21COE-GLOPE

21COE-GLOPE Working Paper Series

Extensive social choice and the problem of paternalism

Kohei Kamaga

December 2006

Revised August 2007

Working Paper No. 10

Extensive social choice and the problem of paternalism

Kohei Kamaga*

Graduate School of Economics, Waseda University,

1-6-1 Nishi-waseda, Shinjuku-ku, Tokyo 169-8050, Japan

e-mail: k-kmg@ruri.waseda.jp

Abstract

There are two different interpretations of extensive social choice: one is social decision making of multiple social planners, and the other is the aggregation of citizens' interpersonal comparisons about their well-being. This paper, based on the latter interpretation, focuses on situations where an individual's opinion is in conflict with the paternalistic opinion of some other individual. We formulate some alternative resolutions to such conflicting situations, and then we propose an admissible (class of) non-dictatorial aggregation rule(s) for each of the resolutions. The axiomatic characterizations of these new (classes of) rule(s) are established.

*I wish to thank Ryo-Ichi Nagahisa, Hiroo Sasaki, Koichi Suga, and Ai Takeuchi for their helpful comments. I am grateful to Waseda University 21 COE-GLOPE for research support.

1 Introduction

As reviewed in Bossert and Weymark [2] and d'Aspremont and Gevers [3], if interpersonal comparisons of individuals' well-being or utilities are made in a certain way, we can construct an ordering of social alternatives in a non-dictatorial way. The intended interpretation of the situation considered there is that we entrust the social decision making to a single social planner who makes interpersonal comparisons of the individuals' well-being, or that all individuals in the society have the same interpersonal comparisons about their well-being, i.e. the assumption of *complete identity* in Sen [11]. Extensive social choice considers a more general situation where multiple social planners or all individuals in society have opinions, perhaps conflicting opinions, about the well-being of each member of the society and explores the way of reconciling these opinions.

This extended framework can be traced back to Suppes' [13] formulation of the grading principle in a two-person model. Although several impossibility results had been obtained in the literature,[1] Ooghe and Lauwers [7] recently established some non-dictatorial rules.[2] In their paper, extensive social choice is interpreted as the framework that aims at the reconciliation of different social planners' opinions. Therefore, their analysis is not specifically based on the interpretation that the extensive social choice deals with the aggregation of all the individuals' interpersonal comparisons about their well-being.[3] According to the difference between these two interpretations of the extensive social choice, an issue to which we should pay particular attention will change.

For example, the following two-opinion conflict is a specific issue to be addressed under the latter interpretation: while an individual, say i, prefers an alternative x to another alternative y, some other individual, say j, thinks that if s/he were in i's position, s/he would be better in y than in x. In this situation a conflict arises between i's opinion about her/his own welfare and j's *paternalistic* opinion about i's welfare. In the literature, the problem of paternalistic opinions has been considered in Sen [11], Roberts [9], [10], and Suzumura [14] but they had, in a way, avoided this problem by restricting the domain of an aggregation rule so as to exclude paternalistic opinions (i.e. the assumption called *identity* or *non-paternalistic unrestricted domain*).

This paper explores a way of reconciling citizens', not social planners', interpersonal comparisons about their well-being, especially focusing on the problem of paternalistic opinions. Instead of assuming the limited domain and avoiding problem, we take a different approach and try to tackle this problem. In

[1] See, for example, Sen [11], Roberts [9], [10], and Suzumura [14].

[2] Some other non-dictatorial possibilities have also been obtained in the literature; e.g. the extension of Borda ranking rule in Gaertner [4] and the egalitarian consensus rules in Ooghe [6].

[3] It should be mentioned, however, that they regarded the latter interpretation as a special case of the former, i.e. the case where every citizen plays the role of a social planner.

analogy to the *two-person situations* considered in Sen [12] and Hammond [5], we focus on the situations where a conflict arises *solely* between the two opinions: one is an individual's, say i's, opinion about her/his own welfare, and the other is some other individual's paternalistic opinion about i's welfare. We formulate some alternative axioms that prescribe resolutions to such conflicting situations. Then, for each of these new axioms, we propose an admissible (class of) non-dictatorial rule(s) and explore the axiomatic characterizations of them.

The paper is organized as follows. The next section introduces the basic framework of extensive social choice. In Section 3, the conflicting situation of our interest is formally stated, and some alternative resolutions to this conflicting situation are formulated. For each of the resolutions, Section 4 proposes an admissible (class of) non-dictatorial rule(s). Section 5 establishes the axiomatic characterizations of these new rules. Section 6 concludes.

2 Framework

Let X be a set of at least three social alternatives. $N = \{1,\ldots,n\}$ is the set of n individuals. We assume $n \geq 2$. In the framework of extensive social choice, each individual plays two roles; one is a provider of opinions about each individual's welfare, and the other is a receiver of welfare. We write N^2 as $N^2 = N \times N$. For each pair of an opinion provider and a welfare receiver $(i, j) \in N^2$, let $U^i_j : X \to \mathbb{R}$ be a real-valued function. For each alternative $x \in X$, $U^i_j(x)$ represents the utility obtained by an individual j according to an individual i when the alternative x is realized. If $i = j$, $U^i_j(x)$ is interpreted as i's utility usually considered in economics. On the other hand, if $i \neq j$, the intended interpretation of $U^i_j(x)$ is i's evaluation about the welfare of j. Let U be a profile of n^2-tuple of utility functions such that $U = (U^1_1, U^1_2, \ldots, U^n_n)$. $\mathscr{U}$ (resp. $\mathscr{U}_{++}$) collects all logically possible real-valued (resp. positive real-valued) utility functions. $\mathscr{R}$ denotes a set of all logically possible orderings on X. An *extensive social welfare functional* is a mapping that assigns a social ordering of the alternatives to each profile in its domain denoted by $\mathscr{D}$; formally, $f : \mathscr{D} \to \mathscr{R}$. In this paper, we consider the following two cases; $\mathscr{D} = \mathscr{U}^{n^2}$, or $\mathscr{D} = \mathscr{U}^{n^2}_{++}$.

As in the papers of Ooghe and Lauwers [7] and Roberts [10], we assume that an extensive social welfare functional satisfies the well-established condition called *strong neutrality* that allows us to focus on a social ordering R defined on the n^2-dimensional attainable utility space $\{(u^1_1, u^1_2, \ldots, u^n_n) : u^i_j = U^i_j(x) \; \forall i, j \in N,\, U \in \mathscr{D},\, x \in X\}$, rather than on an extensive social welfare functional. We call R *extensive social welfare ordering* (hereafter, *ESWO*), and write P (resp. I) as an asymmetric (a symmetric) part of R. In the case of $\mathscr{D} = \mathscr{U}^{n^2}$ (resp. $\mathscr{D} = \mathscr{U}^{n^2}_{++}$), the attainable utility space is $\mathbb{R}^{n^2}$ (resp. $\mathbb{R}^{n^2}_{++}$). In the rest of the paper, we use the term "profile" to mean a utility vector $u = (u^1_1, u^1_2, \ldots, u^n_n)$ of $\mathbb{R}^{n^2}$ (or $\mathbb{R}^{n^2}_{++}$), and axioms will be defined in terms of an ESWO R.

Finally, we introduce two alternative assumptions on measurability and interpersonal comparability of utility. In the literature, such informational assumptions are formalized in terms of an invariance property of social orderings. This paper considers the following two assumptions.

Translation-scale measurability and full comparability (TSF): $\forall u, v, w, z \in \mathbb{R}^{n^2}$, if there exist $\beta_1, \ldots, \beta_n \in \mathbb{R}$ such that $w^i_j = \beta_i + u^i_j$ and $z^i_j = \beta_i + v^i_j$ $\forall i, j \in N$, then $uRv \Leftrightarrow wRz$.
Ratio-scale measurability and full comparability (RSF): $\forall u, v, w, z \in \mathbb{R}^{n^2}_{++}$, if there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{R}_{++}$ such that $w^i_j = \alpha_i \cdot u^i_j$ and $z^i_j = \alpha_i \cdot v^i_j$ $\forall i, j \in N^2$, then $uRv \Leftrightarrow wRz$.

These two assumptions are adopted in Ooghe and Lauwers [7]. In the case of ratio-scale measurable utilities, we restrict our attention to $\mathbb{R}^{n^2}_{++}$. The interpretation of TSF (resp. RSF) is that, within each individual's evaluation, interpersonal comparisons of utility levels and differences are possible, and moreover, the numerical difference between two individuals' utilities (resp. the ratio of two individuals' utilities) has meaning.[4]

3 Two-opinion situations

In this paper, we interpret that the extensive social choice deals with the aggregation of citizens' interpersonal comparisons about their well-being. Under our interpretation, we should pay particular attention to the conflicting situations involving individuals' paternalistic opinions. For example, consider the following two profiles u and v in a three-person society $N = \{i, j, k\}$:

$$\begin{array}{lll} (u^i_i, v^i_i) = (4,2), & (u^i_j, v^i_j) = (5,5), & (u^i_k, v^i_k) = (6,6); \\ (u^j_i, v^j_i) = (3,5), & (u^j_j, v^j_j) = (6,6), & (u^j_k, v^j_k) = (4,4); \\ (u^k_i, v^k_i) = (4,4), & (u^k_j, v^k_j) = (7,7), & (u^k_k, v^k_k) = (8,8). \end{array}$$

The two profiles can be considered as follows: (i) the individual i thinks that s/he is better in u than in v ($u^i_i = 4 > 2 = v^i_i$); (ii) on the other hand, the individual j considers that if s/he were in i's position, s/he would be better in v than in u ($u^j_i = 3 < 5 = v^j_i$); (iii) except these two opinions, no conflict arises in the sense that we have $u^k_l = v^k_l$ $\forall (k,l) \in N^2 \backslash \{(i,i),(j,i)\}$. In other words, in the social decision making of u and v, the conflict arises *solely* between the two opinions: the individual i's opinion about her/his own welfare and j's *paternalistic* opinion concerning i's welfare. This conflicting situation can be generalized as the pair of profiles u and v that satisfies the following condition: there exist $i, j \in N$ such that $u^i_i > v^i_i$, $u^j_i < v^j_i$,

[4]In the case of TSF (resp. RSF), it is meaningful to say, for example, $u^i_j - u^i_k$ (resp. u^i_j / u^i_k) is 5.

and $u^k_l = v^k_l$ $\forall (k,l) \in N^2 \setminus \{(i,i),(j,i)\}$. Such a situation can be called the *two-opinion situation* in analogy to the *two-person situation* considered in Sen [12] and Hammond [5].

In the extensive social choice, any one utility profile in a n-person society consists of all of n^2 utility values. Because of this huge amount of utility information conflicts between opinions that we may have in this extended framework will be very complicated in most case. However, in the two-opinion situation, the conflict arises only between two opinions so that it would be easier to reach an agreement on how to resolve this two-opinion conflict. Since it is the paternalistic opinion with which some individual's opinion is in conflict in the two-opinion situation, it seems plausible to resort to the idea of non-paternalism to resolve such a two-opinion conflict. We now provide some alternative axioms which, based on the idea of non-paternalism, prescribe resolutions to the conflict in the two-opinion situation. We begin with the following axiom.

Anti-paternalistic priority to concerned individual (APCI): $\forall u, v \in \mathbb{R}^{n^2}$, if there exist $i, j \in N$ such that $u^i_i > v^i_i$, $u^j_i < v^j_i$, and $u^k_l = v^k_l$ $\forall (k,l) \in N^2 \setminus \{(i,i),(j,i)\}$, then uPv.

APCI asserts that in the two-opinion situations the opinion of a solely concerned welfare receiver, i in the definition, should be given priority to determine the social ranking as s/he prefers in her/his own position. In APCI the idea of non-paternalism is formalized as the anti-paternalistic priority given to the solely concerned welfare receiver.

Next, we introduce another example of the non-paternalistic resolution to the conflict in the two-opinion situation. In the two-opinion situations, it is also plausible to assert that we should give equal priority to both of the conflicting two opinions. This idea is formalized as follows.

Equal priority to conflicting opinions (EPCO): $\forall u, v \in \mathbb{R}^{n^2}$, if there exist $i, j \in N$ such that $u^i_i > v^i_i$, $u^j_i < v^j_i$, and $u^k_l = v^k_l$ $\forall (k,l) \in N^2 \setminus \{(i,i),(j,i)\}$, then uIv.

In contrast to APCI, EPCO requires that the conflicting two opinions should be treated equally so that the two profiles are concluded to be socially indifferent.

EPCO, as well as APCI, seems reasonable property to be satisfied in the two-opinion situations. However, EPCO leads to the following undesirable result.

Remark 1. If an ESWO R satisfies EPCO, then all utility profiles must be socially indifferent. This can be easily checked as follows. Consider any $u, v \in \mathbb{R}^{n^2}$ such that $u^i_j > v^i_j$ for some $(i,j) \in N^2$ and $u^k_l = v^k_l$ $\forall (k,l) \in N^2 \setminus \{(i,j)\}$. If $i = j$, we can find $w \in \mathbb{R}^{n^2}$ such that (i) $u^i_i > v^i_i > w^i_i$; (ii) $w^m_i > u^m_i = v^m_i$ for some individual $m \neq i$; (iii) $u^k_l = v^k_l = w^k_l$ $\forall (k,l) \in N^2 \backslash \{(i,i),(m,i)\}$. By EPCO, uIw and wIv, and the transitivity of R gives uIv. If $i \neq j$, we can find $w \in \mathbb{R}^{n^2}$ such that (i) $u^j_j = v^j_j > w^j_j$; (ii) $w^i_j > u^i_j > v^i_j$; (iii) $u^k_l = v^k_l = w^k_l$ $\forall (k,l) \in N^2 \backslash \{(i,j),(j,j)\}$. By the same argument as in the case of $i = j$, uIv follows. For any pair of distinct profiles, one of the two profiles can be constructed from the other by finitely applying the above procedure. Thus, by the reflexivity and transitivity of R, any two profiles must be socially indifferent. $\square$

This remark tells that the ESWO satisfying EPCO inevitably violates unanimity conditions such as the Pareto criteria. Because of this unacceptable implication, we do not consider EPCO in the rest of the paper.

Next, we introduce weakened versions of APCI and EPCO.

Incremental anti-paternalistic priority to concerned individual (IAPCI): $\forall u, v \in \mathbb{R}^{n^2}$, if there exist $i, j \in N$ such that $u^i_i - v^i_i = v^j_i - u^j_i > 0$, and $u^k_l = v^k_l$ $\forall (k,l) \in N^2 \backslash \{(i,i),(j,i)\}$, then uPv.
Incremental equal priority to conflicting opinions (IEPCO): $\forall u, v \in \mathbb{R}^{n^2}$, if there exist $i, j \in N$ such that $u^i_i - v^i_i = v^j_i - u^j_i > 0$, and $u^k_l = v^k_l$ $\forall (k,l) \in N^2 \backslash \{(i,i),(j,i)\}$, then uIv.

IAPCI is a weakened version of APCI. It asserts that in the two-opinion situations the solely concerned welfare receiver should be given the priority to determine the social ranking only if the utility differences in the conflicting two opinions are the same in absolute value. In a similar way, IEPCO is defined as the weakened version of EPCO.[5]

Each of IAPCI and IEPCO is a reasonable alternative to APCI or EPCO in the case of translation-scale measurable utilities, but neither of them when utilities are ratio-scale measurable. In the latter case we should consider the ratio-based counterparts of IAPCI and IEPCO. The following axioms are defined in a similar way to IAPCI and IEPCO, and thus require no explanation.

Ratio-based anti-paternalistic priority to concerned individual (RAPCI): $\forall u, v \in \mathbb{R}^{n^2}_{++}$, if there exist $i, j \in N$ such that $u^i_i / v^i_i = v^j_i / u^j_i > 1$ and $u^k_l = v^k_l$ $\forall (k,l) \in N^2 \backslash \{(i,i),(j,i)\}$, then uPv.
Ratio-based equal priority to conflicting opinions (REPCO): $\forall u, v \in \mathbb{R}^{n^2}_{++}$, if there exist $i, j \in N$ such that $u^i_i / v^i_i = v^j_i / u^j_i > 1$ and $u^k_l = v^k_l$ $\forall (k,l) \in N^2 \backslash \{(i,i),(j,i)\}$, then uIv.

For each of the axioms presented above, let us now check whether the non-dictatorial rules established by Ooghe and Lauwers [7] satisfy the axiom. Under our interpretation of the extensive social choice, their

[5] IEPCO can be seen as an extension of *incremental equity* in Blackorby et al. [1].

non-dictatorial rules are defined as follows.[6]

The *utilitarian Kolm-Pollak* orderings (one for each value of $\rho \in \mathbb{R}$): $\forall u, v \in \mathbb{R}^{n^2}$,

$$uRv \Leftrightarrow \begin{cases} \rho \neq 0: & \frac{1}{\rho}\sum_{i\in N}\ln\left(\sum_{j\in N}\exp(\rho u^i_j)\right) \geq \frac{1}{\rho}\sum_{i\in N}\ln\left(\sum_{j\in N}\exp(\rho v^i_j)\right), \\ \rho = 0: & \sum_{(i,j)\in N^2} u^i_j \geq \sum_{(i,j)\in N^2} v^i_j. \end{cases}$$

The *Nash iso-elastic* orderings (one for each value of $\rho \in \mathbb{R}$): $\forall u, v \in \mathbb{R}^{n^2}_{++}$,

$$uRv \Leftrightarrow \begin{cases} \rho \neq 0: & \prod_{i\in N}\left(\sum_{j\in N}(u^i_j)^\rho\right)^{\frac{1}{\rho}} \geq \prod_{i\in N}\left(\sum_{j\in N}(v^i_j)^\rho\right)^{\frac{1}{\rho}}, \\ \rho = 0: & \prod_{(i,j)\in N^2} u^i_j \geq \prod_{(i,j)\in N^2} v^i_j. \end{cases}$$

The Nash iso-elastic orderings are defined on $\mathbb{R}^{n^2}_{++}$. When $\rho = 0$, the utilitarian Kolm-Pollak (resp. Nash iso-elastic) ordering is the *extensive utilitarian* (resp. *extensive Nash*) ordering.[7] First, we examine the utilitarian Kolm-Pollak orderings for each of the three axioms: APCI, IAPCI, and IEPCO. Note that the pair of profiles considered in the beginning of this section satisfies the antecedent of each of the three axioms. Thus, we examine how the utilitarian Kolm-Pollak orderings will rank these two profiles. From easy calculation, we obtain vPu when $\rho > 0$. Therefore, in the utilitarian Kolm-Pollak family, any rule which corresponds to $\rho > 0$ violates all of the three axioms. It is easily checked that we obtain the same conclusion in the case of $\rho < 0$.[8] In the class of the utilitarian Kolm-Pollak orderings, the extensive utilitarian ordering solely satisfies IEPCO among the three axioms. As for the Nash iso-elastic orderings, we obtain the similar conclusion for the three axioms APCI (on $\mathbb{R}^{n^2}_{++}$), RAPCI, and REPCO.[9] This observation motivates us to explore alternative aggregation rules which satisfy APCI or IAPCI (or RAPCI). The next section proposes an admissible (class of) rule(s) for each of the six axioms considered in this section.

[6] In their original definitions of these classes of rules, the set of opinion providers and that of welfare receivers are allowed to be different, and so are the size of them.

[7] The utilitarian Kolm-Pollak (resp. the Nash iso-elastic) ordering can be approximated to the *mean of mins* (resp. the (geometric) *mean of mins*) ordering as ρ approaches to $-\infty$. Formally, the mean of mins ordering is defined as $uRv \Leftrightarrow \sum_{i\in N}\min_{j\in N} u^i_j \geq \sum_{i\in N}\min_{j\in N} v^i_j$. Similarly, in the case of the Nash iso-elastic ordering, the (geometric) mean of mins ordering is defined as $uRv \Leftrightarrow \prod_{i\in N}\min_{j\in N} u^i_j \geq \prod_{i\in N}\min_{j\in N} v^i_j$.

[8] For example, consider the profiles $(-u^i_i, -u^i_j, \ldots, -u^k_k)$ and $(-v^i_i, -v^i_j, \ldots, -v^k_k)$.

[9] Consider the profiles $(\exp(u^1_1), \exp(u^1_2), \ldots, \exp(u^3_3))$ and $(\exp(v^1_1), \exp(v^1_2), \ldots, \exp(v^3_3))$ instead of u and v.

4 Anti-paternalistic priority rules and equal priority rules

We distinguish two types of an aggregation rule, *anti-paternalistic priority* rule and *equal priority* rule, in accordance with which type of the priority, anti-paternalistic or equal, is considered in the two-opinion situations. We begin with the anti-paternalistic priority rules.

4.1 Anti-paternalistic priority rules

We provide four (classes of) anti-paternalistic priority rules, i.e. the rules which satisfy at least one of the anti-paternalistic priority axioms: APCI, IAPCI, and RAPCI.

4.1.1 The *utilitarian* ordering is defined as follows; $\forall u, v \in \mathbb{R}^{n^2}$,

$$uRv \Leftrightarrow \sum_{i \in N} u_i^i \geq \sum_{i \in N} v_i^i.$$

The utilitarian ordering is the direct reformulation of the classical utilitarianism. In this rule, the social rankings depend only on the utilities evaluated in the individuals' own positions. Thus, the possible influence of sympathetic utility information is eliminated altogether. The utilitarian ordering satisfies APCI.

4.1.2 Similarly, the *Nash* ordering is defined on the positive domain; $\forall u, v \in \mathbb{R}^{n^2}_{++}$,

$$uRv \Leftrightarrow \prod_{i \in N} u_i^i \geq \prod_{i \in N} v_i^i.$$

In contrast to the utilitarian ordering, this rule considers the products of utilities evaluated in the individuals' own positions. This rule satisfies APCI (on $\mathbb{R}^{n^2}_{++}$).

4.1.3 Next, we formulate an extension of the extensive utilitarian ordering. An ESWO R is the *anti-paternalistic extensive utilitarian* ordering if there exists a n^2-dimensional vector of weights $(\alpha_1^1, \alpha_2^1, \ldots, \alpha_n^n) \in \mathbb{R}^{n^2}_{++}$ such that $\alpha_i^i = \alpha_j^j > \alpha_j^i = \alpha_l^k$ $\forall i, j, k, l \in N$ ($i \neq j$ and $k \neq l$) and we have, $\forall u, v \in \mathbb{R}^{n^2}$,

$$uRv \Leftrightarrow \sum_{(i,j) \in N^2} \alpha_j^i u_j^i \geq \sum_{(i,j) \in N^2} \alpha_j^i v_j^i.$$

Any rule out of this class determines the social rankings by the comparison of the weighted sums of utilities where utilities in individuals' own positions are more weighted than in the other positions. Each rule of this class satisfies IAPCI, but violates APCI.

4.1.4 An ESWO R defined on $\mathbb{R}^{n^2}_{++}$ is the *anti-paternalistic extensive Nash* ordering if there exists a n^2-dimensional vector of coefficients $(\alpha_1^1, \alpha_2^1, \ldots, \alpha_n^n) \in \mathbb{R}^{n^2}_{++}$ such that $\alpha_i^i = \alpha_j^j > \alpha_j^i = \alpha_l^k \; \forall i, j, k, l \in N$ ($i \neq j$ and $k \neq l$) and we have, $\forall u, v \in \mathbb{R}^{n^2}_{++}$,

$$uRv \Leftrightarrow \prod_{(i,j)\in N^2} (u_j^i)^{\alpha_j^i} \geq \prod_{(i,j)\in N^2} (v_j^i)^{\alpha_j^i}.$$

In the definition of this class, the coefficients are defined in the same way as in the anti-paternalistic extensive utilitarian orderings. Thus, any rule out of this class satisfies RAPCI, but violate APCI.

4.2 Equal priority rules

We move to the equal priority rules, i.e. the rules that satisfies IEPCO or REPCO.

4.2.1 The *Kolm-Pollak of means* orderings are defined as follows (one for each value of $\rho \in \mathbb{R}$); $\forall u, v \in \mathbb{R}^{n^2}$,

$$uRv \Leftrightarrow \begin{cases} \rho \neq 0: & \frac{1}{\rho} \ln\left(\sum_{j\in N} \exp(\rho\, \bar{u}_j)\right) \geq \frac{1}{\rho} \ln\left(\sum_{j\in N} \exp(\rho\, \bar{v}_j)\right), \\ \rho = 0: & \sum_{(i,j)\in N^2} u_j^i \geq \sum_{(i,j)\in N^2} v_j^i, \end{cases}$$

where $\bar{u}_j = \frac{1}{n}\sum_{i\in N} u_j^i$ and $\bar{v}_j = \frac{1}{n}\sum_{i\in N} v_j^i \; \forall j \in N$.

When $\rho = 0$, the Kolm-Pollak of means ordering is defined as the extensive utilitarian ordering.[10] The Kolm-Pollak of means orderings and the utilitarian Kolm-Pollak orderings are formally similar, but different in the order of the two-step calculation whenever $\rho \neq 0$. In the Kolm-Pollak of means orderings, we first derive the representative utility value for each welfare receiver as the average of all individuals' evaluations, and next we calculate the Kolm-Pollak value for each n-dimensional vector of the representative utility values. In contrast to the utilitarian Kolm-Pollak orderings, any rule in this class satisfies IEPCO.

4.2.2 Finally, we provide the *iso-elastic of means* orderings (one for each value of $\rho \in \mathbb{R}$); $\forall u, v \in \mathbb{R}^{n^2}_{++}$,

$$uRv \; \Leftrightarrow \begin{cases} \rho \neq 0: & \left(\sum_{j\in N} (\bar{u}_j)^\rho\right)^{\frac{1}{\rho}} \geq \left(\sum_{j\in N} (\bar{v}_j)^\rho\right)^{\frac{1}{\rho}}, \\ \rho = 0: & \prod_{(i,j)\in N^2} u_j^i \geq \prod_{(i,j)\in N^2} v_j^i, \end{cases}$$

[10] As ρ approaches to $-\infty$, this rule can be approximated to the *min of means* ordering, i.e. $uRv \Leftrightarrow \min_{j\in N} \sum_{i\in N} u_j^i \geq \min_{j\in N} \sum_{i\in N} v_j^i$.

where $\bar{u}_j = \left(\prod_{i\in N} u^i_j\right)^{\frac{1}{n}}$ and $\bar{v}_j = \left(\prod_{i\in N} v^i_j\right)^{\frac{1}{n}}$ $\forall j \in N$.

Each rule of this class is defined on $\mathbb{R}^{n^2}_{++}$. When $\rho = 0$, it is the extensive Nash ordering.[11] The similar discussion to the case of the Kolm-Pollak of means orderings can be applied to explain the difference between the iso-elastic of means orderings and the Nash iso-elastic orderings. It is obvious that any rule in this class satisfies REPCO.

5 Characterizations

This section explores the axiomatic characterizations of the rules proposed in the last section. To this end, let us introduce some usual axioms. When we consider ratio-scale measurable utilities, these axioms are assumed to be defined on $\mathbb{R}^{n^2}_{++}$.

The following four axioms are straightforward and require no explanation.

Weak Pareto (WP): $\forall u, v \in \mathbb{R}^{n^2}$, if $u^i_j > v^i_j$ $\forall (i,j) \in N^2$, then uPv.
Strong Pareto (SP): $\forall u, v \in \mathbb{R}^{n^2}$, if $u^i_j \geq v^i_j$ $\forall (i,j) \in N^2$, then uRv, moreover, if there exists a pair $(k,l) \in N^2$ such that $u^k_l > v^k_l$, then uPv.
Anonymity (AN): $\forall u, v \in \mathbb{R}^{n^2}$, if there exists a bijection $\pi : N \to N$ such that $u^i_j = v^{\pi(i)}_{\pi(j)}$ $\forall i, j \in N$, then uIv.
Continuity (CON): $\forall u \in \mathbb{R}^{n^2}$, both $\{v \in \mathbb{R}^{n^2} : vRu\}$ and $\{v \in \mathbb{R}^{n^2} : uRv\}$ are closed with respect to the Euclidean topology.

Next, we introduce two separability conditions.

Separability between opinion providers (SE^{P}): $\forall u, v, w, z \in \mathbb{R}^{n^2}$, if there exists $M \subseteq N$ such that $u^i_j = v^i_j$ and $w^i_j = z^i_i$ $\forall i \in M$ $\forall j \in N$, and $u^i_j = w^i_j$ and $v^i_j = z^i_j$ $\forall i \in N \backslash M$ $\forall j \in N$, then $uRv \Leftrightarrow wRz$.
Separability between receivers of welfare (SE_{R}): $\forall u, v, w, z \in \mathbb{R}^{n^2}$, if there exists $M \subseteq N$ such that $u^i_j = v^i_j$ and $w^i_j = z^i_j$ $\forall i \in N$ $\forall j \in M$, and $u^i_j = w^i_j$ and $v^i_j = z^i_j$ $\forall i \in N$ $\forall j \in N \backslash M$, then $uRv \Leftrightarrow wRz$.

These two axioms are similar but different in the definition of the subgroup M of N. In SE^{P}, the subgroup M is defined as the set of unconcerned opinion providers who consider that every individual in the society is equally well-off across the two profiles. SE^{P} requires that the opinions of these unconcerned providers

[11] As ρ approaches to $-\infty$, the iso-elastic of means ordering can be approximated to the min of (geometric) means ordering, i.e. $uRv \Leftrightarrow \min_{j\in N} \prod_{i\in N} u^i_j \geq \min_{j\in N} \prod_{i\in N} v^i_j$

have no impact on the social decision making of the two profiles at all.[12] Similarly, $\mathrm{SE_R}$ asserts that the social ranking is determined independently of the unconcerned welfare receivers defined as the subgroup M.

It should be noted that, in the paper of Ooghe and Lauwers [7], the utilitarian Kolm-Pollak (resp. the Nash iso-elastic) family is characterized by the set of seven axioms: TSF (resp. RSF), SP, CON, $\mathrm{SE^P}$, another separability axiom weaker than $\mathrm{SE_R}$, and two anonymity axioms whose conjunction is logically stronger than AN.[13] Table 1 summarizes the properties of the utilitarian Kolm-Pollak orderings and the Nash iso-elastic orderings. For each row in Table 1, each rule (out of the class) in the first column satisfies the two-opinion property and the invariance condition in the second and third columns, and also satisfies the axioms indicated by $+$.

Table 1: Properties of util. Kolm-Pollak and Nash iso-elastic families

	two-opinion	Inv	WP	SP	AN	CON	$\mathrm{SE^P}$	$\mathrm{SE_R}$
util. Kolm-Pollak		TSF	$+$	$+$	$+$	$+$	$+$	
($\rho = 0$) extens. util.	IEPCO	TSF	$+$	$+$	$+$	$+$	$+$	$+$
Nash iso-elastic		RSF	$+$	$+$	$+$	$+$	$+$	
($\rho = 0$) extens. Nash	REPCO	RSF	$+$	$+$	$+$	$+$	$+$	$+$

We are ready to state our characterization results. First, we consider the case of translation-scale measurable utilities.

Proposition 1. *For each row in the table ($n \geq 3$ in the third row), an ESWO satisfies the two-opinion property in combination with the axioms indicated by $\bullet$ if and only if it is the rule (out of the class) in the first column. In addition, $+$ (resp. $-$) means that each rule (out of the class) satisfies (resp. violates) the axiom:*

	two-opinion	TSF	WP	SP	AN	CON	$\mathrm{SE^P}$	$\mathrm{SE_R}$
utilitarian	APCI	$\bullet$	$\bullet$	$-$	$\bullet$	$\bullet$	$+$	$+$
anti-paternalistic extens. util.	IAPCI	$\bullet$	$+$	$\bullet$	$\bullet$	$\bullet$	$+$	$\bullet$
Kolm-Pollak of means ($n \geq 3$)	IEPCO	$\bullet$	$+$	$\bullet$	$\bullet$	$\bullet$		$\bullet$
extensive utilitarian	IEPCO	$+$	$+$	$\bullet$	$\bullet$	$+$	$\bullet$	$+$

Proof. See Appendix A. ■

[12] $\mathrm{SE^P}$ is called *separability between planners* in Ooghe and Lauwers [7].

[13] On the three additional axioms in the list, we refer the reader to Ooghe and Lauwers [7].

As we have seen in Table 1, each rule out of the utilitarian Kolm-Pollak family satisfies TSF, WP, AN, and CON. Thus, given these conditions, the class of admissible ESWOs contains the utilitarian Kolm-Pollak orderings. Proposition 1, however, tells that if we add APCI in the list of axioms, the utilitarian Kolm-Pollak orderings are no longer allowable and the utilitarian ordering is solely admissible. Note that the utilitarian ordering violates SP. Thus, if we replace WP with SP, we are immediately led to the impossibility result that there exists no ESWO that satisfies APCI, SP, AN, and CON. This impossibility is ascribed to the incompatibility between the three axioms APCI, SP, and CON (see Lemma 2 in Appendix A). Our proposition also shows that the anti-paternalistic extensive utilitarian family and the Kolm-Pollak of means family are characterized by the same properties except for the two-opinion axiom. IAPCI is essential in the former result, and IEPCO in the latter. Finally, our proposition provides the axiomatization of the extensive utilitarian ordering without any invariance condition. This is an alternative characterization to the result of Ooghe and Lauwers [7].[14] We should note that if IEPCO is added into the list of axioms that characterizes the utilitarian Kolm-Pollak family, an admissible ESWO is only the extensive utilitarian ordering. Our characterization, however, shows that in the presence of IEPCO it is sufficient to assume only the three additional axioms SP, AN, and SE^{P} to obtain the same result.

Using the logarithmic function, Proposition 1 can be directly applied to the case of ratio-scale measurable utilities. We close this section with providing the following result.

Proposition 2. *For each row in the table ($n \geq 3$ in the third row), an ESWO defined on $\mathbb{R}^{n^2}_{++}$ satisfies the two-opinion property in combination with the axioms indicated by $\bullet$ if and only if it is the rule (out of the class) in the first column. In addition, $+$ (resp. $-$) means that each rule (out of the class) satisfies (resp. violates) the axiom:*

	two-opinion	RSF	WP	SP	AN	CON	SE^{P}	SE_{R}
Nash	APCI	$\bullet$	$\bullet$	$-$	$\bullet$	$\bullet$	$+$	$+$
anti-paternalistic extens. Nash	RAPCI	$\bullet$	$+$	$\bullet$	$\bullet$	$\bullet$	$+$	$\bullet$
iso-elastic of means ($n \geq 3$)	REPCO	$\bullet$	$+$	$\bullet$	$\bullet$	$\bullet$		$\bullet$
extensive Nash	REPCO	$+$	$+$	$\bullet$	$\bullet$	$+$	$\bullet$	$+$

Proof. Let R be an ESWO on $\mathbb{R}^{n^2}_{++}$, and define the following ESWO R' on $\mathbb{R}^{n^2}$;

$$\forall u, v \in \mathbb{R}^{n^2}_{++},\ (\ln(u^1_1), \ln(u^1_2), \ldots, \ln(u^n_n))R'(\ln(v^1_1), \ln(v^1_2), \ldots, \ln(v^n_n)) \Leftrightarrow uRv.$$

[14] On this, see their Lemma 1.

It is easily checked that R satisfies the axioms stated in Proposition 2 if and only if R' satisfies the corresponding axioms in Proposition 1. Applying Proposition 1, easy calculation completes the rest of the proof. ■

6 Conclusion

In the two propositions, we established the axiomatic characterizations of the six new (classes of) rules proposed in Section 4. These axiomatizations help us to classify the rules in accordance with which type of the priority we consider in the two-opinion situations. In the case of translation-scale measurable utilities, if the anti-paternalistic priority defined as APCI is given to a solely concerned welfare receiver, an ESWO that satisfies the three basic conditions: weak Pareto, anonymity and continuity, must be the utilitarian ordering. This rule is the most parsimonious towards utility information among the rules characterized in Proposition 1 in the sense that it eliminates the possible influence of sympathetic utility information altogether. On the other hand, weakening the anti-paternalistic priority, it becomes possible to consider the rules that utilize every component of n^2-dimensional utility vectors. In the case of IAPCI the anti-paternalistic extensive utilitarian orderings are admissible, and in the case of IEPCO the Kolm-Pollak of means orderings. The Kolm-Pollak of means family is formally similar to the utilitarian Kolm-Pollak family. Ooghe et al. [8] provided the discussion about the difference between them in the context of the utilitarian approach to the equality of opportunity.[15] In extensive social choice, the difference between these two classes can be explained in terms of the resolution to the conflict in the two-opinion situation; the former satisfies the two-opinion property IEPCO, but, as we have seen in Section 3, the latter, except the case of $\rho = 0$, violates any two-opinion properties. The similar observation follows for the results in Proposition 2.

Appendix A: Proof of Proposition 1

It is easy to verify that whether each of the rules satisfies or violates the axiom. In what follows, we prove "only if" part of "if and only if" statement. For each characterization, the independence of the axioms will be established in Appendix B.

A.1. An ESWO that satisfies APCI, TSF, WP, AN, and CON must be the utilitarian ordering.

Proof. The proof proceeds through two lemmata.

[15] More precisely, they examined the difference between the mean of mins rule (also called the Roemer rule) and the min of means rule (also called the Van de gaer rule).

Lemma 1. *If an ESWO R satisfies APCI and WP, then,* $\forall u, v \in \mathbb{R}^{n^2}$, $[u_i^i > v_i^i \, \forall i \in N] \Rightarrow uPv$.

Proof of Lemma 1. Fix a pair of distinct individuals (i, j) arbitrarily. First, we will show that uPv follows for any $u, v \in \mathbb{R}^{n^2}$ such that $u_l^k > v_l^k \, \forall (k,l) \in N^2 \backslash \{(i, j)\}$. Consider any $u, v \in \mathbb{R}^{n^2}$ satisfying this condition. We can find $w \in \mathbb{R}^{n^2}$ such that (i) $u_j^j > w_j^j > v_j^j$; (ii) $w_j^i > u_j^i$ and $w_j^i > v_j^i$; (iii) $u_l^k = w_l^k \; (> v_l^k) \; \forall (k,l) \in N^2 \backslash \{(j, j), (i, j)\}$. By APCI and WP, we have uPw and wPv. Since R is transitive, we obtain uPv. Next, using this intermediate result, we complete the proof. Choose any $u, v \in \mathbb{R}^{n^2}$ such that $u_i^i > v_i^i \, \forall i \in N$. We want to show uPv. Let $\varepsilon_i = \frac{u_i^i - v_i^i}{n^2}$ for each $i \in N$, and $\delta \in \mathbb{R}_{++}$. We consider the following operation: choose $(i, j) \in N^2$ arbitrarily, and construct a new vector $z \in \mathbb{R}^{n^2}$ from a vector $w \in \mathbb{R}^{n^2}$ in the following way;

(i) if $i \neq j$ then,

$z_j^i = w_j^i + (n^2 - 1)\delta - (u_j^i - v_j^i)$,

$z_k^k = w_k^k - \varepsilon_k \; \forall k \in N$,

$z_l^k = w_l^k - \delta \; \forall (k,l) \in N^2 \backslash \{(1,1), (2,2), \ldots, (n,n), (i, j)\}$;

(ii) if $i = j$ then,

$z_k^k = w_k^k - \varepsilon_k \; \forall k \in N$,

$z_l^k = w_l^k - \delta \; \forall (k,l) \in N^2 \backslash \{(1,1), (2,2), \ldots, (n,n)\}$.

Applying the intermediate result, we obtain wPz. Invoking the operation once to each pair $(i, j) \in N^2$ repeatedly, we can construct v from u. The transitivity of R gives uPv. ■

Lemma 2. *If an ESWO R satisfies APCI, WP, and CON, then there exists an ordering $\hat{R}$ defined on $\mathbb{R}^n$ such that,* $\forall u, v \in \mathbb{R}^{n^2}$,

$$uRv \Leftrightarrow (u_1^1, u_2^2, \ldots, u_n^n)\hat{R}(v_1^1, v_2^2, \ldots, v_n^n).$$

Proof of Lemma 2. Define $\hat{\mathscr{U}}$ as $\hat{\mathscr{U}} = \{w \in \mathbb{R}^{n^2} : w_i^i = w_i^j \; \forall i, j \in N\}$. Note that there exists a bijection from $\hat{\mathscr{U}}$ to $\{(w_1^1, w_2^2, \ldots, w_n^n) : w \in \hat{\mathscr{U}}\} = \mathbb{R}^n$. Thus, we can define an ordering $\hat{R}$ on $\mathbb{R}^n$ such that; $\forall u, v \in \hat{\mathscr{U}}$,

$$uRv \Leftrightarrow (u_1^1, u_2^2, \ldots, u_n^n)\hat{R}(v_1^1, v_2^2, \ldots, v_n^n).$$

We want to show that, for any $u, v \in \mathbb{R}^{n^2}$, $uRv \Leftrightarrow (u_1^1, u_2^2, \ldots, u_n^n)\hat{R}(v_1^1, v_2^2, \ldots, v_n^n)$ follows. Consider any $u, v \in \mathbb{R}^{n^2}$. We can find $w, z \in \hat{\mathscr{U}}$ such that $w_i^i = u_i^i$ and $z_i^i = v_i^i \; \forall i \in N$. By definition, we have $wRz \Leftrightarrow (u_1^1, u_2^2, \ldots, u_n^n)\hat{R}(v_1^1, v_2^2, \ldots, v_n^n)$. Since Lemma 1 and CON together imply that $u'Iv'$ follows for any $u', v' \in \mathbb{R}^{n^2}$ such that $u_i'^i = v_i'^i \; \forall i \in N$, we obtain uIw and vIz. By the transitivity of R, $uRv \Leftrightarrow wRz$. Hence, combining the equivalence assertions, we obtain $uRv \Leftrightarrow (u_1^1, u_2^2, \ldots, u_n^n)\hat{R}(v_1^1, v_2^2, \ldots, v_n^n)$. ■

Now, applying Lemmata 1 and 2, there exists the ordering $\hat{R}$ defined in Lemma 2. Since R satisfies TSF, WP, and CON, it is easily checked that $\hat{R}$ inherits the following properties P1-P3, P1: $\forall u, v, w, z \in \mathbb{R}^n$, if there exist $\beta_1, \ldots, \beta_n \in \mathbb{R}$ such that $w_i = \beta_i + u_i$ and $z_i = \beta_i + v_i$ $\forall i \in N$, then $u\hat{R}v \Leftrightarrow w\hat{R}z$;

P2: $\forall u, v \in \mathbb{R}^n$, if $u_i > v_i$ $\forall i \in N$, then $u\hat{P}v$;

P3: $\forall u \in \mathbb{R}^n$, both $\{v \in \mathbb{R}^n : v\hat{R}u\}$ and $\{v \in \mathbb{R}^n : u\hat{R}v\}$ are closed with respect to the Euclidean topology.

From Theorem 8.1 in Bossert and Weymark [2], there exists $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ such that, $\forall u, v \in \mathbb{R}^n$, $u\hat{R}v \Leftrightarrow \sum_{i \in N} \alpha_i u_i \geq \sum_{i \in N} \alpha_i v_i$. By AN, we have $\alpha_i = \alpha_j > 0$ $\forall i, j \in N$. ■

A.2. An ESWO that satisfies IAPCI, TSF, SP, AN, CON, and $\mathrm{SE_R}$ must be the anti-paternalistic extensive utilitarian ordering.

Proof. We begin with the following lemma.

Lemma 3. *If an ESWO R satisfies IAPCI, TSF, SP, AN, CON, and SE_R, then there exists a n^2-dimensional vector of weights $(\alpha^1_1, \alpha^1_2, \ldots, \alpha^n_n) \in \mathbb{R}^{n^2}_{++}$ such that $\sum_{i \in N} \alpha^i_j = 1$ and $\alpha^j_j > \alpha^k_j = \alpha^l_j$ $\forall j \in N$ $\forall k, l \neq j$ and the following ordering $\tilde{R}$ on $\mathbb{R}^n$ is well-defined: $\forall u, v \in \mathbb{R}^{n^2}$,*

$$uRv \Leftrightarrow \Big(\textstyle\sum_{i \in N} \alpha^i_1 u^i_1, \ldots, \sum_{i \in N} \alpha^i_n u^i_n\Big) \tilde{R} \Big(\sum_{i \in N} \alpha^i_1 v^i_1, \ldots, \sum_{i \in N} \alpha^i_n v^i_n\Big).$$

Proof of Lemma 3. The proof proceeds in three steps.

Step 1. Fix $m \in N$ arbitrarily. By $\mathrm{SE_R}$, we can define an ordering R_m on $\mathbb{R}^n$ as follows; $\forall u, v \in \mathbb{R}^{n^2}$ such that $u^i_j = v^i_j$ $\forall i \in N$ $\forall j \in N \setminus \{m\}$,

$$uRv \Leftrightarrow (u^1_m, u^2_m, \ldots, u^n_m) R_m (v^1_m, v^2_m, \ldots, v^n_m).$$

Since R satisfies TSF, SP, and CON, it is obvious that R_m inherits the properties P1, P2, and P3 in A.1. Thus, by the same argument as in A.1, there exists $(\alpha^1_m, \ldots, \alpha^n_m) \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ such that, $\forall u, v \in \mathbb{R}^{n^2}$ with $u^i_j = v^i_j$ $\forall i \in N$ $\forall j \in N \setminus \{m\}$,

$$uRv \Leftrightarrow \textstyle\sum_{i \in N} \alpha^i_m u^i_m \geq \sum_{i \in N} \alpha^i_m v^i_m.$$

We can assume that $\sum_{i \in N} \alpha^i_m$ is normalized to 1. Since R satisfies IAPCI and SP, we have $\alpha^m_m > \alpha^i_m > 0$ $\forall i \in N \setminus \{m\}$. Moreover, by AN, $\alpha^i_m = \alpha^j_m$ $\forall i, j \in N \setminus \{m\}$. Since m was arbitrarily chosen, this result can be applied to any $m \in N$.

Step 2. Let $(\alpha^1_1, \alpha^1_2, \ldots, \alpha^n_n) \in \mathbb{R}^{n^2}_{++}$ be composed of the weights considered in Step 1. Choose any $u, v \in \mathbb{R}^{n^2}$ such that $\sum_{i \in N} \alpha^i_j u^i_j = \sum_{i \in N} \alpha^i_j v^i_j$ $\forall j \in N$. We will show that uIv follows. Define the following sequence of utility profiles of $\mathbb{R}^{n^2}$, $\{u_{(l)}\}^{l=n}_{l=0}$;

$u_{(0)} : {u_{(0)}}^i_j = u^i_j$ $\forall i, j \in N$,

$u_{(1)} : {u_{(1)}}_1^i = v_1^i \ \forall i \in N$, and ${u_{(1)}}_l^k = {u_{(0)}}_l^k \ \forall k \in N \ \forall l \in N \backslash \{1\}$,
$u_{(2)} : {u_{(2)}}_2^i = v_2^i \ \forall i \in N$, and ${u_{(2)}}_l^k = {u_{(1)}}_l^k \ \forall k \in N \ \forall l \in N \backslash \{2\}$,
$\vdots$
$u_{(n)} : {u_{(n)}}_n^i = v_n^i \ \forall i \in N$, and ${u_{(n)}}_l^k = {u_{(n-1)}}_l^k \ \forall k \in N \ \forall l \in N \backslash \{n\}$.
By definition, $u_{(n)} = v$. Applying the result of Step 1, we have $u_{(l-1)} I u_{(l)} \ \forall l \in \{1, \dots, n\}$. By the transitivity of R, uIv is obtained.

Step 3. This step completes the proof. Let $\hat{\mathscr{U}}$ be the same set as in the proof of Lemma 2. By the same argument as in the proof of Lemma 2, we can define an ordering $\tilde{R}$ on $\mathbb{R}^n$ such that, $\forall u, v \in \hat{\mathscr{U}}$, $uRv \Leftrightarrow (u_1^1, \dots, u_n^n)\tilde{R}(v_1^1, \dots, v_n^n)$. We want to show that, $\forall u, v \in \mathbb{R}^{n^2}$, $uRv \Leftrightarrow \left(\sum_{i \in N} \alpha_1^i u_1^i, \dots, \sum_{i \in N} \alpha_n^i u_n^i\right)$ $\tilde{R}$ $\left(\sum_{i \in N} \alpha_1^i v_1^i, \dots, \sum_{i \in N} \alpha_n^i v_n^i\right)$, where the weights $\alpha_1^1, \alpha_2^1, \dots, \alpha_n^n$ are the same as in Step 2. Consider any $u, v \in \mathbb{R}^{n^2}$. We can find $w, z \in \hat{\mathscr{U}}$ such that $w_j^j = \sum_{i \in N} \alpha_j^i u_j^i$ and $z_j^j = \sum_{i \in N} \alpha_j^i v_j^i \ \forall j \in N$. From Step 2, uIw and vIz follow. The transitivity of R gives $uRv \Leftrightarrow wRz$. Therefore, by the same argument as in the proof of Lemma 2, we have $uRv \Leftrightarrow \left(\sum_{i \in N} \alpha_1^i u_1^i, \dots, \sum_{i \in N} \alpha_n^i u_n^i\right)\tilde{R}\left(\sum_{i \in N} \alpha_1^i v_1^i, \dots, \sum_{i \in N} \alpha_n^i v_n^i\right)$. ■

From Lemma 3, there exists $(\alpha_1^1, \alpha_2^1, \dots, \alpha_n^n) \in \mathbb{R}_{++}^{n^2}$ such that $\sum_{i \in N} \alpha_j^i = 1$ and $\alpha_j^j > \alpha_j^k = \alpha_j^l \ \forall j \in N$ $\forall k, l \neq j$ and the ordering $\tilde{R}$ is well-defined on $\mathbb{R}^n$. Since R satisfies SP, and CON, it is easily checked that $\tilde{R}$ inherits the properties P2 and P3 in A.1. We will show that $\tilde{R}$ also satisfies the property P1. Consider any utility vectors, $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z} \in \mathbb{R}^n$ such that

$$\tilde{w}_j = \tilde{u}_j + \beta_j \text{ and } \tilde{z}_j = \tilde{v}_j + \beta_j \text{ for some } \beta_j \in \mathbb{R} \ \forall j \in N. \tag{1}$$

We want to show $\tilde{u}\tilde{R}\tilde{v} \Leftrightarrow \tilde{w}\tilde{R}\tilde{z}$. Now, we can find n^2-dimensional vectors $u, v \in \mathbb{R}^{n^2}$ such that

$$\tilde{u}_j = \textstyle\sum_{i \in N} \alpha_j^i u_j^i \text{ and } \tilde{v}_j = \sum_{i \in N} \alpha_j^i v_j^i \ \forall j \in N. \tag{2}$$

By the definition of $\tilde{R}$, $\tilde{u}\tilde{R}\tilde{v} \Leftrightarrow uRv$. Next, suppose that there exist $w, z \in \mathbb{R}^{n^2}$ such that

$$w_j^i = u_j^i + t_i \text{ and } z_j^i = v_j^i + t_i \text{ for some } t_i \in \mathbb{R} \ \forall i \in N \ \forall j \in N, \tag{3}$$

$$\tilde{w}_j = \textstyle\sum_{i \in N} \alpha_j^i w_j^i \text{ and } \tilde{z}_j = \sum_{i \in N} \alpha_j^i z_j^i \ \forall j \in N. \tag{4}$$

Then, we could obtain $uRv \Leftrightarrow wRz$ by TSF and $wRz \Leftrightarrow \tilde{w}\tilde{R}\tilde{z}$ by the definition of $\tilde{R}$, and thus, combining the equivalence assertions, we could have $\tilde{u}\tilde{R}\tilde{v} \Leftrightarrow \tilde{w}\tilde{R}\tilde{z}$ as desired. We now show that such profiles w and z

certainly exist. Substituting the equations in the condition (2) into those in (1), we have

$$\tilde{w}_j = \sum_{i\in N} \alpha_j^i u_j^i + \beta_j \text{ and } \tilde{z}_j = \sum_{i\in N} \alpha_j^i v_j^i + \beta_j \;\forall j \in N. \tag{5}$$

Next, substituting the equations in (3) and (5) into those in (4), the existence of w and z can be equivalently restated as the existence of solution in the following system of equations in matrix form;

$$A\begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}, \text{ where } A = \begin{pmatrix} \alpha_1^1 & \alpha_1^2 & \cdots & \alpha_1^n \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_n^1 & \alpha_n^2 & \cdots & \alpha_n^n \end{pmatrix}.$$

Since we have $\alpha_i^i > \alpha_i^j = \alpha_i^k > 0 \;\forall j,k \in N\backslash\{i\}$ in each i_{th} row of A, we obtain

$$\det(A) = \begin{vmatrix} \alpha_1^1 & \alpha_1^2 - \alpha_1^1 & \cdots & \alpha_1^n - \alpha_1^1 \\ \alpha_2^1 & \alpha_2^2 - \alpha_2^1 & \cdots & \alpha_2^n - \alpha_2^1 \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_n^1 & \alpha_n^2 - \alpha_n^1 & \cdots & \alpha_n^n - \alpha_n^1 \end{vmatrix} = \begin{vmatrix} Y & 0 & \cdots & 0 \\ 0 & \alpha_2^2 - \alpha_2^1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \alpha_n^n - \alpha_n^1 \end{vmatrix}$$

where $Y = \alpha_1^1 - \frac{\alpha_2^1(\alpha_1^2 - \alpha_1^1)}{\alpha_2^2 - \alpha_2^1} - \cdots - \frac{\alpha_n^1(\alpha_1^n - \alpha_1^1)}{\alpha_n^n - \alpha_n^1} > 0$. Hence, we have $\det(A) \neq 0$, which ensures that we can find a solution in the above system. Therefore, $\tilde{R}$ satisfies P1 as well as P2 and P3. By the same argument as in A.1, there exists $(\gamma_1, \ldots, \gamma_n) \in \mathbb{R}_+^n \backslash \{\mathbf{0}\}$ such that, $\forall \tilde{u}, \tilde{v} \in \mathbb{R}^n$, $\tilde{u}\tilde{R}\tilde{v} \Leftrightarrow \sum_{i\in N} \gamma_i \tilde{u}_i \geq \sum_{i\in N} \gamma_i \tilde{v}_i$. Then, we obtain, $\forall u, v \in \mathbb{R}^{n^2}$,

$$\begin{aligned} uRv &\Leftrightarrow \sum_{j\in N} \gamma_j \Big(\sum_{i\in N} \alpha_j^i u_j^i\Big) \geq \sum_{j\in N} \gamma_j \Big(\sum_{i\in N} \alpha_j^i v_j^i\Big) \\ &\Leftrightarrow \sum_{(i,j)\in N^2} \zeta_j^i u_j^i \geq \sum_{(i,j)\in N^2} \zeta_j^i v_j^i, \end{aligned}$$

where $\zeta_j^i = \gamma_j \cdot \alpha_j^i \;\forall (i,j) \in N^2$. By IAPCI and AN, it must be that $\zeta_i^i = \zeta_j^j > \zeta_j^i = \zeta_l^k > 0 \;\forall i,j,k,l \in N$ ($i \neq j$ and $k \neq l$). ■

A.3. An ESWO ($n \geq 3$) that satisfies IEPCO, TSF, SP, AN, CON, and SE_R must be the Kolm-Pollak of means ordering.

Proof. The proof proceeds through the following lemma.

Lemma 4. *If an ESWO R satisfies IEPCO, then there exists an ordering $\bar{R}$ defined on $\mathbb{R}^n$ such that, $\forall u, v \in \mathbb{R}^{n^2}$,*

$$uRv \Leftrightarrow \Big(\tfrac{1}{n}\sum_{i\in N} u_1^i, \ldots, \tfrac{1}{n}\sum_{i\in N} u_n^i\Big)\bar{R}\Big(\tfrac{1}{n}\sum_{i\in N} v_1^i, \ldots, \tfrac{1}{n}\sum_{i\in N} v_n^i\Big).$$

Proof of Lemma 4. Fix $m \in N$ arbitrarily. Applying the same argument as in the proof of Theorem 10 in Blackorby et al. [1], it can be shown that we obtain uIv for any $u, v \in \mathbb{R}^{n^2}$ such that $\frac{1}{n}\sum_{i\in N} u^i_m = \frac{1}{n}\sum_{i\in N} v^i_m$, and $u^i_j = v^i_j \; \forall i \in N \; \forall j \in N \backslash \{m\}$. We omit the rest of the proof that is similar to Steps 2 and 3 in the proof of Lemma 3. ■

From Lemma 4, there exists the ordering $\bar{R}$ on $\mathbb{R}^n$. Since R satisfies CON, $\bar{R}$ inherits the property P3 in A.1. Because of TSF, SP, AN, and $\mathrm{SE_R}$, it is easily checked that $\bar{R}$ also inherits the following properties P4-P7,

P4: $\forall u, v, w, z \in \mathbb{R}^n$, if there exists $\beta \in \mathbb{R}$ such that $w_i = \beta + u_i$ and $z_i = \beta + v_i \; \forall i \in N$, then $u\bar{R}v \Leftrightarrow w\bar{R}z$;
P5: $\forall u, v \in \mathbb{R}^n$, if $u_i \geq v_i \; \forall i \in N$ then $u\bar{R}v$, moreover, if there exists $j \in N$ such that $u_j > v_j$, then $u\bar{P}v$;
P6: $\forall u, v \in \mathbb{R}^n$, if there exists a permutation η on N such that $u_i = v_{\eta(i)} \; \forall i \in N$, then $u\bar{I}v$;
P7: $\forall u, v, w, z \in \mathbb{R}^{n^2}$, if we have $[u_i = v_i$ and $w_i = z_i \; \forall i \in M \subseteq N]$, and $[u_j = w_j$ and $v_j = z_j \; \forall j \in N\backslash M]$, then $u\bar{R}v \Leftrightarrow w\bar{R}z$.
Applying Theorem 13.7 in Bossert and Weymark [2], $\bar{R}$ must be the Kolm-Pollak ordering, i.e. it belongs to the following class (one for each $\rho \in \mathbb{R}$); $\forall u, v \in \mathbb{R}^n$,

$$u\bar{R}v \Leftrightarrow \begin{cases} \rho \neq 0: & \frac{1}{\rho}\ln\left(\sum_{j\in N}\exp(\rho u_j)\right) \geq \frac{1}{\rho}\ln\left(\sum_{j\in N}\exp(\rho v_j)\right), \\ \rho = 0: & \sum_{j\in N} u_j \geq \sum_{j\in N} v_j. \end{cases} \quad ■$$

A.4. An ESWO that satisfies IEPCO, SP, AN, $\mathrm{SE^P}$ must be the extensive utilitarian ordering.

Proof. In view of Lemma 4, we only have to examine the ordering $\bar{R}$ defined in the lemma. It is well known that if an ordering $\bar{R}$ on $\mathbb{R}^n$ satisfies the properties P1 in A.1, and P5 and P6 in A.3, then it must be utilitarianism; $\forall u, v \in \mathbb{R}^n$, $u\bar{R}v \Leftrightarrow \sum_{i\in N} u_i \geq \sum_{i\in N} v_i$ (see, for example, Bossert and Weymark [2]). Thus, it is sufficient to show that $\bar{R}$ satisfies these properties. The properties P5 and P6 are straightforward. We prove that $\bar{R}$ satisfies the property P1. Consider any vectors, $\bar{u}, \bar{v}, \bar{w}, \bar{z} \in \mathbb{R}^n$, that satisfy the antecedent of P1. We want to show $\bar{u}\bar{R}\bar{v} \Leftrightarrow \bar{w}\bar{R}\bar{z}$. Fix some individual $m \in N$, and consider $u, v \in \mathbb{R}^{n^2}$ such that $u^m_j = v^m_j \; \forall j \in N$ and

$$\frac{1}{n}\sum_{i\in N} u^i_j = \bar{u}_j \text{ and } \frac{1}{n}\sum_{i\in N} v^i_j = \bar{v}_j \; \forall j \in N. \tag{6}$$

By the definition of $\bar{R}$, $\bar{u}\bar{R}\bar{v} \Leftrightarrow uRv$. Next, we consider $w, z \in \mathbb{R}^{n^2}$ such that

$$w_j^m = u_j^m + n \cdot \beta_j \text{ and } z_j^m = v_j^m + n \cdot \beta_j \ \forall j \in N; \tag{7}$$

$$w_j^i = u_j^i \text{ and } z_j^i = v_j^i \ \forall i \in N \backslash \{m\} \ \forall j \in N. \tag{8}$$

By SE$^{\mathrm{P}}$, $uRv \Leftrightarrow wRz$. Notice that, from the conditions (6), (7), and (8), we have

$$\tfrac{1}{n} \textstyle\sum_{i \in N} w_j^i = \bar{w}_j \text{ and } \tfrac{1}{n} \sum_{i \in N} z_j^i = \bar{z}_j \ \forall j \in N.$$

Thus, by the definition of $\bar{R}$, $wRz \Leftrightarrow \bar{w}\bar{R}\bar{z}$. Combining the equivalence assertions, we obtain $\bar{u}\bar{R}\bar{v} \Leftrightarrow \bar{w}\bar{R}\bar{z}$. ■

Appendix B: Independence of the axioms

We establish the independence of the axioms. We omit easy proofs to show that rules provided below satisfy the axioms except for one.

B.1. The utilitarian ordering

- Dropping APCI: consider the anti-paternalistic extensive utilitarian ordering.
- Dropping TSF: let $\rho \in \mathbb{R} \backslash \{0\}$, and consider the following rule; $\forall u, v \in \mathbb{R}^{n^2}$,

$$uRv \Leftrightarrow \tfrac{1}{\rho} \ln \Big(\textstyle\sum_{i \in N} \exp(\rho u_i^i) \Big) \geq \tfrac{1}{\rho} \ln \Big(\sum_{i \in N} \exp(\rho v_i^i) \Big).$$

- Dropping WP: consider the following rule; $\forall u, v \in \mathbb{R}^{n^2}$, $uRv \Leftrightarrow \sum_{i \in N} \sum_{j \neq i} u_j^i \leq \sum_{i \in N} \sum_{j \neq i} v_j^i$.
- Dropping AN: consider the utilitarian ordering with non-symmetric weights.
- Dropping CON: consider the following rule; $\forall u, v \in \mathbb{R}^{n^2}$,

$$\begin{cases} uPv \Leftrightarrow (i) \ \sum_{i \in N} u_i^i > \sum_{i \in N} v_i^i, \text{ or} \\ \qquad\quad (ii) \ \sum_{i \in N} u_i^i = \sum_{i \in N} v_i^i \text{ and } \sum_{i \in N} \sum_{j \neq i} u_j^i > \sum_{i \in N} \sum_{j \neq i} v_j^i, \\ uIv \Leftrightarrow \text{otherwise}. \end{cases}$$

B.2. The anti-paternalistic extensive utilitarian orderings

- Dropping IAPCI: consider the Kolm-Pollak of means ordering.
- Dropping TSF: let $\rho \in \mathbb{R} \backslash \{0\}$ and $(\alpha_1^1, \alpha_2^1, \ldots, \alpha_n^n) \in \mathbb{R}_{++}^{n^2}$ with $\alpha_i^i = \alpha_j^j > \alpha_j^i = \alpha_l^k \ \forall i, j, k, l \in N$ $(i \neq j$

and $k \neq l$), and consider the following rule; $\forall u, v \in \mathbb{R}^{n^2}$,

$$uRv \Leftrightarrow \frac{1}{\rho} \ln\left(\sum_{j\in N} \exp(\rho\,\tilde{u}_j)\right) \geq \frac{1}{\rho} \ln\left(\sum_{j\in N} \exp(\rho\,\tilde{v}_j)\right),$$

where $\tilde{u}_j = \sum_{i\in N} \alpha^i_j u^i_j$ and $\tilde{v}_j = \sum_{i\in N} \alpha^i_j v^i_j$ $\forall j \in N$.

• Dropping SP: consider the utilitarian ordering.

• Dropping AN: consider the extensive utilitarian ordering with the strictly positive weights such that (i) $\alpha^i_i = \alpha^j_j$ $\forall i, j \in N$; (ii) $\alpha^i_i > \alpha^i_j$ $\forall i \in N$ $\forall j \neq i$; (iii) $\alpha^i_j \neq \alpha^k_l$ for some $i, j, k, l \in N$ ($i \neq j$ and $k \neq l$).

• Dropping CON: consider the following rule; $\forall u, v \in \mathbb{R}^{n^2}$,

$$\begin{cases} uPv \Leftrightarrow (i)\ \sum_{(i,j)\in N^2} \alpha^i_j u^i_j > \sum_{(i,j)\in N^2} \alpha^i_j v^i_j, \text{ or} \\ \qquad\qquad (ii)\ \sum_{(i,j)\in N^2} \alpha^i_j u^i_j = \sum_{(i,j)\in N^2} \alpha^i_j v^i_j \text{ and } \sum_{i\in N} \alpha^i_i u^i_i > \sum_{i\in N} \alpha^i_i v^i_i, \\ uIv \Leftrightarrow \text{otherwise}, \end{cases}$$

where $\alpha^i_i = \alpha^j_j > \alpha^i_j = \alpha^k_l > 0$ $\forall i, j, k, l \in N$ ($i \neq j$ and $k \neq l$).

• Dropping SE_R: let $(\alpha^1_1, \alpha^1_2, \ldots, \alpha^n_n) \in \mathbb{R}^{n^2}_{++}$ with $\alpha^i_i = \alpha^j_j > \alpha^i_j = \alpha^k_l$ $\forall i, j, k, l \in N$ ($i \neq j$ and $k \neq l$), and also $\gamma \in \left(0, \frac{\alpha^i_i - \alpha^j_i}{\alpha^i_i - \alpha^j_i + 1}\right)$ where $i \neq j$,[16] and consider the following rule; $\forall u, v \in \mathbb{R}^{n^2}$,

$$uRv \Leftrightarrow \sum_{i\in N} \left((1-\gamma) \sum_{j\in N} \alpha^i_j u^i_j + \gamma \min_{j\in N} u^i_j\right) \geq \sum_{i\in N} \left((1-\gamma) \sum_{j\in N} \alpha^i_j v^i_j + \gamma \min_{j\in N} v^i_j\right).$$

B.3. The Kolm-Pollak of means orderings

• Dropping IEPCO: consider the anti-paternalistic extensive utilitarian ordering.

• Dropping TSF: consider the case where the ordering $\bar{R}$ in Lemma 4 is the following rule; $\forall \bar{u}, \bar{v} \in \mathbb{R}^n$,

$$\bar{u}\bar{R}\bar{v} \Leftrightarrow \sum_{i\in N} \exp(\exp(\bar{u}_i)) \geq \sum_{i\in N} \exp(\exp(\bar{v}_i)).$$

• Dropping SP: consider the following rule; $\forall u, v \in \mathbb{R}^{n^2}$, $uRv \Leftrightarrow \sum_{(i,j)\in N^2} u^i_j \leq \sum_{(i,j)\in N^2} v^i_j$.

• Dropping AN: consider the case where the ordering $\bar{R}$ in Lemma 4 is the utilitarian ordering with non-symmetric weights.

• Dropping CON: consider the case where the ordering $\bar{R}$ in Lemma 4 is the leximin rule.

• Dropping SE_R: let $\gamma \in (0, 1)$, and consider the case where the ordering $\bar{R}$ in Lemma 4 is the following

[16]The condition that $\gamma < \frac{\alpha^i_i - \alpha^j_i}{\alpha^i_i - \alpha^j_i + 1}$ ensures this rule satisfies IAPCI.

rule; $\forall \bar{u}, \bar{v} \in \mathbb{R}^n$,

$$\bar{u}\bar{R}\bar{v} \Leftrightarrow (1-\gamma)\sum_{i\in N} \bar{u}_i + \gamma \min_{i\in N} \bar{u}_i \geq (1-\gamma)\sum_{i\in N} \bar{v}_i + \gamma \min_{i\in N} \bar{v}_i.$$

B.4. The extensive utilitarian ordering

• Dropping IEPCO: consider the anti-paternalistic extensive utilitarian ordering.

• Dropping SP: consider the following rule; $\forall u, v \in \mathbb{R}^{n^2}$, $uRv \Leftrightarrow \sum_{(i,j)\in N^2} u^i_j \leq \sum_{(i,j)\in N^2} v^i_j$.

• Dropping AN: consider the case where the ordering $\bar{R}$ in Lemma 4 is the utilitarian ordering with non-symmetric weights.

• Dropping $\mathrm{SE^P}$: consider the Kolm-Pollak of means ordering ($\rho \neq 0$).

References

[1] Blackorby C, Bossert W, Donaldson D (2002) Utilitarianism and the theory of justice. In: Arrow KJ, Sen AK, Suzumura K (eds) Handbook of social choice and welfare vol. I, North-Holland, Amsterdam, pp 543-596

[2] Bossert W, Weymark JA (2004) Utility in social choice. In: Barberà S, Hammond PJ, Seidl C (eds) Handbook of utility theory vol. II. extensions, Kluwer, Dordrecht, pp 1099-1177

[3] d'Aspremont C, Gevers L (2002) Social welfare functionals and interpersonal comparability. In: Arrow KJ, Sen AK, Suzumura K (eds) Handbook of social choice and welfare vol. I, North-Holland, Amsterdam, pp 459-541

[4] Gaertner W (1992) Distributive judgments. In Gaertner W, Klemisch-Ahlert (eds) Social choice and bargaining perspectives on distributive justice, Springer-Verlag, Berlin, pp 17-60

[5] Hammond PJ (1979) Equity in two person situations: some consequences. Econometrica 47: 1127-1135

[6] Ooghe E (2004) Two egalitarian consensus rules. Soc Choice Welf 22: 567-578

[7] Ooghe E, Lauwers L (2005) Non-dictatorial extensive social choice. Econ Theory 25: 721-743

[8] Ooghe E, Schokkaert E, Van de gaer D (2007) Equality of opportunity versus equality of opportunity sets. Soc Choice Welf 28: 209-230

[9] Roberts KWS (1980) Possibility theorems with interpersonal comparable welfare levels. Rev Econ Stud 47: 409-420

[10] Roberts KWS (1995) Valued opinions or opinionated values: the double aggregation problem. In: Basu K, Pattanaik P, Suzumura K (eds) Choice, welfare, and development: a festschrift in honor of Amartya Sen, Oxford University Press, Oxford, pp 141-165

[11] Sen AK (1970) Collective choice and social welfare. Holden-Day, Amsterdam

[12] Sen AK (1977) On weights and measures: informational constraints in social welfare analysis. Econometrica 45: 1539-1572

[13] Suppes P (1966) Some formal models of grading principles. Synthese 6: 284-306

[14] Suzumura K (1996) Interpersonal comparisons of the extended sympathy type and the possibility of social choice. In: Arrow KJ, Sen AK, Suzumura K (eds) Social choice re-examined vol. II, Macmillan, London, pp 202-229