Stable allocations and coalition structures for games with externalities

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1 Introduction

1.1 An overview of games with externalities

Cooperative game theory is one of the theoretical frameworks to study coalition formation and to analyze surplus allocation. Many of the traditional models of cooperative game theory consider the worth of a coalition as the surplus achieved by the members of the coalition without any help from the other players. Although this assumption has contributed to developing the theory of surplus allocations, it has implicitly ruled out many economic/social situations. Recent works attempt to understand more general cases: environments in which there is mutual influence between coalitions. Such mutual influence is commonly called *externalities* among coalitions. The concept of externalities includes a variety of interactions across coalitions. An illustrative example is a competition among firms. In a market consisting of many firms, we suppose that two firms are about to merge and form a large firm. This merger improves their technology and costs to produce commodities. Therefore, the merger benefits the two firms and can be thought of as a beneficial *internal* effect within the coalition of the firms. However, the merger also influences other firms in the market. As the theory of Bertrand competition typically describes, if the improved technology allows the large firm to produce commodities at lower costs, the firm has a cost advantage and must be more likely to win the price competition, which provides the winner with larger profits and may reduce the other firms' profits. Such an *external* influence to other firms is an example of externalities. Besides a competition of firms, the concept of externalities covers a wide range of economic and social topics, *e.g.*, local public goods, customs unions in international trade, pollution, and balance of power among political parties. A practical aspect of incorporating externalities in a cooperative game is to allow us to formulate and understand more various economic behavior and to enlarge the field of cooperative game theory. A cooperative game that admits externalities among coalitions is called a *game with externalities* or a *partition function form game*. In contrast, a game that does not incorporate externalities is called a *game without externalities* or a *coalition function form game*.

The term "externalities" might be an ambiguous word. As described above with the example of an oligopoly, a game with externalities is often derived from a non-cooperative game or an economic model. The externalities among coalitions exhibit a wide range of mutual dependences in those models. One is the externality of a *good* that is often employed in the sense of microeconomics: the effect of a good that causes outside the market mechanism. Spillover of local public goods and pollution of a river are typical examples. The externalities among coalitions can include this sense of external impact if the economic model deals with such a good. Moreover, the *mutual dependence of actions* in a strategic situation is also incorporated into a game with externalities. In the oligopolistic competition above, the good produced by the firms does not necessarily be a good with externalities. Instead of the good, the action of a player (namely, the price a firm sets in the example) affects the payoff of another player in the presence of the mutual dependence of actions. This direct influence among players is also included by the term "externalities" of a game with externalities. This reformulation of a strategic situation in the terms of non-cooperative games can simultaneously handle the issue of competition between coalitions and cooperation within each coalition which was hardly analyzed in the traditional models of cooperative game theory.

A game with externalities was originally introduced by Thrall (1961) and Thrall and Lucas (1963) as a generalization of a game without externalities. Initial attempts in this field were to propose how to allocate a surplus incorporating externalities. These attempts have been achieved by generalizing the Shapley value to games with externalities. Many researchers, Myerson (1977), Bolger (1989), Macho-Stadler et al. (2007), Albizuri et al. (2005), and de Clippel and Serrano (2008), have proposed their distribution concepts. In contrast to the intensive analysis on surplus allocation, few works have been devoted to stable allocations. Of course, in the class of games without externalities, many stability concepts are proposed and analyzed, *e.g.*, the core, the vNM stable set, and the bargaining set. However, extending these concepts to games with externalities is not straightforward. The main reason lies in the fact that these concepts are defined by the notion of deviation, and we can consider multiple definitions for a deviation of pla presence of externalities. To see this, we suppose that some players try to deviate from a large coalition (*.g.*, a large firm) by forming their own coalition (*e.g.*, their own business). In the presence of externalities, a *reaction* from the remaining players of the large coalition is crucial: the remaining players maybe try to "punish" the deviating players so as to reduce their profits; or they maybe behave cooperatively to derive profits even from the deviating players' business. In any case, because of externalities across coalitions, the reaction influences the profit of both sides of the remaining players and the deviating players. Therefore, deviating players should anticipate how the remaining pla . More specifically, a scheme for deviation must include (i) who is a member of the deviation, (ii) what reaction from the other players is expected, and (iii) how beneficial the deviation is. As we will elaborate in the following section, the second factor is the main reason of the multiplicity of deviation. To see this, we begin our discussion by formally defining these concepts.

1.2 Preliminaries

Let $\mathcal{N}$ be a set of all players and $N \subsetneq \mathcal{N}$ be a finite set of players. A *coalition* S is a nonempty subset of N. We denote the number of players in S by $|S|$. For any $S \subseteq N$, a *partition* of S is defined by $\{T_1, ..., T_h\}$ where $1 \leq h \leq |S|$, $T_i \cap T_j = \emptyset$ for $i, j = 1, ..., h$ $(i \neq j)$, $T_i \neq \emptyset$ for $i = 1, ..., h$ and $\bigcup_{i=1}^{h} T_i = S$. We typically use $\mathcal{P}$ or $\mathcal{Q}$ to denote a partition. Assume that the partition of the empty set $\emptyset$ is $\{\emptyset\}$. For any $S \subseteq N$, let $\Pi(S)$ be the set of all partitions of S. For any partition $\mathcal{P}$ and any coalition $S \subseteq N$, the *projection of* $\mathcal{P}$ *on* S is given by $\mathcal{P}|_S = \{S \cap C \mid C \in \mathcal{P},\ S \cap C \neq \emptyset\} \in \Pi(S)$.

An *embedded coalition* of N is a pair of nonempty coalition $S \subseteq N$ and partition $\mathcal{P} \in \Pi(N \setminus S)$. The set of all embedded coalitions of N is given by

$$EC(N) = \{(S, \mathcal{P}) \mid \emptyset \neq S \subseteq N,\ \mathcal{P} \in \Pi(N \setminus S)\}.$$

A *partition function* v assigns a real number to each embedded coalition, namely, $v : EC(N) \to \mathbb{R}$. A *partition function form game* (a *PFF game*) or a *game with externalities* is a pair (N, v). We denote the number of elements of $EC(N)$ by $|EC(N)|$.

For simplicity, we sometimes write $\mathcal{P} \ni S$ instead of $\mathcal{P} \in \Pi(N)$ with $S \in \mathcal{P}$ (if the player set is not N, say N', then $\mathcal{P} \in \Pi(N')$ with $S \in \mathcal{P}$). Similarly, we write $S \ni i$ instead of $S \subseteq N$ with $i \in S$. When there is no danger of confusion, we omit some parenthesis: for example, we write $v(12, \{34, 5\})$ instead of $v(\{1, 2\}, \{\{3, 4\}, \{5\}\})$.

A *coalition function form game* (a *CFF game*) or a *game without externalities* is a pair of the player set N and a characteristic function (also known as a coalitional function) w, namely (N, w). A characteristic function w assigns a real number to each coalition, namely, $w : 2^N \to \mathbb{R}$. Let $w(\emptyset) = 0$. Below, we basically fix N. By N we simply refer to the same player set for a game with externalities and a game without externalities.

We first define the core of a game without externalities. In the absence of externalities, the notion of core was initially proposed by Gillies (1959).

Definition 1.2.1. The core of a game without externalities w is given as

$$C(N, w) = \left\{ x \in \mathbb{R}^N \,\middle|\, \sum_{j \in S} x_j \geq w(S) \text{ for any } S \subseteq N, \sum_{j \in N} x_j = w(N) \right\}.$$

In the definition of the core, the worth of a coalition $w(S)$ plays an important role. When some players try to deviate from an allocation x by forming coalition S, they compare their current payoffs and the payoffs they obtain after the deviation. If the deviation is beneficial, *i.e.*, $\sum_{j \in S} x_j < w(S)$, then they have an incentive to deviate. In other words, the core is the set of allocations from which no players have an incentive to deviate. In contrast, as discussed in the previous section, if externalities exist, the worth of a deviating coalition also depends on a reaction of the non-deviating players and is given as a partition function $v(S, \mathcal{P})$. How can we generalize this notion to games with externalities? To precisely generalize the core, we first introduce an important notion called an expectation function.

The notion of expectation function was formulated by Bloch and van den Nouweland (2014). This concept allows us to generalize the core concept in an unified manner.

Definition 1.2.2. An *expectation function* ψ assigns a partition of N, $\psi(N, v, S) \in \Pi(N \setminus S)$, to a triple consisting of player set N, partition function v, and nonempty coalition $S \subseteq N$ so as to satisfy $S \in \psi(N, v, S)$.

In other words, for any coalition S, an expectation function chooses a partition of $N \setminus S$: $\psi(N, v, S) \in \Pi(N \setminus S)$.[1] We now introduce four important expectation functions. An expectation function is called

- the *optimistic* expectation if $\psi(N, v, S) \in \underset{\mathcal{P}' \in \Pi(N \setminus S)}{\arg\max}\, v(S, \mathcal{P}')$,
- the *pessimistic* expectation if $\psi(N, v, S) \in \underset{\mathcal{P}' \in \Pi(N \setminus S)}{\arg\min}\, v(S, \mathcal{P}')$,

[1] Dutta et al. (2010) formulate a similar notion as *restriction operators*. A restriction operator describes which player moves to which coalition.

- the *singleton*-expectation if $\psi(N, v, S) = \{\{i_{s+1}\}, ..., \{i_n\}\}$,
- the *merge*-expectation if $\psi(N, v, S) = \{N \setminus S\}$.

An expectation function describes an expectation rule for a reaction derived from a deviating coalition S. For example, the pessimistic expectation exhibits the anticipation that after a deviation, the remaining players reorganize their coalition structure to minimize the worth of the deviating players.

Now we define the core for games with externalities. First we define the set of efficient allocations for a game (N, v) as follows:

$$X(N, v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{j \in N} x_j = v(N, \{\emptyset\}) \right\}.$$

Definition 1.2.3. Let (N, v) be a game with externalities. Given an expectation function ψ, the *ψ-core* of game (N, v) is given by

$$C^{\psi}(N, v) = \left\{ x \in X(N, v) \;\middle|\; \sum_{j \in S} x_j \geq v(S, \psi(N, v, S)) \text{ for any } \emptyset \neq S \subseteq N \right\}.$$

Note that an expectation function ψ features the ψ-core. If ψ is the optimistic expectation, then the core is called the optimistic core C^{opt}. In the same manner, we can define the pessimistic core C^{pes}, the s-core C^{s}, and the m-core C^{m}.[2]

Below, we basically focus on these four types of cores. We first offer some basic (and important) properties of the cores.

Proposition 1.2.4. For any game (N, v) and any expectation function ψ, we have

$$C^{\text{opt}}(N, v) \subseteq C^{\psi}(N, v) \subseteq C^{\text{pes}}(N, v).$$

Proof. Let $x \in C^{\text{opt}}(N, v)$. From the definition of the optimistic core, it follows that $x \in X(N, v)$ and $\sum_{j \in S} x_j \geq \max_{\mathcal{P}' \in \Pi(N \setminus S)} v(S, \mathcal{P}')$ for any $\emptyset \neq S \subseteq N$. Hence, for any expectation function ψ, we have

$$\sum_{j \in S} x_j \geq \max_{\mathcal{P}' \in \Pi(N \setminus S)} v(S, \mathcal{P}') \geq v(S, \psi(N, v, S)) \geq \min_{\mathcal{P}' \in \Pi(N \setminus S)} v(S, \mathcal{P}').$$

This completes the proof. □

In words, the optimistic core is the smallest core and the pessimistic core is the largest core of all the ψ-cores. Moreover, conditions of externalities also order the other cores. To see this, we define some specific types of externalities.

Definition 1.2.5. A game with externalities (N, v) has *positive externalities* if for any mutually disjoint nonempty coalitions $S, T_1, T_2 \subseteq N$ and any partition $\mathcal{P}' \in \Pi(N \setminus (S \cup T_1 \cup T_2))$,

$$v(S, \{T_1 \cup T_2\} \cup \mathcal{P}') > v(S, \{T_1, T_2\} \cup \mathcal{P}').$$

Replacing the inequality by $<$, $\geq$, $\leq$, we can define *negative, nonnegative, nonpositive externalities* respectively.

[2] As for the terminology, we follow Hafalir (2007) and Abe (2018a).

Proposition 1.2.6. For any game (N, v), if v has nonnegative externalities, then

$$C^{\mathrm{opt}}(N, v) = C^{\mathrm{m}}(N, v) \subseteq C^{\mathrm{s}}(N, v) = C^{\mathrm{pes}}(N, v).$$

Similarly, if v has nonpositive externalities, then

$$C^{\mathrm{opt}}(N, v) = C^{\mathrm{s}}(N, v) \subseteq C^{\mathrm{m}}(N, v) = C^{\mathrm{pes}}(N, v).$$

Proof. We assume that v has nonnegative externalities. In view of the definition of nonnegative externalities, we have

$$\max_{\mathcal{P}' \in \Pi(N \setminus S)} v(S, \mathcal{P}') = v(S, \{S, N \setminus S\}) \geq v(S, \{S, \{i_{s+1}\}, ..., \{i_n\}\}) = \min_{\mathcal{P}' \in \Pi(N \setminus S)} v(S, \mathcal{P}').$$

Hence the inclusion relationship readily follows. The statement for nonpositive externalities also holds in the same manner. □

Moreover, if $v(S, \mathcal{P}) = v(S, \mathcal{P}')$ holds for every $S \subseteq N$ and $\mathcal{P}, \mathcal{P}' \in \Pi(N \setminus S)$, then we say game v has *no externalities*. In other words, we can consider such a game with (no) externalities as a game without externalities. If v has no externalities, all ψ-cores coincide and reduce to the core for a game without externalities.

Given these core notions, our question should be divided as follows: (i) what condition of games with externalities guarantees the nonemptiness of the cores? (ii) what properties characterize these core notions? In the next chapter, we will examine and answer these questions.

2 Stable allocations for games with externalities

In this chapter, we discuss the nonemptiness and axiomatic characterization of the core for games with externalities. We dedicate Section 2.1 to the nonemptiness conditions and Section 2.2 to the axiomatizations.

Section 2.1 is written based on Abe and Funaki (2017) "The non-emptiness of the core of a partition function form game" published in *International Journal of Game Theory.* Section 2.2 is based on Abe (2018a) "Consistency and the core in games with externalities" published in *International Journal of Game Theory.* In this chapter, the contents analyzed above are unified in terms of stable allocations. Our research conducted after the publications are incorporated, and some new discussions are embedded through the chapter. In particular, Section 2.3 contains new perspectives about some specific topics on the core of a game with externalities.

2.1 Necessary and sufficient conditions

2.1.1 Generalizing the Bondareva-Shapley condition

We first introduce a necessary and sufficient condition for the core of a coalition function form game to be nonempty. The condition is known as the Bondareva-Shapley condition. Let (N, w) be a game without externalities (a CFF game). The core $C(N, w)$ is nonempty if and only if for any $(\delta_S)_{S \in 2^N \setminus \{\emptyset\}}$ satisfying (i) $0 \leq \delta_S \leq 1$ for any $\emptyset \neq S \subseteq N$ and (ii) $\sum_{S \ni i} \delta_S = 1$ for any $i \in N$, we have

$$w(N) \geq \sum_{S \in 2^N \setminus \{\emptyset\}} \delta_S w(S).$$

The right-hand side of the inequality is the weighted sum of every coalition's worth. Therefore, the condition shows that the core is nonempty if and only if the worth of the grand coalition is greater than the weighted sum.

We now offer a generalization of the Bondareva-Shapley condition. In this chapter, we focus on the optimistic core and the pessimistic core. As proved below, the condition for the optimistic core is of a simpler form. Let (N, v) be a game with externalities.

Proposition 2.1.1. The optimistic core $C^{\text{opt}}(N, v)$ is nonempty if and only if

$$\text{for any } (\delta_S^{\mathcal{P}})_{(S,\mathcal{P}) \in EC(N)} \text{ with } \begin{cases} 0 \leq \delta_S^{\mathcal{P}} \leq 1 \text{ for any } (S, \mathcal{P}) \in EC(N), \\ \sum_{S \ni i} \sum_{\mathcal{P} \in \Pi(N \setminus S)} \delta_S^{\mathcal{P}} = 1 \text{ for any } i \in N, \end{cases}$$

$$v(N, \{N\}) \geq \sum_{(S,\mathcal{P}) \in EC(N)} \delta_S^{\mathcal{P}} v(S, \mathcal{P}). \tag{2.1.1}$$

Proof. We begin with the only-if-part. From the definition of the ψ-core and the optimistic expectation, it follows that

$$\begin{aligned} C^{\text{opt}}(N,v) &= \left\{ x \in X(N,v) \;\middle|\; \sum_{j\in S} x_j \geq \max_{\mathcal{P}\in\Pi(N\setminus S)} v(S,\mathcal{P}) \text{ for all } S \subseteq N \right\} \\ &= \left\{ x \in X(N,v) \;\middle|\; \sum_{j\in S} x_j \geq v(S,\mathcal{P}) \text{ for all } (S,\mathcal{P}) \in EC(N) \right\}. \end{aligned}$$

We now consider the following linear programing problem:

$$\min \sum_{j\in N} x_j \text{ subject to } \sum_{j\in S} x_j \geq v(S,\mathcal{P}) \text{ for any } (S,\mathcal{P}) \in EC(N).$$

The optimistic core is nonempty if and only if the value of this prime program is lower than $v(N,\{N\})$. The dual program is given as

$$\max \sum_{(S,\mathcal{P})\in EC(N)} \delta_S^{\mathcal{P}} v(S,\mathcal{P}) \text{ subject to } \sum_{S\ni i} \sum_{\mathcal{P}\in\Pi(N\setminus S)} \delta_S^{\mathcal{P}} = 1 \text{ for any } i \in N.$$

It follows from the duality theorem that if the prime program is feasible and the dual program is also feasible, then both programs have the same value. Hence, we obtain (2.1.1).

We now prove the if-part. If (2.1.1) holds, the dual program (namely, the maximization problem above) is feasible, and its maximum is equal to $v(N,\{N\})$.The prime program (namely, the minimization problem above) is feasible, and its minimum is $v(N,\{N\})$. Hence, again by the duality theorem, there exists a payoff vector x such that for every $(S,\mathcal{P}) \in EC(N)$, $\sum_{j\in S} x_j \geq v(S,\mathcal{P})$ and $\sum_{j\in N} x_j = v(N,\{N\})$, which implies the optimistic core is nonempty. □

The important difference between the original Bondareva-Shapley condition and Proposition 2.1.1 lies in the domain of the summation. As for the original Bondareva-Shapley condition, we sum each weighted worth for all coalitions, while in Proposition 2.1.1, for all embedded coalitions.

We now generalize the condition for the pessimistic core. Unlike the optimistic core, this takes a more complicated form.

Proposition 2.1.2. The pessimistic core $C^{\text{pes}}(N,v)$ is nonempty if and only if $(\lambda_S^{\mathcal{P}})_{(S,\mathcal{P})\in EC(N)}$ exists, satisfying the following three conditions:

(i) for any $(S,\mathcal{P}) \in EC(N),\ \lambda_S^{\mathcal{P}} \in \{0,1\}$,

(ii) for any $S \subseteq N,$ there is exactly one $\mathcal{P} \in \Pi(N\setminus S)$ such that $\lambda_S^{\mathcal{P}} = 1$,

(iii) for any $(\delta_S^{\mathcal{P}})_{(S,\mathcal{P})\in EC(N)}$ with $\begin{cases} 0 \leq \delta_S^{\mathcal{P}} \leq 1 \text{ for any } (S,\mathcal{P}) \in EC(N) \\ \sum_{S\ni i} \sum_{\mathcal{P}\in\Pi(N\setminus S)} \delta_S^{\mathcal{P}} \lambda_S^{\mathcal{P}} = 1 \text{ for any } i \in N, \end{cases}$

$$v(N,\{N\}) \geq \sum_{(S,\mathcal{P})\in EC(N)} \delta_S^{\mathcal{P}} \lambda_S^{\mathcal{P}} v(S,\mathcal{P}). \tag{2.1.2}$$

Proof. In view of the definition of the ψ-core and the pessimistic expectation function, we have

$$C^{\text{pes}}(N,v) = \left\{ x \in X(N,v) \;\middle|\; \text{for any } S \subseteq N, \text{ there is } \mathcal{P} \in \Pi(N\setminus S) \text{ such that } \sum_{j\in S} x_j \geq v(S,\mathcal{P}) \right\}.$$

Now, for all $(S,\mathcal{P}) \in EC(N)$, set

$$\lambda_S^{\mathcal{P}} = \begin{cases} 1 & \text{if } v(S,\mathcal{P}) = \min_{\mathcal{P}' \in \Pi(N\setminus S)} v(S,\mathcal{P}'), \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.3}$$

If there are two or more minimums for a coalition S, we choose one partition, $\mathcal{P}^*$, among these partitions and let $\lambda_S^{\mathcal{P}^*} = 1$. For any $\mathcal{P}' \in \Pi(N \setminus S)$ with $\mathcal{P}' \neq \mathcal{P}^*$, set $\lambda_S^{\mathcal{P}'} = 0$. In view of $(\lambda_S^{\mathcal{P}})_{(S,\mathcal{P})\in EC(N)}$, satisfying (2.1.3), we consider the following linear programing problem:

$$\min \sum_{j\in N} x_j \text{ subject to } \lambda_S^{\mathcal{P}} \sum_{j\in S} x_j \geq \lambda_S^{\mathcal{P}} v(S,\mathcal{P}) \text{ for any } (S,\mathcal{P}) \in EC(N).$$

The pessimistic core is nonempty if and only if the value of this prime program is lower than $v(N,\{N\})$. The dual program is

$$\max \sum_{(S,\mathcal{P})\in EC(N)} \delta_S^{\mathcal{P}} \lambda_S^{\mathcal{P}} v(S,\mathcal{P}) \text{ subject to } \sum_{S\ni i} \sum_{\mathcal{P}\in\Pi(N\setminus S)} \delta_S^{\mathcal{P}} \lambda_S^{\mathcal{P}} = 1 \text{ for any } i \in N.$$

Hence, we obtain (2.1.2).

The if-part is similar to that of Proposition 2.1.1. If there exists $(\lambda_S^{\mathcal{P}})_{(S,\mathcal{P})\in EC(N)}$ satisfying the three conditions (i), (ii), and (iii), then the maximization problem is feasible, and its maximum equals $v(N,\{N\})$. The minimization problem is feasible, and its minimum is $v(N,\{N\})$. From the duality theorem, it follows that there exists a payoff vector x satisfying the following condition: for each $S \subseteq N$, there is $\mathcal{P} \ni S$ such that $\sum_{j\in S} x_j \geq v(S,\mathcal{P})$ and $\sum_{j\in N} x_j = v(N,\{N\})$, which implies that the pessimistic core is nonempty. □

If a game has no externalities, then these conditions coincide with each other and reduce to the original Bondareva-Shapley condition. Note that the condition of Proposition 2.1.1 (*i.e.*, the optimistic core) implies that of Proposition 2.1.2 (*i.e.*, the pessimistic core). To briefly examine this, we choose $(\lambda_S^{\mathcal{P}})_{(S,\mathcal{P})\in EC(N)}$ in Proposition 2.1.2 as follows:

$$\lambda_S^{\mathcal{P}} = \begin{cases} 1 & \text{if } v(S,\mathcal{P}) = \max_{\mathcal{P}' \in \Pi(N\setminus S)} v(S,\mathcal{P}'), \\ 0 & \text{otherwise.} \end{cases}$$

If there are two or more maximums for a coalition S, choose exactly one partition. This vector $(\lambda_S^{\mathcal{P}})_{(S,\mathcal{P})\in EC(N)}$ satisfies the premises of Proposition 2.1.2, and together with (2.1.1), implies (2.1.2). This is consistent with the fact that the optimistic core becomes a subset of the pessimistic core. Moreover, we can interpret $\lambda_S^{\mathcal{P}}$ as an anticipation of coalition S for partition $\mathcal{P}$. If the members of S have a pessimistic view, they believe that exactly one partition forms and that the other possible partitions would not occur.

Example 2.1.3 (Cournot oligopoly)**.** The Cournot oligopoly was initially discussed by Ray and Vohra (1999) as a game with externalities. We apply our conditions to this model. We consider an industry consisting of three identical firms ($i = 1, 2, 3$) with a homogeneous output and constant marginal costs of production c. Let q_i be a level of quantity of firm i. This market has an inverse demand function $p = a - bQ$, where Q is the aggregate quantity in this market. By some calculations, this market is given in partition function form as Table 2.1, where $M = \frac{(a-c)^2}{b}$.

Table 2.1: The Cournot oligopoly with three firms

Partition	{1,2,3}			{1,23}		{2,13}		{3,12}		{123}
Coalition	1	2	3	1	23	2	13	3	12	123
$v(S,\mathcal{P})$	$\frac{1}{16}M$	$\frac{1}{16}M$	$\frac{1}{16}M$	$\frac{1}{9}M$	$\frac{1}{9}M$	$\frac{1}{9}M$	$\frac{1}{9}M$	$\frac{1}{9}M$	$\frac{1}{9}M$	$\frac{1}{4}M$

Clearly, the grand coalition is efficient: $v(N,\{N\}) \geq \sum_{S\in\mathcal{P}} v(S,\mathcal{P})$ for any $\mathcal{P} \in \Pi(N)$. Moreover, this game has a nonempty pessimistic core because there exists $(\lambda_S^{\mathcal{P}}) = (\lambda_1^{\{1,2,3\}},...,\lambda_N^{\{N\}}) = (1,1,1,0,1,0,1,0,1,1)$ such that for any $(\delta_S^{\mathcal{P}})$ satisfying

$$\begin{aligned}
\delta_1^{\{1,2,3\}} + \delta_{13}^{\{2,13\}} + \delta_{12}^{\{3,12\}} + \delta_N^{\{N\}} &= 1,\\
\delta_2^{\{1,2,3\}} + \delta_{23}^{\{1,23\}} + \delta_{12}^{\{3,12\}} + \delta_N^{\{N\}} &= 1,\\
\delta_3^{\{1,2,3\}} + \delta_{23}^{\{1,23\}} + \delta_{13}^{\{2,13\}} + \delta_N^{\{N\}} &= 1,
\end{aligned}$$

we have $v(N,\{N\}) \geq \sum_{(S,\mathcal{P})\in EC(N)} \delta_S^{\mathcal{P}}\lambda_S^{\mathcal{P}} v(S,\mathcal{P})$, equivalently,

$$\frac{M}{4} \geq \frac{M}{16}\left(\delta_1^{\{1,2,3\}}+\delta_2^{\{1,2,3\}}+\delta_3^{\{1,2,3\}}\right)+\frac{M}{9}\left(\delta_{23}^{\{1,23\}}+\delta_{13}^{\{2,13\}}+\delta_{12}^{\{3,12\}}\right)+\frac{M}{4}\delta_N^{\{N\}}.$$

However, the optimistic core is empty because, for example, we can take $(\delta_S^{\mathcal{P}})$ so as to satisfy

$$\begin{aligned}
&\delta_1^{\{1,23\}} = 1,\ \delta_2^{\{2,13\}} = 1,\ \delta_3^{\{3,12\}} = 1 \text{ and}\\
&\delta_S^{\mathcal{P}} = 0 \text{ for all } (S,\mathcal{P}) \notin \{(1,\{1,23\}),(2,\{2,13\}),(3,\{3,12\})\}.
\end{aligned}$$

This $(\delta_S^{\mathcal{P}})$ satisfies the constraints

$$\begin{aligned}
\delta_1^{\{1,2,3\}} + \delta_1^{\{1,23\}} + \delta_{13}^{\{2,13\}} + \delta_{12}^{\{3,12\}} + \delta_N^{\{N\}} &= 1,\\
\delta_2^{\{1,2,3\}} + \delta_{23}^{\{1,23\}} + \delta_2^{\{2,13\}} + \delta_{12}^{\{3,12\}} + \delta_N^{\{N\}} &= 1,\\
\delta_3^{\{1,2,3\}} + \delta_{23}^{\{1,23\}} + \delta_{13}^{\{2,13\}} + \delta_3^{\{3,12\}} + \delta_N^{\{N\}} &= 1.
\end{aligned}$$

However, it yields the inequality $v(N,\{N\}) < \sum_{(S,\mathcal{P})\in EC(N)} \delta_S^{\mathcal{P}} v(S,\mathcal{P})$, namely,

$$\frac{1}{4}M < \frac{1}{9}M\left(\delta_1^{\{1,23\}} + \delta_2^{\{2,13\}} + \delta_3^{\{3,12\}}\right).$$

2.1.2 Balancedness

In this subsection, we define the notion of balanced collection in the presence externalities. Furthermore, we will provide another form of our necessary and sufficient condition for each of the optimistic core and the pessimistic core.

Definition 2.1.4. A collection $\beta \subseteq EC(N)$ is *optimistically balanced* if there are positive numbers $\delta_S^{\mathcal{P}}$ such that for any $i \in N$,

$$\sum_{S\ni i}\sum_{(S,\mathcal{P})\in\beta} \delta_S^{\mathcal{P}} = 1. \tag{2.1.4}$$

The vector $(\delta_S^{\mathcal{P}})_{(S,\mathcal{P})\in\beta}$ is called a system of *optimistically balancing weights*. Similarly, a collection $\beta \subseteq EC(N)$ is *pessimistically balanced* if there are positive numbers $\delta_S^{\mathcal{P}}$ such that

$$
\begin{aligned}
&\text{(i)} \quad \text{for any } i \in N,\ \sum_{S \ni i} \sum_{(S,\mathcal{P})\in\beta} \delta_S^{\mathcal{P}} = 1, \\
&\text{(ii)} \quad \text{for any } S \subseteq N,\ \text{there is a partition } \mathcal{P} \in \Pi(N \setminus S) \text{ such that } (S,\mathcal{P}) \in \beta, \\
&\text{(iii)} \quad \text{for any } (S,\mathcal{P}),(T,Q) \in \beta \text{ with } (S,\mathcal{P}) \neq (T,Q), \mathcal{P} \neq Q \Rightarrow S \neq T.
\end{aligned}
\tag{2.1.5}
$$

We call $(\delta_S^{\mathcal{P}})_{(S,\mathcal{P})\in\beta}$ satisfying the above three conditions a system of *pessimistically balancing weights*. A balanced collection is called a *minimal optimistically (pessimistically) balanced collection* if it contains no smaller optimistically (pessimistically) balanced subcollections.

Clearly, if β is pessimistically balanced, then β is optimistically balanced. Below is a property unique to pessimistically balanced collections.

Proposition 2.1.5. Collection β is pessimistically balanced if and only if β is a minimal pessimistically balanced collection.

Proof. If we remove $(S,\mathcal{P})$ from a pessimistically balanced collection β, then condition (ii) requires that there exists $(S,Q) \in \beta$ such that $Q \neq \mathcal{P}$, which contradicts (iii). □

Optimistically balanced collections do not obey this property. The following example describes this difference.

Example 2.1.6. Let $N = \{1,2,3\}$. In Table 2.2, $\beta_1, ..., \beta_6$ represent collections. Note that they do not cover all the collections of N but some particular illustrative examples. The symbol "+" shows that the embedded coalition is an element of the collection.

Table 2.2: Minimality of balanced collections

Partition	{1,2,3}			{1,23}		{2,13}		{3,12}		$\{N\}$	Opt.		Pes.	
Coalition	1	2	3	1	23	2	13	3	12	N	Balanced	Min.	Balanced	Min.
β_1	+	+	+	-	-	-	-	-	-	-	✓	✓		
β_2	+	+	+	+	-	-	-	-	-	-	✓			
β_3	+	+	+	-	+	-	+	-	+	+	✓		✓	✓
β_4	-	-	-	+	+	+	+	+	+	+	✓		✓	✓
β_5	+	+	-	-	-	-	-	-	-	-				
β_6	+	+	+	+	+	+	+	+	+	+	✓			

Now, in view of the definitions of balanced collections, we reformulate our necessary and sufficient conditions. The following propositions readily follows from Proposition 2.1.1 and Proposition 2.1.2

Proposition 2.1.7. The optimistic core is nonempty if and only if for any optimistically balanced collection β and any of its balancing weights $(\delta_S^{\mathcal{P}})_{(S,\mathcal{P})\in\beta}$,

$$
v(N,\{N\}) \geq \sum_{(S,\mathcal{P})\in\beta} \delta_S^{\mathcal{P}} v(S,\mathcal{P}).
$$

Similarly, the pessimistic core is nonempty if and only if there is a pessimistically balanced collection β' and a balancing weight $(\delta'^{\mathcal{P}}_S)_{(S,\mathcal{P})\in\beta}$ such that

$$v(N, \{N\}) \geq \sum_{(S,\mathcal{P})\in\beta'} \delta'^{\mathcal{P}}_S v(S, \mathcal{P}).$$

These conditions are alternative expressions of Propositions 2.1.1 and 2.1.2. We can straightforwardly refine these conditions through minimal balanced collections. The proof of the following proposition is basically the same with that of games without externalities. See Peleg and Sudhölter (2007) for details. The intuition behind the proof is that replacing an optimistically balanced collection with a minimal optimistically balanced collection uniquely determines the corresponding balancing weight.

Proposition 2.1.8. The optimistic core is nonempty if and only if for any minimal optimistically balanced collection β and its unique balancing weight $(\delta^{\mathcal{P}}_S)_{(S,\mathcal{P})\in\beta}$,

$$v(N, \{N\}) \geq \sum_{(S,\mathcal{P})\in\beta} \delta^{\mathcal{P}}_S v(S, \mathcal{P}).$$

The same replacement for pessimistically balanced collections is redundant: Proposition 2.1.5 indicates that a collection β is pessimistically balanced if and only if β is a minimal pessimistically balanced collection.

2.1.3 Application to the tragedy of the commons

We now apply our necessary and sufficient conditions to the *tragedy of the commons*, which was studied by Funaki and Yamato (1999) in partition function form. In this application, unlike theirs, we introduce a specific production function and offer a more analytical result. Let $N = \{1, ..., n\}$ be a set of fishermen, namely, players ($n \geq 3$). Let x_j denote the amount of labor that fisherman j needs to catch fish. A production function f associates the amount of fish with the amount of labor. We assume that $f(x) = x^\theta$, where $0 \leq \theta \leq 1$. Let $q > 0$ denote the opportunity cost. A fisherman j can obtain the amount of fish represented by $(x_j/\sum_{i\in N} x_i)f(\sum_{i\in N} x_i)$. This amount is not the result of negotiation among fishermen but a technological assumption. Clearly, the amount of fish that the fisherman j can obtain depends on the amount of labor that the other fishermen provide.

We now denote the income of fisherman j by $m_j(x_1, ..., x_n) = (x_j/\sum_{i\in N} x_i)f(\sum_{i\in N} x_i) - qx_j$, where $m_j(0, ..., 0) = 0$. We suppose that fishermen jointly work together as a group. Let $S \subseteq N$ denote a group of fishermen, namely, a coalition. Given the labor input of the fishermen in $N \setminus S$, the fishermen in S choose their total labor input $x_S := \sum_{j\in S} x_j$ to maximize the sum of their incomes:

$$m_S := \sum_{j\in S} m_j(x_1, ..., x_n) = \left(\frac{x_S}{x_N}\right) f(x_N) - qx_S.$$

We fix a partition $\mathcal{P} = \{S_1, ..., S_k\}$ of N. We call a vector $x^*(\mathcal{P}) = (x^*_{S_1}(\mathcal{P}), ..., x^*_{S_k}(\mathcal{P}))$ an equilibrium under $\mathcal{P}$ if for any $i = 1, ..., k$ and any x_{S_i}, $m_{S_i}(x^*_{S_i}, x^*_{-S_i}) \geq m_{S_i}(x_{S_i}, x^*_{-S_i})$, where $x^*_{-S_i} = (x^*_{S_1}, ..., x^*_{S_{i-1}}, x^*_{S_{i+1}}, ..., x^*_{S_k})$. Under the assumptions above, exactly one equilibrium under $\mathcal{P}$ exists for every partition $\mathcal{P}$ (see Funaki and Yamato (1999)).

We now construct a game with externalities. For any $S \in \mathcal{P}$, we define $v(S, \mathcal{P} \setminus \{S\}) := m_S(x^*(\mathcal{P}))$. With some computations, we obtain $v(S, \mathcal{P} \setminus \{S\}) = (\theta - 1 + k)^{\frac{\theta}{1-\theta}} k^{\frac{\theta-2}{1-\theta}} q^{\frac{-\theta}{1-\theta}} (1-\theta)$, where k is the number of coalitions in the partition $\mathcal{P}$. Thus, in view of (2.1.1), the necessary and sufficient condition for the nonempty optimistic core is obtained as follows: for all $\boldsymbol{\delta} = (\delta_S^{\mathcal{P}})_{(S,\mathcal{P}) \in EC(N)}$ satisfying balancedness, we have

$$\theta^{\frac{\theta}{1-\theta}} \geq \sum_{(S,\mathcal{P}) \in EC(N)} \delta_S^{\mathcal{P}} (\theta - 1 + k)^{\frac{\theta}{1-\theta}} k^{\frac{\theta-2}{1-\theta}}. \tag{2.1.6}$$

Removing $\boldsymbol{\delta}$ from (2.1.6), we obtain an explicit form of the necessary and sufficient condition consisting only of n and θ (Proposition 2.1.10). In this end, we first offer the following lemma.

Lemma 2.1.9. In this model, $v(S, \mathcal{P})$ monotonically decreases for k, *i.e.*,

$$\frac{\partial}{\partial k} \left[(\theta - 1 + k)^{\frac{\theta}{1-\theta}} k^{\frac{\theta-2}{1-\theta}} \right] < 0.$$

Proof. This proof is a variant of that of Funaki and Yamato (1999). We have

$$\begin{aligned} \frac{\partial}{\partial k} \left[(\theta - 1 + k)^{\frac{\theta}{1-\theta}} k^{\frac{\theta-2}{1-\theta}} \right] &= \left(\frac{\theta}{\theta - 1 + k} + \frac{\theta - 2}{k} \right) (\theta - 1 + k)^{\frac{\theta}{1-\theta}} k^{-\frac{2-\theta}{1-\theta}} \frac{1}{1-\theta} \\ &= -(\theta - 2 + 2k) \cdot (\theta - 1 + k)^{\frac{2\theta-1}{1-\theta}} k^{\frac{2\theta-3}{1-\theta}} \\ &< 0 \end{aligned}$$

because $k \geq 1$ and $0 < \theta < 1$. □

Lemma 2.1.9 simply states that this game has positive externalities. We define a function $g(\theta, \boldsymbol{\delta})$ with

$$g(\theta, \boldsymbol{\delta}) = \left[\sum_{(S,\mathcal{P}) \in EC(N)} \delta_S^{\mathcal{P}} (\theta - 1 + k)^{\frac{\theta}{1-\theta}} k^{\frac{\theta-2}{1-\theta}} \right] - \theta^{\frac{\theta}{1-\theta}}$$

and obtain the following new condition.

Proposition 2.1.10. The optimistic core of the tragedy of the commons is nonempty if and only if

$$n(\theta + 1)^{\frac{\theta}{1-\theta}} 2^{\frac{\theta-2}{1-\theta}} - \theta^{\frac{\theta}{1-\theta}} \leq 0.$$

Proof. For any $\theta \in (0, 1)$, it holds that

$$\begin{aligned} & \max_{\boldsymbol{\delta}} g(\theta, \boldsymbol{\delta}) \\ = \;& \max_{\boldsymbol{\delta}} \left[\sum_{(S,\mathcal{P}) \in EC(N)} \delta_S^{\mathcal{P}} (\theta - 1 + k)^{\frac{\theta}{1-\theta}} k^{\frac{\theta-2}{1-\theta}} \right] - \theta^{\frac{\theta}{1-\theta}} \\ = \;& \max_{\boldsymbol{\delta}} \left[\left(\sum_{\substack{S \neq N \\ S \neq \emptyset}} \delta_S^{\{S, N \setminus S\}} (\theta + 1)^{\frac{\theta}{1-\theta}} 2^{\frac{\theta-2}{1-\theta}} \right) + \left(\theta^{\frac{\theta}{1-\theta}} \delta_N^{\{N\}} \right) \right] - \theta^{\frac{\theta}{1-\theta}}, \end{aligned} \tag{2.1.7}$$

where (2.1.7) follows from Lemma 2.1.9. Furthermore, we have

$$
\begin{aligned}
(2.1.7) &= \max\left\{\max_{\boldsymbol{\delta}}\left(\sum_{\substack{S\neq N\\ S\neq\emptyset}} \delta_S^{\{S,N\setminus S\}}(\theta+1)^{\frac{\theta}{1-\theta}}2^{\frac{\theta-2}{1-\theta}}\right), \max_{\boldsymbol{\delta}}\left(\theta^{\frac{\theta}{1-\theta}}\delta_N^{\{N\}}\right)\right\} - \theta^{\frac{\theta}{1-\theta}} \\
&= \max\left\{(\theta+1)^{\frac{\theta}{1-\theta}}2^{\frac{\theta-2}{1-\theta}}\max_{\boldsymbol{\delta}}\left(\sum_{\substack{S\neq N\\ S\neq\{\emptyset\}}} \delta_S^{\{S,N\setminus S\}}\right), \theta^{\frac{\theta}{1-\theta}}\right\} - \theta^{\frac{\theta}{1-\theta}} \\
&= \max\left\{(\theta+1)^{\frac{\theta}{1-\theta}}2^{\frac{\theta-2}{1-\theta}}\sum_{i\in N}\delta_i^{\{i,N\setminus i\}}\ , \ \ \theta^{\frac{\theta}{1-\theta}}\right\} - \theta^{\frac{\theta}{1-\theta}} \qquad (2.1.8) \\
&= \max\left\{n(\theta+1)^{\frac{\theta}{1-\theta}}2^{\frac{\theta-2}{1-\theta}} - \theta^{\frac{\theta}{1-\theta}}\ , \ \ 0\right\}.
\end{aligned}
$$

Note that (2.1.8) is attained by $\boldsymbol{\delta}$ such that

$$
\delta_S^{\mathcal{P}} = \begin{cases} 1 & \text{if } S=\{i\},\ \mathcal{P}=\{i, N\setminus i\} \text{ for any } i, \\ 0 & \text{otherwise.} \end{cases}
$$

This completes the proof.[1] □

Figure 2.1 numerically illustrates this proposition. Note that, as Proposition 2.1.10 shows, the nonemptiness of the optimistic core only depends on n and θ. It can be seen from the figure that for any θ, the optimistic core is empty if $n \geq 4$. With some calculations, the optimistic core is nonempty only when $\theta < 0.22...$ and $n = 3$. This fact indicates that the optimistic core is nonempty as long as the size of the society is small and the technology level, *i.e.*, θ, is relatively low and that technological progress may give each player a larger incentive to deviate. In contrast to the conditional existence of the optimistic core, the pessimistic core is always nonempty (Funaki and Yamato, 1999). This difference shows that technology and population, as well as players' expectations, influence the existence of stable allocations.

2.2 Axiomatic characterizations

2.2.1 A reduced game in the presence of externalities

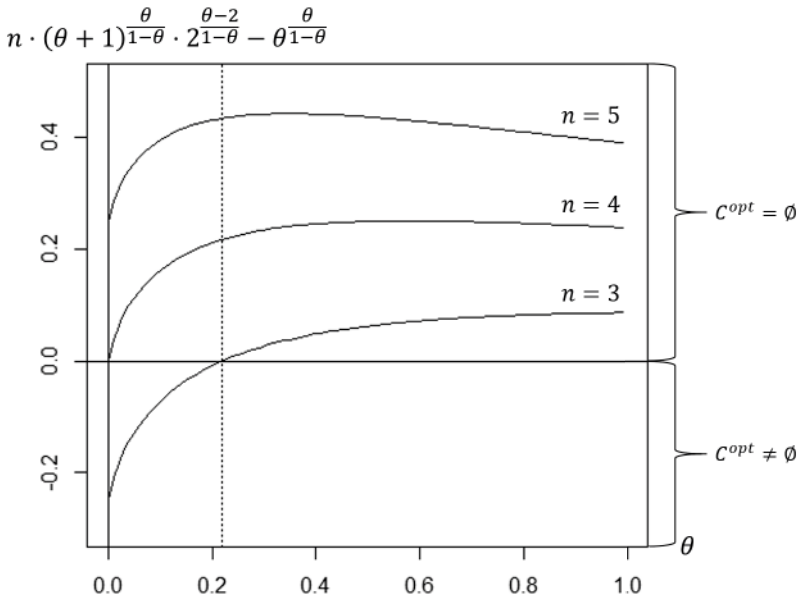
In this section, we characterize the core for games with externalities in an axiomatic approach. Our axiomatizations are based on the concept of reduced game. To introduce reduced games to games with externalities, we need further definitions and preparation.

[1] The constraints for $\boldsymbol{\delta}$ are, in the case of three players, as follows:

$$
\begin{aligned}
\delta_1^{\{1,2,3\}} + \delta_1^{\{1,23\}} + \delta_{13}^{\{2,13\}} + \delta_{12}^{\{3,12\}} + \delta_N^{\{N\}} &= 1, \\
\delta_2^{\{1,2,3\}} + \delta_{23}^{\{1,23\}} + \delta_2^{\{2,13\}} + \delta_{12}^{\{3,12\}} + \delta_N^{\{N\}} &= 1, \\
\delta_3^{\{1,2,3\}} + \delta_{23}^{\{1,23\}} + \delta_{13}^{\{2,13\}} + \delta_3^{\{3,12\}} + \delta_N^{\{N\}} &= 1.
\end{aligned}
$$

If we set, for example, $\delta_{12}^{\{3,12\}} = 1/2, \delta_{13}^{\{2,13\}} = 1/2, \delta_{23}^{\{1,23\}} = 1/2$, and $\delta_S^{\mathcal{P}} = 0$ for any other $(S,\mathcal{P})$, then it satisfies the constraints. However, $\sum_{S\neq N,\ S\neq\{\emptyset\}} \delta_S^{\{S,N\setminus S\}} = 1/2+1/2+1/2 = 3/2$ is less than that of the construction above, which achieves $\delta_1^{\{1,23\}} + \delta_2^{\{2,13\}} + \delta_3^{\{3,12\}} = 1+1+1 = 3$.

Figure 2.1: Proposition 2.1.10 and the optimistic core



Let Γ_A be the set of all partition function form games: $\Gamma_A = \{(N, v) \mid \emptyset \neq N \subseteq \mathcal{N},\ |N| < \infty,\ v : EC(N) \to \mathbb{R}\}$. For any game, we restrict payoff vectors to the following set: $F(N, v) = \left\{x \in \mathbb{R}^N \,\middle|\, \sum_{j \in N} x_j \leq v(N, \{\emptyset\})\right\}$. For a set of games $\Gamma \subseteq \Gamma_A$, a *solution* on Γ is a function σ that associates a subset $\sigma(N, v)$ of $F(N, v)$ with every game $(N, v) \in \Gamma$.

We denote by x_S a *restriction* of $x \in \mathbb{R}^N$ on coalition S, *i.e.*, $x_S = (x_j)_{j \in S} \in \mathbb{R}^S$. For any coalition S and player i, we use $S \cup i$ or $S \setminus i$ to denote $S \cup \{i\}$ or $S \setminus \{i\}$ for simplicity.

In the class of games without externalities, several forms of reduced games are proposed, in many of which remaining players are not influenced by orders of leaving players. However, in the presence of externalities, we may need to consider the possibility that not only an order of leaving players but a partition of leaving pla

litions. To this end, we first focus on the reduced game known as the complement-reduced game. One of the crucial benefits of the complement-reduced game is that it depends neither on the order of leaving players nor on the partition of leaving players even in the presence of externalities. The complement-reduced game might be thought of as the simplest form of reduced game in the sense of externalities. The other types of reduced games are also discussed later.

Now, fix $\Gamma \subseteq \Gamma_A$ and $(N, v) \in \Gamma$. Let $S \subseteq N$ ($S \neq \emptyset$) and $x \in \mathbb{R}^N$.

Definition 2.2.1. The *complement-reduced game with respect to S and x* is the game $(S, v^{S,x})$ defined as follows: for any $\emptyset \neq T \subseteq S$ and any $\mathcal{Q} \in \Pi(S \setminus T)$,

$$v^{S,x}(T, \mathcal{Q}) = v(T \cup (N \setminus S), \mathcal{Q}) - \sum_{j \in N \setminus S} x_j.$$

This form of reduced game describes that a coalition T always obtains the help of all leaving players $N\backslash S$ by paying $(x_j)_{j\in N\backslash S}$ for them. The complement-reduced game was initially introduced by Moulin (1985) for games without externalities. Definition 2.2.1 is the simple extension of the original definition to games with externalities.

In the presence of externalities, the order of leaving players is not trivial. As we mentioned above, the dependence on the order means that different orders result in different worths of each remaining coalition. We now show that the complement-reduced game depends neither on the order of leaving players nor on the partition of leaving players. For notational simplicity, we write $v^{-i} := v^{N\backslash i,x}$, *i.e.*, v^{-i} to denote the complement-reduced game after removing i from the original game. Similarly, we use the following notation: $(v^{-i_1})^{-i_2} := (v^{N\backslash i_1,x})^{(N\backslash i_1)\backslash i_2, x_{N\backslash i_1}}$.

Lemma 2.2.2. For any $x \in \mathbb{R}^N$ and any $i_1, i_2 \in N$ $(i_1 \neq i_2)$,

$$(v^{-i_1})^{-i_2} = (v^{-i_2})^{-i_1} = v^{N\backslash\{i_1,i_2\},x}.$$

Proof. For any $T \subseteq N \setminus i_1$ and any $\mathcal{Q} \in \Pi(N \setminus (T \cup i_1))$, we have

$$v^{-i_1}(T, \mathcal{Q}) \quad = \quad v(T \cup i_1, \mathcal{Q}) - x_{i_1}.$$

For any $T' \subseteq N \setminus \{i_1, i_2\}$ and $\mathcal{Q}' \in \Pi(N \setminus (T \cup \{i_1, i_2\}))$,

$$\begin{aligned} (v^{-i_1})^{-i_2}(T', \mathcal{Q}') &= v^{-i_1}(T' \cup i_2, \mathcal{Q}') - x_{i_2} \\ &= v(T' \cup \{i_1, i_2\}, \mathcal{Q}') - x_{i_1} - x_{i_2}. \end{aligned} \tag{2.2.1}$$

Similarly, we remove them in the order of i_2, i_1 and obtain the same game as (2.2.1).

Now, we assume that players i_1 and i_2 simultaneously leave the game. For any $T' \subseteq N \setminus \{i_1, i_2\}$ and any $\mathcal{Q}' \in \Pi(N \setminus (T \cup \{i_1, i_2\}))$, we have

$$v^{N\backslash\{i_1,i_2\},x}(T', \mathcal{Q}') \quad = \quad v(T' \cup \{i_1, i_2\}, \mathcal{Q}') - x_{i_1} - x_{i_2},$$

which is equal to (2.2.1). □

Lemma 2.2.2 shows that the following two properties hold even in the presence of externalities.

i. The complement-reduced game is independent of the order of leaving players.

ii. The game obtained by removing players one by one is equivalent to the game obtained by removing players simultaneously.

The independence of the order of leaving players relates to the two different *path independence* notions argued by Dutta et al. (2010) and Bloch and van den Nouweland (2014). This topic is elaborated in Subsection 2.3.3. Definition 2.2.1 shows that we can "ignore" this influence in the complement-reduced game, as all leaving players $N \setminus S$ help the remaining players T and form a coalition $T \cup (N \setminus S)$. This property is unique to the complement-reduced game and does not apply to the other types of reduced games. This difference is also elaborated in Subsection 2.2.4.

It is straightforward to extend Lemma 2.2.2 to general coalitions. Consider $T = \{i_1, ..., i_t\} \subsetneq N$. For any permutations π, π' of $T = \{i_1, ..., i_t\}$, by repeating Lemma 2.2.2, we have

$$v^{\pi} = v^{\pi'} = v^{N\backslash T,x},$$

where, for any permutation π'', $v^{\pi''} = (...((v^{-\pi''_1})^{-\pi''_2})...)^{-\pi''_t}$. Player π''_k means the k-th player leaving the game. Hence, we obtain useful notation as follows:

$$v^{-T} := v^{\pi} = v^{\pi'} = v^{N\setminus T,x}.$$

2.2.2 Consistency concepts for an expectation function

Bloch and van den Nouweland (2014) introduce *subset consistency* as a property of an expectation function. We change their original definition slightly to match our framework as follows.

Definition 2.2.3. Let Γ be a set of games and $(N,v) \in \Gamma$. An expectation function ψ satisfies *subset consistency* if for any $S \subseteq N$ and any $T \subseteq S$,

$$\psi(N,v,S) = \psi(N,v,T)|_{(N\setminus S)},$$

Subset consistency describes that for a given N, all players within $S \subseteq N$ share the same expectation on the behavior of outside players $N \setminus S$. As illustrated in the following example, subset consistency does not necessarily imply consistency between player sets in the following sense.

Example 2.2.4. Let $N = \{1,2,3,4\}$. For any $N' \subseteq N$ and any $S \subseteq N'$

$$\psi(N',v,S) = \begin{cases} \{N' \setminus S\}, & \text{if } N' = N, \\ \{\{i_{s+1}\},...,\{i_{n'}\}\}, & \text{if } N' \subsetneq N. \end{cases} \tag{2.2.2}$$

In words, if a player (sub)set N' is equal to N, then (2.2.2) expects a single coalition of $N' \setminus S$; otherwise, it expects a partition of $N' \setminus S$ into singletons. We call this expectation function the *quasi singleton-expectation*. This expectation function satisfies subset consistency because for each $N' \subseteq N$, ψ is the merge- or the singleton-expectations within N'. On the other hand, it is not consistent between player sets (between N and, for example, $N \setminus 4$) since it holds that

$$\psi(N \setminus 4, v^{-4}, \{1\}) = \{\{2\},\{3\}\} \neq \{2,3\} = \{2,3,4\}|_{N\setminus 4} = \psi(N,v,\{1\})|_{N\setminus 4},$$

for any v and x.

Then, what condition guarantees the consistency between the player sets? To answer this question, we introduce a new property for expectation functions. As we will mention later, this condition is a sufficient condition for the ψ-core to satisfy the complement-reduced game property.

Definition 2.2.5. Let Γ be a set of games and $(N,v) \in \Gamma$. An expectation function ψ is *complement-consistent (CC)* if for any $S \subseteq N$ ($|S| \geq 2$), any $h \in S$, and any $x \in \mathbb{R}^N$,

$$\psi(N,v,S) = \psi(N \setminus h, v^{N\setminus h,x}, S \setminus h).$$

Complement-consistency (CC) requires that coalition S's expectation is equal to coalition $S \setminus h$' s expectation. Note that both $\psi(N,v,S)$ and $\psi(N \setminus h, v^{N\setminus h,x}, S \setminus h)$ are partitions of $N \setminus S$ respectively.

In Definition 2.2.5, we define CC by removing one player. We now consider a slight variant of CC. We call it *strong complement-consistency*, $\widehat{CC}$, and define it as follows: an expectation function ψ is $\widehat{CC}$ if for any $S \subseteq N$ and $T \subsetneq S$ ($T \neq \emptyset$),

$$\psi(N, v, S) = \psi(N \setminus T, v^{-T}, S \setminus T).$$

Namely, if ψ is $\widehat{CC}$, then we have

$$v(S, \psi(N, v, S)) = v(S, \psi(N \setminus T, v^{-T}, S \setminus T)). \tag{2.2.3}$$

The following proposition shows that CC is equivalent to $\widehat{CC}$.

Lemma 2.2.6. Expectation function ψ is CC if and only if it is $\widehat{CC}$.

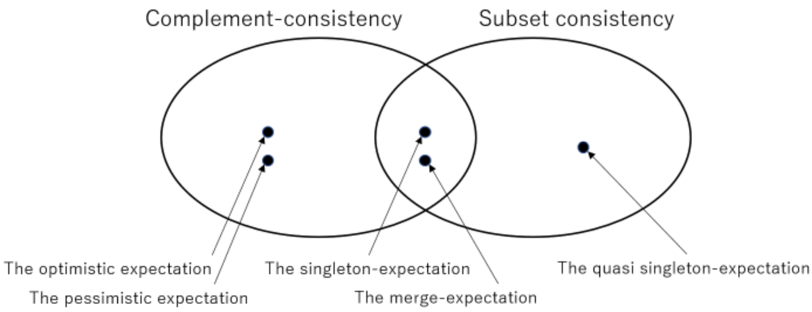
Proof. The if-part is straightforward. We prove the only-if-part. Let $S \subseteq N$ and $T \subsetneq S$ with $T \neq \emptyset$. Define $T = \{h_1, ..., h_t\}$. By CC, we have

$$\begin{aligned}
\psi(N, v, S) &= \psi(N \setminus h_1, v^{-h_1}, S \setminus h_1) \\
&= \psi(N \setminus \{h_1, h_2\}, (v^{-h_1})^{-h_2}, S \setminus \{h_1, h_2\}) \\
&\ldots \\
&= \psi(N \setminus T, (...(v^{-h_1})...)^{-h_t}, S \setminus T) \\
&\overset{\text{Lma. 2.2.2}}{=} \psi(N \setminus T, v^{-T}, S \setminus T).
\end{aligned}$$

□

The four expectation functions we defined in Chapter 1, *i.e.*, the optimistic, pessimistic, merge-, and singleton- expectation functions, are all CC. For the proof, see Proposition 2.2.14 and Corollary 2.2.18. Figure 2.2 illustrates the relationship between the consistency concepts.

Figure 2.2: The relationship between the two consistency concepts



2.2.3 Axiomatization based on the complement-reduced game property

Below, we provide the axioms for our characterization result.

Axiom 1 (comp-RGP)**.** Let Γ be a set of games, $(N, v) \in \Gamma$ and $S \subseteq N$. A solution σ on Γ satisfies the *complement-reduced game property (comp-RGP)* if for every $x \in \sigma(N, v)$, we have $(S, v^{S,x}) \in \Gamma$ and $x_S \in \sigma(S, v^{S,x})$.

Axiom 1 requires solution σ to be consistent with itself: if $x \in \sigma(N, v)$, namely, x solves the game (N, v), then x_S should solve $(S, v^{S,x})$ for any $S \subseteq N$. Therefore, any restriction x_S of x does not contradict the original agreement among all players that x is a solution of (N, v). Note that Axiom 1 is independent of expectation function ψ: the choice of an expectation function does not influence the concept of consistency described in Axiom1.

Axiom 2 (NE on Γ)**.** Let Γ be a set of games. A solution σ on Γ satisfies *nonemptiness on* Γ *(NE on* Γ*)* if for every $(N, v) \in \Gamma$, we have $\sigma(N, v) \neq \emptyset$.

Axiom 2 states that solution σ should be nonempty for any game in a given class Γ. This axiom may indirectly depend on ψ if Γ is specified by ψ.

Axiom 3 (ψ-IR)**.** Let ψ be an expectation function. A solution σ on Γ satisfies ψ*-individual rationality (*ψ*-IR)* if for every $(N, v) \in \Gamma$, any $x \in \sigma(N, v)$, and every player $i \in N$, we have $x_i \geq v(\{i\}, \psi(N, v, \{i\}))$.

In the presence of externalities, each player i's individual worth varies depending on the coalition structure of $N \backslash i$. The axiom ψ-IR requires solution σ to assign to each player at least his individual worth under the expectation ψ and the corresponding partition $\psi(N, v, \{i\})$. Clearly, Axiom 3 depends on ψ.

For any ψ, the ψ-core satisfies ψ-IR because the expectation function ψ is common to both ψ-core and ψ-IR. It is also clear that ψ-core is nonempty on Γ if $\Gamma = \Gamma_{C^\psi}$. For comp-RGP, we have the following result.

Proposition 2.2.7. If an expectation function ψ is CC, the ψ-core satisfies comp-RGP on Γ_{C^ψ}.

Proof. Let $C^\psi(N, v)$ be the ψ-core of (N, v) and $x \in C^\psi(N, v)$. For every nonempty $S \subseteq N$, it suffices to show that $x_S \in C^\psi(S, v^{S,x})$. By Definition 2.2.1, for any $T \subseteq S$ $(T \neq \emptyset)$, we have

$$
\begin{aligned}
& \sum_{j \in T} x_j - v^{S,x}(T, \psi(S, v^{S,x}, T)) \\
= \ & \sum_{j \in T} x_j - \left[v(T \cup (N \setminus S), \psi(S, v^{S,x}, T)) - \sum_{j \in N \setminus S} x_j \right] \\
= \ & \sum_{j \in T \cup (N \setminus S)} x_j - v(T \cup (N \setminus S), \psi(S, v^{S,x}, T)) \\
\geq \ & v(T \cup (N \setminus S), \psi(N, v, T \cup (N \setminus S))) - v(T \cup (N \setminus S), \psi(S, v^{S,x}, T)) \qquad (2.2.4) \\
= \ & v(T \cup (N \setminus S), \psi(S, v^{S,x}, T)) - v(T \cup (N \setminus S), \psi(S, v^{S,x}, T)) \qquad (2.2.5) \\
= \ & 0,
\end{aligned}
$$

where (2.2.4) holds because of $x \in C^\psi(N, v)$, and (2.2.5) follows from (2.2.3). □

Now, we offer the axiomatization below. We first show that a well-known result holds even in games with externalities (Lemma 2.2.8). This result will be useful in the proof of the axiomatization (Proposition 2.2.9).

Lemma 2.2.8. Let ψ be an expectation function and σ be a solution on a set of games Γ. If σ satisfies comp-RGP and ψ-IR, then σ satisfies *efficiency*: for any $x \in \sigma(N, v)$

$$\sum_{j \in N} x_j = v(N, \{\emptyset\}).$$

Proof. This is a simple extension of Peleg (1986) and Tadenuma (1992). Let $(N, v) \in \Gamma$ and $x \in \sigma(N, v)$. Assume that σ is not efficient. Then, there exists $x \in \sigma(N, v)$ such that $\sum_{j \in N} x_j < v(N, \{\emptyset\})$. Let i be a player in N. By comp-RGP, we have $x_i \in \sigma\left(\{i\}, v^{\{i\},x}\right)$. For any ψ, by ψ-IR, we have $x_i \geq v^{\{i\},x}(\{i\}, \{\emptyset\}) = v(N, \{\emptyset\}) - \sum_{j \in N \setminus i} x_j$. Hence, $\sum_{j \in N} x_j \geq v(N, \{\emptyset\})$, and the desired contradiction has been obtained. □

Proposition 2.2.9. Let ψ be an expectation function. If ψ is CC, then ψ-core C^ψ is the unique function on Γ_{C^ψ} that satisfies comp-RGP, NE on Γ_{C^ψ}, and ψ-IR.

Proof. Let σ be a solution satisfying the three conditions. The proof consists of two parts: $\sigma \subseteq C^\psi$ and $C^\psi \subseteq \sigma$.

Part 1:

First, we show that $\sigma(N, v) \subseteq C^\psi(N, v)$ for any $(N, v) \in \Gamma_{C^\psi}$. From Lemma 2.2.8, it follows that σ satisfies efficiency.

Induction base:

For $|N| = 1$, $\sigma(N, v) \subseteq C^\psi(N, v)$ because of efficiency. For $|N| = 2$, let $N = \{i, j\}$. By efficiency, $x_i + x_j = v(N, \{\emptyset\})$ for any $x \in \sigma(N, v)$. By ψ-IR, $x_i \geq v(\{i\}, \{\{j\}\})$ and $x_j \geq v(\{j\}, \{\{i\}\})$. Hence, $\sigma(N, v) \subseteq C^\psi(N, v)$.

Induction proof:

We assume that $\sigma(N, v') \subseteq C^\psi(N, v')$ for any $(N, v') \in \Gamma^\psi$ with $|N| \leq k$ $(k \geq 2)$. We show that for any $(M, v) \in \Gamma^\psi$ with $|M| = k + 1$, we have $\sigma(M, v) \subseteq C^\psi(M, v)$.

Let $x \in \sigma(M, v)$ and $h \in M$. By comp-RGP, we have $x_{M \setminus h} \in \sigma(M \setminus h, v^{M \setminus h, x})$. By the assumption of induction, $\sigma(M \setminus h, v^{M \setminus h, x}) \subseteq C^\psi(M \setminus h, v^{M \setminus h, x})$. Hence, for any nonempty $S \subseteq M \setminus h$,

$$\begin{aligned} \sum_{j \in S} x_j &\geq v^{M \setminus h, x}(S, \psi(M \setminus h, v^{M \setminus h, \mathcal{P}, x}, S)) \\ &= v(S \cup h, \psi(M \setminus h, v^{M \setminus h, \mathcal{P}, x}, S)) - x_h \\ &= v(S \cup h, \psi(M, v, S \cup h)) - x_h, \end{aligned} \tag{2.2.6}$$

where (2.2.6) holds because ψ is CC. Thus, we obtain

$$\sum_{j \in S \cup h} x_j \geq v(S \cup h, \psi(M, v, S \cup h))$$

for any nonempty $S \subseteq M \setminus h$. In addition, by ψ-IR, we have $x_i \geq v(\{i\}, \psi(M, v, \{i\}))$. Hence, $\sigma(M, v) \subseteq C^\psi(M, v)$. By induction, it follows that $\sigma(N, v) \subseteq C^\psi(N, v)$ for all (N, v) in Γ^ψ.

Part 2:

Next, we show that $C^{\psi}(N,v) \subseteq \sigma(N,v)$ for all $(N,v) \in \Gamma_{C^{\psi}}$. To prove this, we construct a game (M,u) by using a game $(N,v) \in \Gamma_{C^{\psi}}$ and a payoff vector $x \in C^{\psi}(N,v)$. Fix $(N,v) \in \Gamma_{C^{\psi}}$ and $x \in C^{\psi}(N,v)$. We define $M := N \cup h$, where $h \in \mathcal{N}$ and $h \notin N$. Define u as follows:

$$\begin{array}{ll} u(\{h\},\mathcal{P}') = 0 & \text{for all } \mathcal{P}' \in \Pi(M \setminus h), \\ u(S \cup h,\mathcal{P}'') = v(S,\mathcal{P}'') & \text{for all } \mathcal{P}'' \in \Pi(M \setminus (S \cup h)), \\ u(S,\mathcal{P}''') = \sum_{j \in S} x_j, & \text{for all } \mathcal{P}''' \in \Pi(M \setminus S). \end{array} \tag{2.2.7}$$

Now, consider $y = (x,0) \in \mathbb{R}^M$. We will prove the following claims.

Claim 1 $y \in C^{\psi}(M,u)$.

Proof. By the definition of y and u, we have

$$\sum_{j \in M} y_j = \sum_{j \in N} x_j = v(N,\{\emptyset\}) = u(M,\{\emptyset\}).$$

First, we show that $v = u^{M \setminus h,y}$. For any $S \subseteq N = M \setminus h$ and any $\mathcal{P}'' \in \Pi(N \setminus S)$, we have

$$\begin{aligned} u^{M \setminus h,y}(S,\mathcal{P}'') &= u(S \cup h,\mathcal{P}'') - y_h \\ &= u(S \cup h,\mathcal{P}'') \\ &= v(S,\mathcal{P}''), \end{aligned}$$

where the last equality holds because of the second line of (2.2.7).

Now, for any $S \subseteq N = M \setminus h$, we have

$$\begin{aligned} \sum_{j \in S \cup h} y_j = \sum_{j \in S} x_j &\geq v(S,\psi(N,v,S)) \\ &= u(S \cup h,\psi(M,u,S \cup h)). \end{aligned}$$

The last equality holds because ψ is CC and v is a complement-reduced game of u. In addition, by the third line of (2.2.7), for any $S \subseteq N = M \setminus h$ and any $\mathcal{P}''' \in \Pi(M \setminus S)$, we have

$$\sum_{j \in S} y_j = \sum_{j \in S} x_j = u(S,\mathcal{P}''').$$

This completes the proof of Claim 1. □

Claim 2 $\{y\} = C^{\psi}(M,u)$.

Proof. If there exists $z \in C^{\psi}(M,u)$ such that $z \neq y$, we must have $\sum_{j \in M} z_j = u(M,\{\emptyset\}) = v(N,\{\emptyset\}) = \sum_{j \in N} x_j = u(N,\{\{h\}\}) \leq \sum_{j \in N} z_j$, and $z_h \geq u(h,\mathcal{P}') = 0$ for any $\mathcal{P}' \in \Pi(M \setminus h)$. Hence, $z_h = 0$.

For any $i \in N$ and any $\mathcal{P}''' \in \Pi(M \setminus i)$, we have $z_i \geq u(i,\mathcal{P}''') = x_i = y_i$ and, also, $\sum_{j \in N} z_j = \sum_{j \in M} z_j = u(M,\{\emptyset\}) = \sum_{j \in M} y_j = \sum_{j \in N} y_j$. Thus, we obtain $z_i = y_i$ for all $i \in N$, *i.e.*, $z = y$. This completes the proof of Claim 2. □

Now, consider $x \in C^{\psi}(N,v)$ and (M,u) again. By the first half of this proof, $\sigma(M,u) \subseteq C^{\psi}(M,u)$. As mentioned above, $C^{\psi}(M,u) = \{y\}$. By connecting them, $\sigma(M,u) \subseteq C^{\psi}(M,u) = \{y\}$. By NE on Γ^{ψ}, we obtain $\sigma(M,u) = C^{\psi}(M,u) = \{y\}$. Furthermore, by comp-RGP and $v = u^{M\setminus h,y}$, we have $x = y_N \in \sigma(N, u^{M\setminus h,y}) = \sigma(N,v)$. Thus, $C^{\psi}(N,v) \subseteq \sigma(N,v)$. □

Proposition 2.2.9 states that we can generalize the axiomatization of the core of games without externalities by using expectation function ψ. In Proposition 2.2.9, ψ is needed to be CC. More specifically, CC is a sufficient condition for the ψ-core to satisfy comp-RGP. To enhance the intuition behind this requirement, we offer the following example.

Example 2.2.10. Let $N = \{1,2,3,4\}$. We consider the expectation function ψ given by (2.2.2) in Example 2.2.4, which satisfies subset consistency but not CC. Consider the following game: for mutually different players $i,j,k,h \in N$,

$$v(\{i,j,k,h\},\emptyset) = 12,\ v(\{i,j,k\},\{\{h\}\}) = 9,\ v(\{i,j\},\{\{k,h\}\}) = 6,\ v(\{i,j\},\{\{k\},\{h\}\}) = 7,$$

$$v(\{i\},\mathcal{P}) = 0 \text{ for any } \mathcal{P} \in \Pi(N \setminus i).$$

The ψ-core of this game is $C^{\psi} = \{(3,3,3,3)\}$, which is equal to the core based on the merge-expectation. We reduce this game to $\{1,2,3\}$ with $x = (3,3,3,3)$. For mutually different players $i,j,k \in N \setminus 4$,

$$v^{-4}(\{i,j,k\},\emptyset) = 9,\ v^{-4}(\{i,j\},\{\{k\}\}) = 6,\ v^{-4}(\{i\},\{\{j,k\}\}) = 3,\ v^{-4}(\{i\},\{\{j\},\{k\}\}) = 4.$$

The ψ-core of this reduced game is empty. This is because the ψ-core is equal to the core based on the singleton-expectation in the player set $N \setminus 4$ ($\subsetneq N$). In contrast, if ψ is CC, we have a consistent expectation function and the corresponding core: if ψ is the merge-expectation within N, then it should be the merge-expectation within $N \setminus 4$, and, similarly, if it is the singleton-expectation within N, then the singleton-expectation within $N \setminus 4$ as well. This property allows the ψ-core to satisfy comp-RGP.

Note that if a game has no externalities, this axiomatization coincides with Tadenuma's approach. Therefore, the following corollary readily follows.

Corollary 2.2.11. The four types of cores, *i.e.*, C^{opt}, C^{pes}, C^{s} and C^{m}, can be axiomatized with axioms 1-3 on each class: $\Gamma_{C^{\mathrm{opt}}}, \Gamma_{C^{\mathrm{pes}}}, \Gamma_{C^{\mathrm{s}}}$ and $\Gamma_{C^{\mathrm{m}}}$, respectively.

An example of expectation function that satisfies neither CC nor subset consistency is

$$\psi(N,v,S) = \begin{cases} \{N \setminus S\} & \text{if } |S| \geq 2, \\ \{\{i_{s+1}\},...,\{i_n\}\} & \text{if } |S| = 1. \end{cases}$$

This expectation function can be seen as a combination of the merge-expectation and the singleton-expectation.

2.2.4 The other reduced games

In view of Proposition 2.2.9, one might consider that the analogous proof probably similarly applies to the other types of reduced games. However, this conjecture is not necessarily true.

To see this, we will extend the max-reduced game (Davis and Maschler, 1965) and the projection-reduced game (Funaki and Yamato, 2001). This extension includes two technical difficulties. First, we need a partition as the additional specifier to define the reduced game. We use $v^{S,\mathcal{P},x}$ to denote the reduced game instead of the previous notation $v^{S,x}$. Second, the generalization of the max-reduced game yields two possible extensions: max-I and max-II. The difference between the two is ascribed to the domain of maximization. For any coalition $S \subseteq N$, the former ignores the partition structure of $N \setminus S$ and chooses $C \subseteq N \setminus S$, while the latter chooses C in the partition of $N \setminus S$.

Fix a set of games $\Gamma \subseteq \Gamma_A$ and a game $(N, v) \in \Gamma$. Let $S \subseteq N$ $(S \neq \emptyset)$, $\mathcal{P} \in \Pi(N \setminus S)$, and $x \in \mathbb{R}^N$.

Definition 2.2.12. The *max-reduced game (I) with respect to* $S, \mathcal{P}$ *and* x is the game $(S, v_{m1}^{S,\mathcal{P},x})$ defined as follows: for any $T \subseteq S$ $(T \neq \emptyset)$ and any $\mathcal{Q} \in \Pi(S \setminus T)$,

$$v_{m1}^{S,\mathcal{P},x}(T, \mathcal{Q}) = \begin{cases} \max\limits_{C \subseteq N \setminus S} \left[v(T \cup C, \mathcal{Q} \cup (\mathcal{P}|_{(N \setminus S) \setminus C})) - \sum\limits_{j \in C} x_j \right], & \text{if } T \subsetneq S \\ v(N, \{\emptyset\}) - \sum_{j \in N \setminus S} x_j, & \text{if } T = S. \end{cases}$$

The *max-reduced game (II)*, $(S, v_{m2}^{S,\mathcal{P},x})$, is also defined by replacing the domain of the maximization $C \subseteq N \setminus S$ with $\mathcal{C} \subseteq \mathcal{P}$,

$$v_{m2}^{S,\mathcal{P},x}(T, \mathcal{Q}) = \begin{cases} \max\limits_{\emptyset \subseteq \mathcal{C} \subseteq \mathcal{P}} \left[v(T \cup \bar{C}, \mathcal{Q} \cup (\mathcal{P} \setminus \mathcal{C})) - \sum\limits_{j \in \bar{C}} x_j \right], & \text{if } T \subsetneq S \\ v(N, \{\emptyset\}) - \sum_{j \in N \setminus S} x_j, & \text{if } T = S, \end{cases}$$

where $\bar{C} = \bigcup_{C_i \in \mathcal{C}} C_i$.

Definition 2.2.13. The *projection-reduced game with respect to* $S, \mathcal{P}$ *and* x is the game $(S, v_p^{S,\mathcal{P},x})$ defined as follows: for any $T \subseteq S$ $(T \neq \emptyset)$ and any $\mathcal{Q} \in \Pi(S \setminus T)$,

$$v_p^{S,\mathcal{P},x}(T, \mathcal{Q}) = \begin{cases} v(T, \mathcal{Q} \cup \mathcal{P}), & \text{if } T \subsetneq S, \\ v(N, \{\emptyset\}) - \sum_{j \in N \setminus S} x_j, & \text{if } T = S. \end{cases}$$

If players leave the game one by one, the max-reduced games (both I and II) and the projection-reduced game are all independent of the order of the leaving players as well as the complement-reduced game. However, if two or more players simultaneously leave the game as a single group, the max-reduced games (I, II) and the projection-reduced game may depend on the partition of the leaving players.[2] This contrasts with the fact that the complement-reduced game is independent of the partition of the leaving players.

The gap between "one-by-one leaving" and "at-once leaving" yields two RGPs. We call them "one-by-one RGP" and "at-once RGP." It is clear that the one-by-one RGP is a weaker property than the at-once RGP. We restrict our attention to the weaker RGP, *i.e.*, one-by-one RGP, and denote it, simply, RGP hereafter.

[2] Formally, as in Lemma 2.2.2, we have $(v_{m1}^{-i_1})_{m1}^{-i_2} = (v_{m1}^{-i_2})_{m1}^{-i_1}$. However, there possibly exist partitions $\mathcal{P}$ and $\mathcal{P}'$ such that $v_{m1}^{N \setminus \{i_1, i_2\}, \mathcal{P}, x} \neq v_{m1}^{N \setminus \{i_1, i_2\}, \mathcal{P}', x}$, where (m1) can be replaced with (m2) or (p).

Now, we return to the main question of this section: can we adapt the technique of Proposition 2.2.9 to the proof of the axiomatizations for the max- and the projection-reduced game? Tables 2.3 and 2.4 describe its difficulty. Table 2.3 describes the relationship between the cores and RGPs, and Table 2.4 shows the relationship between the four expectation functions and consistencies. There is no ambiguity in defining these consistencies in Table 2.4. We define them by simply replacing $v^{N\backslash h,x}$ in Definition 2.2.5 with $v_{m1}^{N\backslash h,\{h\},x}$, $v_{m2}^{N\backslash h,\{h\},x}$ or $v_p^{N\backslash h,\{h\},x}$, namely, we use the weaker definition based on one-by-one leaving. We now consider, for example, the max(I)-reduced game. The max(I)-version of Proposition 2.2.9 should be as follows: If ψ is max(I)-consistent, then ψ-core C^ψ is the unique function on Γ_{C^ψ} that satisfies max(I)-RGP and some axioms.

As Table 2.4 shows, the singleton-expectation and the merge-expectation satisfy the max(I)-consistency. However, as Table 2.3 shows, the s-core and the m-core do not satisfy max(I)-RGP. Therefore, for each expectation function ψ, either the ψ-core violates the max(I)-RGP or ψ violates the max(I)-consistency, except for the complement-RGP and the complement-consistency (CC). This is the difficulty of the straightforward generalization of the axiomatization using reduced game consistency. In other words, the completion of Proposition 2.2.9 is based on the coincidence illustrated in Tables 2.3 and 2.4. Note that all propositions and examples of Tables 2.3 and 2.4 are found in the next subsection.

Table 2.3: The cores and RGPs

	RGP			
	Max(I)	Max(II)	Projection	Complement
Optimistic core	✓	✓	✓	✓
Pessimistic core	-	-	-	✓
s-Core	-	-	-	✓
m-Core	-	-	-	✓

Table 2.4: The expectation functions and consistencies

	Consistency			
	Max(I)	Max(II)	Projection	Complement
Optimistic expectation	-	-	-	✓
Pessimistic expectation	-	-	-	✓
Singleton-expectation	✓	✓	✓	✓
Merge-expectation	✓	✓	✓	✓

2.2.5 Proofs and examples for Tables 2.3 and 2.4

We examine all of the propositions and examples in the two tables above. To distinguish each form of reduced game, we use symbols $v_{m1}^{S,\mathcal{P},x}$, $v_{m2}^{S,\mathcal{P},x}$, $v_p^{S,\mathcal{P},x}$ and $v_c^{S,x}$ to denote max(I), max(II), projection and complement-type of reduced game, respectively. Table A.2.5 and Table A.2.6 correspond to Table 2.3 and Table 2.4, respectively. The number assigned to each cell represents the proposition or example describing the cell, *e.g.*, for the proposition showing that the optimistic core satisfies Max-I RGP, see Proposition 2.2.21.

Table A. 2.5: The cores and RGPs (corresponding to Table 2.3)

	RGP			
	Max(I)	Max(II)	Projection	Complement
Optimistic core	✓ Prop.2.2.21	✓ Prop.2.2.21	✓ Prop.2.2.21	✓ Prop.2.2.7
Pessimistic core	- Ex.2.2.22	- Ex.2.2.22	- Ex.2.2.24	✓ Prop.2.2.7
s-Core	- Ex.2.2.23	- Ex.2.2.23	- Ex.2.2.24	✓ Prop.2.2.7
m-Core	- Ex.2.2.22	- Ex.2.2.22	- Ex.2.2.25	✓ Prop.2.2.7

Table A. 2.6: The expectation functions and consistencies (corresponding to Table 2.4)

	Consistency			
	Max(I)	Max(II)	Projection	Complement
Optimistic expectation	- Ex.2.2.19	- Ex.2.2.19	- Ex.2.2.19	✓ Prop.2.2.14
Pessimistic expectation	- Ex.2.2.19	- Ex.2.2.19	- Ex.2.2.19	✓ Prop.2.2.14
Singleton-expectation	✓ Cor.2.2.18	✓ Cor.2.2.18	✓ Cor.2.2.18	✓ Cor.2.2.18
Merge-expectation	✓ Cor.2.2.18	✓ Cor.2.2.18	✓ Cor.2.2.18	✓ Cor.2.2.18

Proposition 2.2.14. If ψ is optimistic or pessimistic, then ψ is CC.

Proof. We denote by ψ^{opt} the optimistic expectation function. Let (N, v) be a game, and $S \subseteq N$ $(|S| \geq 2)$. We define $\mathcal{P}^*$ as follows:

$$\mathcal{P}^* := \psi^{opt}(N, v, S) = \underset{\mathcal{P}' \in \Pi(N \setminus S)}{\arg\max}\ v(S, \mathcal{P}'). \tag{2.2.8}$$

For any $h \in S$ and $x \in \mathbb{R}^N$, we have

$$\begin{aligned} v_c^{N\setminus h,x}(S \setminus h, \mathcal{P}^*) &= v(S, \mathcal{P}^*) - x_h \\ &= \max_{\mathcal{P}' \in \Pi(N\setminus S)} [v(S, \mathcal{P}') - x_h] \\ &= \max_{\mathcal{P}' \in \Pi(N\setminus S)} [v_c^{N\setminus h,x}(S \setminus h, \mathcal{P}')], \end{aligned}$$

where the first equality holds by the definition of complement reduced games, the second by (2.2.8) and the last by the definition of complement reduced games. Hence, we obtain

$$\mathcal{P}^* = \underset{\mathcal{P}' \in \Pi(N \setminus S)}{\arg\max}\ v_c^{N\setminus h,x}(S \setminus h, \mathcal{P}') = \psi^{opt}(N \setminus h, v_c^{N\setminus h,x}, S \setminus h),$$

and, then, $\psi^{opt}(N, v, S) = \psi^{opt}(N \setminus h, v_c^{N\setminus h,x}, S \setminus h)$, which implies ψ^{opt} is CC. By replacing max with min, we complete the proof of the pessimistic expectation function ψ^{pes} as well. □

Proposition 2.2.15. If ψ satisfies the following condition: for any games (N, v), (M, w), and nonempty coalitions $S \subseteq N$, $T \subseteq M$,

$$N \setminus S = M \setminus T \Longrightarrow \psi(N, v, S) = \psi(M, w, T), \tag{2.2.9}$$

then ψ satisfies all four types of consistencies: Max-I, Max-II, Projection and Complement.

Proof. We prove CC (or, complement consistency). The other types of consistencies are obtained in the same way. Fix a game (N, v). For any $x \in \mathbb{R}^N$ and $h \in N$, we can specify the complement reduced game $(N \setminus h, v_c^{N\setminus h,x})$. For any S such that $h \in S \subseteq N$, we have $N \setminus S = (N \setminus h) \setminus (S \setminus h)$. In view of (2.2.9), we obtain $\psi(N, v, S) = \psi(N \setminus h, v_c^{N\setminus h,x}, S \setminus h,)$ □

Lemma 2.2.16. If ψ is the singleton-expectation, then ψ satisfies (2.2.9).

Proof. We denote by ψ^s the singleton-expectation function. For any nonempty T and S with $T \in S \subseteq N$, and any $w : EC(N\setminus T) \to \mathbb{R}$, we have $\psi^s(N, v, S) = \{\{i\}|i \in N\setminus S\} = \psi^s(N\setminus T, w, S\setminus T)$. □

Lemma 2.2.17. If ψ is the merge-expectation, then ψ satisfies (2.2.9).

Proof. This is similar to Lemma 2.2.16. Let ψ^m denote the merge-expectation function. We have $\psi^m(N, v, S) = \{N \setminus S\} = \psi^m(N \setminus T, w, S \setminus T)$. □

Corollary 2.2.18. If ψ is the singleton-expectation or the merge-expectation, then ψ satisfies all four types of consistencies.

Proof. See Lemmas 2.2.16, 2.2.17 and Proposition 2.2.15. □

Example 2.2.19. Consider the following 4-player game: $N = \{1, 2, 3, 4\}$; for any $i, j, k, h \in N$,

$$v(S, \mathcal{P}) = \begin{cases} 12 & \text{if } (S, \mathcal{P}) = (N, \{\emptyset\}), \\ 5 & \text{if } (S, \mathcal{P}) = (\{i, j, k\}, \{\{h\}\}), \\ 0 & \text{if } (S, \mathcal{P}) = (\{i\}, \{\{j, k, h\}\}), \\ 4 & \text{if } (S, \mathcal{P}) = (\{i, j\}, \{\{k, h\}\}), \\ 3 & \text{if } (S, \mathcal{P}) = (\{i, j\}, \{\{k\}, \{h\}\}), \\ 1 & \text{if } (S, \mathcal{P}) = (\{i\}, \{\{j, k\}, \{h\}\}), \\ 2 & \text{if } (S, \mathcal{P}) = (\{i\}, \{\{j\}, \{k\}, \{h\}\}). \end{cases}$$

Let $x = (3, 3, 3, 3)$, $S = \{1, 2\}$ and player $h = 1$. For the optimistic expectation function, we have

$$\underset{\mathcal{P}'\in\Pi(N\setminus S)}{\arg\max}\ v(S, \mathcal{P}') = \{\{3, 4\}\},$$

because $\max_{\mathcal{P}'\in\Pi(N\setminus S)} v(S, \mathcal{P}') = \max\{v(S, \{\{3, 4\}\}), v(S, \{\{3\}, \{4\}\})\} = \max\{4, 3\}$. However, in the Max-I reduced game, we have $\underset{\mathcal{P}'\in\Pi(N\setminus S)}{\arg\max}\ v_{m1}^{-h}(S \setminus h, \mathcal{P}') = \{\{3\}, \{4\}\}$, because

$$\begin{aligned} & \max_{\mathcal{P}'\in\Pi(N\setminus S)} v_{m1}^{-h}(S \setminus h, \mathcal{P}') \\ = \ & \max \left\{ \begin{array}{ll} v(S, \{\{3, 4\}\}) - x_h, & v(S \setminus h, \{\{1\}, \{3, 4\}\}), \\ v(S, \{\{3\}, \{4\}\}) - x_h, & v(S \setminus h, \{\{1\}, \{3\}, \{4\}\}) \end{array} \right\} \qquad (2.2.10) \\ = \ & \max\{4 - 3, 1, 3 - 3, 2\} \\ = \ & 2, \end{aligned}$$

which is the worth of the bottom-right element in (2.2.10). Hence, $\psi^{opt}(N, v, S) = \{\{3,4\}\} \neq \{\{3\},\{4\}\} = \psi^{opt}(N \setminus h, v_{m1}^{-h}, S \setminus h)$. For the optimistic expectation function, this example is still valid for Max-II and Projection consistencies as well. For the pessimistic expectation function, we can generate the example by swapping $v(\{i,j\},\{\{i,j\},\{k,h\}\})$ for $v(\{i,j\},\{\{i,j\},\{k\},\{h\}\})$.

Lemma 2.2.20. Let $(N, v) \in \Gamma$. Let $S \subseteq N$, $\mathcal{P} \in \Pi(N \setminus S)$ and $x \in \mathbb{R}^N$. We denote each type of reduced game by $v_{m1}^{S,\mathcal{P},x}$, $v_{m2}^{S,\mathcal{P},x}$, $v_p^{S,\mathcal{P},x}$ and $v_c^{S,x}$, respectively. Then, for any $T \subseteq S$ ($T \neq \emptyset$) and $\mathcal{Q} \in \Pi(S \setminus T)$, we have

$$\begin{aligned}
v_{m1}^{S,\mathcal{P},x}(T,\mathcal{Q}) &\geq v_{m2}^{S,\mathcal{P},x}(T,\mathcal{Q}),\\
v_{m2}^{S,\mathcal{P},x}(T,\mathcal{Q}) &\geq v_{p}^{S,\mathcal{P},x}(T,\mathcal{Q}),\\
v_{m2}^{S,\mathcal{P},x}(T,\mathcal{Q}) &\geq v_{c}^{S,x}(T,\mathcal{Q}).
\end{aligned}$$

Proof. The first inequality follows from the domain of maximization: in view of the definitions, for any $\mathcal{P} \in \Pi(N \setminus S)$,

$$\{C | C \in \mathcal{P}\} \text{ (or, Max-II) } \subseteq \{C | C \subseteq N \setminus S\} \text{ (or, Max-I) }.$$

The second (third) inequality holds because we can take $\emptyset$ ($N \setminus S$) as maximizer C. □

Proposition 2.2.21. The optimistic-core satisfies all types of RGP on $\Gamma_{C^{\text{opt}}}$: maxI-RGP, maxII-RGP, projection-RGP and comp-RGP.

Proof. Let $C^{\text{opt}}(N, v)$ be the optimistic core of (N, v) and $x \in C^{\text{opt}}(N, v)$. We show that the optimistic-core satisfies maxI-RGP. For any $S \subseteq N$, $T \subsetneq S$ ($T \neq \emptyset$) and $\mathcal{P} \in \Pi(N \setminus S)$, we have

$$\begin{aligned}
& \sum_{j\in T} x_j - \max_{\mathcal{Q}\in\Pi(S\setminus T)} v_{m1}^{S,\mathcal{P},x}(T,\mathcal{Q}) \\
= \;& \sum_{j\in T} x_j - \max_{\mathcal{Q}\in\Pi(S\setminus T)} \max_{C\subseteq N\setminus S} \left[v(T\cup C, \mathcal{Q}\cup(\mathcal{P}|_{(N\setminus S)\setminus C})) - \sum_{j\in C} x_j \right] \\
= \;& \sum_{j\in T} x_j - \left[v(T\cup C^*, \mathcal{Q}^*\cup(\mathcal{P}|_{(N\setminus S)\setminus C^*})) - \sum_{j\in C^*} x_j \right] \qquad (2.2.11)\\
= \;& \sum_{j\in T\cup C^*} x_j - v(T\cup C^*, \mathcal{Q}^*\cup(\mathcal{P}|_{(N\setminus S)\setminus C^*})) \\
\geq \;& \max_{\mathcal{P}'\in\Pi(N\setminus(T\cup C^*))} v(T\cup C^*, \mathcal{P}') - v(T\cup C^*, \mathcal{Q}^*\cup(\mathcal{P}|_{(N\setminus S)\setminus C^*})) \qquad (2.2.12)\\
\geq \;& 0,
\end{aligned}$$

where $C^*, \mathcal{Q}^*$ in (2.2.11) are maximizers of the target formula, and (2.2.12) holds because $x \in C^{\text{opt}}(N, v)$. Similarly, for $T = S$, we have

$$\sum_{j\in S} x_j - v^{S,\mathcal{P},x}(S,\{\emptyset\}) = \sum_{j\in S} x_j - \left(v(N,\{\emptyset\}) - \sum_{j\in N\setminus S} x_j \right) = 0.$$

By Lemma 2.2.20, we can replace $v_{m1}^{S,\mathcal{P},x}$ with $v_{m2}^{S,\mathcal{P},x}$, $v_p^{S,\mathcal{P},x}$ and $v_c^{S,\mathcal{P},x}$, respectively. Then, we obtain the desired proposition. □

Example 2.2.22. Consider the following four-player game: $N = \{1, 2, 3, 4\}$,

$$v(S, \mathcal{P}) = \begin{cases} 12 & \text{if } (S, \mathcal{P}) = (N, \{\emptyset\}), \\ 6 & \text{if } (S, \mathcal{P}) = (\{i, j\}, \{\{k\}, \{h\}\}), \\ 0 & \text{otherwise.} \end{cases}$$

Let $x = (x_1, x_2, x_3, x_4) = (1, 3, 4, 4)$. Then, $x \in C^{\text{pes}}(N, v) = C^{\text{m}}(N, v)$. Now, for $S = \{1, 2\}$ and $\mathcal{P} = \{\{3\}, \{4\}\}$, we have the following Max-I reduced game:

$$\begin{aligned} v_{m1}^{S,P,x}(\{1, 2\}, \{\emptyset\}) &= 12 - (4 + 4) = 4, \\ v_{m1}^{S,P,x}(\{1\}, \{\{2\}\}) &= 6 - 4 = 2, \\ v_{m1}^{S,P,x}(\{2\}, \{\{1\}\}) &= 6 - 4 = 2. \end{aligned}$$

The restriction of x, $x_S = (1, 3)$, is out of the pessimistic core (and the m-core) of the reduced game: $x_S = (1, 3) \notin \{(2, 2)\} = C^{\text{pes}}(S, v_{m1}^{S,P,x}) = C^{\text{m}}(S, v_{m1}^{S,P,x})$. We have the Max-II reduced game as well as Max-I.

Example 2.2.23. Consider the following five-player game: $N = \{1, 2, 3, 4, 5\}$,

$$v(S, \mathcal{P}) = \begin{cases} 15 & \text{if } (S, \mathcal{P}) = (N, \{\emptyset\}), \\ 7 & \text{if } (S, \mathcal{P}) = (\{i, j\}, \{\{k\}, \{h, l\}\}), \\ 0 & \text{otherwise.} \end{cases}$$

Let $x = (x_1, x_2, x_3, x_4, x_5) = (2, 2, 4, 4, 3)$. Then, $x \in C^{\text{s}}(N, v)$. For $S = \{3, 4\}$ (who obtain 4 in x) and $\mathcal{P} = \{\{1\}, \{2, 5\}\}$, we have the following Max-I reduced game:

$$\begin{aligned} v_{m1}^{S,P,x}(\{3, 4\}, \{\emptyset\}) &= 15 - (2 + 2 + 3) = 8, \\ v_{m1}^{S,P,x}(\{3\}, \{\{4\}\}) &= 7 - 2 = 5, \\ v_{m1}^{S,P,x}(\{4\}, \{\{3\}\}) &= 7 - 2 = 5. \end{aligned}$$

Hence, the s-core is empty. We have the same result in Max-II as well as Max-I.

Example 2.2.24. Consider the following four-player game: $N = \{1, 2, 3, 4\}$,

$$v(S, \mathcal{P}) = \begin{cases} 12 & \text{if } (S, \mathcal{P}) = (N, \{\emptyset\}), \\ 4 & \text{if } (S, \mathcal{P}) = (\{i\}, \{\{j, k, h\}\}), \\ 4 & \text{if } (S, \mathcal{P}) = (\{i\}, \{\{j, k\}, \{h\}\}), \\ 3 & \text{if } (S, \mathcal{P}) = (\{i\}, \{\{j\}, \{k\}, \{h\}\}), \\ 0 & \text{otherwise.} \end{cases}$$

Let $x = (x_1, x_2, x_3, x_4) = (3, 3, 3, 3)$. Then, $x \in C^{\text{pes}}(N, v) = C^{\text{s}}(N, v)$. For $S = \{1, 2\}$ and $\mathcal{P} = \{\{3, 4\}\}$, we have the following projection reduced game:

$$\begin{aligned} v_{p}^{S,P,x}(\{1, 2\}, \{\emptyset\}) &= 12 - (3 + 3) = 6, \\ v_{p}^{S,P,x}(\{1\}, \{\{2\}\}) &= 4, \\ v_{p}^{S,P,x}(\{2\}, \{\{1\}\}) &= 4. \end{aligned}$$

Hence, the pessimistic core and the s-core are empty in the reduced game.

Example 2.2.25. Consider the following four-player game: $N = \{1, 2, 3, 4\}$,

$$v(S, \mathcal{P}) = \begin{cases} 12 & \text{if } (S, \mathcal{P}) = (N, \{\emptyset\}), \\ 3 & \text{if } (S, \mathcal{P}) = (\{i\}, \{\{j, k, h\}\}), \\ 4 & \text{if } (S, \mathcal{P}) = (\{i\}, \{\{j, k\}, \{h\}\}), \\ 4 & \text{if } (S, \mathcal{P}) = (\{i\}, \{\{j\}, \{k\}, \{h\}\}), \\ 0 & \text{otherwise.} \end{cases}$$

Let $x = (x_1, x_2, x_3, x_4) = (3, 3, 3, 3)$. Then, $x \in C^{\mathrm{m}}(N, v)$. For $S = \{1, 2\}$ and $\mathcal{P} = \{\{3, 4\}\}$, we have the following projection reduced game:

$$\begin{aligned} v_p^{S,P,x}(\{1, 2\}, \{\emptyset\}) &= 12 - (3 + 3) = 6, \\ v_p^{S,P,x}(\{1\}, \{\{2\}\}) &= 4, \\ v_p^{S,P,x}(\{2\}, \{\{1\}\}) &= 4. \end{aligned}$$

Hence, the m-core of the reduced game becomes empty.

2.3 Discussions

2.3.1 The credible ψ-core

Throughout this chapter, we discussed *one-shot* deviation and did not consider the case that, after a coalition S deviates, a subcoalition $T \subseteq S$ deviates from S. For coalition function form games, Ray (1989) recursively defines the notion of *credibility* of coalitions and the credible core by restricting deviations to credible coalitions. He showed that the credible core (called the modified core in his paper) coincides with the usual core. In his proof, the following fact plays a central role: if a subcoalition T deviates from S after S's deviation, then T can directly deviate from N. If externalities exist, an incentive for players to deviate depends on the partition their deviation leads to. Does this insight similarly hold for environments with externalities? We examine two approaches to answer this question.

One is a recursive approach, which is summarized in the following equality:

$$\{T\} \cup \psi(S, v, T) \cup \psi(N, v, S) = \{T\} \cup \psi(N, v, T).$$

The left-hand side describes that coalition S first deviates from N and, then, $T \subseteq S$ deviates from S. The right-hand side is the direct deviation from N. If this equality holds, then the corresponding credible core, say the *credible ψ-core*, coincides with the ψ-core.

Although this coincidence clearly does not hold for many cores, it does for the singleton-core. As for the left-hand side, the partition resulted from the first deviation S is $\mathcal{P}_1 = \{S\} \cup \{\{i_{s+1}\}, ..., \{i_n\}\}$ and, hence, the final partition is $\mathcal{P}_2 = \{T\} \cup \{\{i_{t+1}\}, ..., \{i_s\}, \{i_{s+1}\}, ..., \{i_n\}\}$; for the right-hand side, $\mathcal{Q} = \{T\} \cup \{\{i_{t+1}\}, ..., \{i_n\}\}$. Therefore, we have $\mathcal{P}_2 = \mathcal{Q}$, and the statement holds. In contrast, the optimistic, pessimistic, and even merge-expectation functions do not obey this equality. Given Figure 2.2, this consistency-like property differently rules out expectation functions. One might consider that it is artificial to fix the partition of $N \setminus S$ for the second deviation. Of course, we can change the second deviation so as to include all players. However, the property defined by this change rules out no expectation functions, because there is no difference between the second deviation and the direct deviation. In other words, both the partitions resulted from the second deviation and the direct deviation always refer to the same partition $\psi(N, v, T)$. The reason of this lies in the fact that the second partition does not depend on the first partition. To resolve this problem and establish a meaningful property, we should consider the following approach.

The other approach is to extend an expectation function to incorporate a partition into the system of variables. We define $\psi(N, v, s, \mathcal{P})$: an expectation also depends on the partition from which players deviate. Now the equality we examine is

$$\{T\} \cup \psi(N, v, T, \psi(N, v, S, \{N\})) = \{T\} \cup \psi(N, v, T, \{N\}).$$

This might also be seen as a recursive approach. A typical expectation function that matches this extension is the *projective expectation.* Although the same concept appears in the following chapter in a slightly different way, the definition is $\psi(N, v, S, \mathcal{P}) = \mathcal{P}|_{N \setminus S}$. The core based on this concept, however, does not meet the the credible version of the core. This is because as long as the initial partition is $\{N\}$, this core is equal to the merge-core. If we set a different partition as the initial partition, then the discussion drastically changes. We furthermore discuss this extension in Subsections 2.3.2 and 2.3.3.

2.3.2 The core defined on other partitions

Greenberg (1994) defines the *coalition structure core* for games with externalities, which is a set of pairs consisting of a payoff vector x and a partition $\mathcal{P}$ from which no coalition deviates. The core concept described by Kóczy and Lauwers (2004) and the recursive core studied by Kóczy (2007) meet the spirit of the coalition structure core.

Once we consider the coalition structure core, the nonemptiness of each core becomes a more challenging question. As an easy example, we consider the pessimistic (coalition structure) core of the three-person game in Table 2.7. There are many pessimistic core elements. For example,

Table 2.7: A game for the coalition structure core

Partition	{1,2,3}			{1,23}		{2,13}		{3,12}		{123}
Coalition	1	2	3	1	23	2	13	3	12	123
$v(S, \mathcal{P})$	0	0	0	4	7	4	7	4	7	10

$((4, 3.5, 3.5), \{1, 23\})$ is in the pessimistic core (but not in the optimistic core). As this example implicitly describes, if the sum of worths in a partition is larger than or equal to that of the grand coalition, then some nonempty ψ-core appears on coarse partitions. The model we analyze in Chapter 3 incorporates this difficulty.

2.3.3 The path independence of a reduced game

We revisit the extension of expectation function we considered in Subsection 2.3.1. This extension also relates to the path independence of a reduced game. We now compare the independence of a reduced game on the order of leaving players, as is described by Lemma 2.2.2, with similar notions studied by Dutta et al. (2010) and Bloch and van den Nouweland (2014).

Bloch and van den Nouweland (2014) define path independence for expectation functions. Let $\psi(N, v, S, \mathcal{P})$ be an expectation function again. Now, an expectation function satisfies path independence if for any nonempty disjoint coalitions $S, T \subseteq N$ and any $\mathcal{P} \in \Pi(N)$,

$$\psi(N, v, S \cup T, \{S\} \cup \psi(N, v, S, \mathcal{P})) = \psi(N, v, S \cup T, \{T\} \cup \psi(N, v, T, \mathcal{P})).$$

In words, if path independence does not hold, the expectation of a coalition depends on the order in which each member's expectation is aggregated. Bloch and van den Nouweland (2014) show that many "reasonable" expectation functions (including the four expectation functions) obey path independence.

Dutta et al. (2010) define the notion of *restriction operator*. A restriction operator r specifies a subgame (not a reduced game) $v^{-i,r} : EC(N \setminus i) \to \mathbb{R}$ for each player i. More specifically, an operator $r^N_{i,S,\mathcal{Q}}$ determines which coalition S in $\mathcal{Q} \cup \emptyset$ to be merged with player i, namely, $v^{-i,r}(S, \mathcal{Q}) = r^N_{i,S,\mathcal{Q}}(\ v(S, \{(S_1 \cup i), S_2, ..., S_q\}),\ ...,\ v(S, \{S_1, S_2, ..., (S_q \cup i)\}),\ v(S, \{S_1, S_2, ..., S_q, \{i\}\})\ )$. They define path independence as follows: a restriction operator r satisfies path independence if for any (N, v) and any $i, j \in N$, $(v^{-i,r})^{-j,r} = (v^{-j,r})^{-i,r}$. This condition requires the subgame obtained by the order i, j to coincide with that by the order j, i. Note that their path independence is (similar, but) not exactly the same as ours: their independence guarantees the same worth of each coalition between two subgames with different paths, while ours guarantees that between two reduced games for any $x \in \mathbb{R}^N$. In other words, the path considered by Dutta et al. (2010) focuses on which **player** moves to which coalition, whereas our path describes which **game** is specified by player i and agreement x. Therefore, in this sense, their path independence is probably closer to that of expectation functions. An operator r reduces a game with externalities to a game without externalities by defining $w^r_v(S) = v(S, \mathcal{P}_r)$ for any $S \subsetneq N$ and $w^r_v(N) = v(N, \emptyset)$, where $\mathcal{P}_r$ is a partition of $N \setminus S$ specified by r and the $|N| - |S|$ times leavings.

2.3.4 Other expectation functions

As sometimes seen in the papers studying value concepts, a probabilistic approach is also possible, in which $\psi(N, v, S)$ is a probability distribution over $\Pi(N \setminus S)$. It is clear that any convex combination of the four expectation functions listed earlier, for example 50% for the best partition and 50% for the worst partition, is still CC.

Moreover, as Kóczy (2007) and Bloch and van den Nouweland (2014) pointed out, the assumption that a coalition S keeps a single coalition after its deviation is not necessarily general as the coalition may break up into some multiple coalitions after the deviation. Since our assumption of a single coalition can be thought of as a specific case of this general framework, we need to check more deviations under the general setting. For example, an allocation x which prevents players from deviating with a single coalition may allow them to deviate with multiple coalitions. Therefore, the core generally shrinks under the framework of multiple coalitions.

2.3.5 The general condition for the ψ-core to be nonempty

As briefly discussed in Section 2.1, the necessary and sufficient conditions of both the optimistic core and the pessimistic core can be written in terms of $(\lambda_S^{\mathcal{P}})_{(S,\mathcal{P})\in EC(N)}$. Extending this approach, the condition for the ψ-core to be nonempty is also established for any ψ. Fix (N, v) and $S \subseteq N$. For any $\mathcal{P} \in \Pi(N \setminus S)$,

$$\lambda_S^{\mathcal{P}} = \begin{cases} 1 & \text{if } \mathcal{P} = \psi(N, v, S) \\ 0 & \text{otherwise.} \end{cases}$$

Applying this manner to each coalition $S \subseteq N$, we have the vector $(\lambda_S^{\mathcal{P}})_{(S,\mathcal{P})\in EC(N)}$ that exhibits the expectation function ψ. This vector λ allows us to incorporate the function ψ into the linear programming and obtain the condition for the corresponding ψ-core to be nonempty. For example, for the singleton-expectation, we set $\lambda_S^{\{\{i_{s+1}\},\dots,\{i_n\}\}} = 1$ and $\lambda_S^{\mathcal{P}} = 0$ for $\mathcal{P} \in \Pi(N \setminus S)$ with $\mathcal{P} \neq \{\{i_{s+1}\}, \dots, \{i_n\}\}$. Replacing the condition (ii) of Proposition 2.1.2, we obtain the corresponding condition.

3 Stable coalition structures for games with externalities

In this chapter, we analyze stable coalition structures. This chapter is based on Abe (2018b) "Stable coalition structures in symmetric majority games: A coincidence between myopia and farsightedness" published in *Theory and Decision*. In this publication, α-, β-, γ-, δ-stabilities formulated by Hart and Kurz (1984) are hardly investigated. Although these notions are a well-known approach to analyze stable coalition structures, they are difficult to use practically because the definition is composed of not only partition but also strategy terms. In Subsection 3.4.3, we offer simple equivalent formulations for these stability concepts so as not to contain such strategy terms. An extension to weighted majority games is also discussed in the next chapter.

3.1 A motivating question and stable coalition structures

In the previous chapter, we have assumed the formation of the grand coalition and discussed which allocation is stable in the sense of the core. In other words, we have restricted our attention into a specific partition such as the grand coalition and analyzed allocations feasible on the partition. We now analyze partitions by associating an allocation with each partition. Therefore, given that each partition has exactly one payoff allocation, we discuss which partition is stable. In this chapter, we often use the word *coalition structure* to refer to a partition.

A leading attempt to study the problem of stable coalition structure is probably Hart and Kurz (1984)'s work. They use a *coalition structure value* (CS-value) to represent an allocation associated with a partition.[1] A CS-value is a function that assigns a real number to every player in a given coalition structure. One of their interests is to clarify which coalition structure can be seen as a stable coalition structure in a majority voting situation. To understand their question, we assume that there are three identical players (or, voters), say 1, 2, and 3. Each player can form a coalition with some of the other players. We suppose that a coalition consisting of two or three players wins as a majority coalition. All possible coalition structures are $\{\{1,2,3\}\}$, $\{\{1,2\},\{3\}\}$ (and its symmetries), and $\{\{1\},\{2\},\{3\}\}$. The question is: Which coalition structure can we consider as a "stable" coalition structure?

In their analysis, they use the Owen power index as a CS-value to evaluate each player's *power* in a coalition structure, which is an extension of the Shapley-Shubik power index to games with a priori coalition structures. Following Hart and Kurz (1984), we employ the Owen power index

[1] Hart and Kurz (1983) focus on the theoretical aspect of the CS-value, whereas Hart and Kurz (1984) focus on its applications.

to evaluate each player's power.[2] Applying the Owen power index to every coalition structure, we obtain a list of power index profile as Table 3.1 when the number of players n is 3 (this process is elaborated in Section 3.2). It can be seen from the table that each player's power depends on the

Table 3.1: The list of power index profiles for three players

Partition	ϕ_1	ϕ_2	ϕ_3
$\{123\}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\{12, 3\}$	$\frac{1}{2}$	$\frac{1}{2}$	0
$\{1, 2, 3\}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

coalition structure of the other players. For example, player 3's power is 1/3 if the other players are separated, while it is 0 if they form a coalition. This exhibits externalities among coalitions.

In addition to the power index, how should we define a "stable" coalition structure? We focus on the following three stability types: myopic, farsighted, and individual stabilities.

- Myopic stability: We employ the core notions such as the pessimistic core and the optimistic core we have discussed in the previous chapter. The projective core (see Subsection 2.3.1) is also used. These cores can be thought of as *myopic* stability concepts because each notion reflects a myopic anticipation of deviating players: the projective core describes that the players outside a deviating coalition do not react to the deviation and not reorganize their own coalition structure; and the pessimistic (optimistic) core exhibits the deviating players' pessimistic (optimistic) *one-step* expectation for the reaction of the other players.

- Farsighted stability: Farsightedness is featured with a sequence of reactions: when some players try to deviate from a coalition structure, they consider a sequence of reactions their deviation brings about. Chwe (1994), Xue (1997), and Ray and Vohra (1997) have developed the theory of farsightedness, and Diamantoudi and Xue (2003) establish the notion of *farsighted vNM stable set* in the context of hedonic games.[3] We apply this concept to our framework.

- Individual stability: This is a simple variation of the core notions. We restrict deviating coalitions into one-person coalitions. We elaborate this in Subsection 3.3.3.

Hart and Kurz (1984)'s purpose was to study stable coalition structures of a symmetric majority game in the sense of the α-, β-, γ- and δ-stabilities. However, their analysis is available only for

[2]Note that the Owen power index is not the only CS-value. Aumann and Drèze (1974), Wiese (2007), Casajus (2009), and Kamijo (2009) propose their CS-values respectively. We can classify these values into two classes in terms of efficiency. There are two efficiency concepts: component-efficiency and efficiency (to distinguish them, we call the latter N-efficiency). Component-efficiency requires that for each coalition in a coalition structure, the summation of values assigned to its members equals the worth of the coalition. On the other hand, N-efficiency requires that the summation of all players' values coincides with the worth of the grand coalition. The CS-values proposed by Aumann and Drèze (1974), Wiese (2007), and Casajus (2009) satisfy component-efficiency, while Hart and Kurz (1983, 1984) and Kamijo (2009) obey N-efficiency. As mentioned in Hart and Kurz (1984), the values satisfying N-efficiency are mainly applied to measure each player's power for given a priori coalition structures. Therefore Kamijo (2009)'s value can be applied to the same question in the manner we elaborate from now on.

[3]Ray and Vohra (2015) reformulate the farsighted stable set as a modification of Harsanyi (1974)'s formulation and propose a necessary and sufficient condition for the farsighted stable set to exist and contain a single payoff allocation.

some specific number of players: the analysis for general n was left open. Our objective is to offer the general analysis by using the stability concepts mentioned above. As we will observe, the difficulty of this analysis lies in its externalities. More specifically, the externalities in a symmetric majority game are neither positive nor negative; what is more complicated, neither nonnegative nor nonpositive. Therefore, **the typical result such as "the grand coalition or the partition into singletons are stable" is no longer guaranteed.** Since the proofs in this chapter are relatively long, we will provide them in Section 3.5.

3.2 Symmetric majority games

In this chapter, we assume $n \geq 3$. We sometimes write 12|3 to denote partition $\{\{1,2\},\{3\}\}$ for simplicity. An n-person symmetric majority *cooperative* game, $w : 2^N \to \mathbb{R}$, is given by

$$w(S) = \begin{cases} 1 & \text{if } |S| \geq k, \\ 0 & \text{otherwise,} \end{cases} \tag{3.2.1}$$

where k is the "majority," namely, the minimal number of pla . We define k as follows:

$$k = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2}+1 & \text{if } n \text{ is even.} \end{cases} \tag{3.2.2}$$

The word "majority" usually means not only exact k but also any natural number $k^+ \in [k, n]$. Therefore, to avoid confusion, we use "exact majority" to refer to k.

Now we compute the power of each player in a partition. For a given partition, we consider each coalition in the partition to be a player and compute the Shapley-Shubik power index of the corresponding weighted majority game. Below, we demonstrate the computation of each player's power in a partition, for example, $\{12, 3, 4\}$. First, we consider a weighted majority game $[q; w]$ with quota q and weight profile w. For example, consider $[3; 2, 1, 1]$, where 3 is the quota of this game (*i.e.*, the exact majority k) and $2, 1, 1$ is the weight profile based on the number of players of each coalition (*i.e.*, the coalition $\{1, 2\}$ has two votes). The S-S power index ψ of this weighted majority game is $\psi([3; 2, 1, 1]) = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$. Therefore, the coalition $\{1, 2\}$ has power $\psi_{\{1,2\}} = \frac{2}{3}$. We equally divide the S-S index ψ by the number of players of each coalition and obtain the power index ϕ with respect to each individual player: $\phi(12|3|4) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$. Formally, for any partition $\mathcal{P} \in \Pi(N)$ and any player $i \in N$, the power of player i in partition $\mathcal{P}$ is given by

$$\phi_i(\mathcal{P}) = \frac{1}{|S_i|}\psi_{S_i}(\mathcal{P}), \tag{3.2.3}$$

where S_i is the coalition in $\mathcal{P}$ which contains i as its member. For any $j \in S \in \mathcal{P}$, $\phi_S(\mathcal{P}) := \sum_{j \in S} \phi_j(\mathcal{P}) = |S|\phi_j(\mathcal{P})$ $(= \psi_S(\mathcal{P}))$. The power index given by (3.2.3) is the same as the Owen value, that is Owen (1977)'s extension of the Shapley value to games with a priori coalition structures. In general, the Owen value is given, for any cooperative game v and any partition $\mathcal{P}$ and any player i, by $\phi_i(w, \mathcal{P}) = E[v(\rho \cup \{i\}) - v(\rho)]$, where $E[\cdot]$ is the expected value taken over all orders which are consistent with the partition $\mathcal{P}$.[4] Moreover, ρ is the set of predecessors of player

[4] For example, if $N = \{1, 2, 3\}$ and $\mathcal{P} = 12|3$, then all consistent orders are 123, 213, 312, 321.

i in the random order. To obtain the Owen *power index*, we focus on w given by (3.2.1) and k given by (3.2.2). Then, for any partition $\mathcal{P} \in \Pi(N)$ and any player $i \in N$, we obtain the Owen power index $\phi_i(\mathcal{P})$ of i in $\mathcal{P}$.

For any given N, computing $\phi_i(\mathcal{P})$ with respect to every $\mathcal{P} \in \Pi(N)$ uniquely generates a list of power index profiles as Tables 3.1 and 3.2. In the tables, we use a partition to represent its symmetries: for example, by 12|3, we include all partitions with a two-person coalition and a one-person coalition such as 13|2 and 23|1.

In this chapter, we consider the list of power index profiles as a coalition formation game of symmetric majority game. We simply call it a coalition formation game.[5] In our setting, every player does *not* play the underlying symmetric majority game after forming the coalition structure, but simply be evaluated in the sense of the power index. The underlying symmetric majority game is used only to associate ϕ with the power index. Therefore, the coalition formation game ϕ can be thought of as a list that describes "who gets how much power in which coalition structure." In this sense, the main purpose of this framework is not to discuss which voting system is stable, but to analyze which coalition structure is stable given the underlying symmetric majority game and the power index.

Table 3.2: The list of power index profiles for four players

Partition	ϕ_1	ϕ_2	ϕ_3	ϕ_4
1234	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
123|4	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
12|34	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
12|3|4	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$
1|2|3|4	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

As briefly mentioned in Section 3.1, our coalition formation game has *externalities* among coalitions. For example, $\phi_4(1|2|3|4) = \frac{1}{4} \neq \frac{1}{6} = \phi_4(12|3|4)$ in Table 3.2. Note that the game has none of negative, positive, nonpositive, and nonnegative externalities. For example, consider $n = 5$ and a player, say 1. We have $\phi_1(1|2|3|45) = \frac{1}{6}$. If player 3 merges with player 2, player 1's power increases to $\phi_1(1|23|45) = \frac{1}{3}$, whereas if player 3 moves to coalition 45, it decreases to $\phi_1(1|2|345) = 0$.

3.3 Stable coalition structures

3.3.1 Definitions

For any partition $\mathcal{P} \in \Pi(N)$ and every player $i \in N$, we denote i's power in $\mathcal{P}$ by $\phi_i(\mathcal{P})$. For simplicity, we write, for any $\mathcal{P}, \mathcal{P}' \in \Pi(N)$, $\mathcal{P}' \succ_i \mathcal{P} \iff \phi_i(\mathcal{P}') > \phi_i(\mathcal{P})$; $\mathcal{P}' \succsim_i \mathcal{P} \iff \phi_i(\mathcal{P}') \geq \phi_i(\mathcal{P})$; and $\mathcal{P}' \succ_S \mathcal{P} \iff \mathcal{P}' \succ_i \mathcal{P}$ for any $i \in S$.

We first extend the three core concepts discussed in the previous chapter to this model.

[5]The coalition formation game we study in this chapter can be thought of as a cardinal version of games studied by Diamantoudi and Xue (2007). They study hedonic games where each player's preferences depend not only on a coalition but also on a coalition structure.

Definition 3.3.1. A partition $\mathcal{P}$ is in the *projective core*, C^{pro}, if there is no $S \subseteq N$ such that $S \notin \mathcal{P}$ and $\mathcal{P}' \succ_S \mathcal{P}$ where $\mathcal{P}' = \{S\} \cup (\mathcal{P}|_{N \setminus S})$.

Definition 3.3.2. A partition $\mathcal{P}$ is in the *pessimistic core*, C^{pes}, if there is no $S \subseteq N$ such that $S \notin \mathcal{P}$ and for all $\mathcal{P}' \in \Pi(N)$ with $S \in \mathcal{P}'$, $\mathcal{P}' \succ_S \mathcal{P}$.

Definition 3.3.3. A partition $\mathcal{P}$ is in the *optimistic core*, C^{opt}, if there is no $S \subseteq N$ such that $S \notin \mathcal{P}$ and for some $\mathcal{P}' \in \Pi(N)$ with $S \in \mathcal{P}'$, $\mathcal{P}' \succ_S \mathcal{P}$.

By a "*deviating coalition* S from $\mathcal{P}$ to $\mathcal{P}'$," we mean a coalition $S \subseteq N$ satisfying $\mathcal{P}' \succ_S \mathcal{P}$, where we exclude coalitions $S \in \mathcal{P}$ from the definition of deviating coalition. For example, for a partition $\mathcal{P} = 12|34$, coalitions $\{1, 2\}$ and $\{3, 4\}$ are not called a deviating coalition from $\mathcal{P}$ even if there exists a partition $\mathcal{P}'$ such that $\mathcal{P}' \succ_S \mathcal{P}$ and $S \in \mathcal{P}'$. All the other coalitions can be deviating coalitions.[6]

We now introduce the notions based on farsightedness. We adapt Diamantoudi and Xue (2003)'s formulation to our framework.

Definition 3.3.4. A partition $\mathcal{P}'$ *indirectly dominates* $\mathcal{P}$ if there exists a sequence of partitions $\mathcal{P}^1, ..., \mathcal{P}^k$ with $\mathcal{P}^1 = \mathcal{P}$ and $\mathcal{P}^k = \mathcal{P}'$ and a sequence of coalitions $S^1, ..., S^{k-1}$ such that, for every $j = 1, ..., k - 1$,

i. $\mathcal{P}^{j+1} = \{S^j\} \cup (\mathcal{P}^j|_{N \setminus S^j})$, and

ii. $\mathcal{P}' \succ_{S^j} \mathcal{P}^j$.

One might consider the indirect domination as a sequence of projective deviations. However, in each step j, the deviating coalition S^j compares the current partition $\mathcal{P}^j$ with the final destination $\mathcal{P}'$. This is the difference between the indirect domination and an ordinal sequence of projective deviations. Note that the *direct domination*, namely, one-step indirect domination, is the same as the projective deviation. It follows from Definition 3.3.4 that if $\mathcal{P}'$ directly dominates $\mathcal{P}$, then $\mathcal{P}'$ indirectly dominates $\mathcal{P}$. The farsighted vNM stable set is given as follows.

Definition 3.3.5. A *farsighted vNM stable set* V is a set of partitions satisfying the following two conditions:

i. for any $\mathcal{P}$ and $\mathcal{P}' \in V$, $\mathcal{P}'$ does not indirectly dominate $\mathcal{P}$, and

ii. for any $\mathcal{P} \in \Pi(N) \setminus V$, there exists $\mathcal{P}' \in V$ such that $\mathcal{P}'$ indirectly dominates $\mathcal{P}$.

The first condition is known as *internal stability*: any partition in V does not dominate any other partition in V. The second condition is *external stability*, which requires that every partition outside V is dominated by some partition in V. Note that there can be some farsighted vNM stable sets for a game.

[6] Strictly speaking, we do not have to exclude $S \in \mathcal{P}$ from the definition of deviation for the projective core and the pessimistic core. However, this difference does not matter in this chapter.

3.3.2 Coincidence between myopia and farsightedness

We begin with our main proposition.

Proposition 3.3.6. For any symmetric majority game, there exists a farsighted vNM stable set V such that

$$V = C^{\text{pes}}.$$

As discussed in Section 3.2, a list of power index profiles is uniquely determined for each n. Therefore, Proposition 3.3.6 states that for any number of players, some farsighted vNM stable set coincides with the pessimistic core, which is the largest myopic core. Diamantoudi and Xue (2007) argue that the farsighted vNM stable set reflects optimism of a deviating coalition in a sense: players form a deviating coalition with expecting that a beneficial coalition structure that is one of the outcomes the deviation may lead to realizes as a result. Although our optimism and pessimism are introduced as myopic concepts, Proposition 3.3.6 describes a coincidence between the (myopic) pessimism and the (farsighted) optimism.[7] Table 3.3 is an example that shows which coalition structure satisfies which stability concept for $n = 7$ (Nash and IS are discussed in Subsection 3.3.3). In this example, partitions 1234|567, 1234|56|7, and 1234|5|6|7 belong to the pessimistic core and a farsighted vNM stable set.

Table 3.3: The seven-player symmetric majority game

Partition	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	C^{pro}	C^{pes}	C^{opt}	V	Nash	IS
1234567	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$					✓	✓
123456|7	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0						✓
12345|67	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	0	0						✓
12345|6|7	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	0	0						✓
1234|567	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0		✓		✓		
1234|56|7	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	✓	✓		✓		✓
1234|5|6|7	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	✓	✓		✓		✓
123|456|7	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$						
123|45|67	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$						
123|45|6|7	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$						
123|4|5|6|7	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$						
12|34|56|7	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0						✓
12|34|5|6|7	$\frac{3}{20}$	$\frac{3}{20}$	$\frac{3}{20}$	$\frac{3}{20}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{2}{15}$						
12|3|4|5|6|7	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{2}{15}$						
1|2|3|4|5|6|7	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$						

Given the coincidence between the pessimistic core and a farsighted vNM stable set, how can we characterize the coalition structures satisfying the above two stabilities? Proposition 3.3.7 answers

[7] Diamantoudi and Xue (2007) consider *caution* as another behavioral assumption on deviating coalitions. Mauleon and Vannetelbosch (2004) also define the *largest cautious consistent set*, LCCS, as a refinement of the largest consistent set defined by Chwe (1994). They examine LCCS in games with positive externalities. The relationships between LCCS and our myopic cores are still open.

this question. Let K be a coalition whose size is exactly k, *i.e.*, $|K| = k$.

Proposition 3.3.7. For any symmetric majority game and any partition $\mathcal{P} \in \Pi(N)$,

$$\mathcal{P} \in C^{\text{pes}} \iff K \in \mathcal{P}.$$

This proposition shows that containing an exact majority coalition is a necessary and sufficient condition for a coalition structure to belong to the pessimistic core and the farsighted vNM stable set. In view of Proposition 3.3.6, Proposition 3.3.7 is equivalent to the statement $\mathcal{P} \in V \iff K \in \mathcal{P}$. Hence, internal stability implies that for any two coalition structures containing an exact majority coalition, one does not farsightedly dominate the other. Moreover, external stability indicates that every coalition structure with no exact majority coalition is farsightedly dominated by some coalition structure with exact majority coalition. In Table 3.3, partitions 1234|567, 1234|56|7, and 1234|5|6|7 are the partitions containing an exact majority coalition. Another implication of this proposition is that any partition containing a coalition strictly larger than K, say K^+, no longer satisfies the pessimistic stability and farsighted stability, because each member of K^+ has an incentive to deviate by forming an exact majority coalition K to obtain $\frac{1}{k} > \frac{1}{k^+}$.

As mentioned in Chapter 1, the pessimistic core is the largest (myopic) core. Hence, the farsighted vNM stable set is also a superset of the optimistic core and the projective core. From this fact and the following result, we can derive the nonemptiness of the pessimistic core and the farsighted vNM stable set. For any nonempty coalition $T \subseteq N$, let $[T]$ be a partition of coalition T into singletons.

Proposition 3.3.8. For any symmetric majority game,

$$\{K\} \cup [N \setminus K] \in C^{\text{pro}}.$$

Proposition 3.3.8 guarantees the nonemptiness of the projective core, the pessimistic core, and the farsighted vNM stable set. Moreover, in view of the definition of the projective core, it implies that an exact majority coalition K prevents the players in $N \setminus K$ from deviating from the partition $\{K\} \cup [N \setminus K]$ by holding K. The following remark describes a detailed inclusion relationship between the projective core and the pessimistic core.

Remark 3.3.9. For $n = 3, 4, 5, 6, 8, 10$, $C^{\text{pro}} = C^{\text{pes}} = V$. For $n = 7, 9, 11, ...$, $C^{\text{pro}} \subsetneq C^{\text{pes}} = V$.

If n is small enough, then the pessimistic core (equivalently, the vNM farsighted stable set) and the projective core coincide. See, for example, Table 3.4. In contrast, Table 3.3 describes the gap between these stability notions. In Table 3.3, partition 1234|567 is not in the projective core, because a player, say 4, has an incentive to deviate and form his one-player coalition. The resulting partition is 123|4|567. Player 4 obtains $\frac{1}{3}$ in 123|4|567, which is strictly greater than $\frac{1}{4}$ in 1234|567. This result implies that even if a coalition structure has an exact majority coalition, some myopic players may try to deviate from the coalition structure.

Moreover, the gap between the projective core and the vNM farsighted stable set describes the difference between a projective deviation and an indirect dominance. As mentioned in Definition 3.3.4, there is a similarity between the two notions: each step of indirect domination can be thought of as the projection of a partition on the deviating coalition; while the difference lies in which coalition structure is the destination the deviation leads to. For a projective deviation, the deviating players myopically consider the "next" coalition structure as their destination. In contrast, as for an indirect dominance, the deviating players farsightedly anticipate the coalition structure that results from a sequence of multi-step deviations. The gap between the projective core and the vNM farsighted stable set can be seen as the result of this difference.

Unlike the preceding stability concepts, the nonemptiness of the optimistic core deeply depends on the number of players.

Proposition 3.3.10. For any symmetric majority game with $n \geq 5$, the optimistic core is empty. For $n = 3, 4$, we have $C^{\text{opt}} = C^{\text{pro}} = C^{\text{pes}} = V$.

In short, the optimistic core is basically empty. As exceptions, for games with few players ($n = 3$ or 4), the scarcity of possible partitions allows the optimistic core to be nonempty and to coincide with the other concepts as described in Table 3.4. For $n = 3, 4$, partition $\{K, \{i\}\}$ satisfies all of the myopic core stabilities and the farsighted vNM stability.

Table 3.4: The four-player symmetric majority game

Partition	ϕ_1	ϕ_2	ϕ_3	ϕ_4	C^{pro}	C^{pes}	C^{opt}	V	Nash	IS
1234	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$					✓	✓
123|4	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	✓	✓	✓	✓		✓
12|34	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$						
12|3|4	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$						
1|2|3|4	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$						

3.3.3 Individual stability concepts

All of the stability concepts that we have studied admit two or more players to jointly deviate as a coalition. We now restrict our attention into individual deviations. Our individual stability concepts include Nash stability and individual stability.[8]

Definition 3.3.11. A partition $\mathcal{P}$ is *Nash stable* if there is no pair of $i \in N$ and $S \in \mathcal{P} \cup \{\emptyset\}$ such that $\mathcal{P}_{i \to S} \succ_i \mathcal{P}$, where $\mathcal{P}_{i \to S} = \{S \cup \{i\}\} \cup (\mathcal{P}|_{N \setminus (S \cup \{i\})})$. We denote the set of Nash stable partitions by Nash.

Definition 3.3.12. A partition $\mathcal{P}$ is *individually stable* if there is no pair of $i \in N$ and $S \in \mathcal{P} \cup \{\emptyset\}$ such that $\mathcal{P}_{i \to S} \succ_i \mathcal{P}$ and for every $j \in S$, $\mathcal{P}_{i \to S} \succsim_j \mathcal{P}$. We denote the set of individually stable partitions by IS.

Clearly, we have Nash $\subseteq$ IS. The importance of these concepts lies in the fact that they tend to contain partitions radically different from the myopic cores and the farsighted vNM stable set. We begin with a simple result.

[8] These notions are intensively studied in early works of hedonic games, such as Banerjee *et al.* (2001) and Bogomolnaia and Jackson (2002). Diamantoudi and Xue (2003) call these concepts "non-cooperative" stability concepts.

Proposition 3.3.13. For every symmetric majority game, $\{N\}$ is Nash stable.

Proof. For any $i \in N$, we have $\phi_i(\{N\}) = \frac{1}{n}$ and $\phi_i(\{N \setminus \{i\}, \{i\}\}) = 0$. Hence, $\{N\} \succ_i \{N \setminus \{i\}, \{i\}\}$. □

The grand coalition is Nash stable but not the unique Nash stable coalition structure for some n. The first example is $n = 9$. Consider the partition $\mathcal{P} = 123|456|789$. For any player $i \in N$, we have $\phi_i(\mathcal{P}) = \frac{1}{9}$. Without loss of generality, let 1 be the player who tries to move. Let $\mathcal{P}' = 23|456|1789$ and $\mathcal{P}'' = 1|23|456|789$. We have $\phi_1(\mathcal{P}') = \frac{1}{12} < \frac{1}{9}$ and $\phi_1(\mathcal{P}'') = 0 < \frac{1}{9}$, which means that $\mathcal{P}$ is Nash stable. As n increases, Nash stable coalition structures should increase. This example also shows that even a coalition structure with no majority coalition can be seen as a stable coalition structure.

As for the individual stability, the following result holds.

Proposition 3.3.14. For every symmetric majority game, $\{K\} \cup [N \setminus K] \in \text{IS}$. Moreover, if $\mathcal{P} \ni K^+$, then $\mathcal{P} \in \text{IS}$.

In view of Proposition 3.3.8, Proposition 3.3.14 guarantees that partition $\{K\} \cup [N \setminus K]$ satisfies all stability concepts we discuss in this chapter except for the optimistic core and the Nash stability. Similar to the Nash stability, the partitions mentioned in Proposition 3.3.14 are not the only individually stable coalition structures for some n. For example, in Table 3.3, partition 12|34|56|7 is individually stable but not supported by Proposition 3.3.14. In the partition 12|34|56|7, player 7 has an incentive to merge with each two-player coalition. Hence, 12|34|56|7 is not Nash stable. However, individual stability takes into account the incentive of accepting players. If player 7 moves to coalition $\{1, 2\}$, each of players 1 and 2 gets $\frac{1}{9} < \frac{1}{6}$ and, hence, rejects player 7.

3.4 Discussions

3.4.1 Comparisons with Hart and Kurz (1984) and Bloch (1996)

We compare our results with two eminent works: Hart and Kurz (1984) and Bloch (1996). The following two paragraphs summarize their approaches, respectively.

- Hart and Kurz (1984): As briefly mentioned in Section 3.1, their approach is basically the same as ours. However, there are three differences. The first difference is the stability concept. They use the α-, β-, γ- and δ-stability concepts. Second, they analyze stable coalition structures only for $n \leq 10$. Our results hold for general n. Third, they analyze not only exact majority k but also $k^+ \in [k, n]$ for some n.

- Bloch (1996): His approach is very different from ours. He models the process of coalition formation as a noncooperative game and studies "stable" coalition structures of symmetric majority games as an application of his model. His stability concept is described as the *equilibrium coalition structure* (*EQ*), which is an equilibrium concept of the noncooperative game. The general result of symmetric majority game is not offered in his analysis, either. He also analyzes $k^+ \in [k, n]$ for some n.

We compare our results with theirs for the case of $n = 5$. Hart and Kurz (1984) show that the partitions 123|4|5 and 123|45 satisfy all the α-, β-, γ- and δ-stabilities, while other partitions satisfy none of them. Moreover, Bloch (1996) also shows that exactly the same partitions are the equilibrium coalition strictures. In our analysis, according to Table 3.5, the pessimistic core, the projective core, and the farsighted vNM stable set coincide with their stability concepts. We have the same coincidence for $n = 6$.

Table 3.5: The five-pla

Partition	1	2	3	4	5	C^{pro}	C^{pes}	C^{opt}	V	Nash	IS	α-δ	EQ
12345	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$					✓	✓		
1234|5	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0						✓		
123|45	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	✓	✓		✓		✓	✓	✓
123|4|5	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	✓	✓		✓		✓	✓	✓
12|34|5	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$								
12|3|4|5	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$								
1|2|3|4|5	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$								

3.4.2 The uniqueness of the farsighted vNM stable set

As briefly mentioned, the farsighted vNM stable set is not necessarily unique in general. Below, we discuss its uniqueness (and difficulties) in our model. Let V be a set of the partitions containing K, namely, the farsighted vNM stable set. Assume that V' is another farsighted stable set. From internal stability and external stability, the following two statements follow: $V \not\subseteq V'$ and $V' \not\subseteq V$. Hence, there exists a partition, say $\mathcal{P}$, such that $\mathcal{P} \in V$ and $\mathcal{P} \notin V'$. Similarly, there exists $\mathcal{P}'$ such that $\mathcal{P}' \notin V$ and $\mathcal{P}' \in V'$. The difficulty lies in the existence of partitions $\mathcal{P}'$ such as, for example, $\mathcal{P}' := 123|456|7$ in the seven-player game. In this partition, player 7 receives $\frac{1}{3}$, while he receives $\frac{1}{4}(< \frac{1}{3})$ in partition $123|4567 =: \mathcal{P}$. Note that $\mathcal{P}$ contains $K = \{4, 5, 6, 7\}$ and player 7 is its member. In other words, even a member of an exact majority coalition (player 7) has an incentive to deviate from $\mathcal{P}$ to $\mathcal{P}'$. If there is no such partition $\mathcal{P}'$, the uniqueness is straightforward. To see this, let B be the set of partitions containing an exact majority coalition and $\mathcal{P} \in B(= V)$. (i) Any member of K has no incentive to deviate from $\mathcal{P}$, and (ii) the players of $N \setminus K$ have an incentive to deviate from $\mathcal{P}$, but (iii) the players of $N \setminus K$ cannot deviate from B to outside B without help of some members of K. Hence, $\mathcal{P}$ contradicts to external stability of V'. However, if $\mathcal{P}'$ exists, as in our game, the statement (i) does not hold because of the existence of such $\mathcal{P}'$.[9] Exploring the condition for the uniqueness is one of our future works.

3.4.3 The α-δ stabilities

As formulated and popularized by Hart and Kurz (1983, 1984), α-, β-, γ-, and δ-stabilities are a well-known approach to analyze stable coalition structures. However, since their definitions are

[9]What is worse, the number of partitions such as $\mathcal{P}'$ increases as n increases.

based on strategy terms, as provided below, they are difficult to practically use. In this subsection, we attempt to offer simple equivalent formulations for these stability concepts in terms of partition instead of strategy.

We first define the stabilities by following Hart and Kurz (1983, 1984). For every player $i \in N$, let $\mathcal{A}_i$ denote the set of coalitions containing player i, namely, $\mathcal{A}_i = \{S \subseteq N | i \in S\}$. Let $\mathcal{A}_S = \times_{j\in S}\mathcal{A}_j$. For any $S \subseteq N$, we write $\sigma_S = (\sigma_i)_{i\in S} \in \mathcal{A}_S$. In particular, we simply use σ to denote a strategy profile in $\mathcal{A}_N$, namely, $\sigma = (\sigma_i)_{i\in N} \in \mathcal{A}_N$. For any $\mathcal{P} \in \Pi(N)$, $\sigma^{\mathcal{P}}$ is the strategy profile satisfying $\sigma_i^{\mathcal{P}} = \mathcal{P}(i)$ for every $i \in N$, where $\mathcal{P}(i)$ is the coalition in $\mathcal{P}$ that contains player i. In the same manner, for any $S \subseteq N$, $\sigma_S^{\mathcal{P}}$ is the strategy profile given by $(\sigma_i^{\mathcal{P}})_{i\in S}$.

For convenience, we first define γ-stability and δ-stability. Let $\sigma_i \in \mathcal{A}_i$ be player i's strategy: player i chooses a coalition which contains i as a member. Let $\mathcal{B}^\gamma : \mathcal{A}_N \to \Pi(N)$ be a function that assigns a partition to a strategy profile. The function $\mathcal{B}^\gamma$ is given as $\mathcal{B}^\gamma(\sigma) = \{T_\sigma^i | i \in N\}$ where T_σ^i is σ_i if $\sigma_i = \sigma_j$ for all $j \in \sigma_i$ and is $\{i\}$ otherwise. Consider a function ϕ that assigns a payoff profile to a partition, namely, a general version of the function ϕ discussed in this chapter. For any $\mathcal{P} \in \Pi(N)$, $\phi_i(\mathcal{P})$ is player i's payoff in partition $\mathcal{P}$. The composite function $\phi \circ \mathcal{B}^\gamma$ assigns a payoff profile to every strategy profile. In Hart and Kurz (1983, 1984), this function is called model γ. Together with the player set N and the strategy sets $(\mathcal{A}_i)_{i\in N}$, we consider $\phi\circ\mathcal{B}^\gamma$ as a strategic game. We use G_ϕ^γ to denote the model γ, $G_\phi^\gamma := \phi \circ \mathcal{B}^\gamma$. Now, let $\mathcal{B}^\delta(\sigma) = \{T \subseteq N | i, j \in T \iff \sigma_i = \sigma_j\}$. Similar to the model γ, the model δ is defined as $G_\phi^\delta := \phi \circ \mathcal{B}^\delta$. The γ-stability and δ-stability are defined by Hart and Kurz (1983, 1984) as follows: a partition $\mathcal{P}$ is

- γ-*stable* if strategy profile $\sigma^{\mathcal{P}}$ is a strong equilibrium in G_ϕ^γ,
- δ-*stable* if strategy profile $\sigma^{\mathcal{P}}$ is a strong equilibrium in G_ϕ^δ.

Now, we define α-stability and β-stability. To this end, we need to construct a NTU-game. For any $S \subseteq N$, define $V^\alpha(S) = \{(x_i)_{i\in S} \in \mathbb{R}^S |$ there is $\sigma_S \in \mathcal{A}_S$ such that for all $\sigma_{N\setminus S} \in \mathcal{A}_{N\setminus S}$, $\phi_i(\mathcal{B}^*(\sigma)) \geq x_i$ for all $i \in S\}$. Now, for any $S \subseteq N$, $V^\beta(S) = \{(x_i)_{i\in S} \in \mathbb{R}^S |$ for all $\sigma_{N\setminus S} \in \mathcal{A}_{N\setminus S}$, there is $\sigma_S \in \mathcal{A}_S$ such that $\phi_i(\mathcal{B}^*(\sigma)) \geq x_i$ for all $i \in S\}$. Note that this formulation is provided by Hart and Kurz (1983, 1984). In the definitions above, $\mathcal{B}^*$ means either $\mathcal{B}^\gamma$ or $\mathcal{B}^\delta$. We can replace $\mathcal{B}^*$ by either $\mathcal{B}^\gamma$ or $\mathcal{B}^\delta$. In any case, the definitions of V^α and V^β are the same, respectively. We now define the core of an NTU-game. Let V denote an NTU-game. The core of V is defined as follows: $C(V) = \{x \in \mathbb{R}^N |$ there is no $T \subseteq N$ and no $y \in V(T)$ such that $y_i > x_i$ for every $i \in T\}$. We now define the remaining two stability concepts: a partition $\mathcal{P}$ is

- α-*stable* if $\phi(\mathcal{P})$ is in the core of V^α,
- β-*stable* if $\phi(\mathcal{P})$ is in the core of V^β.

As described above, these definitions are elaborate and might be difficult to practically use in an application. Our conjecture is that we can "remove" the strategy terms from the above definitions. To this end, we introduce some useful notions. Let $\overline{\Pi}(N) = \{\mathcal{P} | \mathcal{P} \in \Pi(S), S \subseteq N\}$. For any nonempty coalition $S \subseteq N$ and any partition $\mathcal{P} \in \overline{\Pi}(T)$, let $\overline{\mathcal{P}_S} = \{C | C \in \mathcal{P}, S \cap C \neq \emptyset\}$. Let $\widehat{\mathcal{P}_S} = \bigcup_{C\in\overline{\mathcal{P}_S}} C$. The equivalent expression is given as follows: $\mathcal{P} \in \Pi(N)$ is α-stable if and only if there exist no $T \subseteq N$ and no $\mathcal{Q} \in \Pi(T)$ such that for any $\mathcal{P}' \in \Pi(N \setminus T)$, $\phi_i(\mathcal{Q} \cup \mathcal{P}') > \phi_i(\mathcal{P})$

for every $i \in T$; similarly, $\mathcal{P}$ is β-stable if and only if there exist no $T \subseteq N$ such that for any $\mathcal{P}' \in \Pi(N \setminus T)$, there exists $\mathcal{Q} \in \Pi(T)$ such that $\phi_i(\mathcal{Q} \cup \mathcal{P}') > \phi_i(\mathcal{P})$ for every $i \in T$. Associating the strategies of coalition S with the partition $\mathcal{Q}$ of S is the intuition behind these expressions. In this sense, as suggested by Bloch and van den Nouweland (2014), each of α- and β-stabilities exhibits another form of pessimistic expectation. Moreover, in the same vein, γ-stability can be seen as a variation of the singleton-expectation: there exist no $T \subseteq N$ and no $\mathcal{Q} \in \Pi(T)$ such that $\phi_i(\mathcal{Q} \cup [\widehat{\mathcal{P}_T} \setminus T] \cup (\mathcal{P} \setminus \overline{\mathcal{P}_T})) > \phi_i(\mathcal{P})$ for every $i \in T$. In addition, δ-stability is an analog of the merge-expectation: there exist no $T \subseteq N$ and no $\mathcal{Q} \in \Pi(T)$ such that $\phi_i(\mathcal{Q} \cup (\mathcal{P}|_{N \setminus T})) > \phi_i(\mathcal{P})$ for every $i \in T$. By using these equivalent expressions, we can more smoothly analyze the four stability concepts and might derive the inclusion relationship among these notions from externalities. The further investigation is our next work.

3.5 Proofs

We first prove Proposition 3.3.7 and, next, Proposition 3.3.6.

Proof of Proposition 3.3.7

Proof. For any partition $\mathcal{P}$, consider the following three cases:

A. $\mathcal{P}$ contains a majority coalition whose size is strictly greater than k;

B. $\mathcal{P}$ contains an exact majority coalition;

C. For every coalition in $\mathcal{P}$, its size is strictly less than k.

We show that the partitions satisfying B are in the pessimistic core and the partitions satisfying A or C are not. Let K be an exact majority coalition.

Case A. Let $\mathcal{P}$ be be a partition satisfying A. Let $K^+ \in \mathcal{P}$ be a coalition such that $|K^+| = k^+ > k$. Let $\mathcal{P}^*$ be be a partition satisfying B such that $K \in \mathcal{P}^*$ and $K \subseteq K^+$. Since for any $i \in K \cap K^+ = K$, $\phi_i(\mathcal{P}^*) = \frac{1}{k} > \frac{1}{k^+} = \phi_i(\mathcal{P})$, the players in $K \subsetneq K^+$ have an incentive to deviate to $\mathcal{P}^*$. Whatever partition the other players, $N \setminus K$, forms, the players in K get $\frac{1}{k}$. Thus, $\mathcal{P}$ is not in the pessimistic core.

Case B. Let $\mathcal{P}$ be a partition satisfying B and $K \in \mathcal{P}$. Let $S \subseteq N$ be a deviating coalition. We typically use $\mathcal{P}^*$ with $S \in \mathcal{P}^*$ to denote a partition after S's deviation. First, any coalition $S \subseteq N$ with $|S| \geq k$ does not deviate from $\mathcal{P}$, because for some player $i \in K \cap S$, $\phi_i(\mathcal{P}) = \frac{1}{k} \geq \frac{1}{|S|} = \phi_i(\mathcal{P}^*)$ for any partition $\mathcal{P}^*$ with $S \in \mathcal{P}^*$. Next, consider a coalition S with $|S| < k$. We focus on the case of $S \cap K \neq \emptyset$, because if $S \cap K = \emptyset$, then $S \subseteq N \setminus K$ and any player i in S gets zero both before and after their deviation: $\phi_i(\mathcal{P}) = 0 = \phi_i(\mathcal{P}^*)$.

In the case of odd n, $|N \setminus S| \geq k$. Hence, in the pessimistic view, $\phi_S(\mathcal{P}^*) = 0$, because the players in $N \setminus S$ form their coalition $N \setminus S$, and the coalition $N \setminus S$ is a majority coalition, K or K^+, in $\mathcal{P}^*$. In other words, any S with $|S| < k$ has no incentive to deviate from $\mathcal{P}$.

In the case of even n, if $|S| \leq k-2$, then it is the same as the case of odd n. If $|S| = k-1 = \frac{n}{2}$, then the power of coalition S increases as the number of coalitions except S increases.[10] Therefore, in the pessimistic view, the players in coalition S expect that the players in $N \setminus S$ form their coalition $\{N \setminus S\}$, where $|S| = |N \setminus S| = \frac{n}{2} = k-1$ and $\mathcal{P}^* = \{S, N \setminus S\}$. Hence, each player in coalition S expects their minimum $\frac{1}{2} \cdot \frac{1}{|S|}$ per capita after their deviation. We have, for any $i \in K \cap S$,

$$\phi_i(\mathcal{P}) - \phi_i(\mathcal{P}^*) = \frac{1}{k} - \frac{1}{2}\frac{1}{|S|} = \frac{2}{n+2} - \frac{1}{n} = \frac{n-2}{n(n+2)} \geq 0,$$

where the last inequality holds as $n \geq 3$.

Case C. Let $\mathcal{P}$ be a partition satisfying C. We consider the players who obtain at least $\frac{1}{k}$ in $\mathcal{P}$. Let $\hat{K} = \{j \in N | \phi_j(\mathcal{P}) \geq \frac{1}{k}\}$ and $\hat{k} = |\hat{K}|$. We must have $\hat{k} \leq k$ because of $\sum_{j \in N} \phi_j(\mathcal{P}) = 1$. Moreover, we claim that $\hat{k} \neq k$ as follows.

Claim 1 $\hat{k} \leq k-1$.

Proof. Assume that $\hat{k} = k$. Let $\mathcal{Z}$ be a partition of $\hat{K}$, namely, $\mathcal{Z}$ is a subpartition of $\mathcal{P}$. Now, $\mathcal{Z}$ must consist of at least two coalitions, because $\mathcal{Z}$ is a partition of exactly k players and for every coalition in $\mathcal{P}$ its size is strictly less than k. We consider the following order of coalitions in $\mathcal{P}$. First, we arrange the coalitions in $\mathcal{Z}$ in order of their size (the biggest coalition in $\mathcal{Z}$ is located at the top) and remove (one of) the smallest coalition(s) in $\mathcal{Z}$ from the order. At least $\frac{k}{2}$ players are now lined up. Next, we arrange the coalitions which are not in $\mathcal{Z}$. The number of the players who are not in $\mathcal{Z}$ is $n - \hat{k} = n - k = \frac{n}{2} - 1$. Hence, there exists a coalition which becomes a pivot and is not in $\mathcal{Z}$. In other words, the power index, ϕ, assigns a positive value to this coalition. This, however, contradicts $\hat{k} = k$, because $\hat{k} = k$ implies that any player not in $\hat{K}$ gets zero. □

We assume $\hat{k} \leq k-1$ hereafter.

In the case of odd n, the fact $\hat{k} \leq k-1$ implies that $n - \hat{k} \geq n - (k-1)$: in $\mathcal{P}$, the number of players who get strictly less than $\frac{1}{k}$ is greater than $n - (k-1) = n - (\frac{n+1}{2} - 1) = k$. Hence, these players have an incentive to form coalition S such that $|S| = k$ and deviate from $\mathcal{P}$.

In the case of even n, if $\hat{k} \leq k-2$, then it is the same as the case of odd n. If $\hat{k} = k-1 = \frac{n}{2}$, the other $\frac{n}{2}$ players (namely, $N \setminus \hat{K}$) obtain strictly less than $\frac{1}{k}$ in $\mathcal{P}$. We show that, in the pessimistic view, the players in $N \setminus \hat{K}$ have an incentive to form a deviating coalition S such that $S = N \setminus \hat{K}$ with $|S| = n - \hat{k} = \frac{n}{2}$. If $S = N \setminus \hat{K}$ and $|S| = \frac{n}{2}$, then the deviating players in S expect to get $\frac{1}{2}\frac{1}{|S|} = \frac{1}{n}$ per capita after their deviation (see the case of even n in Case B). Therefore, it suffices to show that $\phi_i(\mathcal{P}) < \frac{1}{n}$ for any $i \in N \setminus \hat{K}$.

Claim 2 If $\hat{k} = k-1 = \frac{n}{2}$, then $\phi_i(\mathcal{P}) < \frac{1}{n}$ for any $i \in N \setminus \hat{K}$.

Proof. Let $T = \{i \in N \setminus \hat{K} | \phi_i(\mathcal{P}) \geq \frac{1}{n}\}$ and $t = |T|$. We have

$$\begin{aligned} \hat{k} \cdot \frac{1}{k} + t \cdot \frac{1}{n} + 0 &\leq \sum_{j \in \hat{K}} \phi_j(\mathcal{P}) + \sum_{j \in T} \phi_j(\mathcal{P}) + \sum_{j \in N \setminus (\hat{K} \cup T)} \phi_j(\mathcal{P}) = 1, \text{ and} \\ \hat{k} \cdot \frac{1}{k} + t \cdot \frac{1}{n} &= \frac{n}{2} \cdot \frac{2}{n+2} + \frac{t}{n} = \frac{n}{n+2} + \frac{t}{n}. \end{aligned}$$

[10] In general, for any S with $|S| = \frac{n}{2}$, we have $\phi_S(\mathcal{P}) = 1 - \frac{r!}{(r+1)!} = \frac{r}{r+1}$, where r is the number of coalitions in $\mathcal{P} \setminus S$.

We obtain $t \leq \frac{2n}{n+2}$. As $n \geq 3$, $1 \leq \frac{2n}{n+2} < 2$. Hence, t should be 0 or 1. We assume $t = 1$ and call the player t. The player t forms his singleton coalition in $\mathcal{P}$, because if he is in a coalition consisting of more than two players, *symmetry within a coalition* implies that his partners also obtain at least $\frac{1}{n}$, which contradicts $t = 1$.[11] Moreover, similarly, *symmetry across coalitions* implies that the player t's singleton coalition is the only singleton coalition in $\mathcal{P}$.[12] In other words, the other coalitions consist of at least two players. We take a coalition $S' \in \mathcal{P}$ such that $|S'| \geq 2$, $S' \subseteq N \setminus \hat{K}$. Each player in S' gets strictly less than $\frac{1}{k}$ in $\mathcal{P}$ because they are not the members of $\hat{K}$. By *local monotonicity* and inequality $|S'| > |\{t\}| = 1$, we obtain $\phi_{S'}(\mathcal{P}) \geq \phi_t(\mathcal{P}) = \frac{1}{n}$.[13] Hence, we have

$$\begin{aligned} \hat{k} \cdot \frac{1}{k} + \frac{1}{n} + \frac{1}{n} + 0 \quad &\leq \quad \sum_{j \in \hat{K}} \phi_j(\mathcal{P}) + \phi_t(\mathcal{P}) + \phi_{S'}(\mathcal{P}) + \sum_{j \notin (\hat{K} \cup \{t\} \cup S')} \phi_j(\mathcal{P}) = 1, \text{ and} \\ \hat{k} \cdot \frac{1}{k} + \frac{1}{n} + \frac{1}{n} \quad &= \quad \frac{n}{n+2} + \frac{2}{n} = \frac{n^2 + 2n + 4}{n^2 + 2n} > 1. \end{aligned}$$

This is a contradiction. Hence, we have $t = 0$, *i.e.*, every player in $N \setminus \hat{K}$ gets strictly less than $\frac{1}{n}$ in $\mathcal{P}$. □

Thus, in $\mathcal{P}$, the players in $N \setminus \hat{K}$ have an incentive to from their coalition $N \setminus \hat{K}$ and deviate to get $\frac{1}{n}$ even in the pessimistic view. □

Proof of Proposition 3.3.6

Proof. For any partition $\mathcal{P}$, consider the following three cases:

A. $\mathcal{P}$ contains a majority coalition whose size is strictly greater than k;

B. $\mathcal{P}$ contains an exact majority coalition;

C. For every coalition in $\mathcal{P}$, its size is strictly less than k.

Let V be a set of partitions satisfying B (the set of partitions satisfying B is the pessimistic core. See Proposition 3.3.7).

Internal stability. Let $\mathcal{P}, \mathcal{P}'$ be any two different partitions in V. Let $K \in \mathcal{P}$ and $K' \in \mathcal{P}'$ ($K \neq K'$ and $k = k'$) be exact majority coalitions. Since K and K' are exact majority coalitions, we have $K \cap K' \neq \emptyset$. Now, assume that there exists a path from $\mathcal{P}$ to $\mathcal{P}'$ satisfying the conditions of Definition 3.3.4.

As K is an exact majority coalition in $\mathcal{P}$, we have $\phi_i(\mathcal{P}) = 0$ for any $i \in N \setminus K$ and $\phi_i(\mathcal{P}) = \frac{1}{k}$ for any $i \in K$. Hence, in the path, the first deviating coalition S^1 should be a subset of $N \setminus K$. Let the resulting partition be $\mathcal{P}^1$. As $S^1 \subseteq N \setminus K$, K is still in $\mathcal{P}^1$. We have $\phi_i(\mathcal{P}^1) = 0$ for any $i \in N \setminus K$ and $\phi_i(\mathcal{P}^1) = \frac{1}{k}$ for any $i \in K$. Hence, the second deviating coalition S^2 is a subset of $N \setminus K$. In

[11] This property and the following two properties are ascribed to the Owen value. For any partition $\mathcal{P}$, any coalition $S \in \mathcal{P}$ ($|S| \geq 2$) and any $i, j \in S$, we have $\phi_i(\mathcal{P}) = \phi_j(\mathcal{P})$. See Hart and Kurz (1984).

[12] For any partition $\mathcal{P}$ and any coalition $S, T \in \mathcal{P}$, if $|S| = |T|$ then $\sum_{j \in S} \phi_j(\mathcal{P}) = \sum_{j \in T} \phi_j(\mathcal{P})$. See Hart and Kurz (1984).

[13] For any partition $\mathcal{P}$ and any coalition $S, T \in \mathcal{P}$, if $|S| \geq |T|$ then $\sum_{j \in S} \phi_j(\mathcal{P}) \geq \sum_{j \in T} \phi_j(\mathcal{P})$. Note that this implies symmetry across coalitions. See Alonso-Meijide et al. (2009).

general, for any step j, we have $\phi_i(\mathcal{P}^j) = \frac{1}{k}$ for any $i \in K$. On the other hand, for the path to reach partition $\mathcal{P}'$, there must be at least one coalition S^{j^*} and partition $\mathcal{P}^{j^*}$ (at step j^*) such that $K \cap K' \subseteq S^{j^*}$ and $\mathcal{P}' \succ_{S^{j^*}} \mathcal{P}^{j^*}$; otherwise, $K = K'$. However, we have $\phi_i(\mathcal{P}^{j^*}) = \frac{1}{k} = \frac{1}{k'} = \phi_i(\mathcal{P}')$ for any $i \in K \cap K'$. This is a contradiction.

For any K and any two different partitions $\mathcal{P}, \mathcal{P}'$ such that $K \in \mathcal{P}$ and $K \in \mathcal{P}'$, we have $\phi_i(\mathcal{P}) = \phi_i(\mathcal{P}')$ for any player $i \in N$. Thus, there is no indirect dominance within V.

External stability. We show that for any partition $\mathcal{P}$ satisfying A or C, there exists a partition $\mathcal{P}^* \in V$ such that $\mathcal{P}^*$ indirectly dominates $\mathcal{P}$.

Case A. For any $\mathcal{P}$ satisfying A, let $K^+ \in \mathcal{P}$ denote a majority coalition whose size k^+ is strictly greater than k. There exists a partition $\mathcal{P}^* \in V$ and an exact majority coalition $K \in \mathcal{P}$ such that $K \subsetneq K^+$ and $\phi_i(\mathcal{P}^*) = \frac{1}{k} > \frac{1}{k^+} = \phi_i(\mathcal{P})$ for any $i \in K$. Thus, $\mathcal{P}^*$ (in)directly dominates $\mathcal{P}$.

Case C. Let $\mathcal{P}$ be a partition satisfying C. We consider the set of players who obtain at least $\frac{1}{k}$ in $\mathcal{P}$. Let $\hat{K} = \{j \in N | \phi_j(\mathcal{P}) \geq \frac{1}{k}\}$ and $\hat{k} = |\hat{K}|$. As Claim 1 in the proof of Proposition 3.3.7, we have $\hat{k} \leq k - 1$.

In the case of odd n, $\hat{k} \leq k - 1$ means that at least k players get strictly less than $\frac{1}{k}$ in $\mathcal{P}$. They form coalition S consisting of k players and directly deviate to partition $\{S\} \cup (\mathcal{P}|_{N \setminus S}) \in V$.

In the case of even n, if $\hat{k} \leq k - 2$, then it is the same as the case of odd n. If $\hat{k} = k - 1 = \frac{n}{2}$, then the other $\frac{n}{2}$ players (namely, $N \setminus \hat{K}$) obtain strictly less than $\frac{1}{k}$. Let $\mathcal{Q}$ be a partition of $N \setminus \hat{K}$, namely $\mathcal{Q}$ is a subpartition of $\mathcal{P}$. We first show the following claim.

Claim 3 $|\mathcal{Q}| \geq 2$.

Proof. Assume that $|\mathcal{Q}| = 1$ (*i.e.*, $\mathcal{Q} = \{N \setminus \hat{K}\}$, which contains $k - 1$ players). We have

$$\phi_{N \setminus \hat{K}}(\mathcal{P}) = 1 - (|\mathcal{P}| - 1)\frac{(|\mathcal{P}| - 2)!}{|\mathcal{P}|!} = 1 - (|\mathcal{P}| - 1)\frac{1}{|\mathcal{P}|(|\mathcal{P}| - 1)} = \frac{|\mathcal{P}| - 1}{|\mathcal{P}|}, \tag{3.5.1}$$

for $2 \leq |\mathcal{P}| \leq k$.[14] This attains its minimum at $|\mathcal{P}| = 2$. Hence, $\phi_{N \setminus \hat{K}}(\mathcal{P}) \geq \frac{1}{2}$. However, since $\phi_i(\mathcal{P}) \geq \frac{1}{k}$ for any $i \in \hat{K}$, we have $\phi_{N \setminus \hat{K}}(\mathcal{P}) = 1 - \phi_{\hat{K}}(\mathcal{P}) \leq 1 - (\frac{1}{k} \cdot \hat{k}) = 1 - (\frac{k-1}{k}) = \frac{1}{k}$, which contradicts $\phi_{N \setminus \hat{K}}(\mathcal{P}) \geq \frac{1}{2}$ as $n \geq 4$ ($k \geq 3$ when $n \geq 4$, and 4 is the smallest even n). □

Next, we construct a path from $\mathcal{P}$ to $\mathcal{P}^* \in V$ via a partition $\bar{\mathcal{P}}$ given by

$$\begin{aligned} \bar{\mathcal{P}} &= (\mathcal{P}|_{\hat{K}}) \cup \{N \setminus \hat{K}\}, \\ \mathcal{P}^* &= (\mathcal{P}|_{\hat{K} \setminus \{i^*\}}) \cup \{(N \setminus \hat{K}) \cup \{i^*\}\}, \end{aligned}$$

where i^* is any player in $\hat{K}$. Note that $\mathcal{P}^* \in V$ because coalition $(N \setminus \hat{K}) \cup \{i^*\}$ is an exact majority.

The first deviating coalition (from $\mathcal{P}$ to $\bar{\mathcal{P}}$) is $N \setminus \hat{K}$. For every player $i \in N \setminus \hat{K}$, $\phi_i(\mathcal{P}^*) > \phi_i(\mathcal{P})$, because $\phi_i(\mathcal{P}) < \frac{1}{k}$ and $\phi_i(\mathcal{P}^*) = \frac{1}{k}$. Moreover, by $|\mathcal{Q}| \geq 2$, the all players in $\mathcal{Q}$ can form one coalition $\{N \setminus \hat{K}\}$ and move to $\bar{\mathcal{P}}$.

The second deviating coalition (from $\bar{\mathcal{P}}$ to $\mathcal{P}^*$) is $(N \setminus \hat{K}) \cup \{i^*\}$. We show that for any $i \in N \setminus \hat{K}$, $\phi_i(\bar{\mathcal{P}}) < \frac{1}{k}$ (for i^*, see Claim 5).

Claim 4 In $\bar{\mathcal{P}}$, for any $i \in N \setminus \hat{K}$, $\phi_i(\bar{\mathcal{P}}) < \frac{1}{k} = \phi_i(\mathcal{P}^*)$.

[14] In general, for any partition $\mathcal{P}$ and any coalition $S \in \mathcal{P}$ with $|S| = k - 1$, we have $\phi_S(\mathcal{P}) = \frac{|\mathcal{P}|-1}{|\mathcal{P}|}$.

Proof. We first show that $\phi_i(\bar{\mathcal{P}}) \le \frac{1}{k}$. We have

$$\frac{1}{k} - \phi_i(\bar{\mathcal{P}}) = \frac{1}{k} - \frac{|\bar{\mathcal{P}}|-1}{|\bar{\mathcal{P}}|} \cdot \frac{1}{k-1} = \frac{1}{k(k-1)}\left(\frac{k}{|\bar{\mathcal{P}}|} - 1\right) \ge 0,$$

where the first equality holds in the same manner with (3.5.1) and the final inequality follows from $k \ge |\bar{\mathcal{P}}|$. Hence, if $k = |\bar{\mathcal{P}}|$, then $\phi_i(\bar{\mathcal{P}}) = \frac{1}{k}$. Below, we show that this case does not occur. Assume that $|\bar{\mathcal{P}}| = k$. Note that $\hat{k} = k-1$. From $\bar{\mathcal{P}} = (\mathcal{P}|_{\hat{K}}) \cup \{N \setminus \hat{K}\}$, it follows that $\mathcal{P}|_{\hat{K}}$ is a partition of $\hat{K}$ into $\hat{k}$ singletons, which means that the players in $\hat{K}$ are singletons in the first partition $\mathcal{P}$, because the first deviating coalition is $\{N \setminus \hat{K}\}$. Local monotonicity of the Owen power index implies that for any $i \in \hat{K}$ and any $j \in N \setminus \hat{K}$, $\phi_i(\mathcal{P}) \le \phi_j(\mathcal{P})$, because $i \in \hat{K}$ forms his singleton coalition in $\mathcal{P}$ and $j \in N \setminus \hat{K}$ belongs to a coalition consisting of at least one player in $\mathcal{P}$. This contradicts $\hat{K}$, namely, $\phi_i(\mathcal{P}) \ge \frac{1}{k}$ for $i \in \hat{K}$ and $\phi_j(\mathcal{P}) < \frac{1}{k}$ for $j \in N \setminus \hat{K}$. $\square$

We next show that in $\bar{\mathcal{P}}$, there exists $i^* \in \hat{K}$ who agrees with the second deviation.

Claim 5 In $\bar{\mathcal{P}}$, there exists $i^* \in \hat{K}$ such that $\phi_{i^*}(\bar{\mathcal{P}}) < \frac{1}{k} = \phi_{i^*}(\mathcal{P}^*)$.

Proof. Assume that for every $i \in \hat{K}$, $\phi_i(\bar{\mathcal{P}}) \ge \frac{1}{k}$. We have $\phi_{N\setminus\hat{K}}(\bar{\mathcal{P}}) = \frac{|\bar{\mathcal{P}}|-1}{|\bar{\mathcal{P}}|}$ $(2 \le |\bar{\mathcal{P}}| \le k)$ in the same manner with (3.5.1). This attains its minimum $\frac{1}{2}$ at $|\bar{\mathcal{P}}| = 2$ (namely, $\phi_{N\setminus\hat{K}}(\bar{\mathcal{P}}) = \sum_{j\in N\setminus\hat{K}} \phi_j(\bar{\mathcal{P}}) \ge \frac{1}{2}$). Hence,

$$\sum_{j\in N} \phi_j(\bar{\mathcal{P}}) = \sum_{j\in\hat{K}} \phi_j(\bar{\mathcal{P}}) + \sum_{j\in N\setminus\hat{K}} \phi_j(\bar{\mathcal{P}}) \ge (k-1)\frac{1}{k} + \frac{1}{2} = \frac{3}{2} - \frac{1}{k}.$$

This, however, contradicts $\sum_{j\in N} \phi_j(\bar{\mathcal{P}}) = 1$ as $k \ge 3$ (*i.e.*, $n \ge 4$). $\square$

Thus, for every player $i \in (N \setminus \hat{K}) \cup \{i^*\}$, $\phi_i(\mathcal{P}^*) > \phi_i(\bar{\mathcal{P}})$. $\square$

Proof of Proposition 3.3.8

Proof. We show that $\mathcal{P}^* = \{K\} \cup [N \setminus K]$ is in the projective core. We consider the following two cases: odd n and even n. Let S be a deviating coalition and $\mathcal{P}$ be the partition after S's deviation, formally,

$$\mathcal{P} := \{K \setminus S\} \cup \{S\} \cup [(N \setminus K) \setminus S]. \tag{3.5.2}$$

Odd, Condition (i): We show that for any $S \subseteq N$ such that $S \cap K \ne \emptyset$ and $|S| \le k-1$, we have $\phi_i(\mathcal{P}^*) \ge \phi_i(\mathcal{P})$.

for any player $i \in S \cap K$.[15] To compute $\phi_S(\mathcal{P})$, let $l = |S \cap K|$ and $r = |(N \setminus K) \setminus S|$. Note that

$$1 \le |S| \le k-1, \tag{3.5.3}$$
$$1 \le l \le k-1,$$
$$0 \le r \le n-k. \tag{3.5.4}$$

[15]If $|S| \ge k$, then the coalition S should be a winning coalition and gets 1 in $\mathcal{P}$. Hence, $\phi_i(\mathcal{P}) \le \frac{1}{k}$ for any $i \in S$. Since $\phi_i(\mathcal{P}^*) = \frac{1}{k}$ for any $i \in K$, we have $\mathcal{P}^* \succsim_i \mathcal{P}$ for any $i \in K \cap S$.

The number of orders that S becomes a pivot as a coalition in $\mathcal{P}$ (formally, a pivot in its corresponding weighted majority game) is

$$\begin{aligned}\sum_{j=0}^{l-1}\left[(1+j)!(r-j)!\begin{pmatrix} r \\ j \end{pmatrix}\times 2\right] &= 2\sum_{j=0}^{l-1}[(j+1)r!] \\ &= 2r!\sum_{j=1}^{l} j \\ &= r!l(1+l). \end{aligned} \tag{3.5.5}$$

The number of all orders (of coalitions in $\mathcal{P}$) is $(r+2)!$. Hence, by (3.5.5), we have

$$\phi_S(\mathcal{P}) = \frac{r!l(1+l)}{(r+2)!} = \frac{l(1+l)}{(r+2)(r+1)}.$$

Since $|S| = l+n-k-r$, we have, for any $i \in S$,

$$\phi_i(\mathcal{P}) = \frac{l(1+l)}{(r+2)(r+1)} \cdot \frac{1}{l+n-k-r}. \tag{3.5.6}$$

This is increasing with respect to l. By $|S| = l+n-k-r$, the right hand side of (3.5.3) implies $l \leq 2k+r-1-n$. By $k = \frac{n+1}{2}$, we have $l \leq r$. Hence, with respect to l, (3.5.6) attains its maximum at $l = r$. Assuming $l = r$, (3.5.6) is

$$\frac{r(1+r)}{(r+2)(r+1)} \cdot \frac{1}{r+n-k-r} = \frac{r}{r+2} \cdot \frac{1}{n-k}. \tag{3.5.7}$$

This is increasing with respect to r. By (3.5.4), substituting $r = n-k$ for (3.5.7), we have

$$\frac{1}{n-k+2} = \frac{2}{n+3} < \frac{2}{n+1} = \frac{1}{k}. \tag{3.5.8}$$

Thus, (3.5.6), (3.5.7) and (3.5.8) imply that for any $i \in S \cap K$, $\phi_i(\mathcal{P}) < \frac{1}{k} = \phi_i(\mathcal{P}^*)$.
Odd, Condition (ii): For any $S \subseteq N \setminus K$, by (3.5.2), we have $\mathcal{P} = \{K\} \cup \{S\} \cup [(N \setminus K) \setminus S]$. Since K is an exact majority coalition, we have $\phi_i(\mathcal{P}^*) = 0 = \phi_i(\mathcal{P})$ for any i with $i \in S \subseteq N \setminus K$.
Even, Condition (i): The approach is the same as the case of odd n. We have (3.5.3) to (3.5.4). We change the computation of (3.5.5) as follows:

$$\begin{aligned}\sum_{j=0}^{l-2}\left[(1+j)!(r-j)!\begin{pmatrix} r \\ j \end{pmatrix}\times 2\right] + l!(r-l+1)!\begin{pmatrix} r \\ l-1 \end{pmatrix} &= 2\sum_{j=0}^{l-2}[(j+1)r!] + l \cdot r! \\ &= r! \cdot l^2. \end{aligned} \tag{3.5.9}$$

The number of all orders is $(r+2)!$. Hence, we obtain

$$\phi_S(\mathcal{P}) = \frac{r! \cdot l^2.}{(r+2)!} = \frac{l^2}{(r+2)(r+1)}.$$

Since $|S| = (l+n-k-r)$, for any $i \in S$,

$$\phi_i(\mathcal{P}) = \frac{l^2}{(r+2)(r+1)} \cdot \frac{1}{l+n-k-r}. \tag{3.5.10}$$

This is increasing with respect to l. From $|S| = l + n - k - r$ and (3.5.3), we have $l \leq 2k + r - 1 - n$. Moreover, because of $k = \frac{n}{2} + 1$, we obtain $l \leq r + 1$. Hence, with respect to l, (3.5.10) attains its maximum at $l = r + 1$. Assuming $l = r + 1$, (3.5.10) is

$$\frac{(r+1)^2}{(r+2)(r+1)} \cdot \frac{1}{r+1+n-k-r} = \frac{r+1}{r+2} \cdot \frac{1}{n-k+1}. \tag{3.5.11}$$

This is increasing with respect to r. By (3.5.4), substituting $r = n - k$ for (3.5.11), we have

$$\frac{1}{n-k+2} = \frac{2}{n+2} < \frac{2}{n+1} = \frac{1}{k}. \tag{3.5.12}$$

Thus, from (3.5.10), (3.5.11) and (3.5.12), we have the same conclusion as the case of odd n.
Even, Condition (ii): This part is exactly the same as the case of odd n. □

Proof of Proposition 3.3.10

Proof. From Proposition 3.3.7 and $C^{\text{opt}} \subseteq C^{\text{pes}}$, it follows that if a partition $\mathcal{P}$ is in the optimistic core, then $\mathcal{P} \in C^{\text{pes}}$ and $\mathcal{P}$ contains an exact majority coalition K. In $\mathcal{P}$, players in $N \setminus K$ get zero. Consider $n \geq 5$. If $\mathcal{P} \neq \{K\} \cup [N \setminus K]$, then there is a coalition S in $\mathcal{P}$ such that $S \neq K$ and $|S| \geq 2$. Hence, any player $i \in S$, who receives zero in $\mathcal{P}$ because of K, can deviate to obtain a positive value by deviating alone, because there exists a partition including a one-person coalition but neither K nor K^+ (*e.g.*, $[N]$), in which the deviating player obtains a positive value because of the absence of coalitions K and K^+. If $\mathcal{P} = \{K\} \cup [N \setminus K]$, then the players in $[N \setminus K]$ form coalition $N \setminus K$. There exists a partition including $N \setminus K$ but neither K nor K^+ (*e.g.*, $\{N \setminus K\} \cup [K]$), in which the deviating players obtain a positive value because of the absence of coalitions K and K^+. For $n = 3$ or 4, $N \setminus K$ consists of just one player. Each player in K has no incentive to join any deviation. □

Proof of Proposition 3.3.14

Proof. We first consider the partition $\{K\} \cup [N \setminus K]$. Although every player in $[N \setminus K]$ has an incentive to join coalition K, all players in K refuse it. Moreover, every player in $[N \setminus K]$ has no incentive to merge with any other player in $[N \setminus K]$, because he gets zero even after the merge. Every player in K has no incentive to deviate alone from K, because he gets $\frac{n-2k+3}{(n-k+1)(n-k+2)}$ $(< \frac{1}{k})$ after his deviation.[16] Every player in K has no incentive to merge with any player in $[N \setminus K]$, because he gets $\frac{1}{(n-k+1)(n-k)}$ $(\leq \frac{1}{k}$ for $n \geq 3$, odd) or $\frac{1}{2(n-k+1)(n-k)}$ $(< \frac{1}{k}$ for $n \geq 4$, even) after the merge.[17] Next, consider a partition which contains a winning coalition whose size is strictly

[16] We have the following inequalities: for $n \geq 3$,

$$\frac{1}{k} - \frac{n-2k+3}{(n-k+1)(n-k+2)} = \begin{cases} \frac{2}{n+3} > 0 & \text{if } n \text{ is odd } (k = \frac{n+1}{2}), \\ \frac{2(n-2)}{n(n+2)} > 0 & \text{if } n \text{ is even } (k = \frac{n}{2} + 1). \end{cases}$$

[17] We have the following inequalities: for $n \geq 3$,

$$\frac{1}{k} - \frac{1}{(n-k+1)(n-k)} = \frac{1}{4k(n-k+1)(n-k)}(n+1)(n-3) \geq 0, \ (n \geq 3);$$
$$\frac{1}{k} - \frac{1}{2(n-k+1)(n-k)} = \frac{1}{4k(n-k+1)(n-k)}(n^2-3n-2) > 0, \ (n \geq 4).$$

greater than k. The partition $\{N\}$ is individually stable because it is Nash stable. For a partition which contains a winning coalition K^+ with $k^+ > k$, every player i in K^+ has no incentive to deviate from K^+ because $K^+ \setminus \{i\}$ is still winning. Although every singleton player in $N \setminus K^+$ has an incentive to join coalition K^+, all players in K^+ refuse it. $\square$

4 Conclusion

Summaries: Stability is a central theme in this thesis and contains two key topics: stable allocations and stable coalition structures.

- In Chapter 2, we analyze stable allocations. We generalize the Bondareva-Shapley condition to games with externalities. A system of endogenous variables λ reflects a difference between the core concepts. An extension of balanced collections is also discussed. A contrastive difference between optimistically balanced collections and pessimistically balanced collections lies in their minimality: a collection is pessimistically balanced if and only if it is a minimal pessimistically balanced collection, while such an equivalence does not hold for optimistically balanced collections. Applications to the Cournot oligopoly and the tragedy of the commons are also provided. Moreover, we provide characterization results for the cores by extending a form of consistency, called the complement-reduced game property. Consistency notions, called the max-reduced game property and the projection-reduced game property are also analyzed. We show that the consistency concepts for the cores are derived from some properties for expectation functions. We provide some asymmetric results among these different forms of reduced games in the presence of externalities.

- In Chapter 3, we study stable coalition structures. To answer Hart and Kurz (1984)'s question, we clarify which coalition structure satisfies which stability concept in a symmetric majority game. We show that the pessimistic core coincides with some farsighted vNM stable set. Moreover, we prove that a coalition structure satisfies the above two stability concepts if and only if the partition contains an exact majority coalition. The other stability concepts are also analyzed. One of important facts is that even if a coalition structure contains an exact majority coalition, some myopic players may have an incentive to deviate from the coalition structure. In addition, even a coalition structure that contains a coalition lager than an exact majority coalition admits a deviation consisting of two or more players.

Future work:

<u>The coalition structure core:</u> We have developed the basic properties of the core concepts for games with externalities in Chapter 2. However, each of the core concepts might be thought of as a "naive" core, because an expectation function, that features each core, only depends on a coalition and a game. The projective core is an approach to improve this weakness. As discussed in Subsection 2.3.1 and Chapter 3, this core plays a key role to analyze the stability of a coalition structure that is not the grand coalition. Therefore, following the spirit of the coalition structure core (see Subsection 2.3.2), we should define the projective core as a set of pairs of an allocation

and a partition. More specifically, we can define the projective core as

$$C^{\text{pro}}(N,v) = \left\{ (x,\mathcal{P}) \;\middle|\; \begin{array}{ll} \sum_{j\in S} x_j \geq v(S,\{S\}\cup\mathcal{P}|_{N\setminus S}), & \text{for all } S \notin \mathcal{P} \\ \sum_{j\in T} x_j = v(T,\mathcal{P}), & \text{for all } T \in \mathcal{P} \end{array} \right\}.$$

The projective core above is no longer a set of allocations available only for the grand coalition, but a set of allocations for all possible partitions. This extension allows us to study which allocation is stable for which coalition structure. Oligopolistic competitions, cooperative provision of public goods, and the tragedy of the commons should be good examples to use this core.

<u>The dual of a partition function:</u> Oishi et al. (2016) study the relationship between the dual game and the axiomatization of the core in games without externalities. Can we apply their results to games with externalities? We first follow Oishi et al. (2016) and consider the dual game of a coalition function form game. Let w be a coalition function form game. The dual of w, w^*, is given by

$$w^*(S) = w(N) - w(N \setminus S)$$

for any $S \subseteq N$. Let σ be a solution on Γ. The anti-dual of σ, σ^{ad}, is a solution that satisfies $\sigma^{ad}(w) = -\sigma(-w^*)$ for all $w \in \Gamma$. We say that σ is self-anti-dual if $\sigma(w) = \sigma^{ad}(w)$ for any $w \in \Gamma$. Moreover, two axioms A and A' are anti-dual to each other if for all solutions that satisfy A, their anti-duals also satisfy A', and vice versa. An axiom is self-anti-dual if the axiom is anti-dual to itself.

Now we attempt to define the dual game of a partition function form game. However, because of the partition term of $v(S,\mathcal{P})$, the extension is not straightforward. To avoid this problem, we might use an expectation function as follows: for any $(S,\mathcal{P}) \in EC(N)$,

$$v^*(S,\mathcal{P}) = v(N,\{\emptyset\}) - v(N \setminus S, \psi(N,v,N\setminus S)).$$

However, this is not well-defined because self-duality is violated as follows: for any $(S,\mathcal{P}) \in EC(N)$,

$$\begin{aligned} v^{**}(S,\mathcal{P}) &= v^*(N,\{\emptyset\}) - v^*(N\setminus S, \psi(N,v^*,N\setminus S)) \\ &= v(S,\psi(N,v,S)), \end{aligned}$$

where we set $v^*(N,\{\emptyset\}) = v(N,\{\emptyset\})$. Hence, for the equality to hold, we must have $\psi(N,v,S) = \mathcal{P} \neq \mathcal{P}' = \psi(N,v,S)$ for each S, and there is no such an expectation function.

If we established the dual game in partition function form, then their Proposition 1: complement consistency and projection consistency are anti-dual to each other, could hold in some sense, and the duality connection between axiomatization of the ψ-core with complement RGP and that with projection RGP might be established. To achieve this, we need another formulation of the dual game. This is also a future work.

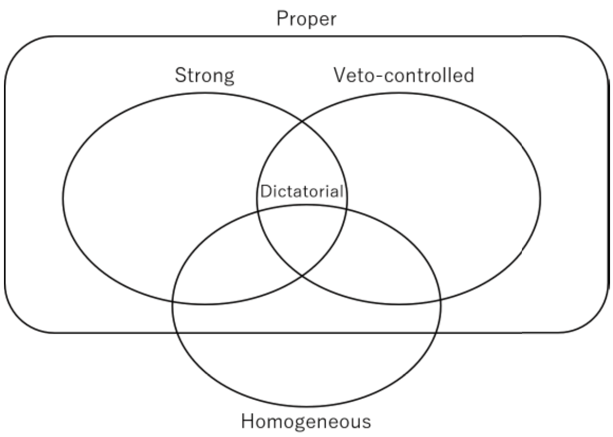
<u>An extension to weighted majority games:</u> In Chapter 3, we focus on the class of symmetric majority games. An extension of our study is to consider the class of weighted majority games. The class of weighted majority games contains the class of symmetric majority games and covers a wide variety of voting situations. Below we offer some results that must be useful for the further investigation.

A weighted majority game is an $(n+1)$-tuple $(q; w_1, ..., w_n)$, where w_i is player i's weight and q is a quota. A coalition S wins if and only if $\sum_{j \in S} w_j \geq q$. Let $\mathcal{W}$ be the set of winning coalitions. Let $\mathcal{W}^m$ denote the set of minimal winning coalitions: $\mathcal{W}^m = \{S \in \mathcal{W} | T \subsetneq S \Rightarrow T \notin \mathcal{W}\}$. Below is the list of basic properties of weighted majority games: a weighted majority game is

- *proper* if $S \in \mathcal{W} \Longrightarrow N \setminus S \notin \mathcal{W}$,
- *strong* if $S \in \mathcal{W} \iff N \setminus S \notin \mathcal{W}$,
- *veto-controlled* if $\bigcap_{S \in \mathcal{W}} S \neq \emptyset$,
- *homogeneous* if $S \in \mathcal{W}^m \Longrightarrow \sum_{j \in S} w_j = q$,
- *dictatorial* if there exists $j \in N$ such that $S \in \mathcal{W} \iff j \in S$.

The relationship is summarized in Figure 4.1. A weighted majority game is dictatorial if and only if it is strong and veto-controlled.

Figure 4.1: The properties of weighted majority games



Given a weighted majority game and a partition of N, we can compute the Owen power index in the same manner. However, the stability result is not straightforward. We first observe the following results.

- There are many examples with an empty optimistic core.
- There is a five-player example with an empty projective core.
- There is a six-player example with an empty pessimistic core.

The following game is the six-player example: $(q; w_1, ..., w_6) = (22; 10, 10, 8, 1, 1, 1)$. This game is veto-controlled. Therefore, this fact implies that veto-control property is not a sufficient condition for each core to be nonempty, because the pessimistic core is empty. One might consider that veto-control property can be a sufficient condition for each core to be *empty*. However, $(10; 10, 1, 1, 1)$ is veto-controlled (dictatorial) and has a nonempty optimistic core. Therefore, veto-control property cannot be a condition that distinguishes the emptiness of the cores. Similarly, the same observation holds even for properness and homogeneity. The remaining conditions are strongness and dictator property. We have computed the three cores for all weighted majority games with $n \leq 7$ (the two weighted majority games which share the same collection of winning sets is considered as the same game). As a result, strongness guarantees the nonempty pessimistic core as long as $n \leq 7$. This statement for general n is still an open question and is our future work.

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