Verified numerical computation for solutions to semilinear heat equations using semigroup theory

半群理論を用いた半線形熱方程式の解に対する精度保証付き数値計算に関する研究

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Contents

CHAPTER 1

INTRODUCTION

Let $\Omega \subset \mathbb{R}^d$ $(d = 1, 2)$ be a bounded convex domain and $J := (t_0, t_1]$ $(0 \leq t_0 < t_1 \leq \infty)$. The functional spaces $L^p(\Omega)$ $(p \geq 1)$, $H^k(\Omega)$ $(k \in \mathbb{N})$, and $H_0^1(\Omega)$ are defined in Chapter 2. For a function $u : J \times \Omega \to \mathbb{R}$, let us denote $u(t) = u(t, \cdot)$. In this thesis, we are concerned with the following semilinear heat equation:

$$\begin{cases} \partial_t u - \Delta u = f(x, u) & t \in J,\ x \in \Omega, \qquad \text{(1a)} \\ u = 0 & t \in J,\ x \in \partial\Omega, \qquad \text{(1b)} \\ u(0, x) = u_0 & x \in \Omega, \qquad \text{(1c)} \end{cases}$$

where $\partial_t u = \frac{\partial u}{\partial t}$, $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian, whose domain is $\mathcal{D}(\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$, $u_0 \in H_0^1(\Omega)$ is an initial function, and f is a function from $\Omega \times \mathbb{R}$ to $\mathbb{R}$. This thesis proposes an algorithm for numerically verifying the existence of a global-in-time solution to (1). Here, the global-in-time solution to (1) is defined by a solution, which exists in $(0, \infty)$, to (1). Moreover, this thesis also provides an explicit bound of the Sobolev embedding constant for the fractional power of the Laplacian. In next section, we present several previous works related to solutions for the parabolic equation of (1) and estimation of the embedding constant regarding to the fractional power of the Laplacian.

1.1. PREVIOUS WORKS

Many researchers have been studying the existence of solutions to parabolic equations because the results of the studies have been applicable to many fields such as physic, chemistry, and biology. We focus on methods using the semigroup theory for solving problems of partial differential equations.

K. Yoshida [**34**] and E. Hille [**12**] independently showed a necessary and sufficient condition for generating a C_0 semigroup. Their pioneering works have contributed to the progress of studies of the solutions of parabolic equations. In particular, the semigroup theory helps to derive solutions to parabolic equations. We consider the following abstract Cauchy problem in a Banach space X:

$$\begin{cases} \partial_t u(t) + Au(t) = G(u(t)), \\ u(0) = u_0, \end{cases} \tag{2}$$

where $u_0 \in X$ and G is a nonlinear operator from $\mathcal{D}(G) \subset X$ to X. Suppose that a linear operator $-A : \mathcal{D}(A) \subset X \to X$ generates a C_0 semigroup e^{-tA} over X. Fixed point theorems such as Banach's fixed-point theorem and Brouwer's fixed-point theorem provide a sufficient condition for the existence of a solution to (2). Then, the existence of the solution is guaranteed in a neighborhood, which satisfies the sufficient condition (see e.g., [**22, 24**]). The solution $u(t)$ for $t \in J$ is expressed by

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} G(u(s))ds,$$

which is called a mild solution (see e.g., [**22**]).

It has been proven that global-in-time solutions of (1) exist for special cases of f. In one example, for the semilinear heat equation (1a) and (1c) with $f(x,u) = u^p$ ($p > 1$) and $\Omega = \mathbb{R}^m$ ($m \in \mathbb{N}$), H. Fujita [**9**] has shown that a global-in-time solution exists if $p > 1 + 2/m$. Following his work, there has been proven the existence of a global-in-time solution for parabolic equations of the form (1a) and (1c) with $f(x,u) = u^p$ ($p \in \mathbb{R}$) if imposing several assumptions on the exponent p ([**10, 15, 18, 19**] and references therein). In another example, for the equation (1), there has been shown the existence of a global-in-time solution which converges to zero as $t \to \infty$ if a suitable norm of the initial function u_0 is enough small (see e.g., [**13, 24, 27, 28**] and references therein).

Recently, there have been proposed several methods of verified numerical computations to find solutions for a class of parabolic equations. M.T. Nakao, T. Kinoshita, and T. Kimura [**14, 20, 21**] have proposed a computer-assisted method for proving the existence of an inverse operator related to the parabolic equations. Moreover, their method rigorously estimates a norm of the inverse operator if the existence of the inverse operator is shown. S. Cai [**4**] has derived a sufficient condition for the existence of a global-in-time solution to a system of reaction-diffusion equations using verified numerical computations.

We also focus on studies regarding to the optimal embedding constant $C_{p,\alpha}$ such that for $0 \leq \alpha \leq 1$ and $1 \leq p \leq \infty$,

$$C_{p,\alpha} := \sup_{u \in X_\alpha \setminus \{0\}} \frac{\|u\|_{L^p}}{\|u\|_{X_\alpha}}, \tag{3}$$

where the detailed definition of X_α is given in Chapter 5. The inequality (3) is known as Sobolev type inequality for fractional derivatives. The constant for (3) has been applied to studies of partial differential equations (see e.g., [**1, 2, 3, 5, 6, 11, 16, 22, 25, 29, 31**] and references therein). Moreover, several estimation for obtaining upper bounds of $C_{p,\alpha}$ have been proposed in special cases (see e.g., [**3, 6, 32**]).

1.2. ABSTRACT

In this thesis, we will propose a numerical method for validating the existence of solutions to the semilinear heat equation (1) using semigroup theory. In particular, we provide a numerical verification algorithm for enclosing a global-in-time solution to (1).

In Section 3.1, we will provide a sufficient condition for the existence of a global-in-time solution to (1) with $J = (t', \infty)$ for $t' > 0$. Here, assume that the solution η of (1) exists at time $t = t'$ in a neighborhood of a stationary solution ϕ to (1). The equation (1) is transformed into an equivalent form of a semilinear heat equation with initial function $\eta - \phi$. We define a nonlinear operator S using the semigroup theory. The operator S and Banach's fixed-point theorem derive a sufficient condition for the existence of the global-in-time solution to (1). Its condition is described in Theorem 3.1.1.

In Section 3.2, we will give a sufficient condition for the existence of a solution to the initial-boundary value problem to (1) with a certain interval $J = (t_0, t_1]$ $(0 \leq t_0 < t_1 < \infty)$. The solution is called the local-in-time solution to (1) in this thesis. Let $\omega_0(t)$ $(t \in J)$ be an approximate solution of (1). The equation (1) is also transformed into an equivalent form of a semilinear heat equation with initial function $\xi - \omega_0(t_0)$, where ξ is a solution of (1) at time $t = t_0$. A nonlinear operator $\tilde{S}$ is introduced by using the semigroup theory. The operator $\tilde{S}$ and Banach's fixed-point theorem yield a sufficient condition for the existence of the local-in-time solution to (1) with $J = (t_0, t_1]$. Its condition is described in Theorem 3.2.1.

In Section 3.3, we will present a verification algorithm for enclosing a global-in-time solution to (1). The verification algorithm works as follows. The algorithm verifies that the existence of the stationary solution ϕ is guaranteed or not. If the solution ϕ exists, the algorithm rigorously checks whether the sufficient condition in Theorem 3.1.1 holds or not. If the condition satisfies, the global-in-time solution is validated. When we cannot confirm that the condition holds, we consider a solution of the initial-boundary value problem to (1) with $J = (0, \tau]$ $(0 < \tau < \infty)$. We put an approximate solution ω_0 of (1) and make the algorithm try to enclose the solution in a neighborhood of ω_0. The algorithm verifies whether the sufficient condition in Theorem 3.2.1 holds or not. If the condition in Theorem 3.2.1 holds, it turns out that there exists the solution to (1) with $J = [0, \tau)$ in the neighborhood of ω_0. Then, we consider the global-in-time solution to (1) with $J = (\tau, \infty)$. The algorithm also verifies whether the condition in Theorem 3.1.1 satisfies or not. If the condition in Theorem 3.1.1 holds, we can finally prove the existence of a global-in-time solution, which are enclosed in the neighborhood of ω_0 in time $t \in (0, \tau]$ and converges to ϕ for $t > \tau$. The verification algorithm encloses a global-in-time solution by repeatedly verifying that conditions of Theorems 3.1.1 and 3.2.1 are satisfied. The detailed procedure of the algorithm is summarized in Algorithm 1.

In Section 4.1, 4.2, and 4.3, we will provide several estimation required to use the verification algorithm. In Section 4.4, we present several semilinear heat equations and enclose the global-in-time solutions of these equations using the verification algorithm.

This thesis also proposes quantitative estimation of the embedding constant $C_{p,\alpha}$ satisfying (3). To obtain the estimation, we provide an explicit expression corresponding with the inverse operator of the fractional power of Laplacian in Lemma 5.1.1 and an estimate using the heat kernel over $\mathbb{R}^d$ in Lemma 5.1.2. The estimation of the embedding constant is presented in Theorem 5.1.1 and several explicit bounds of the embedding constant over a square domain and a L-shape domain are provided in Section 5.2.

1.3. OUTLINES

The outlines of this thesis are as follows. In Chapter 2, we provide notation and several lemmas required to present the verification algorithm. In Chapter 3, we present the numerical verification algorithm for enclosing a global-in-time solution, which exponentially converges to a stationary solution. First, in Section 3.1, we derive a sufficient condition (Theorem 3.1.1) to enclose a global-in-time solution. Next, in Section 3.2, we present a sufficient condition for the existence and the local-in-time uniqueness (Theorem 3.2.1) of a mild solution to (1) in a certain interval $t \in J = (t_0, t_1]$ $(0 \leq t_0 < t_1 < \infty)$. Furthermore, we give an a posteriori error estimate in Corollary 3.2.1. Finally, in Section 3.3, a procedure of the verification algorithm is given on the basis of Theorems 3.1.1 and 3.2.1, and Corollary 3.2.1. In Chapter 4, we present several semilinear heat equations of the form (1). Then, we derive global-in-time solutions of these equations by the algorithm. In Chapter 5, we propose estimation of the constant $C_{p,\alpha}$ defined by (3) in Theorem 5.1.1. Moreover, we show several values of the estimation in Theorem 5.1.1 over a square domain and a L-shape domain.

CHAPTER 2

PREPARATION

Unless otherwise specified, let Ω be a bounded convex domain. The functional space L^p ($p \geq 1$) is the space of p-th power Lebesgue integrable functions over Ω endowed with norm $\|f\|_{L^p} = (\int_\Omega |f(x)|^p dx)^{\frac{1}{p}}$ if $p < \infty$ and $\|f\|_{L^\infty} = \operatorname{ess\,sup}_{x\in\Omega} |u(x)|$ if $p = \infty$. The inner product of $L^2(\Omega)$ is given by

$$(u, v)_{L^2} := \int_\Omega u(x)v(x)dx.$$

For any positive integer m, let $H^m(\Omega)$ be the m-th order Sobolev space of $L^2(\Omega)$ and $H_0^1(\Omega) := \{u \in H^1(\Omega) \mid u = 0 \text{ on } \partial\Omega \text{ in the trace sense}\}$

Let J be an interval in $(0, \infty)$ and Y denotes a Banach space endowed with the norm $\|\cdot\|_Y$. We define the functional space $L^\infty(J; Y)$ as

$$L^\infty(J; Y) := \left\{u : J \times \Omega \to \mathbb{R} \mid u(t, \cdot) \in Y,\ \operatorname*{ess\,sup}_{t\in J} \|u(t, \cdot)\|_Y < \infty\right\}$$

endowed with norm $\|u\|_{L^\infty(J;Y)} := \operatorname{ess\,sup}_{t\in J} \|u(t, \cdot)\|_Y$. Let $C^0(J)$ be the function space of all continuous functions from J to $\mathbb{R}$. A function space $C^0(J; Y)$ denotes

$$C^0(J; Y) := \{u : J \times \Omega \to \mathbb{R} \mid t \in J \to u(t, \cdot) \text{ is continuous with respect to } Y \text{ norm}\}.$$

Let P and Q be Banach spaces endowed with the norms $\|\cdot\|_P$ and $\|\cdot\|_Q$, respectively. For a bounded linear operator $B : P \to Q$, the operator norm of B is defined by $\|B\|_{P,Q} = \sup_{x\in P\setminus\{0\}} \|Bx\|_Q/\|x\|_P$.

For $x > 0$, the error function $\operatorname{erf}(x)$ is defined by

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds.$$

By an elemental calculation it follows for $\alpha > 0$ and $x > 0$,

$$\int_0^x s^{-1/2} e^{-\alpha s} ds = \sqrt{\frac{\pi}{\alpha}} \operatorname{erf}(\sqrt{\alpha x}). \tag{4}$$

Let $\rho > 0$ and J be any interval. For $v \in L^\infty(J; H_0^1(\Omega))$, a closed ball $B_{L^\infty(J;H_0^1(\Omega))}(v, \rho)$ is defined by

$$B_{L^\infty(J;H_0^1(\Omega))}(v, \rho) := \left\{ y \in L^\infty\left(J; H_0^1(\Omega)\right) \mid \|y - v\|_{L^\infty\left(J;H_0^1(\Omega)\right)} \le \rho \right\}.$$

Assume that the function f, which appears in (1), satisfies $f(\cdot, v(\cdot)) \in L^2(\Omega)$ for each $v \in H_0^1(\Omega)$. An operator $F : H_0^1(\Omega) \to L^2(\Omega)$ is also defined by $F(v) := f(\cdot, v(\cdot))$ for $v \in H_0^1(\Omega)$. Furthermore, suppose that the operator $F : H_0^1(\Omega) \to L^2(\Omega)$ is the twice Fréchet differentiable. The operators $F'[v]$ and $F''[v]$ denote the first and the second order Fréchet derivatives of F at $v \in H_0^1(\Omega)$, respectively.

Unless otherwise specified, we denote $A = -\Delta : \mathcal{D}(A) \subset L^2(\Omega) \to L^2(\Omega)$ with domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Let $\rho(A)$ be the resolvent set of A defined by

$$\rho(A) := \{z \in \mathbb{C} \mid (zI - A)^{-1} : L^2(\Omega) \to L^2(\Omega) \text{ exists and is a bounded operator.}\}.$$

The spectrum of A denotes $\sigma(A) = \mathbb{C} \setminus \rho(A)$. Since the inverse of the operator A is a compact and self-adjoint operator, the spectral theorem shows that the operator A has positive discrete spectrum (see e.g., [**7**]). For $i \in \mathbb{N}$, let λ_i be the i-th eigenvalue of A satisfying $0 < \lambda_1 \le \lambda_2 \le \cdots$. Let $\psi_i \in H_0^1(\Omega)$ be an eigenfunction of A corresponding to λ_i: It is well known that λ_i and ψ_i satisfy $(\nabla \psi_i, \nabla v)_{L^2} = \lambda_i (\psi_i, v)_{L^2}$, $\forall v \in H_0^1(\Omega)$. Here, we choose $\{\psi_i\}_{i \in \mathbb{N}}$ so that ψ_i satisfies $(\psi_i, \psi_j)_{L^2} = \delta_{i,j}$, where $\delta_{i,j}$ is Kronecker's delta. For $u \in L^2(\Omega)$, we can express $u = \sum_{j=1}^\infty c_j \psi_j$ using the spectral decomposition, where $c_i = (u, \psi_i)_{L^2}$. For $0 \le \alpha \le 1$, a fractional operator of A is defined by

$$A^\alpha u := \sum_{j=1}^\infty \lambda_j^\alpha c_j \psi_j,\ \mathcal{D}(A^\alpha) = \left\{ u = \sum_{j=1}^\infty c_j \psi_j \in L^2(\Omega) \mid \sum_{j=1}^\infty c_j^2 \lambda_j^{2\alpha} < \infty \right\}. \tag{5}$$

Hereafter, we set $\lambda_{\min} := \lambda_1$.

The linear operator $-A$ generates the analytic semigroup $\left\{e^{-tA}\right\}_{t \ge 0}$ over $L^2(\Omega)$ (see e.g., [**22, 33**]).

Definition 1. *Let* $J = (t_0, t_1]$ $(0 \le t_0 < t_1 \le \infty)$. *For the semilinear parabolic equation:*

$$\begin{cases} \partial_t u - \Delta u = f(x,u) & t \in J, x \in \Omega, \\ u = 0 & t \in J, x \in \partial\Omega, \\ u(t_0, x) = u_0(x) & x \in \Omega, \end{cases} \tag{6}$$

the function $u \in C^0(J; L^2(\Omega))$ *given by*

$$u(t) = e^{-(t-t_0)A} u_0 + \int_{t_0}^{t} e^{-(t-s)A} F(u(s)) ds, \ t \in J$$

is called a mild solution of (6) *on* J.

In order to enclose a global-in-time solution of (1), the following lemmas are required. We provide Lemmas 2.0.1 and 2.0.2.

Lemma 2.0.1 (see e.g., [**33**]). $\mathcal{D}(A^{1/2}) = H_0^1(\Omega)$ *and*

$$\|w\|_{H_0^1} = \|A^{1/2} w\|_{L^2}, \ \forall w \in H_0^1(\Omega)$$

hold.

Lemma 2.0.2 (see e.g., [**22**]). *Let* $\alpha \in (0, 1]$. *If* $u \in \mathcal{D}(A^\alpha)$, *then*

$$A^\alpha e^{-tA} u = e^{-tA} A^\alpha u, \ t > 0$$

holds.

Furthermore, we present the following estimate:

Lemma 2.0.3. *Let* $\lambda_{\min}$ *be the least eigenvalue of* A. *For* $\alpha \in (0, 1)$ *and* $\beta \in (0, 1)$, *the following estimate holds:*

$$\|A^\alpha e^{-tA}\|_{L^2, L^2} \le \left(\frac{\alpha}{e\beta}\right)^\alpha t^{-\alpha} e^{-(1-\beta)t\lambda_{\min}}, \ t > 0. \tag{7}$$

PROOF. Since the least eigenvalue of A is positive, we have

$$\sup_{x\in[\lambda_{\min},\infty)} x^\alpha e^{-\beta tx} \le \left(\frac{\alpha}{e\beta t}\right)^\alpha \quad \text{and} \quad \sup_{x\in[\lambda_{\min},\infty)} e^{-(1-\beta)tx} \le e^{-(1-\beta)t\lambda_{\min}}$$

for any $\alpha \in (0,1)$ and $\beta \in (0,1)$. From the spectral mapping theorem (see e.g., [**8**]), the following inequality holds:

$$\begin{aligned} \|A^\alpha e^{-tA}\|_{L^2,L^2} &\le \sup_{x\in[\lambda_{\min},\infty)} x^\alpha e^{-tx} \\ &\le \sup_{x\in[\lambda_{\min},\infty)} x^\alpha e^{-\beta tx} \sup_{x\in[\lambda_{\min},\infty)} e^{-(1-\beta)tx}. \end{aligned}$$

This indicates that the inequality (7) holds. □

Chapter 3

Numerical verification for a global-in-time solution

3.1. NUMERICAL VERIFICATION FOR A GLOBAL-IN-TIME SOLUTION

Let $\phi \in \mathcal{D}(A)$ be a stationary solution of (1), i.e., ϕ satisfies

$$\begin{cases} -\Delta\phi(x) = f(x, \phi(x)), & x \in \Omega, \\ \phi(x) = 0, & x \in \partial\Omega. \end{cases}$$

Let $V_h \subset \mathcal{D}(A)$ be a finite dimensional subspace depending on a parameter $h > 0$. Assume that ϕ is the stationary solution unique in the ball:

$$B_{H_0^1}(\hat{\phi}, \rho') := \left\{ \mu \in H_0^1(\Omega) \mid \|\mu - \hat{\phi}\|_{H_0^1} \le \rho' \right\} \text{ for } \rho' > 0, \tag{8}$$

where $\hat{\phi} \in V_h$ is a certain numerical approximation of ϕ.

We are concerned with the existence of a mild solution of

$$\begin{cases} \partial_t u - \Delta u = f(x, u), & t \in (t', \infty),\ x \in \Omega, \\ u = 0, & t \in (t', \infty),\ x \in \partial\Omega, \\ u(t', x) = \eta, & x \in \Omega, \end{cases} \tag{9}$$

satisfying

$$u(t) = e^{-(t-t')A}\eta + \int_{t'}^{t} e^{-(t-s)A} F(u(s)) ds,$$

where $\eta \in B_{H_0^1}(\hat{u}, \varepsilon)$ for a certain $\hat{u} \in V_h$.

For $\lambda \ge 0$, we define a functional space E_λ as

$$E_\lambda := \left\{ u \in L^\infty((t', \infty); H_0^1(\Omega)) \mid \operatorname*{ess\,sup}_{t \in (t', \infty)} e^{(t-t')\lambda} \|u(t, \cdot)\|_{H_0^1} < \infty \right\},$$

where E_λ becomes a Banach space with norm

$$\|u\|_{E_\lambda} := \operatorname*{ess\,sup}_{t \in (t', \infty)} e^{(t-t')\lambda} \|u(t, \cdot)\|_{H_0^1}.$$

The following theorem gives a sufficient condition for enclosing the mild solution of (9) in E_λ.

THEOREM 3.1.1. *Let* $\phi \in \mathcal{D}(A)$ *be a locally unique stationary solution of* (9) *in* $B_{H_0^1}(\hat{\phi}, \rho')$. *Assume that there exists a non-decreasing function* $L_\phi : \mathbb{R} \to \mathbb{R}$ *such that for* $y \in B_{L^\infty((t',\infty);H_0^1(\Omega))}(\phi, \rho)$

$$\|F'[y]u\|_{L^\infty((t',\infty);L^2(\Omega))} \leq L_\phi(\rho)\|u\|_{L^\infty((t',\infty);H_0^1(\Omega))}, \ \forall u \in L^\infty((t',\infty);H_0^1(\Omega)). \quad (10)$$

Let λ *satisfy* $0 \leq \lambda < \lambda_{\min}/2$. *If there exists* $\rho > 0$ *such that*

$$\|\eta - \phi\|_{H_0^1} + L_\phi(\rho)\rho\sqrt{\frac{2\pi}{e(\lambda_{\min} - 2\lambda)}} < \rho, \quad (11)$$

then (9) *has a unique mild solution* $u \in B_{E_\lambda}(\phi, \rho)$, *which is defined by*

$$B_{E_\lambda}(\phi, \rho) := \{u \in E_\lambda \mid \|u - \phi\|_{E_\lambda} \leq \rho\}.$$

Therefore, the solution u *satisfies*

$$\|u(t) - \phi\|_{H_0^1} \leq \rho e^{-(t-t')\lambda}, \ t \in (t', \infty).$$

REMARK 3.1.1. *Since* $\|\eta - \hat{u}\|_{H_0^1} \leq \varepsilon$ *and a stationary solution* ϕ *exists in* $B_{H_0^1}(\hat{\phi}, \rho')$, *it follows*

$$\begin{aligned} \|\eta - \phi\|_{H_0^1} &\leq \|\eta - \hat{u}\|_{H_0^1} + \|\hat{u} - \hat{\phi}\|_{H_0^1} + \|\hat{\phi} - \phi\|_{H_0^1} \\ &\leq \varepsilon + \|\hat{u} - \hat{\phi}\|_{H_0^1} + \rho', \end{aligned}$$

where we remark that $\|\hat{u} - \hat{\phi}\|_{H_0^1}$ *is rigorously computable by using interval arithmetic. Therefore,* $\|\eta - \phi\|_{H_0^1}$ *in Theorem 3.1.1 can be estimated rigorously.*

REMARK 3.1.2. *In order to obtain* $\rho > 0$ *satisfying* (11), $L_\phi(\rho)$ *is required some assumptions, e.g.,* $L_\phi(\rho) \to 0$ *as* $\rho \to 0$. *Therefore, the inequality* (11) *provides a sufficient condition for the existence of the global-in-time solution* u *for only special cases of* f *in* (9).

Proof of Theorem 3.1.1. A nonlinear operator $S : L^{\infty}((t', \infty); H_0^1(\Omega)) \to L^{\infty}((t', \infty); H_0^1(\Omega))$ is defined by

$$(Sz)(t) := e^{-(t-t')A}(\eta - \phi) + \int_{t'}^{t} e^{-(t-s)A}\left(F(z(s)+\phi) - F(\phi)\right) ds,\ t > t',$$

where we denote $z(s) = z(s, \cdot) \in H_0^1(\Omega)$ for $s > t'$. Let $Z := \{z \in E_\lambda : \|z\|_{E_\lambda} \leq \rho\}$ for $\rho > 0$. We derive a condition based on Banach's fixed-point theorem so that S has a fixed-point in Z. We note that the solution $u(t) := z(t) + \phi$ is a mild solution of (9) if and only if z is a fixed point of S.

Let $z \in Z$. Then, Lemma 2.0.1 yields

$$\begin{aligned}
e^{(t-t')\lambda}\|(Sz)(t)\|_{H_0^1} &\leq e^{(t-t')\lambda}\|e^{-(t-t')A}(\eta - \phi)\|_{H_0^1} \\
&\quad + e^{(t-t')\lambda}\int_{t'}^{t} \|e^{-(t-s)A}\|_{L^2, H_0^1}\|F(z(s)+\phi) - F(\phi)\|_{L^2} ds \\
&= e^{(t-t')\lambda}\|A^{1/2}e^{-(t-t')A}(\eta - \phi)\|_{L^2} \\
&\quad + \int_{t'}^{t} e^{(t-s)\lambda}\|A^{1/2}e^{-(t-s)A}\|_{L^2, L^2} e^{(s-t')\lambda}\|F(z(s)+\phi) - F(\phi)\|_{L^2} ds.
\end{aligned}$$

From $\lambda < \lambda_{\min}/2$ and Lemma 2.0.3 with $\alpha = \beta = 1/2$, we have

$$\begin{aligned}
&e^{(t-t')\lambda}\|(Sz)(t)\|_{H_0^1} \\
&\leq e^{(t-t')\lambda}\|A^{1/2}e^{-(t-t')A}(\eta - \phi)\|_{L^2} \\
&+ \operatorname*{ess\,sup}_{s \in (t', \infty)}\left(e^{(s-t')\lambda}\left\|F(z(s)+\phi) - F(\phi)\right\|_{L^2}\right) e^{-1/2}\int_{t'}^{t} (t-s)^{-1/2} e^{-(t-s)\frac{\lambda_{\min}-2\lambda}{2}} ds.
\end{aligned}$$

For $s \in (t', \infty)$, it follows from the mean-value theorem and (10) that

$$\begin{aligned}
e^{(s-t')\lambda}\left\|F(z(s)+\phi) - F(\phi)\right\|_{L^2} &\leq \int_0^1 \left\|F'[\phi + \theta z(s)] e^{(s-t')\lambda} z(s)\right\|_{L^2} d\theta \\
&\leq L_\phi(\rho)\|z\|_{E_\lambda}.
\end{aligned}$$

From Lemma 2.0.1, Lemma 2.0.2, and the spectral mapping theorem (see e.g., [**8**]), we obtain

$$e^{(t-t')\lambda}\|(Sz)(t)\|_{H_0^1} \leq e^{(t-t')(\lambda-\lambda_{\min})}\|\eta-\phi\|_{H_0^1} + e^{-1/2}L_\phi(\rho)\|z\|_{E_\lambda}\int_{t'}^{t}(t-s)^{-1/2}e^{-(t-s)\frac{\lambda_{\min}-2\lambda}{2}}ds.$$

It follows from (4) and $\lambda < \lambda_{\min}/2$ that

$$e^{(t-t')\lambda}\|(Sz)(t)\|_{H_0^1} \leq \|\eta-\phi\|_{H_0^1} + L_\phi(\rho)\rho\frac{\sqrt{2\pi}\,\mathrm{erf}\left(\sqrt{\frac{(\lambda_{\min}-2\lambda)(t-t')}{2}}\right)}{\sqrt{e(\lambda_{\min}-2\lambda)}}.$$

Since $\mathrm{erf}(x) < 1$ for $x \geq 0$ and $\mathrm{erf}(x) \to 1$ as $x \to \infty$ hold, we have

$$\|S(z)\|_{E_\lambda} \leq \|\eta-\phi\|_{H_0^1} + L_\phi(\rho)\rho\sqrt{\frac{2\pi}{e(\lambda_{\min}-2\lambda)}}.$$

Therefore, if $\rho > 0$ satisfies (11), $S(z) \in Z$ holds. For any $z_i \in Z$ $(i = 1, 2)$, we have

$$\begin{aligned}
&e^{(t-t')\lambda}\left\|\left(Sz_1\right)(t)-(Sz_2)(t)\right\|_{H_0^1}\\
&\leq \int_{t'}^{t} e^{(t-s)\lambda}\|A^{1/2}e^{-(t-s)A}\|_{L^2,L^2}e^{(s-t')\lambda}\|F(z_1(s)+\phi)-F(z_2(s)+\phi)\|_{L^2}ds\\
&\leq \operatorname*{ess\,sup}_{s\in(t',\infty)}\left(e^{(s-t')\lambda}\left\|F(z_1(s)+\phi)-F(z_2(s)+\phi)\right\|_{L^2}\right)\\
&\quad\times e^{-1/2}\int_{t'}^{t}(t-s)^{-1/2}e^{-(t-s)\frac{\lambda_{\min}-2\lambda}{2}}ds\\
&\leq L_\phi(\rho)\|z_1-z_2\|_{E_\lambda}e^{-1/2}\int_{t'}^{t}(t-s)^{-1/2}e^{-(t-s)\frac{\lambda_{\min}-2\lambda}{2}}ds.
\end{aligned}$$

From (4), we obtain

$$e^{(t-t')\lambda}\left\|\left(Sz_1\right)(t)-(Sz_2)(t)\right\|_{H_0^1} \leq L_\phi(\rho)\frac{\sqrt{2\pi}\,\mathrm{erf}\left(\sqrt{\frac{(\lambda_{\min}-2\lambda)(t-t')}{2}}\right)}{\sqrt{e(\lambda_{\min}-2\lambda)}}\|z_1-z_2\|_{E_\lambda}.$$

Then, it turns out that

$$\|S(z_1) - S(z_2)\|_{E_\lambda} \leq L_\phi(\rho)\sqrt{\frac{2\pi}{e(\lambda_{\min} - 2\lambda)}}\|z_1 - z_2\|_{E_\lambda}.$$

If $\rho > 0$ satisfies (11), $L_\phi(\rho)\sqrt{\frac{2\pi}{e(\lambda_{\min}-2\lambda)}} < 1$ holds. Therefore, S becomes a strictly contraction mapping on Z and Banach's fixed-point theorem proves that a fixed point of S uniquely exists in Z. □

3.2. NUMERICAL VERIFICATION FOR A LOCAL-IN-TIME SOLUTION

For real numbers t_0 and t_1 satisfying $0 \leq t_0 < t_1 < \infty$, set $J := (t_0, t_1]$ and $\tau := t_1 - t_0$. In this section, we provide a sufficient condition for the existence and uniqueness (Theorem 3.2.1) of a local mild solution to (1). Furthermore, we give an a posteriori error estimate in Corollary 3.2.1. Then, we consider a mild solution of

$$\begin{cases} \partial_t u - \Delta u = f(x, u), & t \in J,\ x \in \Omega, \\ u = 0, & t \in J,\ x \in \partial\Omega, \\ u(t_0, x) = \xi, & x \in \Omega, \end{cases} \tag{12}$$

satisfying

$$u(t) = e^{-(t-t_0)A}\xi + \int_{t_0}^{t} e^{-(t-s)A} F(u(s))ds,$$

where $\xi \in B_{H_0^1}(\hat{u}_0, \varepsilon)$ for $\varepsilon > 0$ and $\hat{u}_0 \in V_h$.

Let $\hat{u}_1 \in V_h$ and $l_k(t)$ $(t \in J)$ denotes a linear Lagrange basis satisfying $l_k(t_j) = \delta_{kj}$ $(j = 0, 1)$, where δ_{kj} is Kronecker's delta. We define $\omega_0(t)$ as

$$\omega_0(t) = \hat{u}_0 l_0(t) + \hat{u}_1 l_1(t),\ t \in J. \tag{13}$$

In the following, we give a sufficient condition for the existence and the local uniqueness of a mild solution in $B_{L^\infty(J;H_0^1(\Omega))}(\omega_0, \rho)$ with a certain $\rho > 0$.

THEOREM 3.2.1. *For $J = (t_0, t_1]$ with $0 \leq t_0 < t_1 < \infty$, define ω_0 by* (13). *Set*

$$\delta = \left\| \int_{t_0}^{t} e^{-(t-s)A}(\partial_s \omega_0(s) + A\omega_0(s) - F(\omega_0(s)))ds \right\|_{L^\infty\left(J;H_0^1(\Omega)\right)}. \tag{14}$$

Assume that there exists a non-decreasing function $L_{\omega_0} : \mathbb{R} \to \mathbb{R}$ such that for $y \in B_{L^\infty(J;H_0^1(\Omega))}(\omega_0, \rho)$

$$\|F'[y]u\|_{L^\infty(J;L^2(\Omega))} \leq L_{\omega_0}(\rho)\|u\|_{L^\infty(J;H_0^1(\Omega))},\ \forall u \in L^\infty(J; H_0^1(\Omega)). \tag{15}$$

For given $\xi \in H_0^1(\Omega)$, assume that there exist $\tau := t_1 - t_0$ and $\rho > 0$ satisfying

$$\|\xi - \hat{u}_0\|_{H_0^1} + \sqrt{\frac{2\pi}{\lambda_{\min} e}}\,\mathrm{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right) L_{\omega_0}(\rho)\rho + \delta < \rho. \tag{16}$$

Then, the (12) admits a unique mild solution $u \in B_{L^\infty(J;H_0^1(\Omega))}(\omega_0, \rho)$.

Remark 3.2.1. *In order to be given $\rho > 0$ satisfying (16), $\tau > 0$ and the value of $\|\xi - \hat{u}_0\|_{H_0^1}$ are required to be enough small. Therefore, the mild solution u is not always validated using Theorem 3.2.1 even if the solution u exists.*

Proof of Theorem 3.2.1. An operator $\tilde{S} : L^\infty(J; H_0^1(\Omega)) \to L^\infty(J; H_0^1(\Omega))$ is defined by

$$(\tilde{S}z)(t) := e^{-(t-t_0)A}(\xi - \hat{u}_0) + \int_{t_0}^{t} e^{-(t-s)A} G(z(s))ds,$$

where let $z(s) = z(s, \cdot) \in H_0^1(\Omega)$ and $G(z(s)) := F(z(s) + \omega_0(s)) - \partial_t \omega_0(s) - A\omega_0(s)$ for $s \in J$. We note that $u(t) = z(t) + \omega_0(t)$ is a mild solution of (12) if and only if z is a fixed point of $\tilde{S}$. We will derive a condition based on Banach's fixed-point theorem so that $\tilde{S}$ has a fixed-point in $B_{L^\infty(J;H_0^1(\Omega))}(0, \rho)$.

From Lemma 2.0.1, Lemma 2.0.2, and the spectral mapping theorem (see e.g., [**8**]), it follows that for $t \in J$,

$$\begin{aligned} \left\|e^{-(t-t_0)A}(\xi - \hat{u}_0)\right\|_{H_0^1} &= \left\|e^{-(t-t_0)A} A^{1/2}(\xi - \hat{u}_0)\right\|_{L^2} \\ &\le e^{-(t-t_0)\lambda_{\min}} \|\xi - \hat{u}_0\|_{H_0^1} \\ &\le \|\xi - \hat{u}_0\|_{H_0^1}. \end{aligned} \tag{17}$$

We express as $G(z(s)) = G_1(z(s)) + G_2(z(s))$ for $s \in J$ with $G_1(z(s)) = F(z(s) + \omega_0(s)) - F(\omega_0(s))$ and $G_2(z(s)) = F(\omega_0(s)) - A\omega_0(s) - \partial_s \omega(s)$.

For $s \in J$, it follows from the mean-value theorem and (15) that

$$\|F(\omega_0(s)+z(s))-F(\omega_0(s))\|_{L^2} \leq \int_0^1 \|(F'[\omega_0(s)+\theta z(s)]z(s)\|_{L^2}\, d\theta$$
$$\leq L_{\omega_0}(\rho)\|z\|_{L^\infty(J;H_0^1(\Omega))}.$$

Lemma 2.0.3 with $\alpha=\beta=1/2$ and Lemma 2.0.1 imply

$$\begin{aligned}
\left\|\int_{t_0}^{t} e^{-(t-s)A}G_1(s)ds\right\|_{H_0^1} &\leq \int_{t_0}^{t} \|e^{-(t-s)A}\|_{L^2,H_0^1}\|G_1(s)\|_{L^2}ds \\
&= \int_{t_0}^{t} \|A^{\frac{1}{2}}e^{-(t-s)A}\|_{L^2,L^2}\|G_1(s)\|_{L^2}ds \\
&\leq e^{-1/2}\nu(t)\,\|G_1(z)\|_{L^\infty(J;L^2(\Omega))} \\
&\leq \sqrt{\frac{2\pi}{\lambda_{\min}e}}\operatorname{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right)L_{\omega_0}(\rho)\|z\|_{L^\infty(J;H_0^1(\Omega))},
\end{aligned}$$

where $\nu(t)$ denotes

$$\nu(t) := \int_{t_0}^{t}(t-s)^{-1/2}e^{-(t-s)\lambda_{\min}/2}ds$$

and the last inequality follows from (4). The inequalities (17) and (14) imply

$$\|\tilde{S}(z)\|_{L^\infty\left(J;H_0^1(\Omega)\right)} \leq \|\xi-\hat{u}_0\|_{H_0^1} + \sqrt{\frac{2\pi}{\lambda_{\min}e}}\operatorname{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right)L_{\omega_0}(\rho)\rho+\delta.$$

Therefore,

$$\tilde{S}\left(B_{L^\infty(J;H_0^1(\Omega))}(0,\rho)\right) \subset B_{L^\infty(J;H_0^1(\Omega))}(0,\rho)$$

holds if $\rho>0$ satisfies (16).

Let $z_1, z_2 \in B_{L^\infty(J;H_0^1(\Omega))}(0,\rho)$. It follows that

$$(\tilde{S}z_1)(t)-(\tilde{S}z_2)(t) = \int_{t_0}^{t} e^{-(t-s)A}\left\{F(z_1(s)+\omega_0(s))-F(z_2(s)+\omega_0(s))\right\}ds.$$

From Lemma 2.0.1, (15), and Lemma 2.0.3 with $\alpha = 1/2$ and $\beta = 1/2$, we estimate as

$$\left\|\tilde{S}(z_1) - \tilde{S}(z_2)\right\|_{L^\infty\left(J;H_0^1(\Omega)\right)} \leq \sqrt{\frac{2\pi}{\lambda_{\min} e}}\,\mathrm{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right) L_{\omega_0}(\rho)\|z_1 - z_2\|_{L^\infty\left(J;H_0^1(\Omega)\right)}.$$

If (16) holds, since we have $\sqrt{\frac{2\pi}{\lambda_{\min} e}}\,\mathrm{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right) L_{\omega_0}(\rho) < 1$, $\tilde{S}$ becomes a strictly contraction mapping on $B_{L^\infty(J;H_0^1(\Omega))}(0,\rho)$. Banach's fixed-point theorem yields that a fixed point of $\tilde{S}$ uniquely exists in $B_{L^\infty(J;H_0^1(\Omega))}(0,\rho)$. □

Moreover, we obtain the following a posteriori error estimate at $t = t_1$ if Theorem 3.2.1 holds.

Corollary 3.2.1. *Under the assumption in Theorem 3.2.1, let*

$$\tilde{\delta} = \left\|\int_{t_0}^{t_1} e^{-(t-s)A}(\partial_s\omega_0(s) + A\omega_0(s) - F(\omega_0(s)))ds\right\|_{H_0^1(\Omega)}. \tag{18}$$

Then, the mild solution u of (12) *satisfies*

$$\|u(t_1) - \hat{u}_1\|_{H_0^1} \leq e^{-\tau\lambda_{\min}}\varepsilon + \sqrt{\frac{2\pi}{\lambda_{\min} e}}\,\mathrm{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right) L_{\omega_0}(\rho)\rho + \tilde{\delta}. \tag{19}$$

Proof. Let z be a fixed point of $\tilde{S}$ in the proof of Theorem 3.2.1. Then,

$$z(t_1) = u(t_1) - \hat{u}_1 = e^{-(t_1-t_0)A}(\xi - \hat{u}_0) + \int_{t_0}^{t_1} e^{-(t_1-s)A}G(z(s))ds,$$

where $G(z(s)) = F(z(s) + \omega_0(s)) - A\omega_0(s) - \partial_s\omega_0(s)$. Similar discussions in those in the proof of Theorem 3.2.1 provide

$$\|u(t_1) - \hat{u}_1\|_{H_0^1} \leq e^{-\tau\lambda_{\min}}\|\xi - \hat{u}_0\|_{H_0^1} + \sqrt{\frac{2\pi}{\lambda_{\min} e}}\,\mathrm{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right) L_{\omega_0}(\rho)\rho + \tilde{\delta}.$$

The inequality (19) follows from the assumption $\xi \in B_{H_0^1}(\hat{u}_0, \varepsilon)$ in (12). □

3.3. Verification algorithm

In this section, on the basis of Theorem 3.1.1, Theorem 3.2.1, and Corollary 3.2.1, we will provide a verification algorithm to show the existence of a global-in-time solution in Algorithm 1.

Algorithm 1 Verification algorithm

```
Set $\hat{\phi} \in V_h$;
Verify the existence and the local uniqueness of a stationary solution $\phi$ in $B_{H_0^1}(\hat{\phi}, \rho')$;
if Failed in enclosing $\phi$ then
    error ("Failed in enclosing $\phi$");
end if
Set $\hat{u}_0 \in V_h$ and compute $\varepsilon$ satisfying $\|u_0 - \hat{u}_0\|_{H_0^1} \leq \varepsilon$;
$t' = 0$; $\eta = u_0$; $\hat{u} = \hat{u}_0$; $k = 0$;
while true do
    Compute $\|\eta - \phi\|_{H_0^1}$ based on Remark 3.1.1;
    Choose $\lambda$ satisfying $0 \leq \lambda < \lambda_{\min}/2$;
    if There exists $\rho > 0$ satisfying (11) in Theorem 3.1.1 then
        break;
    end if
    $k = k + 1$;
    $\hat{u}_0 = \hat{u}$; $t_0 = t'$; $\xi = \eta$;
    Set $\tau > 0$. Let $t_1 = t_0 + \tau$ and $T_k = (t_0, t_1]$;
    Choose $\hat{u}_1 \in V_h$ and set $\omega_0(t) = \hat{u}_0 l_0(t) + \hat{u}_1 l_1(t)$ for $t \in T_k$ ;
    Compute $\delta$ defined by (14);
    if there exists $\rho > 0$ satisfying (16) in Theorem 3.2.1 then
        there exists a mild solution $u(t)$ for $t \in (t_0, t_1]$ satisfying (12).
        Define a ball $C_{T_k}$ as $B_{L^\infty(J;H_0^1(\Omega))}(\omega_0, \rho)$ and $\rho_k = \rho$;
        Compute $\tilde{\delta}$ defined by (18);
        Substituting $\rho$ for the right-hand side of (19), update $\varepsilon > 0$ as $\varepsilon = e^{-\tau\lambda_{\min}}\varepsilon + \sqrt{\frac{2\pi}{\lambda_{\min}e}}\,\mathrm{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right) L_{\omega_0}(\rho)\rho + \tilde{\delta}$;
    else
        error ("Verification failed for $t \in T_k$.");
    end if
    $t' = t_1$; $\eta = u(t_1)$; $\hat{u} = \hat{u}_1$;
end while
$n = k$;
disp ("The solution for $t \in (0, \infty)$ exists and $\|u(t) - \phi\|_{H_0^1} \leq \rho e^{-\lambda(t-t')}$ holds for $t > t'$ ");
```

Remark 3.3.1. *We can freely choose* $\hat{u}_1 \in V_h$ *in Algorithm 1. However, note that it is possible that* $\rho > 0$*, which satisfies* (16)*, does not exists depending on the selected* $\hat{u}_1$.

In Algorithm 1, each ball C_{T_k} $(k = 1, 2, \ldots, n)$ is an enclosure of the solution to (1) for $t \in T_k$. We denote $T := \bigcup_{1 \le k \le n} T_k$. Let us define C_T as

$$C_T := \left\{ y \in L^\infty \left(T; H_0^1(\Omega) \right) \mid y \in C_{T_k},\ k = 1, 2, \ldots, n \right\}.$$

If Algorithm 1 finishes successfully, we can show that a solution $u(t)$ of (1) for $t \in T$ is enclosed in C_T. Moreover, the solution is asymptotically approaching to ϕ for $t \in (t', \infty)$. Therefore, in this case, the existence of a global-in-time solution to (1) can be proved by verified numerical computations.

Chapter 4

Numerical results

Let $\Omega := \{x = (x_1, x_2) : 0 < x_1, x_2 < 1\} \subset \mathbb{R}^2$ be an unit square domain. Using Algorithm 1, we verify the existence of global-in-time solutions for the following semilinear parabolic equations:

$$\begin{cases} \partial_t u - \Delta u = f(x, u), & t \in (0, \infty), x \in \Omega, \\ u = 0, & t \in (0, \infty), x \in \partial\Omega, \\ u(0, x) = 2\sin(\pi x)\sin(\pi y), & x \in \Omega, \end{cases} \tag{20}$$

where we consider the cases f being given by

$$\begin{aligned} \text{(Case 1) } f(x,u) &= u^2 + 4\sin(\pi x_1)\sin(\pi x_2), \\ \text{(Case 2) } f(x,u) &= u^2 + 4(\sin(\pi x_1)\sin(\pi x_2) \\ &+ \sin(2\pi x)\sin(2\pi y) + \sin(\pi x_1)\sin(2\pi x_2)), \\ \text{(Case 3) } f(x,u) &= u^2 + 4\sum_{1\le k,l\le 2}\sin(k\pi x_1)\sin(l\pi x_2), \end{aligned}$$

and

$$\text{(Case 4) } f(x,u) = u^2 + 4\sum_{1\le k,l\le 3}\sin(k\pi x_1)\sin(l\pi x_2).$$

In Section 4.1, 4.2, 4.3, we provide estimation of $\lambda_{\min}$, a local Lipschitz constant, and residuals since these estimation are required when using Algorithm 1. In Section 4.4, we present several global-in-time solutions of (20) numerically enclosed by Algorithm 1.

4.1. ENCLOSING THE LEAST EIGENVALUE

We provide estimation of $\lambda_{\min}$. Let λ_i $(i \in \mathbb{N})$ be all eigenvalues of A defined by Chapter 2. Let $m \in \mathbb{N}$ be the dimension of V_h. For $1 \le i \le m$, we define an eigenpair $(\lambda_i^h, \psi_i^h) \in \mathbb{R} \times V_h$ satisfying

$$(\nabla\psi_i^h, \nabla v_h)_{L^2} = \lambda_i^h(\psi_i^h, v_h)_{L^2}, \ \forall v_h \in V_h, \tag{21}$$

where $0 < \lambda_1^h \le \lambda_2^h \le \cdots \le \lambda_m^h$ satisfies. The orthogonal projection $R_h : H_0^1(\Omega) \to V_h$ is defined by

$$(\nabla(u - R_h u), \nabla v_h)_{L^2} = 0, \ \forall v_h \in V_h.$$

A constant C_h denotes

$$\|u - R_h u\|_{L^2} \le C_h \|\nabla(u - R_h u)\|_{L^2} \tag{22}$$

for any $u \in H_0^1(\Omega)$. Then, we have the following theorem:

THEOREM 4.1.1 (see e.g., [**17**]). *Let* λ_i $(i \in \mathbb{N})$ *be all eigenvalues of* A *provided in Chapter 2. Let* λ_i^h $(1 \le i \le m)$ *and* C_h *be defined by* (21) *and* (22)*, respectively. Then,*

$$\frac{\lambda_i^h}{1 + C_h^2 \lambda_i^h} \le \lambda_i \le \lambda_i^h$$

holds.

From $\lambda_1 = \lambda_{\min}$, Theorem 4.1.1 derives both a lower bound and an upper bound of $\lambda_{\min}$.

4.2. LOCAL LIPSCHITZ CONSTANTS

We derive $L_\phi(\rho)$ in Theorem 3.1.1 and $L_{\omega_0}(\rho)$ in Theorem 3.2.1. Let q be a natural number. Let $C_{e,q} > 0$ be a constant satisfying

$$\|u\|_{L^q} \le C_{e,q} \|u\|_{H_0^1}, \ \forall u \in H_0^1(\Omega), \tag{23}$$

where the constant $C_{e,p}$ is called Sobelev embedding constant (see e,g., [**1**]). Note that the constant can be numerically estimated (see e.g., Lemma 2 in [**23**]). Let J be any interval. For $\rho > 0$ and a given $v \in L^\infty(J; H_0^1(\Omega))$, let $w \in B_{L^\infty(J;H_0^1(\Omega))}(v, \rho)$.

Here, for $u \in L^\infty(J; H_0^1(\Omega))$ and any $s \in J$, we can obtain

$$\begin{aligned}
\|F'[w(s)]u(s)\|_{L^2} &= 2\|w(s)u(s)\|_{L^2} \\
&\leq 2\|w(s)\|_{L^4}\|u(s)\|_{L^4} \\
&\leq 2C_{e,4}^2\|w(s)\|_{H_0^1}\|u\|_{L^\infty(J;H_0^1(\Omega))} \\
&\leq 2C_{e,4}^2(\rho + \|v\|_{L^\infty(J;H_0^1(\Omega))})\|u\|_{L^\infty(J;H_0^1(\Omega))}.
\end{aligned}$$

Therefore

$$L_\phi(\rho) = 2C_{e,4}^2(\rho + \|\phi\|_{L^\infty(J;H_0^1(\Omega))})$$

and

$$L_{\omega_0}(\rho) = 2C_{e,4}^2(\rho + \|\omega_0\|_{L^\infty(J;H_0^1(\Omega))})$$

hold. Furthermore, we estimate

$$\|\phi\|_{L^\infty(J;H_0^1(\Omega))} \leq \rho' + \|\hat{\phi}\|_{H_0^1}$$

and

$$\|\omega_0\|_{L^\infty(J;H_0^1(\Omega))} \leq \max\left\{\|\hat{u}_0\|_{H_0^1}, \|\hat{u}_1\|_{H_0^1}\right\}.$$

4.3. RESIDUAL ESTIMATES

We provide estimates of residuals δ and $\tilde{\delta}$ such that

$$\delta \geq \left\| \int_{t_0}^{t} e^{-(t-s)A}(\partial_s\omega_0(s) + A\omega_0(s) - F(\omega_0(s)))ds \right\|_{L^\infty\left(J;H_0^1(\Omega)\right)}$$

defined by Theorem 3.2.1 and

$$\tilde{\delta} \geq \left\| \int_{t_0}^{t_1} e^{-(t-s)A}(\partial_s\omega_0(s) + A\omega_0(s) - F(\omega_0(s)))ds \right\|_{H_0^1}$$

defined by Corollary 3.2.1.

For real numbers t_0 and t_1 such that $0 \leq t_0 < t_1 < \infty$, let $J = (t_0, t_1]$ and $\tau = t_1 - t_0$. Furthermore, let $u_0 \in H_0^1(\Omega)$ be a solution of (20) at time $t = t_0$ and

$\hat{u}_0 \in V_h$ be a numerical approximation of u_0. We employ the Crank-Nicolson scheme; for $\hat{w}_0 \in V_h$, we find $w_1 \in V_h$ such that

$$\left(\frac{w_1 - \hat{w}_0}{\tau}, v_h\right)_{L^2} + \frac{1}{2}\left(A(\hat{w}_0 + w_1), v_h\right)_{L^2} = \frac{1}{2}\left(F(\hat{w}_0) + F(w_1), v_h\right)_{L^2} \qquad (24)$$

for any $v_h \in V_h$. Let $\hat{u}_1 \in V_h$ be a numerical approximation of $w_1 \in V_h$ of the equation (24) replaced $\hat{w}_0 \in V_h$ by $\hat{u}_0$. Let l_k $(k = 0, 1)$ be a linear Lagrange basis satisfying $l_k(t_j) = \delta_{k,j}$ $(k, j = 0, 1)$, where $\delta_{k,j}$ is Kronecker's delta. Then, we define $\omega_0 \in L^\infty(J; V_h)$ as

$$\omega_0(t) = \hat{u}_0 l_0(t) + \hat{u}_1 l_1(t),\ t \in J. \qquad (25)$$

For a real number θ satisfying $0 \leq \theta \leq 1$, we define $\mathcal{C}_\theta \in L^2(\Omega)$ as

$$\mathcal{C}_\theta := \frac{\hat{u}_1 - \hat{u}_0}{\tau} + (1-\theta)A\hat{u}_0 + \theta A\hat{u}_1 - (1-\theta)F(\hat{u}_0) - \theta F(\hat{u}_1).$$

Let $\Phi(t) := F(\hat{u}_1)l_1(t) + F(\hat{u}_0)l_0(t)$ for $t \in J$. Then, we have

$$\begin{aligned}
&\left\| \int_{t_0}^{t} e^{-(t-s)A}\left(F(\omega_0(s)) - \partial_s \omega_0(s) - A\omega_0(s)\right) ds \right\|_{H_0^1} \\
&\leq \int_{t_0}^{t} \left\| e^{-(t-s)A}(F(\omega_0(s)) - \Phi(s)) \right\|_{H_0^1} ds \\
&\quad + \int_{t_0}^{t} \left\| e^{-(t-s)A}\left(\Phi(s) - \partial_s \omega_0(s) - A\omega_0(s)\right) \right\|_{H_0^1} ds. \qquad (26)
\end{aligned}$$

We estimate the first term of the right hand side of (26). Since both $\hat{u}_0$ and $\hat{u}_1$ are in $V_h \subset L^\infty(\Omega)$, a classical error bound of linear interpolation yields for $x \in \Omega$,

$$\begin{aligned}
|f(x, \omega_0(t,x)) - \Phi(t)| &\leq \frac{\tau^2}{8} \max_{t \in J} \left| \frac{d^2}{dt^2} f(x, \omega_0(t,x)) \right| \\
&= \frac{\tau^2}{8} \max_{t \in J} \left| \left(F''[\omega_0(t)] \left(\frac{d\omega_0}{dt}\right)^2 \right)(x) \right| \\
&= \frac{1}{8} \max_{t \in J} \left| \frac{\partial^2 f}{\partial u^2}(x, \omega_0(t,x))\,(\hat{u}_1 - \hat{u}_0)(x))^2 \right|.
\end{aligned}$$

Integrating over Ω, it follows from (23) that

$$\|F(\omega_0(t)) - \Phi(t)\|_{L^2} \leq \frac{C_{e,4}^2}{8}\|f''(\omega_0)\|_{L^\infty(J;L^\infty(\Omega))}\|\hat{u}_1 - \hat{u}_0\|_{H_0^1}^2, \tag{27}$$

where $\|f''(\omega_0)\|_{L^\infty(J;L^\infty(\Omega))} := \operatorname{ess\,sup}_{t\in J}\|\frac{\partial^2 f}{\partial u^2}(\cdot,\omega_0(t,\cdot))\|_{L^\infty(\Omega)}$.

From (27) and Lemma 2.0.3 with $\alpha = \beta = 1/2$,

$$\begin{aligned}
&\int_{t_0}^{t}\|e^{-(t-s)A}(F(\omega_0(s)) - \Phi(s))\|_{H_0^1}ds\\
&= \int_{t_0}^{t}\|A^{1/2}e^{-(t-s)A}(F(\omega_0(s)) - \Phi(s))\|_{L^2}ds\\
&\leq e^{-1/2}\int_{t_0}^{t}(t-s)^{-1/2}e^{-1/2(t-s)\lambda_{\min}}\left\|F(\omega_0(s)) - \Phi(s)\right\|_{L^2}ds\\
&\leq \sqrt{\frac{2\pi}{\lambda_{\min}e}}\operatorname{erf}\left(\sqrt{\frac{\lambda_{\min}(t-t_0)}{2}}\right)\|F(\omega_0) - \Phi\|_{L^\infty(J;L^2(\Omega))}
\end{aligned}$$

holds. Therefore, we obtain the following upper bound:

$$\left\|\int_{t_0}^{t}e^{-(t-s)A}(F(\omega_0(s)) - \Phi(s))ds\right\|_{L^\infty\left(J;H_0^1(\Omega)\right)} \leq C_p\alpha^2\sqrt{\frac{2\pi}{\lambda_{\min}e}}\operatorname{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right),$$

where we put

$$C_p := \frac{C_{e,4}^2}{8}\|f''(\omega_0)\|_{L^\infty(J;L^\infty(\Omega))} \text{ and } \alpha := \|\hat{u}_1 - \hat{u}_0\|_{H_0^1}.$$

We estimate the second term of the right hand side of (26). From $l_1(s) + l_0(s) = 1$ for $s \in J$, we have

$$\begin{aligned}
\Phi(s) - \partial_s\omega_0(s) - A\omega_0(s) &= -\left(\mathcal{C}_1 l_1(s) + \mathcal{C}_0 l_0(s)\right)\\
&= -\left((\mathcal{C}_1 - \mathcal{C}_\theta)l_1(s) + (\mathcal{C}_0 - \mathcal{C}_\theta)l_0(s) + \mathcal{C}_\theta\right)\\
&= -\left(\mathcal{C}_\theta + (\mathcal{C}_1 - \mathcal{C}_0)((1-\theta)l_1(s) - \theta l_0(s)\right).
\end{aligned}$$

Then, for $t \in J$, it sees that

$$\begin{aligned}
&\int_{t_0}^{t} \|e^{-(t-s)A}(\Phi(s) - \partial_s\omega_0(s) - A\omega_0(s))\|_{H_0^1} ds \\
&= \int_{t_0}^{t} \left\|A^{1/2}e^{-(t-s)A}(\Phi(s) - \partial_s\omega_0(s) - A\omega_0(s))\right\|_{L^2} ds \\
&\leq \int_{t_0}^{t} e^{-1/2}\|\mathcal{C}_\theta\|_{L^2}(t-s)^{-1/2}e^{-(t-s)\frac{\lambda_{\min}}{2}} ds \\
&+ \|\mathcal{C}_1 - \mathcal{C}_0\|_{L^2} \max_{s\in J}|(1-\theta)l_1(s) - \theta l_0(s))| \int_{t_0}^{t} (t-s)^{-1/2}e^{-(t-s)\frac{\lambda_{\min}}{2}} ds \\
&\leq \sqrt{\frac{2\pi}{\lambda_{\min}e}} \operatorname{erf}\left(\sqrt{\frac{\lambda_{\min}(t-t_0)}{2}}\right) \left(\|\mathcal{C}_\theta\|_{L^2} + \max(\theta, 1-\theta)\|\mathcal{C}_1 - \mathcal{C}_0\|_{L^2}\right).
\end{aligned}$$

Therefore, when $\theta = 1/2$, both δ and $\tilde{\delta}$ are bounded by

$$\sqrt{\frac{2\pi}{\lambda_{\min}e}} \operatorname{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right) \left(C_p\alpha^2 + \left\|\mathcal{C}_{\frac{1}{2}}\right\|_{L^2} + \frac{\|\mathcal{C}_1 - \mathcal{C}_0\|_{L^2}}{2}\right).$$

4.4. NUMERICAL RESULTS

We present several global-in-time solutions of (20) numerically enclosed by Algorithm 1. All computations are carried out on CentOS 6.3 with 3.10GHz Intel(R) Xeon(R) CPU E5-2687W, 128GB RAM. We use MATLAB 2012b with INTLAB ver.7.1 [**26**]. The spectrum method is employed for discretizing the spatial variable. Namely, we construct a numerical solution by using the Fourier basses. For $N \in \mathbb{N}$, a finite dimensional subspace $V_N \subset \mathcal{D}(A)$ is defined by

$$V_N := \left\{ u \in \mathcal{D}(A) : u(x,y) = \sum_{k,l=1}^{N} a_{k,l} \sin(k\pi x)\sin(l\pi y), a_{k,l} \in \mathbb{R} \right\}.$$

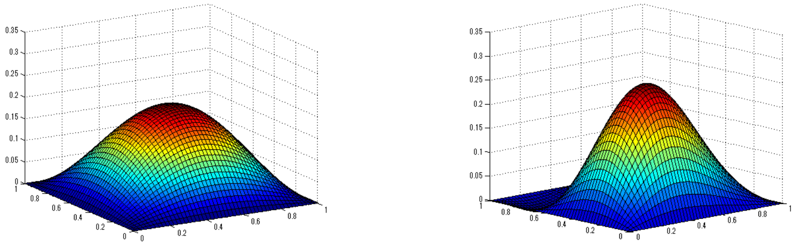
We fix $N = 10$, $\tau = 2^{-8}$, and $\lambda = 1/40$ ($< \lambda_{\min} = 2\pi^2$) in Algorithm 1. Then, we try to verify the existence of global-in-time solutions to (20) by using Algorithm 1.

Let ϕ denotes a stationary solution of (20). We verify the existence and the local uniqueness of ϕ in a neighborhood of a numerical solution $\hat{\phi} \in V_h$ by using the verification method based on [**30**]. A radius of the neighborhood is denoted by ρ'

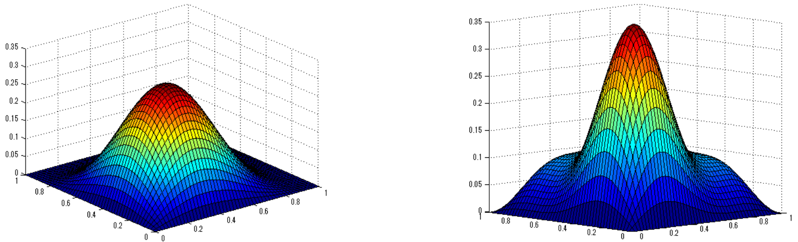
satisfying $\|\phi - \hat{\phi}\|_{H_0^1} \leq \rho'$. Each ρ' is shown in Table 4.1. The numerical solutions $\hat{\phi}$ are displayed in Figure 4.1.

TABLE 4.1. Radii of the neighborhood enclosing ϕ when $N = 10$.

Case	ρ'
1	0.002706328809
2	0.003861742749
3	0.004967902695
4	0.00724564522



(A) Case 1 (B) Case 2



(C) Case 3 (D) Case 4

FIGURE 4.1. The numerical solutions $\hat{\phi}$ for the four cases.

For simplicity, we consider (20) for Case 1. We define a numerical solution ω_0 of (20) as

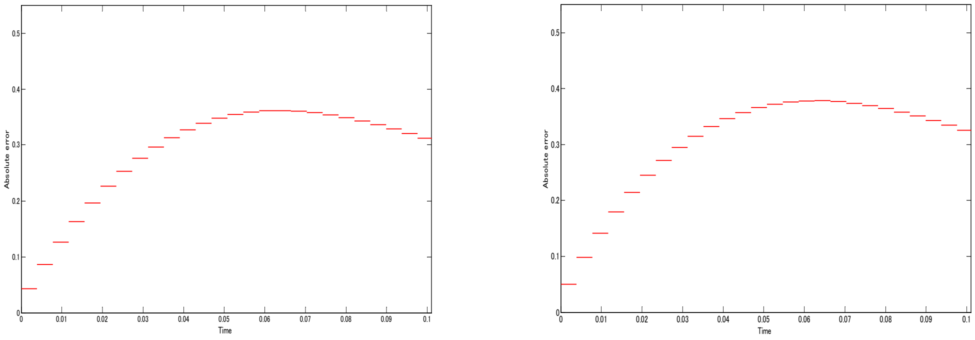
$$\omega_0(t) = \hat{u}_0 l_0(t) + \hat{u}_1 l_1(t),\ t \in T_k,$$

where $\hat{u}_0 \in V_h$ and $\hat{u}_1 \in V_h$ are provided in (25). Then, Algorithm 1 gives $\rho_k > 0$ satisfying

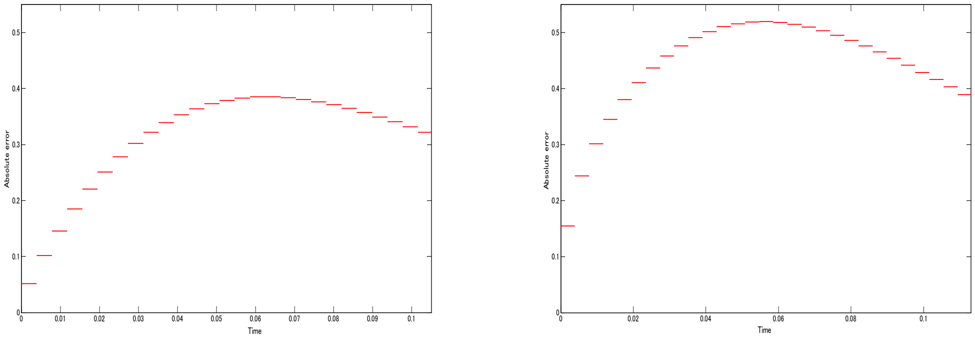
$$\|u - \omega_0\|_{L^\infty(T_k;H_0^1(\Omega))} \leq \rho_k.$$

Figure 4.2a displays each ρ_k for T_k when $N = 10$ and $\tau = 2^{-8}$.

For the Cases 2, 3, and 4, Figure 4.2 also shows each ρ_k for T_k when $N = 10$ and $\tau = 2^{-8}$. Furthermore, the algorithm 1 gives the following estimates:



(A) Case 1 (B) Case 2



(C) Case 3 (D) Case 4

FIGURE 4.2. Each ρ_k for T_k for the four cases.

$$\|u(t) - \phi\|_{H_0^1} \leq \rho e^{-(t-t')/40},\ t \in (t', \infty). \tag{28}$$

Table 4.2 also shows each error estimate ρ and t' of (28).

TABLE 4.2. Error estimate ρ and t' are presented when $N = 10$ and $\tau = 2^{-8}$.

Cases	ρ	t'
1	0.973712650429328	0.1015625
2	0.939460907598910	0.10546875
3	0.953394626139478	0.10546875
4	0.954276545574080	0.11328125

CHAPTER 5

ESTIMATION OF THE EMBEDDING CONSTANT FROM X_α TO L^p SPACE

5.1. ESTIMATION OF THE EMBEDDING CONSTANT FROM X_α TO L^p SPACE

In this chapter, we update the domain Ω as a bounded domain in $\mathbb{R}^N$ $(N \in \mathbb{N})$ and the operator $A : \mathcal{D}(A) \subset L^2(\Omega) \to L^2(\Omega)$ as

$$(Au, v)_{L^2} := (\nabla u, \nabla v)_{L^2},\ \forall v \in H^1_0(\Omega),$$

where the domain of A denotes $\mathcal{D}(A) := \{u \in H^1_0(\Omega) \mid Au \in L^2(\Omega)\}$. Then, for $0 \le \alpha \le 1$, the operator A^α and the domain of A^α are defined by (5) in Chapter 2. The operator A^α is a closed and invertible operator (see e.g., [**22**]). The closedness of A^α implies that $\mathcal{D}(A^\alpha)$ endowed with the graph norm: $\|u\|_{L^2} + \|A^\alpha u\|_{L^2}$ is a Banach space. Since A^α is invertible, the graph norm is equivalent to the norm $\|A^\alpha u\|_{L^2}$. For $0 \le \alpha \le 1$, the functional space X_α is defined by

$$X_\alpha := \{u \in L^2(\Omega) \mid A^\alpha u \in L^2(\Omega)\}$$

endowed with norm $\|u\|_{X_\alpha} = \|A^\alpha u\|_{L^2}$. In this chapter, we provide estimation for obtaining an explicit upper bound of $C_{p,\alpha}$ such that

$$C_{p,\alpha} := \sup_{u \in X_\alpha \backslash \{0\}} \frac{\|u\|_{L^p}}{\|u\|_{X_\lambda}}$$

for $\alpha > N(1/2 - 1/p)/2$. The following theorem gives the estimation:

THEOREM 5.1.1. *Let $\Omega \subset \mathbb{R}^N$ $(N \in \mathbb{N})$ be a bounded domain. The least eigenvalue of A is denoted by $\lambda_{\min}$. For $2 < p \le \infty$, let r and α be real numbers such that $1/r = 1/2 - 1/p$ and $N/(2r) < \alpha \le 1$, where $1/p = 0$ if $p = \infty$. Then, we can estimate $C_{p,\alpha}$ as*

$$C_{p,\alpha} \le \begin{cases} \dfrac{\alpha^\alpha \Gamma\left(\alpha - \frac{N}{2r}\right)}{(4\pi)^{\frac{N}{2r}} (\frac{N}{2r})^{\frac{N}{2r}} (\alpha - \frac{N}{2r})^{\alpha - \frac{N}{2r}} \Gamma(\alpha)} \lambda_{\min}^{-\left(\alpha - \frac{N}{2r}\right)} & (p > 2) \\ \lambda_{\min}^{-\alpha} & (p = 2). \end{cases} \tag{29}$$

We introduce two fundamental lemmas in order to prove Theorem 5.1.1. For $u \in L^2(\Omega)$ and $0 < \gamma \leq 1$, the function $(A^\gamma)^{-1}u$ can be expressed by using the Dunford integral (see e.g., [**33**]). The resulting expression corresponds with the right hand side of (30) (see e.g., [**22**]). Therefore, the following lemma holds:

Lemma 5.1.1. *Let* $\Omega \subset \mathbb{R}^N$ $(N \in \mathbb{N})$ *be a bounded domain. For* $0 < \gamma \leq 1$, $A^\gamma : \mathcal{D}(A^\gamma) \subset L^2(\Omega) \to L^2(\Omega)$ *is invertible and*

$$(A^\gamma)^{-1}u = \Gamma(\gamma)^{-1} \int_0^\infty t^{\gamma-1} e^{-tA} u \; dt \tag{30}$$

for $u \in L^2(\Omega)$.

Hereafter, for $0 \leq \alpha \leq 1$, set $A^{-\alpha} := (A^\alpha)^{-1}$. Moreover, some properties of the Dirichlet heat kernel give the following lemma:

Lemma 5.1.2 (see e.g., [**24**]). *Let* $\Omega \subset \mathbb{R}^N$ $(N \in \mathbb{N})$ *be a bounded domain. For* $1 \leq p < q \leq \infty$, *put* $1/r = 1/p - 1/q$, *where* $1/q = 0$ *if* $q = \infty$. *For all* $t \in (0, \infty)$ *and* $u \in L^p(\Omega)$,

$$\|e^{-tA}u\|_{L^q} \leq (4\pi t)^{-\frac{N}{2r}} \|u\|_{L^p}.$$

Proof of Theorem 5.1.1. First, we show that Theorem 5.1.1 holds for $2 < p \leq \infty$. Put $u \in \mathcal{D}(A^\alpha)$. From Lemma 5.1.1,

$$\begin{aligned}
\|u\|_{L^p} &= \|A^{-\alpha}A^\alpha u\|_{L^p} \\
&\leq \Gamma(\alpha)^{-1} \int_0^\infty t^{\alpha-1} \|e^{-tA}A^\alpha u\|_{L^p(\Omega)} dt \\
&\leq \Gamma(\alpha)^{-1} \int_0^\infty t^{\alpha-1} \|e^{-tA}\|_{L^2, L^p} \|A^\alpha u\|_{L^2} dt \\
&\leq \Gamma(\alpha)^{-1} \int_0^\infty t^{\alpha-1} \|e^{-\beta tA}\|_{L^2, L^p} \|e^{-(1-\beta)tA}\|_{L^2, L^2} \|A^\alpha u\|_{L^2} dt
\end{aligned}$$

holds valid for $0 < \beta < 1$. The spectral mapping theorem (see e.g., [**8**]) and Lemma 5.1.2 state that

$$\begin{aligned}
\|u\|_{L^p} &\leq \Gamma(\alpha)^{-1} \int_0^\infty t^{\alpha-1}(4\pi\beta t)^{-\frac{N}{2r}} e^{-t(1-\beta)\lambda_{\min}} \|A^\alpha u\|_{L^2} dt \\
&= (4\pi\beta)^{-\frac{N}{2r}} \Gamma(\alpha)^{-1} \int_0^\infty t^{\alpha-1-\frac{N}{2r}} e^{-t(1-\beta)\lambda_{\min}} dt \|A^\alpha u\|_{L^2} \\
&= (4\pi\beta)^{-\frac{N}{2r}} \Gamma(\alpha)^{-1} \left(\frac{1}{(1-\beta)\lambda_{\min}}\right)^{\alpha-1-\frac{N}{2r}} \\
&\quad \times \int_0^\infty s^{\alpha-1-\frac{N}{2r}} e^{-s} \left(\frac{1}{(1-\beta)\lambda_{\min}}\right) ds \|A^\alpha u\|_{L^2} \\
&= \frac{\Gamma\left(\alpha - \frac{N}{2r}\right)}{(4\pi)^{\frac{N}{2r}} g(\beta)\Gamma(\alpha)} \lambda_{\min}^{-\left(\alpha-\frac{N}{2r}\right)} \|A^\alpha u\|_{L^2}. \qquad (31)
\end{aligned}$$

Here, let $g(\beta) := \beta^{\frac{N}{2r}}(1-\beta)^{\alpha-\frac{N}{2r}}$ $(0 < \beta < 1)$ and note that $\Gamma(\alpha - N/2r) < \infty$ for $\alpha > N/(2r)$. Since the estimate (31) holds valid for any $0 < \beta < 1$, we put $\beta = \frac{N}{2r\alpha} (< 1)$ so that the function g admits the maximal value. It follows that

$$\|u\|_{L^p} \leq \frac{\alpha^\alpha \Gamma\left(\alpha - \frac{N}{2r}\right)}{(4\pi)^{\frac{N}{2r}} (\frac{N}{2r})^{\frac{N}{2r}} (\alpha - \frac{N}{2r})^{\alpha - \frac{N}{2r}} \Gamma(\alpha)} \lambda_{\min}^{-\left(\alpha-\frac{N}{2r}\right)} \|A^\alpha u\|_{L^2}.$$

Next, we prove Theorem 5.1.1 for $p = 2$. For $0 \leq \alpha \leq 1$ and $u \in \mathcal{D}(A^\alpha)$, the spectral mapping theorem and Lemma 5.1.1 yield

$$\begin{aligned}
\|u\|_{L^2} &= \|A^{-\alpha} A^\alpha u\|_{L^2} \\
&\leq \Gamma(\alpha)^{-1} \int_0^\infty t^{\alpha-1} \|e^{-tA} A^\alpha u\|_{L^2(\Omega)} dt \\
&\leq \Gamma(\alpha)^{-1} \int_0^\infty t^{\alpha-1} \|e^{-tA}\|_{L^2,L^2} dt \ \|A^\alpha u\|_{L^2} \\
&\leq \Gamma(\alpha)^{-1} \int_0^\infty t^{\alpha-1} e^{-t\lambda_{\min}} dt \ \|A^\alpha u\|_{L^2} \\
&= \lambda_{\min}^{-\alpha} \|A^\alpha u\|_{L^2}
\end{aligned}$$

Therefore, $C_{2,\alpha} = \lambda_{\min}^{-\alpha}$ holds valid for $0 \leq \alpha \leq 1$. □

5.2. Explicit bounds of the embedding constant from X_α to L^p space

We provide some numerical examples to estimate the constant $C_{p,\alpha}$ in Theorem 5.1.1. All computations were carried out on a computer with CentOS 7.2, CPU intel Core i7-6950X 3.0GHz, and 128 GByte RAM. We used MATLAB R2016a with INTLAB ver. 9 [**26**].

For the first case, we selected an unit square domain $\Omega := (0,1) \times (0,1)$. Varying $p = 3,\ 4\ ,5\ ,6$, and α such that $1/2 - 1/p < \alpha \leq 1$, the values of the right hand side of (29) in Theorem 5.1.1 are plotted on the domain $\Omega = (0,1) \times (0,1)$ in Figure 5.1, Figure 5.2, Figure 5.3, and Figure 5.4. Moreover, we recall that Theorem 5.1.1 enables us to estimate $C_{\infty,\alpha}$ for $\alpha > 1/2$. Figure 5.5 shows the values of the right hand side of (29) with $p = \infty$ in Theorem 5.1.1 for $1/2 < \alpha \leq 1$ on $\Omega = (0,1) \times (0,1)$.

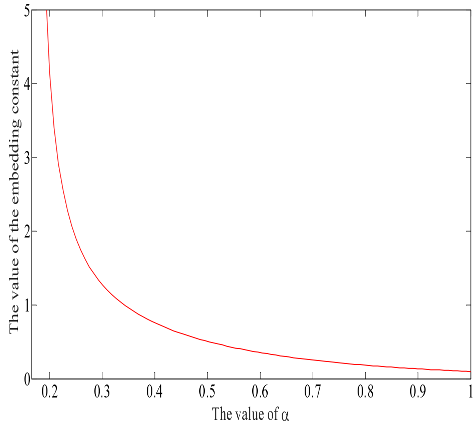


FIGURE 5.1. Values of $C_{3,\alpha}$ on $\Omega = (0,1) \times (0,1)$

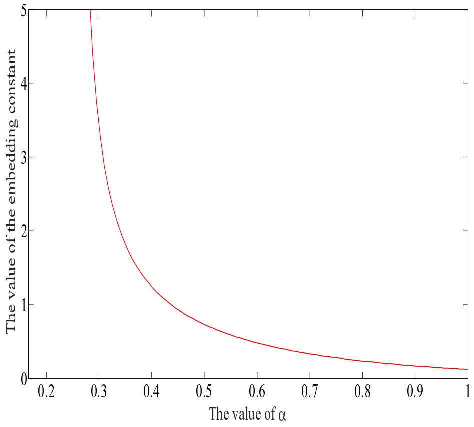


FIGURE 5.2. Values of $C_{4,\alpha}$ on $\Omega = (0,1) \times (0,1)$

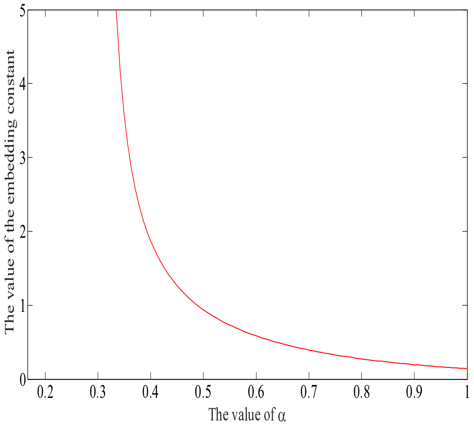


FIGURE 5.3. Values of $C_{5,\alpha}$ on $\Omega = (0,1) \times (0,1)$

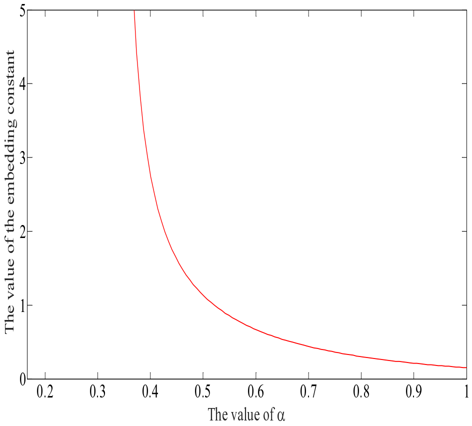


FIGURE 5.4. Values of $C_{6,\alpha}$ on $\Omega = (0,1) \times (0,1)$

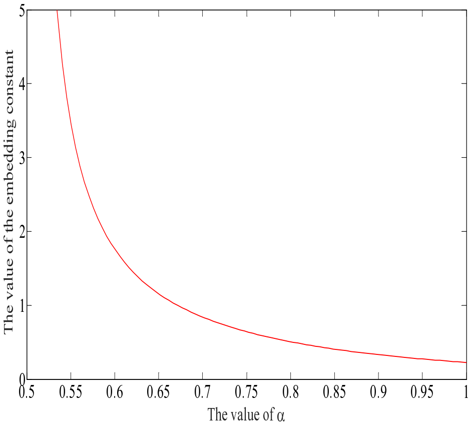


FIGURE 5.5. Values of $C_{\infty,\alpha}$ on $\Omega = (0,1) \times (0,1)$

For the second case, we selected a L-shape domain $\Omega = (0,2)\times(0,2)\setminus[1,2]\times[1,2]$. Then, the least eigenvalue over the domain Ω is included in $[9.639717, 9.639724]$ [**17**]. Varying $p = 3,\ 4\ ,5\ ,6$ and α such that $1/2-1/p < \alpha \leq 1$, the values of the right hand side of (29) in Theorem 5.1.1 are plotted on the domain $\Omega = (0,2)\times(0,2)\setminus[1,2]\times[1,2]$ in Figure 5.6, Figure 5.7, Figure 5.8, and Figure 5.9.

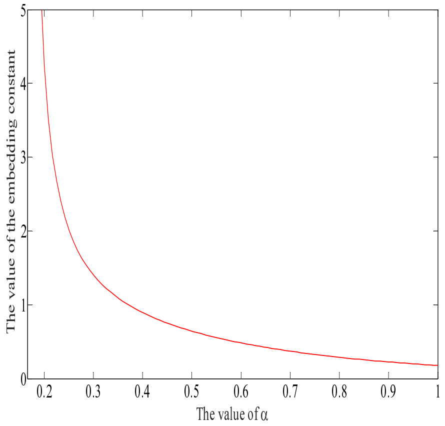


FIGURE 5.6. Values of $C_{3,\alpha}$ on $\Omega = (0,2)\times(0,2)\setminus[1,2]\times[1,2]$

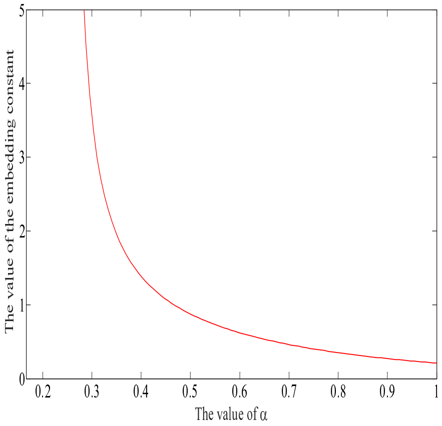


FIGURE 5.7. Values of $C_{4,\alpha}$ on $\Omega = (0,2)\times(0,2)\setminus[1,2]\times[1,2]$

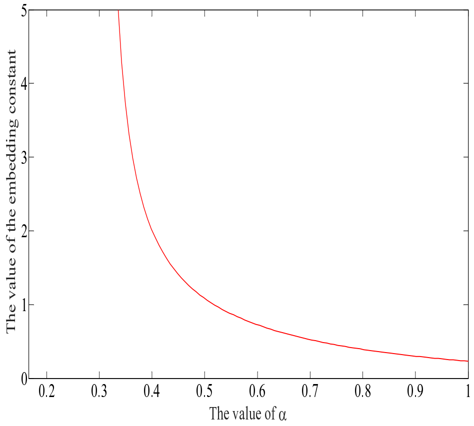


FIGURE 5.8. Values of $C_{5,\alpha}$ on $\Omega = (0,2)\times(0,2)\setminus[1,2]\times[1,2]$

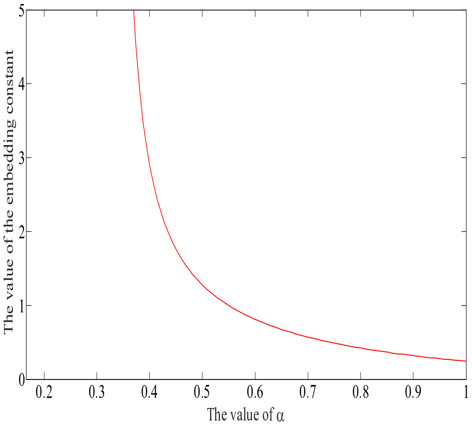


FIGURE 5.9. Values of $C_{6,\alpha}$ on $\Omega = (0,2)\times(0,2)\setminus[1,2]\times[1,2]$

Furthermore, we recall that Theorem 5.1.1 enables us to estimate $C_{\infty,\alpha}$ for $\alpha > 1/2$. Figure 5.10 shows the values of the right hand side of (29) with $p = \infty$ in Theorem 5.1.1 for $1/2 < \alpha \leq 1$ on the domain $\Omega = (0,2) \times (0,2) \setminus [1,2] \times [1,2]$.

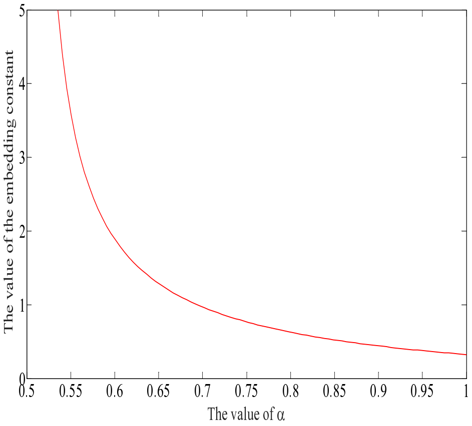


FIGURE 5.10. Values of $C_{\infty,\alpha}$ on $\Omega = (0,2) \times (0,2) \setminus [1,2] \times [1,2]$

CHAPTER 6

CONCLUSION

This thesis presents a verification algorithm for enclosing a global-in-time solution of (1) and estimation of the constant $C_{p,\alpha}$ satisfying (3). The organization of this thesis is given in the followings.

In Chapter 1, we provide the abstract of this thesis and the background regarding to the author's study. In Chapter 2, we also provide notation and several lemmas required to present the verification algorithm. In Chapter 3, we present the numerical verification algorithm to enclose a global-in-time solution, which exponentially converges to a stationary solution. First, in Section 3.1, we propose a sufficient condition (Theorem 3.1.1) to enclose a global-in-time solution. Next, in Section 3.2, we present a sufficient condition to guarantee the existence and the local-in-time uniqueness (Theorem 3.2.1) of a mild solution to (1) in a certain time interval $t \in J = (t_0, t_1]$ $(0 \leq t_0 < t_1 < \infty)$. Then, we give an a posteriori error estimate in Corollary 3.2.1. Finally, in Section 3.3, a procedure of the verification algorithm is given on the basis of Theorems 3.1.1 and 3.2.1, and Corollary 3.2.1. In Chapter 4, we present several semilinear heat equations of the form (1). Then, we derive global-in-time solutions of these equations by the algorithm. In Chapter 5, we propose estimation of the constant $C_{p,\alpha}$ defined by (3) in Theorem 5.1.1. Moreover, we show several values of the estimation in Theorem 5.1.1 over a square domain and a L-shape domain.

Consequently, we show the existence of the global-in-time solution to several semilinear heat equations of the form (1) using the verification algorithm. Furthermore, we provide the quantitative estimation of the constant $C_{p,\alpha}$.

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