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Max-min slackness and the effect of new workers in job-matching markets

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Abstract

In the Kelso and Crawford (1982) model of job matching, we investigate agents' demand/supply behavior at the doctor-optimal equilibrium price vector. We prove a striking property, termed max-min slackness, which states that the maximum labor supply is *strictly* greater than the minimum labor demand. Building on this property, we prove that the doctor-optimal equilibrium price vector is also an equilibrium price vector in the market where one (arbitrary) doctor is absent. Combining this finding with the lattice property, we demonstrate that adding new doctors reduces wages, which in turn harms the original doctors and helps the hospitals, at the doctor-optimal stable outcome.

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1 Introduction

The Kelso and Crawford (1982) model of job matching has been a basis for the analysis of two-sided markets with monetary transfers. The model is suitable for analyzing situations in which (i) objects are traded in discrete units, and (ii) prices can be finely adjusted and indifferences over objects enter the analysis. The auction model with heterogeneous commodities (Gul and Stacchetti 1999, Ausubel 2006) and the trading network model (Hatfield et al. 2013, Fleiner et al. 2018) can be viewed as a special case and a generalization

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of the Kelso-Crawford model, respectively.[1] The model has three essential results: (i) the Walrasian equilibria coincide with the stable outcomes (Kelso and Crawford 1982); (ii) the equilibrium price vectors always exist (Kelso and Crawford 1982); and (iii) the set of equilibrium price vectors forms a lattice (Kojima et al. 2018a). The last two results imply that there exists a doctor-optimal equilibrium price vector (henceforth, D-vector). The properties of the D-vector have been studied extensively, especially in terms of strategic behaviors.

The purpose of the present paper is to uncover the salient features of the D-vector in terms of the agents' demand/supply behavior and derive new theoretical results. A job-matching market can be viewed as a commodity market by identifying the "pairs" with commodities. If a doctor chooses a pair (representing labor supply) and a hospital also chooses it (representing labor demand), then this is interpreted as realization of a matching. Equilibrium price vectors bring demand and supply into balance.[2] Our first result shows that, at the D-vector, the maximum labor supply is *strictly* greater than the minimum labor demand at any commodity bundle (in a sense to be specified later). We call this property *max-min slackness* to emphasize that the max-min inequality is not tight.

To develop some intuition for the slackness result, consider an auction model in which, contrary to the job-matching model, the "one" side represents the demand side. Suppose that there are one seller of one commodity and many buyers. In a second-price auction, at least *two* buyers (possibly) demand the commodity, while only *one* seller supplies it. Namely, demand is strictly greater than supply.

We apply max-min slackness to derive new results with economic implications. In real job-matching markets, the set of job candidates often changes due to policy reforms.[3] We examine how this change affects the D-vector; to simplify the discussion, suppose that one doctor leaves the market. Intuitively, this change leads to a decrease in labor supply and induces excess demand. This is, however, not the case at the D-vector because supply is strictly $(+1)$ greater than demand (by max-min slackness), and removing one doctor induces only a one-unit (-1) decrease in labor supply. Based on this observation, we prove that the D-vector remains an equilibrium price vector in the market where one doctor is absent. By combining this finding with the lattice property, we demonstrate that adding new doctors always decreases wages, which in turn harms the original doctors and helps the hospitals.

Our new results make two contributions to the literature. First, the effect of adding/removing

[1] Furthermore, the Kelso-Crawford model has a close connection to the matching-with-contracts model due to Hatfield and Milgrom (2005); see Echenique (2012).

[2] This sentence refers to a seemingly obvious fact, but defining the "balance" of demand and supply is more complex than it appears, as combinatorial problems and indifferences over commodities enter the analysis. We overcome this difficulty using the techniques in discrete convex analysis (see Theorem 3).

[3] For example, some developed countries have recently begun to accept new workers from developing countries. Anti-discrimination laws intended to accelerate the hiring of minority/disabled people would promote labor force participation.

agents observed here is parallel to that in marriage markets (see Gale and Sotomayor (1985)) and enables a unified understanding of stable outcomes in two-sided markets. The second contribution concerns application. Recently, market designers have emphasized the advantages of implementing a stable outcome (a matching and match-specific salaries) to real job-matching markets (see Crawford (2008) or Kojima et al. (2018a)). Our results help us understand the welfare effect of adding new doctors at the doctor-optimal stable outcome.

Related literature

There is a stream of research on two-sided markets with monetary transfers. Previous studied have analyzed the mechanism that implements the best equilibrium outcome for the agents on one side. Most notably, such a mechanism is strategy-proof for unit-demand agents. This fact was first observed by Vickrey (1961) in a second-price auction and later extended to more general cases; see Demange (1982), Leonard (1983), Pérez-Castrillo and Sotomayor (2017), Hatfield et al. (2018), Jagadeesan et al. (2018), and Schlegel (2019).[4] Hatfield et al. (2014) and Hatfield et al. (2018) prove that, in the Kelso-Crawford model, implementing the D-vector is not only strategy-proof for doctors but also induces efficient investment choices.

The effect of adding new agents was first discovered by Gale and Sotomayor (1985) in marriage markets. They prove that, at the woman-optimal stable matching, the entrance of new women harms the original women and helps the men. In two-sided markets with monetary transfers, parallel results have been obtained for some, albeit limited, cases. Section 7.1 of Roth and Sotomayor (1990) discusses the effect in a market with one seller and many buyers.[5] Our result generalizes this finding to many-to-one markets with non-quasi-linear preferences.

The key analytical tool in this paper is discrete convex analysis developed by Murota (2003). This theory enables us to convert complicated combinatorial problems to tractable mathematical operations. Some recent studies utilize this strength. Kojima et al. (2018b) study the framework of matching under constraints and prove that $\mathrm{M}^\natural$-convexity, a notion of discrete convexity, is essential for implementing the deferred acceptance algorithm. Candogan et al. (2016) translate trading network problems into $\mathrm{M}^\natural$-convex submodular flow problems and conduct a refined analysis of competitive equilibria. See Murota (2016) for a survey of applications of discrete convex analysis to economics.

The remainder of this paper is organized as follows. Section 2 introduces our model.

[4] Schlegel (2019) deals with a general trading network model with non-quasi-linear preferences.

[5] Although not as closely related to our study, Bulow and Klemperer (1996) investigate the effect of adding new bidders on the seller's revenue in an auction model. They prove that adding a new bidder is more profitable for the seller in expectation than holding an optimal auction.

Section 3 presents the main results. Section 4 presents concluding remarks. All proofs are relegated to Section 5.

2 Model

Let D denote the set of **doctors** and H the set of **hospitals**. Let $I \equiv D \cup H$. In the standard interpretation of a job-matching model, these agents have preference relations over the agents on the opposite side and participate in bilateral contracts. In this paper, we identify the job-matching market with a *commodity market* by regarding the set $\Omega \equiv D \times H$ as objects.[6] For example, for a doctor $d \in D$, being matched to $h \in H$ is translated into choosing the commodity $(d, h) \in \Omega$. We further describe the consumption bundles by 0-1 vectors; let $\tilde{\Omega} \equiv \{0, 1\}^{\Omega}$.

For each $d \in D$, we define d's **consumption set** by

$$\tilde{\Omega}^d = \Big\{\alpha \in \tilde{\Omega} : |\{h \in H : \alpha_{(d,h)} = 1\}| \leq 1 \text{ and } \alpha_{(d',h)} = 0 \text{ for all } (d', h) \in (D\backslash\{d\}) \times H\Big\}. \tag{1}$$

Note that d can consume at most one commodity, representing that d can work for at most one hospital in the job-matching context.

Each doctor d has a **utility function** $U^d : \tilde{\Omega}^d \times \mathbb{R} \to \mathbb{R}$ that satisfies the following two conditions, the second of which is due to Kojima et al. (2018a):

- **Monotonicity and continuity:** For any $\alpha \in \tilde{\Omega}^d$, $U^d(\alpha, \cdot)$ is continuous and strictly monotonic with respect to the second argument.
- **Bounded compensability:** There is a constant $\Delta_d > 0$ such that, for any $\alpha, \alpha' \in \tilde{\Omega}^d$ and $s \in \mathbb{R}$, we have $U^d(\alpha', s + \Delta_d) > U^d(\alpha, s)$.

The complete vector of prices for all commodities is denoted by $p \in \mathbb{R}^{\Omega}$. We define the **supply correspondence** $X^d : \mathbb{R}^{\Omega} \to \tilde{\Omega}^d$ by

$$X^d(p) = \big\{\alpha \in \tilde{\Omega}^d : U^d(\alpha, p \cdot \alpha) \geq U^d(\beta, p \cdot \beta) \text{ for all } \beta \in \tilde{\Omega}^d\big\} \text{ for all } p \in \mathbb{R}^{\Omega}.$$

For each $h \in H$, we define h's **consumption set** by

$$\tilde{\Omega}^h = \Big\{\alpha \in \tilde{\Omega} : \alpha_{(d,h')} = 0 \text{ for all } (d, h') \in D \times (H\backslash\{h\})\Big\}.$$

[6] This approach to the market was previously considered by Hatfield et al. (2013).

Each hospital h has a **revenue function**[7] $u^h : \tilde{\Omega}^h \to \mathbb{R} \cup \{-\infty\}$ that satisfies $u^h(\mathbf{0}) = 0$ and the following condition:

- **Monotonicity:** For any $\alpha, \beta \in \tilde{\Omega}^h$ with $\alpha \leq \beta$, we have $u^h(\alpha) \leq u^h(\beta)$.

For each $p \in \mathbb{R}^\Omega$ and $\alpha \in \tilde{\Omega}^h$, h's **profit** is given by $u^h[p](\alpha) \equiv u^h(\alpha) - p \cdot \alpha$; note that the quasi-linearity assumption is imposed on firms. We define the **demand correspondence** $X^h : \mathbb{R}^\Omega \to \tilde{\Omega}^h$ by

$$X^h(p) = \left\{\alpha \in \tilde{\Omega}^h : u^h[p](\alpha) \geq u^h[p](\beta) \text{ for all } \beta \in \tilde{\Omega}^h\right\} \text{ for all } p \in \mathbb{R}^\Omega.$$

We make the following assumption due to Kelso and Crawford (1982):

- **Substitutability:** For any $p, p' \in \mathbb{R}^\Omega$ with $p \leq p'$, and any $\alpha \in X^h(p)$, there exists $\beta \in X^h(p')$ such that $p_\omega = p'_\omega$ implies $\alpha_\omega \leq \beta_\omega$.

A **commodity market** derived from a job-matching market can be summarized as $\mathcal{E} = \langle D, H, (U^d)_{d\in D}, (u^h)_{h\in H}\rangle$.

3 Main results

3.1 Equilibrium price vector and max-min slackness

We say that $p \in \mathbb{R}^\Omega$ is an **equilibrium price vector for $\boldsymbol{\mathcal{E}}$** if[8]

$$\sum_{d\in D} X^d(p) \cap \sum_{h\in H} X^h(p) \neq \emptyset. \tag{2}$$

Suppose that the above set is non-empty and choose a 0-1 vector from the set. Then, any doctor-hospital pair (d, h) to which the vector assigns 1 (res. 0) is chosen from both (res. neither) of them, representing the coincidence between labor supply and labor demand. As this choice maximizes utility (profit), the vector represents a Walrasian equilibrium allocation.

We cite two fundamental theorems:[9]

Theorem 1 (Existence (Kelso and Crawford 1982; Kojima et al. 2018a))**.** *There exists an equilibrium price vector for $\mathcal{E}$.*

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[7]The value $-\infty$ captures technological constraints (Hatfield et al. 2013, p.973) or institutional constraints (Kojima et al. 2018a).

[8]The summation over sets represents the Minkowski sum.

[9]Existence also follows from Fleiner et al. (2018). The lattice structure and boundedness also follow from Schlegel (2019). Both of these papers deal with a general trading network model.

Theorem 2 (Lattice structure (Kojima et al. 2018a)). *The set of equilibrium price vectors for $\mathcal{E}$ is a complete sublattice.*

By these theorems, there always exist unique maximum/minimum equilibrium price vectors. We write $\overline{p}$ to denote the maximum equilibrium price vectors for $\mathcal{E}$ and call it the doctor-optimal equilibrium price vector, shortly **D-vector**.

Next, we translate the above set-language definition of equilibrium price vectors into the inequality-language definition. To this end, we introduce some preliminaries. For $i \in I$, we define the **min-requirement function** $\hat{R}^i(\cdot,\cdot)$ and the **max-requirement function** $\check{R}^i(\cdot,\cdot)$ as follows:[10]

$$\begin{aligned} \hat{R}^i(\lambda, p) &= \min\{\alpha \cdot \lambda : \alpha \in X^i(p)\} \text{ for all } \lambda \in \tilde{\Omega} \text{ and } p \in \mathbb{R}^{\Omega}, \\ \check{R}^i(\lambda, p) &= \max\{\alpha \cdot \lambda : \alpha \in X^i(p)\} \text{ for all } \lambda \in \tilde{\Omega} \text{ and } p \in \mathbb{R}^{\Omega}. \end{aligned} \tag{3}$$

To see the intended meaning of these functions, consider the set of commodities to which λ assigns 1. Then, $\hat{R}^i(\lambda, p)$ (res. $\check{R}^i(\lambda, p)$) represents the minimum (res. maximum) number of commodities that i requires from the set to form an optimal consumption bundle.

Theorem 3. *$p \in \mathbb{R}^{\Omega}$ is an equilibrium price vector for $\mathcal{E}$ if and only if*

$$\sum_{h \in H} \hat{R}^h(\lambda, p) \leq \sum_{d \in D} \check{R}^d(\lambda, p) \text{ and } \sum_{d \in D} \hat{R}^d(\lambda, p) \leq \sum_{h \in H} \check{R}^h(\lambda, p) \text{ for all } \lambda \in \tilde{\Omega}.$$

The above system of inequalities can be interpreted as representing the balance of demand and supply. To see this point, suppose that there exists $\lambda \in \tilde{\Omega}$ such that the former inequality fails, i.e.,

$$\sum_{h \in H} \hat{R}^h(\lambda, p) > \sum_{d \in D} \check{R}^d(\lambda, p).$$

Placing the above into the job-matching context, the left-hand side represents the minimum number of doctors that the hospitals must hire. This number exceeds the right-hand side, the maximum number of doctors who can work for the hospitals. Namely, excess demand is present. In a similar vein, violation of the latter inequality is interpreted as excess supply. Conversely, if all the inequalities hold, then no excess demand/supply occurs, which is the essence of the notion of an equilibrium.

Remark 1. Yokote (2017) prove the inequality-language characterization of equilibrium price vectors in the auction model due to Gul and Stacchetti (1999). Theorem 3 is a straightforward generalization of Yokote's (2017) result and the proof is omitted. The *only if* part

[10] Gul and Stacchetti (2000) define $\hat{R}^i(\cdot,\cdot)$ in the auction model and call it a requirement function.

follows from the definition of an equilibrium price vector and the *if* part follows from the discrete separation theorem. ∎

We now turn our attention to the D-vector $\bar{p}$. The salient feature of this vector becomes transparent in terms of the inequality-language characterization.

Theorem 4 (Max-min slackness)**.** *At $\bar{p}$, we have*

$$\sum_{h\in H}\hat{R}^h(\lambda,\bar{p}) < \sum_{d\in D}\check{R}^d(\lambda,\bar{p}) \textit{ for all } \lambda\in\{0,1\}^\Omega \textit{ with } \lambda\neq\mathbf{0}.$$

Proof. See Section 5.2. □

The intuition for this theorem is as follows: $\bar{p}$ is the best equilibrium price vector for the doctors (salaries are sufficiently high), and the maximum labor supply becomes large enough to *strictly* exceed the minimum labor demand.

3.2 Effect of adding/removing doctors

In real job-matching markets, the set of job candidates often changes due to policy reforms. We investigate how $\bar{p}$ changes when a doctor leaves/enters the market.

Fix $d'\in D$, who is to be removed from the market. For each $x\in\mathbb{R}^\Omega$, let $x_{-d'}$ denote the **projection** of x on $\mathbb{R}^{(D\setminus\{d'\})\times H}$. For each $\tilde{\Psi}\subseteq\tilde{\Omega}$, we define

$$\tilde{\Psi}_{-d'}=\left\{\alpha'\in\{0,1\}^{(D\setminus\{d'\})\times H}:\alpha'=\alpha_{-d'} \text{ for some } \alpha\in\tilde{\Psi}\right\}.$$

For notational consistency, let $\mathbb{R}^\Omega_{-d'}\equiv\mathbb{R}^{(D\setminus\{d'\})\times H}$. We define the **reduced market** $\mathcal{E}_{-d'}\equiv\langle D\setminus\{d'\},H,(U^d_{-d'})_{d\in D\setminus\{d'\}},(u^h_{-d'})_{h\in H}\rangle$, where

- for each $d\in D$, $U^d_{-d'}(\cdot,\cdot)$ denotes the restriction of $U^d(\cdot,\cdot)$ on $\tilde{\Omega}^d_{-d'}\times\mathbb{R}$; and
- for each $h\in H$, $u^h_{-d'}(\cdot)$ denotes the restriction of $u^h(\cdot)$ on $\tilde{\Omega}^h_{-d'}$.

We remark that all the assumptions on utility/revenue functions are preserved in the reduced market $\mathcal{E}_{-d'}$. We define equilibrium price vectors for $\mathcal{E}_{-d'}$ in the same way as for $\mathcal{E}$.

Bearing Theorems 3 and 4 in mind, we investigate the effect of removing d' on $\bar{p}$; consider $\bar{p}_{-d'}$ in the reduced market. Clearly, excess supply never occurs for this vector, as only the agent on the supply side is removed. What about excess demand? By Theorem 4, labor supply is strictly (+1) larger than labor demand, while eliminating a unit-demand agent yields only a one-unit decrease (−1) of supply. As a result, the max-min inequality of Theorem 3 remains true in the reduced market, which leads us to the following theorem:

Theorem 5. *$\bar{p}_{-d'}$ is an equilibrium price vector for $\mathcal{E}_{-d'}$.*

Proof. See Section 5.3. □

We combine this theorem with Theorem 2 (lattice structure). For the D-vector $\bar{q} \in \mathbb{R}^{\Omega}_{-d'}$ for $\mathcal{E}_{-d'}$, we have $\bar{p}_{-d'} \leq \bar{q}$. Equivalently,

$$\bar{p}_\omega \leq \bar{q}_\omega \text{ for all } \omega \in (D \backslash \{d'\}) \times H.$$

That is, the entrance of new doctors always decreases wages. In view of utility/profit maximization, this change harms the doctors and helps the hospitals. This result helps us understand the welfare effect of adding new doctors at the doctor-optimal stable outcome.

4 Concluding remarks

In Theorem 4, we revealed max-min slackness at the D-vector. Under the quasi-linearity assumption, we can prove a parallel result for the hospital-optimal equilibrium price vector; the maximum labor *demand* is strictly grater than the minimum labor *supply*. Note further that Theorem 5 has a close connection to strategy-proofness of the doctor-optimal stable mechanism. We will discuss these issues in an updated version of this paper.

5 Proofs

5.1 Preliminaries

5.1.1 Discrete convex analysis

We introduce basic concepts in discrete convex analysis (Murota 2003). For $x \in \mathbb{R}^\Omega$, we define

$$\mathrm{supp}^+ x = \{\omega \in \Omega : \alpha_\omega > 0\}, \ \mathrm{supp}^- x = \{\omega \in \Omega : \alpha_\omega < 0\}.$$

For $\omega \in \Omega$, let $\mathbb{1}^\omega \in \tilde{\Omega}$ denote the ω-th unit vector. Let $\mathbb{1}^0 \equiv \mathbf{0}$.

We say that $u^h(\cdot)$ is an **M$^\natural$-concave function** if for any $\alpha, \beta \in \tilde{\Omega}^h$ and any $\omega \in \mathrm{supp}^+(\alpha - \beta)$, there exists $\psi \in \mathrm{supp}^-(\alpha - \beta) \cup \{0\}$ such that

$$u^h(\alpha) + u^h(\beta) \leq u^h(\alpha - \mathbb{1}^\omega + \mathbb{1}^\psi) + u^h(\beta + \mathbb{1}^\omega - \mathbb{1}^\psi).$$

We say that $\tilde{\Psi} \subseteq \tilde{\Omega}$ with $\tilde{\Psi} \neq \emptyset$ is an **M$^\natural$-convex set** if for any $\alpha, \beta \in \tilde{\Psi}$ and any $\omega \in$

$\mathrm{supp}^+(\alpha - \beta)$, there exists $\psi \in \mathrm{supp}^-(\alpha - \beta) \cup \{0\}$ such that

$$\alpha - \mathbb{1}^\omega + \mathbb{1}^\psi \in \tilde{\Psi},\ \beta + \mathbb{1}^\omega - \mathbb{1}^\psi \in \tilde{\Psi}.$$

Theorem 6 (Fujishige and Yang 2003; Murota 2003, Theorem 6.30). *Let $h \in H$. The following are equivalent:*

- *$u^h(\cdot)$ satisfies substitutability.*
- *$u^h(\cdot)$ is an M$^\natural$-concave function.*
- *$X^h(p)$ is an M$^\natural$-convex set for all $p \in \mathbb{R}^\Omega$.*

We remark that, for any $d \in D$ and $p \in \mathbb{R}^\Omega$, $X^d(p)$ consists of unit-vectors and thus forms an M$^\natural$-convex set. Hence, in our economy, every agent's demanded (supplied) commodities form an M$^\natural$-convex set.

5.1.2 Additional notations and definitions for $\mathcal{E}$

For $\alpha \in \tilde{\Omega}$ and $d \in D$, we define $\alpha^d \in \tilde{\Omega}$ by

$$\alpha^d_\omega = \begin{cases} \alpha_\omega & \text{if } \omega = (d, h) \text{ for some } h \in H, \\ 0 & \text{otherwise.} \end{cases}$$

For $h \in H$, we define $\alpha^h \in \tilde{\Omega}$ analogously.

Let $p \in \mathbb{R}^\Omega$ be an equilibrium price vector for $\mathcal{E}$ (i.e., (2) holds). We say that $\alpha \in \tilde{\Omega}$ is an **equilibrium allocation at $\boldsymbol{p}$** if

$$\alpha \in \sum_{d \in D} X^d(p) \cap \sum_{h \in H} X^h(p).$$

For each $i \in I$, we define

$$\hat{Q}^i(\lambda, p) = \{\alpha \in X^i(p) : \alpha \cdot \lambda = \hat{R}^i(\lambda, p)\},\ \check{Q}^i(\lambda, p) = \{\alpha \in X^i(p) : \alpha \cdot \lambda = \check{R}^i(\lambda, p)\}.$$

In words, $\hat{Q}^i(\lambda, p)$ (res. $\check{Q}^i(\lambda, p)$) represents the set of optimal consumption bundles that attain the value of $\hat{R}^i(\lambda, p)$ (res. $\check{R}^i(\lambda, p)$).

For $\lambda, \mu \in \tilde{\Omega}$, we define $\lambda \backslash \mu \in \tilde{\Omega}$, $\lambda \vee \mu \in \tilde{\Omega}$ and $\lambda \wedge \mu \in \tilde{\Omega}$ by

$$(\lambda \backslash \mu)_\omega = \begin{cases} 1 & \text{if } \lambda_\omega = 1 \text{ and } \mu_\omega = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\lambda \vee \mu)_\omega = \begin{cases} 1 & \text{if } \lambda_\omega = 1 \text{ or } \mu_\omega = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad (\lambda \wedge \mu)_\omega = \begin{cases} 1 & \text{if } \lambda_\omega = 1 \text{ and } \mu_\omega = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $\lambda \in \tilde{\Omega}$, let $\epsilon(\lambda) > 0$ denote a sufficiently small number that satisfies, for any $i \in I$,

$$\alpha \notin X^i(\bar{p}) \Longrightarrow \alpha \notin X^i\big(\bar{p} + \epsilon(\lambda) \cdot \lambda\big). \tag{4}$$

Such an $\epsilon(\cdot)$ always exists by continuity of $U^d(\alpha, \cdot)$ for any $d \in D$ and $\alpha \in \tilde{\Omega}^d$.

For $x \in \mathbb{R}^\Omega$, we define the $\boldsymbol{\ell_1}$**-norm** of x by

$$|x| = \sum_{\omega \in \Omega} |x_\omega|.$$

5.1.3 Characterization of equilibrium price vectors for $\mathcal{E}_{-d'}$

Fix $d' \in D$. For $d \in D \backslash \{d'\}$, let $X^d_{-d'}(\cdot)$ denote the supply coorespondence induced from $U^d_{-d'}(\cdot)$. Similarly, for $h \in H$, let $X^h_{-d'}(\cdot)$ denote the demand coorespondence induced from $u^h_{-d'}(\cdot)$. For $i \in I \backslash \{d'\}$, let $\hat{R}^i_{-d'}(\cdot, \cdot)$ and $\check{R}^i_{-d'}(\cdot, \cdot)$ denote the max- and min-requirement functions induced from $X^i_{-d'}(\cdot)$, respectively; formally, for $\lambda \in \tilde{\Omega}_{-d'}$ and $p \in \mathbb{R}^\Omega_{-d'}$,

$$\hat{R}^i_{-d'}(\lambda, p) = \min\{\alpha \cdot \lambda : \alpha \in X^i_{-d'}(p)\}, \ \check{R}^i_{-d'}(\lambda, p) = \max\{\alpha \cdot \lambda : \alpha \in X^i_{-d'}(p)\}.$$

By definition, for any $d \in D \backslash \{d'\}$, $p \in \mathbb{R}^\Omega$ and $\lambda \in \tilde{\Omega}$, we have

$$\hat{R}^d(\lambda, p) = \hat{R}^d_{-d'}(\lambda_{-d'}, p_{-d'}), \tag{5}$$
$$\check{R}^d(\lambda, p) = \check{R}^d_{-d'}(\lambda_{-d'}, p_{-d'}). \tag{6}$$

The following is an immediate corollary of Theorem 3:

Corollary 1. *Let $d' \in D$. Then, $p \in \mathbb{R}^\Omega_{-d'}$ is an equilibrium price vector for $\mathcal{E}_{-d'}$ if and only if*

$$\sum_{h \in H} \hat{R}^h_{-d'}(\lambda, p) \leq \sum_{d \in D} \check{R}^d_{-d'}(\lambda, p) \ \textit{and} \ \sum_{d \in D} \hat{R}^d_{-d'}(\lambda, p) \leq \sum_{h \in H} \check{R}^h_{-d'}(\lambda, p) \ \textit{for all} \ \lambda \in \tilde{\Omega}_{-d'}.$$

5.2 Proof of Theorem 4

5.2.1 The modified economy

We modify each doctor's utility function in $\mathcal{E}$ so that increasing prices of the D-vector has a "constant" effect on utility among optimal consumption bundles (see (10) below).

For each $d \in D$, we define $U_*^d : \tilde{\Omega}^d \times \mathbb{R} \to \mathbb{R}$ by

$$U_*^d(\alpha, s) = \begin{cases} U^d(\alpha, \bar{p} \cdot \alpha) + \min\limits_{\beta \in X^d(\bar{p})} \left\{ U^d\big(\beta, \bar{p} \cdot \beta + (s - \bar{p} \cdot \alpha)\big) - U^d(\beta, \bar{p} \cdot \beta) \right\} & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \alpha \in X^d(\bar{p}) \text{ and } s \geq \bar{p} \cdot \alpha, & \\ U^d(\alpha, s) \qquad\qquad\qquad\qquad\quad \text{otherwise.} & \end{cases} \tag{7}$$

We enumerate key properties of $U_*^d(\cdot, \cdot)$:

$$\text{For any } \alpha \in \tilde{\Omega}^d, \text{ we have } U_*^d(\alpha, \bar{p} \cdot \alpha) = U^d(\alpha, \bar{p} \cdot \alpha). \tag{8}$$

$$\text{For any } (\alpha, s) \in \tilde{\Omega}^d \times \mathbb{R}, \text{ we have } U^d(\alpha, s) \geq U_*^d(\alpha, s). \tag{9}$$

$$\text{For any } \alpha, \beta \in X^d(\bar{p}) \text{ and } \epsilon > 0, \text{ we have } U_*^d(\alpha, \bar{p} \cdot \alpha + \epsilon) = U_*^d(\beta, \bar{p} \cdot \beta + \epsilon). \tag{10}$$

As $U_*^d(\cdot, \cdot)$ is defined by the minimum of $U^d(\cdot, \cdot)$, it inherits the assumptions on $U^d(\cdot, \cdot)$. Let $X_*^d(\cdot)$ denote the supply correspondence induced from $U_*^d(\cdot, \cdot)$. Let $\mathcal{E}_* \equiv \langle D, H, (U_*^d)_{d \in D}, (u^h)_{h \in H} \rangle$. Recall that $\bar{p}$ denotes the D-vector for $\mathcal{E}$.

Claim 1. *$\bar{p}$ is the D-vector for $\mathcal{E}_*$.*

Proof. By (8), $\bar{p}$ is an equilibrium price vector. Suppose to the contrary that $\bar{p}$ is not the D-vector for $\mathcal{E}_*$. Then, by Theorems 1 and 2, there exists $p_* \in \mathbb{R}^\Omega$ such that p_* is the D-vector for $\mathcal{E}_*$ and satisfies

$$p_* \geq \bar{p}, \text{with strict inequality holding for at least one } \omega \in \Omega. \tag{11}$$

Let $\bar{\alpha}, \alpha_* \in \tilde{\Omega}$ denote the equilibrium allocations at $\bar{p}$ for $\mathcal{E}$ and at p_* for $\mathcal{E}_*$, respectively.

For any $d \in D$, we have

$$U^d(\alpha_*^d, p_* \cdot \alpha_*^d) \geq U_*^d(\alpha_*^d, p_* \cdot \alpha_*^d) \geq U_*^d(\bar{\alpha}^d, p_* \cdot \bar{\alpha}^d) \geq U_*^d(\bar{\alpha}^d, \bar{p} \cdot \bar{\alpha}^d) = U^d(\bar{\alpha}^d, \bar{p} \cdot \bar{\alpha}^d), \tag{12}$$

where the first inequality follows from (9), the second inequality follows from $\alpha_*^d \in X_*^d(p_*)$, the third inequality follows from (11) and monotonicity of utility in money, and the last equality follows from (8). If the above inequality is strict for at least one $d \in D$, then we obtain a contradiction to group strategy-proofness of the doctor-optimal stable mechanism for doctors (see Schlegel (2019)). Hence, we must have

$$U^d(\alpha_*^d, p_* \cdot \alpha_*^d) = U^d(\bar{\alpha}^d, \bar{p} \cdot \bar{\alpha}^d) \text{ for all } d \in D. \tag{13}$$

Let $d \in D$ and $\alpha^d \in \tilde{\Omega}^d$ be arbitrarily chosen. We prove that the following inequality holds:

$$U^d(\alpha^d, p_* \cdot \alpha^d) \leq U^d(\alpha_*^d, p_* \cdot \alpha_*^d). \tag{14}$$

To prove this, we consider two cases.

Case 1: Suppose $\alpha^d \in X^d(\bar{p})$. Supose to the contrary that $\bar{p} \cdot \alpha^d < p_* \cdot \alpha^d$. Then,

$$\begin{aligned} U^d(\bar{\alpha}^d, \bar{p} \cdot \bar{\alpha}^d) = U^d(\alpha^d, \bar{p} \cdot \alpha^d) &= U_*^d(\alpha^d, \bar{p} \cdot \alpha^d) \\ &< U_*^d(\alpha^d, p_* \cdot \alpha^d) \leq U_*^d(\alpha_*^d, p_* \cdot \alpha_*^d) \leq U^d(\alpha_*^d, p_* \cdot \alpha_*^d), \end{aligned}$$

where the first equality follows from $\alpha^d, \bar{\alpha}^d \in X^d(\bar{p})$, the second equality follows from (8), the strict inequality follows from $\bar{p} \cdot \alpha^d < p_* \cdot \alpha^d$ and monotonicity of utility in money, the penultimate inequality follows from $\alpha_*^d \in X_*^d(p_*)$, and the last inequality follows from (9). Thus, we obtain a contradiction to (13). Hence, $\bar{p} \cdot \alpha^d = p_* \cdot \alpha^d$. It follows that

$$U^d(\alpha^d, p_* \cdot \alpha^d) = U^d(\alpha^d, \bar{p} \cdot \alpha^d) = U^d(\bar{\alpha}^d, \bar{p} \cdot \bar{\alpha}^d) = U^d(\alpha_*^d, p_* \cdot \alpha_*^d),$$

where the first equality follows from $\bar{p} \cdot \alpha^d = p_* \cdot \alpha^d$, the second equality follows from $\alpha^d, \bar{\alpha}^d \in X^d(\bar{p})$, and the last equality follows from (13).

Case 2: Suppose $\alpha^d \notin X^d(\bar{p})$. Then,

$$U^d(\alpha^d, p_* \cdot \alpha^d) = U_*^d(\alpha^d, p_* \cdot \alpha^d) \leq U_*^d(\alpha_*^d, p_* \cdot \alpha_*^d) \leq U^d(\alpha_*^d, p_* \cdot \alpha_*^d),$$

where the first equality follows from the definition of $U_*(\cdot, \cdot)$, the first inequality follows from $\alpha_*^d \in X_*^d(p_*)$, and the second inequality follows from (9).

Hnece, in either case, we have (14). This means that p_* is an equilibrium price vector for $\mathcal{E}$, with the corresponding equilibrium allocation α_*. Together with (11), we obtain a contadiction to the maximality of $\bar{p}$. □

5.2.2 Proof of Theorem 4 for $\mathcal{E}_*$

By Claim 1, $\bar{p}$ is the D-vector for $\mathcal{E}_*$. We prove Theorem 4 for $\mathcal{E}_*$. For $d \in D$, let $\hat{R}_*^d(\cdot, \cdot)$ and $\check{R}_*^d(\cdot, \cdot)$ denote the max- and min-requirement functions induced from $X_*^d(\cdot)$, respectively (see (3)).

We first prove two claims.

Claim 2. *Let $\mu \in \tilde{\Omega}$ and $\bar{p}' \equiv \bar{p} + \epsilon(\mu) \cdot \mu$. Then, for any $h \in H$, $\alpha \in \hat{Q}^h(\mu, \bar{p})$ implies $\alpha \in X^h(\bar{p}')$. In particular, for any $\lambda \in \tilde{\Omega}$, we have $\alpha \cdot \lambda \geq \hat{R}^h(\lambda, \bar{p}')$.*

Proof. For simplicity, we write ϵ instead of $\epsilon(\mu)$. Let $\alpha \in \hat{Q}^h(\mu, \bar{p})$. Then, for any $\beta \in X^h(\bar{p})$,

$$\begin{aligned} u^h[\bar{p}'](\beta) &= u^h[\bar{p}](\beta) - \beta \cdot (\epsilon \cdot \mu) \\ &\leq u^h[\bar{p}](\alpha) - \beta \cdot (\epsilon \cdot \mu) \\ &\leq u^h[\bar{p}](\alpha) - \alpha \cdot (\epsilon \cdot \mu) \\ &= u^h[\bar{p}'](\alpha), \end{aligned}$$

where the first inequality follows from $\alpha \in X^h(\bar{p})$ and the second inequality follows from $\alpha \cdot \mu = \hat{Q}^h(\mu, \bar{p}) \leq \beta \cdot \mu$. Moreover, by (4), $X^h(\bar{p}') \subseteq X^h(\bar{p})$. Together with the above inequality, we obtain $\alpha \in X^h(\bar{p}')$. The last part of the statement follows from $\alpha \in X^h(\bar{p}')$ and the definition of $\hat{R}^h(\cdot, \cdot)$. □

Claim 3. *Let $\lambda, \mu \in \tilde{\Omega}$ and $\bar{p}' \equiv \bar{p} + \epsilon(\mu) \cdot \mu$. Then, for any $d \in D$ and $h \in H$, the following inequalities hold:*

(i) $\hat{R}^d_*(\lambda, \bar{p}') \leq \check{R}^d_*(\mu, \bar{p}) - \check{R}^d_*(\mu \backslash \lambda, \bar{p}) + \hat{R}^d_*(\lambda \backslash \mu, \bar{p})$.

(ii) $\check{R}^h(\lambda, \bar{p}') \geq \hat{R}^h(\mu, \bar{p}) - \hat{R}^h(\mu \backslash \lambda, \bar{p}) + \check{R}^h(\lambda \backslash \mu, \bar{p})$.

(iii) $\hat{R}^h(\lambda, \bar{p}') \leq \hat{R}^h(\lambda \vee \mu, \bar{p}) - \hat{R}^h(\mu, \bar{p}) + \hat{R}^h(\lambda \wedge \mu, \bar{p})$.

(iv) $\check{R}^d_*(\lambda, \bar{p}') \geq \check{R}^d_*(\lambda \vee \mu, \bar{p}) - \check{R}^d_*(\mu, \bar{p}) + \check{R}^d_*(\lambda \wedge \mu, \bar{p})$.

Proof. For simplicity, we write ϵ instead of $\epsilon(\mu)$.
Proof of (i): Suppose not, i.e.,

$$\hat{R}^d_*(\lambda, \bar{p}') > \check{R}^d_*(\mu, \bar{p}) - \check{R}^d_*(\mu \backslash \lambda, \bar{p}) + \hat{R}^d_*(\lambda \backslash \mu, \bar{p}).$$

Since $\check{R}^d_*(\mu, \bar{p}) \geq \check{R}^d_*(\mu \backslash \lambda, \bar{p})$, for the above inequality to hold, we must have one of the following two cases:

Case 1: $\hat{R}^d_*(\lambda, \bar{p}') = 1$, $\check{R}^d_*(\mu, \bar{p}) = 1$, $\check{R}^d_*(\mu \backslash \lambda, \bar{p}) = 1$, and $\hat{R}^d_*(\lambda \backslash \mu, \bar{p}) = 0$.

Case 2: $\hat{R}^d_*(\lambda, \bar{p}') = 1$, $\check{R}^d_*(\mu, \bar{p}) = 0$, $\check{R}^d_*(\mu \backslash \lambda, \bar{p}) = 0$, and $\hat{R}^d_*(\lambda \backslash \mu, \bar{p}) = 0$.

We consider each case and derive a contradiction.
Case 1: By $\check{R}^d_*(\mu\backslash\lambda, \bar{p}) = 1$, there exists $\alpha \in X^d_*(\bar{p})$ with $\alpha \cdot \lambda = 0$. By (4) and (10),[11] we have $\alpha \in X^d_*(\bar{p}')$. This implies that $\hat{R}^d_*(\lambda, \bar{p}') = 0$, a contradiction to $\hat{R}^d_*(\lambda, \bar{p}') = 1$.
Case 2: By $\check{R}^d_*(\mu, \bar{p}) = 0$ and $\hat{R}^d_*(\lambda\backslash\mu, \bar{p}) = 0$, there exists $\alpha \in X^d_*(\bar{p})$ such that $\alpha \cdot \lambda \vee \mu = 0$. By $\check{R}^d_*(\mu, \bar{p}) = 0$, we have $\beta \cdot \mu = 0$ for all $\beta \in X^d_*(\bar{p})$. Together with (4), we obtain $\alpha \in X^d_*(\bar{p}')$. This implies that $\hat{R}^d_*(\lambda, \bar{p}') = 0$, a contradiction to $\hat{R}^d_*(\lambda, \bar{p}') = 1$.
Proof of (ii): Let $\alpha \in \check{Q}^h(\lambda, \bar{p}')$; equivalently, $\alpha \in X^h(\bar{p}')$ and $\check{R}^h(\lambda, \bar{p}') = \alpha \cdot \lambda$. Then, it suffices to prove that

$$\begin{aligned}\alpha \cdot \lambda &= \alpha \cdot \mu - \alpha \cdot \mu\backslash\lambda + \alpha \cdot \lambda\backslash\mu \\ &\geq \hat{R}^h(\mu, \bar{p}) - \hat{R}^h(\mu\backslash\lambda, \bar{p}) + \check{R}^h(\lambda\backslash\mu, \bar{p}).\end{aligned} \tag{15}$$

By $\alpha \in X^h(\bar{p}')$ and (4), we have $\alpha \in X^h(\bar{p})$. Together with the definition of $\hat{R}^h(\mu, \bar{p})$, we have $\alpha \cdot \mu \geq \hat{R}^h(\mu, \bar{p})$. Hence, for (15) to hold, it suffices to prove that

$$\alpha \cdot \mu\backslash\lambda \leq \hat{R}^h(\mu\backslash\lambda, \bar{p}), \tag{16}$$
$$\alpha \cdot \lambda\backslash\mu \geq \check{R}^h(\lambda\backslash\mu, \bar{p}). \tag{17}$$

Proof of (16): Suppose to the contrary that $\alpha \cdot \mu\backslash\lambda > \hat{R}^h(\mu\backslash\lambda, \bar{p})$. Let $\beta \in \hat{Q}^h(\mu\backslash\lambda, \bar{p})$ be such that

$$|\beta - \alpha| \leq |\beta' - \alpha| \text{ for all } \beta' \in \hat{Q}^h(\mu\backslash\lambda, \bar{p}). \tag{18}$$

By the supposition, there exists $\omega \in \mathrm{supp}^+(\alpha - \beta) \cap \mathrm{supp}^+\mu\backslash\lambda$. By M$^\natural$-concavity, there exists $\psi \in \mathrm{supp}^-(\alpha - \beta) \cup \{0\}$ such that

$$\begin{aligned} u^h(\alpha) + u^h(\beta) &\leq u^h(\alpha - \mathbb{1}^\omega + \mathbb{1}^\psi) + u^h(\beta + \mathbb{1}^\omega - \mathbb{1}^\psi), \\ u^h[\bar{p}'](\alpha) + u^h[\bar{p}](\beta) &\leq u^h[\bar{p}'](\alpha - \mathbb{1}^\omega + \mathbb{1}^\psi) + u^h[\bar{p}](\beta + \mathbb{1}^\omega - \mathbb{1}^\psi) - \bar{p}'_\omega + \bar{p}'_\psi + \bar{p}_\omega - \bar{p}_\psi. \end{aligned} \tag{19}$$

Since $\omega \in \mathrm{supp}^+\mu\backslash\lambda$, we have

$$\bar{p}_\omega - \bar{p}'_\omega = -\epsilon. \tag{20}$$

By the definition of $\bar{p}'$,

$$-\bar{p}_\psi + \bar{p}'_\psi \leq \epsilon. \tag{21}$$

[11] We need (10) to complete the logic here. To see this point, suppose that $X^d_*(\bar{p}) = \{\mathbb{1}^\omega, \mathbb{1}^\psi\}$, where $(\lambda \wedge \mu)_\omega = 1$ and $(\mu\backslash\lambda)_\psi = 1$. Then, without (10), increasing the prices in $\mathrm{supp}^+\mu$ might entail $\mathbb{1}^\omega \in X^d_*(\bar{p}')$ and $\mathbb{1}^\psi \notin X^d_*(\bar{p}')$. Then, we have $\hat{R}^d_*(\lambda, \bar{p}') = 1$, which cannot yield a contracition.

Since $\alpha \in X^h(\bar{p}')$ and $\beta \in X^h(\bar{p})$, we have

$$u^h[\bar{p}'](\alpha) \geq u^h[\bar{p}'](\alpha - \mathbb{1}^\omega + \mathbb{1}^\psi),\ u^h[\bar{p}](\beta) \geq u^h[\bar{p}](\beta + \mathbb{1}^\omega - \mathbb{1}^\psi). \tag{22}$$

(19)-(22) together imply that all the inequalities reduce to equalities. In particular, by (22),

$$\alpha - \mathbb{1}^\omega + \mathbb{1}^\psi \in X^h(\bar{p}'),\ \beta + \mathbb{1}^\omega - \mathbb{1}^\psi \in X^h(\bar{p}).$$

If $\psi \in \text{supp}^+\mu\backslash\lambda$, then $\beta + \mathbb{1}^\omega - \mathbb{1}^\psi \in \hat{Q}^h(\mu\backslash\lambda, \bar{p})$ and $|\beta - \alpha| > |(\beta + \mathbb{1}^\omega - \mathbb{1}^\psi) - \alpha|$, a contradiction to (18). Hence, $\psi \notin \text{supp}^+\mu\backslash\lambda$. Since $\mu_\psi = 1$ by (21), we must have $\lambda_\psi = 1$. Since $\omega \in \text{supp}^+\mu\backslash\lambda$, we have $\lambda_\omega = 0$. Then,

$$(\alpha - \mathbb{1}^\omega + \mathbb{1}^\psi) \cdot \lambda > \alpha \cdot \lambda,$$

a contradiction to $\alpha \in \check{Q}^h(\lambda, \bar{p}')$.

Proof of (17): Suppose to the contrary that $\alpha \cdot \lambda\backslash\mu < \check{R}^h(\lambda\backslash\mu, \bar{p})$. Let $\beta \in \check{Q}^h(\lambda\backslash\mu, \bar{p})$ be such that

$$|\beta - \alpha| \leq |\beta' - \alpha| \text{ for all } \beta' \in \check{Q}^h(\lambda\backslash\mu, \bar{p}). \tag{23}$$

By the supposition, there exists $\omega \in \text{supp}^+(\beta - \alpha) \cap \text{supp}^+\lambda\backslash\mu$. By M$^\natural$-concavity, there exists $\psi \in \text{supp}^-(\beta - \alpha) \cup \{0\}$ such that

$$\begin{aligned} u^h(\beta) + u^h(\alpha) &\leq u^h(\beta - \mathbb{1}^\omega + \mathbb{1}^\psi) + u^h(\alpha + \mathbb{1}^\omega - \mathbb{1}^\psi), \\ u^h[\bar{p}](\beta) + u^h[\bar{p}'](\alpha) &\leq u^h[\bar{p}](\beta - \mathbb{1}^\omega + \mathbb{1}^\psi) + u^h[\bar{p}'](\alpha + \mathbb{1}^\omega - \mathbb{1}^\psi) - \bar{p}_\omega + \bar{p}_\psi + \bar{p}'_\omega - \bar{p}'_\psi. \end{aligned} \tag{24}$$

Since $\omega \in \text{supp}^+\lambda\backslash\mu$, we have

$$\bar{p}'_\omega - \bar{p}_\omega = 0. \tag{25}$$

By the definition of $\bar{p}'$,

$$-\bar{p}'_\psi + \bar{p}_\psi \leq 0, \tag{26}$$

Since $\beta \in X^h(\bar{p})$ and $\alpha \in X^h(\bar{p}')$, we have

$$u^h[\bar{p}](\beta) \geq u^h[\bar{p}](\beta - \mathbb{1}^\omega + \mathbb{1}^\psi),\ u^h[\bar{p}'](\alpha) \geq u^h[\bar{p}'](\alpha + \mathbb{1}^\omega - \mathbb{1}^\psi). \tag{27}$$

(24)-(27) together imply that all the inequalities reduce to equalities. In particular, by (27),

$$\beta - \mathbb{1}^{\omega} + \mathbb{1}^{\psi} \in X^h(\bar{p}),\ \alpha + \mathbb{1}^{\omega} - \mathbb{1}^{\psi} \in X^h(\bar{p}').$$

If $\psi \in \mathrm{supp}^+\lambda\backslash\mu$, then $\beta - \mathbb{1}^{\omega} + \mathbb{1}^{\psi} \in \check{Q}^h(\lambda\backslash\mu, \bar{p})$ and $|\beta - \alpha| > |(\beta - \mathbb{1}^{\omega} + \mathbb{1}^{\psi}) - \alpha|$, a contradiction to (23). Hence, $\psi \notin \mathrm{supp}^+\lambda\backslash\mu$. Since $\mu_\psi = 0$ by (26), we must have $\lambda_\psi = 0$. Since $\omega \in \mathrm{supp}^+\lambda\backslash\mu$, we have $\lambda_\omega = 1$. Then,

$$(\alpha + \mathbb{1}^{\omega} - \mathbb{1}^{\psi}) \cdot \lambda > \alpha \cdot \lambda,$$

a contradiction to $\alpha \in \check{Q}^h(\lambda, \bar{p}')$.
Proof of (iii): Let $\alpha \in \tilde{\Omega}^h$ be such that

$$\alpha \in \hat{Q}^h(\lambda \vee \mu, \bar{p}), \tag{28}$$
$$\alpha \cdot \mu \le \alpha' \cdot \mu \text{ for all } \alpha' \in \hat{Q}^h(\lambda \vee \mu, \bar{p}), \tag{29}$$
$$\alpha \cdot \lambda \wedge \mu \le \alpha' \cdot \lambda \wedge \mu \text{ for all } \alpha' \in \hat{Q}^h(\lambda \vee \mu, \bar{p}) \text{ with } \alpha' \cdot \mu = \alpha \cdot \mu. \tag{30}$$

To prove the claim, it suffices to prove that

$$\alpha \cdot \lambda \ge \hat{R}^h(\lambda, \bar{p}'), \tag{31}$$
$$\alpha \cdot \lambda \le \hat{R}^h(\lambda \vee \mu, \bar{p}) - \hat{R}^h(\mu, \bar{p}) + \hat{R}^h(\lambda \wedge \mu, \bar{p}). \tag{32}$$

<u>Proof of (31):</u> In view of Claim 2, it suffices to prove that $\alpha \in \hat{Q}^h(\mu, \bar{p})$. Suppose not. Let $\beta \in \hat{Q}^h(\mu, \bar{p})$ be such that

$$|\beta - \alpha| \le |\beta' - \alpha| \text{ for all } \beta' \in \hat{Q}^h(\mu, \bar{p}). \tag{33}$$

By $\alpha \notin \hat{Q}^h(\mu, \bar{p})$, we have $\alpha \cdot \mu > \beta \cdot \mu$. Hence, there exists $\omega \in \mathrm{supp}^+(\alpha - \beta) \cap \mathrm{supp}^+\mu$. By M$^\natural$-convexity, there exists $\psi \in \mathrm{supp}^-(\alpha - \beta) \cup \{0\}$ such that

$$\alpha - \mathbb{1}^{\omega} + \mathbb{1}^{\psi} \in X^h(\bar{p}),\ \beta + \mathbb{1}^{\omega} - \mathbb{1}^{\psi} \in X^h(\bar{p}).$$

If $\psi \in \mathrm{supp}^+\mu$, then $\beta + \mathbb{1}^{\omega} - \mathbb{1}^{\psi} \in \hat{Q}^h(\mu, \bar{p})$ and

$$|(\beta + \mathbb{1}^{\omega} - \mathbb{1}^{\psi}) - \alpha| < |\beta - \alpha|,$$

a contradiction to (33). Hence, we have $\psi \notin \mathrm{supp}^+\mu$. If $\psi \notin \mathrm{supp}^+\lambda$, together with

$\omega \in \text{supp}^+\mu$, we have

$$(\alpha - \mathbb{1}^\omega + \mathbb{1}^\psi) \cdot \lambda \vee \mu < \alpha \cdot \lambda \vee \mu,$$

a contradiction to (28). Hence, $\psi \in \text{supp}^+\lambda \backslash \mu$. Then, together with $\omega \in \text{supp}^+\mu$, we have $\alpha - \mathbb{1}^\omega + \mathbb{1}^\psi \in \hat{Q}^h(\lambda \vee \mu, \bar{p})$ and

$$(\alpha - \mathbb{1}^\omega + \mathbb{1}^\psi) \cdot \mu < \alpha \cdot \mu,$$

a contradiction to (29).
Proof of (32): Suppose not, i.e.,

$$\begin{aligned} \alpha \cdot \lambda &= \alpha \cdot \lambda \vee \mu - \alpha \cdot \mu + \alpha \cdot \lambda \wedge \mu \\ &> \hat{R}^h(\lambda \vee \mu, \bar{p}) - \hat{R}^h(\mu, \bar{p}) + \hat{R}^h(\lambda \wedge \mu, \bar{p}). \end{aligned} \tag{34}$$

By (28), $\alpha \cdot \lambda \vee \mu = \hat{R}^h(\lambda \vee \mu, \bar{p})$. By the definition of $\hat{R}^h(\cdot, \cdot)$, we have $\alpha \cdot \mu \geq \hat{R}^h(\mu, \bar{p})$. Hence, for (34) to hold, we must have $\alpha \cdot \lambda \wedge \mu > \hat{R}^h(\lambda \wedge \mu, \bar{p})$. Let $\beta \in \hat{Q}^h(\lambda \wedge \mu, \bar{p})$ be such that

$$|\beta - \alpha| \leq |\beta' - \alpha| \text{ for all } \beta' \in \hat{Q}^h(\lambda \wedge \mu, \bar{p}). \tag{35}$$

Since $\alpha \cdot \lambda \wedge \mu > \beta \cdot \lambda \wedge \mu$, there exists $\omega \in \text{supp}^+(\alpha - \beta) \cap \text{supp}^+\lambda \wedge \mu$. By M$^\natural$-convexity, there exists $\psi \in \text{supp}^-(\alpha - \beta) \cup \{0\}$ such that

$$\alpha - \mathbb{1}^\omega + \mathbb{1}^\psi \in X^h(\bar{p}), \ \beta + \mathbb{1}^\omega - \mathbb{1}^\psi \in X^h(\bar{p}).$$

If $\psi \in \text{supp}^+\lambda \wedge \mu$, then $\beta + \mathbb{1}^\omega - \mathbb{1}^\psi \in \hat{Q}^h(\lambda \wedge \mu, \bar{p})$ and

$$|(\beta + \mathbb{1}^\omega - \mathbb{1}^\psi) - \alpha| < |\beta - \alpha|,$$

a contradiction to (35). Hence, $\psi \notin \text{supp}^+\lambda \wedge \mu$. If $\psi \notin \text{supp}^+\lambda \vee \mu$, together with $\omega \in \text{supp}^+\lambda \wedge \mu$, we have

$$(\alpha - \mathbb{1}^\omega + \mathbb{1}^\psi) \cdot \lambda \vee \mu < \alpha \cdot \lambda \vee \mu,$$

a contradiction to (28). Hence, $\psi \in \text{supp}^+\lambda \vee \mu$, which together with $\omega \in \text{supp}^+\lambda \wedge \mu$ implies $\alpha - \mathbb{1}^\omega + \mathbb{1}^\psi \in \hat{Q}^h(\lambda \vee \mu, \bar{p})$. If $\psi \notin \text{supp}^+\mu$, together with $\omega \in \text{supp}^+\lambda \wedge \mu$, we have

$$(\alpha - \mathbb{1}^\omega + \mathbb{1}^\psi) \cdot \mu < \alpha \cdot \mu,$$

a contradiction to (29). Hence, the remaining possibility is that $\psi \in \mathrm{supp}^+\mu\backslash\lambda$. In this case, again together with $\omega \in \mathrm{supp}^+\lambda \wedge \mu$, we have

$$(\alpha - \mathbb{1}^\omega + \mathbb{1}^\psi) \cdot \mu = \alpha \cdot \mu,\ (\alpha - \mathbb{1}^\omega + \mathbb{1}^\psi) \cdot \lambda \wedge \mu < \alpha \cdot \lambda \wedge \mu,$$

a contradiction to (30).

Proof of (iv): Suppose not, i.e.,

$$\check{R}^d_*(\lambda, \bar{p}') < \check{R}^d_*(\lambda \vee \mu, \bar{p}) - \check{R}^d_*(\mu, \bar{p}) + \check{R}^d_*(\lambda \wedge \mu, \bar{p}).$$

Since $\check{R}^d_*(\mu, \bar{p}) \geq \check{R}^d_*(\lambda \wedge \mu, \bar{p})$, for the above inequality to hold, we must have one of the following two cases:

Case 1: $\check{R}^d_*(\lambda, \bar{p}') = 0$, $\check{R}^d_*(\lambda \vee \mu, \bar{p}) = 1$, $\check{R}^d_*(\mu, \bar{p}) = 1$, and $\check{R}^d_*(\lambda \wedge \mu, \bar{p}) = 1$.

Case 2: $\check{R}^d_*(\lambda, \bar{p}') = 0$, $\check{R}^d_*(\lambda \vee \mu, \bar{p}) = 1$, $\check{R}^d_*(\mu, \bar{p}) = 0$, and $\check{R}^d_*(\lambda \vee \mu, \bar{p}) = 0$.

We consider each case and derive a contradiction.

Case 1: By $\check{R}^d_*(\lambda \wedge \mu, \bar{p}) = 1$, there exists $\alpha \in X^d_*(\bar{p})$ with $\alpha \cdot \lambda = 1$. By (4) and (10),[12] we have $\alpha \in X^d_*(\bar{p}')$. This implies that $\check{R}^d_*(\lambda, \bar{p}') = 1$, a contradiction to $\check{R}^d_*(\lambda, \bar{p}') = 0$.

Case 2: By $\check{R}^d_*(\lambda \vee \mu, \bar{p}) = 1$ and $\check{R}^d_*(\mu, \bar{p}) = 0$, there exists $\alpha \in X^d_*(\bar{p})$ such that $\alpha \cdot \lambda\backslash\mu = 1$. By $\check{R}^d_*(\mu, \bar{p}) = 0$, we have $\beta \cdot \mu = 0$ for all $\beta \in X^d_*(\bar{p})$. Together with (4), we obtain $\alpha \in X^d_*(\bar{p}')$. This implies that $\check{R}^d_*(\lambda, \bar{p}') = 1$, a contradiction to $\check{R}^d_*(\lambda, \bar{p}') = 0$. □

We resume the proof of Theorem 4 for $\mathcal{E}_*$. Suppose to the contrary that there exists $\mu \in \tilde{\Omega}$ such that $\mu \neq \mathbf{0}$ and

$$\sum_{h \in H} \hat{R}^h(\mu, \bar{p}) \geq \sum_{d \in D} \check{R}^d_*(\mu, \bar{p}).$$

Together with Theorem 3 for $\mathcal{E}_*$,

$$\sum_{h \in H} \hat{R}^h(\mu, \bar{p}) = \sum_{d \in D} \check{R}^d_*(\mu, \bar{p}). \tag{36}$$

Let $\bar{p}' \equiv \bar{p} + \epsilon(\mu) \cdot \mu$. In view of the maximality of $\bar{p}$, it suffices to prove that

$$\bar{p}' \text{ is an equilibrium price vector for } \mathcal{E}_*. \tag{37}$$

[12] We need (10) to complete the logic here. To see this point, suppose that $X^d_*(\bar{p}) = \{\mathbb{1}^\omega, \mathbb{1}^\psi\}$, where $(\lambda \wedge \mu)_\omega = 1$ and $(\mu\backslash\lambda)_\psi = 1$. Then, without (10), increasing the prices in $\mathrm{supp}^+\mu$ might entail $\mathbb{1}^\omega \notin X^d_*(\bar{p}')$ and $\mathbb{1}^\psi \in X^d_*(\bar{p}')$. Then, we have $\check{R}^d_*(\lambda, \bar{p}') = 0$, which cannot yield a contracition.

In view of Theorem 3 for $\mathcal{E}_*$, to prove (37), it suffices to prove that, for an arbitrarily chosen $\lambda \in \tilde{\Omega}$,

$$\sum_{h\in H} \hat{R}^h(\lambda, \bar{p}') \leq \sum_{d\in D} \check{R}^d_*(\lambda, \bar{p}'), \text{ and} \tag{38}$$

$$\sum_{d\in D} \hat{R}^d_*(\lambda, \bar{p}') \leq \sum_{h\in H} \check{R}^h(\lambda, \bar{p}'). \tag{39}$$

Proof of (38):

$$\begin{aligned}
&\sum_{h\in H} \hat{R}^h(\mu, \bar{p}) + \sum_{h\in H} \hat{R}^h(\lambda, \bar{p}') - \sum_{h\in H} \hat{R}^h(\lambda \wedge \mu, \bar{p}) \\
\leq &\sum_{h\in H} \hat{R}^h(\lambda \vee \mu, \bar{p}) \\
\leq &\sum_{d\in D} \check{R}^d_*(\lambda \vee \mu, \bar{p}) \\
\leq &\sum_{d\in D} \check{R}^d_*(\mu, \bar{p}) + \sum_{d\in D} \check{R}^d_*(\lambda, \bar{p}') - \sum_{d\in D} \check{R}^d_*(\lambda \wedge \mu, \bar{p}) \\
= &\sum_{h\in H} \hat{R}^h(\mu, \bar{p}) + \sum_{d\in D} \check{R}^d_*(\lambda, \bar{p}') - \sum_{d\in D} \check{R}^d_*(\lambda \wedge \mu, \bar{p}),
\end{aligned}$$

where the first inequality follows from Claim 3(iii), the second inequality follows from Theorem 3 for $\mathcal{E}_*$, the third inequality follows from Claim 3(iv), and the equality follows from (36). The above inequality implies

$$\sum_{h\in H} \hat{R}^h(\lambda, \bar{p}') - \sum_{h\in H} \hat{R}^h(\lambda \wedge \mu, \bar{p}) \leq \sum_{d\in D} \check{R}^d_*(\lambda, \bar{p}') - \sum_{d\in D} \check{R}^d_*(\lambda \wedge \mu, \bar{p}). \tag{40}$$

By Theorem 3 for $\mathcal{E}_*$,

$$\sum_{h\in H} \hat{R}^h(\lambda \wedge \mu, \bar{p}) \leq \sum_{d\in D} \check{R}^d_*(\lambda \wedge \mu, \bar{p}). \tag{41}$$

(40) and (41) imply (38).

Proof of (39):

$$\begin{aligned}\sum_{h\in H}\check{R}^h(\lambda\backslash\mu,\bar{p}) &\leq \sum_{h\in H}\check{R}^h(\lambda,\bar{p}')-\sum_{h\in H}\hat{R}^h(\mu,\bar{p})+\sum_{h\in H}\hat{R}^h(\mu\backslash\lambda,\bar{p})\\ &=\sum_{h\in H}\check{R}^h(\lambda,\bar{p}')-\sum_{d\in D}\check{R}^d_*(\mu,\bar{p})+\sum_{h\in H}\hat{R}^h(\mu\backslash\lambda,\bar{p})\\ &\leq\sum_{h\in H}\check{R}^h(\lambda,\bar{p}')-\sum_{d\in D}\check{R}^d_*(\mu,\bar{p})+\sum_{d\in D}\check{R}^d_*(\mu\backslash\lambda,\bar{p})\\ &\leq\sum_{h\in H}\check{R}^h(\lambda,\bar{p}')+\sum_{d\in D}\hat{R}^d_*(\lambda\backslash\mu,\bar{p})-\sum_{d\in D}\hat{R}^d_*(\lambda,\bar{p}'),\end{aligned}$$

where the first inequality follows from Claim 3(ii), the equality follows from (36), the second inequality follows from Theorem 3 for $\mathcal{E}_*$, and the last inequality follows from Claim 3(i). The above inequality implies

$$\sum_{h\in H}\check{R}^h(\lambda\backslash\mu,\bar{p})+\sum_{d\in D}\hat{R}^d_*(\lambda,\bar{p}')\leq\sum_{h\in H}\check{R}^h(\lambda,\bar{p}')+\sum_{d\in D}\hat{R}^d_*(\lambda\backslash\mu,\bar{p}). \tag{42}$$

By Theorem 3 for $\mathcal{E}_*$,

$$\sum_{h\in H}\check{R}^h(\lambda\backslash\mu,\bar{p})\geq\sum_{d\in D}\hat{R}^d_*(\lambda\backslash\mu,\bar{p}). \tag{43}$$

(42) and (43) imply (39). □

5.2.3 Proof of Theorem 4 for $\mathcal{E}$

By (8), for any $d\in D$ and $\lambda\in\tilde{\Omega}$, we have

$$\check{R}^d_*(\lambda,\bar{p})=\check{R}^d(\lambda,\bar{p}).$$

As proven in Subsection 5.2.2, Theorem 4 holds for $\mathcal{E}_*$. Together with the above equation, we obtain the desired claim. □

5.3 Proof of Theorem 5

As $\overline{p}$ is an equilibrium price vector for $\mathcal{E}$, there exists a corresponding equilibrium allocation $\alpha\in\tilde{\Omega}$. If $\alpha^{d'}=\mathbf{0}$, then we immediately conclude that $\overline{p}_{-d'}$ is an equilibrium price vector for $\mathcal{E}_{-d'}$. Thus, in the remaining part, we assume $\alpha^{d'}\neq\mathbf{0}$, i.e., there exists $h'\in H$ such that $\alpha^{d'}_{(d',h')}=1$. Let $\omega'\equiv(d',h')$, $\epsilon\equiv\epsilon(\mathbb{1}^{\omega'})$, and $\overline{p}'\equiv\overline{p}+\epsilon\cdot\mathbb{1}^{\omega'}$.[13]

[13] See (4) for the definition of $\epsilon(\cdot)$.

Claim 4. $\alpha^{h'} \notin X^{h'}(\overline{p}')$.

Proof. At $\bar{p}'$, only the price of ω' increases from $\bar{p}$. Together with $\alpha^{d'}_{\omega'} = 1$ and monotonicty of utility in money, we have $\alpha^{d'} \in X^{d'}(\overline{p}')$. Moreover, as the change in $\overline{p}_{\omega'}$ does not affect the demand/supply correspondences of the agents except d' and h', we obtain $\alpha^i \in X^i(\overline{p}')$ for all $i \in I\backslash\{d', h'\}$. Thus, if $\alpha^{h'} \in X^{h'}(\overline{p}')$, then $\overline{p}'$ is an equilibrium price vector for $\mathcal{E}$ with the corresponding equilibrium allocation α, a contradiction to the maximality of $\bar{p}$. Hence, we must have $\alpha^{h'} \notin X^{h'}(\overline{p}')$. $\square$

Claim 5. *There exists* $\beta \in X^{h'}(\overline{p})$ *such that* $\beta_{\omega'} = 0$.

Proof. Suppose not, i.e., suppose $\beta_{\omega'} = 1$ for all $\beta \in X^{h'}(\overline{p})$. This implies

$$v^{h'}[\overline{p}'](\alpha^{h'}) = v^{h'}[\overline{p}'](\beta) \text{ for all } \beta \in X^{h'}(\overline{p}).$$

Moreover, by (4), we have $X^h(\bar{p}') \subseteq X^h(\bar{p})$. Together with thet above equation, we obtain $\alpha^{h'} \in X^{h'}(\overline{p}')$, a contradiction to Claim 4. $\square$

Claim 6. *For any* $h \in H$*, there exists* $\beta \in X^h(\overline{p})$ *such that* $\beta_{(d',h)} = 0$.

Proof. For any $h \in H\backslash\{h'\}$, by $\alpha^{d'}_{\omega'} = \alpha^{h'}_{\omega'} = 1$, we have $\alpha^h_{(d',h)} = 0$. Since $\alpha^h \in X^h(\overline{p})$, the claim holds for $h \in H\backslash\{h'\}$. Together with Claim 5, we obtain the desired condition. $\square$

Claim 7. *Let* $\mu \in \tilde{\Omega}$ *be such that* $\mu_{(d',h)} = 0$ *for all* $h \in H$*. Then, for any* $h \in H$*,*

$$\check{R}^h_{-d'}(\mu_{-d'}, \overline{p}_{-d'}) \geq \check{R}^h(\mu, \overline{p}).$$

Proof. Let $h \in H$ be arbitrarily chosen. Let $\psi \equiv (d', h)$ and $\beta \in \check{Q}^h(\mu, \overline{p})$ be such that

$$\beta_\psi \leq \beta'_\psi \text{ for all } \beta' \in \check{Q}^h(\mu, \overline{p}). \tag{44}$$

Suppose to the contrary that $\beta_\psi = 1$. By Claim 6, there exists $\beta' \in X^h(\overline{p})$ such that $\beta'_\psi = 0$. By M$^\natural$-convexity, for $\psi \in \mathrm{supp}^+(\beta - \beta')$, there exists $\psi' \in \mathrm{supp}^-(\beta - \beta') \cup \{0\}$ such that

$$\beta - \mathbb{1}^\psi + \mathbb{1}^{\psi'} \in X^h(\overline{p}),\ \beta' + \mathbb{1}^\psi - \mathbb{1}^{\psi'} \in X^h(\overline{p}).$$

Let $\beta^* \equiv \beta - \mathbb{1}^\psi + \mathbb{1}^{\psi'}$. Since $\mu_\psi = 0$, we have $\beta^* \cdot \mu \geq \beta \cdot \mu$, which implies $\beta^* \in \check{Q}^h(\mu, \overline{p})$. Moreover, by the choice of ψ', we have $\psi' \neq \psi$. Then, $\beta^*_\psi = 0 < 1 = \beta_\psi$, a contradiction to (44). Hence, $\beta_\psi = 0$. This implies $\beta_{-d'} \in X^h_{-d'}(\overline{p}_{-d'})$ and $\beta_{-d'} \cdot \mu_{-d'} = \beta \cdot \mu = \check{R}^h(\mu, \overline{p})$. Together with the definition of $\check{R}^h_{-d'}(\cdot, \cdot)$, we obtain the desired condition. $\square$

Claim 8. $\sum_{d \in D\backslash\{d'\}} \hat{R}^d_{-d'}(\lambda, \overline{p}_{-d'}) \leq \sum_{h \in H} \check{R}^h_{-d'}(\lambda, \overline{p}_{-d'})$ *for all* $\lambda \in \tilde{\Omega}_{-d'}$.

Proof. Suppose to the contrary that there exists $\lambda \in \tilde{\Omega}_{-d'}$ such that

$$\sum_{d \in D \backslash \{d'\}} \hat{R}^d_{-d'}(\lambda, \overline{p}_{-d'}) > \sum_{h \in H} \check{R}^h_{-d'}(\lambda, \overline{p}_{-d'}). \tag{45}$$

We define $\mu \in \tilde{\Omega}$ by $\mu_\omega = \lambda_\omega$ for all $\omega \in (D \backslash \{d'\}) \times H$ and $\mu_\omega = 0$ for all $\omega \in \{d'\} \times H$; note that $\mu_{-d'} = \lambda$. Then, by (5),

$$\hat{R}^d_{-d'}(\lambda, \overline{p}_{-d'}) = \hat{R}^d(\mu, \overline{p}) \text{ for all } d \in D \backslash \{d'\}. \tag{46}$$

By $\mu_\omega = 0$ for all $\omega \in \{d'\} \times H$,

$$\hat{R}^{d'}(\mu, \overline{p}) = 0. \tag{47}$$

By Claim 7,

$$\check{R}^h_{-d'}(\lambda, \overline{p}_{-d'}) \geq \check{R}^h(\mu, \overline{p}) \text{ for all } h \in H. \tag{48}$$

By (45)-(48), we obtain

$$\sum_{d \in D} \hat{R}^d(\mu, \overline{p}) > \sum_{h \in H} \check{R}^h(\mu, \overline{p}),$$

which is a contradiction to Theorem 3. □

Claim 9. *$\sum_{h \in H} \hat{R}^h_{-d'}(\lambda, \overline{p}_{-d'}) \leq \sum_{d \in D \backslash \{d'\}} \check{R}^d_{-d'}(\lambda, \overline{p}_{-w'})$ for all $\lambda \in \tilde{\Omega}_{-d'}$.*

Proof. Let $\lambda \in \tilde{\Omega}_{-d'}$. Define $\mu \in \tilde{\Omega}$ by $\mu_\omega = \lambda_\omega$ for all $\omega \in (D \backslash \{d'\}) \times H$ and $\mu_\omega = 1$ for all $\omega \in \{d'\} \times H$; note that $\mu_{-d'} = \lambda$. By Theorem 4,

$$\sum_{h \in H} \hat{R}^h(\mu, \overline{p}) \leq \sum_{d \in D} \check{R}^d(\mu, \overline{p}) - 1. \tag{49}$$

By (6),

$$\check{R}^d_{-d'}(\lambda, \overline{p}_{-d'}) = \check{R}^d(\mu, \overline{p}) \text{ for all } d \in D \backslash \{d'\}.$$

By $\alpha^{d'} \neq \mathbf{0}$ and $\mu_\omega = 1$ for all $\omega \in \{d'\} \times H$,

$$\check{R}^{d'}(\mu, \overline{p}) = 1.$$

The above two equations imply

$$\sum_{d\in D}\check{R}^d(\mu,\overline{p})-1=\sum_{d\in D\setminus\{d'\}}\check{R}^d_{-d'}(\lambda,\overline{p}_{-d'}).$$

Bearing this equation and (49) in mind, to prove the desired inequality, it suffices to prove that

$$\sum_{h\in H}\hat{R}^h_{-d'}(\lambda,\overline{p}_{-d'})\leq\sum_{h\in H}\hat{R}^h(\mu,\overline{p}).$$

Let $h\in H$ be arbitrarily chosen. To prove the above inequality, it suffices to prove that

$$\hat{R}^h_{-d'}(\lambda,\overline{p}_{-d'})\leq\hat{R}^h(\mu,\overline{p}). \tag{50}$$

<u>Proof of (50)</u>: Let $\psi\equiv(d',h)$. Let $\beta\in\hat{Q}^h(\mu,\overline{p})$ be such that.

$$\beta_\psi\leq\beta'_\psi\text{ for all }\beta'\in\hat{Q}^h(\mu,\overline{p}). \tag{51}$$

Suppose to the contrary that $\beta_\psi=1$. By Claim 6, there exists $\beta'\in X^h(\overline{p})$ such that $\beta'_\psi=0$. By M$^\natural$-convexity, for $\psi\in\text{supp}^+(\beta-\beta')$, there exists $\psi'\in\text{supp}^-(\beta-\beta')\cup\{0\}$ such that

$$\beta-\mathbb{1}^\psi+\mathbb{1}^{\psi'}\in X^h(\overline{p}),\ \beta'+\mathbb{1}^\psi-\mathbb{1}^{\psi'}\in X^h(\overline{p}).$$

Let $\beta^*\equiv\beta-\mathbb{1}^\psi+\mathbb{1}^{\psi'}$. Since $\mu_\psi=1$, we have $\beta^*\cdot\mu\leq\beta\cdot\mu$, which implies $\beta^*\in\hat{Q}^h(\mu,\overline{p})$. Moreover, by the choice of ψ', we have $\psi'\neq\psi$. Then, $\beta^*_\psi=0<1=\beta_\psi$, a contradiction to (51). Hence, $\beta_\psi=0$. This implies $\beta_{-d'}\in X^h_{-d'}(\overline{p}_{-d'})$ and $\beta_{-d'}\cdot\mu_{-d'}=\beta\cdot\mu=\hat{R}^h(\mu,\overline{p})$. Together with the definition of $\hat{R}^h_{-d'}(\cdot,\cdot)$, we obtain (50). $\square$

Combining Corollary 1 with Claims 8 and 9, we conclude that $\overline{p}_{-d'}$ is an equilibrium price vector for $\mathcal{E}_{-d'}$. $\square$

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