Study on the additive representation of numbers from the view point of generalized Thue-Morse sequence

一般化されたThue-Morse数列から見た数の加法的表示について

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Chapter 1

Introduction

The Thue-Morse sequence $(t(n))_{n=0}^{\infty}$ (or Prouhet-Thue-Morse sequence) is the following well-known binary sequence

$$(t(n))_{n=0}^{\infty} = 0110100110010110100101100110100110010110011010010 11\cdots.$$

The origin of this sequence is Prouhet's investigation on the following problem, known as Prouhet-Tarry-Escott problem [AlS2, BoPL, P]: For any integer $k > 0$, find two distinct sets of integers A and B satisfying the equations,

$$\sum_{x\in A} x^j = \sum_{y\in B} y^j$$

for $j \in \{0, 1, \cdots, k\}$. Prouhet [P] discovered the solutions to the above problem by using the following identity,

Prouhet's identity *Set* $f_1(x) := x - (x+1) - (x+2) + (x+3) = 0$. *We define the polynomial* $f_k(x)$ *recursively as*

$$f_k(x) := \sum_{n=0}^{2^k-1} \epsilon(n)(x+n)^k - \sum_{n=0}^{2^k-1} \epsilon(n)(x+2^k+n)^k = 0$$

where $f_{k-1}(x) = \sum_{n=0}^{2^k-1} \epsilon(n)(x+n)^{k-1}$, $\epsilon(n) \in \{1, -1\}$.

Letting k tend to infinity, we can define the sequence $(\epsilon(n))_{n=1}^{\infty}$. This sequence is the origin of the Thue-Morse sequence. Fifty years later, Thue [Th], independently of Prouhet, introduced the Thue-Morse sequence as the infinite binary sequence that contains no cube, i.e, no three consecutive identical blocks. This property of the Thue-Morse sequence made change the rule for infinite play in chess. Moreover, independently of them, Morse [Mors] (famous for Morse theory) rediscovered this sequence in 1921. Morse studied geodesics on a surface of negative curvature by using this sequence. After pioneering works on this sequence, the Thue-Morse sequence and its several generalizations have been investigated in the following areas [AlS2]; combinatorial number theory, transcendental number theory, differential geometry, dynamical system, finite automata, etc.

In this thesis, we study the Thue-Morse sequence and its generalizations which is deeply related with number theory. This thesis consists of two chapters as follows.

In Chapter 2, we investigate transcendental numbers related to the Thue-Morse sequence and its generalizations. In 1929, Mahler [Ma] proved the transcendence of the series $\sum_{n=0}^{\infty} t(n)\beta^{-n-1}$, where β is an integer greater than 1. Let *per* be the permutation $0 \to 1$, $1 \to -1$ and $\bar{t}(n) := per(t(n))$. For the proof of the transcendence of the series, Mahler used the following functional equation of this sequence's $(\bar{t}(n))_{n=0}^{\infty}$ generating function $f(z) := \sum_{n=0}^{\infty} \bar{t}(n)z^n$,

$$f(z) = (1-z)f(z^2).$$

Recently, this method is known as Mahler functions theory [N], which is extensively investigated by many researchers. On the other hand, Morton and Mourant [MortM], Adamczewski, Bugeaud and Luca [AdBL] and Adamczewski and Bugeaud [AdB] proved the transcendence of the series by the combinatorial study on transcendental numbers: Let k be an integer greater than 1. We define the k-adic expansion of non-negative integer n as follows

$$n = \sum_{q=1}^{\text{finite}} s_{n,q} k^{w_n(q)},$$

where $1 \le s_{n,q} \le k-1, 0 \le w_n(q) < w_n(q+1)$. For any integer s in $\{1, \ldots, k-1\}$, let $e_s(n)$ denote the number of occurrences of s in the base k representation of n. For an integer L greater than 1, we define a sequence $(e_s^L(n))_{n=0}^{\infty}$ by

$$e_s^L(n) \equiv e_s(n) \pmod{L},$$

where $0 \le e_s^L(n) \le L-1$, $e_s(0) = 0$. Then $(e_1^2(n))_{n=0}^{\infty}$, where $k = 2$, is the Thue-Morse sequence.

Morton and Mourant [MortM] introduced a new sequence as follows; Let K be a map,

$$K : \{1, \ldots, k-1\} \longrightarrow \{0, 1, \ldots, L-1\}.$$

We define $(a(n))_{n=0}^{\infty}$ as

$$a(n) \equiv \sum_{s=1}^{k-1} K(s) e_s^L(n) \pmod{L}, \tag{1.1}$$

where $0 \le a(n) \le L-1$. Now we introduce a class of sequences as follows.

Definition 6 The sequence $(a(n))_{n=0}^{\infty}$ is called a *k-automatic sequence* if the set of sequences $\{(a(k^e n + j))_{n=0}^{\infty} |$ where $e \ge 0$ and $0 \le j \le k^e - 1\}$ is the finite set.

Morton and Mourant proved that these sequences are k-*automatic sequences*. Furthermore, the necessary-sufficient condition for the non-periodicity of these sequences is given in [MortM]. Later, Adamczewski-Bugeaud-Luca [AdBL] discovered the combinatorial transcendence criterion in 2004; Adamczewski, Bugeaud, and Luca [AdBL] introduced a new class of sequences *stammering sequence*, as follows. For any positive number $y, \lfloor y \rfloor$ and $\lceil y \rceil$ are the floor and ceiling functions. Let W be a finite word on $\{a_0, a_1, \ldots, a_{L-1}\}$ and let $|W|$ be the length of W. For any positive number x, we let W^x defined the word $W^{\lfloor x \rfloor} W^{\text{`}}$, where $W^{\text{`}}$ is a prefix of W of length $\lceil (x - \lfloor x \rfloor)|W| \rceil$.

Definition 3 $(a(n))_{n=0}^{\infty}$ is called a *stammering sequence* if $(a(n))_{n=0}^{\infty}$ satisfies the following conditions:

(1) The sequence $(a(n))_{n=0}^{\infty}$ is a non-periodic sequence.

(2) There exist two sequences of finite words, $(U_m)_{m\geq 1}$ and $(V_m)_{m\geq 1}$, such that,

(A) there exists a real number $w > 1$ independent of n such that the word ${U_m V_m}^w$ is a prefix of the word $(a(n))_{n=0}^{\infty}$,

(B) $\lim_{m\to\infty} |U_m|/|V_m| < +\infty$, and

(C) $\lim_{m\to\infty} |V_m| = +\infty$.

Adamczewski, Bugeaud, Luca [AdBL] proved the following combinatorial transcendence criterion that follows by the Schmidt subspace theorem.

Combinatorial transcendence criterion ([AdBL]) *If β is an integer greater than 1 and $(a(n))_{n=0}^{\infty}$ is a stammering sequence on $\{0, 1, \ldots, \beta-1\}$, then $\sum_{n=0}^{\infty} \frac{a(n)}{\beta^{n+1}}$ is a transcendental number.*

By using this criterion, Adamczewski and Bugeaud [AdB] gives the following affirmative answer to the Cobham conjecture [Co, W]

The Cobham conjecture (AdB) *If β is an integer greater than 1 and $(a(n))_{n=0}^{\infty}$ is a non-periodic k-automatic sequence on $\{0, 1, \ldots, \beta - 1\}$, then $\sum_{n=0}^{\infty} \frac{a(n)}{\beta^{n+1}}$ is a transcendental number.*

Morton and Mourant [MortM] and Adamczewski and Bugeaud [AdB] have proved the following result.

Theorem 1 ([MortM, AdB]) *Let $\beta \geq L$ be an integer. Then $\sum_{n=0}^{\infty} \frac{a(n)}{\beta^{n+1}}$ is a transcendental number unless*

$$sK(1) \equiv K(s) \pmod{L} \text{ for all } 1 \leq s \leq k-1 \text{ and } K(k-1) \equiv 0 \pmod{L}.$$

The investigations [AdBL, AdB, MortM] can suggest the combinatorial study on transcendental numbers. In this way, Mahler functions theory and combinatorial transcendental number theory often give the analogous results of these arguments.

In Chapter 2, we give a generalization of Morton and Mourant' investigation. Specifically, we generalize the Thue-Morse sequence by more detailed digit counting; For any integer s in $\{1, \ldots, k-1\}$ and any non-negative integer y, letting $d(n; sk^y)$ be 1 or 0, and $d(n; sk^y)$ satisfies that $d(n; sk^y) = 1$ if and only if there exists an integer q such that $s_{n,q} k^{w_n(q)} = sk^y$. Let μ be a map,

$$\mu : \{1, \ldots, k-1\} \times \mathbb{N} \longrightarrow \{0, 1, \ldots, L-1\},$$

where $\mathbb{N}$ denotes the set of non-negative integers. We define $(a(n))_{n=0}^{\infty}$ as

$$a(n) \equiv \sum_{y=0}^{\infty} \sum_{s=1}^{k-1} \mu(s, y) d(n; sk^y) \pmod{L}, \tag{1.2}$$

where $0 \leq a(n) \leq L-1$ and $a(0) = 0$. We call $(a(n))_{n=0}^{\infty}$ a generalized Thue-Morse sequence of type (L, k, μ), abbreviated as the (L, k, μ)-TM sequence. Thus the Thue-Morse sequence is the generalized Thue-Morse sequence of type $(2, 2, \mu)$ with $\mu(1, y) = 1$ for all $y \in \mathbb{N}$. Moreover, if a generalized Thue-Morse sequence $(a(n))_{n=0}^{\infty}$ is of type (L, k, μ) with

$$\mu(s, y) = \mu(s, y+1)$$

for all s with $1 \leq s \leq k-1$ and for all $y \in \mathbb{N}$, then $(a(n))_{n=0}^{\infty}$ coincides with the sequence defined by (1.1), which satisfies the conditions $K(s) = \mu(s, y)$ for all s with $1 \leq s \leq k-1$. In Chapter 2, we generalize Theorem 1 as follows.

Theorem 2 *Let $(a(n))_{n=0}^{\infty}$ be an (L, k, μ)-TM sequence. Let $\beta \geq L$ be an integer. If there is not an integer A such that*

$$\mu(s, A+y) \equiv \mu(1, A) sk^y \pmod{L}$$

for all s with $1 \leq s \leq k-1$ and for all $y \in \mathbb{N}$, then $\sum_{n=0}^{\infty} \frac{a(N+nl)}{\beta^{n+1}}$ (for all $N \geq 0$ and for all $l > 0$) is a transcendental number.

We can find an uncountable quantity of new transcendental numbers in the series generated by (L, k, μ)-TM sequences. Moreover, we show that any arithmetical subsequence of a non-periodic (L, k, μ)-TM sequence gives a transcendental number. This theorem can be regarded as a combinatorial analogy of Tachiya's investigation [Ta] in Mahler functions.

Now we explain the outline of proof of Theorem 2. This proof consists of the two parts, the irrationality of the series and the transcendence of the series. For the analysis of the irrationality, we give the following key lemma about the k-adic expansion of non-negative integers.

Lemma 4 *If $k > 1$ and $l > 0$ be integers and t be a non-negative integer, then there exists an integer x such that*

$$xl = \sum_{q=1}^{\text{finite}} s_{xl,q} k^{w_{xl}(q)},$$

where $s_{xl,1} = 1, w_{xl}(2) - w_{xl}(1) > t, w_{xl}(q+1) > w_{xl}(q) \geq 0.$

Furthermore, if t' *be other non-negative integer, then there exists an integer* X *such that*

$$Xl = \sum_{q=1}^{\text{finite}} s_{Xl,q} k^{w_X l(q)},$$

where $s_{Xl,1} = 1, w_{Xl}(2) - w_{Xl}(1) > t', w_{Xl}(q+1) > w_{Xl}(q) \geq 0, w_{xl}(1) = w_{Xl}(1).$

We prove the following theorem by using this lemma.

Theorem 3 *Let* $A_\infty = (a(n))_{n=0}^{\infty}$ *be an* (L, k, μ)*-TM sequence. The sequence* $A_\infty = (a(n))_{n=0}^{\infty}$ *is ultimately periodic if and only if there exists an integer* A *such that*

$$\mu(s, A+y) \equiv \mu(1, A) s k^y \pmod{L},$$

for all s *with* $1 \leq s \leq k-1$ *and for all* $y \in \mathbb{N}$.

Moreover, if the (L, k, μ)*-TM sequence is not ultimately periodic, then no arithmetical subsequence of* (L, k, μ)*-TM sequence is ultimately periodic.*

On the other hands, for the analysis of the transcendence, we give the recursively word definition of (L, k, μ)-TM sequences as follows.

Definition 2 Let L be an integer greater than 1, and let $a_0, a_1, \ldots, a_{L-1}$ be L distinct complex numbers. We let $\{a_0, a_1 \ldots a_{L-1}\}^*$ denote the free monoid generated by $\{a_0, a_1 \ldots a_{L-1}\}$. We define a morphism f from $\{a_0, a_1 \ldots a_{L-1}\}^*$ to $\{a_0, a_1 \ldots a_{L-1}\}^*$ as follows:

$$f(a_i) = a_{i+1},$$

where the index i is defined to be $\mod L$. Let f^j be the j times composed mapping of f, and let f^0 be an identity mapping. Let A and B be two finite words on $\{a_0, a_1, \ldots, a_{L-1}\}$, and let AB denote the concatenation of A and B.

Let $A_0 = a_0$, k be an integer greater than 1, and let μ be a map μ:$\{1, \ldots, k-1\} \times \mathbb{N} \to \{0, \ldots, L-1\}$. For a non-negative integer m, we define a space of words W_m by

$$W_m := \{a_{i_1} a_{i_2} \ldots a_{i_m} \,|a_{i_1}, a_{i_2}, \cdots, a_{i_m} \in \{a_0, a_1, \ldots, a_{L-1}\}\}.$$

We define $A_{n+1} \in W_{k^{n+1}}$ recursively as

$$A_{n+1} := A_n f^{\mu(1,n)}(A_n) \cdot \cdots \cdot f^{\mu(k-1,n)}(A_n),$$

and we let

$$A_\infty := \lim_{n \to \infty} A_n$$

denote the limit of A_n. The sequence (or infinite word) A_∞ is called the *generalized Thue-Morse sequence of type* (L, k, μ), abbreviated as the (L, k, μ)-TM sequence.

The following lemma shows that the (L,k,μ)-TM sequence defined by digit counting corresponds to the (L,k,μ)-TM sequence defined by the recursively word.

Lemma 3 *Let $A_\infty = (b(n))_{n=0}^{\infty}$ be a (L,k,μ)-TM sequence with $a_j = \exp\frac{2\pi\sqrt{-1}j}{L}$ (for all j with $0 \le\ j \le L-1$). Let $G_{A_\infty}(z)$ be the generating function of $(b(n))_{n=0}^{\infty}$,*

$$G_{A_\infty}(z) := \sum_{n=0}^{\infty} b(n)z^n.$$

The generating function $G_{A_\infty}(z)$ will have the infinite product on $|z|<1$,

$$G_{A_\infty}(z) = \prod_{y=0}^{\infty}(1+\sum_{s=1}^{k-1}\exp\frac{2\pi\sqrt{-1}\mu(s,y)}{L}z^{sk^y}).$$

From the recursively word definition of (L,k,μ)-TM sequence $(a(n))_{n=0}^{\infty}$, we deduce that any arithmetical subsequence $(a(N+ln))_{n=0}^{\infty}$ of a non-periodic (L,k,μ)-TM sequence is a *stammering sequence*. Therefore, we complete the proof by combing this fact with Theorem 3.

We also give the following necessary-sufficient condition that an (L,k,μ)-TM sequence is a k-automatic sequence.

Proposition 3 *An (L,k,μ)-TM sequence is a k-automatic sequence if and only if there exist integers N and t such that*

$$\mu(s,y) = \mu(s,y+t),$$

for all s with $1 \le s \le k-1$ and for all $y \ge N$.

From this proposition, almost no (L,k,μ)-TM sequence is a k-automatic sequence.

In Chapter 3, we investigate the additive representation of integers related to the Thue-Morse sequence (This representation of integers is not unique). The results in Chapter 3 is the application of the special class of (L,k,μ)-TM sequences. Erdös-Surányi [ErS] and Prielipp [Bl] independently prove the following result; For any integers n, there are an integer N and a map $\epsilon : \{1,\ldots,N\} \to \{-1,1\}$ such that

$$n = \sum_{j=1}^{N}\epsilon(j)j^2.$$

They deduce the above result from the identity

$$x^2-(x+1)^2-(x+2)^2+(x+3)^2 = 4$$

where x is a real variable. Erdös-Surányi and Prielipp suggested to study the following problem: For any integers $k > 0$ and n, are there an integer N and a map $\epsilon : \{1, \ldots, N\} \to \{-1, 1\}$ such that

$$n = \sum_{j=1}^{N} \epsilon(j) j^k \ ?$$

Mitek [Mi], Kaja [K] and Bleicher [Bl] independently give the positive answer to this problem. The key point of this problem is the identity generated by the 2^k length prefix of the Thue-Morse sequence. The identity is related to Prouhet-Tarry-Escott problem.

In Chapter 3, we give a generalization of the Erdös-Surányi problem Mitek [Mi], Kaja [K] and Bleicher [Bl]. We study the case that the values of ϵ are L-th roots of unity, where L is a positive integer. Since, by Mitek [Mi], Kaja [K] and Bleicher [Bl], we already know that the answer is positive if L is even, we restrict our attention to odd L. Let U be the set of L-th roots of unity. Then we consider the following problem. For any integers $k > 0$ and n, are there an integer N and a map $\epsilon : \{1, \ldots, N\} \to U$ such that

$$n = \sum_{j=1}^{N} \epsilon(j) j^k \ ?$$

We prove the following three results.

Theorem 10 *Let L be a positive odd integer with $L \geq 2$ which is not a prime power and let U be the set of L-th roots of unity.*
Then for any integers $k > 0$ and n, there are an integer N and a map $\epsilon : \{1, \ldots, N\} \to U$ such that

$$n = \sum_{j=1}^{N} \epsilon(j) j^k.$$

The following result shows that the statement of Theorem 10 is valid if L is an odd prime power p^m and k is a multiple of $p - 1$.

Theorem 11 *Let p be an odd prime number, m be a positive integer and let U be the set of p^m-th roots of unity. Then for any integers $k > 0$ with $p - 1 \mid k$ and n, there are an integer N and a map $\epsilon : \{1, \ldots, N\} \to U$ such that*

$$n = \sum_{j=1}^{N} \epsilon(j) j^k.$$

Moreover, the following result shows that the statement of Theorem 10 is not valid if L is an odd prime power p^m and k is not a multiple of $p - 1$.

Theorem 12 *Let p be an odd prime number, m be a positive integer and let U be the set of p^m-th roots of unity. Then for any integer $k > 0$ with $p - 1 \nmid k$, there are infinitely many integers n such that n cannot be represented as*

$$n = \sum_{j=1}^{N} \epsilon(j) j^k, \tag{1.3}$$

where N is a positive integer and $\epsilon : \{1, \dots, N\} \to U$.

Now we explain the outlines of proofs. For the proofs of Theorem 10 and 11, we give the following identity as follows.

Lemma 5 *Let $L \geq 2$ be an integer and let U be the set of L-th roots of unity. Let x be a real variable. Then for any integer $k > 0$ there exist an integer M, a map $\epsilon : \{0, \dots, M-1\} \to U$ and a non-zero integer P such that*

$$\sum_{j=0}^{M-1} \epsilon(j)(x + j)^k = P.$$

Moreover for P, M as above and any integer m, there exist a multiple N of M and a map $\epsilon_m : \{0, \dots, M-1\} \to U$ such that

$$\sum_{j=0}^{N-1} \epsilon_{m(j)}(x + j)^k = mP.$$

We prove this lemma by induction on k. The key point of the proof is the facts $\sum_{j=0}^{L-1} \zeta_L^j = 0$ and $\sum_{j=1}^{L-1} \zeta_L^j = -1$, where $\zeta_L := \exp \frac{2\pi\sqrt{-1}}{L}$. Letting k tend to infinity, we can define the sequence $(\epsilon(j))_{j=1}^{\infty}$. The sequence $(\epsilon(j))_{j=1}^{\infty}$ can be regarded as the $(L, L(L-1), \mu)$-TM sequence where some μ. By the second statement of Lemma 5, we prove that, for each l with $0 \leq l \leq P - 1$, there exist an integer N and a map $\epsilon : \{1, \dots, N\} \to U$ such that

$$l \equiv \sum_{j=1}^{N} \epsilon(j) j^k \pmod{P}.$$

This proves Theorem 10 and 11. For the proof of Theorem 12, we show the following lemma.

Lemma 6 *Let p be an odd prime number, m be a positive integer and k be a positive integer with $p - 1 \nmid k$. Let U be the set of p^m-th roots of unity. Put $S_k(M) := \sum_{j=1}^{M} j^k$. Assume an integer n is represented by*

$$n = \sum_{j=1}^{N} \epsilon(j) j^k,$$

where N is an integer and $\epsilon : \{1, \ldots, N\} \to U$.
Then

$$S_k(N) \equiv n \pmod{p}.$$

By Lemma 6, we prove that there is at least one residue class modulo p of numbers n which cannot be represented as (1.3) for any positive integer N and any map. This proves Theorem 12.

Chapter 2

Transcendence of digital expansions generated by a generalized Thue-Morse sequence

2.1 Introduction

First we introduce the Thue-Morse sequence, defined by digit counting. Let k be an integer greater than 1. We define the k-adic expansion of non-negative integer n as follows:

$$n = \sum_{q=1}^{\text{finite}} s_{n,q} k^{w_n(q)}, \tag{2.1}$$

where $1 \leq s_{n,q} \leq k-1, 0 \leq w_n(q) < w_n(q+1)$. For any integer s in $\{1, \ldots, k-1\}$, let $e_s(n)$ denote the number of occurrences of s in the base k representation of n. For an integer L greater than 1, we define a sequence $(e_s^L(n))_{n=0}^{\infty}$ by

$$e_s^L(n) \equiv e_s(n) \pmod{L}, \tag{2.2}$$

where $0 \leq e_s^L(n) \leq L-1$, $e_s(0) = 0$. Then $(e_1^2(n))_{n=0}^{\infty}$, where $k = 2$, is known as the Thue-Morse sequence. The Thue-Morse sequence has several definitions. See [AlS2, Em, BerR, Fo].

Now we introduce a new sequence. Let K be a map,

$$K : \{1, \ldots, k-1\} \longrightarrow \{0, 1, \ldots, L-1\}.$$

We define $(a(n))_{n=0}^{\infty}$ as

$$a(n) \equiv \sum_{s=1}^{k-1} K(s) e_s^L(n) \pmod{L}, \tag{2.3}$$

where $0 \leq a(n) \leq L-1$. Morton and Mourant [MortM] and Adamczewski and Bugeaud [AdB] have proved the following result.

Theorem 1 ([MortM, AdB]) *Let $\beta \geq L$ be an integer. Then $\sum_{n=0}^{\infty} \frac{a(n)}{\beta^{n+1}}$ is a transcendental number unless*

$$sK(1) \equiv K(s) \pmod{L} \text{ for all } 1 \leq s \leq k-1 \text{ and } K(k-1) \equiv 0 \pmod{L}. \tag{2.4}$$

The proof of Theorem 1 rests on the periodicity of $(a(n))_{n=0}^{\infty}$ [MortM] and the Cobham conjecture [Co, W] that was settled by Adamczewski and Bugeaud [AdB]. More precisely, Morton and Mourant [MortM] proved that $(a(n))_{n=0}^{\infty}$ is a k-automatic sequence for any map K (see Definition 6 in Section 2.5 for the full definition). Furthermore, they proved that $(a(n))_{n=0}^{\infty}$ is periodic if and only if $(a(n))_{n=0}^{\infty}$ is purely periodic which enabled them to prove that $(a(n))_{n=0}^{\infty}$ is periodic if and only if the map K satisfies (2.4). Later, Adamczewski and Bugeaud [AdB] proved the Cobham conjecture by using the Schmidt subspace theorem. Thus they deduced Theorem 1 by combining the results of Morton and Mourant with the Cobham conjecture.

Let us define a generalized Thue-Morse sequence as follows: For any integer s in $\{1, \ldots, k-1\}$ and any non-negative integer y, letting $d(n; sk^y)$ be 1 or 0, and $d(n; sk^y)$ satisfies that $d(n; sk^y) = 1$ if and only if there exists an integer q such that $s_{n,q} k^{w_n(q)} = sk^y$. Let μ be a map,

$$\mu : \{1, \ldots, k-1\} \times \mathbb{N} \longrightarrow \{0, 1, \ldots, L-1\},$$

where $\mathbb{N}$ denotes the set of non-negative integers. We define $(a(n))_{n=0}^{\infty}$ as

$$a(n) \equiv \sum_{y=0}^{\infty} \sum_{s=1}^{k-1} \mu(s, y) d(n; sk^y) \pmod{L}, \tag{2.5}$$

where $0 \leq a(n) \leq L-1$ and $a(0) = 0$. We call $(a(n))_{n=0}^{\infty}$ a generalized Thue-Morse sequence of type (L, k, μ). Thus the Thue-Morse sequence is the generalized Thue-Morse sequence of type $(2, 2, \mu)$ with $\mu(1, y) = 1$ for all $y \in \mathbb{N}$. Moreover, if a generalized Thue-Morse sequence $(a(n))_{n=0}^{\infty}$ is of type (L, k, μ) with

$$\mu(s, y) = \mu(s, y+1) \tag{2.6}$$

for all s with $1 \leq s \leq k-1$ and for all $y \in \mathbb{N}$, then $(a(n))_{n=0}^{\infty}$ coincides with the sequence defined by (2.3), which satisfies the conditions $K(s) = \mu(s, y)$ for all s with $1 \leq s \leq k-1$. In Chapter 2, we generalize Theorem 1 as follows.

Theorem 2 *Let $(a(n))_{n=0}^{\infty}$ be a generalized Thue-Morse sequence of type (L, k, μ). Let $\beta \geq L$ be an integer. If there is not an integer A such that*

$$\mu(s, A+y) \equiv \mu(1, A)sk^y \pmod{L} \tag{2.7}$$

for all s with $1 \leq s \leq k-1$ and for all $y \in \mathbb{N}$, then $\sum_{n=0}^{\infty} \frac{a(N+nl)}{\beta^{n+1}}$ (for all $N \geq 0$ and for all $l > 0$) is a transcendental number.

By Theorem 2, one can find an uncountable quantity of transcendental numbers. Moreover, if $\sum_{n=0}^{\infty} \frac{a(n)}{\beta^{n+1}}$ is a transcendental number, then $\sum_{n=0}^{\infty} \frac{a(N+nl)}{\beta^{n+1}}$ (for all $N \geq 0$ and for all $l > 0$) is also a transcendental number. The proof of Theorem 2 does not rest on pure periodicity of the periodic generalized Thue-Morse sequence $(a(n))_{n=0}^{\infty}$ and the Cobham conjecture. Here we study non-periodicity of the subsequence $(a(N+nl))_{n=0}^{\infty}$ (for all $N \geq 0$ and for all $l > 0$) of a generalized Thue-Morse sequence $(a(n))_{n=0}^{\infty}$. See also Morgenbesser, Shallit, and Stoll [MorgSS]. Almost no generalized Thue-Morse sequence $(a(n))_{n=0}^{\infty}$ is k-automatic (see Proposition 3 in Section 2.5). Therefore, the proof of Theorem 2 is different from the proof of Theorem 1. We prove Theorem 2 by combining Theorem 3 in Section 3 with the combinatorial transcendence criterion established by Adamczewski, Bugeaud, and Luca [AdBL].

This part is organized as follows. In Section 2.2, we review the basic concepts of the periodicity of sequences, and give the formal definition of the generalized Thue-Morse sequences. For a sequence $(a(n))_{n=0}^{\infty}$, we set its generating function $g(z) \in \mathbb{C}[[z]]$ to be

$$g(z) := \sum_{n=0}^{\infty} a(n)z^n.$$

For a generalized Thue-Morse sequence, one can prove that the generating function is convergent on the open unit disk and that it has an infinite product expansion. In Section 2.3, first we prove the key lemma on the k-adic expansion of non-negative integers. Next, we use this lemma and the infinite product expansion of the generating function of a generalized Thue-Morse sequence to prove a necessary-sufficient condition for the non-periodicity of the generalized Thue-Morse sequence. Furthermore, we prove that if the generalized Thue-Morse sequence is not periodic then no subsequence $(a(N+nl))_{n=0}^{\infty}$ (for all $N \geq 0$ and for all $l > 0$) of the generalized Thue-Morse sequences is periodic. In Section 2.4, we introduce the concept of the stammering sequence, introduced by Adamczewski, Bugeaud, and Luca [AdBL], and the combinatorial transcendence criterion, established by Adamczewski, Bugeaud, Luca [AdBL] and Bugeaud [Bu1]. By applying this combinatorial transcendence criterion to the generalized non-periodic Thue-Morse sequence $(a(n))_{n=0}^{\infty}$, which takes its values from $\{0, 1, \ldots, \beta-1\}$, we show that $\sum_{n=0}^{\infty} a(N+nl)\beta^{-n-1}$ is a transcendental number. Furthermore by applying this combinatorial transcendence criterion to the generalized non-periodic Thue-Morse sequence $(a(n))_{n=0}^{\infty}$, which takes its values in bounded positive integers, we show that the continued fraction $[0 : a(N), a(N+l), \ldots, a(N+nl), \ldots]$ is also transcendental number. This result includes Theorem 2. In Section 2.5, we consider the necessary-sufficient condition that a generalized Thue-Morse sequence is a k-automatic sequence. Then we find many transcendental numbers whose irrationality exponent is finite in all arithmetical subsequences of the corresponding generalized Thue-Morse sequence by applying the Adamczewski and Cassaigne result on k-automatic irrational numbers [AdC]. Furthermore, we consider the transcendence of the value at the algebraic point of the generating function $\sum_{n=0}^{\infty} a(N+nl)z^{-n-1}$ by applying Becker's result on k-automatic power series.

2.2 Generalized Thue-Morse sequences and their generating functions

Let $(a(n))_{n=0}^{\infty}$ be a sequence with values in $\mathbb{C}$. $(a(n))_{n=0}^{\infty}$ is called *ultimately periodic* if there exist non-negative integers N and $l > 0$ such that

$$a(n) = a(n+l) \qquad (\forall n \geq N). \tag{2.8}$$

An *arithmetical subsequence* of $(a(n))_{n=0}^{\infty}$ is defined to be a subsequence such as $(a(N+tl))_{t=0}^{\infty}$, where $N \geq 0$ and $l > 0$.

Definition 1 Let $(a(n))_{n=0}^{\infty}$ be a sequence with values in $\mathbb{C}$. The sequence $(a(n))_{n=0}^{\infty}$ is called *everywhere non-periodic* if no arithmetical subsequence of $(a(n))_{n=0}^{\infty}$ takes on only one value.

Now we present some lemmas about the everywhere non-periodic sequences.

Lemma 1 *If $(a(n))_{n=0}^{\infty}$ is everywhere non-periodic, then $(a(n))_{n=0}^{\infty}$ is not ultimately periodic.*

Proof. We prove contraposition. Assume that $(a(n))_{n=0}^{\infty}$ is ultimately periodic. From the definition of everywhere non-periodic, there exist non-negative integers N and $l > 0$ such that

$$a(n) = a(n+l) \qquad (\forall n \geq N). \tag{2.9}$$

It follows from (2.9) that the arithmetical subsequence $(a(N+tl))_{t=0}^{\infty}$ takes on only one value. □

Lemma 2 *If $(a(n))_{n=0}^{\infty}$ is everywhere non-periodic, then all arithmetical subsequences of $(a(n))_{n=0}^{\infty}$ are everywhere non-periodic.*

Proof. We prove contraposition. If $(a(N+tl))_{t=0}^{\infty}$ is not everywhere non-periodic, then there exist non-negative integers k and $J > 0$ such that $(a(N+kl+mJl))_{m=0}^{\infty}$ takes on only one value. The subsequence $(a(N+kl+mJl))_{m=0}^{\infty}$ is also an arithmetical subsequence of $(a(n))_{n=0}^{\infty}$. Therefore, $(a(n))_{n=0}^{\infty}$ is not everywhere non-periodic. □

Corollary 1 *$(a(n))_{n=0}^{\infty}$ is everywhere non-periodic if and only if no arithmetical subsequence of $(a(n))_{n=0}^{\infty}$ is ultimately periodic.*

Proof. Assume $(a(n))_{n=0}^{\infty}$ is everywhere non-periodic. By Lemma 1 and Lemma 2, no arithmetical subsequence of the sequence $(a(n))_{n=0}^{\infty}$ is periodic.

We show the sufficient condition. Assume $(a(n))_{n=0}^{\infty}$ is not everywhere non-periodic. Then there exist non-negative integers N and $l > 0$ such that $(a(N + tl))_{t=0}^{\infty}$ takes on only one value. This sequence is ultimately periodic. $\square$

Next, we generalize the Thue-Morse sequence of Emmanuel [Em].

Definition 2 Let L be an integer greater than 1, and let $a_0, a_1, \ldots, a_{L-1}$ be L distinct complex numbers. We let $\{a_0, a_1 \ldots a_{L-1}\}^*$ denote the free monoid generated by $\{a_0, a_1 \ldots a_{L-1}\}$. We define a morphism f from $\{a_0, a_1 \ldots a_{L-1}\}^*$ to $\{a_0, a_1 \ldots a_{L-1}\}^*$ as follows:

$$f(a_i) = a_{i+1}, \tag{2.10}$$

where the index i is defined to be $\mod L$. Let f^j be the j times composed mapping of f, and let f^0 be an identity mapping. Let A and B be two finite words on $\{a_0, a_1, \ldots, a_{L-1}\}$, and let AB denote the concatenation of A and B.

Let $A_0 = a_0$, k be an integer greater than 1, and let μ be a map μ:$\{1, \ldots, k-1\} \times \mathbb{N} \to \{0, \ldots, L-1\}$. For a non-negative integer m, we define a space of words W_m by

$$W_m := \{a_{i_1} a_{i_2} \ldots a_{i_m} \,| a_{i_1}, a_{i_2}, \cdots, a_{i_m} \in \{a_0, a_1, \ldots, a_{L-1}\}\}. \tag{2.11}$$

We define $A_{n+1} \in W_{k^{n+1}}$ recursively as

$$A_{n+1} := A_n f^{\mu(1,n)}(A_n) \cdot \cdots \cdot f^{\mu(k-1,n)}(A_n), \tag{2.12}$$

and we let

$$A_\infty := \lim_{n \to \infty} A_n \tag{2.13}$$

denote the limit of A_n. The sequence (or infinite word) A_∞ is called the *generalized Thue-Morse sequence of type* (L, k, μ), abbreviated as the (L, k, μ)-TM sequence.

Example 1 ([Em]) Let $L = 2$, $a_0 = 0$, $a_1 = 1$ and $\mu(1, y) = 1$ for all $y \in \mathbb{N}$. The $(2, 2, 1)$-TM sequence will be as follows:

$$A_0 = 0,\ A_1 = 01,\ A_2 = 0110,\ A_3 = 01101001,$$
$$A_\infty = 0110100110010110100101100110100110010110011010010110100110 \cdots.$$

This example is the Thue-Morse sequence of Emmanuel [Em].

Example 2 Let $L = 2$, $a_0 = 0$, $a_1 = 1$ and

$$\mu(1, y) = \begin{cases} 1, & y \text{ is a prime number;} \\ 0, & \text{otherwise.} \end{cases}$$

The $(2, 2, \mu)$-TM sequence will be

$$A_0 = 0,\ A_1 = 00,\ A_2 = 0000,\ A_3 = 00001111,$$
$$A_\infty = 00001111000011111111000011110000 \cdots .$$

Example 3 Let $L = 2$, $a_0 = 0$, $a_1 = 1$ and

$$\mu(1, y) = \begin{cases} 1, & y \text{ is a square number and } s = 2; \\ 0, & \text{otherwise.} \end{cases}$$

The $(2, 3, \mu)$-TM sequence will be

$$A_0 = 0,\ A_1 = 001,\ A_2 = 001001001,$$
$$A_\infty = 001001001001001001001001001001001001001001001001001001110110 \cdots .$$

Let $(a(n))_{n=0}^{\infty}$ be a sequence with values in $\mathbb{C}$. The *generating function* of $(a(n))_{n=0}^{\infty}$ is the formal power series $g(z) \in \mathbb{C}[[z]]$, defined as

$$g(z) := \sum_{n=0}^{\infty} a(n) z^n.$$

The following lemma clarifies the meaning of an (L, k, μ)-TM sequence.

Lemma 3 *Let $A_\infty = (b(n))_{n=0}^{\infty}$ be a (L, k, μ)-TM sequence with $a_j = \exp \frac{2\pi\sqrt{-1}j}{L}$ (for all j with $0 \le j \le L - 1$). Let $G_{A_\infty}(z)$ be the generating function of $(b(n))_{n=0}^{\infty}$,*

$$G_{A_\infty}(z) := \sum_{n=0}^{\infty} b(n) z^n.$$

The generating function $G_{A_\infty}(z)$ will have the infinite product on $|z| < 1$,

$$G_{A_\infty}(z) = \prod_{y=0}^{\infty} (1 + \sum_{s=1}^{k-1} \exp \frac{2\pi\sqrt{-1}\mu(s, y)}{L} z^{sk^y}). \tag{2.14}$$

Proof. From the assumption $a_j = \exp \frac{2\pi\sqrt{-1}j}{L}$ for all j with $0 \le j \le L - 1$, we have

$$f(a_j) = \exp \frac{2\pi\sqrt{-1}}{L} a_j \tag{2.15}$$

for all j with $0 \le j \le L-1$. The (L, k, μ)-TM sequence takes on only finite values and the Cauchy-Hadamard theorem, $G_{A_\infty}(z)$ and $\prod_{y=0}^{\infty}(1+\sum_{s=1}^{k-1} \exp \frac{2\pi\sqrt{-1}\mu(s,y)}{L} z^{sk^y})$

converge absolutely on the unit disk. Let $G_{A_n}(z)$ be the generating function of A_n; We identify the infinite word $A_n 0 \cdots 0 \cdots =: A_n 0^\infty$ with A_n.

We will show by induction that the following equality holds on n,

$$G_{A_n}(z) = \prod_{y=0}^{n-1}(1 + \sum_{s=1}^{k-1} \exp \frac{2\pi\sqrt{-1}\mu(s,y)}{L} z^{sk^y}). \tag{2.16}$$

First, we check the case $n = 1$. From the definition of A_1, we have

$$G_{A_1}(z) = 1 + \sum_{s=1}^{k-1} \exp \frac{2\pi\sqrt{-1}\mu(s,0)}{L} z^s. \tag{2.17}$$

Thus, the $n = 1$ case is true. By the induction hypothesis we may assume that

$$G_{A_j}(z) = \prod_{y=0}^{j-1}(1 + \sum_{s=1}^{k-1} \exp \frac{2\pi\sqrt{-1}\mu(s,y)}{L} z^{sk^y}). \tag{2.18}$$

Therefore, we have

$$G_{A_{j+1}}(z) = G_{A_j}(z) + \sum_{s=1}^{k-1} G_{f^{\mu(s,j)}(A_j)}(z) z^{sk^j}. \tag{2.19}$$

Alternatively,

$$G_{f^{\mu(s,j)}(A_j)}(z) = \exp \frac{2\pi\sqrt{-1}\mu(s,j)}{L} G_{A_j}(z). \tag{2.20}$$

From (2.18)-(2.20), we get

$$\begin{aligned} G_{A_{j+1}}(z) &= G_{A_j}(z)(1 + \sum_{s=1}^{k-1} \exp \frac{2\pi\sqrt{-1}\mu(s,y)}{L} z^{sk^j}) \\ &= \prod_{y=0}^{j}(1 + \sum_{s=1}^{k-1} \exp \frac{2\pi\sqrt{-1}\mu(s,y)}{L} z^{sk^y}). \end{aligned} \tag{2.21}$$

Therefore (2.14) is true. Finally, we will compare the coefficients of z^j on both sides of (2.14). On the right-hand side of (2.14), the coefficient of z^j are determined by $G_{A_N}(z)$ for sufficiently large N. From the definition of A_∞, the prefix word, p^N, of A_∞ is A_N. From the above argument and (2.16), the coefficients of z^j on both sides of (2.14) must coincide. $\square$

Proposition 1 *Let $A_\infty = (b(n))_{n=0}^\infty$ be a (L, k, μ)-TM sequence with $a_j = \exp \frac{2\pi\sqrt{-1}j}{L}$ (for all j with $0 \le j \le L-1$). Let $(a(n))_{n=0}^\infty$ be a sequence defined by* (2.5)*. Then*

$$\frac{L}{2\pi\sqrt{-1}} \log b(n) \equiv a(n) \pmod{L}. \tag{2.22}$$

Proof. Let the k-adic expansion of n be as follows:

$$n = \sum_{q=1}^{n(k)} s_{n,q} k^{w_n(q)}, \tag{2.23}$$

where $1 \leq s_{n,q} \leq k-1$, $0 \leq w_n(q) < w_n(q+1)$. By uniqueness of the k-adic expansion and Lemma 3, we have

$$\begin{aligned} b(n) &= \textstyle\prod_{q=1}^{n(k)} \exp \frac{2\pi\sqrt{-1}\mu(s_{n,q}, w_n(q))}{L} \\ &= \exp \frac{2\pi\sqrt{-1}(\sum_{q=1}^{n(k)} \mu(s_{n,q}, w_n(q)))}{L} \\ &= \exp \frac{2\pi\sqrt{-1}(\sum_{q=1}^{n(k)} \mu(s_{n,q}, w_n(q)) \pmod{L})}{L}. \end{aligned} \tag{2.24}$$

By (2.23), (2.24) and the definition of $a(n)$, the equality (2.22) is obtained. $\square$

Now we give other representations of Example 2 and Example 3 by using Proposition 1.

We begin Example 2. Let the 2-adic expansion of non-negative integer n be

$$n = \sum_{q=1}^{\text{finite}} 2^{w_n(q)}, \tag{2.25}$$

where $0 \leq w_n(q) < w_n(q+1)$. We define the number $A(n)$ to be

$$A(n) = \#\{ \ w_n(q) \mid w_n(q) \text{ is a prime number } \},$$

and we define $(a(n))_{n=0}^{\infty}$ as

$$a(n) = \begin{cases} 1, & A(n) \equiv 1 \pmod{2} \\ 0, & A(n) \equiv 0 \pmod{2}; \end{cases} \tag{2.26}$$

e.g., $a(44) = a(2^2 + 2^3 + 2^5) = 1$, $a(12) = a(2^2 + 2^3) = 0$). The sequence $(a(n))_{n=0}^{\infty}$ is the generalized Thue-Morse sequence of type $(2, 2, \mu)$ with

$$\mu(1, y) = \begin{cases} 1, & y \text{ is a prime number;} \\ 0, & \text{otherwise.} \end{cases}$$

Next, we give another representation of Example 3. Let the 3-adic expansion of non-negative integer n be

$$n = \sum_{q=1}^{\text{finite}} s_{n,q} 3^{w_n(q)}, \tag{2.27}$$

where $1 \leq s_{n,q} \leq 2, 0 \leq w_n(q) < w_n(q+1)$. We define the number $B(n)$ as

$$B(n) = \#\{ \ w_n(q) \mid w_n(q) \text{ is a square number and } s_{n,q} = 2 \ \},$$

and we define $(a(n))_{n=0}^{\infty}$ as

$$a(n) = \begin{cases} 1, & B(n) \equiv 1 \pmod 2 \\ 0, & B(n) \equiv 0 \pmod 2; \end{cases} \tag{2.28}$$

e.g., $a(169) = a(1 + 2 \times 3 + 2 \times 3^4) = 0$, $a(7) = a(1 + 2 \times 3) = 1)$. The sequence $(a(n))_{n=0}^{\infty}$ is the generalized Thue-Morse sequence of type $(2, 3, \mu)$ with

$$\mu(s, y) = \begin{cases} 1 & y \text{ is a square number and } s = 2; \\ 0 & \text{otherwise.} \end{cases}$$

2.3 Necessary-sufficient condition for the non-periodicity of a generalized Thue-Morse sequence

We begin by presenting the following key lemma about the k-adic expansion of non-negative integers.

Lemma 4 *If $k > 1$ and $l > 0$ be integers and t be a non-negative integer, then there exists an integer x such that*

$$xl = \sum_{q=1}^{\text{finite}} s_{xl,q} k^{w_{xl}(q)}, \tag{2.29}$$

where $s_{xl,1} = 1, w_{xl}(2) - w_{xl}(1) > t, w_{xl}(q + 1) > w_{xl}(q) \geq 0$.

Furthermore, if t' be other non-negative integer, then there exists an integer X such that

$$Xl = \sum_{q=1}^{\text{finite}} s_{Xl,q} k^{w_{Xl}(q)}, \tag{2.30}$$

where $s_{Xl,1} = 1, w_{Xl}(2) - w_{Xl}(1) > t', w_{Xl}(q + 1) > w_{Xl}(q) \geq 0, w_{xl}(1) = w_{Xl}(1)$.

Proof. Let us assume the factorization of k into prime factors is

$$k = \prod_{t=1}^{N} {p_t}^{y_t}, \tag{2.31}$$

where $p_1, p_2, \cdots p_N$ are N distinct prime numbers and y_t for p_t $(1 \leq t \leq N)$ are N positive integers. Let l be represented as

$$l = G \prod_{u=1}^{n} {p_{t_u}}^{x_u}, \tag{2.32}$$

where G and k are coprime, $p_{t_u} \in \{p_t | 1 \leq t \leq N\}$ and x_u are n positive integers. As G and k are coprime, there exist integers D and E such that

$$DG = 1 - k^{t+1}E. \tag{2.33}$$

We set

$$F = \max\{A | x_u = y_{t_u} A + H,\ 0 \leq H < y_{t_u},\ 1 \leq u \leq n\}.$$

From the definition of F, $k^{F+1} \prod_{u=1}^{n} p_{t_u}{}^{-x_u}$ is a non-negative integer. Thus we have

$$lD^2 G k^{F+1} \prod_{u=1}^{n} p_{t_u}{}^{-x_u} = k^{F+1} D^2 G^2. \tag{2.34}$$

On the other hand, by (2.33) we have

$$D^2 G^2 = 1 + k^{t+1} E (k^{t+1} E - 2). \tag{2.35}$$

Thus $E(k^{t+1}E - 2)$ is a non-negative integer. If $E(k^{t+1}E - 2) > 0$, it follows from the k-adic expansion of $E(k^{t+1}E - 2)$ that $k^{F+1}D^2G^2$ satisfies the Lemma. If $E(k^{t+1}E - 2) = 0$, then $G = 1$. The integer $k^{F+1}(1 + k^{t+1})$ also satisfies the Lemma. As $F + 1$ is independent of t, the second claim is trivial. □

Now we will show the everywhere non-periodic result by the previous lemma.

Proposition 2 *Let $A_\infty = (a(n))_{n=0}^{\infty}$ be a sequence with values in $\mathbb{C}$, and let $G_{A_\infty}(z)$ denote the generating function of $(a(n))_{n=0}^{\infty}$. Assume that $G_{A_\infty}(z)$ has the following infinite product expansion for an integer k greater than 1 and $t_{s,y} \neq 0$ for all s with $1 \leq s \leq k-1$ and for all $y \in \mathbb{N}$,*

$$G_{A_\infty}(z) = \prod_{y=0}^{\infty} (1 + \sum_{s=1}^{k-1} t_{s,y} z^{sk^y}). \tag{2.36}$$

If there exists a periodic arithmetical subsequence of $(a(n))_{n=0}^{\infty}$, then $G_{A_\infty}(z)$ has the following infinite product expansion

$$G_{A_\infty}(z) = (\sum_{n=0}^{k^A - 1} a(n) z^n) \prod_{y=0}^{\infty} (1 + \sum_{s=1}^{k-1} h^{sk^y} z^{sk^{A+y}}), \tag{2.37}$$

where A is a non-negative integer and h is a complex number.

Proof. Let n and m be two non-negative integers and their respective k-adic expansions are as follows:

$$n = \sum_{q}^{\text{finite}} s_{n,q} k^{w_n(q)}, \quad m = \sum_{p}^{\text{finite}} s_{m,p} k^{w_m(p)}, \tag{2.38}$$

where $1 \leq s_{n,q}, s_{m,p} \leq k-1$, $0 \leq w_n(q) < w_n(q+1)$, and $0 \leq w_m(p) < w_m(p+1)$. If $w_n(q) \neq w_n(p)$ for all pairs (q, p), then

$$a(n+m) = a(n)a(m) \tag{2.39}$$

by the assumption of $G_{A_\infty}(z)$ and the uniqueness of the k-adic expansion of non-negative integers. If $(a(n))_{n=0}^{\infty}$ has a periodic arithmetical subsequence, then by Corollary 1 $(a(n))_{n=0}^{\infty}$ is not everywhere non-periodic. Thus there exist two non-negative integers, N and $l > 0$, such that

$$a(N) = a(N+tl) \qquad (\forall t \in \mathbb{N}). \tag{2.40}$$

Let the k-adic expansion of N be

$$N = \sum_{q=1}^{N(k)} s_{N,q} k^{w_N(q)} \qquad \textit{where } 1 \leq s_{N,q} \leq k-1, 0 \leq w_N(q) < w_N(q+1). \tag{2.41}$$

By the assumption of $G_{A_\infty}(z)$ and (2.39), we have

$$a(N) = a(N + k^r tl) = a(N)a(k^r tl) \qquad (\forall r > w_N(N(k))). \tag{2.42}$$

$$a(N) \neq 0. \tag{2.43}$$

From (2.42) and (2.43), we get

$$a(k^r tl) = 1 \qquad (\forall r > w_N(N(k))). \tag{2.44}$$

By Lemma 4, there exists an integer x greater than zero such that

$$xl = \sum_{q=1}^{xl(k)} s_{xl,q} k^{w_{xl}(q)}, \tag{2.45}$$

where $s_{xl,1} = 1$ and $w_{xl}(2) - w_{xl}(1) > 1$.

Moreover, there exists an integer X greater than zero such that

$$Xl = \sum_{q=1}^{Xl(k)} s_{Xl_q} k^{w_{Xl}(q)}, \tag{2.46}$$

where $s_{Xl,1} = 1$, $w_{Xl}(2) - w_{Xl}(1) > w_{xl}(xl(k))$ and $w_{Xl}(1) = w_{xl}(1)$.

Let $xlk^{-w_{xl}(1)}$ and $Xlk^{-w_{xl}(1)}$ be replaced by xl and Xl, respectively. Let r be an integer greater than $w(N(k)) + w_{xl}(1)$ and s be an integer in $\{1, \ldots, k-1\}$.

By the definition of Xl and (2.39), we have

$$a(k^r sXl) = a(sk^r)a(k^r sXl - sk^r). \tag{2.47}$$

From (2.43), we get

$$1 = a(k^r xl), \tag{2.48}$$
$$1 = a(k^r sXl), \tag{2.49}$$
$$1 = a(k^r xl + k^r sXl). \tag{2.50}$$

By (2.39), (2.47)-(2.50) and the definitions of xl and Xl, we have

$$a(k^r)a(k^r xl - k^r) = 1, \tag{2.51}$$
$$a(lsk^r)a(sXlk^r - sk^r) = 1, \tag{2.52}$$
$$a(k^r(s+1))a(xlk^r - k^r)a(sXlk^r - sk^r) = 1. \tag{2.53}$$

From (2.51)-(2.53), we get

$$a(k^r(s+1)) = a(k^r)a(k^r s). \tag{2.54}$$

Put $h := a(k^{w(N(k))+w_{xl}(1)+1})$.

By (2.39), we have

$$a(sk^y) = t_{s,y}, \tag{2.55}$$

for all s with $1 \le s \le k-1$ and for all $y \in \mathbb{N}$.

By (2.43), (2.54), (2.55) and inductive computation, we get the relations

$$t_{s,w(N(k))+w_{xl}(1)+1+y} = h^{sk^y}, \tag{2.56}$$

for all s with $1 \le s \le k-1$ and for all $y \in \mathbb{N}$. From the assumption of $G_{A_\infty}(z)$, the proof is complete. □

Finally, we prove the main theorem in Section 3.

Theorem 3 *Let $A_\infty = (a(n))_{n=0}^\infty$ be an (L,k,μ)-TM sequence. The sequence $A_\infty = (a(n))_{n=0}^\infty$ is ultimately periodic if and only if there exists an integer A such that*

$$\mu(s, A+y) \equiv \mu(1, A)sk^y \pmod{L}, \tag{2.57}$$

for all s with $1 \le s \le k-1$ and for all $y \in \mathbb{N}$.

Moreover, if the (L,k,μ)-TM sequence is not ultimately periodic, then no arithmetical subsequence of (L,k,μ)-TM sequence is ultimately periodic.

Proof. We assume, without loss of generality, that $A_\infty = (a(n))_{n=0}^\infty$ is an (L,k,μ)-TM sequence with $a_j = \exp\frac{2\pi\sqrt{-1}j}{L}$ (for all $0 \le j \le L-1$). From this assumption and Lemma 3, $(a(n))_{n=0}^\infty$ satisfies the assumption of Proposition 2. Therefore, (2.57) is the necessary condition.

Now, we show the sufficient condition. Let $G_{A_\infty}(z)$ be the generating function of $(a(n))_{n=0}^\infty$. Notation is the same as for Proposition 2. If we assume that $(a(n))_{n=0}^\infty$ satisfies (2.57), then there exists a non-negative integer A such that

$$t_{s,A+y} = h^{sk^y} \qquad (\forall y \in \mathbb{N}). \tag{2.58}$$

Thus $G_{A_\infty}(z)$ has the infinite product expansion

$$G_{A_\infty}(z) = (\sum_{n=0}^{k^A-1} b(n)z^n)\prod_{y=0}^{\infty}(1+\sum_{s=1}^{k-1}(hz^{k^A})^{sk^y}). \tag{2.59}$$

Let $Z = hz^{k^A}$. As h is the L-th root of 1 and μ is a zero map in Lemma 3, we find

$$\prod_{y=0}^{\infty}(1+\sum_{s=1}^{k-1} Z^{sk^y}) = \sum_{n=0}^{\infty} Z^n \qquad on\ |Z|<1. \tag{2.60}$$

We put $G(z) = \sum_{n=0}^{k^A-1} a(n)z^n$. From (2.59) and (2.60),

$$G_{A_\infty}(z) = G(z)(\sum_{n=0}^{\infty}(hz^{k^A})^n). \tag{2.61}$$

As h is the L-th root of 1,

$$G_{A_\infty}(z) = (G(z)(\sum_{n=0}^{L-1}(hz^{k^A})^n))(1+\sum_{s=1}^{\infty} z^{sLk^A}) = \frac{G(z)(\sum_{n=0}^{L-1}(hz^{k^A})^n)}{1-z^{Lk^A}}. \tag{2.62}$$

As the degree of $G(z)$ is k^A-1, and using (2.62), we find that the sequence $(a(n))_{n=0}^{\infty}$ that satisfies (2.57) has a period Lk^A. Moreover, if the (L,k,μ)-TM sequence is not ultimately periodic, then no arithmetical sequence of (L,k,μ)-TM sequence will be ultimately periodic by the above argument and by Proposition 2. $\square$

If an (L,k,μ)-TM sequence satisfies

$$\mu(s,y) = \mu(s,y+1) \tag{2.63}$$

for all s with $1 \le s \le k-1$ and for all $y \in \mathbb{N}$, then $(\mu(1),\mu(2),\ldots,\mu(k-1))$-$L$ will denote the (L,k,μ)-TM sequence.

The weak version of the corollary that follows is given as Theorem 2 in Morton and Mourant [MortM]. See also Allouche and Shallit [AlS3], Frid [Fr].

Corollary 2 *The sequence $(\mu(1),\mu(2),\ldots,\mu(k-1))$-$L$ is periodic if and only if $\mu(s)$ (for all s with $1 \le s \le k-1$) satisfies*

$$s\mu(1) \equiv \mu(s), \mu(k-1) \equiv 0 \pmod{L}. \tag{2.64}$$

Moreover, if $(\mu(1),\mu(2),\ldots,\mu(k-1))$-$L$ is not periodic, then no arithmetical subsequence of $(\mu(1),\mu(2),\ldots,\mu(k-1)) - L$ will be periodic.

Proof. By Theorem 3, the necessary-sufficient condition for the periodicity of $(\mu(1),\mu(2),\ldots,\mu(k-1))$-$L$ comprises the following relations:

$$\mu(1,A+1) \equiv \mu(1,A)k \pmod{L}, \mu(k-1) \equiv (k-1)\mu(1) \equiv 0 \pmod{L}. \tag{2.65}$$

. $\square$

2.4 Transcendence results of the generalized Thue-Morse sequences

Adamczewski, Bugeaud, and Luca [AdBL] introduced a new class of sequences, as follows. For any positive number y,$\lfloor y \rfloor$ and $\lceil y \rceil$ are the floor and ceiling functions. Let W be a finite word on $\{a_0, a_1, \ldots, a_{L-1}\}$ and let $|W|$ be the length of W. For any positive number x, we let W^x defined the word $W^{\lfloor x \rfloor} W^{`}$, where $W^{`}$ is a prefix of W of length $\lceil (x - \lfloor x \rfloor)|W| \rceil$.

Definition 3 $(a(n))_{n=0}^{\infty}$ is called a *stammering sequence* if $(a(n))_{n=0}^{\infty}$ satisfies the following conditions:

(1) The sequence $(a(n))_{n=0}^{\infty}$ is a non-periodic sequence.

(2) There exist two sequences of finite words, $(U_m)_{m\geq 1}$ and $(V_m)_{m\geq 1}$, such that,

(A) there exists a real number $w > 1$ independent of n such that the word $U_m {V_m}^w$ is a prefix of the word $(a(n))_{n=0}^{\infty}$,

(B) $\lim_{m\to\infty} |U_m|/|V_m| < +\infty$, and

(C) $\lim_{m\to\infty} |V_m| = +\infty$.

Let $(a(n))_{n=0}^{\infty}$ be a sequence of positive integers. We define the *continued fraction* of $(a(n))_{n=0}^{\infty}$ as

$$[0 : a(0), a(1), \ldots, a(n), \ldots] := 0 + \cfrac{1}{a(0) + \cfrac{1}{a(1) + \cfrac{1}{\cdots + \cfrac{1}{a(n) + \cfrac{1}{\cdots}}}}}. \tag{2.66}$$

Adamczewski, Bugeaud, Luca [AdBL] and Bugeaud [Bu1] proved the result that follows by the Schmidt subspace theorem.

Theorem 4 ([AdBL, Bu1]) *If β is an integer greater than 1 and $(a(n))_{n=0}^{\infty}$ is a stammering sequence on $\{0, 1, \ldots, \beta - 1\}$, then $\sum_{n=0}^{\infty} \frac{a(n)}{\beta^{n+1}}$ is a transcendental number. Moreover, if $(a(n))_{n=0}^{\infty}$ is a stammering sequence on bounded positive integers, then the continued fraction $[0 : a(0), a(1) \ldots, a(n) \ldots]$ is also a transcendental number.*

We will prove the next theorem using Theorem 3 and 4.

Theorem 5 *Let $A_\infty = (a(n))_{n=0}^{\infty}$ be an (L, k, μ)-TM sequence and β be an integer greater than 1. We assume that $(a(n))_{n=0}^{\infty}$ takes its input from $\{0, 1, \ldots, \beta - 1\}$. If there is no integer A such that*

$$\mu(s, A + y) \equiv \mu(1, A)sk^y \pmod{L} \tag{2.67}$$

for all s with $1 \le s \le k-1$ *and for all* $y \in \mathbb{N}$*, then* $\sum_{n=0}^{\infty} \frac{a(N+nl)}{\beta^{n+1}}$ (*for all* $N \ge 0$ *and for all* $l > 0$) *is a transcendental number.*

Moreover, if we assume that $(a(n))_{n=0}^{\infty}$ *takes its input from the positive integers, and if there is no integer A such that*

$$\mu(s, A+y) \equiv \mu(1, A)sk^y \pmod{L} \tag{2.68}$$

for all s with $1 \le s \le k-1$ *and for all* $y \in \mathbb{N}$*, then* $[0 : a(N), a(N+s), \ldots, a(N+nl)\ldots]$ (*for all* $N \ge 0$ *and for all* $l > 0$) *is a transcendental number.*

Proof. Let N and $l > 0$ be positive integers. By Theorem 3, $(a(N+nl))_{n=0}^{\infty}$ is non-periodic. Therefore, we only have to prove that (L, k, μ)-TM satisfies the condition (2) of Definition 3.

We choose an integer M such that $k^M > 2(N+l)$, and assume that $m > M$. As f is a cyclic permutation of order L and by Definition 2, the $(Ll+1)k^m$ prefix word of $(a(n))_{n=0}^{\infty}$ is as follows

$$A_\infty = (a(n))_{n=0}^{\infty} = A_m f^{i_1}(A_m) \cdots f^{i_{Ll}}(A_m) \cdots , \tag{2.69}$$

where A_m is the k^m prefix word of $(a(n))_{n=0}^{\infty}$, $i_j (1 \le j \le Ll) \in \{0, \ldots, L-1\}$.

By (2.69), we have

$$f^{i_{tl}}(a(n)) = a(n + k^m tl) \tag{2.70}$$

for all $0 \le n \le k^m - 1$ and for all $1 \le t \le L$.

As f is a cyclic permutation of order L, by (2.69), (2.70) and the Dirichlet Schubfachprinzip, we have

$$(a(N+nl))_{n=0}^{\infty} = W_{1,m}W_{2,m}W_{3,m}W_{2,m} \cdots , \tag{2.71}$$

where $W_{i,m}$ ($i \in \{1,2,3\}$) are finite words such that

$$|W_{1,m}| \le ((Ll+1)k^m - N)/l + 1, \tag{2.72}$$

$$|W_{2,m}| \ge (k^m - N)/l - 1, \tag{2.73}$$

$$|W_{2,m}| + |W_{3,m}| \le ((Ll+1)k^m - N)/l + 1. \tag{2.74}$$

We put $U_m := W_{1,m}$, $V_m := W_{2,m}W_{3,m}$ and $w := 1 + \frac{1}{2Ll+3}$.

By (2.72)-(2.74) and the assumption of m, we obtain

$$\lceil (w-1)|V_m| \rceil = \lceil \tfrac{1}{2Ll+3}(|W_{2,m}| + |W_{3,m}|) \rceil \le \tfrac{1}{2Ll+3}((Ll+1)k^m - N + l)/l \le \tfrac{k^m}{2l} < |W_{2,m}|. \tag{2.75}$$

From (2.75), $(a(n))_{n=0}^{\infty}$ satisfies Condition (A).

Furthermore,

$$|U_m|/|V_m| = |W_{1,m}|/|W_{2,m}W_{3,m}| \le ((Ll+1)k^m - N + l)/l \times l/(k^m - N - l) \le 2Ll + 3. \tag{2.76}$$

From (2.76), $(a(n))_{n=0}^{\infty}$ satisfies Condition (B).

It follows directly that $(V_m)_{m \ge 1}$ satisfies Condition (C). $\square$

Corollary 3 *Let $(a(n))_{n=0}^{\infty}$ be an (L,k,μ)-TM sequence and β be an integer greater than 1. If $(a(n))_{n=0}^{\infty}$ takes its input from $\{0,1,\ldots,\beta-1\}$ and there is no integer A such that*

$$\mu(s,A+y)\equiv\mu(1,A)sk^{y}\pmod{L}\tag{2.77}$$

for all s with $1\le s\le k-1$ and for all $y\in\mathbb{N}$, then the generating function $f(z):=\sum_{n=0}^{\infty}\frac{a(N+nl)}{z^{n+1}}$ (for all $N\ge 0$ and for all $l>0$) is transcendental over $\mathbb{C}(z)$.

Proof. We assume $f(z)$ is algebraic over $\mathbb{C}(z)$. As $f(z)$ is algebraic over $\mathbb{Q}(z)$ if and only if $f(z)$ is algebraic over $\mathbb{C}(z)$ (see the Remark in Theorem 1.2 in Nishioka [N]), then $f(z)$ satisfies the equation

$$c_n(z)f^{n}(z)+c_{n-1}(z)f^{n-1}(z)+\cdots+c_0(z)=0,\tag{2.78}$$

where $c_i(z)\in\mathbb{Q}[z]$ $(0\le i\le n)$, $c_n(z)c_0(z)\ne 0$ and $c_i(z)$ $(0\le i\le n)$ are coprime. From Theorem 5, $f(\frac{1}{\beta})$ is a transcendental number. From the above argument and by (2.78), $c_i(\frac{1}{\beta})=0$ (for all $0\le i\le n$). This contradicts the assumption that $c_i(z)$ $(0\le i\le n)$ are coprime. □

2.5 k-Automatic generalized Thue-Morse sequences and some results

First, we introduce some definitions.

Definition 4 ([Bu2]) Let α be an irrational real number. The irrationality exponent $\mu(\alpha)$ of α is the supremum of the real numbers μ such that the inequality

$$\left|\alpha-\frac{p}{q}\right|<\frac{1}{q^{\mu}}\tag{2.79}$$

has infinitely many solutions in non-zero integers p and q.

Definition 5 ([AlS1], [N]) The *k-kernel* of $(a(n))_{n=0}^{\infty}$ is the set of all subsequences of the form $(a(k^{e}n+j))_{n=0}^{\infty}$, where $e\ge 0$ and $0\le j\le k^{e}-1$.

Definition 6 ([AlS1]) The sequence$(a(n))_{n=0}^{\infty}$ is called a *k-automatic sequence* if the k-kernel of $(a(n))_{n=0}^{\infty}$ is the finite set.

Definition 7 ([Bec]) The power series $\sum_{n=0}^{\infty}a(n)z^{n}\in\mathbb{C}[[x]]$ is called a *k-automatic power series* if $(a(n))_{n=0}^{\infty}$ is a k-automatic sequence.

Definition 8 An (L, k, μ)-TM sequence is called *y-periodic* if there exist non-negative integers N and $t(0 < t)$ such that

$$\mu(s, y) = \mu(s, y + t), \tag{2.80}$$

for all s with $1 \leq s \leq k - 1$ and for all $y \geq N$.

Now we introduce two results.

Theorem 6 ([AdC]) *If β is an integer greater than 1 and $(a(n))_{n=0}^{\infty}$ is a non-periodic k-automatic sequence on $\{0, 1, \ldots, \beta - 1\}$, then $\mu(\sum_{n=0}^{\infty} \frac{a(n)}{\beta^{n+1}})$ is finite.*

Theorem 7 ([Bec]) *If $f(z) \in \mathbb{Q}[[z]] \setminus \mathbb{Q}(z)$ is a k-automatic power series and $0 < R < 1$, then $f(\alpha)$ is transcendental for all but finitely many algebraic numbers α with $|\alpha| \leq R$.*

Now we consider the necessary-sufficient condition that an (L, k, μ)-TM sequence is a k-automatic sequence.

Proposition 3 *An (L, k, μ)-TM sequence is y-periodic if and only if it is a k-automatic sequence.*

Proof. We assume, without loss of generality, that $A_\infty = (a(n))_{n=0}^{\infty}$ is an (L, k, μ)-TM sequence with $a_j = \exp \frac{2\pi\sqrt{-1}j}{L}$ (for all $0 \leq j \leq L - 1$).

Let us assume that $(a(n))_{n=0}^{\infty}$ is a k-automatic sequence. As the k-kernel of $(a(n))_{n=0}^{\infty}$ is a finite set, there exist integers e for $0 < t$ such that

$$a(k^e n) = a(k^{e+t} n) \qquad (\forall n \geq 0). \tag{2.81}$$

Let s be any integer in $\{1, 2, \ldots, k - 1\}$, and let y be any integer in $\mathbb{N}$. By Lemma 3 with (2.39) and (2.81), and substituting sk^y for n, we have

$$\exp \frac{2\pi\sqrt{-1}\mu(s, e + y)}{L} = a(k^e sk^y) = a(k^{e+t} sk^y) = \exp \frac{2\pi\sqrt{-1}\mu(s, e + y + t)}{L}. \tag{2.82}$$

By the definition of the (L, k, μ)-TM sequence and (2.82), $(a(n))_{n=0}^{\infty}$ is y-periodic.

Now we show the converse. If an (L, k, μ)-TM sequence $A_\infty = (a(n))_{n=0}^{\infty}$ is y-periodic, then there exist non-negative integers e for $0 < t$ such that

$$\mu(s, e + y) = \mu(s, e + y + t), \tag{2.83}$$

for all y being any integer in $\mathbb{N}$ and for all s with $1 \leq s \leq k - 1$. Let l be any integer greater than $t - 1$ and let $(a(k^{e+l} n + j))_{n=0}^{\infty}$ (where $0 \leq j \leq k^{e+l} - 1$) be any sequence in the k-kernel of $(a(n))_{n=0}^{\infty}$.

Therefore, from Lemma 3 with (2.39), we get

$$a(k^{e+l} n + j) = a(k^{e+l} n) a(j). \tag{2.84}$$

As $(a(n))_{n=0}^{\infty}$ takes on only finitely many values, then $a(j)$ also takes on only finitely many values.

Let the k-adic expansion of n be

$$n = \sum_{q=1}^{N(n)} s_{n,q} k^{w(j)} \qquad \text{where } 1 \leq s_{n,q} \leq k-1, w(q+1) > w(q) \geq 0. \quad (2.85)$$

Let $l(t) \equiv l \pmod{t}$, where $0 \leq l(t) \leq t-1$. By Lemma 3 with (2.39), we have

$$a(k^{e+l}n) = a(\sum_{q=1}^{N(n)} s_q k^{w(q)+e+l}) = \prod_{q=1}^{N(n)} a(s_q k^{w(q)+e+l}). \quad (2.86)$$

From (2.85), (2.86), and Lemma 3 with (2.39), we get

$$a(k^{e+l}n) = \prod_{q=1}^{N(n)} a(s_q k^{w(q)+e+l}) = \prod_{q=1}^{N(n)} a(s_q k^{w(q)+e+l(t)})$$
$$= a(\sum_{q=1}^{N(n)} s_q k^{w(q)+e+l(t)}) = a(k^{e+l(t)}n). \quad (2.87)$$

As $a(j)$ takes on only finitely many values, and by (2.84) and (2.87), it follows that the k-kernel of $(a(n))_{n=0}^{\infty}$ is a finite set. □

Theorem 8 *Let $(a(n))_{n=0}^{\infty}$ be an (L,k,μ)-TM and β be an integer greater than 1. If $(a(n))_{n=0}^{\infty}$ takes on the values $\{0,1,\ldots,\beta-1\}$, is y-periodic and there is no integer A such that*

$$\mu(s, A+y) \equiv \mu(1,A) s k^y \pmod{L} \quad (2.88)$$

for all s with $1 \leq s \leq k-1$ and for all $y \in \mathbb{N}$, then $\mu(\sum_{n=0}^{\infty} \frac{a(N+nl)}{\beta^{n+1}})$ (for all $N \geq 0$ and for all $l > 0$) is finite.

Proof. By the previous proposition, $(a(n))_{n=0}^{\infty}$ is a k-automatic sequence. As the arithmetical subsequence of a k-automatic sequence is k-automatic, see Theorem 2.3 and Theorem 2.6 in Allouche and Shallit [AlS1], and by Theorems 5 and 6, $\mu(\sum_{n=0}^{\infty} \frac{a(N+nl)}{\beta^{n+1}})$ is finite. □

Theorem 9 *Let $(a(n))_{n=0}^{\infty}$ be an (L,k,μ)-TM, β be an integer greater than 1, $f(z) := \sum_{n=0}^{\infty} \frac{a(N+nl)}{z^{n+1}}$ (for all $N \geq 0$ and for all $l > 0$), and $0 < R < 1$. If $(a(n))_{n=0}^{\infty}$ takes on the values $\{0,1,\ldots,\beta-1\}$, is y-periodic and there is no integer A such that*

$$\mu(s, A+y) \equiv \mu(1,A) s k^y \pmod{L} \quad (2.89)$$

for all s with $1 \leq s \leq k-1$ and for all $y \in \mathbb{N}$, then $f(\alpha)$ is a transcendental number for all but finitely many algebraic numbers α with $|\alpha| \leq R$.

Proof. By Corollary 3, $f(z)$ is transcendental over $\mathbb{Q}(z)$. From Proposition 3, $(a(N+nl))_{n=0}^{\infty}$ (for all $N \geq 0$ and for all $l > 0$) is a k-automatic sequence. Therefore, $f(z)$ is a k-automatic power series. Theorem 7 implies that $f(\alpha)$ is transcendental for all but finitely many algebraic numbers α with $|\alpha| \leq R$. $\square$

Chapter 3

A generalization of the Erdös-Surányi problem

3.1 Introduction

Erdös-Surányi [ErS] and Prielipp [Bl] suggested to study the following problem: For any integers $k > 0$ and n, are there an integer N and a map $\epsilon : \{1, \ldots, N\} \to \{-1, 1\}$ such that

$$n = \sum_{j=1}^{N} \epsilon(j) j^k \ ? \tag{3.1}$$

Mitek [Mi], Kaja[K] and Bleicher [Bl] independently solved this problem affirmatively. Later many people investigated analogies and generalizations of this problem (see [AnT, BaW, ChC1]). Some researchers replaced the function ϵ by another function ([BoC, ChC1]).

We study the case that the values of ϵ are L-th roots of unity, where L is a positive integer. Since, by [Bl] and [Mi], we already know that the answer is positive if L is even, we restrict our attention to odd L. Let U be the set of L-th roots of unity. Then we consider the following problem (cf. [BoC]). For any integers $k > 0$ and n, are there an integer N and a map $\epsilon : \{1, \ldots, N\} \to U$ such that

$$n = \sum_{j=1}^{N} \epsilon(j) j^k \ ? \tag{3.2}$$

We prove the following result.

Theorem 10 *Let L be a positive odd integer with $L \geq 2$ which is not a prime power and let U be the set of L-th roots of unity.*

Then for any integers $k > 0$ and n, there are an integer N and a map $\epsilon : \{1, \dots, N\} \to U$ such that

$$n = \sum_{j=1}^{N} \epsilon(j) j^k. \tag{3.3}$$

The following result shows that the statement of Theorem 10 is valid if L is an odd prime power p^m and k is a multiple of $p - 1$.

Theorem 11 *Let p be an odd prime number, m be a positive integer and let U be the set of p^m-th roots of unity. Then for any integers $k > 0$ with $p - 1 \mid k$ and n, there are an integer N and a map $\epsilon : \{1, \dots, N\} \to U$ such that*

$$n = \sum_{j=1}^{N} \epsilon(j) j^k. \tag{3.4}$$

Moreover, the following result shows that the statement of Theorem 10 is not valid if L is an odd prime power p^m and k is not a multiple of $p - 1$.

Theorem 12 *Let p be an odd prime number, m be a positive integer and let U be the set of p^m-th roots of unity. Then for any integer $k > 0$ with $p - 1 \nmid k$, there are infinitely many integers n such that n cannot be represented as*

$$n = \sum_{j=1}^{N} \epsilon(j) j^k, \tag{3.5}$$

where N is a positive integer and $\epsilon : \{1, \dots, N\} \to U$.

Remark 1 Theorem 12 contradicts Theorem 5.3 in [BoC]. The proof of Proposition 4.2 of [BoC] contains a serious error. Let μ_K, $\mathcal{R}$, $\varepsilon D_m[f](l)$ and $\varepsilon \overline{D_m}[f](l)$ be defined as in [BoC]. Since μ_K need not contain -1, it may be that $\varepsilon D_m[f](l)$ and $\varepsilon \overline{D_m}[f](l)$ are not contained in $\mathcal{R}$.

This part is organized as follows. In Section 3.2, we discover the key identity generated by $(L, L(L-1), \mu)$-TM sequence. In Section 3.3, we give the proof of Theorem 10. In Section 3.4, we give the proof of Theorem 11. In Section 3.5, we give the proof of Theorem 12.

3.2 The identity generated by $(L, L(L-1), \mu)$-TM sequence

In this section we generalize Lemma 3 in [Bl] as follows:

Lemma 5 *Let $L \geq 2$ be an integer and let U be the set of L-th roots of unity. Let x be a real variable. Then for any integer $k > 0$ there exist an integer M, a map $\epsilon : \{0, \ldots, M-1\} \to U$ and a non-zero integer P such that*

$$\sum_{j=0}^{M-1} \epsilon(j)(x+j)^k = P. \tag{3.6}$$

Moreover for P, M as above and any integer m, there exist a multiple N of M and a map $\epsilon_m : \{0, \ldots, M-1\} \to U$ such that

$$\sum_{j=0}^{N-1} \epsilon_{m(j)}(x+j)^k = mP. \tag{3.7}$$

Proof. We show the first statement of this lemma by induction on k. Set $\zeta_L := \exp \frac{2\pi\sqrt{-1}}{L}$. First, we check the case of $k = 1$. From $\sum_{j=0}^{L-1} \zeta_L^j = 0$, we have that

$$\sum_{j=0}^{L-1} \zeta_L^j (x+j) = C_1, \tag{3.8}$$
$$x + \sum_{j=1}^{L-t} \zeta_L^{j+t-1}(x+j) + \sum_{j=L-t+1}^{L-1} \zeta_L^{j+t}(x+j) = C_t,$$

where C_1 and C_t $(2 \leq t \leq L-1)$ are complex numbers. By (3.8) and $\sum_{j=1}^{L-1} \zeta_L^j = -1$, we get

$$\sum_{t=1}^{L-1} C_t = -\frac{(L-1)L}{2}. \tag{3.9}$$

By substituting $x + (t-1)L$ for x in (3.8) and (3.9), we obtain

$$\sum_{j=0}^{L-1} \zeta_L^j (x+j) + \sum_{t=2}^{L-1} \left\{ x + (t-1)L + \sum_{j=1}^{L-t} \zeta_L^{j+t-1}(x+(t-1)L+j) \right. \tag{3.10}$$
$$\left. + \sum_{j=L-t+1}^{L-1} \zeta_L^{j+t}(x+(t-1)L+j) \right\} = -\frac{(L-1)L}{2}.$$

Thus, the case of $k = 1$ is true.

By the induction hypothesis, we may assume that

$$\sum_{j=0}^{M'-1} \epsilon(j)(x+j)^{k-1} = P', \tag{3.11}$$

where M' is an integer, P' is a non-zero integer and $\epsilon : \{0, \ldots, M'-1\} \to U$. By integrating both sides of (3.11) in x and substituting $x+tM'$ for x in (3.11), we obtain

$$\frac{1}{k}\{\sum_{j=0}^{M'-1} \epsilon(j)(x+tM'+j)^k\} = P'(x+tM') + C_0, \tag{3.12}$$

where C_0 is an integration constant and $0 \le t \le L-1$.
From $\sum_{j=0}^{L-1} \zeta_L^j = 0$, we get that for any integer $1 \le t \le L-1$ and for any C

$$\sum_{j=0}^{L-1} \zeta_L^j k\{P'(x+jM)+C\} = D_1, \tag{3.13}$$

$$k(P'x+C) + \sum_{j=1}^{L-t} \zeta_L^{j+t-1} k\left\{P'(x+jM')+C\right\} + \sum_{j=L-t+1}^{L-1} \zeta_L^{j+t} k\left\{P'(x+jM')+C\right\} = D_t,$$

where D_1 and D_t $(2 \le t \le L-1)$ are complex numbers.
By (3.13) and $\sum_{j=1}^{L-1} \zeta_L^j = -1$, we get

$$\sum_{t=1}^{L-1} D_t = -kP'M'\frac{(L-1)L}{2}. \tag{3.14}$$

By (3.13) and (3.14), we obtain

$$\sum_{j=0}^{L-1} \zeta_L^j k\{P'(x+jM)+C\} + \sum_{t=2}^{L-1} \left\{ kP'(x+(t-1)LM') + \sum_{j=1}^{L-t} \zeta_L^{j+t-1} kP'(x+(t-1)LM'+jM') \right. \tag{3.15}$$

$$\left. + \sum_{j=L-t+1}^{L-1} \zeta_L^{j+t} kP'(x+(t-1)LM'+jM') \right\} = -kP'M'\frac{(L-1)L}{2}.$$

From (3.13)-(3.15), the proof of the first statement is completed.

Finally we prove the second statement of this lemma. By the first statement, for any integer $k > 0$ there exist an integer M, a map $\epsilon : \{0, \ldots, M-1\} \to U$ and a non-zero integer P such that for every x

$$\sum_{j=0}^{M-1} \epsilon(j)(x+j)^k = P. \tag{3.16}$$

By $\sum_{j=1}^{L-1} \zeta_L^j = -1$ and substituting $x+(t-1)M$ for x in (3.16), we have

$$\sum_{t=1}^{L-1} \zeta_L^t \sum_{j=0}^{M-1} \epsilon(j)\{x+(t-1)M+j\}^k = -P. \tag{3.17}$$

Put $N = mM$ if $m \geq 0$ and $N = -mM$ if $m < 0$. By substituting $x + sM$ for x in (3.16) if $m \geq 0$ and in (3.17) if $m < 0$, where $0 \leq s \leq |m| - 1$ and adding the results, there exists a map $\epsilon_m\{0, \ldots, M-1\} \to U$ such that

$$\sum_{j=0}^{N-1} \epsilon_{m(j)}(x+j)^k = mP. \tag{3.18}$$

This proves the second statement of Lemma 5.

Remark 2 Letting k tend to infinity, we can define the sequence $(\epsilon(j))_{j=1}^{\infty}$. The sequence $(\epsilon(j))_{j=1}^{\infty}$ can be regarded as the $(L, L(L-1), \mu)$-TM sequence where some μ.

3.3 The case that L is an odd composite number

In this section we deal with the case where L is an odd composite number. Now we prove Theorem 10. By the second statement of Lemma 5, we only have to prove that, for each l with $0 \leq l \leq P-1$, there exist an integer N and a map $\epsilon : \{1, \ldots, N\} \to U$ such that

$$l \equiv \sum_{j=1}^{N} \epsilon(j) j^k \qquad (\text{mod } P). \tag{3.19}$$

First we prove that for any positive integer c and each prime factor q of L, there exist a multiple $N_{c,q}$ of P and a map $\epsilon : \{1, \ldots, N\} \to U$ such that

$$q \equiv \sum_{j=1}^{N_{c,q}} \epsilon(j)(Pc+j)^k \qquad (\text{mod } P). \tag{3.20}$$

Set $\zeta_q := \exp \frac{2\pi\sqrt{-1}}{q}$. (Note $\zeta_q^t \in U$ for any integer t.) From $\sum_{j=0}^{q-1} \zeta_q^j = 0$, we have

$$\sum_{t=0}^{q-1}\left((tP+Pc+1)^k + \sum_{j=2}^{P} \zeta_q^t (tP+Pc+j)^k\right) = q + \sum_{t=0}^{q-1} \zeta_q^t a_t P \tag{3.21}$$

where a_t's are integers. We put $A := \max\{a_t | 0 \leq t \leq q-1\}$. By Lemma 5 we have that, for any t with $0 \leq t \leq q-1$ there exist a multiple M_t of M and a map $\epsilon_t : \{1, \ldots, M_t\} \to U$ such that

$$\sum_{j=0}^{M_t - 1} \epsilon_t(j)(x+j)^k = (A - a_t)P. \tag{3.22}$$

From (3.21) and (3.22), substituting $Pc + qP + 1 + \sum_{s=0}^{t-1} M_s$ for x, we have

$$\begin{aligned} B := \sum_{t=0}^{q-1} (tP + Pc + 1)^k + \sum_{t=0}^{q-1} \sum_{j=2}^{P} \zeta_q^t (tP + Pc + j)^k \\ + \sum_{t=0}^{q-1} \zeta_q^t \sum_{j=0}^{M_t - 1} \epsilon_t(j)(Pc + qP + 1 + \sum_{s=0}^{t-1} M_s + j)^k \equiv q \pmod{P}. \end{aligned} \tag{3.23}$$

Put

$$K := \frac{\sum_{s=0}^{q-1} M_s}{M}.$$

Hence K is an integer. By (3.16) and (3.23), we have

$$B + \sum_{t=0}^{(P-1)K-1} \sum_{j=0}^{M-1} \epsilon(j)\{Pc + qP + (K+t)M + 1 + j\}^k \equiv q \pmod{P}. \tag{3.24}$$

Moreover, we have,

$$qP + \{K + (P-1)K - 1\} M + 1 + M - 1 = P(KM + q). \tag{3.25}$$

Therefore we have completed the proof of (3.20) with $N_{c,q} = P(KM + q)$.

Since L is a not prime power, L has two distinct prime factors r and u. Let x and y be any integers. We choose an integer I such that $x + IP > 0$ and $y + IP > 0$. By using $x + IP$ times (3.20) for r (Note $N_{c,r}$ is a multiple of P), there exist a multiple N' of P, and a map $\epsilon : \{1, \ldots, N'\} \to U$ such that

$$rx \equiv r(x + IP) \equiv \sum_{j=1}^{N'} \epsilon(j) j^k \pmod{P}. \tag{3.26}$$

Combining (3.26) with using $y + IP$ times (3.20) for u (Note N' and $N_{c,u}$ are multiples of P), there exist an integer N'' and a map $\epsilon : \{1, \ldots, N''\} \to U$ such that

$$rx + uy \equiv r(x + IP) + u(y + IP) \equiv \sum_{j=1}^{N''} \epsilon(j) j^k \pmod{P}. \tag{3.27}$$

Since r and u are coprime, for any integer $0 \leq l \leq P - 1$ there exist integers x and y such that

$$rx + uy = l. \tag{3.28}$$

The proof of Theorem 10 is completed by combining (3.19) with (3.27) and (3.28).

3.4 The case that L is an odd prime power p^m and k is a multiple of $p-1$

In this section we deal with the case where L is an odd prime power p^m and k is a multiple of $p-1$. Choose M, ϵ, P as in Lemma 5. By the second statement of Lemma 5, it suffices to prove that, for all integers x and l with $0 \leq l \leq p-1$, there exist an integer N and a map $\epsilon : \{1, \ldots, N\} \to U$ such that

$$px + l \equiv \sum_{j=1}^{N} \epsilon(j) j^k \qquad (\mathrm{mod}\ P). \tag{3.29}$$

By Fermat's little theorem and $p-1 \mid k$, we have

$$\sum_{j=1}^{l} j^k = l + ps_l \tag{3.30}$$

for some integers s_l. Let x be an integer. By (3.26) , there exist a multiple N' of P and a map $\epsilon : \{1, \ldots, N'\} \to U$ such that

$$p(x - s_l) \equiv \sum_{j=1}^{N'} \epsilon(j)(P + j)^k \qquad (\mathrm{mod}\ P). \tag{3.31}$$

Set $\zeta_p := \exp \frac{2\pi\sqrt{-1}}{p}$. (Note $\zeta_p^t \in U$ for any integer t.) From $\sum_{j=0}^{p-1} \zeta_p^j = 0$, (122) and $P \mid N'$, we have

$$\begin{aligned} &\sum_{j=1}^{l} j^k + \sum_{j=l+1}^{P} j^k + \sum_{t=0}^{p-1} \zeta_p^t \sum_{j=1}^{l} \{N' + (t+1)P + j\}^k \\ &+ \sum_{t=1}^{p-1} \zeta_p^t \sum_{j=l+1}^{P} (N' + tP + j)^k = ps_l + l + \sum_{t=0}^{p-1} \zeta_p^t b_t P \end{aligned} \tag{3.32}$$

where b_t's are integers. We put $C := \max\{b_t | 0 \leq t \leq p-1\}$. By Lemma 5, we have that, for any t with $0 \leq t \leq p-1$, there exist an integer M_t and a map $\epsilon_t : \{1, \ldots, M_t\} \to U$ such that

$$\sum_{j=0}^{M_t - 1} \epsilon_t(j)(x + j)^k = (C - b_t)P. \tag{3.33}$$

From (3.30)-(3.32) and (3.33), substituting $N' + pP + l + 1 + \sum_{s=0}^{t-1} M_s$ for x, we have

$$\sum_{j=1}^{l} j^k + \sum_{j=l+1}^{P} j^k + \sum_{j=1}^{N'} \epsilon(j)(P+j)^k \tag{3.34}$$
$$+ \sum_{t=0}^{p-1} \zeta_p^t \sum_{j=1}^{l} \{N' + (t+1)P + j\}^k + \sum_{t=1}^{p-1} \zeta_p^t \sum_{j=l+1}^{P} (N' + tP + j)^k$$
$$+ \sum_{t=0}^{p-1} \zeta_p^t \sum_{j=0}^{M_t - 1} \epsilon_t(j)(N' + pP + l + 1 + \sum_{s=0}^{t-1} M_s + j)^k \equiv px + l \pmod{P}.$$

The proof of Theorem 11 is completed by combining (3.29) with (3.34) .

3.5 The case that L is an odd prime power p^m and k is not a multiple of $p-1$

In this section we deal with the case where L is an odd prime power p^m and k is not a multiple of $p-1$.

Lemma 6 *Let p be an odd prime number, m be a positive integer and k be a positive integer with $p-1 \nmid k$. Let U be the set of p^m-th roots of unity. Put $S_k(M) := \sum_{j=1}^{M} j^k$. Assume an integer n is represented by*

$$n = \sum_{j=1}^{N} \epsilon(j) j^k, \tag{3.35}$$

where N is an integer and $\epsilon : \{1, \ldots, N\} \to U$.
Then

$$S_k(N) \equiv n \pmod{p}. \tag{3.36}$$

Proof. Set $\zeta_{p^m} := \exp \frac{2\pi\sqrt{-1}}{p^m}$. We assume that $\sum_{j=0}^{p^m-1} a_j \zeta_{p^m}^j = 0$, where the a_j's are integers. Since p is a prime number, $\sum_{j=0}^{p-1} x^{jp^{m-1}}$ is the minimal polynomial of ζ_{p^m}. Therefore there exists a polynomial $\sum_{j=0}^{p^{m-1}-1} c_j x^j$ with rational coefficients c_j such that

$$\sum_{j=0}^{p^m-1} a_j x^j = (\sum_{j=0}^{p-1} x^{jp^{m-1}})(\sum_{j=0}^{p^{m-1}-1} c_j x^j). \tag{3.37}$$

By the uniqueness of p-adic expansion and (3.37), we have

$$a_t = a_{t+p^{m-1}} \tag{3.38}$$

for every t with $0 \leq t \leq p^m - 1 - p^{m-1}$. By (3.35), we have

$$\sum_{1\leq j\leq N,\ \epsilon(j)=1} j^k - n + \sum_{l=1}^{p-1} \sum_{1\leq j\leq N,\ \epsilon(j)=\zeta_{p^m}^{lp^{m-1}}} \zeta^{lp^{m-1}} j^k \tag{3.39}$$
$$+ \sum_{t=1}^{p^{m-1}-1} \sum_{l=0}^{p-1} \sum_{1\leq j\leq N,\ \epsilon(j)=\zeta_{p^m}^{t+lp^{m-1}}} \zeta_{p^m}^{t+lp^{m-1}} j^k = 0.$$

By (3.38) and (3.39), we get

$$\sum_{1\leq j\leq N,\ \epsilon(j)=\zeta_{p^m}^{t+lp^{m-1}}} j^k = \sum_{1\leq j\leq N,\ \epsilon(j)=\zeta_{p^m}^{t}} j^k \tag{3.40}$$

for every t and l with $1 \leq t \leq p^{m-1}$ and $1 \leq l \leq p-1$.
Moreover, by (3.38) and (3.39), we get

$$\sum_{1\leq j\leq N,\ \epsilon(j)=\zeta_{p^m}^{lp^{m-1}}} j^k = \sum_{1\leq j\leq N,\ \epsilon(j)=1} j^k - n \tag{3.41}$$

for every l with $1 \leq l \leq p-1$. We put $Q_t := \sum_{1\leq j\leq N,\ \epsilon(j)=\zeta_{p^m}^{t}} j^k$ where $1 \leq t \leq p^{m-1} - 1$ and $Q_0 := \sum_{1\leq j\leq N,\ \epsilon(j)=1} j^k - n$. By the definition of $S_k(N)$, (3.39)-(3.41) we have

$$S_k(N) = \sum_{1\leq j\leq N,\ \epsilon(j)=1} j^k + \sum_{l=1}^{p-1} \sum_{1\leq j\leq N,\ \epsilon(j)=\zeta_{p^m}^{lp^{m-1}}} j^k \tag{3.42}$$
$$+ \sum_{t=1}^{p^{m-1}-1} \sum_{l=0}^{p-1} \sum_{1\leq j\leq N,\ \epsilon(j)=\zeta_{p^m}^{t+lp^{m-1}}} j^k$$
$$= \sum_{1\leq j\leq N,\ \epsilon(j)=1} j^k + (p-1)Q_0 + \sum_{t=1}^{p^{m-1}-1} pQ_t.$$

From the definition of Q_0 and (3.42), we get

$$S_k(N) - n = \sum_{t=0}^{p^{m-1}-1} pQ_t. \tag{3.43}$$

The proof of Lemma 6 is completed.

Now we prove Theorem 12 by combining Lemma 6 with the the argument of the case 1 of Theorem 1.2's proof in [ChC1].

By $p-1 \nmid k$, we have

$$\sum_{j=1}^{p}(cp+j)^k \equiv 0 \pmod{p} \tag{3.44}$$

for any integer c. By (3.44), we have

$$\#\{0 \leq l \leq p-1 \mid S_k(N) \equiv l \pmod{p}\} \leq p-1. \tag{3.45}$$

Hence, by (3.45) and Lemma 6, there is at least one residue class modulo p of numbers n which cannot be represented as (3.5) for any positive integer N and any map $\epsilon : \{1, \ldots, N\} \to U$. This proves Theorem 12.

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