!)!)!)!$. Since $7! + 1 = 71^2$, we obtain that $\{10^d, \ldots, \underbrace{9\ldots9}_{d \text{ digits}}\} \subseteq \mathcal{X}$. Hence, $\mathrm{card}(\mathcal{X}) \geqslant 9 \cdot 10^{d-1}$. By this and Lemma 10, condition `(1*)` holds. Conditions `(2)-(3)` and `(5)` hold trivially. By Theorem 13, the statement Ψ_6 implies that $\underbrace{9\ldots9}_{d \text{ digits}}$ is a threshold number of $\mathcal{X}$. Thus condition `(4)` holds. □

References

[1] C. Dorr, *"Natural Properties"*, in: The Stanford Encyclopedia of Philosophy (Fall 2019 edition), Edward N. Zalta (ed.), `http://plato.stanford.edu/archives/fall2019/entries/natural-properties`.

[2] M. Erickson, A. Vazzana, D. Garth, *Introduction to number theory,* 2nd ed., CRC Press, Boca Raton, FL, 2016.

[3] J.-M. De Koninck and F. Luca, *Analytic number theory: Exploring the anatomy of integers,* American Mathematical Society, Providence, RI, 2012.

[4] M. Křížek, F. Luca, L. Somer, *17 lectures on Fermat numbers: from number theory to geometry,* Springer, New York, 2001.

[5] W. Narkiewicz, *Rational number theory in the 20th century: From PNT to FLT,* Springer, London, 2012.

[6] M. Overholt, *The Diophantine equation $n! + 1 = m^2$,* Bull. London Math. Soc. 25 (1993), no. 2, 104.

[7] `PARI/GP` *online documentation,* `http://pari.math.u-bordeaux.fr/dochtml/html/Arithmetic_functions.html`.

[8] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences, A002496, Primes of the form $n^2 + 1$,* `http://oeis.org/A002496`.

[9] E. W. Weisstein, *CRC Concise Encyclopedia of Mathematics,* 2nd ed., Chapman & Hall/CRC, Boca Raton, FL, 2002.

[10] S. Y. Yan, *Number theory for computing,* 2nd ed., Springer, Berlin, 2002.

[11] A. A. Zenkin, *Super-induction method: logical acupuncture of mathematical infinity,* Twentieth World Congress of Philosophy, Boston, MA, August 10–15, 1998, `http://www.bu.edu/wcp/Papers/Logi/LogiZenk.htm`.

[12] A. A. Zenkin, *Superinduction: new logical method for mathematical proofs with a computer,* in: J. Cachro and K. Kijania-Placek (eds.), Volume of Abstracts, 11th International Congress of Logic, Methodology and Philosophy of Science, August 20–26, 1999, Cracow, Poland, p. 94, The Faculty of Philosophy, Jagiellonian University, Cracow, 1999.

Apoloniusz Tyszka
University of Agriculture
Faculty of Production and Power Engineering
Balicka 116B, 30-149 Kraków, Poland
E-mail: `rttyszka@cyf-kr.edu.pl`